AN EXPLICIT FORMULA FOR THE LODAY ASSEMBLY

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Dedicated to Mathematics,

my one true love.

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Finally, one parting thought—I know from experience that the only people who will ever read through this dissertation in the years to come are depressed graduate students. They wander over to the internet and library to find out how things were like for their predecessors. Believe me, I know. With this in mind, I feel obliged to admit that the most important result of this dissertation was obtained at the beginning of my final year. Therefore, please remember that things are not hopeless! Working in a chaotic place is not necessarily so bad! Miracles can happen!

One final thought: I am free!!!

TABLE OF CONTENTS

ABSTRACT

Chan, Virgil Ph.D., Purdue University, May 2020. An Explicit Formula for the Loday Assembly. Major Professor: Daniel A. Ramras.

We give an explicit description of the Loday assembly map on homotopy groups when restricted to a subgroup coming from the Atiyah-Hirzebruch spectral sequence. This proves and generalises a formula about the Loday assembly map on the first homotopy group that originally appeared in work of Waldhausen. Furthermore, we show that the Loday assembly map is injective on the second homotopy groups for a large class of integral group rings. Finally, we show that our methods can be used to compute the universal assembly map on homotopy.

1. INTRODUCTION

Classically, algebraic K-theory deals with three Abelian groups $K_0(R)$, $K_1(R)$ and $K_2(R)$ associated to a ring R. See [Ros95] for their definitions. For a topological space X, topologists have found geometric applications of the groups $K_i(\mathbb{Z}[\pi_1(X)])$ for $i = 0, 1, 2$, where $\mathbb{Z}[\pi_1(X)]$ is the integral group ring of the fundamental group of X. For example, an important application of K_0 was found by Wall [Wal65]. He defined an element $\chi(X)$ in a quotient group $\mathrm{Wh}_0(\pi_1(X))$ of $K_0(\mathbb{Z}[\pi_1(X)])$, known as *Wall's finiteness obstruction*, and showed that a sufficiently nice X is homotopy equivalent to a finite CW-complex if and only if $\chi(X) = 0$. (See [Ros05, Theorem 1 on page 579] for the precise statement.) The second example, which is due to Whitehead, is an attempt to classify manifolds of dimension at least five. In a collection of work [Whi39, Whi41, Whi50], he defined the *Whitehead torsion* $\tau(f) \in \mathrm{Wh}_1(\pi_1(X))$ of a continuous map f, where $\mathrm{Wh}_1(\pi_1(X))$ is a quotient of $K_1(\mathbb{Z}[\pi_1(X)])$. In particular, the vanishing of the Whitehead torsion allowed Smale to prove the Poincaré Conjecture in dimensions greater than four [Sma61].

For an ideal $I \subseteq R$, there is a natural exact sequence

$$\begin{tikzcd}
K_2(R) \arrow[r] & K_2(R/I) \arrow[r] & K_1(R,I) \arrow[r] & K_1(R) \arrow[dll, out=0, in=180] \\
 & K_1(R/I) \arrow[r] & K_0(R,I) \arrow[r] & K_0(R) \arrow[dll, out=0, in=180] \\
 & K_0(R/I) & &
\end{tikzcd} \tag{1.1}$$

(see [Ros95, Theorem 4.3.1 on page 200]), and a lot of effort went into the search for definitions for $K_i(R)$ with $i \geq 3$ to extend this exact sequence to the left. The first widely accepted definition was due to Quillen [Qui72, Qui73], for which he was awarded the Fields Medal in 1978. His idea was that one should try to construct the groups $K_i(R)$ not one at a time but all at once, as homotopy groups

$$K_i(R) = \pi_i(K(R)),$$

of a certain topological space $K(R)$ constructed to reflect structures of the category of finitely generated projective R-modules. It was later discovered by Gersten and Wagoner that the space $K(R)$ is an infinite loop space [Ger72,Wag72], and hence is the 0-th space of an Ω-spectrum $\mathbb{K}_R^{\mathrm{GW}}$ (see Definition 2.3.2), making algebraic K-theory part of stable homotopy theory. Several definitions of algebraic K-theory came out to account for this property, notably Waldhausen's $S_\bullet$-construction [Wal78c]. Nowadays, people understand algebraic K-theory as a machine that takes in nice categorical data to produce Ω-spectra, or more precisely, a functor

$$\mathbb{K} : \mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \to \Omega\text{-}\mathcal{S}\mathrm{pectra} \tag{1.2}$$

from symmetric monoidal categories to Ω-spectra such that the homotopy groups

$$\pi_i\left(\mathbb{K}\left(\mathcal{P}\mathrm{roj}_R^{\mathrm{fg}}\right)\right)$$

of the Ω-spectrum, obtained by evaluating at the category $\mathcal{P}\mathrm{roj}_R^{\mathrm{fg}}$ of finitely generated projective R-modules, recover the classical K-groups of R for $i = 0, 1, 2$ and the higher K-groups defined by Quillen.

However, the general consensus is that computing homotopy groups is difficult. Hence, investigating algebraic K-theory of rings is not an easy task. On the other hand, homology is more accessible—there are excisions, the Mayer-Vietoris sequence, and even spectral sequences in homology that are more user-friendly than they would be in homotopy. This is the story of assembly—to approximate homotopy theory by a generalised homology theory.

1.1 History and Motivations

In his dissertation [Lod76], Loday defined a map

$$\alpha_{\text{Loday}} : BG_+ \wedge \mathbb{K}_R^{\text{GW}} \to \mathbb{K}_{R[G]}^{\text{GW}} \tag{1.3}$$

of spectra, which is now known as the Loday assembly (see Definition 3.1.2), to unify the classical Whitehead groups $\text{Wh}_i(G)$ for $i = 0, 1, 2$ studied by Wall [Wal65], Whitehead [Whi39, Whi41, Whi50], and Hatcher-Wagoner [HW73]. These groups are isomorphic to the cokernel

$$\text{Wh}_i(G) \cong \text{coker}\left(\pi_i(\alpha_{\text{Loday}}) : \pi_i\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\text{GW}}\right) \to K_i(\mathbb{Z}[G])\right) \tag{1.4}$$

for $i = 0, 1, 2$. See [Lod76, page 357–364], or Corollary 3.3.2 for the case $i = 1$. Before this work appeared, Wall defined a version of assembly

$$A_i : H_i(BG; \mathbb{Q}) \to L_i(\mathbb{Z}[G]) \otimes \mathbb{Q} \tag{1.5}$$

for L-theory. Furthermore, he showed that the injectivity of A_i implies the classical Novikov Conjecture about homotopy invariance of the higher signature of a closed, oriented manifold M when $G = \pi_1(M)$ [Wal70, 17H]. The converse is also true [KM81], and therefore the classical Novikov Conjecture is equivalent to the injectivity of the map A_i for all i. Motivated by this, Hsiang cast the following conjecture in his 1984 ICM address:

Conjecture 1.1.1 (K-theoretic Novikov Conjecture, [Hsi84]) *Let R be a regular ring, and G be a torsion-free group. Then the map*

$$\pi_i(\alpha_{\text{Loday}}) \otimes \text{id}_{\mathbb{Q}} : \pi_i(BG_+ \wedge \mathbb{K}_R^{\text{GW}}) \otimes \mathbb{Q} \to K_i(R[G]) \otimes \mathbb{Q} \tag{1.6}$$

is injective for all i.

Attempts at answering the K-theoretic Novikov Conjecture 1.1.1 often lead to the creation of new mathematics, most notably the Topological Cyclic Homology, which allowed Bökstedt-Hsiang-Madsen to prove the following:

Theorem 1.1.1 ([BHM93, Theorem 9.13 on page 535]) *The K-theoretic Novikov Conjecture 1.1.1 is true if the homology groups $H_i(BG; \mathbb{Z})$ are all finitely generated.*

A stronger statement is also possible:

Conjecture 1.1.2 (Classical Farrell-Jones Conjecture) *Let R be a regular ring, and G be a torsion-free group. Then the map*

$$\pi_i(\alpha_{\text{Loday}}) : \pi_i(BG_+ \wedge \mathbb{K}_R^{\text{GW}}) \to K_i(R[G]) \tag{1.7}$$

is an isomorphism for all i.

This conjecture can be understood as a conceptual approach to computing algebraic K-theory of a group ring via a homology theory, but it has important implications for a range of topics, notably the Borel Conjecture concerning the topological rigidity of closed spherical manifolds, and the Kaplansky Conjecture about the idempotents in a group ring.

The following case has been verified:

Theorem 1.1.2 ([BLR08, Theorem 1.1 (i) on page 58]) *The Classical Farrell-Jones Conjecture 1.1.2 is true if the group G is word-hyperbolic.*

The work presented here is motivated by the Classical Farrell-Jones Conjecture 1.1.2.

1.2 What Are We Trying to Do?

The purpose of this dissertation is to help the author to obtain his doctoral degree by studying the Loday assembly. More precisely, the source of the Loday assembly

$$\alpha_{\mathrm{Loday}} : BG_+ \wedge \mathbb{K}_R^{\mathrm{GW}} \to \mathbb{K}_{R[G]}^{\mathrm{GW}}$$

represents a generalised homology theory. Hence, its homotopy groups can be computed by the Atiyah-Hirzebruch spectral sequence

$$\begin{aligned} E^2_{p,q} &\cong H_p(BG_+; K_q(R)) \\ &\Rightarrow \pi_{p+q}\left(BG_+ \wedge \mathbb{K}_R^{\mathrm{GW}}\right). \end{aligned}$$

From the construction of the spectral sequence, we know there is a subgroup $E^\infty_{1,i}$ of the homotopy group $\pi_{i+1}\left(BG_+ \wedge \mathbb{K}_R^{\mathrm{GW}}\right)$ coming from the 1-skeleton of the classifying space BG. We provide a formula for the restriction $\pi_{i+1}(\alpha_{\mathrm{Loday}})|_{E^\infty_{1,i}}$:

Theorem 1.2.1 (See Theorem 3.3.1 and Corollary 4.1.1 for the formula on π_2) *Let R be a ring and G be a group. For $i \geq 0$, the filler of the diagram*

$$\begin{tikzcd} E^2_{1,i} \arrow[d, two heads] & \cong & H_1(BG; K_i(R)) & \cong & G_{ab} \otimes K_i(R) \arrow[r, dashed, "\Phi_{i+1}"] & K_{i+1}\left(R[G]\right) \\ E^\infty_{1,i} \arrow[rrrrr, hook] & & & & & \pi_{i+1}\left(BG \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \arrow[u, "\pi_{i+1}\left(\alpha_{\mathrm{Loday}}\right)"'] \end{tikzcd} \tag{1.8}$$

is induced by the bilinear map

$$\begin{aligned} G \times K_i(R) &\to K_{i+1}(R[G]) \\ (g, [f]) &\mapsto \{g\} \star' [f], \end{aligned}$$

for which on the right-hand side, the element g is considered as an element in $GL(1, R[G])$, and $\star'$ is the extended Loday product (see Definition 2.2.1).

In particular, when $R = \mathbb{Z}$, the map

$$\Phi_2 : G_{ab} \otimes K_1(\mathbb{Z}) \to K_2(\mathbb{Z}[G]) \tag{1.9}$$

is induced by the bilinear map

$$\begin{aligned} G \times K_1(\mathbb{Z}) &\to K_2(\mathbb{Z}[G]) \\ (g, \pm 1) &\mapsto \{-1, g\}_{\mathrm{St}}\,, \end{aligned} \tag{1.10}$$

where $\{-1, g\}_{\mathrm{St}} \in K_2(\mathbb{Z}[G])$ is the Steinberg symbol of $\{-1\} \in K_1(\mathbb{Z})$ and $\{g\} \in K_1(\mathbb{Z}[G])$.

The derivation of our formula involves extending the original Loday pairing

$$\gamma_{\mathrm{Loday}} : BGL(R)^+ \wedge BGL(S)^+ \to BGL(R \otimes S)^+ \tag{1.11}$$

to the full K-theory space

$$\gamma'_{\mathrm{Loday}} : \left[K_0(R) \times BGL(R)^+\right] \wedge \left[K_0(S) \times BGL(S)^+\right] \to K_0(R \otimes S) \times BGL(R \otimes S)^+ \tag{1.12}$$

in a non-obvious way (see Definition 2.2.1). Here, the superscript "+" denotes Quillen's plus construction of the classifying space $BGL\,(R)$ relative to the subgroup $E(R)$ generated by elementary matrices. In particular,

(i) we relate the product map

$$\star' : K_i(R) \otimes K_j(S) \to K_{i+j}(R \otimes S)$$

induced by the extended pairing γ'_{Loday} to the classical product maps defined by Milnor in [Mil72] (see Theorem 2.3.1).

(ii) We recover (and generalise) a formula about the Loday assembly on π_1 written down in a survey article [LR05] by Lück-Reich (see Corollary 3.3.1).

(iii) We obtain injectivity results on π_2 for a large class of integral group rings (See Corollary 4.1.2).

(iv) We show that our formula can be used to compute the universal assembly in the sense of Weiss-Williams

$$\widehat{\alpha}_{BG} : BG_+ \wedge \mathbb{K}_R^{\mathrm{PW}} \to \mathbb{K}_{R[G]}^{\mathrm{PW}}$$

written in terms of the non-connective Pedersen-Weibel K-theory spectrum $\mathbb{K}_R^{\mathrm{PW}}$ of a ring R (see Theorem 5.6.1).

We elaborate more on item (ii). The assembly on fundamental groups

$$\pi_1\left(\widehat{\alpha}_{BG}\right) : \pi_1\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{PW}}\right) \to K_1(\mathbb{Z}[G]) \tag{1.13}$$

was first described by Waldhausen. It is induced by the bilinear map

$$\begin{aligned} \{\pm 1\} \times G &\to K_1(\mathbb{Z}[G]) \\ (\pm 1, g) &\mapsto \{\pm g\} \end{aligned} \tag{1.14}$$

under the identification

$$\begin{aligned} \pi_1\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{GW}}\right) &\cong K_1(\mathbb{Z}) \oplus H_1(G; K_0(\mathbb{Z})) \\ &\cong K_1(\mathbb{Z}) \oplus [G_{ab} \otimes K_0(\mathbb{Z})] \\ &\cong \{\pm 1\} \oplus G_{ab} \end{aligned} \tag{1.15}$$

resulting from the vanishing of differentials in the Atiyah-Hirzebruch spectral sequence [Wal78b, Assertion 15.8 on page 229]. On the left-hand side of Equation (1.14), we think of $\{\pm g\}$ as represented by $\pm g \in GL(1, \mathbb{Z}[G])$. A formula for arbitrary regular rings was later written down in the survey article by Lück-Reich [LR05, page 708].

To prove (1.13) is as given by (1.14), Waldhausen reformulates the Loday assembly in terms of Quillen's Q-construction, and interprets the source of the assembly as a generalised homology theory for simplicial sets. Then, he verifies the formula on the simplicial level. The work presented here provides a different approach to prove Waldhausen's formula by working in Loday's original setting—using Quillen's plus construction. This new approach allows us to get higher degree results, and in particular, an elegant formula on π_2 (see Corollary 4.1.1).

1.3 Organisation

This dissertation consists of five parts, including the introduction. In Chapter 2, we review the construction of the original Loday pairing map and extend it to the full K-theory space using the Gersten-Wagoner delooping. We then relate the induced product maps on the K-groups with the classical product maps defined by Milnor and use the extended Loday pairing to construct the non-connective Gersten-Wagoner Algebraic K-theory spectrum $\mathbb{K}_R^{\mathrm{GW}}$ of a ring R.

In Chapter 3, we use the extended Loday pairing to define the Loday assembly and identify the subgroup of the source that we are interested in by studying the Atiyah-Hirzebruch spectral sequence. It also contains the statement and the proof of our main result.

Chapter 4 deals with the injectivity problem for the Loday assembly on π_2. We study the extension problem for the second homotopy group of the source of the Loday assembly coming from the Atiyah-Hirzebruch spectral sequence. Then we provide injectivity results for a large class of integral group rings.

Chapter 5 is a quick summary of Sperber's work on proving the Loday assembly is the universal assembly [Spe04]. We show that our version of the Loday assembly can be used to compute the universal assembly on homotopy even with the non-obvious extension of the original Loday pairing. We begin by discussing a model for the universal assembly

$$\widehat{\alpha}_{BG} : BG_+ \wedge \mathbb{K}_R^{\mathrm{PW}} \to \mathbb{K}_{R[G]}^{\mathrm{PW}} \tag{1.16}$$

written in terms of the Pedersen-Weibel K-theory spectrum (see Definition 5.4.5). Then, we construct two intermediate spectra and their versions of assembly

$$\alpha_{\mathrm{proj}} : BG_+ \wedge \mathbb{K}_R^{\mathrm{proj}} \to \mathbb{K}_{R[G]}^{\mathrm{proj}} \tag{1.17}$$

$$\alpha_{\mathrm{free}} : BG_+ \wedge \mathbb{K}_R^{\mathrm{free}} \to \mathbb{K}_{R[G]}^{\mathrm{free}} \tag{1.18}$$

(see Definition 5.5.1), which allows us to move from a theoretically-friendly model for the assembly to a computation-friendly model. More specifically, there is zig-zag diagram

$$\begin{tikzcd}
BG_+ \wedge \mathbb{K}_R^{\mathrm{PW}} \arrow[r, "\widehat{\alpha}_{BG}"] \arrow[d, "\mathrm{id} \wedge (5.68)"'] & \mathbb{K}_{R[G]}^{\mathrm{PW}} \arrow[d, "(5.68)"] \\
BG_+ \wedge \mathbb{K}_R^{\mathrm{proj}} \arrow[r, "\alpha_{\mathrm{proj}}"] & \mathbb{K}_{R[G]}^{\mathrm{proj}} \\
BG_+ \wedge \mathbb{K}_R^{\mathrm{free}} \arrow[r, "\alpha_{\mathrm{free}}"'] \arrow[u, "\mathrm{id} \wedge (5.65)"] & \mathbb{K}_{R[G]}^{\mathrm{free}} \arrow[u, "(5.65)"']
\end{tikzcd} \tag{1.19}$$

of spectra that commutes up after passing to homotopy groups. We show that our formula for the restriction $\pi_{i+1}(\alpha_{\mathrm{Loday}})|_{E^{\infty}_{1,i}}$ can be used to compute the restriction $\pi_{i+1}(\alpha_{\mathrm{free}})|_{E^{\infty}_{1,i}}$ for $i \geq 2$ (see Theorem 5.6.1) and therefore, computes the universal assembly on higher homotopy when restricted onto the subgroup coming from the 1-skeleton of BG.

1.4 Notations and Conventions

1.4.1 Groups

For a group G, we write G_{ab} to be its *Abelianisation*. All groups in this article are discrete. We write BG to be the *classifying space* for G, constructed from the *bar construction*. Therefore, BG admits a canonical base-point. For $n \in \mathbb{N}$, we write C_n to be the cyclic group of order n. We write $\langle t \rangle$ to be the infinite cyclic group with generator t, while reserving $\mathbb{Z}$ to be the ring of integers.

1.4.2 Rings and Modules

All *rings* in this article are associative and unital, but not necessarily commutative. A *module* over a ring will always be understood as a left module.

We define the *cone* of the integers $\mathbb{Z}$ to be the ring $\operatorname{cone}(\mathbb{Z})$ of locally finite matrices over $\mathbb{Z}$, that is,

$$\operatorname{cone}(\mathbb{Z}) := \left\{ A \in M(\mathbb{Z}) \;\middle|\; \begin{array}{c} \text{Each column and each row of } A \\ \text{has only finitely many non-zero entries} \end{array} \right\}. \tag{1.20}$$

There is an ideal $m(\mathbb{Z})$ of $\operatorname{cone}(\mathbb{Z})$ consisting of infinite matrices with only finitely many non-zero entries. The *suspension of* $\mathbb{Z}$ is defined to be the quotient

$$\Sigma\mathbb{Z} := \frac{\operatorname{cone}(\mathbb{Z})}{m(\mathbb{Z})}. \tag{1.21}$$

We also define the *suspension of a ring* to be the tensor product

$$\Sigma R := \Sigma\mathbb{Z} \otimes R, \tag{1.22}$$

and inductively

$$\Sigma^0 R := R, \tag{1.23}$$

$$\Sigma^n R := \Sigma\left(\Sigma^{n-1} R\right). \tag{1.24}$$

Therefore, for any two rings R, S, and for any m, $n \in \mathbb{N} \cup \{0\}$, we have a natural isomorphism

$$\Sigma^m R \otimes \Sigma^n S \cong \Sigma^{m+n}(R \otimes S). \tag{1.25}$$

1.4.3 Topological Spaces and Spectra

In Chapter 2–4, the term *topological space* (or simply *space*) will always mean a pointed topological space that is homotopy equivalent to a (not necessary finite) CW-complex. Maps between spaces are always pointed and continuous. In Chapter 5, we will work with un-based topological spaces that are homotopy equivalent to (not necessary finite) CW-complexes. Maps between them are always continuous but not necessary pointed.

We say two maps $f, g : X \to Y$ of spaces are *weakly homotopic* if the restrictions $f|_K$, $g|_K$ are homotopic for every compact subset $K \subseteq X$. Weakly homotopic maps between infinite loop spaces induce the same group homomorphisms on homotopy.

By *spectrum*, we mean a sequence $\mathbb{E} = \{\mathbb{E}_i \mid i \in I\}$ of spaces, where $I = \mathbb{N} \cup \{0\}$ or $\mathbb{Z}$, together with maps

$$f_i : \mathbb{S}^1 \wedge \mathbb{E}_i \to \mathbb{E}_{i+1}$$

for each $i \in I$, called the *structure maps*. An Ω-*spectrum* is a spectrum for which the adjoints

$$\overline{f}_i : \mathbb{E}_i \to \Omega\mathbb{E}_{i+1}$$

of the structure maps are all homotopy equivalences. A *morphism* or a *map* (resp. *weak morphism* or a *weak map*) $\varphi : \mathbb{E} \to \mathbb{F}$ of spectra is a collection $\{\varphi_i : \mathbb{E}_i \to \mathbb{F}_i \mid i \in I\}$ of maps that commutes with the structure maps (resp. up to weak homotopy).

If $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$ are spectra, then a pairing (resp. weak pairing) $\mu : \mathbb{E} \wedge \mathbb{F} \to \mathbb{G}$ of spectra is a collection of maps $\mu_{m,n} : \mathbb{E}_m \wedge \mathbb{F}_n \to \mathbb{G}_{m+n}$ that commute with the structure maps up to homotopy (resp. weak homotopy). We abuse notation here to use the smash product $\mathbb{E} \wedge \mathbb{F}$ in writing down a pairing of spectra.

1.4.4 Algebraic K-theory

Let R be a ring. We write $K_0(R)$ to be *projective class group* of R, which is an Abelian group whose generators are isomorphism classes $[P]$ of finitely generated projective R-modules P, and whose relations are

$$[P_0] + [P_2] = [P_1] \tag{1.26}$$

for every short exact sequence

$$0 \to P_0 \to P_1 \to P_2 \to 0. \tag{1.27}$$

We write $BGL(R)^+$ to be *Quillen's plus construction* for the classifying space $BGL(R)$ for the general linear group $GL(R)$, relative to the subgroup $E(R)$ generated by elementary matrices. For finite matrices, our convention is that $BGL(p,R)^+$ denotes $BGL\,(p,R)$ for $p \leq 2$; for $p \geq 3$, it denotes the plus construction of $BGL\,(p,R)$ relative to the subgroup $E(p,R)$ generated by elementary matrices.

The *K-theory space* of R is

$$K_R := K_0(R) \times BGL(R)^+\,, \tag{1.28}$$

where we think of $K_0(R)$ as a discrete group. In particular, the homotopy groups $\pi_i(K_R)$ agree with the classical K-groups for $i = 0, 1, 2$, and with the higher K-groups defined by Quillen.

If $u \in GL(n, R)$, we write $\{u\} \in K_1(R)$ to be the class represented by u. If G is a group and $g \in G$, then we write $\{g\} \in K_1(R[G])$ to be the class represented by g, thinking of it as an element in $GL(1, R[G])$.

