RICCI CURVATURE OF FINSLER METRICS

BY WARPED PRODUCT

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To everyone who has ever taught me,
even the smallest lesson.

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ABSTRACT

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In the present work, we consider a class of Finsler metrics using the warped product notion introduced by Chen, Shen and Zhao [1], with another "warping", one that is consistent with the form of metrics modeling static spacetimes and simplified by spherical symmetry over spatial coordinates, which emerged from the Schwarzschild metric in isotropic coordinates. We will give the PDE characterization for the proposed metrics to be Ricci-flat and construct explicit examples. Whenever possible, we describe both positive-definite solutions and solutions with Lorentz signature. For the latter, the 4-dimensional metrics may also be studied as Finsler spacetimes.

1. INTRODUCTION

Consider $\mathbb{R}$, $\mathbb{R}^n$ with their Euclidean metrics dt^2, α^2 (respectively). On the Cartesian product $\mathbb{R} \times \mathbb{R}^n$, a Riemannian metric of the form

$$ds^2 = dt^2 + f^2(t)\alpha^2\,, \tag{1.1}$$

where f is some smooth function on $\mathbb{R}$, is called a warped product. Similarly, a Riemannian metric given by

$$ds^2 = g^2(x)dt^2 + \alpha^2\,, \tag{1.2}$$

for a smooth function g on $\mathbb{R}^n$, is also a warped product. These metrics differ by their "warping" type.

From (1.1), we obtain

$$ds = \alpha\sqrt{\left(\frac{dt}{\alpha}\right)^2 + f^2(t)}\,,$$

which inspired Chen, Shen and Zhao to define a class of Finsler metrics by

$$F = \alpha\sqrt{\phi\left(\frac{dt}{\alpha}, t\right)}\,, \tag{1.3}$$

where ϕ must be a suitable function on $\mathbb{R}^2$. [1]

We define a Finsler metric by the same idea as (1.3) with the "warping" type of (1.2), that is,

$$F = \alpha\sqrt{\phi\left(\frac{dt}{\alpha}, \rho\right)}\,, \tag{1.4}$$

where $\rho = |x|$ for $x \in \mathbb{R}^n$ and ϕ must be again a suitable function on $\mathbb{R}^2$. [2]

We give the PDE characterization for the metric (1.4) to be Ricci-flat in the Theorem 5.2.1. Afterwards, we explicitly construct two non-Riemannian solutions. For $n \geq 3$, they are:

$$\phi(z,\rho) = (Az^m + B\rho^{-2m})^{\frac{2}{m}}\,,\ A, B > 0 \tag{5.23b}$$

$$\phi(z,\rho) = (\sqrt{Az^2 + B\rho^{-4}} + \varepsilon\sqrt{A}z)^2\,,\ A, B > 0\,,\ 0 < |\varepsilon| < 1\,,\ C \in \mathbb{R} \tag{5.28b}$$

Preliminary to our work, we will take some time to introduce the theory of Finsler Geometry with the desire to be as friendly as possible towards fellow graduate students; particularly because we keep in mind the following consideration by Bao, Chern and Shen:

> "It is true that Finsler geometry has not been nearly as popular as its progeny – Riemannian geometry. One reason is that deceptively simple formulas can quickly give rise to complicated expressions and mind-boggling computations. With the effort of many dedicated practitioners, this situation is slowly being turned around." [3]

For what follows, we will constantly reference the books – "An Introduction to Riemann-Finsler Geometry" [3], and "Lectures on Finsler Geometry" [4].

2. FINSLER SPACES

The origin of Finsler Geometry might be considered – on a technicality – the same as that of Riemannian Geometry, namely Riemann's habilitation thesis "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen" in 1854, which was published posthumously in 1868 by Dedekind [5] and in 1873 as an English version "On the Hypotheses which lie at the Bases of Geometry" translated by Clifford [6]. This is because Riemann himself noted the line-element to measure the length of a curve on a manifold need not be quadratic. In his words:

> "[...] the linear element may be any homogeneous function of the first degree of the quantities dx, which is unchanged when we change the signs of all the dx, and in which the arbitrary constants are continuous functions of the quantities x. [...] For Space, when the position of points is expressed by rectilinear co-ordinates, $ds = \sqrt{\sum (dx)^2}$; Space is therefore included in this simplest case. The next case in simplicity, includes those manifoldnesses in which the line-element may be expressed as the fourth root of a quartic differential expression. The investigation of this more general kind would require no really different principles, but would take considerable time and throw little new light on the theory of space, especially as the results cannot be geometrically expressed; I restrict myself, therefore, to those manifoldnesses in which the line element is expressed as the square root of a quadric differential expression." [6]

His decision to consider only the quadratic case seems to be the right one at the time, once we recognize how rich and impactful Riemannian Geometry has been. However, Riemann's comment laid dormant the general case for over half-century, until Finsler's dissertation "Ueber Kurven und Flächen in allgemeinen Räumen" [7] in

1918, which translates to “About curves and surfaces in general spaces” but has never been published in English. This advance in the study of general metrics arose from a geometrical approach to Calculus of Variations by Carathéodory, the dissertation supervisor. In a loose translation, Finsler wrote in the introduction:

> “The present work deals with different parts of Differential Geometry in multidimensional spaces based on a generalized measurement. Namely, the length of a curve is to be measured by the integral of a substantially arbitrary function of the coordinates and their first derivatives. The Euclidean geometry and the one in spaces of arbitrary curvature are the most important special cases to which these investigations can be applied.” [7]

Regardless of the importance of this work to Differential Geometry, Finsler soon after turned his attention to Set Theory. Fortunately, other mathematicians became interested in this theory. Around 1925, Berwald [8], Synge [9] and Taylor [10] independently applied methods of Tensor Calculus to the study of general metrics. In 1934, Cartan published the book “Les espaces de Finsler” [11], which established the the terminology “Finsler spaces”. For the historically curious, a comprehensive introduction was given by Rund [12].

In fewer words, let M be an n-manifold and $U \subset M$ an open with local coordinates $(x^1, \dots, x^n)$. If $x(t)$ is a curve on U, then the arc-length of the curve is

$$s = \int_{t_0}^{t_1} F\left(x^i, \frac{dx^i}{dt}\right) dt \tag{2.1}$$

where the function $F(x^i, y^i)$ must be positively homogeneous of degree one in $y = (y^i)$ and $F(x^i, y^i) > 0$ unless $y = 0$. These conditions ensure the arc-length is well-defined and independent of the speed of the curve, although velocity is not necessarily reversible. So $\tilde{x}(t) = x(at)$ with $a > 0$ satisfy

$$s = \int_{at_0}^{at_1} F\left(x^i, \frac{dx^i}{dt}\right) dt = \int_{t_0}^{t_1} F\left(\tilde{x}^i, \frac{d\tilde{x}^i}{dt}\right) dt\,,$$

but this is not always true for $a < 0$. Moreover, the convexity of F is fundamental for the existence of extrema in Calculus of Variations.

Integrals in the form of (2.1) appear naturally in several contexts. If we think of $x(t)$ as the position of a particle varying with time, then $\frac{dx}{dt}$ represents the velocity of the particle and $F\left(x, \frac{dx}{dt}\right)$ the speed, so s measures distance traveled. For most the examples that easily come to mind, such as the distance covered by a car on the road or a thrown ball, F is the square root of a quadratic expression, that is, Riemannian. However, the general case also applies to how long it takes to navigate a path on a hillside, the amount of energy it takes to swim in flowing waters, the time it takes light to travel across an anisotropic medium, the energy cost for a species or an ecosystem to evolve from one state to another.

2.1 Definitions and Conventions

Let M be an n-dimensional smooth manifold. For each point $x \in M$, the tangent space of M at x is the n-dimensional vector space composed of velocity at x of curves in M, denoted by T_xM. The tangent bundle, denoted TM, is comprised of all elements (x, y) with $x \in M$ and $y \in T_xM$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$, and a map $\sigma : M \to TM$ is a section if $\pi \circ \sigma = Id_M$. Particularly, the correspondence $x \mapsto 0 \in T_xM$ defines the zero section. The slit tangent bundle, denoted $TM \setminus 0$, is obtained from TM by excluding the zero section. The dual space of T_xM is denoted T_x^*M, and the union of all these spaces composes the cotangent bundle, denoted T^*M. The natural projection and sections are defined similarly.

A function $F : TM \to [0, \infty)$ is a **Finsler metric** on M if it satisfies the following properties:

(i) **Regularity:** F is C^∞ on $TM \setminus 0$;

(ii) **Positive Homogeneity:** $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$;

(iii) **Strong convexity:** $\forall (x, y) \in TM \setminus 0$, the symmetric bilinear form

$$g_y(u, v) := \frac{\partial^2}{\partial s \partial t} \left[\frac{1}{2} F^2(x, y + su + tv) \right] \Bigg|_{s=t=0}$$

is positive-definite.

The pair (M, F) is called a **Finsler space**. Moreover, for each $x \in M$, the restriction $F|_{T_xM}$ defines what is called a **Minkowski norm** on T_xM. Thus, a Finsler metric consists of a smoothly varying family of Minkowski norms, one on each tangent space. Generally, this family is no more than C^1 along the zero section of the tangent bundle. In fact, F^2 is C^2 or smoother along the zero section of TM if and only if it defines a smoothly varying family of inner products, i.e. it is a Riemannian metric. For a proof of this result, refer to section 6.4 of [3].

Let $(x^1, \ldots, x^n) = (x^i) : U \to \mathbb{R}^n$ be a local coordinate system for an open $U \subset M$. The standard coordinate frame $\left\{\frac{\partial}{\partial x^i}\right\}$ are local sections of TM defined on U such that evaluated at x they form a basis for T_xM, $\forall x \in U$. Similarly, the coordinates (x^i) determine a natural local frame $\{dx^i\}$ for T^*M. Throughout the work, lower case Latin indices range from 1 to n and Einstein summation convention is adopted, so repeated indices are implicitly summed over. Now, for any $x \in U$ and $y \in T_xM$, we can write $y = y^i \left.\frac{\partial}{\partial x^i}\right|_x$, whence we obtain local coordinates (x^i, y^i) on $\pi^{-1}U \subset TM$. With this setting, for each $(x, y) \in TM \setminus 0$, we have:

$$
\begin{aligned}
g_y(u, v) &= \frac{\partial^2}{\partial s \partial t}\left[\frac{1}{2}F^2(x, y + su + tv)\right]\bigg|_{s=t=0} \\
&= \left[\frac{1}{2}F^2(x^k, y^k)\right]_{y^i y^j} u^i v^j
\end{aligned}
$$

Define:

$$
g_{ij}(x^k, y^k) := \left[\frac{1}{2}F^2(x^k, y^k)\right]_{y^i y^j} \tag{2.2}
$$

Hence, condition (iii) – strong convexity – is equivalent to the $n \times n$ Hessian matrix (g_{ij}) being positive-definite at every point of $TM \setminus 0$. Particularly, it has an inverse, denoted (g^{ij}).

Those familiar with metric spaces are most likely expecting three basic properties: positivity, triangle inequality and reversibility. In general, reversibility does not hold. When $F(x, -y) = F(x, y)$, the Finsler metric is called **reversible**. In this case, F is absolutely homogeneous of degree one in y:

$$
F(x, \lambda y) = |\lambda| F(x, y), \ \forall \lambda \in \mathbb{R}
$$

Interestingly, positivity and the triangle inequality are consequences of the defining properties of Minkowski norms. Before we can prove this, we will need a technical result known as **Euler's homogeneous function theorem**, which will be used repeatedly sometimes without mention.

Theorem 2.1.1 (Theorem 1.2.1 of [3]) *Suppose* $H : \mathbb{R}^n \to \mathbb{R}$ *is differentiable away from the origin. The following are equivalent:*

- H *is positively homogeneous of degree* r*, i.e.* $H(\lambda y) = \lambda^r H(y)$, $\forall \lambda > 0$.
- *The radial directional derivative of* H *is* r *times* H*, i.e.* $H_{y^i}(y)y^i = rH(y)$.

Proof Suppose H is positively homogeneous of degree r. Let y be fixed and differentiate the equation $H(\lambda y) = \lambda^r H(y)$ with respect to λ:

$$H_{y^i}(\lambda y)y^i = r\lambda^{r-1}H(y)$$

Take $\lambda = 1$ to obtain the desired equality.

Conversely, assume $H_{y^i}(y)y^i = rH(y)$. Fixed y, consider the function $H(\lambda y)$ with $\lambda > 0$. By the chain rule,

$$\frac{d}{d\lambda}H(\lambda y) = H_{y^i}(\lambda y)y^i = \frac{1}{\lambda}H_{y^i}(\lambda y)(\lambda y)^i = \frac{r}{\lambda}H(\lambda y)\,,$$

where the last equality follows from the hypothesis. So $H(\lambda y)$ satisfies the following linear ODE:

$$\frac{d}{d\lambda}H(\lambda y) - \frac{r}{\lambda}H(\lambda y) = 0\,,$$

whose solution is $H(\lambda y) = C\lambda^r$ for some constant C depending on the fixed y. Take $\lambda = 1$ to conclude $C = H(y)$. ■

Euler's theorem applied to a Finsler metric F implies:

$$F_{y^i}(y)y^i = F(y) \tag{2.3a}$$

$$F_{y^iy^j}(y)y^j = 0 \tag{2.3b}$$

Moreover, applying these identities to (2.2) gives:

$$g_{ij}(y)y^i y^j = F^2(y) \tag{2.4}$$

Here and often after, the dependence of F on x is left implicit to simplify notation.