2. LODAY PAIRING

2.1 The Loday Pairing

Let R, S be rings, and fix isomorphisms

$$\xi_{m,n} : R^m \otimes S^n \to (R \otimes S)^{mn} . \tag{2.1}$$

for every $m, n \in \mathbb{N} \cup \{0\}$. In this setting, the tensor product of matrices gives a group homomorphism

$$GL\,(m, R) \times GL\,(n, S) \to GL\,(mn, R \otimes S)\,,$$

sending elementary matrices to elementary matrices. Hence we have an induced map

$$f^{R,S}_{m,n} : BGL(m, R)^+ \times BGL(n, S)^+ \to BGL(mn, R \otimes S)^+ . \tag{2.2}$$

The convention is that $BGL(p, R)^+$ denotes $BGL\,(p, R)$ for $p \leq 2$; for $p \geq 3$, it denotes the plus construction of $BGL\,(p, R)$ relative to the subgroup $E(p, R)$ generated by elementary matrices. Write

$$i_{mn} : BGL(mn, R \otimes S)^+ \to BGL(R \otimes S)^+$$

to be map induced by the canonical inclusion

$$GL\,(mn, R \otimes S) \to GL\,(R \otimes S)\,.$$

Definition 2.1.1 (The map $\gamma_{m,n}$, [Lod76, page 332]) *The map*

$$\gamma_{m,n} : BGL(m,R)^+ \times BGL(n,S)^+ \to BGL(R \otimes S)^+ \tag{2.3}$$

is defined by the formula:

$$\gamma_{m,n}(x,y) := i_{mn} \circ f^{R,S}_{m,n}(x,y) - i_{mn} \circ f^{R,S}_{m,n}(x,y_0) - i_{mn} \circ f^{R,S}_{m,n}(x_0,y)\,. \tag{2.4}$$

Here, the minus sign on the right-hand side comes from the H-group structure of $BGL(R \otimes S)^+$; and x_0 (resp. y_0) is the base-point in $BGL(m,R)^+$ (resp. $BGL(n,S)^+$). (I.e., represented by the identity matrices.)

One gets a map

$$\gamma : BGL(R)^+ \times BGL(S)^+ \to BGL(R \otimes S)^+ \tag{2.5}$$

by letting $m,n \to \infty$. If we choose a different collection

$$\left\{\xi'_{m,n} : R^m \otimes S^n \to (R \otimes S)^{mn} \;\middle|\; m,n \in \mathbb{N} \cup \{0\}\right\}$$

of isomorphisms, we get a different collection

$$\left\{\gamma'_{m,n} : BGL(m,R)^+ \times BGL(n,S)^+ \to BGL(R \otimes S)^+ \;\middle|\; m,n \in \mathbb{N} \cup \{0\}\right\}$$

of continuous maps, and therefore, a different map

$$\gamma' : BGL(R)^+ \times BGL(S)^+ \to BGL(R \otimes S)^+$$

by letting $m,n \to \infty$. It turns out that the maps γ and γ' are weakly homotopic, that is, the restrictions $\gamma|_K$ and $\gamma'|_K$ are homotopic for every compact subset $K \subseteq BGL(R)^+ \times BGL(S)^+$ (see [Lod76, Lemma 2.1.6 on page 333]), that is, γ and γ' are weakly homotopic. As a result, the choice of the collection $\{\xi_{m,n} \mid m, n \in \mathbb{N} \cup \{0\}\}$ becomes irrelevant after passing to homotopy groups.

The map γ is homotopy trivial on the wedge product $BGL(R)^+ \vee BGL(S)^+$. Hence, we obtain the following definition.

Definition 2.1.2 (The Loday pairing map γ_{Loday}, [Lod76, Section 2.1.7 on page 333]) *The map*

$$\gamma_{\text{Loday}} : BGL(R)^+ \wedge BGL(S)^+ \to BGL(R \otimes S)^+ \tag{2.6}$$

is defined to be the filler of the following homotopy commutative diagram:

$$\begin{tikzcd}
BGL(R)^+ \times BGL(S)^+ \arrow[r, "\gamma"] \arrow[d, "\text{proj}"'] & BGL(R \otimes S)^+ \\
BGL(R)^+ \wedge BGL(S)^+. \arrow[ur, dashed, "\gamma_{\text{Loday}}"'] &
\end{tikzcd} \tag{2.7}$$

This allows us to define a multiplicative structure on algebraic K-theory.

Definition 2.1.3 (Loday product $\star$, [Lod76, Section 2.1.10 on page 335]) *For each integers $i, j \geq 1$, we define the product*

$$\star : K_i(R) \otimes K_j(S) \to K_{i+j}(R \otimes S) \tag{2.8}$$

by

$$[f] \star [g] := [\gamma_{\text{Loday}} \circ (f \wedge g)] , \tag{2.9}$$

where f and g are spheroids

$$f : \mathbb{S}^i \to BGL(R)^+ ,$$
$$g : \mathbb{S}^j \to BGL(S)^+ .$$

Proposition 2.1.1 (Properties of the Loday Product $\star$) *The product map $\star$ is*

(i) natural in R and S, associative and bilinear [Lod76, Theorem 2.1.11 on page 335];

(ii) graded commutative: for every $[f] \in K_i(R)$ and $[g] \in K_j(S)$,

$$[f] \star [g] = (-1)^{ij} ([g] \star [f]) \in K_{i+j} (R \otimes S) . \tag{2.10}$$

Here, we think of $[g] \star [f]$ as represented by the composition

$$\mathbb{S}^j \wedge \mathbb{S}^i \xrightarrow{g \wedge f} BGL(S)^+ \wedge BGL(R)^+ \xrightarrow{\gamma_{\text{Loday}}} BGL(S \otimes R)^+ \xrightarrow{\simeq} BGL (R \otimes S) , \tag{2.11}$$

where the last homotopy equivalence is induced by the natural isomorphism $S \otimes R \cong R \otimes S$ [Lod76, Theorem 2.1.12 on page 335];

(iii) when $i = j = 1$, the product $\{u\} \star \{v\}$ is the inverse to the Steinberg symbol:

$$\{u\} \star \{v\} = - \{u, v\}_{\text{St}} \in K_2 (R \otimes S) \tag{2.12}$$

[Lod76, Proposition 2.2.3 on page 337]. Here, we write the Abelian group $K_2(R \otimes S)$ additively.

2.2 The Extended Loday Pairing

We want to extend the Loday pairing map

$$\gamma_{\text{Loday}} : BGL(R)^+ \wedge BGL(S)^+ \to BGL(R \otimes S)^+$$

to a map

$$\gamma'_{\text{Loday}} : K_R \wedge K_S \to K_{R \otimes S},$$

where

$$K_R := K_0(R) \times BGL(R)^+ \,. \tag{2.13}$$

This will yield an extended external product

$$K_i(R) \otimes K_j(S) \to K_{i+j}(R \otimes S)$$

for all i, $j \geq 0$. Moreover, this external product should be coherent with the isomorphism

$$K_i(R) \xrightarrow{\cong} K_{i+1}(\Sigma R) \tag{2.14}$$

promised by the **Gersten-Wagoner delooping** [Ger72, Wag72]:

$$K_R \simeq \Omega K_{\Sigma R}. \tag{2.15}$$

At this point, we need the closed form of the isomorphism in Equation (2.14). Define the element $\tau \in GL\,(\Sigma\mathbb{Z}) = GL\,(1, \Sigma\mathbb{Z})$ by

$$\tau := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.16}$$

The isomorphism in Equation (2.14) is given by

$$\begin{cases} [f] \in K_i(R) & \mapsto & [f] \star \{\tau\} \text{ if } i \geq 1 \text{ ([Lod76, Corollary 2.3.6 on page 345])}, \\ [P] \in K_0(R) & \mapsto & [P]\#\{\tau\} := \{p \otimes \tau + (1-p) \otimes 1\} \text{ ([Lod76, page 328])}, \end{cases} \tag{2.17}$$

where p is the projection operator associated to the finitely generated projective R-module P. Equally important, the extended external product should be related to the classical products

$$\# : K_i(R) \otimes K_j(S) \to K_{i+j}(R \otimes S) \tag{2.18}$$

defined by Milnor [Mil72] for i, $j \geq 0$, and $i + j \leq 2$.

Definition 2.2.1 (The Extended Loday pairing γ'_{Loday}) *The map*

$$\gamma'_{\mathrm{Loday}} : K_R \wedge K_S \to K_{R \otimes S}$$

is defined to be the filler of the following homotopy commutative diagram:

$$\begin{tikzcd} K_R \wedge K_S \arrow[r, dashed, "\gamma'_{\mathrm{Loday}}"] \arrow[d, "\simeq"'] & K_{R\otimes S} \arrow[d, "\simeq"] \\ \Omega BGL(\Sigma R)^+ \wedge \Omega BGL(\Sigma S)^+ \arrow[r, "\Omega^2 \gamma_{\mathrm{Loday}}"'] & \Omega^2 BGL\big(\Sigma^2 (R \otimes S)\big)^+. \end{tikzcd} \tag{2.19}$$

The bottom arrow is given by sending the pair

$$\big(f : \mathbb{S}^1 \to BGL(\Sigma R)^+\big) \wedge \big(g : \mathbb{S}^1 \to BGL(\Sigma S)^+\big)$$

to the composition

$$\mathbb{S}^2 \cong \mathbb{S}^1 \wedge \mathbb{S}^1 \xrightarrow{f\wedge g} BGL(\Sigma R)^+ \wedge BGL(\Sigma S)^+ \xrightarrow{\gamma_{\mathrm{Loday}}} BGL\big(\Sigma^2 (R \otimes S)\big)^+ .$$

Furthermore, we define a new product map

$$\star' : K_i(R) \otimes K_j(S) \to K_{i+j}(R \otimes S) \tag{2.20}$$

for all $i, j \geq 0$ *by*

$$[f] \star' [g] := \left[\gamma'_{\mathrm{Loday}} \circ (f \wedge g)\right] . \tag{2.21}$$

Corollary 2.2.1 *Let* R *and* S *be rings. The pairing map*

$$\gamma'_{\mathrm{Loday}} : K_R \wedge K_S \to K_{R\otimes S}$$

defined in Definition 2.2.1 is

(i) natural in R *and* S*;*

(ii) associative;

up to weak homotopy.

The proof follows from the corresponding results for the pairing γ_{Loday} as in [Lod76, Proposition 2.1.8 on page 334]. The two products are related in the following way:

Proposition 2.2.1 *For each* $i, j \geq 1$*, if* $[f] \in K_i(R)$ *and* $[g] \in K_j(S)$*, then*

$$[f] \star' [g] = [f] \star \big((-1)^j [g]\big) = (-1)^j \left([f] \star [g]\right) . \tag{2.22}$$

In particular, for $\{u\} \in K_1(R)$ and $\{v\} \in K_1(S)$, the $\star'$-product is given by the Steinberg symbol

$$\{u\} \star' \{v\} = \{u, v\}_{\mathrm{St}} \,. \tag{2.23}$$

Proof. We consider the commutative diagram

$$\begin{tikzcd}
K_i(R) \otimes K_j(S) \arrow[r, "\star'"] \arrow[d, "(- \star \{\tau\}) \otimes (- \star \{\tau\})"', "\cong"] & K_{i+j}(R \otimes S) \arrow[d, "\cong"', "(- \star \{\tau\}) \star \{\tau\}"] \\
K_{i+1}(\Sigma R) \otimes K_{j+1}(\Sigma S) \arrow[r, "\star"'] & K_{i+j+2}\left(\Sigma^2 \left(R \otimes S\right)\right)
\end{tikzcd} \tag{2.24}$$

induced by Diagram (2.19).

If $[f] \in K_i(R)$ and $[g] \in K_j(S)$, then the lower part of the diagram gives

$$\begin{aligned}
([f] \star \{\tau\}) \star ([g] \star \{\tau\}) &= [f] \star (\{\tau\} \star [g]) \star \{\tau\} && \text{(associativity)} \\
&= [f] \star \left((-1)^j \left([g] \star \{\tau\}\right)\right) \star \{\tau\} && \text{(graded-commutativity)} \\
&= \left([f] \star \left((-1)^j [g]\right) \star \{\tau\}\right) \star \{\tau\} \,. && \text{(associativity)}
\end{aligned}$$

So we must have

$$[f] \star' [g] = [f] \star \left((-1)^j [g]\right) .$$

□

Note that Proposition 2.2.1 says the product $\star'$ is not graded-commutative in general.

2.3 Properties of the Extended Loday Pairing

2.3.1 Relationship With the Classical Pairings in Algebraic K-Theory

Milnor defined multiplicative structures on lower K-groups

$$\# : K_i(R) \otimes K_j(S) \to K_{i+j}(R \otimes S) \tag{2.25}$$

for i, $j \geq 0$, and $i + j \leq 2$ in [Mil72]. Actually, Milnor only defined the multiplication internally (i.e., when $R = S$, and is commutative). But one can mimic his definition to obtain an external product as in Equation (2.25), so that when $R = S$ it becomes Milnor's version.

The multiplication in Equation (2.25) is given by the formulas:

$$\begin{cases} [P]\#[Q] & = & [P \otimes Q] & \text{if } i = j = 0, \\ [P]\#\{v\} & = & \{p \otimes v + (1-p) \otimes 1\} & \text{if } i = 0, \text{ and } j = 1, \\ \{u\}\#[P] & = & \{u \otimes p + 1 \otimes (1-p)\} & \text{if } i = 1,\ j = 0, \\ \{u\}\#\{v\} & = & \{u, v\}_{\mathrm{St}} & \text{if } i = j = 1. \end{cases} \tag{2.26}$$

Here, p is the idempotent operator associated to the projective module P. We omit the case $i = 0$, $j = 2$ here. The main tool we need is

Proposition 2.3.1 ([Mil72, Lemma 8.9 on page 70]) *The multiplication $\#$ in Equation (2.25) is associative and bilinear for i, $j \geq 0$, and $i + j \leq 2$.*

Note that Milnor's proofs extend to the external product case. It is also straight forward to verify that $\#$ is graded-commutative. We now relate $\#$ and $\star'$.

Theorem 2.3.1 *For every $[P] \in K_0(R)$, $[Q] \in K_0(S)$, $\{u\} \in K_1(R)$, and $\{v\} \in K_1(S)$, we have*

(i) $[P] \star' [Q] = [P]\#[Q]$.

(ii) $[P] \star' \{v\} = [P]\#(-\{v\}) = -([P]\#\{v\})$, *and* $\{u\} \star' [Q] = \{u\}\#[Q]$.

(iii) $\{u\} \star' \{v\} = \{u\} \# \{v\}$.

Proof. (i) When $i = j = 0$, Diagram (2.19) induces the following commutative diagram

$$\begin{tikzcd} K_0(R) \otimes K_0(S) \arrow[rr, "\star'"] \arrow[d, "(-\#\{\tau\}) \otimes (-\#\{\tau\})"', "\cong"] & & K_0(R \otimes S) \arrow[d, "\cong"', "((-\#\{\tau\}) \star \{\tau\})"] \\ K_1(\Sigma R) \otimes K_1(\Sigma S) \arrow[rr, "\star"'] & & K_2(\Sigma^2(R \otimes S)). \end{tikzcd} \tag{2.27}$$

on homotopy. We then compute

$$
\begin{aligned}
&([P]\# \{\tau\}) \star ([Q]\# \{\tau\}) \\
&= -\{[P]\# \{\tau\}, [Q]\# \{\tau\}\}_{\mathrm{St}} && \text{(Equation (2.12))} \\
&= \{[P]\# \{\tau\}, (-[Q]) \# \{\tau\}\}_{\mathrm{St}} && \text{(Bilinearity of } \{-,-\}_{\mathrm{St}}) \\
&= ([P]\# \{\tau\}) \# ((-[Q]) \# \{\tau\}) && \text{(Equation (2.26))} \\
&= [P]\# (\{\tau\} \# (-[Q])) \# \{\tau\} && \text{(Associativity of } \#) \\
&= [P]\# (-[Q]\# \{\tau\}) \# \{\tau\} && \text{(Graded-commutativity of } \#) \\
&= -([P]\# ([Q]\# \{\tau\}) \# \{\tau\}) && \text{(Bilinearity of } \#)
\end{aligned}
$$

$$= -\{[P]\#([Q]\#\{\tau\}),\tau\}_{\mathrm{St}} \qquad \text{(Equation (2.26))}$$

$$= ([P]\#([Q]\#\{\tau\}))\star\{\tau\} \qquad \text{(Equation (2.12))}$$

$$= (([P]\#[Q])\#\{\tau\})\star\{\tau\}, \qquad \text{(Associativity of } \#\text{)}$$

proving

$$[P]\star'[Q] = [P]\#[Q].$$

(ii) When $i = 0$, $j = 1$, Diagram (2.19) induces the following commutative diagram

$$\begin{tikzcd}
K_0(R)\otimes K_1(S) \arrow[r, "\star'"] \arrow[d, "(-\#\{\tau\})\otimes(-\star\{\tau\})"', "\cong"] & K_1(R\otimes S) \arrow[d, "\cong"', "((-\star\{\tau\})\star\{\tau\})"] \\
K_1(\Sigma R)\otimes K_2(\Sigma R') \arrow[r, "\star"'] & K_3(\Sigma^2(R\otimes S)).
\end{tikzcd} \tag{2.28}$$

on homotopy groups. We then compute

$$([P]\#\{\tau\})\star(\{v\}\star\{\tau\})$$

$$= (([P]\#\{\tau\})\star\{v\})\star\{\tau\} \qquad \text{(Associativity of } \star\text{)}$$

$$= (-\{[P]\#\{\tau\},v\}_{\mathrm{St}})\star\{\tau\} \qquad \text{(Proposition 2.1.1)}$$

$$= (-(([P]\#\{\tau\})\#\{v\}))\star\{\tau\} \qquad \text{(Equation (2.26))}$$

$$= (-([P]\#(\{\tau\}\#\{v\}))) \star \{\tau\} \qquad \text{(Associativity of } \#)$$

$$= ([P]\#(\{v\}\#\{\tau\})) \star \tau \qquad \text{(Graded-commutativity of } \#)$$

$$= (([P]\#\{v\})\#\{\tau\}) \star \{\tau\} \qquad \text{(Associativity of } \#)$$

$$= \{[P]\#\{v\}, \{\tau\}\}_{\mathrm{St}} \star \{\tau\} \qquad \text{(Equation (2.26))}$$

$$= (-(([P]\#\{v\}) \star \{\tau\})) \star \{\tau\} \qquad \text{(Proposition 2.1.1)}$$

$$= ((-([P]\#\{v\}) \star \{\tau\})) \star \{\tau\} \qquad \text{(Bilinearity of } \star)$$

$$= (([P]\#(-\{v\})) \star \{\tau\}) \star \{\tau\} \qquad \text{(Graded-commutativity of } \#),$$

proving

$$[P] \star' \{v\} = [P]\#(-\{v\}).$$

The proof for

$$\{u\} \star' [Q] = \{u\}\#[Q]$$

is similar.

(iii)

$$\{u\} \star' \{v\} = \{u, v\}_{\mathrm{St}} \qquad \text{(Proposition 2.2.1)}$$

$$= \{u\} \# \{v\} .$$

□

The following case is still unknown.

Question 2.3.1 *Is the product map*

$$\star' : K_0(R) \otimes K_2(S) \to K_2(R \otimes S) \tag{2.29}$$

compatible with classical product map

$$\# : K_0(R) \otimes K_2(S) \to K_2(R \otimes S). \tag{2.30}$$

defined by Milnor in [Mil72, page 51]?

2.3.2 The Non-Connective Gersten-Wagoner Algebraic K-Theory Spectrum $\mathbb{K}_R^{\mathrm{GW}}$

We relate our extended Loday pairing

$$\gamma'_{\mathrm{Loday}} : K_R \wedge K_S \to K_{R \otimes S}$$

to the structure map

$$\mathbb{S}^1 \wedge K_R \to K_{\Sigma R}$$

of the Gersten-Wagoner spectrum $\mathbb{K}_R^{\mathrm{GW}}$. This will be used later in proving our version of the Loday assembly is a map of spectra.

In [Lod76, page 341–343], Loday gave an explicit description of the Gersten-Wagoner delooping

$$K_R \xrightarrow{\simeq} \Omega K_{\Sigma R},$$

so that the induced isomorphisms on homotopy groups in Equation (2.14) are given as in Equation (2.17). We will use our extended Loday pairing to define a map

$$\mathbb{S}^1 \wedge K_R \to K_{\Sigma R},$$

whose adjoint

$$K_R \to \Omega K_{\Sigma R}$$

will induce isomorphisms on homotopy groups, and hence is a homotopy equivalence by the Whitehead Theorem. We begin by thinking of the circle $\mathbb{S}^1$ as the classifying space $B\langle t\rangle$ of the infinite cyclic group generated by t, and define the following group homomorphism:

Definition 2.3.1 (The map $\mathbf{t}^+ : B\langle t\rangle \to K_{\Sigma\mathbb{Z}}$) *Let $\langle t\rangle$ be the infinite cyclic group generated by t, and $\tau \in GL(\Sigma\mathbb{Z})$ be the element defined in Equation (2.16). Define the group homomorphism*

$$\begin{aligned}\langle t\rangle &\to GL(\Sigma\mathbb{Z})\\ t &\mapsto \tau^{-1}.\end{aligned}$$

The induced map

$$B\langle t\rangle \to K_{\Sigma\mathbb{Z}} \tag{2.31}$$

is denoted by $\mathbf{t}^+$.

Proposition 2.3.2 (Another Gersten-Wagoner Delooping) *Let R be a ring, and $\langle t\rangle$ be the infinite cyclic group generated by t. The adjoint of the composition*

$$B\langle t\rangle \wedge K_R \xrightarrow{\mathrm{t}^+\wedge \mathrm{id}} K_{\Sigma\mathbb{Z}} \wedge K_R \xrightarrow{\gamma'_{\mathrm{Loday}}} K_{\Sigma R} \tag{2.32}$$

is a weak equivalence, and hence a homotopy equivalence.

Proof. A class $[f] \in K_i(R)$ is represented by a spheroid

$$f : \mathbb{S}^i \to K_R.$$

Suspending it yields the following composition

$$\mathbb{S}^1 \wedge \mathbb{S}^i \xrightarrow{\mathrm{id}\wedge f} \mathbb{S}^1 \wedge K_R \xrightarrow{\mathrm{t}^+\wedge \mathrm{id}} K_{\Sigma\mathbb{Z}} \wedge K_R \xrightarrow{\gamma'_{\mathrm{Loday}}} K_{\Sigma R}, \tag{2.33}$$

which represents the element $\left\{\tau^{-1}\right\} \star' [f] \in K_{i+1}(\Sigma R)$. In particular, we have

$$\left\{\tau^{-1}\right\} \star' [f] = \begin{cases} [f] \star \{\tau\} & \text{if } i > 0 \quad (\text{Proposition 2.2.1}) \\ \left\{\tau^{-1}\right\} \#[f] = -\left(\{\tau\} \#[f]\right) & \text{if } i = 0 \quad (\text{Theorem 2.3.1 (ii)}) \end{cases} \tag{2.34}$$

The map on homotopy groups

$$K_i(R) \to K_{i+1}(\Sigma R)$$

induced by the adjoint of (2.32) sends

$$[f] \mapsto [f] \star' \left\{\tau^{-1}\right\},$$

which is an isomorphism for $i = 0$ (resp. $i > 0$) by [Lod76, after Theorem 1.4.7 on page 328] (resp. [Lod76, Corollary 2.3.6 on page 345]). Therefore, Whitehead Theorem asserts the homotopy equivalence

$$K_R \simeq \Omega K_{\Sigma R}.$$

□

Consequently, we have the following Ω-spectrum

Definition 2.3.2 (The Non-connective Gersten-Wagoner Algebraic K-theory Spectrum $\mathbb{K}_R^{\mathrm{GW}}$) *Let R be a ring. Define the spectrum $\mathbb{K}_R^{\mathrm{GW}}$ by having n-th space as*

$$\left(\mathbb{K}_R^{\mathrm{GW}}\right)_n := \begin{cases} K_{\Sigma^n R} & \textit{if } n \geq 0, \\ \Omega^{-n} K_R & \textit{if } n < 0, \end{cases} \tag{2.35}$$

with the convention that $\Sigma^0 R := R$. The structure maps

$$f_n : \mathbb{S}^1 \wedge \left(\mathbb{K}_R^{\mathrm{GW}}\right)_n \to \left(\mathbb{K}_R^{\mathrm{GW}}\right)_{n+1} \tag{2.36}$$

are given by

$$f_n := \begin{cases} (2.32) & \textit{if } n \geq 0, \\ \textit{adjoint of } \mathrm{id}_{\left(\mathbb{K}_R^{\mathrm{GW}}\right)_n} & \textit{if } n < 0. \end{cases} \tag{2.37}$$

By construction and Proposition 2.3.2, $\mathbb{K}_R^{\mathrm{GW}}$ is an Ω-spectrum.