Theorem 2.1.2 (Adapted from Theorem 1.2.2 of [3]) *If $F : TM \to [0, \infty)$ is a Finsler metric, then for each $x \in M$ $F(x, y) = F(y)$ satisfies:*

- **Positivity:** $F(y) > 0$ *whenever* $y \neq 0$*;*
- **Triangle inequality:** $F(y_1 + y_2) \leq F(y_1) + F_(y_2)$ *and equality holds if and only if* $y_2 = \lambda y_1$ *or* $y_1 = \lambda y_2$ *for some* $\lambda \geq 0$*.*

Proof By condition (iii) – strong convexity, the left-hand side of equation (2.4) is positive whenever $y \neq 0$. Since F is non-negative, positivity holds.

To prove the triangle inequality we will first prove the following inequality:

$$F_{y^i}(y)w^i \leq F(w) \text{ at all } y \neq 0 \tag{2.5}$$

and equality holds if and only if $w = \lambda y$ for some $\lambda \geq 0$.

When $w = 0$, the statement is trivial. The inequality also holds when $w = \lambda y$ for $\lambda < 0$, because the left-hand side is going to be negative, while the right-hand side is positive. If $w \neq 0$ is not a negative multiple of y, then we can apply the mean value theorem to obtain:

$$F(w) = F(y) + F_{y^i}(y)(w^i - y^i) + \frac{1}{2}F_{y^i y^j}(v)(w^i - y^i)(w^j - y^j)$$

where $v = (1 - t)y + tw$ for some $t \in (0, 1)$. By (2.3a), the above equation simplifies to:

$$F(w) = F_{y^i}(y)w^i + \frac{1}{2}F_{y^i y^j}(v)(w^i - y^i)(w^j - y^j) \tag{2.6}$$

First notice that $v \neq 0$. Next, consider that, for each $v \in T_xM \setminus \{0\}$, $(g_{ij}(v))$ defines an inner product on T_xM. So the following Cauchy-Schwarz type inequality is valid:

$$[g_{ij}(v)u_1^i u_2^j]^2 \leq [g_{kl}(v)u_1^k u_1^l][g_{pq}(v)u_2^p u_2^q], \forall u_1, u_2 \in T_xM \tag{2.7}$$

where equality holds if and only if u_1 and u_2 are collinear. By (2.7) and (2.4), we obtain:

$$[g_{ij}(v)u^i v^j]^2 \leq F^2(v)[g_{kl}(v)u^k u^l], \; \forall u \in T_x M \tag{2.8}$$

where equality holds if and only if u and v are collinear. Meanwhile, by its definition, $g_{ij} = FF_{y^i y^j} + F_{y^i}F_{y^j}$, and using Euler's theorem we may write:

$$F_{y^i y^j}(v)u^i u^j = \frac{1}{F^3(v)}\left(F^2(v)[g_{ij}(v)u^i u^j] - [g_{ij}(v)v^i u^j]^2\right) \tag{2.9}$$

Putting together (2.8) and (2.9), we conclude:

$$F_{y^i y^j}(v)u^i u^j \geq 0, \; \forall u \in T_x M \tag{2.10}$$

where equality holds if and only if u and v are collinear. Finally, (2.6) and (2.10) provide (2.5) with equality exactly when $w - y$ and $(1-t)y + tw$ are collinear, which under our hypotheses is equivalent to $w = \lambda y$ for some $\lambda > 0$.

Now, the triangle inequality follows from (2.3a) and (2.5):

$$\begin{aligned} F(y_1 + y_2) &= F_{y^i}(y_1 + y_2)(y_1 + y_2)^i \\ &= F_{y^i}(y_1 + y_2)y_1^i + F_{y^i}(y_1 + y_2)y_2^i \\ &\leq F(y_1) + F(y_2) \end{aligned}$$

and equality holds if and only if $y_2 = \lambda y_1$ or $y_1 = \lambda y_2$ for some $\lambda \geq 0$. ■

The inequality (2.5) proved in the previous theorem is referred to as the **fundamental inequality**. It may be viewed as an extension of Euler's Theorem from an equation to an inequality, since (2.5) applied to $w = \lambda y$ for $\lambda > 0$ results in (2.3a). Moreover, this inequality generates a geometric interpretation for the graph of Minkowski norms, for (2.5) together with (2.3a) gives

$$F(y) + F_{y^i}(y)(w-y)^i \leq F(w)\,,$$

where equality holds if and only if $w = \lambda y$ with $\lambda \geq 0$. When $y \in T_x M \setminus 0$ is fixed, the above inequality shows that the tangent space to the graph of $F|_{T_x M}$ at $(y, F(y))$ lies below the graph and it intersects the graph exclusively along $(\lambda y, \lambda F(y))$ for $\lambda \geq 0$. So the graph of $F|_{T_xM}$ is a convex cone with its vertex at the origin of T_xM, as the figure bellow.

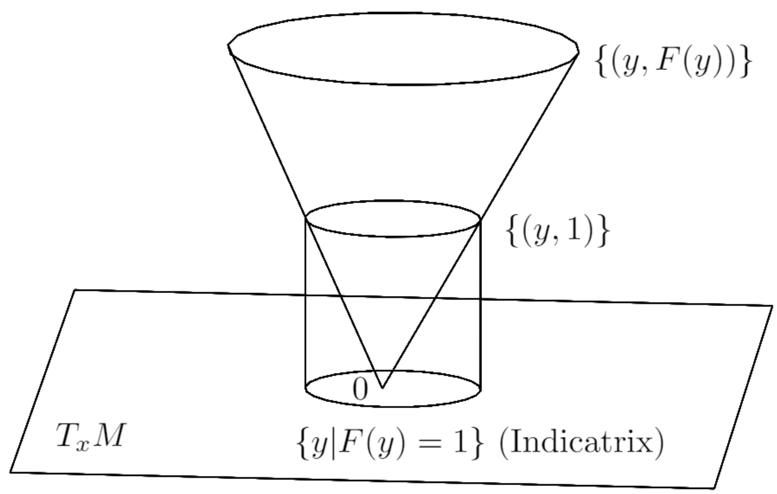


Fig. 2.1. The graph of $F|_{T_xM}$. Here the indicatrix is an ellipse and the cone leans slightly to the right.