3. THE MAIN THEOREM

3.1 Definition of the Loday Assembly

We use the extended Loday pairing γ'_{Loday} to define a version of the Loday assembly map:

Definition 3.1.1 (The Loday Assembly Map α_{Loday} in Reduced Case) *Let R be a ring, and G be a group. The* **Loday assembly map** *is the weak map of spectra*

$$\alpha_{\mathrm{Loday}} : BG \wedge \mathbb{K}_R^{\mathrm{GW}} \to \mathbb{K}_{R[G]}^{\mathrm{GW}}, \tag{3.1}$$

whose components are given by the compositions of maps of spaces:

$$BG \wedge K_{\Sigma^m R} \xrightarrow{j^+ \wedge \mathrm{id}} K_{\mathbb{Z}[G]} \wedge K_{\Sigma^m R} \xrightarrow{\gamma'_{\mathrm{Loday}}} K_{\mathbb{Z}[G] \otimes \Sigma^m R} \simeq K_{\Sigma^m R[G]}. \tag{3.2}$$

Here, the map

$$j^+ : BG \to K_{\mathbb{Z}[G]} \tag{3.3}$$

is induced by the group homomorphism

$$\begin{aligned} j : G &\to GL\left(\mathbb{Z}[G]\right) \\ g &\mapsto \begin{bmatrix} g & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned} \tag{3.4}$$

The last homotopy equivalence is induced by the natural isomorphism of rings in Equation (1.25).

We need to justify our definition.

Proposition 3.1.1 (Cf [Lod76, Proposition 4.1.1 on page 356]) *Let R be a ring, and G be a group. The Loday assembly map*

$$\alpha_{\text{Loday}} : BG \wedge \mathbb{K}_R^{\text{GW}} \to \mathbb{K}_{R[G]}^{\text{GW}}$$

defined in Definition 3.1.1 is a weak map of spectra.

Proof. We need to show the components of α_{Loday}, as defined in Equation (3.2), commute with the structure maps of $\mathbb{K}_R^{\text{GW}}$ as defined in Equation (2.32). Let $\langle t \rangle$ be the infinite cyclic group generated by t. By unwrapping the definitions of α_{Loday} and the structure maps given in Proposition 2.3.2, the following diagram

$$\begin{tikzcd}
\mathbb{S}^1 \wedge (BG \wedge K_R) \arrow[rr, "\mathrm{id} \wedge \left(j^+ \wedge \mathrm{id}\right)"] \arrow[d, "\wr"] & & \mathbb{S}^1 \wedge \left(K_{\mathbb{Z}[G]} \wedge K_R\right) \arrow[r, "\mathrm{id} \wedge \gamma'_{\text{Loday}}"] \arrow[d, "\wr"] & \mathbb{S}^1 \wedge K_{R[G]} \arrow[d, "\wr"] \\
BG \wedge B\langle t \rangle \wedge K_R \arrow[r, "j^+ \wedge \mathrm{id} \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge \mathrm{t}^+ \wedge \mathrm{id}"'] & K_{\mathbb{Z}[G]} \wedge B\langle t \rangle \wedge K_R \arrow[r, "\mathrm{twist}", "\cong"'] \arrow[d, "\mathrm{id} \wedge \mathrm{t}^+ \wedge \mathrm{id}"'] & B\langle t \rangle \wedge K_{\mathbb{Z}[G]} \wedge K_R \arrow[r, "\mathrm{id} \wedge \gamma'_{\text{Loday}}"] \arrow[d, "\mathrm{t}^+ \wedge \mathrm{id} \wedge \mathrm{id}"'] & B\langle t \rangle \wedge K_{R[G]} \arrow[d, "\mathrm{t}^+ \wedge \mathrm{id}"] \\
BG \wedge K_{\Sigma\mathbb{Z}} \wedge K_R \arrow[r, "j^+ \wedge \mathrm{id} \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge \gamma'_{\text{Loday}}"'] & K_{\mathbb{Z}[G]} \wedge K_{\Sigma\mathbb{Z}} \wedge K_R \arrow[r, "\mathrm{twist}", "\cong"'] \arrow[d, "\mathrm{id} \wedge \gamma'_{\text{Loday}}"'] & K_{\Sigma\mathbb{Z}} \wedge K_{\mathbb{Z}[G]} \wedge K_R \arrow[r, "\mathrm{id} \wedge \gamma'_{\text{Loday}}"] \arrow[d, "\gamma'_{\text{Loday}} \circ \gamma'_{\text{Loday}}"'] & K_{\Sigma\mathbb{Z}} \wedge K_{R[G]} \arrow[d, "\gamma'_{\text{Loday}}"] \\
BG \wedge K_{\Sigma R} \arrow[r, "j^+ \wedge \mathrm{id}"'] & K_{\mathbb{Z}[G]} \wedge K_{\Sigma R} \arrow[r, "\gamma'_{\text{Loday}}"'] & K_{\Sigma R[G]} \arrow[r, "\mathrm{id}"'] & K_{\Sigma R[G]}
\end{tikzcd} \tag{3.5}$$

commutes up to weak homotopy by Corollary 2.2.1. Here, the columns are the structure maps, and the rows are the components of the Loday assembly. □

The homotopy groups $\pi_i(BG \wedge \mathbb{K}_R^{\mathrm{GW}})$ of the spectrum $BG \wedge \mathbb{K}_R^{\mathrm{GW}}$ are the reduced homology groups of the classifying space BG of G with coefficients in $\mathbb{K}_R^{\mathrm{GW}}$. So the induced map

$$\pi_i(\alpha_{\mathrm{Loday}}) : \pi_i(BG \wedge \mathbb{K}_R^{\mathrm{GW}}) \to K_i(R[G])$$

approximates the algebraic K-theory of the group ring $R[G]$ by a reduced homology theory. We want to lift the domain to an unreduced homology theory. Recall for any (based) topological space X, there is a split cofibre sequence

$$\begin{tikzcd} \mathbb{S}^0 \arrow[r, hook, bend right] & X_+ \arrow[l, dashed, bend right] \arrow[r] & X \end{tikzcd} \tag{3.6}$$

of spaces, where $X_+ := X \sqcup \text{point}$ is the space obtained by adding a disjoint point to X. If $\mathbb{E}$ is a spectrum, then we have a split cofibre sequence

$$\begin{tikzcd} \mathbb{S}^0 \wedge \mathbb{E} \arrow[r, hook, bend right] & X_+ \wedge \mathbb{E} \arrow[l, dashed, bend right] \arrow[r] & X \wedge \mathbb{E} \end{tikzcd} \tag{3.7}$$

of spectra. Consequently, we have a splitting

$$\begin{aligned} X_+ \wedge \mathbb{E} &\simeq (X \wedge \mathbb{E}) \vee \left(\mathbb{S}^0 \wedge \mathbb{E}\right) \\ &\cong (X \wedge \mathbb{E}) \vee \mathbb{E} \end{aligned} \tag{3.8}$$

of spectra. This allows us to extend the Loday assembly to the unreduced case.

Definition 3.1.2 (The Loday Assembly Map α_{Loday} in Unreduced Case) *Let R be a ring and G be a group. The canonical inclusion $R \to R[G]$ induces a map*

$$i : \mathbb{K}_R^{\mathrm{GW}} \to \mathbb{K}_{R[G]}^{\mathrm{GW}} \tag{3.9}$$

of spectra. The Loday assembly in unreduced case is defined by

$$BG_+ \wedge \mathbb{K}_R^{\mathrm{GW}} \simeq \left(BG \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \vee \mathbb{K}_R^{\mathrm{GW}} \xrightarrow{\text{(Definiton 3.1.1)}\vee i} \mathbb{K}_{R[G]}^{\mathrm{GW}}. \tag{3.10}$$

By abuse of terminology and notation, we call this map the Loday assembly and denote it by α_{Loday} as well.

3.2 A Subgroup of the Source of Loday Assembly

Fix a ring R, and write $\mathrm{AHSS}\,(X)^r_{p,q}$ to be the $E^r_{p,q}$-term of the Atiyah-Hirzebruch spectral sequence for the space X with coefficients in the spectrum $\mathbb{K}_R^{\mathrm{GW}}$. This spectral sequence is concentrated on the right-half plane. If in addition R is regular, then this spectral sequence is concentrated on the first quadrant. It follows that the differentials

$$\begin{aligned} d^r_{0,q} &: \mathrm{AHSS}\,(X)^r_{0,q} \to \mathrm{AHSS}\,(X)^r_{-r,q+r-1} \\ d^r_{1,q} &: \mathrm{AHSS}\,(X)^r_{1,q} \to \mathrm{AHSS}\,(X)^r_{1-r,q+r-1} \end{aligned}$$

are trivial for all q whenever $r > 1$. Consequently, the term $\mathrm{AHSS}\,(X)^\infty_{0,q}$ (resp. $\mathrm{AHSS}\,(X)^\infty_{1,q}$) is a quotient of $\mathrm{AHSS}\,(X)^2_{0,q}$ (resp. $\mathrm{AHSS}\,(X)^2_{1,q}$).

When $X = BG_+$, Definition 3.1.2 guarantees a natural splitting

$$\mathrm{AHSS}\,(BG_+)^r_{p,q} \cong \mathrm{AHSS}\left(\mathbb{S}^0\right)^r_{p,q} \oplus \mathrm{AHSS}\,(BG)^r_{p,q} \tag{3.11}$$

of spectral sequences, where the left-hand side converges to the homotopy group $\pi_{p+q}\left(BG_+ \wedge \mathbb{K}_R^{\mathrm{GW}}\right)$. Because

$$\mathrm{AHSS}\left(\mathbb{S}^0\right)^2_{p,q} \cong \begin{cases} K_q(R) & \text{if } p = 0, \\ 0 & \text{if else,} \end{cases} \tag{3.12}$$

a dimension argument says

$$\mathrm{AHSS}\left(\mathbb{S}^0\right)^{\infty}_{p,q} = \mathrm{AHSS}\left(\mathbb{S}^0\right)^{2}_{p,q} \tag{3.13}$$

On the other hand, we also have

$$\mathrm{AHSS}\,(BG)^{\infty}_{0,q} = \mathrm{AHSS}\,(BG)^{2}_{0,q} = 0 \tag{3.14}$$

because the homology group $\widetilde{H}_0\,(BG;\mathbb{Z})$ is trivial due to BG being path-connected. As a result, the source of the Loday assembly

$$\pi_{i+1}\left(\alpha_{\mathrm{Loday}}\right) : \pi_{i+1}(BG_+ \wedge \mathbb{K}_R^{\mathrm{GW}}) \to K_{i+1}(R[G])$$

contains

$$\begin{aligned}
&\mathrm{AHSS}\,(BG_+)^{\infty}_{0,1+i} \oplus \mathrm{AHSS}\,(BG_+)^{\infty}_{1,i} \\
\cong&\left(\mathrm{AHSS}\left(\mathbb{S}^0\right)^{\infty}_{0,1+i} \oplus \mathrm{AHSS}\,(BG)^{\infty}_{0,1+i}\right) \oplus \left(\mathrm{AHSS}\left(\mathbb{S}^0\right)^{\infty}_{1,i} \oplus \mathrm{AHSS}\,(BG)^{\infty}_{1,i}\right) \\
=&\,\mathrm{AHSS}\left(\mathbb{S}^0\right)^{\infty}_{0,1+i} \oplus \mathrm{AHSS}\,(BG)^{\infty}_{1,i} \\
=&K_{i+1}(R) \oplus \mathrm{AHSS}\,(BG)^{\infty}_{1,i}
\end{aligned} \tag{3.15}$$

as a subgroup. In fact, we already know what the Loday assembly is doing when restricted onto the summand $K_{i+1}(R)$—it is induced by the inclusion

$$i : R \to R[G]$$

of rings by Definition 3.1.2 before. What we want now is to study the Loday assembly when restricted onto the summand $\mathrm{AHSS}\,(BG)^{\infty}_{1,i}$.

3.3 Statement of the Main Theorem and the Proof

We have a diagram

$$\begin{tikzcd} \mathrm{AHSS}\,(BG)^2_{1,i} \cong G_{ab} \otimes K_i(R) \arrow[rr, dashed, "\Phi_{i+1}"] \arrow[d, two heads] & & K_{i+1}\,(R[G]) \\ \mathrm{AHSS}\,(BG)^\infty_{1,i} \arrow[r, hook] & \pi_{i+1}\left(BG \wedge \mathbb{K}_R^{\mathrm{GW}}\right). \arrow[ur, "\pi_{i+1}\,(\alpha_{\mathrm{Loday}})"'] & \end{tikzcd} \tag{3.16}$$

We would like to study the filler Φ_{i+1} for $i \geq 0$. Let us go back to the E^1-page of the Atiyah-Hirzebruch spectral sequence. Unlike the later pages, the E^1-page is not homotopy invariant—it depends on the choice of cellular structure on BG. **For our purpose, we are using the skeletal filtration on the bar construction for $\boldsymbol{BG}$**, so that

$$\mathrm{AHSS}\,(BG)^1_{1,i} \cong \pi_{i+1}\left(\left(\bigvee_{g\in G} \mathbb{S}^1\right) \wedge \mathbb{K}_R^{\mathrm{GW}}\right). \tag{3.17}$$

The filler we want to study is then induced by the composition

$$\pi_{i+1}\left(\left(\bigvee_{g\in G} \mathbb{S}^1\right) \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \xrightarrow{(\mathfrak{i}_G)_*} \pi_{i+1}\left(BG \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \xrightarrow{\pi_{i+1}\,(\alpha_{\mathrm{Loday}})} K_{i+1}\,(R[G]), \tag{3.18}$$

for which the first arrow is induced by the inclusion

$$\mathfrak{i}_G : \bigvee_{g\in G} \mathbb{S}^1 \hookrightarrow BG \tag{3.19}$$

of the 1-skeleton into BG. This follows from the construction of the Atiyah-Hirzebruch spectral sequence (see [Wic15] for example). When $G = \langle t \rangle$ is the infinite cyclic group, we know the circle $\mathbb{S}^1$ is a model for $B\langle t \rangle$. Moreover,

Lemma 3.3.1 *The inclusion*

$$\mathfrak{i}_t : \mathbb{S}^1 \to B\langle t \rangle \tag{3.20}$$

that sends $\mathbb{S}^1$ to the 1-cell of $B\langle t \rangle$ labelled by the generator t is a weak equivalence.

Proof. It is clear that the induced group homomorphism

$$(\mathfrak{i}_t)_* : \pi_i\left(\mathbb{S}^1\right) \to \pi_i\left(B\langle t \rangle\right) \tag{3.21}$$

is an isomorphism for $i \neq 1$, because the source and target are both trivial. So we need to check the map

$$(\mathfrak{i}_t)_* : \pi_1\left(\mathbb{S}^1\right) \to \pi_1\left(B\langle t \rangle\right) \tag{3.22}$$

is an isomorphism.

The map in Equation (3.22) is just an endomorphism of the infinite cyclic group. In particular, it is surjective by the definition of $\mathfrak{i}_t$. (The generator is in the image.) So the First Isomorphism Theorem says the map in Equation (3.22) must have trivial kernel, and hence an isomorphism. $\square$

The Whitehead Theorem then asserts $\mathfrak{i}_t$ is a homotopy equivalence. Thus when $G = \langle t \rangle$ is the infinite cyclic group, we have the following commutative diagram

$$\begin{tikzcd} \bigvee_{g \in \langle t \rangle} \mathbb{S}^1 \arrow[r, "\mathfrak{i}_{\langle t \rangle}"] & B\langle t \rangle \\ & \mathbb{S}^1 \arrow[ul, "\mathfrak{j}_t"] \arrow[u, "\mathfrak{i}_t"', "\simeq"] \end{tikzcd} \tag{3.23}$$

The map $\mathfrak{j}_t$ sends $\mathbb{S}^1$ to the loop in $\bigvee_{g\in\langle t\rangle} \mathbb{S}^1$ labelled by the generated t. The upshot is that we are able to describe the composition in Equation (3.18) when $G = \langle t\rangle$.

Proposition 3.3.1 *When $G = \langle t\rangle$ is the infinite cyclic group and $i \geq 0$, the composition in Equation (3.18) can be identified as*

$$
\begin{tikzcd}
\pi_{i+1}\left(\left(\bigvee\limits_{g\in\langle t\rangle} \mathbb{S}^1\right) \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \arrow[r, "(\mathfrak{i}_{\langle t\rangle})"] & \pi_{i+1}\left(B\langle t\rangle \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \arrow[r, "\pi_{i+1}(\alpha_{\mathrm{Loday}})"] & K_{i+1}\left(R[t^{\pm 1}]\right) \\
\pi_{i+1}\left(\mathbb{S}^1 \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \arrow[u, "(\mathfrak{j}_t)_*"] \arrow[r, "\mathrm{id}"'] \arrow[d, phantom, "\cong" rotate=90] & \pi_{i+1}\left(\mathbb{S}^1 \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \arrow[u, "(\mathfrak{i}_t)_*", "\cong"'] \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & K_{i+1}\left(R[t^{\pm 1}]\right) \arrow[u, equal] \arrow[d, equal] \\
K_i(R) \arrow[r, "\mathrm{id}"'] & K_i(R) \arrow[r] & K_{i+1}\left(R[t^{\pm 1}]\right),
\end{tikzcd}
\tag{3.24}
$$

where the bottom composition sends $[f] \in K_i(R)$ to the element

$$\{t\} \star' [f] \in K_{i+1}\left(R[t^{\pm 1}]\right),$$

(We remind readers the commutativity of the top-left square comes from Diagram (3.23).)

Proof. The class $[f] \in K_i(R)$ is represented by the spheroid

$$f : \mathbb{S}^i \to K_R,$$

where

$$K_R := K_0(R) \times BGL(R)^+ .$$

Taking suspension yields

$$\mathbb{S}^1 \wedge \mathbb{S}^i \xrightarrow{\mathrm{id} \wedge f} \mathbb{S}^1 \wedge K_R \xrightarrow{j^+ \wedge \mathrm{id}} K_{\mathbb{Z}[t^{\pm 1}]} \wedge K_R \xrightarrow{\gamma'_{\mathrm{Loday}}} K_{R[t^{\pm 1}]}.$$

This composition represents the element $\pi_{i+1}\left(\alpha_{\mathrm{Loday}}\right)([f]) \in K_{i+1}\left(R[t^{\pm 1}]\right)$, as well as the product

$$\left\{j^+\right\} \star' [f] \in K_{i+1}\left(R[t^{\pm 1}]\right).$$

Now, j^+ is induced by the group homomorphism

$$\begin{aligned} j : \langle t \rangle &\to GL\left(\mathbb{Z}[t^{\pm 1}]\right) \\ t &\mapsto \begin{bmatrix} t & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned}$$

So $\left\{j^+\right\} = \{t\}$, and thus

$$\left\{j^+\right\} \star' [f] = \{t\} \star' [f]$$

as desired. $\square$

For arbitrary group G, we recall the isomorphism

$$\begin{aligned} G &\cong \mathrm{Hom}\left(\langle t \rangle, G\right) \\ g &\mapsto \left(\varphi_g : t \mapsto g\right), \end{aligned} \tag{3.25}$$

allowing us to extend Proposition 3.3.1 to arbitrary G.

Proposition 3.3.2 *For an arbitrary group G and $i \geq 0$, under the canonical isomorphism*

$$\pi_{i+1}\left(\left(\bigvee_{g\in G}\mathbb{S}^1\right)\wedge\mathbb{K}_R^{\mathrm{GW}}\right)\cong\bigoplus_{g\in G}K_i(R), \tag{3.26}$$

the composition in (3.18) can be identified as

$$\begin{tikzcd}
\pi_{i+1}\left(\left(\bigvee_{g\in G}\mathbb{S}^1\right)\wedge\mathbb{K}_R^{\mathrm{GW}}\right) \arrow[r, "(\mathfrak{i}_G)_*"] \arrow[d, "\wr", phantom, sloped] & \pi_{i+1}\left(BG\wedge\mathbb{K}_R^{\mathrm{GW}}\right) \arrow[r, "\pi_{i+1}(\alpha_{\mathrm{Loday}})"] \arrow[d, equal] & K_{i+1}\left(R[G]\right) \arrow[d, equal] \\
\bigoplus_{g\in G}K_i(R) \arrow[r] & \pi_{i+1}\left(BG\wedge\mathbb{K}_R^{\mathrm{GW}}\right) \arrow[r] & K_{i+1}\left(R[G]\right),
\end{tikzcd} \tag{3.27}$$

where the bottom composition sends the element $[f]\in K_i(R)$ in the summand in $\bigoplus_{g\in G}K_i(R)$ labelled by $g\in G$ to the element

$$\{g\}\star'[f]\in K_{i+1}\left(R[G]\right).$$

Proof. Fix $g \in G$, and consider the commutative diagram

$$\begin{tikzcd}
\pi_{i+1}\left(\mathbb{S}^1 \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \arrow[r, "(\varphi_g)_*"] \arrow[d, "(\mathfrak{i}_t)_*"', "\cong"] & \pi_{i+1}\left(\left(\bigvee_{g\in G} \mathbb{S}^1\right) \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \arrow[d, "(\mathfrak{i}_G)_*"] \\
\pi_{i+1}\left(B\langle t\rangle \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \arrow[r, "(\varphi_g)_*"'] \arrow[d, "\pi_{i+1}(\alpha_{\mathrm{Loday}})"'] & \pi_{i+1}\left(BG \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \arrow[d, "\pi_{i+1}(\alpha_{\mathrm{Loday}})"] \\
K_{i+1}\left(R[t^{\pm 1}]\right) \arrow[r, "(\varphi_g)_*"'] & K_{i+1}\left(R[G]\right).
\end{tikzcd} \tag{3.28}$$

The top square commutes by chasing the definitions of the arrows. The bottom square commutes by the naturality of α_{Loday} via the group homomorphism

$$\varphi_g : \langle t\rangle \to G.$$

Now, Proposition 3.3.1 says the left vertical composition in Diagram (3.28) sends $[f] \in K_i(R)$ to

$$\{t\} \star' [f] = \in K_{i+1}\left(R[t^{\pm 1}]\right).$$

So the commutativity of Diagram (3.28) says the right vertical composition sends $[f] \in K_i(R)$ in the summand $\bigoplus_{g\in G} K_i(R)$ indexed by $g \in G$ to the element

$$\{g\} \star' [f] \in K_{i+1}\left(R[G]\right).$$

$\square$

Theorem 3.3.1 (An Explicit Formula for the Loday Assembly) *For $i \geq 0$, the filler Φ_{i+1} in Diagram (3.16) is induced by the bilinear map*

$$
\begin{aligned}
G \times K_i(R) &\to K_{i+1}(R[G]) \\
(g, [f]) &\mapsto \{g\} \star' [f].
\end{aligned} \tag{3.29}
$$

Proof. Follows immediately from Proposition 3.3.2. □

Corollary 3.3.1 (The Loday Assembly on π_1, [Wal78b, Assertion 15.8 on page 229], [LR05, page 708]) *For a regular ring R, and a group G, the Loday assembly on π_1 is given by*

$$
K_1(i) \oplus \Phi_1 : K_1(R) \oplus [G_{ab} \otimes K_0(R)] \to K_1(R[G]), \tag{3.30}
$$

for which $i : R \to R[G]$ is the inclusion, and Φ_1 is induced by the bilinear map

$$
\begin{aligned}
G \times K_0(R) &\to K_1(R[G]) \\
(g, [P]) &\mapsto \{\mathfrak{h}_g\},
\end{aligned} \tag{3.31}
$$

for which $\mathfrak{h}_g : R[G] \otimes_R P \to R[G] \otimes_R R$ is the automorphism given by

$$
\mathfrak{h}_g(u \otimes x) := gu \otimes x.
$$

Proof. If the ring R is regular, then the Atiyah-Hirzebruch spectral sequence for $\pi_i(BG_+ \wedge \mathbb{K}_R^{\mathrm{GW}})$ is concentrated in the first quadrant. Hence,

$$
\pi_1(BG_+ \wedge \mathbb{K}_R^{\mathrm{GW}}) \cong K_1(R) \oplus [G_{ab} \otimes K_0(R)].
$$

If $g \in G$, and $[P] \in K_0(R)$, then Theorem 3.3.1 says the Loday assembly sends $(g, [P])$ to

$$\begin{aligned}\{g\} \star' [P] &= \{g\} \#[P] && \text{(Theorem 2.3.1)} \\ &= \{g \otimes p + 1 \otimes (1-p)\}. && \text{(Equation (2.26))}\end{aligned}$$

If $u \otimes x \in R[G] \otimes_R P$, then

$$\begin{aligned}(g \otimes p + 1 \otimes (1-p))\,(u \otimes x) &= (g \otimes p)\,(u \otimes x) + (1 \otimes (1-p))\,(u \otimes x) \\ &= (g \otimes p)\,(u \otimes x) \\ &\quad \text{(since } (1-p)(x) = 0\text{)} \\ &= gu \otimes x.\end{aligned}$$

Therefore,

$$g \otimes p + 1 \otimes (1-p) = \mathfrak{h}_g.$$

□

Corollary 3.3.2 ([Wal78b, Proposition 15.7 (1) on page 229]) *Let G be a group. The cokernel of the Loday assembly*

$$\operatorname{coker}\left(\pi_1(\alpha_{\text{Loday}}) : \pi_1\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\text{GW}}\right) \to K_1(\mathbb{Z}[G])\right) \tag{3.32}$$

on π_1 is isomorphic to the first Whitehead group

$$\operatorname{Wh}_1(G) := \frac{K_1(\mathbb{Z}[G])}{\pm G}. \tag{3.33}$$

Proof. We need to check the image $\operatorname{im}(\pi_1(\alpha_{\text{Loday}}))$ of the Loday assembly on π_1 is the subgroup $\pm G$ of $K_1(\mathbb{Z}[G])$.

Note that the ring $\mathbb{Z}$ has no negative K-groups for being regular. Thus, the vanishing of the differentials of the Atiyah-Hirzebruch spectral sequences says

$$\pi_1\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{GW}}\right) \cong K_1(\mathbb{Z}) \oplus [G_{ab} \otimes K_0(\mathbb{Z})] \tag{3.34}$$
$$\cong C_2 \oplus [G_{ab} \otimes \mathbb{Z}]$$
$$\cong \{\pm 1\} \oplus G_{ab} \tag{3.35}$$

We already know, from Definition 3.1.2, the Loday assembly $\pi_1\left(\alpha_{\text{Loday}}\right)|_{K_1(\mathbb{Z})}$ when restricted onto the summand $K_1(\mathbb{Z})$ of $\pi_1\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{GW}}\right)$ is the group homomorphism

$$K_1(\mathbb{Z}) \to K_1(\mathbb{Z}[G])$$

induced by the canonical inclusion $i : \mathbb{Z} \to \mathbb{Z}[G]$. Therefore, under the identification

$$K_1(\mathbb{Z}) \cong \{\pm 1\},$$

the restriction $\pi_1\left(\alpha_{\text{Loday}}\right)|_{K_1(\mathbb{Z})}$ sends ± 1 to $\pm 1 \in K_1(\mathbb{Z}[G])$.

On the other hand, we observe that the map

$$\Phi_1 : G_{ab} \otimes K_0(\mathbb{Z}) \to K_1(\mathbb{Z}[G])$$

from Corollary 3.3.1 sends the simple tensor $g \otimes [\mathbb{Z}] \in G_{ab} \otimes K_0(\mathbb{Z})$ to the element $\{g\} \in K_1(\mathbb{Z}[G])$, and therefore, sends the simple tensor $g \otimes [\mathbb{Z}^n] \in G_{ab} \otimes K_0(\mathbb{Z})$ to the element $\{g^n\} \in K_1(\mathbb{Z}[G])$. As a result, under the identification

$$G_{ab} \otimes K_0(\mathbb{Z}) \cong G_{ab},$$

the map Φ_1 sends $g \in G_{ab}$ to $\{g\} \in K_1(\mathbb{Z}[G])$.

Combining everything together, we see that, under the identification

$$\pi_1\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{GW}}\right) \cong \{\pm 1\} \oplus G_{ab},$$

the Loday assembly $\pi_1(\alpha_{\mathrm{Loday}})$ sends the element $(\pm 1) \oplus g \in \{\pm 1\} \oplus G_{ab}$ to the element $\{\pm g\} \in K_1(\mathbb{Z}[G])$. This proves $\mathrm{im}\,(\pi_1(\alpha_{\mathrm{Loday}}))$ is the subgroup $\pm G$ of $K_1(\mathbb{Z}[G])$ as desired. $\square$

4. THE INJECTIVITY PROBLEM FOR THE INTEGRAL LODAY ASSEMBLY

One might ask whether torsion-free is a necessary condition in the Classical Farrell-Jones Conjecture 1.1.2. Integrally, the cokernel

$$\operatorname{coker}\left(\pi_i\left(\alpha_{\text{Loday}}\right) : \pi_i\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{GW}}\right) \to K_i(\mathbb{Z}[G])\right) \cong \mathrm{Wh}_i\left(G\right) \tag{4.1}$$

can be identified with the classical Whitehead groups $\mathrm{Wh}_i\left(G\right)$ for $i = 0$, 1 and 2. See [Lod76, page 357–364], or Corollary 3.3.2 for the case $i = 1$. As a result, non-vanishing $\mathrm{Wh}_i\left(G\right)$ implies the non-surjectivity of $\pi_i(\alpha_{\text{Loday}})$. For example, if p is an odd prime and C_p is the cyclic group of order p, then $\mathrm{Wh}_1\left(C_p\right) \neq 0$ [Coh73, 11.5 on page 45]. Consequently, torsion-free is necessary to guarantee surjectivity. What about injectivity?

Question 4.0.1 (Non-injectivity Problem for the Loday Assembly) *Let R be a regular ring. Is there a group G with torsion, such that the Loday assembly*

$$\pi_i\left(\alpha_{\text{Loday}}\right) : \pi_i\left(BG_+ \wedge \mathbb{K}_R^{\mathrm{GW}}\right) \to K_i(R[G])$$

is not injective for some i?

It turns out that under suitable conditions, the Loday assembly is injective in lower degrees.

Proposition 4.0.1 ([LR05, Lemma 2 on page 709]) *Let R be a regular ring and G be a group.*

(i) The Loday assembly

$$\pi_0\left(\alpha_{\text{Loday}}\right):\pi_0\left(BG_+\wedge\mathbb{K}_R^{\text{GW}}\right)\to K_0(R[G])$$

on π_0 is always injective. In particular, its left inverse is induced by the augmentation map

$$R[G]\to R.$$

(ii) If also R is commutative, and the natural map

$$\begin{aligned}\mathbb{Z}&\to K_0(R)\\ 1&\mapsto [R]\end{aligned}$$

is an isomorphism, then the Loday assembly

$$\pi_1\left(\alpha_{\text{Loday}}\right):\pi_1\left(BG_+\wedge\mathbb{K}_R^{\text{GW}}\right)\to K_1(R[G])$$

on π_1 is injective.

Consequently, the first place to look for non-injectivity phenomena for Loday assembly would be the second homotopy group.

4.1 Second Homotopy Group

Question 4.0.1 was answered by Ullmann-Wu in the case when R is a finite field.

Theorem 4.1.1 ([UW17, Theorem 2 on page 461]) *Let G be a finite group such that $H_2(G;\mathbb{Z})$ is non-trivial, and $\mathbb{F}$ a finite field with characteristic p which does not divide the order of G, then the Loday assembly*

$$\pi_2\left(\alpha_{\text{Loday}}\right):\pi_2\left(BG_+\wedge\mathbb{K}_{\mathbb{F}}^{\text{GW}}\right)\to K_2(\mathbb{F}[G])$$

is not injective.

For example, we can take $G = C_2 \oplus C_2$ and $\mathbb{F}$ to be any finite field with characteristic $p > 2$.

However, no such example is known when $R = \mathbb{Z}$. In this case, the Atiyah-Hirzebruch spectral sequence for $\pi_2\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{GW}}\right)$ yields the following short exact sequence

$$0 \to K_2(\mathbb{Z}) \oplus [G_{ab} \otimes K_1(\mathbb{Z})] \to \pi_2\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{GW}}\right) \to H_2(BG; K_0(\mathbb{Z})) \to 0. \tag{4.2}$$

See [Leh18, Theorem 12.2 on page 29]. The key point is that the differential

$$d^2_{3,0} : H_3(BG; K_0(\mathbb{Z})) \to H_1(BG; K_1(\mathbb{Z})) \cong G_{ab} \otimes K_1(\mathbb{Z}) \tag{4.3}$$

is trivial [Leh18, page 28]. Together with Theorem 3.3.1, we have the following corollary:

Corollary 4.1.1 (A Formula for the Loday Assembly on π_2 of an Integral Group Ring) *Let G be a group. The Loday assembly map*

$$\alpha_{\mathrm{Loday}} : BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{GW}} \to \mathbb{K}_{\mathbb{Z}[G]}^{\mathrm{GW}} \tag{4.4}$$

for the integral group ring $\mathbb{Z}[G]$ on π_2, when restricted onto the subgroup

$$K_2(\mathbb{Z}) \oplus [G_{ab} \otimes K_1(\mathbb{Z})] \leq \pi_2\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{GW}}\right),$$

is given by the formula:

$$K_2(i) \oplus \Phi_2 : K_2(\mathbb{Z}) \oplus [G_{ab} \otimes K_1(\mathbb{Z})] \to K_2(\mathbb{Z}[G]), \tag{4.5}$$

where $i : \mathbb{Z} \to \mathbb{Z}[G]$ is the inclusion, and Φ_2 is induced by the bilinear map

$$\begin{aligned} G \times K_1(\mathbb{Z}) &\to K_2(\mathbb{Z}[G]) \\ (g, \pm 1) &\mapsto \{\pm 1, g\}_{\mathrm{St}}\,, \end{aligned} \tag{4.6}$$

where we have identified $K_1(\mathbb{Z}) \cong \{\pm 1\}$*. In particular, the Steinberg symbol* $\{1, g\}_{\mathrm{St}}$ *is always the identity element in* $K_2(\mathbb{Z}[G])$ *for all* $g \in G$*.*

Proof. Only the description of Φ_2 needs to be justified. Theorem 3.3.1 says Φ_2 is induced by the bilinear map

$$\begin{aligned} G \times K_1(\mathbb{Z}) &\to K_2(\mathbb{Z}[G]) \\ (g, \pm 1) &\mapsto \{g\} \star' \{\pm 1\} \end{aligned} \tag{4.7}$$

and we have

$$\begin{aligned} \{g\} \star' \{\pm 1\} &= \{g, \pm 1\}_{\mathrm{St}} && \text{(Proposition 2.2.1)} \\ &= -\{\pm 1, g\}_{\mathrm{St}} && \text{(Skew-symmetric)} \\ &= \left\{(\pm 1)^{-1}, g\right\}_{\mathrm{St}} && \text{(Bilinearity)} \\ &= \{\pm 1, g\}_{\mathrm{St}} \end{aligned}$$

as desired. □

It turns out that injectivity can happen for group with torsion.

Example 4.1.1 (Bijectivity on $K_2(\mathbb{Z}[C_2])$). Denote by C_2 the cyclic group of order two, with generator g. Dunwoody showed that the second K-group of $\mathbb{Z}[C_2]$ is generated by two Steinberg symbols [Dun75, Theorem on page 482]:

$$\begin{aligned} K_2\left(\mathbb{Z}[C_2]\right) &= \left\langle \{-1,-1\}_{\mathrm{St}}, \{-1,g\}_{\mathrm{St}} \right\rangle \\ &\cong C_2 \oplus C_2. \end{aligned} \tag{4.8}$$

Because

$$\begin{aligned} H_2(BC_2; K_0(\mathbb{Z})) &\cong H_2(BC_2; \mathbb{Z}) \\ &= 0, \end{aligned}$$

we have

$$\pi_2\left(BC_{2+} \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{GW}}\right) \cong K_2(\mathbb{Z}) \oplus \left[(C_2)_{ab} \otimes K_1(\mathbb{Z})\right].$$

Corollary 4.1.1 then says the Loday assembly for C_2

$$\pi_2\left(\alpha_{\mathrm{Loday}}\right) = K_2(i) \oplus \Phi_2 \tag{4.9}$$

is bijective on π_2.

The following result was proven in collaboration with Daniel Ramras.

Theorem 4.1.2 (An Injectivity Result for the Loday Assembly on π_2) *Let G be a group. The composition*

$$K_2\left(\mathbb{Z}\right) \oplus \left[G_{ab} \otimes K_1\left(\mathbb{Z}\right)\right] \xrightarrow{(4.2)} \pi_2\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{GW}}\right) \xrightarrow{\pi_2\left(\alpha_{\mathrm{Loday}}\right)} K_2\left(\mathbb{Z}[G]\right) \tag{4.10}$$

is injective.

Proof. By Corollary 4.1.1, the composition in Equation (4.10) is given by the group homomorphism

$$K_2(i) \oplus \Phi_2 : K_2(\mathbb{Z}) \oplus (G_{ab} \otimes \{\pm 1\}) \to K_2(\mathbb{Z}[G]), \tag{4.11}$$

where $i : \mathbb{Z} \to \mathbb{Z}[G]$ is the canonical inclusion, and

$$\Phi_2 : [g] \otimes \{\pm 1\} \mapsto \{\pm 1, g\}_{\mathrm{St}} \tag{4.12}$$

for $[g] \in G_{ab}$ represented by $g \in G$. Here, we have identified

$$K_1(\mathbb{Z}) \cong \{\pm 1\}.$$

We know the group homomorphism $K_2(i)$ is injective, with left inverse induced by the augmentation map

$$\mathbb{Z}[G] \to \mathbb{Z}.$$

As a result, we only need to verify injectivity for Φ_2.

At this point, we will identify $K_1(\mathbb{Z})$ with the additive group of the field with two elements. We denote by $\bullet$ the field multiplication on $K_1(\mathbb{Z})$. We define the group homomorphism

$$\begin{aligned} \mathfrak{r} : G_{ab} &\to G_{ab} \otimes K_1(\mathbb{Z}) \\ [g] &\mapsto [g] \otimes (-1). \end{aligned} \tag{4.13}$$

Then, we note that the group homomorphism

$$\mathfrak{r} \otimes \mathrm{id} : G_{ab} \otimes K_1(\mathbb{Z}) \to (G_{ab} \otimes K_1(\mathbb{Z})) \otimes K_1(\mathbb{Z}) \tag{4.14}$$

is an isomorphism, with inverse given by

$$\begin{aligned}\mathfrak{s} : (G_{ab} \otimes K_1(\mathbb{Z})) \otimes K_1(\mathbb{Z}) &\to G_{ab} \otimes K_1(\mathbb{Z}) \\ ([g] \otimes \varepsilon_1) \otimes \varepsilon_2 &\mapsto [g] \otimes (\varepsilon_1 \bullet \varepsilon_2)\,. \end{aligned} \tag{4.15}$$

Write $V := G_{ab} \otimes K_1(\mathbb{Z})$. Then we have a group homomorphism given by the composition

$$\begin{tikzcd} G \arrow[r, two heads] \arrow[rrr, dashed, bend left, "\mathfrak{v}"] & G_{ab} \arrow[r] & G_{ab} \otimes K_1(\mathbb{Z}) & =: V \\[-20pt] \Cup & \Cup & \Cup & \\[-20pt] g \arrow[r, mapsto] & {[g]} \arrow[r, mapsto] & {[g] \otimes (-1),} & \end{tikzcd} \tag{4.16}$$

which induces the following commutative diagram

$$\begin{tikzcd} G_{ab} \otimes K_1(\mathbb{Z}) \arrow[rr, "\Phi_2^G"] \arrow[d, "\mathfrak{r} \otimes \mathrm{id}"', "\cong"] & & K_2(\mathbb{Z}[G]) \arrow[d, "\mathfrak{v}_*"] \\ V \otimes K_1(\mathbb{Z}) \arrow[r, "\mathrm{id}"'] & V_{ab} \otimes K_1(\mathbb{Z}) \arrow[r, "\Phi_2^V"'] & K_2(\mathbb{Z}[V]) \end{tikzcd} \tag{4.17}$$

by naturality of the Loday assembly. We wish to show the group homomorphism Φ_2^V is injective. Then, it will follow that Φ_2^G is also injective.

Now, because we can think of $K_1(\mathbb{Z})$ as the field with two elements, V is then a vector space over this field. Therefore for every element $x \in V$, there exists a group homomorphism

$$\mathfrak{f}_x : V \to C_2 \tag{4.18}$$

that sends x to the generator. A non-zero element in $V \otimes K_1(\mathbb{Z})$ is of the form $x \otimes (-1)$, where $x \in V$ is also non-zero. Hence, the group homomorphism $\mathfrak{f}_x$ induces the following commutative diagram

$$\begin{tikzcd} x \otimes (-1) \arrow[d, mapsto] & \in & V \otimes K_1(\mathbb{Z}) \arrow[r, "\Phi_2^V"] \arrow[d, "\mathfrak{f}_x \otimes \mathrm{id}"'] & K_2(\mathbb{Z}[V]) \arrow[d, "(\mathfrak{f}_x)_*"] \\ \mathfrak{f}_x(x) \otimes (-1) & \in & C_2 \otimes K_1(\mathbb{Z}) \arrow[r, "\Phi_2^{C_2}"'] & K_2(\mathbb{Z}[C_2]) \end{tikzcd} \tag{4.19}$$

by naturality of the Loday assembly. Now, Example 4.1.1 says the element $\Phi_2^{C_2}(\mathfrak{f}_x(x) \otimes (-1)) \in K_2(\mathbb{Z}[C_2])$ is non-zero. Therefore, the commutativity of Diagram (4.19) says the element $\Phi_2^V(x \otimes (-1)) \in K_2(\mathbb{Z}[V])$ is non-zero as well. This proves the injectivity of Φ_2^V and therefore, the claim holds as desired. □

Corollary 4.1.2 (A Bijectivity Result for the Loday Assembly on π_2) *Let G be a group such that the second homology group $H_2(BG;\mathbb{Z})$ is trivial. Then the Loday assembly*

$$\pi_2(\alpha_{\text{Loday}}) : \pi_2\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\text{GW}}\right) \to K_i(\mathbb{Z}[G])$$

on π_2 is injective.

Proof. If $H_2(BG;\mathbb{Z})$ is trivial, then the short exact sequence in Equation (4.2) says

$$\pi_2(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\text{GW}}) \cong K_2(\mathbb{Z}) \oplus [G_{ab} \otimes K_1(\mathbb{Z})].$$

Corollary 4.1.1 then tells us that the Loday assembly on π_2 is given by

$$\pi_2(\alpha_{\text{Loday}}) = K_2(i) \oplus \Phi_2.$$

The claim then follows from Theorem 4.1.2. □

We can use our result to give a new proof of

Corollary 4.1.3 (Cf [Leh18, page 28]) *The differential*

$$d^2_{3,0} : \mathrm{AHSS}\,(BG)^2_{3,0} \to \mathrm{AHSS}\,(BG)^2_{1,1} \cong G_{ab} \otimes K_1(\mathbb{Z}) \tag{4.20}$$

for the Atiyah-Hirzebruch spectral sequence of BG with coefficients in the spectrum $\mathbb{K}^{\mathrm{GW}}_{\mathbb{Z}}$ *is trivial.*

Proof. We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{AHSS}\,(BG)^2_{1,1} \arrow[r, "\Phi_2"] \arrow[d, "\mathrm{coker}\,\left(d^2_{3,0}\right)"'] & K_2\,(\mathbb{Z}[G]). \\
\mathrm{AHSS}\,(BG)^{\infty}_{1,1} \arrow[ur, "\pi_2\,(\alpha_{\mathrm{Loday}})\,|_{\mathrm{AHSS}(BG)^{\infty}_{1,1}}"'] &
\end{tikzcd} \tag{4.21}$$

Since we know Φ_2 is injective from Theorem 4.1.2, the arrow $\mathrm{coker}\left(d^2_{3,0}\right)$ is also injective. Therefore, the differential $d^2_{3,0}$ is trivial. $\square$

Example 4.1.2 (Cyclic Groups of Odd Orders). Our formula does not give any information about the K-theory of the integral group ring $\mathbb{Z}[C_n]$ of cyclic group C_n of odd order n. This is because $C_n \otimes C_2 = 0$, so the domain of the map Φ_2 in Equation (4.5) is trivial. Because $H_2\,(BC_n; \mathbb{Z}) = 0$ for all odd numbers n, the short exact sequence in Equation (4.2) gives

$$\pi_2\left(BC_{n+} \wedge \mathbb{K}^{\mathrm{GW}}_{\mathbb{Z}}\right) \cong K_2(\mathbb{Z}), \tag{4.22}$$

and therefore we learn nothing about structures of $K_2(\mathbb{Z}[C_n])$ from this approach, other than it contains $K_2(\mathbb{Z})$ as a subgroup.

4.1.1 The Steinberg Symbol $\{-1, g\}_{\mathrm{St}}$

We study the Steinberg symbol $\{-1, g\}_{\mathrm{St}} \in K_2(\mathbb{Z}[G])$, and obtain an expression for it in terms of generators of the Steinberg group $\mathrm{St}\,(\mathbb{Z}[G])$.

Theorem 4.1.3 *Let G be a group. The Steinberg symbol $\{-1, g\}_{\mathrm{St}} \in K_2(\mathbb{Z}[G])$ for $g \in G$ is given by*

$$\{-1, g\}_{\mathrm{St}} = w_{12}(-1)\; w_{12}(-1)\; w_{12}(g)\; w_{12}(g), \tag{4.23}$$

where

$$w_{ij}(u) := x_{ij}(u)\; x_{ji}(-u^{-1})\; x_{ij}(u), \tag{4.24}$$

and the $x_{ij}(u)$s' are the generators of the Steinberg group $\mathrm{St}\,(\mathbb{Z}[G])$.

Proof. The proof involves playing with the presentation of the Steinberg group.

We recall from [Ros95, Lemma 4.2.15 on page 195] the identities

$$w_{ij}(u)^{-1} = w_{ij}(-u), \tag{4.25}$$

$$w_{ij}(u) = w_{ji}(-u^{-1}), \tag{4.26}$$

$$w_{k\ell}(u)\; w_{ij}(v)\; w_{k\ell}(u)^{-1} = \begin{cases} w_{ij}(v) & \text{if } i,\ j,\ k,\ \ell \text{ are all distinct,} \\ w_{\ell j}(-u^{-1}v) & \text{if } k = i, \text{ and } i,\ j,\ \ell \text{ are all distinct,} \\ w_{i\ell}(-vu) & \text{if } k = j, \text{ and } i,\ j,\ k \text{ are all distinct,} \\ w_{ji}(-u^{-1}vu^{-1}) & \text{if } k = i, \text{ and } j = \ell. \end{cases} \tag{4.27}$$

In particular, Equation (4.27) gives

$$w_{13}(u)\; w_{12}(v) = w_{23}(v^{-1}u)\; w_{13}(u), \tag{4.28}$$

$$w_{13}(u)\ w_{23}(v) = w_{12}(-uv^{-1})\ w_{13}(u). \tag{4.29}$$

Put

$$h_{ij}(u) := w_{ij}(u)\ w_{ij}(-1), \tag{4.30}$$

then a long and boring computation shows

$$\{-1, g\}_{\mathrm{St}} := [h_{12}(-1), h_{13}(g)]$$

$$= h_{12}(-1)\ h_{13}(g)\ h_{12}(-1)^{-1}\ h_{13}(g)^{-1}$$

$$= w_{12}(-1)\ w_{12}(-1)\ w_{13}(g)\ w_{13}(-1)\ w_{12}(-1)^{-1}\ w_{12}(-1)^{-1}\ w_{13}(-1)^{-1}\ w_{13}(g)^{-1}$$

$$= w_{12}(-1)\ w_{12}(-1)\ w_{13}(g)\ w_{13}(-1)\ w_{12}(1)\ w_{12}(1)\ w_{13}(1)\ w_{13}(-g)$$

(Equation (4.25))

$$= w_{12}(-1)\ w_{12}(-1)\ w_{13}(g)\ \underbrace{w_{13}(-1)\ w_{12}(1)}_{w_{23}(-1)\ w_{13}(-1)}\ w_{12}(1)\ w_{13}(1)\ w_{13}(-g)$$

(Equation (4.28))

$$= w_{12}(-1)\ w_{12}(-1)\ w_{13}(g)\ w_{23}(-1)\ \underbrace{w_{13}(-1)\ w_{12}(1)}_{w_{23}(-1)\ w_{13}(-1)}\ w_{13}(1)\ w_{13}(-g)$$

(Equation (4.28))

$$= w_{12}(-1)\ w_{12}(-1)\ w_{13}(g)\ w_{23}(-1)\ w_{23}(-1)\ \underbrace{w_{13}(-1)\ w_{13}(1)}_{1}\ w_{13}(-g)$$

(Equation (4.25))

$$= w_{12}(-1)\ w_{12}(-1)\ \underbrace{w_{13}(g)\ w_{23}(-1)}_{w_{12}(g)\ w_{13}(g)}\ w_{23}(-1)\ w_{13}(-g)$$

(Equation (4.29))

$$= w_{12}(-1)\ w_{12}(-1)\ w_{12}(g)\ \underbrace{w_{13}(g)\ w_{23}(-1)\ w_{13}(-g)}_{w_{31}(-g^{-1})\ w_{23}(-1)\ w_{31}(g^{-1})}$$

(Equation (4.26))

$$= w_{12}(-1)\ w_{12}(-1)\ w_{12}(g)\ \underbrace{w_{31}(-g^{-1})\ w_{23}(-1)\ w_{31}(g^{-1})}_{w_{21}(-g^{-1})}$$

(Equation (4.27))

$$= w_{12}(-1)\ w_{12}(-1)\ w_{12}(g)\ \underbrace{w_{21}(-g^{-1})}_{w_{12}(g)}$$

(Equation (4.26))

$$= w_{12}(-1)\ w_{12}(-1)\ w_{12}(g)\ w_{12}(g)$$

as desired. □

Theorem 4.1.3 motivates the following definition:

Definition 4.1.1 (The group $W_{12}(G)$) *Let G be a group. The subgroup $W_{12}(G)$ of $K_2(\mathbb{Z}[G])$ is generated by the Steinberg symbols:*

$$W_{12}(G) := \langle \{-1, g\}_{\mathrm{St}} \mid g \in G \rangle\,. \tag{4.31}$$

Notice the following:

(1) The image of the map

$$\Phi_2 : G \times K_1(\mathbb{Z}) \to K_2(\mathbb{Z}[G])$$

in Equation (4.5) is precisely $W_{12}(G)$.