The fundamental inequality is also equivalent to the inequality

$$g_{ij}(y)w^i y^j \leq F(w)F(y) \tag{2.11}$$

which is considered a **generalization of the Cauchy-Schwarz inequality** from inner products to Minkowski norms. We point out that (2.11) may be expressed as

$$[g_{ij}(y)w^i y^j]^2 \leq [g_{kl}(w)w^k w^l][g_{pq}(y)y^p y^q]$$

by use of (2.4). This is similar but not the same as (2.7), since here the first term of the right contains $g_{kl}(w)$ instead of $g_{kl}(y)$.

Nuances such as the above express a crucial aspect of Finsler metrics – in general, the formal object $g_{ij}dx^i \otimes dx^j$ does not define an inner product on each tangent space

T_xM, because of the dependence on $y \neq 0$. In fact, this holds only when the metric is Riemannian. Nonetheless, this object defines a Riemannian metric on π^*TM, the pullback tangent bundle over $TM \setminus 0$, that is, the subspace of $TM \setminus 0 \times TM$ with the commutative diagram:

$$\begin{array}{ccc} \pi^*TM & \xrightarrow{\pi_2} & TM \\ \big\downarrow{\scriptstyle \pi_1} & & \big\downarrow{\scriptstyle \pi} \\ TM \setminus 0 & \xrightarrow{\pi} & M \end{array}$$

where π_1 (resp. π_2) denotes projection onto first (resp. second) factor. Simply put, over each point $(x, y) \in TM \setminus 0$, the fiber of π^*TM is the vector space T_xM. A natural local frame $\{\frac{\partial}{\partial x^i}\}$ of TM determines a local frame for π^*TM, still denoted $\{\frac{\partial}{\partial x^i}\}$, which is defined locally in x and globally in y. In the same way, $\{dx^i\}$ generates a local frame of π^*T^*M with same notation. Thus, $g = g_{ij}dx^i \otimes dx^j$ defines a symmetric section of $\pi^*T^*M \otimes \pi^*T^*M$, called the **fundamental tensor** on π^*TM. Similarly, if we let

$$C_{ijk}(y) := \left[\frac{1}{4}F^2\right]_{y^iy^jy^k}(y)\,, \tag{2.12}$$

then $C = C_{ijk}dx^i \otimes dx^j \otimes dx^k$ defines a symmetric section of $\otimes^3\pi^*T^*M$, called the **Cartan tensor** on π^*TM. The following is a direct consequence of this definition, which is frequently used without mention in published writings.

Proposition 2.1.1 *A Finsler metric F is Riemannian if and only if the Cartan tensor vanishes.*

Proof A Finsler metric $F(y) = \sqrt{g_{ij}y^iy^j}$ is Riemannian if and only if the g_{ij} are independent of $y \neq 0$, or equivalently,

$$\frac{\partial g_{ij}}{\partial y^k} = 0,\ \forall i, j, k\,.$$

Since

$$C_{ijk} = \left[\frac{1}{4}F^2\right]_{y^iy^jy^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k},\ \forall i, j, k\,,$$

the result follows. $\blacksquare$

Specifically, the Cartan tensor is a non-Riemannian quantity, also called the Cartan torsion. It appeared in Finsler's dissertation [7], but Cartan was the first to give this quantity a geometric interpretation [11]. Before we dive any further into tensors, let us complete this chapter with the notion for pseudo-Finsler metrics and some basic examples (in the positive–definite case).

2.2 Pseudo-metrics

A function $L : TM \to \mathbb{R}$ is a **pseudo-Finsler metric** on M if:

(i) L is C^∞ on $TM \setminus 0$;

(ii) $L(x, \lambda y) = \lambda^2 L(x, y)$, $\forall \lambda > 0$;

(iii) $\forall (x, y) \in TM \setminus 0$, the Hessian matrix

$$g_{ij}(x, y) := \left[\frac{1}{2} L(x, y)\right]_{y^i y^j}$$

is non-degenerate.

The **signature** of L is the list of signs of eigenvalues of the matrix (g_{ij}) with respect to some basis. For example, the Lorentz signature is $(+, -, \ldots, -)$ or $(-, +, \ldots, +)$.

A distance function on M comes from the Finsler function $F = |L|^{\frac{1}{2}}$ associated to L. In general, F is not differentiable when $L = 0$.

This notion was first introduced by Beem [13]. A generalization was given by Pfeifer and Wohlfarth [14], taking L to be positively homogeneous of degree $r \geq 2$ in y with associated Finsler function $F = |L|^{\frac{1}{r}}$.

Commonly, a spacetime is a 4-dimensional manifold with Lorentz signature, where the first coordinate x^0 represents time and the remaining three (x^1, x^2, x^3) the spacial coordinates. The equivalent for the Euclidean metric in this case is the pseudo-metric

$$L(y) = (y^0)^2 - (y^1)^2 - (y^2)^2 - (y^3)^2 \,.$$

The resulting space $(\mathbb{R}^4, L)$ is reffered to as Minkowski spacetime; not to be confused with a Minkowski space (V, F) composed of a vector space V and a general Minkowski norm F.

2.3 Examples

For simplicity, we study the Minkowski space $(T_xM, F|_{T_xM})$ for some fixed $x \in M$. In light of figure 2.1, we want to describe the indicatrix of the Minkowki norm, i.e. $\{y \in T_xM | F(y) = 1\}$. To allow visualization, we are particularly interested in Minkowski norms on the plane. In this case, as a result of strong convexity, the indicatrix must be a close, strictly convex, smooth curve that encloses the origin.

2.3.1 Shimada

We start with Riemann's idea of a "line-element that is the fourth root of a quartic differential expression". Specifically, consider the quartic metric:

$$F(y) = \sqrt[4]{(y^1)^4 + (y^2)^4}\,.$$

Its indicatrix (fig. 2.2) is strictly convex. However, its Hessian matrix (g_{ij}) is singular on the y^1 and y^2 axes. So strict convexity does not imply strong convexity, and despite its name F is not a well-defined Minkowski norm.

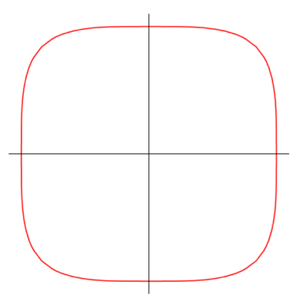
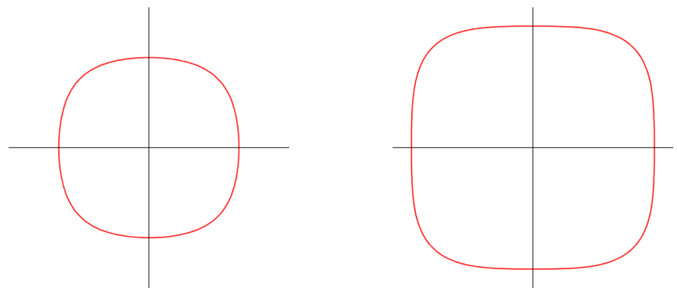


Fig. 2.2. The indicatrix of the quartic metric on the plane.

This can be fixed with a perturbation of the quartic metric; namely, take

$$F_\lambda = \sqrt{\sqrt{(y^1)^4 + (y^2)^4} + \lambda[(y^1)^2 + (y^2)^2]}$$

for any nonnegative constant λ. Then $F_0 = F$ and F_λ is a well defined Minkowski norm for all $\lambda > 0$. In other words, the perturbation has regularized the quartic metric.



(a) $\lambda = 1$ (b) $\lambda = \frac{1}{10}$

Fig. 2.3. The indicatrix of a perturbation of the quartic metric on the plane.

Notice the indicatrix of F_λ (fig. 2.3) is symmetric with respect to the origin of the plane, which shows the metric is reversible.

On a general manifold, Finsler functions expressed in local coordinates as

$$F(y) = \sqrt[m]{a_{i_1 i_2 \dots i_m} y^{i_1} y^{i_2} \cdots y^{i_m}}\,,$$

with $m \geq 3$ and $a_{i_1 i_2 \dots i_m}$ symmetric in all indices, are know as **$\mathbf{m^{th}}$-root metrics**. They were first studied by Shimada [15].

2.3.2 Randers

> "Perhaps the most characteristic property of the physical world is the unidirection of time-like intervals. Since there is no obvious reason why this asymmetry should disappear in the mathematical description it is of

> interest to consider the possibility of a metric with asymmetrical properties." [16]

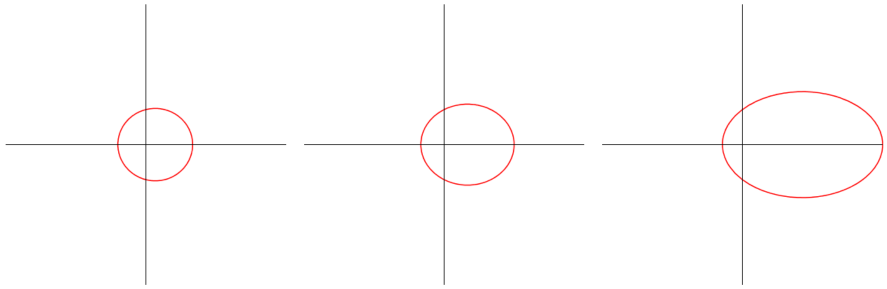
This is the principle Randers pointed out in the article "On an Asymmetrical Metric in the Four-Space of General Relativity" [16] to introduce a simple asymmetrical generalization of Riemannian metrics. Here is Randers approach:

> "The only way of introducing an asymmetry while retaining the quadratic indicatrix, is to displace the center of the indicatrix. In other words, we adopt as indicatrix an eccentric quadratic (hyper-) surface. This involves the definition of a vector at each point of the space, determining the displacement of the center of the indicatrix. The formula for the length ds of a line-element dx^μ must necessarily be homogeneous of first degree in dx^μ. The simplest "eccentric" line-element possessing this property, and of course being invariant, is
>
> $$ds = k_\mu dx^\mu + (g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}} \, ,$$
>
> where $g_{\mu\nu}$ is the fundamental tensor of the Riemannian affine connection, and k_μ is a covariant vector determining the displacement of the center of the indicatrix." [16]

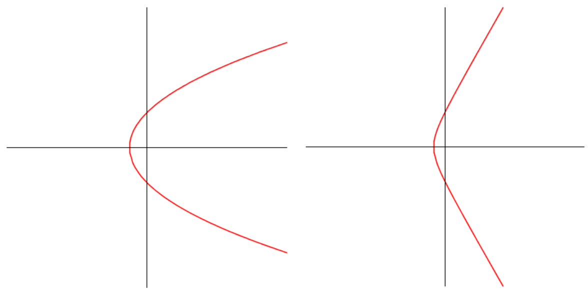
In our notation, for the Euclidean plane and a horizontal displacement, we have

$$F(y) = \sqrt{(y^1)^2 + (y^2)^2} + b y^1 \, .$$

The positivity of F is equivalent to $|b| < 1$, in which case F is strongly convex. For this case, strong convexity of the norm coincides with strict convexity of the indicatrix (see fig. 2.4).



(a) $|b|=\frac{1}{4}$ (b) $|b|=\frac{1}{2}$ (c) $|b|=\frac{3}{4}$



(d) $|b|=1$ (e) $|b|=2$

Fig. 2.4. The indicatrix of Randers norm over the plane.

In general, if (M, α^2) is a Riemannian manifold, and β is some 1-form over M, then $\alpha + \beta$ defines a **Randers norm** on M, which is positive if and only if $\|\beta\| < 1$, and this criterion also ensures strong convexity.

2.3.3 Matsumoto

In 1969, Matsumoto sent letters to several mathematicians asking for their opinion on "models of Finsler spaces". Amongst others, Finsler replied. He wrote:

> "In astronomy we measure the distance in a time, in particular, in the light-year. When we take a second as the unit, the unit surface is a sphere with the radius of $300,000$km. To each point of our space is associated such a sphere; this defines the distance (measured in a time) and the geometry of our space is the simplest one, namely, the euclidean

> geometry. Next, when a ray of light is considered as the shortest line in the gravitational field, the geometry of our space is Riemannian geometry. Furthermore, in an anisotropic medium the speed of the light depends on its direction, and the unit surface is not any longer a sphere.
>
> Now, on a slope of the earth surface we sometimes measure the distance in a time, namely, the time required such as seen on a guidepost. Then the unit curve, taken a minute as the unit, will be a general closed curve without a centre, because we can walk only a shorter distance in an uphill than in a downhill road. This defines a general geometry, although it is not exact. The shortest line along which we can reach the goal, for instance, the top of a mountain as soon as possible will be a complicated curve." [17]

Matsumoto gave a precise formulation for the model described in the second paragraph in the article "A slope of a mountain is a Finsler surface with respect to a time measure" [17], from where we obtained the letter excerpt. He determined that the indicatrix with respect to the time measure of a plane with an angle α of inclination is a limaçon given by

$$r = v + a\cos\theta,$$

in polar coordinates (r, θ) with pole at the origin and the downhill ray as polar axis, where $a = w\sin\theta$ and v, w are non-zero constants. The construction of the limaçon results from the Euclidean indicatrix $r = v$, when the plane is horizontal, with the slide caused by gravity, represented by the circle $r = a\cos\theta$ (fig. 2.5).