(2) Readers who are familiar with the classical definition

$$\mathrm{Wh}_2(G) := \frac{K_2(\mathbb{Z}[G])}{K_2(\mathbb{Z}[G]) \cap W(G)} \tag{4.32}$$

of the second Whitehead group proposed by Hatcher-Wagoner in [HW73, page 10] will see our group $W_{12}(G)$ follows their spirit. The group $W(G)$ is generated by the elements $w_{ij}(\pm g)$ for all $g \in G$, and for all i, j, where

$$w_{ij}(u) := x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u), \tag{4.33}$$

and the $x_{ij}(u)$s' are the free generators of the Steinberg group. See [Ros05, Definition 4.2.1 on page 187, and the end of page 192]. Together with Theorem 4.1.3, we see that $W_{12}(G)$ is a subgroup of $W(G)$.

(3) From the functoriality of the Steinberg group, we get a functor

$$W_{12}(-) : \mathcal{G}\text{roups} \to \mathcal{A}\text{belian} \tag{4.34}$$

from groups to Abelian groups. Moreover, for every group G, there is a natural group homomorphism

$$\Psi_G : G \to W_{12}(G)$$

$$g \mapsto \{-1, g\}_{\mathrm{St}}\,, \tag{4.35}$$

which satisfies

$$\Psi_G = \Phi_2|_{\{-1\}\times G} \tag{4.36}$$

for Φ_2 as in Equation (4.5). Non-injectivity of Ψ_G will imply the non-injectivity of the integral Loday assembly on π_2.

4.1.2 Kähler Differentials and de Rham Cohomology

Let k be a unital, commutative ring, and A be a unital, commutative k-algebra. We define the **A-module of Kähler differentials** to be the free A-module generated by the symbols da for each $a \in A$, modulo linearity and the Leibniz rule:

$$\Omega^1_{A|k} := \frac{\langle da \mid a \in A\rangle}{\left\langle \begin{aligned} d(\lambda a + \mu b) &= \lambda da + \mu db, \\ d(ab) &= a(db) + b(da) \end{aligned} \right\rangle}, \tag{4.37}$$

where λ, $\mu \in k$ and a, $b \in A$. Let 1_k be the multiplicative identity in k. The Leibniz rule implies $d(1_k) = 0$, and consequently, $du = 0$ for any $u \in k$.

We then define the module $\Omega^n_{A|k}$ of **differential n-forms** to be the exterior product

$$\Omega^n_{A|k} := \Lambda^n_A \Omega^1_{A|k}\,, \tag{4.38}$$

where the exterior product is over A, not k. It is spanned by the elements $a_0 da_1 \wedge \cdots \wedge da_n$, for $a_i \in A$, that we usually write $a_0 da_1 \cdots da_n$.

Let us put $\Omega^0_{A|k} := A$. Then for each $n \geq 0$, the **exterior differential operator** is defined by

$$\begin{aligned} d : \Omega^n_{A|k} &\to \Omega^{n+1}_{A|k} \\ a_0 da_1 \cdots da_n &\mapsto da_0 da_1 \cdots da_n. \end{aligned} \tag{4.39}$$

Since $d(1_k) = 0$, it follows that $d \circ d = 0$, and the following sequence

$$A := \Omega^0_{A|k} \xrightarrow{d} \Omega^1_{A|k} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{A|k} \xrightarrow{d} \cdots \tag{4.40}$$

is a chain complex, called the **de Rham chain complex of A over k**. The homology groups of the de Rham chain complex are denoted by $H^n_{\mathrm{dR}}(A|k)$, and are the **de Rham cohomology groups of A over k**.

Using the theory of cyclic homology and Chern character, Loday constructed a map

$$K_2(A) \to H^2_{\mathrm{dR}}(A|k) \tag{4.41}$$

that sends the Steinberg symbol $\{x, y\}_{\mathrm{St}}$, where x, $y \in A$, to the cohomology class represented by the differential form $x^{-1}y^{-1}dxdy$ [Lod97, page 275]. One may hope that this map will be useful for studying the Steinberg symbol $\{-1, g\}_{\mathrm{St}}$ when $A = \mathbb{Z}[G]$ is an integral group ring. However, because

$$\begin{aligned} d(-1) &= -d(1) \\ &= 0, \end{aligned}$$

the image of the Steinberg symbol $\{-1, g\}_{\mathrm{St}}$ is always trivial in $H^2_{\mathrm{dR}}(\mathbb{Z}[G]|\mathbb{Z})$. This means we need other methods to analyse $\{-1, g\}_{\mathrm{St}}$.

5. COMPARING ASSEMBLY MAPS

In this section, we show our definition of Loday assembly can be used to compute the Weiss-Williams assembly map (i.e., the universal assembly map).

5.1 Two Intermediate Spectra

5.1.1 Symmetric Monoidal Categories and the $S^{-1}S$-Construction

Definition 5.1.1 (Symmetric Monoidal Category) *A* ***symmetric monoidal category*** *is a category $\mathcal{C}$ together with*

(SMC 1) a functor

$$\begin{aligned} \square : \mathcal{C} \times \mathcal{C} &\to \mathcal{C} \\ (A, B) &\mapsto A \square B, \end{aligned}$$

(SMC 2) a distinguished object $e \in \mathcal{C}$,

(SMC 3) and four basic natural isomorphisms

$$\begin{aligned} e \square A &\cong A, \\ A \square e &\cong A, \\ (A \square B) \square C &\cong A \square (B \square C), \\ A \square B &\cong B \square A. \end{aligned}$$

subject to the coherence conditions given in [ML98].

*A **monoidal functor** between symmetric monoidal categories is a functor that respects all the axioms. The category of symmetric monoidal categories and monoidal functors will be denoted as* $\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}$.

Example 5.1.1 . The category $\mathcal{F}\mathrm{in}\mathcal{S}\mathrm{et}$ of finite sets is a symmetric monoidal category under disjoint union. The distinguished object is the empty set.

Example 5.1.2 . If $(\mathcal{C}, \square)$ is a symmetric monoidal category, then its category iso $(\mathcal{C})$ of isomorphisms is also a symmetric monoidal category under $\square$.

The axioms (SMC 1)–(SMC 3) make the set $\pi_0(B\mathcal{C})$ of the path components of the classifying space of the symmetric monoidal category $\mathcal{C}$ a monoid. The $S^{-1}S$-construction is a categorical method to study the group completion of this monoid.

Definition 5.1.2 (The Category $S^{-1}S$, [Wei13, Definition 4.2 on page 328]) *Let $(S, \square)$ be a symmetric monoidal category. We define a new category $S^{-1}S$ from S by the following data:*

(SIS 1) Objects in $S^{-1}S$ are pairs (m, n) of objects in S.

(SIS 2) A morphism $(m_1, m_2) \to (n_1, n_2)$ in $S^{-1}S$ is an equivalence class of composites

$$(m_1, m_2) \xrightarrow{s\square} (s\square m_1, s\square m_2) \xrightarrow{(f,g)} (n_1, n_2).$$

This composite is equivalent to

$$(m_1, m_2) \xrightarrow{t\square} (t\square m_1, t\square m_2) \xrightarrow{(f',g')} (n_1, n_2)$$

if and only if there is an isomorphism $\alpha : s \to t$ in S so that the composition with $\alpha \square \mathrm{id}_{m_i}$ sends f' and g' to f and g.

We note that a monoidal functor $S \to T$ induces a functor

$$S^{-1}S \to T^{-1}T.$$

Moreover, the category $S^{-1}S$ has a natural symmetric monoidal structure induced by the symmetric monoidal structure on S (see [Wei13, Remark 4.2.2 on page 329]). Thus the $S^{-1}S$-construction gives a functor

$$\begin{aligned} \mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} &\to \mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \\ S &\mapsto S^{-1}S, \end{aligned}$$

and $\pi_0\left(BS^{-1}S\right)$ is an Abelian group. Under the conditions as in [Wei13, Theorem 4.8 on page 333], $\pi_0\left(BS^{-1}S\right)$ is the group completion of $\pi_0\left(BS\right)$.

Definition 5.1.3 (The Category $\mathcal{S}_{\mathcal{A}}$) *Let $\mathcal{A}$ be a symmetric monoidal category. We define the category $\mathcal{S}_{\mathcal{A}}$ to be the $S^{-1}S$-construction of the category $\mathrm{iso}\left(\mathcal{A}\right)$ of isomorphisms in $\mathcal{A}$, as defined in Definition 5.1.2:*

$$\mathcal{S}_{\mathcal{A}} := \mathrm{iso}\left(\mathcal{A}\right)^{-1}\mathrm{iso}\left(\mathcal{A}\right) \tag{5.1}$$

In particular, we write

$$\mathcal{S}_R := \mathcal{S}_{\mathcal{F}\mathrm{ree}_R^{\mathrm{fg}}} \tag{5.2}$$

for a ring R.

We say a ring R satisfies the **Invariant Basis Property (IBP)** when $R^n \cong R^m$ if and only if $m = n$. The importance of IBP is that we have explicit descriptions for the classifying space $B\,\mathrm{iso}\left(\mathcal{F}\mathrm{ree}_R^{\mathrm{fg}}\right)$ and its homotopy group completion $B\mathcal{S}_R$.

Theorem 5.1.1 ([Wei13, Theorem 4.9 on page 334]) *Let R be a ring satisfying the IBP. Then the classifying space of the category* $\mathrm{iso}\left(\mathfrak{Free}_R^{\mathrm{fg}}\right)$ *is given by*

$$B\,\mathrm{iso}\left(\mathfrak{Free}_R^{\mathrm{fg}}\right) = \bigsqcup_{p\geq 1} BGL(p,R). \tag{5.3}$$

Moreover, the space $\mathbb{Z}\times BGL(R)^+$ is a model for the classifying space of the $S^{-1}S$-construction of the category $\mathrm{iso}\left(\mathfrak{Free}_R^{\mathrm{fg}}\right)$*:*

$$B\mathcal{S}_R \simeq \mathbb{Z}\times BGL(R)^+\,. \tag{5.4}$$

The natural homotopy group completion map

$$\mathfrak{gc}_R : \bigsqcup_{p\geq 1} BGL(p,R) \to \mathbb{Z}\times BGL(R)^+\,, \tag{5.5}$$

induced by the maps sending a matrix $A_p \in GL(p,R)$ to $(p,A_p)\in\mathbb{Z}\times GL(R)$, makes the diagram

$$\begin{tikzcd} B\,\mathrm{iso}\left(\mathfrak{Free}_R^{\mathrm{fg}}\right) \arrow[r] \arrow[d, equal] & B\mathcal{S}_R \arrow[d, "\lightning"] \\ \bigsqcup_{p\geq 1} BGL(p,R) \arrow[r, "\mathfrak{gc}_R"'] & \mathbb{Z}\times BGL(R)^+ \end{tikzcd} \tag{5.6}$$

commute up to homotopy. The lightning bolt in the diagram is a zig-zag of homotopy equivalences.

5.1.2 A Categorical Description of the Loday Pairing and the Spectrum $\mathbb{K}_R^{\mathrm{free}}$

Under Theorem 5.1.1, we give another formalism of the Loday pairing γ_{Loday} defined in Definition 2.1.2 using category theory. This formalism is well-documented in [Wei81] and we shall review it below.

Again, let us choose an isomorphism $R^m \otimes S^n \cong (R \otimes S)^{mn}$. Tensor product of modules gives a functor

$$\otimes : \mathrm{iso}\left(\mathfrak{Free}_R^{\mathrm{fg}}\right) \times \mathrm{iso}\left(\mathfrak{Free}_S^{\mathrm{fg}}\right) \to \mathrm{iso}\left(\mathfrak{Free}_{R\otimes S}^{\mathrm{fg}}\right). \tag{5.7}$$

We then have the following homotopy commutative diagram

$$\begin{tikzcd}
B\,\mathrm{iso}\left(\mathfrak{Free}_R^{\mathrm{fg}}\right) \wedge B\,\mathrm{iso}\left(\mathfrak{Free}_S^{\mathrm{fg}}\right) \arrow[r, "B\otimes"] \arrow[d, equal] & B\,\mathrm{iso}\left(\mathfrak{Free}_{R\otimes S}^{\mathrm{fg}}\right) \arrow[d, equal] \\
\left(\coprod_{p,q\geq 1} BGL(p,R) \times BGL(q,S)\right)_+ \arrow[r, "B\otimes"] \arrow[d] & \coprod_{r\geq 0} BGL(r, R\otimes S) \arrow[d] \\
\left(\coprod_{p,q\geq 1} BGL(p,R)^+ \times BGL(q,S)^+\right)_+ \arrow[r, "f"] \arrow[d, equal] & \coprod_{r\geq 0} BGL(r, R\otimes S)^+ \arrow[dd, "\mathfrak{gc}_{R\otimes S}^+"] \\
\left(\coprod_{p\geq 0} BGL(p,R)^+\right) \wedge \left(\coprod_{q\geq 0} BGL(q,S)^+\right) \arrow[d, "\mathfrak{gc}_R^+ \wedge \mathfrak{gc}_S^+"'] & \\
\left(\mathbb{Z} \times BGL(R)^+\right) \wedge \left(\mathbb{Z} \times BGL(S)^+\right) \arrow[r, dashed, "\gamma_{\mathrm{free}}"'] & \mathbb{Z} \times BGL(R\otimes S)^+.
\end{tikzcd} \tag{5.8}$$

The convention is that $BGL(p,R)^+$ denotes $BGL\,(p,R)$ for $p \leq 2$; for $p \geq 3$, it denotes the plus construction of $BGL\,(p,R)$ relative to the subgroup $E(p,R)$ generated by elementary matrices. The map f (resp. the maps $\mathfrak{gc}^+$) is induced by $B\otimes$ (resp. $\mathfrak{gc}$) via the universal property of the +-construction, which is well-defined up to homotopy.

The group completion $\left(\mathbb{Z} \times BGL(R)^+, \mathfrak{gc}_R\right)$ satisfies the following **cofinality condition**:

Condition 5.1.1 (Cofinality Condition) *For every $x := (\xi, \mathcal{M}) \in \mathbb{Z} \times BGL(R)^+$, there exists $m \in \mathbb{N}$ and $x_0 \in BGL(p,R)^+$ such that*

$$(\xi, \mathcal{M}) + (m, *_\infty) = \mathfrak{gc}_R(x_0),$$

where $*_\infty$ is the base-point of $BGL(R)^+$, and the addition operation is the H-group operation of $\mathbb{Z} \times BGL(R)^+$.

We want to define a map

$$\gamma_{\text{free}} : \left(\mathbb{Z} \times BGL(R)^+\right) \wedge \left(\mathbb{Z} \times BGL(S)^+\right) \to \mathbb{Z} \times BGL(R \otimes S)^+$$

so that Diagram (5.8) commutes up to weak homotopy.

Definition 5.1.4 (The map $\widetilde{\gamma_{\text{free}}}$) *If $x \in \mathbb{Z} \times BGL(R)^+$ (resp. $y \in \mathbb{Z} \times BGL(S)^+$), let $m \in \mathbb{N}$, $x_0 \in BGL(p,R)^+$ (resp. $n \in \mathbb{N}$, $y_0 \in BGL(q,S)^+$) be as in Cofinality Condition (5.1.1).*

We define a map

$$\widetilde{\gamma_{\text{free}}} : \left(\mathbb{Z} \times BGL(R)^+\right) \times \left(\mathbb{Z} \times BGL(S)^+\right) \to \mathbb{Z} \times BGL(R \otimes S)^+$$

by

$$\widetilde{\gamma_{\text{free}}}(x,y) := \mathfrak{gc}_{R\otimes S} \circ f(x_0, y_0) - \mathfrak{gc}_{R\otimes S} \circ f(*_m, y_0) - \mathfrak{gc}_{R\otimes S} \circ f(x_0, *_n) + \mathfrak{gc}_{R\otimes S} \circ f(*_m, *_n)\,.$$

Here, $*_m$ (resp. $*_n$) is the base-point of $BGL(m,R)^+$ (resp. $BGL(n,S)^+$). The plus and minus are from the H-group structure of $\mathbb{Z} \times BGL(R \otimes S)^+$.

The choice of isomorphism $R^m \otimes S^n \cong (R \otimes S)^{mn}$ implies the pairing $\widetilde{\gamma_{\text{free}}}$ is well-defined up to weak homotopy.

Lemma 5.1.1 *Under the notations in Diagram (5.8), the following diagram*

$$
\begin{tikzcd}
\left(\coprod_{p\geq 0} BGL(p,R)^+\right) \times \left(\coprod_{q\geq 0} BGL(q,S)^+\right) \arrow[r, "f"] \arrow[d, "\mathfrak{gc}_R \wedge \mathfrak{gc}_S"'] & \coprod_{r\geq 0} BGL(r, R\otimes S)^+ \arrow[d, "\mathfrak{gc}_{R\otimes S}"] \\
\left(\mathbb{Z}\times BGL(R)^+\right) \times \left(\mathbb{Z}\times BGL(S)^+\right) \arrow[r, "\widetilde{\gamma_{\mathrm{free}}}"'] & \mathbb{Z}\times BGL(R\otimes S)^+.
\end{tikzcd}
$$

commutes up to homotopy.

Proof. This is just a matter of diagram-chasing. Consider a point x in the image of $\mathfrak{gc}_R$. We know

$$x = (p, [M_p])$$

where $[M_p]$ is an equivalence class in $BGL(R)^+$, represented by some $M_p \in BGL(p,R)^+$. The Cofinality Condition (5.1.1) says there exists $m \in \mathbb{N}$ such that

$$(p, [M_p]) + (m, *_\infty) = (p+m, [M_p])$$

is in the image of $\mathfrak{gc}_R$. In fact, we have

$$(p+m, [M_p \boxplus *_m]) = \mathfrak{gc}_R(M_p \boxplus *_m)\,,$$

where the operation "$\boxplus$" is induced by block sum of matrices. Similarly, for $y = (q, N_q) \in \mathbb{Z}\times BGL(S)^+$, there exists $n \in \mathbb{N}$ such that

$$(q+n, [N_q \boxplus *_n]) = \mathfrak{gc}_S(N_q \boxplus *_n)\,.$$

We then compute

$$
\begin{aligned}
\widetilde{\gamma_{\mathrm{free}}}(x,y) &= \widetilde{\gamma_{\mathrm{free}}}((p,M_p),(q,N_q)) \\[2em]
&:= \mathfrak{gc}_{R\otimes S}\circ f(M_p \boxplus *_m, N_q \boxplus *_n) - \mathfrak{gc}_{R\otimes S}\circ f(*_m, N_q \boxplus *_n) \\
&\quad - \mathfrak{gc}_{R\otimes S}\circ f(M_p \boxplus *_m, *_n) + \mathfrak{gc}_{R\otimes S}\circ f(*_m, *_n) \\[2em]
&= \mathfrak{gc}_{R\otimes S}\circ f(M_p \boxplus *_m, N_q \boxplus *_n) - \mathfrak{gc}_{R\otimes S}\circ f(*_m, N_q \boxplus *_n) \\
&\quad - \mathfrak{gc}_{R\otimes S}\circ f(M_p \boxplus *_m, *_n) + \mathfrak{gc}_{R\otimes S}\circ f(*_m, *_n) \\[2em]
&\simeq \mathfrak{gc}_{R\otimes S}\circ f(M_p \boxplus *_m, N_q \boxplus *_n) - \mathfrak{gc}_{R\otimes S}\circ f(N_q \boxplus *_n, *_m) \\
&\quad - \mathfrak{gc}_{R\otimes S}\circ f(M_p \boxplus *_m, *_n) + \mathfrak{gc}_{R\otimes S}\circ f(*_m, *_n) \\[2em]
&= \mathfrak{gc}_{R\otimes S}\big(M_p \otimes N_q \boxplus M_p^{\boxplus n} \boxplus N_q^{\boxplus m} \boxplus *_{mn}\big) - \mathfrak{gc}_{R\otimes S}\big(N_q^{\boxplus m} \boxplus *_{mn}\big) \\
&\quad - \mathfrak{gc}_{R\otimes S}\big(M_p^{\boxplus n} \boxplus *_{mn}\big) + \mathfrak{gc}_{R\otimes S}(*_{mn}) \\[2em]
&\simeq \mathfrak{gc}_{R\otimes S}(M_p \otimes N_q)
\end{aligned}
$$

as desired. □

From the definition, it is clear that $\widetilde{\gamma_{\mathrm{free}}}$ is homotopically trivial on the wedge

$$\left(\mathbb{Z}\times BGL(R)^+\right) \vee \left(\mathbb{Z}\times BGL(S)^+\right),$$

hence $\widetilde{\gamma_{\mathrm{free}}}$ factors through the smash product to give the map

$$\gamma_{\mathrm{free}} : \left(\mathbb{Z}\times BGL(R)^+\right) \wedge \left(\mathbb{Z}\times BGL(S)^+\right) \to \mathbb{Z}\times BGL(R\otimes S)^+, \tag{5.9}$$

making Diagram (5.8) commute up to weak homotopy. In particular, when restricting onto the base-point component $\big(\{0\} \times BGL(R)^+\big) \wedge \big(\{0\} \times BGL(S)^+\big)$, we have

$$\gamma_{\mathrm{free}}\big|_{\left(\{0\}\times BGL(R)^+\right)\wedge\left(\{0\}\times BGL(S)^+\right)} = \gamma_{\mathrm{Loday}} \tag{5.10}$$

We are now able to define an intermediate spectrum that allows us to compare the Loday assembly and the universal assembly. Let us mimic the construction given in Proposition 2.3.2 and recall the map

$$\mathfrak{t}^+ : B\langle t\rangle \to K_{\Sigma\mathbb{Z}}$$

from in Definition 2.3.1.

Definition 5.1.5 (The Spectrum $\mathbb{K}_R^{\mathrm{free}}$) *Let R be a ring satisfying the IBP. Define the spectrum $\mathbb{K}_R^{\mathrm{free}}$ by having*

$$\begin{aligned}\left(\mathbb{K}_R^{\mathrm{free}}\right)_n &:= \mathbb{Z} \times BGL(\Sigma^n R)^+ \\ &\simeq B\mathcal{S}_{\Sigma^n R}\end{aligned} \tag{5.11}$$

for $n \geq 0$. Recall our notation that

$$K_R := K_0(R) \times BGL(R)^+ .$$

We write

$$K_R^{\mathrm{free}} := \mathbb{Z} \times BGL(R)^+ \tag{5.12}$$

The structure maps are given by the composition

$$B\langle t\rangle \wedge K_{\Sigma^n R}^{\mathrm{free}} \xrightarrow{\widehat{\mathfrak{t}}^+\wedge \mathrm{id}} K_{\Sigma\mathbb{Z}}^{\mathrm{free}} \wedge K_{\Sigma^n R}^{\mathrm{free}} \xrightarrow{\gamma_{\mathrm{free}}} K_{\Sigma^{n+1}R}^{\mathrm{free}}, \tag{5.13}$$

where $\widehat{\mathfrak{t}}^+$ *is induced by the group homomorphism*

$$\begin{aligned} \langle t \rangle &\to GL(\Sigma\mathbb{Z}) \\ t &\mapsto \tau, \end{aligned} \tag{5.14}$$

and $\tau \in GL(\Sigma\mathbb{Z})$ *be the element defined in Equation (2.16).*

It is worth pointing out that

$$\pi_i(\mathbb{K}_R^{\text{free}}) \cong K_i(R)$$

for all $i \geq 0$. This can be seen from the definition of homotopy groups of a spectrum, or Theorem 5.1.2 below. However, we do not know if there is a map of spectra

$$\mathbb{K}_R^{\text{free}} \to \mathbb{K}_R^{\text{GW}}$$

or in the other direction that induces isomorphisms on homotopy groups. It is worth pointing out the following. There is a map of spaces

$$K_R^{\text{free}} \to K_R \tag{5.15}$$

given by

$$\begin{aligned} K_R^{\text{free}} := \mathbb{Z} \times BGL(R)^+ &\to K_0(R) \times BGL(R)^+ =: K_R \\ (m, x) &\mapsto ([R^m], x). \end{aligned}$$

This map induces the identity maps on π_n for $n \geq 1$. However, it does not extend to a map of spectra. Otherwise, we would have a commutative diagram

$$\begin{tikzcd}
B\langle t\rangle \wedge K_R^{\text{free}} \arrow[r, "\mathrm{t}^+ \wedge \mathrm{id}"] \arrow[d] & K_{\Sigma\mathbb{Z}}^{\text{free}} \wedge K_R^{\text{free}} \arrow[r, "\gamma_{\text{free}}"] \arrow[d] & K_{\Sigma R}^{\text{free}} \arrow[d] \\
B\langle t\rangle \wedge K_R \arrow[r, "\mathrm{t}^+ \wedge \mathrm{id}"'] & K_{\Sigma\mathbb{Z}} \wedge K_R \arrow[r, "\gamma'_{\text{Loday}}"'] & K_{\Sigma R}
\end{tikzcd}$$

of spaces. The right square then induces the following commutative diagram

$$\begin{tikzcd}
K_i(\Sigma\mathbb{Z}) \otimes K_j(R) \arrow[r, "\star"] \arrow[d] & K_{i+j}(\Sigma R) \arrow[d] \\
K_i(\Sigma\mathbb{Z}) \otimes K_j(R) \arrow[r, "\star'"'] & K_{i+j}(\Sigma R)
\end{tikzcd}$$

on homotopy groups. When $i,\, j \geq 1$, the vertical arrows are identities. The commutativity of this square would imply that

$$[f] \star [g] = [f] \star' [g] \in K_{i+j}(\Sigma R)$$

for all $[f] \in K_i(\Sigma\mathbb{Z})$ and $[g] \in K_j(R)$, which is absurd by Proposition 2.2.1. As a result, the map in Equation (5.15) does not extend to a map of spectra.