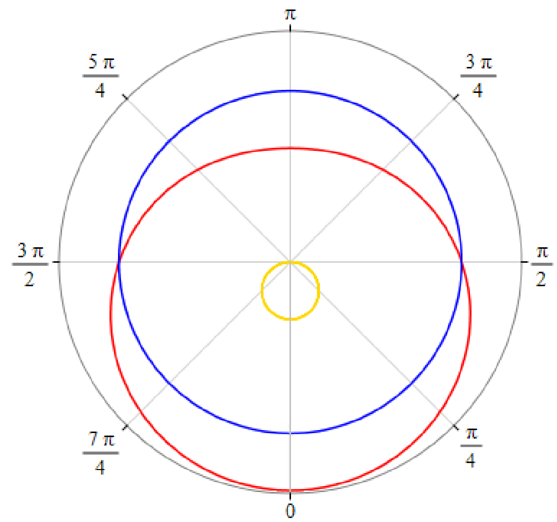
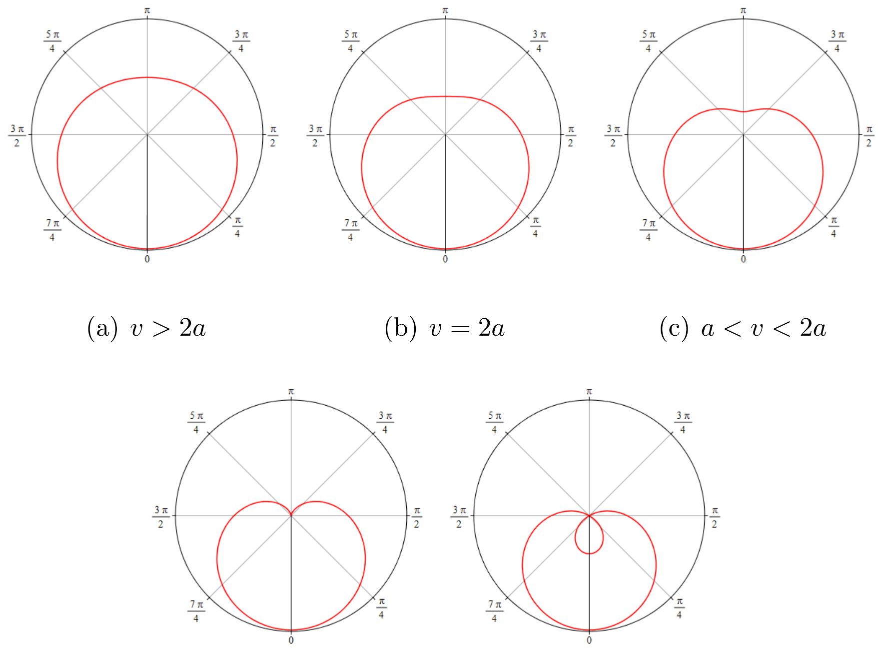


Fig. 2.5. With respect to the time measure, the indicatrix of a plane with an angle α of inclination (in red), the Euclidean indicatrix (in blue), and the slide caused by gravity (in yellow).

For $0 < v < a$, the limaçon has a self-closed part and does not describe the indicatrix of a Minkowki norm. When $v > 2a$, the limaçon is strictly convex and it is indeed the indicatrix of a well-defined Minkowski norm. The remaining cases need to be carefully considered around the uphill direction. These results are pictorially summarized in the figure 2.6.



(a) $v > 2a$ (b) $v = 2a$ (c) $a < v < 2a$

(d) $v = a$ (e) $0 < v < a$

Fig. 2.6. The indicatrix of a plane with an angle α of inclination with respect to the time measure.

They show it is impossible to walk a path straight uphill when $v \leq a$ and it may be faster to get uphill by zigzagging for $a < v < 2a$, which explains why Lombard Street in San Francisco was built with hairpin turns.

3. SPRAY AND CONNECTIONS

3.1 Nonlinear Connection

Consider $TM \setminus 0$ as a manifold with local coordinates (x^i, y^i). The tangent bundle of $TM \setminus 0$ has a local frame $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i} \right\}$. However, such frame is not natural with respect to the transformation on $TM \setminus 0$ induced by a change of coordinates on M. Namely, if local coordinates on M change by $x^i = x^i(\tilde{x}^p)$ and its inverse $\tilde{x}^p = \tilde{x}^p(x^i)$, then:

$$\frac{\partial}{\partial \tilde{x}^p} = \frac{\partial x^i}{\partial \tilde{x}^p}\frac{\partial}{\partial x^i} + \frac{\partial^2 x^i}{\partial \tilde{x}^p \partial \tilde{x}^q}\tilde{y}^q \frac{\partial}{\partial y^i}$$
$$\frac{\partial}{\partial \tilde{y}^p} = \frac{\partial x^i}{\partial \tilde{x}^p}\frac{\partial}{\partial y^i}$$

The same is true for the cotangent bundle of $TM \setminus 0$, where $\{dx^i, dy^i\}$ behave as follows:

$$d\tilde{x}^p = \frac{\partial \tilde{x}^p}{\partial x^i} dx^i$$
$$d\tilde{y}^p = \frac{\partial \tilde{x}^p}{\partial x^i} dy^i + \frac{\partial^2 \tilde{x}^p}{\partial x^i \partial x^j} y^j dx^i$$

To introduce natural local frames for the tangent and the cotangent bundles of $TM \setminus 0$, define the **formal Christoffel symbols** of the second kind associated to the components g_{ij} of the fundamental tensor:

$$\gamma^i_{jk} := \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} + \frac{\partial g_{kl}}{\partial x^j} \right) \tag{3.1}$$

Next, define the quantities:

$$N^i_j := \gamma^i_{jk} y^k - C^i_{jk} \gamma^k_{lm} y^l y^m \tag{3.2}$$

where $C^i_{jk} := g^{il} C_{ljk}$ and C_{ijk} are the components of the Cartan tensor.

The transformation law for N^i_j under a local change of coordinates on M is:

$$\tilde{N}^p_q = \frac{\partial \tilde{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \tilde{x}^q}N^i_j + \frac{\partial \tilde{x}^p}{\partial x^i}\frac{\partial^2 x^i}{\partial \tilde{x}^q \partial \tilde{x}^r}\tilde{y}^r$$

By replacing $\frac{\partial}{\partial x^i}$ with

$$\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$$

and dy^i by

$$\delta y^i := dy^i + N^i_j dx^j$$

we obtain the natural local bases $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$ for the tangent bundle of $TM \setminus 0$ and $\{dx^i, \delta y^i\}$ for the cotangent bundle of $TM \setminus 0$, which are dual to each other. Moreover,

$$\mathcal{H}TM := \text{span}\left\{\frac{\delta}{\delta x^i}\right\} \ , \quad \mathcal{V}TM := \text{span}\left\{\frac{\partial}{\partial y^i}\right\}$$

are well-defined subbundles of $T(TM \setminus 0)$ and, similarly,

$$\mathcal{H}^*TM := \text{span}\left\{dx^i\right\} \ , \quad \mathcal{V}^*TM := \text{span}\left\{\delta y^i\right\}$$

are well-defined subbundles of $T^*(TM \setminus 0)$. They give the decompositions:

$$T(TM \setminus 0) = \mathcal{H}TM \oplus \mathcal{V}TM$$

$$T^*(TM \setminus 0) = \mathcal{H}^*TM \oplus \mathcal{V}^*TM$$

The **Sasaki (type) metric** [18]

$$g_{ij}dx^i \otimes dx^j + g_ij\delta y^i \otimes \delta y^j$$

is a natural Riemannian metric on the manifold $TM \setminus 0$ with respect to which $\mathcal{H}TM$ is orthogonal to $\mathcal{V}TM$. So $TM \setminus 0$ admits an **Ehresmann connection** [19]. Its existence is a direct consequence of the quantities N^i_j, that are, hence, called the **nonlinear connection**.

3.2 Spray Coefficients

The local functions

$$G^i := \frac{1}{2}\gamma^i_{jk} y^j y^k \tag{3.4}$$

give rise to a globally defined vector field on $TM \setminus 0$:

$$G := y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}\,,$$

called the **spray** induced by F. It turns out that a curve $x = x(t)$ on (M, F) is a geodesic if and only if it is the projection of an integral curve of G. For details, check section (5.1) of [4].

The quantities G^i, called **spray coefficients**, are positively homogeneous of degree two in y, but they are not quadratic in general, for γ^i_{jk} are dependent on y. When in standard local coordinates (x^i, y^i) the spray coefficients G^i are quadratic in y, the Finsler metric is called a **Berwald metric**. In particular, if F is Riemannian, then $\gamma^i_{jk}(x)$ are the usual Christoffel coefficients and the metric is Berwald.

By (2.2) and (3.1), we may also express:

$$G^i = \frac{1}{4} g^{il} \left(2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right) y^j y^k \tag{3.5a}$$

$$G^i = \frac{1}{4} g^{il} \left([F^2]_{x^k y^l} y^k - [F^2]_{x^k} \right) \tag{3.5b}$$

Finally, it is straightforward to prove that:

$$N^i_j = \frac{\partial G^i}{\partial y^j}$$

So the nonlinear connection may be calculated without having to compute the Cartan tensor C^i_{jk} and the formal Christoffel symbols γ^i_{jk}.

3.3 Linear Connections

Let E be a vector bundle over a manifold M and $C^\infty(E)$ the vector space of smooth sections of E.

A **linear connection** on E is a linear mapping

$$\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)\,,$$

satisfying the Leibniz rule:

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma\,,\ \forall f \in C^\infty(M,\mathbb{R})\,,\ \forall \sigma \in C^\infty(E)\,.$$

This operator determines how to measure the rate of change $\nabla_v \sigma$ of σ along a direction v at some point of the manifold, know as the **covariant derivative**. By requiring the Leibniz rule, ∇ also defines the covariant derivative for any smooth section on tensor products of E and its dual bundle E^*.

In local coordinates, a linear connection ∇ may be specified by its **connection 1-forms** ω^i_j, with respect to which:

$$\nabla_v \frac{\partial}{\partial x^i} := \omega^j_i(v)\frac{\partial}{\partial x^j}$$
$$\nabla_v dx^i := -\omega^i_j(v) dx^j$$

For a Finsler manifold (M, F), we choose E to be the pullback tangent bundle π^*TM over the manifold $TM \setminus 0$, where the fundamental tensor $g = g_{ij}dx^i \otimes dx^j$ is defined. Bellow, we introduce a very simple connection on π^*TM that was discovered by Chern in [20].

Theorem 3.3.1 (Theorem 2.4.1 of [3]) *Let (M, F) be a Finsler manifold. The pullback bundle π^*TM admits a unique linear connection, called the* **Chern connection**. *Its connection forms are characterized by the structural equations:*

- **Torsion freeness:**

$$dx^j \wedge \omega^i_j = 0 \tag{3.6}$$

- **Almost g-compatibility:**

$$dg_{ij} - g_{kj}\omega^k_i - g_{ik}\omega^k_j = 2C_{ijk}\delta y^k \tag{3.7}$$

Proof We solve the structural equations to obtain the connection 1-forms ω_j^i.

In the local basis $\{dx^i, dy^i\}$, the connection forms are written as

$$\omega_j^i = \Gamma_{jk}^i dx^k + Z_{jk}^i dy^k \,.$$

Torsion freeness is equivalent to the vanishing of Z_{jk}^i:

$$\omega_j^i = \Gamma_{jk}^i dx^k \,,$$

together with the symmetry:

$$\Gamma_{jk}^i = \Gamma_{kj}^i \,.$$

Almost g-compatibility then implies:

$$\Gamma_{jk}^i = \gamma_{jk}^i - g^{il}\left(C_{ljm}N_k^m - C_{jkm}N_l^m + C_{klm}N_l^m\right) \,,$$

which may be re-written in the following elegant form:

$$\Gamma_{jk}^i = \frac{1}{2}g^{il}\left(\frac{\delta g_{lj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^l} + \frac{\delta g_{kl}}{\delta x^j}\right) \tag{3.8}$$

■

When the Finsler metric F is Riemannian, the components of the fundamental tensor g_{ij} are functions on the manifold M and the Cartan tensor vanishes. So the components of the Chern connection Γ_{jk}^i are equal to the Christoffel symbols of the second kind $\gamma_{jk}^i(x)$. In this case, the Chern connection is nothing but the pullback of the Levi-Civita connection of the Riemannian manifold (M, g).

Notice that the uniqueness of the Chern connection is limited to its structural equations. In general, there are other linear connections on π^*TM, none considered natural for a Finsler manifold. We highlight the **Cartan connection**, given by the connection forms $\omega_j^i + C_{jk}^i \delta y^k$, and the **Berwald connection**, given by $\omega_j^i + \dot{C}_{jk}^i dx^k$, where ω_j^i are the Chern connection forms and $\dot{C} := \nabla_v C$ is the covariant derivative of the Cartan tensor along the direction $v := y^i \frac{\delta}{\delta x^i}$. The Cartan connection is metric-compatible but has torsion, while the Berwald connection is torsion-free but not necessarily metric compatible. In truth, there exists a torsion-free g-compatible linear connection on π^*TM if and only if the Finsler metric F is Riemannian. However, when the Finsler metric is of Berwald type, the Chern and the Berwald connections reduce to a linear connection on TM (both to the same).