Finally, because of the construction of the structure maps in Equation (5.13), the pairing γ_{free} constructed in Equation (5.9) extends to a weak pairing of spectra. By abuse of notation, we shall denote it as

$$\gamma_{\text{free}} : \mathbb{K}_R^{\text{free}} \wedge \mathbb{K}_S^{\text{free}} \to \mathbb{K}_{R\otimes S}^{\text{free}} \tag{5.16}$$

for any two rings R, S satisfying the IBP.

5.1.3 Idempotent Completion and the Spectrum $\mathbb{K}_R^{\text{proj}}$

A quick inspection of the classifying space $B\mathcal{S}_R$ given in Theorem 5.1.1 says

$$\pi_0\left(B\mathcal{S}_R\right) \ncong K_0(R).$$

The spectrum $\mathbb{K}_R^{\mathrm{free}}$ defined in Definition 5.1.5 fixes this problem at the spectrum-level, and we describe a space-level solution to this problem here.

Definition 5.1.6 (Idempotent Completion, [Wei13, page 143]) *The* ***idempotent completion*** *of a category $\mathcal{C}$ is the category $\widehat{\mathcal{C}}$ whose objects are pairs (C,p) with $p: C \to C$ an idempotent endomorphism of an object C of $\mathcal{C}$.*

A morphism $(C,p) \to (C',p')$ in $\widehat{\mathcal{C}}$ is a map $f: C \to C'$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "p"'] & C' \arrow[d, "p'"] \\ C \arrow[r, "f"'] & C' \end{tikzcd} \tag{5.17}$$

commutes.

Example 5.1.3 . Because any projective module is a direct summand of a free module, the idempotent completion of $\mathcal{F}\mathrm{ree}_R^{\mathrm{fg}}$ is a (small) category equivalent to the category of finitely generated projective R-modules, and we denote this category by

$$\mathcal{P}\mathrm{roj}_R^{\mathrm{fg}} := \widehat{\mathcal{F}\mathrm{ree}_R^{\mathrm{fg}}} \tag{5.18}$$

and call it the category of finitely generated projective R-modules by abuse of terminology.

If $\mathcal{A}$ is a symmetric monoidal category, then so is $\widehat{\mathcal{A}}$. Thus, we define:

Definition 5.1.7 (The Category $\mathcal{P}_{\mathcal{A}}$) *Let $\mathcal{A}$ be a symmetric monoidal category. We define the category $\mathcal{P}_{\mathcal{A}}$ to be the $S^{-1}S$-construction of the category $\operatorname{iso}\left(\widehat{\mathcal{A}}\right)$ of isomorphisms in the idempotent completion $\widehat{\mathcal{A}}$, as defined in Definition 5.1.2:*

$$\mathcal{P}_{\mathcal{A}} := \operatorname{iso}\left(\widehat{\mathcal{A}}\right)^{-1} \operatorname{iso}\left(\widehat{\mathcal{A}}\right) \tag{5.19}$$

In other words, $\mathcal{P}_{\mathcal{A}} := \mathcal{S}_{\widehat{\mathcal{A}}}$.

In particular, we write

$$\mathcal{P}_R := \mathcal{P}_{\mathcal{F}\mathrm{ree}_R^{\mathrm{fg}}} \tag{5.20}$$

for a ring R.

The following theorem tells us how $\mathcal{S}_R$ and $\mathcal{P}_R$ are related.

Theorem 5.1.2 ([CP97]) *If $\mathcal{C}$ is a category, then there is a canonical embedding*

$$\begin{aligned} \mathcal{C} &\to \widehat{\mathcal{C}} \\ C &\mapsto (C, \mathrm{id}). \end{aligned} \tag{5.21}$$

Therefore, we have an induced functor

$$\mathcal{S}_{\mathcal{C}} \to \mathcal{P}_{\mathcal{C}} \tag{5.22}$$

between the categories defined in Definition 5.1.3 and Definition 5.1.7.

When $\mathcal{C} = \mathcal{F}\mathrm{ree}_R^{\mathrm{fg}}$ is the category of finitely generated free modules over the ring R, the induced group homomorphism

$$\pi_n\left(B\mathcal{S}_R\right) \to \pi_n\left(B\mathcal{P}_R\right) \tag{5.23}$$

is an isomorphism for $n \geq 1$.

Definition 5.1.8 (Pairing of Symmetric Monoidal Categories, [May80, page 308]) *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be symmetric monoidal categories. A* ***pairing of symmetric monoidal categories*** *is a functor*

$$\begin{aligned} \otimes : \mathcal{A} \times \mathcal{B} &\to \mathcal{C} \\ (A, B) &\mapsto A \otimes B \end{aligned} \tag{5.24}$$

satisfying the following condition:

(PSMC 1) For any objects $A \in \mathcal{A}$, $B \in \mathcal{B}$, we have

$$0_{\mathcal{A}} \otimes B = 0_{\mathcal{C}} = A \otimes 0_{\mathcal{B}} \tag{5.25}$$

(PSMC 2) For any objects A, $A' \in \mathcal{A}$ and B, $B' \in \mathcal{B}$, there is a coherent natural bi-distributivity isomorphism:

$$(A \Box_{\mathcal{A}} A') \otimes (B \Box_{\mathcal{B}} B') \cong (A \otimes B) \Box_{\mathcal{C}} (A \otimes B') \Box_{\mathcal{C}} (A' \otimes B) \Box_{\mathcal{C}} (A' \otimes B') \,. \tag{5.26}$$

Example 5.1.4 . The standard example of a pairing of symmetric monoidal categories is the tensor product of modules.

Theorem 5.1.3 ([May80, Theorem 1.6 on page 307, and Theorem 2.1 on page 310]) *A pairing $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ of symmetric monoidal categories determines a pairing*

$$B\mathcal{S}_{\mathcal{A}} \wedge B\mathcal{S}_{\mathcal{B}} \to B\mathcal{S}_{\mathcal{C}} \tag{5.27}$$

of homotopy group completions. Moreover, this map fits into the following homotopy commutative diagram:

$$\begin{tikzcd}
B\operatorname{iso}(\mathcal{A}) \wedge B\operatorname{iso}(\mathcal{B}) \arrow[r] \arrow[d] & B\operatorname{iso}(\mathcal{C}) \arrow[d] \\
B\mathcal{S}_{\mathcal{A}} \wedge B\mathcal{S}_{\mathcal{A}} \arrow[r] & B\mathcal{S}_{\mathcal{C}},
\end{tikzcd} \tag{5.28}$$

where the vertical arrows are the homotopy group completion maps.

Example 5.1.5 . According to [Wei81, Diagram 3.1 on page 500], the map

$$\gamma_{\text{free}} : K_R^{\text{free}} \wedge K_S^{\text{free}} \to K_{R\otimes S}^{\text{free}}$$

is the pairing of homotopy group completions determined by the tensor product

$$\otimes : \operatorname{iso}\left(\mathfrak{Free}_R^{\text{fg}}\right) \times \operatorname{iso}\left(\mathfrak{Free}_S^{\text{fg}}\right) \to \operatorname{iso}\left(\mathfrak{Free}_{R\otimes S}^{\text{fg}}\right)$$

of modules.

Definition 5.1.9 (The Pairing γ_{proj}) *Let R, S be rings.*

We define the pairing

$$\gamma_{\text{proj}} : B\mathcal{P}_R \wedge B\mathcal{P}_S \to B\mathcal{P}_{R\otimes S} \tag{5.29}$$

to be the pairing that is functorially determined by the tensor product

$$\otimes : \operatorname{iso}\left(\mathfrak{Proj}_R^{\text{fg}}\right) \times \operatorname{iso}\left(\mathfrak{Proj}_S^{\text{fg}}\right) \to \operatorname{iso}\left(\mathfrak{Proj}_{R\otimes S}^{\text{fg}}\right)$$

of modules in the sense of Theorem 5.1.3.

Because

$$
\begin{aligned}
B\mathcal{P}_R &\simeq K_0(R) \times BGL(R)^+ \\
&=: K_R
\end{aligned}
\tag{5.30}
$$

by [Wei13, Corollary 4.11.1 on page 337], the pairing γ_{proj} induces a map

$$
\gamma_{\mathrm{proj}} : K_R \wedge K_S \to K_{R\otimes S}. \tag{5.31}
$$

If in addition that the rings R, S both satisfy the IBP, then functoriality of the pairings says the following diagram

$$
\begin{tikzcd}
B\,\mathrm{iso}\left(\mathcal{F}\mathrm{ree}_R^{\mathrm{fg}}\right) \wedge B\,\mathrm{iso}\left(\mathcal{F}\mathrm{ree}_S^{\mathrm{fg}}\right) \arrow[rr, "B\otimes"] \arrow[dr, "B\,(\mathrm{inclusion})"] \arrow[dd] & & B\,\mathrm{iso}\left(\mathcal{F}\mathrm{ree}_{R\otimes S}^{\mathrm{fg}}\right) \arrow[dr, "B\,(\mathrm{inclusion})"] \arrow[dd] & \\
& B\,\mathrm{iso}\left(\mathcal{P}\mathrm{roj}_R^{\mathrm{fg}}\right) \wedge B\,\mathrm{iso}\left(\mathcal{P}\mathrm{roj}_S^{\mathrm{fg}}\right) \arrow[rr, "B\otimes"] \arrow[dd] & & B\,\mathrm{iso}\left(\mathcal{P}\mathrm{roj}_{R\otimes S}^{\mathrm{fg}}\right) \arrow[dd] \\
B\mathcal{S}_R \wedge B\mathcal{S}_S \arrow[rr, "\gamma_{\mathrm{free}}"] \arrow[dr, "B(5.22)"'] & & B\mathcal{S}_{R\otimes S} \arrow[dr, "B(5.22)"'] & \\
& B\mathcal{P}_R \wedge B\mathcal{P}_S \arrow[rr, "\gamma_{\mathrm{proj}}"'] & & B\mathcal{P}_{R\otimes S}
\end{tikzcd}
\tag{5.32}
$$

commutes up to weak homotopy, where the vertical arrows are the homotopy group completion maps. However, Proposition 2.2.1 says the induced product maps are different on homotopy groups. Therefore, the pairings γ'_{Loday} and γ_{proj} are not homotopic.

Definition 5.1.10 (The Spectrum $\mathbb{K}_R^{\mathrm{proj}}$) *Let R be a ring. Define the spectrum $\mathbb{K}_R^{\mathrm{proj}}$ by setting*

$$\left(\mathbb{K}_R^{\text{proj}}\right)_n := K_{\Sigma^n R} \simeq B\mathcal{P}_{\Sigma^n R} \tag{5.33}$$

for $n \geq 0$. The structure maps are given by the composition

$$B\langle t\rangle \wedge K_R \xrightarrow{\hat{\mathfrak{t}}^+ \wedge \mathrm{id}} K_{\Sigma\mathbb{Z}} \wedge K_R \xrightarrow{\gamma_{\text{proj}}} K_{\Sigma R}, \tag{5.34}$$

where $\hat{\mathfrak{t}}$ is induced by the group homomorphism in Equation (5.14).

Again, the pairing γ_{proj} constructed in Equation (5.29) extends to a weak pairing of spectra. By abuse of notation, we shall denote it as

$$\gamma_{\text{proj}} : \mathbb{K}_R^{\text{proj}} \wedge \mathbb{K}_S^{\text{proj}} \to \mathbb{K}_{R\otimes S}^{\text{proj}} \tag{5.35}$$

for any two rings R, S.

The following result is an immediate consequence of Theorem 5.1.3, Definition 5.1.5 and Definition 5.1.10.

Theorem 5.1.4 *Let R, S be rings satisfying the IBP.*

The inclusion functor

$$\mathcal{F}\mathrm{ree}_R^{\mathrm{fg}} \to \mathcal{P}\mathrm{roj}_R^{\mathrm{fg}}$$

induces the following diagram

$$\begin{CD} \mathbb{K}_R^{\text{free}} \wedge \mathbb{K}_S^{\text{free}} @>{\gamma_{\text{free}}}>> \mathbb{K}_{R\otimes S}^{\text{free}} \\ @VVV @VVV \\ \mathbb{K}_R^{\text{proj}} \wedge \mathbb{K}_S^{\text{proj}} @>>{\gamma_{\text{proj}}}> \mathbb{K}_{R\otimes S}^{\text{proj}} \end{CD} \tag{5.36}$$

of weak pairings of spectra. More precisely, for each pair (m, n) of non-negative integers, the diagram

$$\begin{tikzcd} K^{\mathrm{free}}_{\Sigma^m R} \wedge K^{\mathrm{free}}_{\Sigma^n S} \arrow[r, "\gamma_{\mathrm{free}}"] \arrow[d] & K^{\mathrm{free}}_{\Sigma^{m+n}(R\otimes S)} \arrow[d] \\ K_{\Sigma^m R} \wedge K_{\Sigma^n S} \arrow[r, "\gamma_{\mathrm{proj}}"'] & K_{\Sigma^{m+n}(R\otimes S)} \end{tikzcd} \tag{5.37}$$

commutes up to weak homotopy.

The presence of the pairing map γ_{free} allows one to define a multiplication map

$$\begin{aligned} \star_{\mathrm{free}} : K_i(R) \otimes K_j(S) &\to K_{i+j}(R \otimes S) \\ [f] \otimes [g] &\mapsto [\gamma_{\mathrm{free}} \circ (f \wedge g)] \end{aligned} \tag{5.38}$$

It is clear from Diagram (5.8) that $\star_{\mathrm{free}}$ recovers tensor product of modules when $i = j = 0$, and coincides with Loday's multiplication $\star$ by Equation (5.10) when $i, j \geq 1$. However, the following cases are still open:

Question 5.1.1 *What is the product map*

$$\star_{\mathrm{free}} : K_0(R) \otimes K_j(S) \to K_j(R \otimes S)? \tag{5.39}$$

In particular, can we relate this product map with the classical product map

$$\# : K_0(R) \otimes K_1(S) \to K_j(R \otimes S)$$

defined by Milnor as in Equation (2.26) when $j = 1, 2$?

5.2 Categories Over Metric Spaces and the Non-Connective Pedersen-Weibel Algebraic K-Theory Spectrum

Recall the definition of an additive category.

Definition 5.2.1 (Additive Category) *A category $\mathcal{A}$ is* **additive** *if all of the following are satisfied:*

(A1) The collection of morphisms $\mathrm{Hom}\,(A, B)$ *from A to B is an Abelian group for each object $A,\ B \in \mathcal{A}$.*

(A2) The composition of morphisms

$$\mathrm{Hom}\,(A, B) \times \mathrm{Hom}\,(B, C) \to \mathrm{Hom}\,(A, C)$$

is a bilinear map.

(A3) There is a distinguished object $0_{\mathcal{A}} \in \mathcal{A}$ such that

$$\mathrm{Hom}\,(A, 0_{\mathcal{A}}) = 0 = \mathrm{Hom}\,(0_{\mathcal{A}}, A)$$

for all objects $A \in \mathcal{A}$.

(A4) There is a binary operation

$$\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

which is both the categorical product and categorical coproduct.

Example 5.2.1 . Our standard example is the category $\mathfrak{Free}_R^{\mathrm{fg}}$ of finitely generated free modules over the ring R. The binary operation in (A4) is the direct sum of modules. Note that the ring R does not have to satisfy the IBP for $\mathfrak{Free}_R^{\mathrm{fg}}$ being additive.

Definition 5.2.2 (Filtered Additive Category, [PW85, Definition 1.1 on page 168]) *An additive category $\mathcal{A}$ is said to be* ***filtered*** *if there is an increasing filtration*

$$F_0(A,B) \subseteq \cdots \subseteq F_n(A,B) \subseteq \cdots \subseteq \mathrm{Hom}\,(A,B)$$

on $\mathrm{Hom}\,(A,B)$ *for every pair of objects A, $B \in \mathcal{A}$, satisfying the following properties:*

(F1) Each $F_i(A,B)$ is a subgroup of $\mathrm{Hom}\,(A,B)$.

(F2) $\bigcup\limits_i F_i(A,B) = \mathrm{Hom}\,(A,B)$.

(F3) The identity map id_A *and the zero map 0_A are in $F_0(A,A)$.*

(F4) The canonical maps

$$A \oplus B \to A,$$
$$A \to A \oplus B$$

and all coherence isomorphisms are in $F_0(A,B)$.

(F5) The composition law in $\mathcal{A}$ respects the filtration, meaning that if $f \in F_i(A,B)$ and $g \in F_j(B,C)$, then $g \circ f \in F_{i+j}(A,C)$.

Definition 5.2.3 (The Category $\mathcal{C}_M(\mathcal{A})$, [Spe04, Definition 4.2 on page 14]) *Let $\mathcal{A}$ be a category and (M,d) be a metric space. We definition the category $\mathcal{C}_M(\mathcal{A})$ as follows*

(i) objects are sets $\{A_x\}_{x\in M}$ of objects $A_x \in \mathcal{A}$, with the condition that the collection $\{x \in M \mid A_x \neq 0\}$ is locally finite.

(ii) A morphism $\varphi : A \to B$ in $\mathcal{C}_M(\mathcal{A})$ is a collection of morphisms

$$\varphi^x_y : A_x \to B_y$$

*in $\mathcal{A}$ with a bound $R \geq 0$, so that $\varphi^x_y = 0$ whenever $d(x, y) > R$. In this case, we say the morphism $\varphi : A \to B$ is **bounded** by R.*

Note that if $\mathcal{A}$ is an additive category, then so is $\mathcal{C}_M(\mathcal{A})$. Secondly, the boundedness condition on morphisms gives a natural filtration on the Hom-sets of $\mathcal{C}_M(\mathcal{A})$—for each pair of objects $A,\ B \in \mathcal{C}_M(\mathcal{A})$, the collection $F_i(A, B)$ consists of morphisms $A \to B$ in $\mathcal{C}_M(\mathcal{A})$ that are bounded by i. Thus the category $\mathcal{C}_M(\mathcal{A})$ is filtered additive in the sense of Definition 5.2.2. Thirdly, if the category $\mathcal{A}$ we start with is already filtered, we require all components $\varphi^x_y : A_x \to B_y$ of the morphism $\varphi : A \to B$ have filtration degree i as well.

Example 5.2.2 ($\mathcal{C}_i(\mathcal{A})$). Our standard example for $\mathcal{C}_M(\mathcal{A})$ is the case when $M = \mathbb{Z}^i$. Let us equip $\mathbb{Z}^i$ with the metric induced by the ℓ^∞-norm, so that

$$\begin{aligned} d(\vec{x}, \vec{y}) &:= \|\vec{x} - \vec{y}\|_{\ell^\infty} \\ &:= \max_{1 \leq j \leq i} |x_j - y_j| \end{aligned}$$

for $\vec{x} = (x_1, \cdots, x_i)$, $\vec{y} = (y_1, \cdots, y_i) \in \mathbb{Z}^i$. In this case, we write

$$\mathcal{C}_i(\mathcal{A}) := \begin{cases} \mathcal{A} & \text{if } i = 0, \\ \mathcal{C}_{\mathbb{Z}^i}(\mathcal{A}) & \text{if } i > 0 \end{cases} \tag{5.40}$$

and we get a sequence $\{\mathcal{C}_i(\mathcal{A})\}_{i=0}^\infty$ of categories. Note that the categories $\mathcal{C}_i(\mathcal{C}_j(\mathcal{A}))$ and $\mathcal{C}_{i+j}(\mathcal{A})$ are isomorphic. The sequence $\{\mathcal{C}_i(\mathcal{A})\}_{i=0}^\infty$ was first constructed by Pedersen-Weibel to produce non-connective spectra from additive categories [PW85]. In particular, when $\mathcal{A}$ is the category of finitely generated free modules, the resulting spectrum has homotopy groups isomorphic to algebraic K-groups of the underlying ring. The categories $\mathcal{C}_M(\mathcal{A})$ were also studied by Carlsson-Pedersen to prove the K-theoretic Novikov Conjecture for a large class of groups [CP95].

Finally, we give $\mathcal{C}_i(\mathcal{A})$ the split exact structure as in [PW85, page 171]. More precisely, We say a chain

$$A \to C \to B$$

of morphisms in $\mathcal{C}_1(\mathcal{A})$ is an exact sequence if there is an isomorphism $A \oplus B \xrightarrow{\cong} C$ in $\mathcal{C}_1(\mathcal{A})$ such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{id}"'] & A \oplus B \arrow[r] \arrow[d, "\cong"'] & B \arrow[d, "\mathrm{id}"] \\ A \arrow[r] & C \arrow[r] & B \end{tikzcd} \tag{5.41}$$

commutes. Because $\mathcal{C}_i(\mathcal{C}_j(\mathcal{A})) \cong \mathcal{C}_{i+j}(\mathcal{A})$, this gives an exact category structure on $\mathcal{C}_i(\mathcal{A})$ inductively.

Definition 5.2.4 (The Category $\mathcal{P}_{\mathcal{A},i}$) *Let $\mathcal{A}$ be a split exact, additive category. For each integer $i \geq 0$, we define*

$$\mathcal{P}_{\mathcal{A},i} := \mathcal{P}_{\mathcal{C}_i(\mathcal{A})} \tag{5.42}$$

as in Definition 5.1.7. In particular, we write:

$$\mathcal{P}_{R,i} := \mathcal{P}_{\mathcal{C}_i\left(\mathcal{F}\mathrm{ree}_R^{\mathrm{fg}}\right)} \tag{5.43}$$

for a ring R.

Pedersen-Weibel proved the following:

Theorem 5.2.1 ([PW85, Theorem B on page 167]) *Let $\mathcal{A}$ be a split exact, additive category.*

For each $i \geq 0$, we define

$$\left(\mathbb{K}_{\mathcal{A}}^{\mathrm{PW}}\right)_i := B\mathcal{P}_{\mathcal{A},i}. \tag{5.44}$$

Then for each $i \geq 0$, we have the delooping

$$\left(\mathbb{K}_{\mathcal{A}}^{\mathrm{PW}}\right)_i \simeq \Omega\left[\left(\mathbb{K}_{\mathcal{A}}^{\mathrm{PW}}\right)_{i+1}\right]. \tag{5.45}$$

Thus the sequence $\left\{\left(\mathbb{K}_{\mathcal{A}}^{\mathrm{PW}}\right)_i\right\}_{i=0}^{\infty}$ of spaces forms an Ω-spectrum.

Theorem 5.2.2 (The Non-connective Pedersen-Weibel K-theory Spectrum of Rings, [PW85, Theorem A on page 166]) *Let R be a ring.*

The spectrum

$$\mathbb{K}_R^{\mathrm{PW}} := \mathbb{K}_{\mathcal{F}\mathrm{ree}_R^{\mathrm{fg}}}^{\mathrm{PW}} \tag{5.46}$$

is an Ω-spectrum, the so-called ***non-connective Pedersen-Weibel algebraic K-theory spectrum of*** *R, whose homotopy groups are*

$$\pi_i\left(\mathbb{K}_R^{\mathrm{PW}}\right) \cong \begin{cases} K_i(R) & \text{if } i \geq 0, \\ \begin{array}{c}\text{negative } K\text{-groups of } R \\ \text{defined by Bass in [Bas68]}\end{array} & \text{if } i < 0. \end{cases} \tag{5.47}$$

5.3 The Universal Assembly Map

Definition 5.3.1 (Homotopy invariant, excisive, and strongly excisive functors) *Let $\mathcal{S}\mathrm{paces}$ be the category of (un-based) topological spaces homotopy equivalent to (not necessary finite) CW-complexes and continuous maps. We say a functor*

$$\mathbb{F} : \mathcal{S}\mathrm{paces} \to \mathcal{S}\mathrm{pectra}$$

is **homotopy invariant** *if it takes homotopy equivalences to homotopy equivalences.*

A homotopy invariant functor $\mathbb{F}$ is

(1) **excisive** *if it is excisive and preserves homotopy push-out squares, and* $\mathbb{F}(\emptyset)$ *is contractible;*

(2) **strongly excisive** *if it preserves arbitrary coproducts, up to homotopy equivalence.*

Theorem 5.3.1 (Weiss-Williams/Universal Assembly, [WW95, Theorem 1.1. on page 333]) *For any homotopy invariant functor*

$$\mathbb{F} : \mathcal{S}\mathrm{paces} \to \mathcal{S}\mathrm{pectra},$$

there exists a strongly excisive functor

$$\mathbb{F}^{\%} : \mathcal{S}\mathrm{paces} \to \mathcal{S}\mathrm{pectra}$$

and a homotopy natural transformation

$$\alpha_{\mathrm{WW}} : \mathbb{F}^{\%} \Rightarrow \mathbb{F}, \tag{5.48}$$

the so-called **Weiss-Williams assembly***, or the* **universal assembly***, such that the component*

$$\alpha_{\mathrm{WW}} : \mathbb{F}^{\%}\,(\mathrm{point}) \to \mathbb{F}\,(\mathrm{point})$$

of the homotopy natural transformation is a homotopy equivalence. Moreover, $\mathbb{F}^{\%}$ *and* α_{WW} *can be made to depend functorially on* $\mathbb{F}$*.*

Proof. (Outline)

The point is that the functor

$$X \mapsto X_+ \wedge \mathbb{F}\,(\mathrm{point}) \tag{5.49}$$

is a model for $\mathbb{F}^{\%}$; and the homotopy natural transformation

$$\alpha_{\mathrm{WW}} : \mathbb{F}^{\%} \Rightarrow \mathbb{F}$$

is induced by the constant map $X \to \text{point}$. $\square$

Theorem 5.3.2 ([WW95, Observation 1.3 on page 336]) *If the homotopy invariant functor*

$$\mathbb{F} : \mathcal{S}\mathrm{paces} \to \mathcal{S}\mathrm{pectra}$$

is already excisive, then the component

$$(\alpha_{\mathrm{WW}})_X : \mathbb{F}^{\%}(X) \to \mathbb{F}(X)$$

of the homotopy natural transformation

$$\alpha_{\mathrm{WW}} : \mathbb{F}^{\%} \Rightarrow \mathbb{F}$$

is a homotopy equivalence for every compact X.

If $\mathbb{F}$ is strongly excisive, then $(\alpha_{\mathrm{WW}})_X$ is a homotopy equivalence for all X.

Theorem 5.3.3 (Universal Property of the Weiss-Williams Assembly, [WW95, page 336]) *Suppose the homotopy invariant functor*

$$\mathbb{F} : \mathcal{S}\mathrm{paces} \to \mathcal{S}\mathrm{pectra}$$

admits another homotopy natural transformation

$$\beta : \mathbb{E} \Rightarrow \mathbb{F}$$

from a strongly excisive functor $\mathbb{E}$. *Then the diagram*

$$\begin{tikzcd} \mathbb{E}^{\%} \arrow[r, "\alpha^{\mathbb{E}}_{\mathrm{WW}}"] \arrow[d, "\beta^{\%}"'] & \mathbb{E} \arrow[d, "\beta"] \\ \mathbb{F}^{\%} \arrow[r, "\alpha^{\mathbb{F}}_{\mathrm{WW}}"'] & \mathbb{F} \end{tikzcd} \tag{5.50}$$

commutes.

We conclude by the following remark. In Diagram (5.50), if the component

$$\beta_{\mathrm{point}} : \mathbb{E}(\mathrm{point}) \to \mathbb{F}(\mathrm{point})$$

of the homotopy natural transformation β is a homotopy equivalence, then

$$\beta^{\%}_X : \mathbb{E}^{\%}(X) \to \mathbb{F}^{\%}(X)$$

is a homotopy equivalence for all X by the Eilenberg-Steenrod Uniqueness Theorem ([ES52, Theorem 10.1 on page 100–101]). So β is homotopic to $\alpha^{\mathbb{F}}_{\mathrm{WW}}$ after identifying the source and target.

5.4 A Model for the Universal Assembly

We recall the notions of ringoids and modules over them as in [WW95, page 337–338].

Definition 5.4.1 (Ringoid, [WW95, page 337]) *A* **ringoid** *is a small category in which*

(1) all morphism sets come equipped with an Abelian group structure,

(2) composition of morphisms is bilinear.

Example 5.4.1 (The Ringoid $R[\mathcal{C}]$ of the Category $\mathcal{C}$ Over the Ring R). Let R be a ring, and $\mathcal{C}$ be a small category. We define the ringoid $R[\mathcal{C}]$ to be the category having the same objects as $\mathcal{C}$, and the morphism set

$$\mathrm{mor}_{R[\mathcal{C}]}(x, y) := R\left\langle \mathrm{mor}_{\mathcal{C}}(x, y)\right\rangle$$

is the free left R-module generated by the set $\mathrm{mor}_{\mathcal{C}}(x, y)$. When $\mathcal{C} = G$ is a group, considered as a category with one object, then the ringoid $R[\mathcal{C}]$ is the group ring $R[G]$, considered as a category with one object, hence justifying the notation.

The category $R[\mathcal{C}]$ is also referred as the **R-category associated to** $\mathcal{C}$ in [DL98, page 212].

Definition 5.4.2 (Modules over Ringoids, [WW95, page 338]) *Let $\mathcal{R}$ be a ringoid.*

A **left $\mathcal{R}$-module** *is a functor*

$$f : \mathcal{R} \to \mathcal{A}\mathrm{belian}$$

from $\mathcal{R}$ to the category of Abelian groups, such that the induced map

$$f : \mathrm{mor}_{\mathcal{R}}(x, y) \to \mathrm{mor}_{\mathcal{A}\mathrm{belian}}(f(x), f(y))$$

is a group homomorphism. A **right $\mathcal{R}$-module** *is a left $\mathcal{R}^{op}$-module.*

A left $\mathcal{R}$-module f is

(1) **free on one generator** *if it is representable:*

$$f(-) \cong \mathrm{mor}_{\mathcal{R}}(x, -)$$

for some $x \in \mathcal{R}$;

(2) **finitely generated free** *if it is isomorphic to a finite direct sum of representable ones;*

(3) **finitely generated projective** *if it is a direct summand of a finitely generated free one.*

The category $\mathcal{M}\mathrm{odule}_{\mathcal{R}}$ of left $\mathcal{R}$-modules and natural transformations forms an Abelian category. The subcategory $\mathcal{F}\mathrm{ree}^{\mathrm{fg}}_{\mathcal{R}}$ of finitely generated free left $\mathcal{R}$-modules is a split exact, additive category. We can then use the Pedersen-Weibel construction as in Theorem 5.2.1 to get an Ω-spectrum

$$\mathbb{K}^{\mathrm{PW}}_{\mathcal{R}} := \mathbb{K}^{\mathrm{PW}}_{\mathcal{F}\mathrm{ree}^{\mathrm{fg}}_{\mathcal{R}}}. \tag{5.51}$$

Now, when $\mathcal{R} = R$ is a ring, considered as a category with one object, there is a natural equivalence

$$\mathcal{F}\mathrm{ree}^{\mathrm{fg}}_{\mathcal{R}} \simeq \mathcal{F}\mathrm{ree}^{\mathrm{fg}}_{R}$$

of categories. In this case, the spectra $\mathbb{K}^{\mathrm{PW}}_{\mathcal{R}}$ and $\mathbb{K}^{\mathrm{PW}}_{R}$ are weakly equivalent.

Let us construct a functor $\mathbb{Y} : \mathcal{S}\mathrm{paces} \to \mathcal{S}\mathrm{pectra}$. Let $\Pi(X)$ be the fundamental groupoid of the topological space X. Given a ring R, we can form the ringoid $R[\Pi(X)]$ as in Example 5.4.1. Our functor is then given by:

Definition 5.4.3 (The Functor $\mathbb{Y}$) *Let R be a ring. We define the functor $\mathbb{Y}$ by*

$$\begin{aligned} \mathbb{Y} : \mathcal{S}\mathrm{paces} &\to \mathcal{S}\mathrm{pectra} \\ X &\mapsto \mathbb{K}^{\mathrm{PW}}_{R[\Pi(X)]} \end{aligned} \tag{5.52}$$

Clearly, the functor $\mathbb{Y}$ is homotopy invariant, so Theorem 5.3.2 asserts the existence of the universal assembly map

$$\alpha_{\mathrm{WW}} : \mathbb{Y}^{\%} \to \mathbb{Y}.$$

We will construct this assembly map explicitly following [Spe04]. The key ingredient is an analogue of the Loday pairing for the Pedersen-Weibel spectrum.

Lemma 5.4.1 ([Spe04, Lemma 4.6 on page 16 and Lemma 4.7 on page 17]) *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be split exact, additive categories, considered as symmetric monoidal categories in the canonical way.*

Suppose $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a pairing of symmetric monoidal categories. Then

(i) for all i, $j \geq 0$, there is an induced pairing

$$\otimes_j^i : \mathcal{C}_i(\mathcal{A}) \times \mathcal{C}_j(\mathcal{B}) \to \mathcal{C}_{i+j}(\mathcal{C}) \tag{5.53}$$

of symmetric monoidal categories.

(ii) Moreover, the collection $\{\otimes\}_{i,j=0}^{\infty}$ of pairings assemble to give a pairing

$$\widehat{\otimes} : \mathbb{K}_{\mathcal{A}}^{\mathrm{PW}} \wedge \mathbb{K}_{\mathcal{B}}^{\mathrm{PW}} \to \mathbb{K}_{\mathcal{C}}^{\mathrm{PW}}$$

of spectra.

Let us apply this result to ringoids.

Definition 5.4.4 (Tensor Product of Ringoids) *Let $\mathcal{R}$, $\mathcal{S}$ be ringoids. The tensor product of $\mathcal{R}$ and $\mathcal{S}$ is the ringoid $\mathcal{R}\otimes\mathcal{S}$ whose objects are given by pairs (r, s) of objects in $\mathcal{R}$ and $\mathcal{S}$, and the collection of morphisms is given by*

$$\mathrm{Hom}_{\mathcal{R}\otimes\mathcal{S}}\left((r_1, s_1), (r_2, s_2)\right) := \mathrm{Hom}_{\mathcal{R}}\left(r_1, r_2\right) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{S}}\left(s_1, s_2\right). \tag{5.54}$$

Composition is given by

$$(f_1 \otimes f_2) \circ (g_1 \otimes g_2) := (f_1 \circ g_1) \otimes (f_2 \circ g_2), \tag{5.55}$$

and then extended linearly.

When the ringoids are rings considered as categories with one object, this tensor product becomes tensor product of rings in the usual sense. Furthermore, the tensor product of ringoids extends to a tensor product of modules over ringoids as defined in Definition 5.4.2.

Example 5.4.2 . Let R be a ring, and $\mathcal{C}$ be a small category. We can then form the ringoids $\mathbb{Z}[\mathcal{C}]$, $R[\mathcal{C}]$ as in Example 5.4.1. Tensor product of ringoids then gives

$$R \otimes \mathbb{Z}[\mathcal{C}] \cong R[\mathcal{C}]. \tag{5.56}$$

Moreover, it also gives a pairing

$$\otimes : \mathcal{F}\mathrm{ree}_R^{\mathrm{fg}} \times \mathcal{F}\mathrm{ree}_{\mathbb{Z}[\mathcal{C}]}^{\mathrm{fg}} \to \mathcal{F}\mathrm{ree}_{R[\mathcal{C}]}^{\mathrm{fg}} \tag{5.57}$$

of symmetric monoidal categories as in Definition 5.1.8. Note that in this case, we are considering modules over ringoids as in Definition 5.4.2.

We are now ready to construct a model for the universal assembly map associated to the functor $\mathbb{Y}$. Let us recall from the proof of Theorem 5.3.1 that the functor

$$X \mapsto X_+ \wedge \mathbb{Y}\,(\mathrm{point})$$

is a model for the functor $\mathbb{Y}^{\%} : \mathcal{S}\mathrm{paces} \to \mathcal{S}\mathrm{pectra}$.

Definition 5.4.5 (The Assembly $\widehat{\alpha}$) *Write $\Pi(X)$ to be the fundamental groupoid for the topological space X. Let R be a ring.*

Define a map of spectra

$$\widehat{\alpha}_X : X_+ \wedge \mathbb{K}_R^{\mathrm{PW}} \to \mathbb{K}_{R[\Pi(X)]}^{\mathrm{PW}}$$

having components

$$
\begin{tikzcd}
X_+ \wedge \left(\mathbb{K}_R^{\mathrm{PW}}\right)_n \arrow[r, "c_+ \wedge \mathrm{id}"] \arrow[rrd, dashed, "(\widehat{\alpha}_X)_n"'] & B\Pi(X)_+ \wedge \left(\mathbb{K}_R^{\mathrm{PW}}\right)_n \arrow[r, "J \wedge \mathrm{id}"] & \left(\mathbb{K}_{\mathbb{Z}[\Pi(X)]}^{\mathrm{PW}}\right)_{0+} \wedge \left(\mathbb{K}_R^{\mathrm{PW}}\right)_n \arrow[d, "\widehat{\otimes}"] \\
 & & \left(\mathbb{K}_{R[\Pi(X)]}^{\mathrm{PW}}\right)_n .
\end{tikzcd}
\tag{5.58}
$$

Here,

(i) *the map* $c : X \to B\Pi(X)$ *is the classifying map of the universal cover of* X*;*

(ii) *the map* $J : B\Pi(X)_+ \to \left(\mathbb{K}_{\mathbb{Z}[\Pi(X)]}^{\mathrm{PW}}\right)_{0+}$ *is induced by the inclusion functor*

$$\Pi(X) \to \mathrm{iso}\left(\mathbb{Z}[\Pi(X)]\right);$$

(iii) *and the map* $\widehat{\oplus}$ *is the pairing map constructed in Lemma 5.4.1. In particular, it is induced by the tensor product of ringoids as constructed in Definition 5.4.4.*

The collection of maps $\{\widehat{\alpha}_X \mid X \in \mathcal{S}\mathrm{paces}\}$ *then gives a homotopy natural transformation*

$$\widehat{\alpha} : \mathbb{Y}^{\%} \Rightarrow \mathbb{Y} \tag{5.59}$$

of the functor $\mathbb{Y} : \mathcal{S}\mathrm{paces} \to \mathcal{S}\mathrm{pectra}$ *constructed in Definition 5.4.3.*

Theorem 5.4.1 ([HP04, Theorem 4.3 on page 39], [Spe04, Theorem 4.9 on page 18]) *The homotopy natural transformation* $\widehat{\alpha} : \mathbb{Y}^{\%} \Rightarrow \mathbb{Y}$ *in Definition 5.4.5 is (homotopic to) the universal assembly map in the sense of Theorem 5.3.1.*

5.5 The Comparison

We mimic Definition 3.1.1 to create two Loday-like assemblies.

Definition 5.5.1 (The Assembly Maps α_{free} and α_{proj}) *Let R be a ring and G be a group.*

We define two maps of spectra

$$BG \wedge \mathbb{K}_R^{\text{free}} \to \mathbb{K}_{R[G]}^{\text{free}} \tag{5.60}$$

$$BG \wedge \mathbb{K}_R^{\text{proj}} \to \mathbb{K}_{R[G]}^{\text{proj}} \tag{5.61}$$

having components

$$\begin{tikzcd} BG \wedge \left(\mathbb{K}_R^{\text{free}}\right)_n \arrow[r, "j^{\text{free}} \wedge \mathrm{id}"] \arrow[dr, dashed, "(\alpha_{\text{free}})_n"'] & \left(\mathbb{K}_{\mathbb{Z}[G]}^{\text{free}}\right)_0 \wedge \left(\mathbb{K}_R^{\text{free}}\right)_n \arrow[d, "\gamma_{\text{free}}"] \\ & \left(\mathbb{K}_{R[G]}^{\text{free}}\right)_n, \end{tikzcd} \quad \begin{tikzcd} BG \wedge \left(\mathbb{K}_R^{\text{proj}}\right)_n \arrow[r, "j^{+} \wedge \mathrm{id}"] \arrow[dr, dashed, "(\alpha_{\text{proj}})_n"'] & \left(\mathbb{K}_{\mathbb{Z}[G]}^{\text{proj}}\right)_0 \wedge \left(\mathbb{K}_R^{\text{proj}}\right)_n \arrow[d, "\gamma_{\text{proj}}"] \\ & \left(\mathbb{K}_{R[G]}^{\text{proj}}\right)_n \end{tikzcd} \tag{5.62}$$

respectively. Here, the map j^+ is defined in Equation (3.3), and the map j^{free} is defined analogously. We then use the construction in Definition 3.1.2 to extend these two maps and get:

$$\alpha_{\text{free}} : BG_+ \wedge \mathbb{K}_R^{\text{free}} \to \mathbb{K}_{R[G]}^{\text{free}}, \tag{5.63}$$

$$\alpha_{\text{proj}} : BG_+ \wedge \mathbb{K}_R^{\text{proj}} \to \mathbb{K}_{R[G]}^{\text{proj}}. \tag{5.64}$$

In light of their definitions and Proposition 3.1.1, the maps α_{free} and α_{proj} are maps of spectra, well-defined up to weak-homotopy. The following result is an immediate consequence of the definitions.

Proposition 5.5.1 *Let R be a ring satisfying the IBP, and let G be a group.*

The canonical map

$$\mathcal{S}_R \to \mathcal{P}_R$$

given in Equation (5.22) gives a weak equivalence

$$\mu : \mathbb{K}_R^{\text{free}} \to \mathbb{K}_R^{\text{proj}} \tag{5.65}$$

of spectra. Moreover, this weak equivalence fits into the following diagram

$$\begin{tikzcd} BG_+ \wedge \mathbb{K}_R^{\text{free}} \arrow[r, "\alpha_{\text{free}}"] \arrow[d, "\mu"'] & \mathbb{K}_{R[G]}^{\text{free}} \arrow[d, "\mu"] \\ BG_+ \wedge \mathbb{K}_R^{\text{proj}} \arrow[r, "\alpha_{\text{proj}}"'] & \mathbb{K}_{R[G]}^{\text{proj}} \end{tikzcd} \tag{5.66}$$

of spectra, which commutes up to weak homotopy.

The proof of the following result was outlined in [HP04], and later completed in [Spe04].

Theorem 5.5.1 *Let R be a ring satisfying the IBP.*

For each $i \geq 0$, there is a functor

$$G_i : \mathcal{C}_i \left(\mathcal{F}\text{ree}_R^{\text{fg}} \right) \to \mathcal{F}\text{ree}_{\Sigma^i R}^{\text{fg}}, \tag{5.67}$$

such that

(a) the collection $\{G_i\}_{i=0}^{\infty}$ of functors gives a homotopy equivalence

$$\mathfrak{g} : \mathbb{K}_R^{\text{PW}} \to \mathbb{K}_R^{\text{proj}} \tag{5.68}$$

of spectra [Spe04, Theorem 5.14 on page 26 and Proposition 5.15 on page 28].

(b) *Moreover, the map $\mathfrak{g}$ is compatible with the pairings of spectra. More precisely, for each pair of non-negative integers m, n, there is a diagram*

$$\begin{tikzcd} \left(\mathbb{K}_R^{\mathrm{PW}}\right)_m \wedge \left(\mathbb{K}_R^{\mathrm{PW}}\right)_n \arrow[r, "\widehat{\otimes}"] \arrow[d, "\mathfrak{g}_m \wedge \mathfrak{g}_n"'] & \left(\mathbb{K}_R^{\mathrm{PW}}\right)_{m+n} \arrow[d, "\mathfrak{g}_{m+n}"] \\ \left(\mathbb{K}_R^{\mathrm{proj}}\right)_m \wedge \left(\mathbb{K}_R^{\mathrm{proj}}\right)_n \arrow[r, "\gamma_{\mathrm{proj}}"'] & \left(\mathbb{K}_R^{\mathrm{proj}}\right)_{m+n} \end{tikzcd} \tag{5.69}$$

that commutes up to weak homotopy [Spe04, Corollary 6.3 on page 40].

Proof. (Outline) We outline the construction of the functors G_n here since this theorem is the main result of [Spe04]. We begin by constructing the functors

$$G_n : \mathfrak{C}_n\left(\mathfrak{Free}_R^{\mathrm{fg}}\right) \to \mathfrak{Free}_{\Sigma^n R}^{\mathrm{fg}}$$

inductively.

<u>**Step 1:**</u> When $n = 0$, we have

$$\mathfrak{C}_0\left(\mathfrak{Free}_R^{\mathrm{fg}}\right) = \mathfrak{Free}_R^{\mathrm{fg}}.$$

Thus, we define

$$G_0 := \mathrm{id}_{\mathfrak{Free}_R^{\mathrm{fg}}}. \tag{5.70}$$

<u>**Step 2:**</u> Secondly, let $\mathfrak{Free}_R^{\mathbb{N}}$ be the category of countably generated free R-modules and locally finite matrices over R. It is proven that the categories

$$\frac{\mathfrak{Free}_R^{\mathbb{N}}}{\mathfrak{Free}_R^{\mathrm{fg}}} \simeq \mathfrak{Free}_{\Sigma R}^{\mathrm{fg}}$$

are equivalent [PW89, Proposition 6.1 on page 359].

The idea is as follows: choose an infinitely based R-module R^∞ in $\mathfrak{Free}_R^{\mathbb{N}}$, and observe that

$$\mathrm{End}_{\mathfrak{Free}_R^{\mathbb{N}}}\left(R^\infty\right) \cong \mathrm{cone}\left(R\right).$$

Now, the category $\mathfrak{Free}_R^{\mathbb{N}}$ is Karoubi-filtered by its subcategory $\mathfrak{Free}_R^{\mathrm{fg}}$, and the completely continuous endomorphisms of R^∞ form the ideal $m(R)$ of $\mathrm{cone}\left(R\right)$. Thus, we have

$$\mathrm{End}_{\frac{\mathfrak{Free}_R^{\mathbb{N}}}{\mathfrak{Free}_R^{\mathrm{fg}}}}\left(R^\infty\right) \cong \Sigma R.$$

The canonical additive functor

$$\begin{aligned} \mathfrak{Free}_{\Sigma R}^{\mathrm{fg}} &\to \frac{\mathfrak{Free}_R^{\mathbb{N}}}{\mathfrak{Free}_R^{\mathrm{fg}}} \\ \Sigma R &\mapsto R^\infty \end{aligned} \tag{5.71}$$

is therefore full and faithful. But every object of $\frac{\mathfrak{Free}_R^{\mathbb{N}}}{\mathfrak{Free}_R^{\mathrm{fg}}}$ is either isomorphic to the zero module or to R^∞, so this functor is also an equivalence.

We denote by

$$\alpha : \frac{\mathfrak{Free}_R^{\mathbb{N}}}{\mathfrak{Free}_R^{\mathrm{fg}}} \to \mathfrak{Free}_{\Sigma R}^{\mathrm{fg}} \tag{5.72}$$

the inverse equivalence functor of Equation (5.71).

Step 3: Thirdly, we define the functor

$$\beta : \mathcal{C}_+\left(\mathfrak{Free}_R^{\mathrm{fg}}\right) \to \mathfrak{Free}_R^{\mathbb{N}}$$

$$\{A_i\}_{i=0}^{\infty} \mapsto \bigoplus_{i=0}^{\infty} A_i,$$

for which $\mathcal{C}_+\left(\mathfrak{Free}_R^{\mathrm{fg}}\right)$ is the subcategory of $\mathcal{C}_1\left(\mathfrak{Free}_R^{\mathrm{fg}}\right)$ consisting objects $\{A_i\}_{i=-\infty}^{\infty}$ with $A_i = 0$ whenever $i < 0$.

Note that each A_i is finitely generated, so the direct sum $\bigoplus_{i=0}^{\infty} A_i$ is an object in $\mathfrak{Free}_R^{\mathbb{N}}$. Moreover, this functor induces a functor

$$\widehat{\beta} : \frac{\mathcal{C}_+\left(\mathfrak{Free}_R^{\mathrm{fg}}\right)}{\mathfrak{Free}_R^{\mathrm{fg}}} \to \frac{\mathfrak{Free}_R^{\mathbb{N}}}{\mathfrak{Free}_R^{\mathrm{fg}}}. \tag{5.73}$$

We comment that $\mathfrak{Free}_R^{\mathrm{fg}}$ is regarded as a subcategory of $\mathcal{C}_+\left(\mathfrak{Free}_R^{\mathrm{fg}}\right)$ via the embedding

$$A \mapsto (A, 0, 0, \cdots).$$

Step 4: Next, for a filtered additive category $\mathcal{A}$ in the sense of Definition 5.2.2, we define the functor

$$\begin{aligned} \tau : \mathcal{C}_1(\mathcal{A}) &\to \frac{\mathcal{C}_+(\mathcal{A})}{\mathcal{A}} \\ \{A_i\}_{i=-\infty}^{\infty} &\mapsto \{A_i\}_{i=0}^{\infty}. \end{aligned} \tag{5.74}$$

Recall that a morphism ϕ in $\mathcal{C}_1(\mathcal{A})$ is an infinite matrix $(\phi_{i,j})_{i,j=-\infty}^{\infty}$. We define $\tau(\phi)$ to be the sub-matrix $(\phi_{i,j})_{i,j=1}^{\infty}$.

Step 5: We now define the functor

$$G_n : \mathcal{C}_n\left(\mathfrak{Free}_R^{\mathrm{fg}}\right) \to \mathfrak{Free}_{\Sigma^n R}^{\mathrm{fg}}$$

for $n \geq 1$ inductively. When $n = 1$, we define G_1 to be the composition

$$\begin{tikzcd} \mathcal{C}_1\left(\mathfrak{Free}_R^{\mathrm{fg}}\right) \arrow[r, "\tau"] \arrow[rrr, dashed, bend right, "G_1"'] & \dfrac{\mathcal{C}_+\left(\mathfrak{Free}_R^{\mathrm{fg}}\right)}{\mathfrak{Free}_R^{\mathrm{fg}}} \arrow[r, "\widehat{\beta}"] & \dfrac{\mathfrak{Free}_R^{\mathbb{N}}}{\mathfrak{Free}_R^{\mathrm{fg}}} \arrow[r, "\alpha"] & \mathfrak{Free}_{\Sigma R}^{\mathrm{fg}}. \end{tikzcd} \tag{5.75}$$

Now suppose the functor

$$G_n : \mathcal{C}_n\left(\mathfrak{Free}_R^{\mathrm{fg}}\right) \to \mathfrak{Free}_{\Sigma^n R}^{\mathrm{fg}}$$

is defined. As pointed out in Example 5.2.2, the categories $\mathcal{C}_i\left(\mathcal{C}_j\left(\mathfrak{Free}_R^{\mathrm{fg}}\right)\right)$ and $\mathcal{C}_{i+j}\left(\mathfrak{Free}_R^{\mathrm{fg}}\right)$ are isomorphic. We define the functor G_{n+1} to be the composition:

$$\begin{tikzcd} \mathcal{C}_{n+1}\left(\mathfrak{Free}_R^{\mathrm{fg}}\right) \arrow[r, "\cong"] \arrow[rrdddd, dashed, "G_{n+1}"'] & \mathcal{C}_1\left(\mathcal{C}_n\left(\mathfrak{Free}_R^{\mathrm{fg}}\right)\right) \arrow[r, "\mathcal{C}_1(G_n)"] & \mathcal{C}_1\left(\mathfrak{Free}_{\Sigma^n R}^{\mathrm{fg}}\right) \arrow[d, "\tau"] \\ & & \dfrac{\mathcal{C}_+\left(\mathfrak{Free}_{\Sigma^n R}^{\mathrm{fg}}\right)}{\mathfrak{Free}_{\Sigma^n R}^{\mathrm{fg}}} \arrow[d, "\widehat{\beta}"] \\ & & \dfrac{\mathfrak{Free}_{\Sigma^n R}^{\mathbb{N}}}{\mathfrak{Free}_{\Sigma^n R}^{\mathrm{fg}}} \arrow[d, "\alpha"] \\ & & \mathfrak{Free}_{\Sigma^{n+1} R}^{\mathrm{fg}}. \end{tikzcd} \tag{5.76}$$

This completes the construction of the functors

$$G_i : \mathcal{C}_i\left(\mathfrak{Free}_R^{\mathrm{fg}}\right) \to \mathfrak{Free}_{\Sigma^i R}^{\mathrm{fg}},$$

and we refer readers to [Spe04, Theorem 5.14 on page 26, Proposition 5.15 on page 28 and Corollary 6.3 on page 40] for checking the remaining properties. □

A consequence of Theorem 5.5.1 is the following corollary:

Corollary 5.5.1 *Let R be a ring satisfying IBP, and G be a group.*

Consider G as a groupoid in the canonical way. Then the component

$$\widehat{\alpha}_{BG} : BG_+ \wedge \mathbb{K}_R^{\mathrm{PW}} \to \mathbb{K}_{R[G]}^{\mathrm{PW}} \tag{5.77}$$

of the universal assembly map $\widehat{\alpha}$, as constructed in Definition 5.4.5, is weakly homotopic to the assembly map

$$\alpha_{\mathrm{proj}} : BG_+ \wedge \mathbb{K}_R^{\mathrm{proj}} \to \mathbb{K}_{R[G]}^{\mathrm{proj}} \tag{5.78}$$

constructed in Definition 5.5.1.

Proof. We need to check the diagram

$$\begin{tikzcd}
BG_+ \wedge \left(\mathbb{K}_R^{\mathrm{PW}}\right)_n \arrow[r, "c_+ \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge \mathfrak{g}_n"'] & BG_+ \wedge \left(\mathbb{K}_R^{\mathrm{PW}}\right)_n \arrow[r, "J \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge \mathfrak{g}_n"'] & \left(\mathbb{K}_{\mathbb{Z}[G]}^{\mathrm{PW}}\right)_0 \wedge \left(\mathbb{K}_R^{\mathrm{PW}}\right)_n \arrow[r, "\widehat{\otimes}"] \arrow[d, "\mathrm{id} \wedge \mathfrak{g}_n"'] & \left(\mathbb{K}_{R[G]}^{\mathrm{PW}}\right)_n \arrow[d, "\mathfrak{g}_n"] \\
BG_+ \wedge \left(\mathbb{K}_R^{\mathrm{proj}}\right)_n \arrow[r, "c_+ \wedge \mathrm{id}"'] & BG_+ \wedge \left(\mathbb{K}_R^{\mathrm{proj}}\right)_n \arrow[r, "j^+ \wedge \mathrm{id}"'] & \left(\mathbb{K}_{\mathbb{Z}[G]}^{\mathrm{proj}}\right)_0 \wedge \left(\mathbb{K}_R^{\mathrm{proj}}\right)_n \arrow[r, "\gamma_{\mathrm{proj}}"'] & \left(\mathbb{K}_{R[G]}^{\mathrm{proj}}\right)_n
\end{tikzcd} \tag{5.79}$$

commutes up to weak homotopy for all $n \geq 0$. Here,

(1) the classifying maps

$$c_+ : BG_+ \to BG_+$$

for the universal cover of BG_+ is just the identity, so the left square commutes up to homotopy automatically.

(2) The middle square commutes up to homotopy by the definitions of J and j^+.

(3) The right square commutes up to weak homotopy by part (b) of Theorem 5.5.1.

Finally, the map $\mathfrak{g}_n$ is a homotopy equivalence by Theorem 5.5.1 (a). So the claim holds as desired. $\square$

Corollary 5.5.2 (Corollary of Corollary 5.5.1) *Let R be a ring satisfying IBP, and G be a group.*

Consider G as a groupoid in the canonical way, then the component

$$\widehat{\alpha}_{BG} : BG_+ \wedge \mathbb{K}_R^{\mathrm{PW}} \to \mathbb{K}_{R[G]}^{\mathrm{PW}} \tag{5.80}$$

of the universal assembly map $\widehat{\alpha}$, as constructed in Definition 5.4.5, is weakly homotopic to the assembly map

$$\alpha_{\mathrm{free}} : BG_+ \wedge \mathbb{K}_R^{\mathrm{free}} \to \mathbb{K}_{R[G]}^{\mathrm{free}} \tag{5.81}$$

constructed in Definition 5.5.1.

5.6 An Explicit Formula for the Universal Assembly

We now relate Theorem 3.3.1 to the universal assembly map.

Theorem 5.6.1 *Let R be a ring and G be a group.*

There is a subgroup $\mathrm{AHSS}\,(BG)_{1,i}^{\infty}$ of $\pi_{i+1}\left(BG_+ \wedge \mathbb{K}_R^{\mathrm{PW}}\right)$ which is a quotient of the E^2-term

$$\mathrm{AHSS}\,(BG)_{1,i}^{2} \cong G_{ab} \otimes K_i(R)$$

from the Atiyah-Hirzebruch spectral sequence of the classifying space BG of G with coefficients in the Pedersen-Weibel K-theory spectrum $\mathbb{K}_R^{\mathrm{PW}}$ of R.

For $i > 0$, the filler of the diagram

$$\begin{tikzcd}
\mathrm{AHSS}\,(BG)^2_{1,i} \cong G_{ab} \otimes K_i(R) \arrow[r, dashed, "\widehat{\Phi_{i+1}}"] \arrow[d, two heads] & K_{i+1}\,(R[G]) \\
\mathrm{AHSS}\,(BG)^\infty_{1,i} \arrow[r, hook] & \pi_{i+1}\left(BG \wedge \mathbb{K}_R^{\mathrm{PW}}\right) \arrow[u, "\pi_{i+1}\,(\widehat{\alpha}_{BG})"']
\end{tikzcd} \tag{5.82}$$

is induced by

$$\begin{aligned} G \times K_i(R) &\to K_{i+1}\,(R[G]) \\ (g, [f]) &\mapsto \{g\} \star [f]. \end{aligned} \tag{5.83}$$

This provides an explicit formula for the universal assembly map when restricted onto the subgroup $\mathrm{AHSS}\,(BG)^\infty_{1,i}$.

Proof. From Corollary 5.5.2, we know the universal assembly $\widehat{\alpha}_{BG}$ induces the same map on homotopy groups as the assembly map α_{free}. So we need to verify the formula for α_{free}.

As pointed out in Equation (5.10), the pairing map γ_{free} used in constructing the assembly α_{free} is homotopic to the original Loday pairing γ_{Loday}. Therefore, by repeating the same proofs as in Proposition 3.3.1 and Proposition 3.3.2, we see that the bottom composition of

$$\begin{tikzcd}
\pi_{i+1}\left(\left(\bigvee_{g\in G}\mathbb{S}^1\right)\wedge\mathbb{K}_R^{\mathrm{free}}\right) \arrow[r, "(\mathfrak{i}_G)_*"] & \pi_{i+1}\left(BG\wedge\mathbb{K}_R^{\mathrm{free}}\right) \arrow[r, "\alpha_{\mathrm{free}}"] & K_{i+1}\left(R[G]\right) \\
\bigoplus_{g\in G}K_i(R) \arrow[u, phantom, "\cong" rotate=90] \arrow[r] & \pi_{i+1}\left(BG\wedge\mathbb{K}_R^{\mathrm{free}}\right) \arrow[u, equal] \arrow[r] & K_{i+1}\left(R[G]\right) \arrow[u, equal]
\end{tikzcd} \tag{5.84}$$

sends the element $[f]\in K_i(R)$ in the summand in $\bigoplus_{g\in G}K_i(R)$ labelled by $g\in G$ to the element

$$\{g\}\star[f]\in K_{i+1}\left(R[G]\right).$$

This completes the proof. $\square$

The universal assembly then admits the following version of Corollary 4.1.1.

Corollary 5.6.1 (The Universal Assembly on π_2 of an Integral Group Ring) *Let G be a group. The universal assembly map*

$$\widehat{\alpha}_{BG}:BG_+\wedge\mathbb{K}_{\mathbb{Z}}^{\mathrm{PW}}\to\mathbb{K}_{\mathbb{Z}[G]}^{\mathrm{PW}} \tag{5.85}$$

for the integral group ring $\mathbb{Z}[G]$ on π_2, when restricted onto the subgroup

$$K_2\left(\mathbb{Z}\right)\oplus\left[G_{ab}\oplus K_1(\mathbb{Z})\right],$$

is given by the formula:

$$K_2(i)\oplus\widehat{\Phi}_2:K_2\left(\mathbb{Z}\right)\oplus\left[G_{ab}\oplus K_1(\mathbb{Z})\right]\to K_2(\mathbb{Z}[G]), \tag{5.86}$$

for which $i:\mathbb{Z}\to\mathbb{Z}[G]$ is the inclusion, and $\widehat{\Phi}_2$ is induced by the map

$$\begin{aligned} G \times K_1(\mathbb{Z}) &\to K_2(\mathbb{Z}[G]) \\ (g, \pm 1) &\mapsto -\{\pm 1, g\}_{\mathrm{St}} . \end{aligned} \tag{5.87}$$

Proof. We have $-\{\pm 1, g\}_{\mathrm{St}}$ instead of $\{\pm 1, g\}_{\mathrm{St}}$ because of Proposition 2.1.1 (iii). ☐

Our proof for Theorem 4.1.2 applies to the following result.

Theorem 5.6.2 (An Injectivity Result for the Universal Assembly on π_2) *Let G be a group. The composition*

$$K_2(\mathbb{Z}) \oplus [G_{ab} \otimes K_1(\mathbb{Z})] \xrightarrow{(4.2)} \pi_2\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{PW}}\right) \xrightarrow{\pi_2(\widehat{\alpha}_{BG})} K_2(\mathbb{Z}[G]) \tag{5.88}$$

is injective. Moreover, if $H_2(BG;\mathbb{Z})$ is trivial, then the universal assembly $\widehat{\alpha}_{BG}$ is injective on π_2.

Because of Question 5.1.1, it is unclear if Corollary 3.3.1 (or its variants) holds for the universal assembly. However, Waldhausen showed that when $R = \mathbb{Z}$, the assembly

$$\pi_1(\alpha_{\mathrm{free}}) : \pi_1(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{free}}) \to K_1(\mathbb{Z}[G])$$

on π_1 is the usual map

$$\begin{aligned} \{1, -1\} \oplus G_{ab} &\to K_1(\mathbb{Z}[G]) \\ (\pm 1) \oplus g &\mapsto \{\pm g\} \end{aligned}$$

under the identification

$$\pi_1\left(BG_+ \wedge \mathbb{K}_{\mathbb{Z}}^{\mathrm{free}}\right) \cong \{1, -1\} \oplus G_{ab}.$$

See [Wal78b, Assertion 15.8 on page 229]. His proof involves rewriting the assembly map in terms of Quillen's Q-construction and then verifying the formula at the simplicial level. Therefore, it seems likely that the following conjecture is true. But a rigorous proof is not known to the author.

Conjecture 5.6.1 (The Universal Assembly on Fundamental Group) *Let R be a regular ring, and G be a group. The universal assembly map on π_1 is given by the formula*

$$K_1(i) \oplus \Phi_1 : K_1(R) \oplus [G_{ab} \otimes K_0(R)] \to K_1\left(R[G]\right), \tag{5.89}$$

where $i : R \to R[G]$ is the inclusion, and the map Φ_1 is induced by the map

$$\begin{aligned} G \times K_0(R) &\to K_1(R[G]) \\ (g, [P]) &\mapsto \left\{ \begin{array}{cccc} \mathfrak{h}_g : & P \otimes_R R[G] & \to & P \otimes_R R[G] \\ & x \otimes u & \mapsto & x \otimes ug \end{array} \right\}. \end{aligned} \tag{5.90}$$

5.7 A Final Remark on Extending the Loday Pairing

One might ask why we use the Gersten-Wagoner delooping to extend the Loday pairing

$$\gamma_{\text{Loday}} : BGL(R)^+ \wedge BGL(S)^+ \to BGL(R \otimes S)^+$$

to

$$\gamma'_{\text{Loday}} : \left[K_0(R) \times BGL(R)^+\right] \wedge \left[K_0(S) \times BGL(S)^+\right] \to K_0(R \otimes S) \times BGL(R \otimes S)^+,$$

instead of using the "naive approach":

$$\gamma_{\text{naive}} : \left[K_0(R) \times BGL(R)^+\right] \wedge \left[K_0(S) \times BGL(S)^+\right] \to K_0(R \otimes S) \times BGL(R \otimes S)^+$$
$$([P], x) \wedge ([Q], y) \mapsto ([P \otimes Q], \gamma_{\text{Loday}}(x, y)) \,. \tag{5.91}$$

However, if we look at the induced product map

$$\begin{aligned} \star_{\text{naive}} : K_i(R) \otimes K_j(S) &\to K_{i+j}(R \otimes S) \\ [f] \otimes [g] &\mapsto [\gamma_{\text{naive}} \circ (f \wedge g)] \,, \end{aligned} \tag{5.92}$$

we immediately see that it is the zero map when $i = 0$ and $j > 0$. Therefore, algebraically the naive pairing gives the wrong thing.

Secondly, if the naive pairing extends to a pairing (or weak pairing)

$$\gamma_{\text{naive}} : \mathbb{K}_R^{\text{GW}} \wedge \mathbb{K}_S^{\text{GW}} \to \mathbb{K}_{R \otimes S}^{\text{GW}} \tag{5.93}$$

of spectra, then one can define an assembly map

$$\alpha_{\text{naive}} : BG_+ \wedge \mathbb{K}_R^{\text{GW}} \to \mathbb{K}_{R[G]}^{\text{GW}} \tag{5.94}$$

by mimicking Definition 3.1.2. However, the cokernel $\operatorname{coker}(\pi_1(\alpha_{\text{naive}}))$ of this assembly on π_1 will not be isomorphic to the classical Whitehead group $\mathrm{Wh}_1(G)$ when $R = \mathbb{Z}$. Since Waldhausen's work shows the cokernel of the universal assembly map on π_1 is isomorphic to $\mathrm{Wh}_1(G)$ when $R = \mathbb{Z}$ [Wal78b, Assertion 15.8 on page 229], the assembly map α_{naive} is not the correct one. Therefore, topologically the naive pairing gives the wrong thing.

REFERENCES

[Bas68] Hyman Bass. *Algebraic K-Theory*. W. A. Benjamin;, 1 edition, 1968.

[BHM93] M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Invent. Math.*, 111(3):465–539, 1993.

[BLR08] Arthur Bartels, Wolfgang Lück, and Holger Reich. On the Farrell-Jones conjecture and its applications. *J. Topol.*, 1(1):57–86, 2008.

[Coh73] Marshall M. Cohen. *A Course in Simple-Homotopy Theory*. Number 10 in Graduate Texts in Mathematics. Springer, 1973.

[CP95] Gunnar Carlsson and Erik Kjær Pedersen. Controlled algebra and the Novikov conjectures for K- and L-theory. *Topology*, 34(3):731–758, 1995.

[CP97] Manuel Cárdenas and Erik Kjær Pedersen. On the Karoubi filtration of a category. *K-Theory*, 12(2):165–191, 1997.

[DL98] James F. Davis and Wolfgang Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory*, 15(3):201–252, 1998.

[Dun75] M. J. Dunwoody. $K_2(\mathbf{Z}\Pi)$ for Π a group of order two or three. *J. London Math. Soc. (2)*, 11(4):481–490, 1975.

[ES52] Samuel Eilenberg and Norman E. Steenrod. *Foundations of Algebraic Topology*. Princeton University Press, 1 edition, 1952.

[Ger72] Stephen M. Gersten. On the spectrum of algebraic k-theory. *Bull. Amer. Math. Soc.*, 78(2):216–219, 03 1972.

[Gra76] Daniel Grayson. Higher algebraic K-theory. II (after Daniel Quillen). In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 217–240. Lecture Notes in Math., Vol. 551, 1976.

[Hat01] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2001.

[HP04] Ian Hambleton and Erik K. Pedersen. Identifying assembly maps in K- and L-theory. *Math. Ann.*, 328(1-2):27–57, 2004.

[Hsi84] Wu Chung Hsiang. Geometric applications of algebraic K-theory. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 99–118. PWN, Warsaw, 1984.

[HW73] Allen Hatcher and John Wagoner. *Pseudo-isotopies of compact manifolds*. Société Mathématique de France, Paris, 1973. With English and French prefaces, Astérisque, No. 6.

[KM81] Jerome Kaminker and John G. Miller. A comment on the Novikov conjecture. *Proc. Amer. Math. Soc.*, 83(3):656–658, 1981.

[Leh18] Georg Lehner. The atiyah-hirzebruch spectral sequence and k-theory of group rings, 2018.

[Lod76] Jean-Louis Loday. k-théorie algébrique et représentations de groupes. *Annales scientifiques de l'École Normale Suprieure*, 9(3):309–377, 1976.

[Lod97] Jean-Louis Loday. *Cyclic Homology.* Number 301 in Grundlehren der mathematischen Wissenschaften. Springer, 2 edition, 1997.

[LR05] Wolfgang Lück and Holger Reich. *The Baum–Connes and the Farrell–Jones Conjectures in K- and L-Theory*, pages 703–842. Springer Berlin Heidelberg, Berlin, Heidelberg, 2005.

[May80] J. Peter May. Pairings of categories and spectra. *Journal of Pure and Applied Algebra*, 19:299–346, 1980.

[Mil72] John W. Milnor. *Introduction to Algebraic K-Theory.* Number 71 in Annals of Mathematics Studies. Princeton University Press, 1972.

[ML98] Saunders Mac Lane. *Categories for the Working Mathematician.* Number 5 in Graduate Texts in Mathematics. Springer, 2 edition, 1998.

[Ped84] Erik Kjaer Pedersen. On the K_{-i}-functors. *J. Algebra*, 90(2):461–475, 1984.

[PW85] Erik K. Pedersen and Charles A. Weibel. A nonconnective delooping of algebraic K-theory. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 166–181. Springer, Berlin, 1985.

[PW89] Erik K. Pedersen and Charles A. Weibel. K-theory homology of spaces. In *Algebraic topology (Arcata, CA, 1986)*, volume 1370 of *Lecture Notes in Math.*, pages 346–361. Springer, Berlin, 1989.

[Qui72] Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Ann. of Math. (2)*, 96:552–586, 1972.

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

[Ros95] Jonathan Rosenberg. *Algebraic K-Theory and Its Applications.* Number 147 in Graduate Texts in Mathematics. Springer, 2 edition, 1995.

[Ros05] Jonathan Rosenberg. K-theory and geometric topology. In *Handbook of K-theory. Vol. 1, 2*, pages 577–610. Springer, Berlin, 2005.

[Shi88] Kazuhisa Shimakawa. Uniqueness of products in higher algebraic K-theory. *Publ. Res. Inst. Math. Sci.*, 24(1):99–120, 1988.

[Sma61] Stephen Smale. Generalized Poincaré's conjecture in dimensions greater than four. *Ann. of Math. (2)*, 74:391–406, 1961.

[Spe04] Ron Sperber. *A comparison of assembly maps in algebraic K-theory*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–State University of New York at Binghamton.

[UW17] Mark Ullmann and Xiaolei Wu. Note on the injectivity of the Loday assembly map. *J. Algebra*, 489:460–462, 2017.

[Wag72] John B. Wagoner. Delooping classifying spaces in algebraic k-theory. *Topology*, 11(4):349 – 370, 1972.

[Wal65] C. T. C. Wall. Finiteness conditions for CW-complexes. *Ann. of Math. (2)*, 81:56–69, 1965.

[Wal70] C. T. C. Wall. *Surgery on compact manifolds*. Academic Press, London-New York, 1970. London Mathematical Society Monographs, No. 1.

[Wal78a] Friedhelm Waldhausen. Algebraic k-theory of generalized free products, part 1. *Annals of Mathematics*, 108(2):135–204, 1978.

[Wal78b] Friedhelm Waldhausen. Algebraic k-theory of generalized free products, part 2. *Annals of Mathematics*, 108(2):205–256, 1978.

[Wal78c] Friedhelm Waldhausen. Algebraic K-theory of topological spaces. I. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, Proc. Sympos. Pure Math., XXXII, pages 35–60. Amer. Math. Soc., Providence, R.I., 1978.

[Wei81] Charles A. Weibel. A survey of products in algebraic K-theory. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 494–517. Springer, Berlin, 1981.

[Wei13] Charles A. Weibel. *The K-Book: An Introduction to Algebraic K-Theory*. Number 145 in Graduate Studies in Mathematics. American Mathematical Society, 2013.

[Whi39] J. H. C. Whitehead. Simplicial Spaces, Nuclei and m-Groups. *Proc. London Math. Soc. (2)*, 45(4):243–327, 1939.

[Whi41] J. H. C. Whitehead. On incidence matrices, nuclei and homotopy types. *Ann. of Math. (2)*, 42:1197–1239, 1941.

[Whi50] J. H. C. Whitehead. Simple homotopy types. *Amer. J. Math.*, 72:1–57, 1950.

[Wic15] Kirsten Wickelgren. Atiyah-hirzebruch spectral sequence, 2015.

[WW95] Michael Weiss and Bruce Williams. *Assembly*, volume 2 of *London Mathematical Society Lecture Note Series*, page 332352. Cambridge University Press, 1995.

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