4. CURVATURE

4.1 Curvature Tensors

The **curvature** 2**-forms** associated to a linear connection are

$$\Omega_j^i := d\omega_j^i - \omega_j^k \wedge \omega_k^i\,. \tag{4.1}$$

The curvature forms for a connection on π^*TM are 2-forms on $TM \setminus 0$. So they can be generically expanded as:

$$\Omega_j^i = \frac{1}{2} R_j{}^i{}_{kl} dx^k \wedge dx^l + P_j{}^i{}_{kl} dx^k \wedge \delta y^l + \frac{1}{2} Q_j{}^i{}_{kl} \delta y^k \wedge \delta y^l\,, \tag{4.2}$$

where, without loss of generality, we can assume that:

$$R_j{}^i{}_{lk} = -R_j{}^i{}_{kl}\,; \tag{4.3a}$$

$$Q_j{}^i{}_{lk} = -Q_j{}^i{}_{kl}\,. \tag{4.3b}$$

Their components define tensors R, P, Q on π^*TM, which are called **hh-, hv-, vv-curvature tensors** of the connection, respectively.

Consider the Chern connection. By torsion freeness (3.6), we obtain

$$dx^j \wedge d\omega_j^i = 0\,,$$

and consequently,

$$dx^j \wedge \Omega_j^i = 0\,, \tag{4.4}$$

with the use of (4.1). Substituting (4.2) into (4.4), we have

$$\frac{1}{2} R_j{}^i{}_{kl} dx^j \wedge dx^k \wedge dx^l + P_j{}^i{}_{kl} dx^j \wedge dx^k \wedge \delta y^l + \frac{1}{2} Q_j{}^i{}_{kl} dx^j \wedge \delta y^k \wedge \delta y^l = 0\,.$$

Since the three terms on the left have different types, each must vanish.

From $\frac{1}{2}Q_{j\,kl}^{\,i}dx^j \wedge \delta y^k \wedge \delta y^l = 0$ we get

$$Q_{j\,lk}^{\,i} = Q_{j\,kl}^{\,i}\,. \tag{4.5}$$

Comparing (4.3b) and (4.5), we conclude $Q_{j\,kl}^{\,i} = 0$. So the vv-curvature Q vanishes and the curvature forms of the Chern connection simplify to

$$\Omega_j^i = \frac{1}{2}R_{j\,kl}^{\,i}dx^k \wedge dx^l + P_{j\,kl}^{\,i}dx^k \wedge \delta y^l\,. \tag{4.6}$$

From $P_{j\,kl}^{\,i}dx^j \wedge dx^k \wedge \delta y^l = 0$, we get

$$P_{k\,jl}^{\,i} = P_{j\,kl}^{\,i}\,. \tag{4.7}$$

Lastly, from $\frac{1}{2}R_{j\,kl}^{\,i}dx^j \wedge dx^k \wedge dx^l = 0$, we obtain

$$R_{j\,kl}^{\,i} + R_{k\,lj}^{\,i} + R_{l\,jk}^{\,i} = 0\,, \tag{4.8}$$

which is know as the **first Bianchi identity** for R.

Putting together (4.1) and (4.6), a somewhat easy computation yields formulas for R and P in natural coordinates:

$$R_{j\,kl}^{\,i} = \frac{\delta\Gamma_{jl}^i}{\delta x^k} - \frac{\delta\Gamma_{jk}^i}{\delta x^l} + \Gamma_{mk}^i\Gamma_{jl}^m - \Gamma_{ml}^i\Gamma_{jk}^m \tag{4.9a}$$

$$P_{j\,kl}^{\,i} = -\frac{\partial\Gamma_{jk}^i}{\partial y^l} \tag{4.9b}$$

In particular, if the Finsler metric is Riemannian, the coefficients Γ_{jk}^i become the Christoffel symbols of second kind $\gamma_{jk}^i(x)$. Then $P = 0$ and

$$R_{j\,kl}^{\,i} = \frac{\partial\gamma_{jl}^i}{\partial x^k} - \frac{\partial\gamma_{jk}^i}{\partial x^l} + \gamma_{mk}^i\gamma_{jl}^m - \gamma_{ml}^i\gamma_{jk}^m\,,$$

the components of the Riemannian curvature tensor of (M, g).

Our choice to define curvature through a connection and, in particular, the Chern connection, is due to the theoretical resemblance with Riemannian Geometry. We lose, however, the historical perspective of the development of Fisler Geometry. The Riemann curvature tensor for Finsler metrics given by (4.9a), for instance, was first defined by Berwald in 1926 [21].

4.2 Riemann Curvature

For each $(x, y) \in TM \setminus 0$, let

$$R^i_k := y^j R_j{}^i{}_{kl} y^l \tag{4.10}$$

and define a linear transformation

$$R_y := R^i_k(y) \frac{\partial}{\partial x^i} \otimes dx^k .$$

The family of transformations $\{R_y : T_xM \to T_xM | (x, y) \in TM \setminus 0\}$ is called the **Riemann curvature**.

Here, there is another downside of our approach to curvature – the formula it produces for the Riemannn components (4.10) is unnecessarily hard to compute. The quantities R^i_k may be expressed entirely in terms of the spray coefficients G^i; namely,

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial x^k}\frac{\partial G^i}{\partial x^k} . \tag{4.11}$$

This formula is again due to the work of Berwald [22].

From (4.11) and Euler's theorem, is now easy to see

$$R_y(y) = 0 .$$

Moreover, R_y is self-adjoint with respect to the inner product $g_y = g_{ij}(y) dx^i \otimes dx^j$, i.e.

$$g_y(R_y(u), v) = g_y(u, R_y(v)) \, , \forall u, v \in T_xM \, ;$$

although the proof of this statement is not immediate. By linear algebra, for each $(x, y) \in TM \setminus 0$, R_y is diagonalizable and at least one of its eigenvalues is zero.

4.3 Ricci Curvature

For each $(x, y) \in TM \setminus 0$, let

$$\mathrm{Ric}(y) := \sum_i R^i_i(y) . \tag{4.12}$$

Then Ric is a scalar function on $TM \setminus 0$, called the **Ricci curvature**. It is positively homogeneous of degree two in y, by formula (4.11) and the homogeneity of G^i. Hence, it satisfies

$$\mathrm{Ric} = \frac{1}{2}\,[\mathrm{Ric}]_{y^i y^j}\, y^i y^j\,.$$

Furthermore, it represents the trace of R_y at each point. So it corresponds to the sum of the $(n-1)$ possibly non-zero eigenvalues of R_y. For this reason, some authors choose to define $\frac{1}{n-1}\,\mathrm{Ric}$ as the Ricci curvature.

5. FINSLER METRICS BY WARPED PRODUCT

If (M, ds_1^2), (N, ds_2^2) are Riemannian manifolds, then a warped product is the manifold $M \times N$ endowed with a Riemannian metric of the form

$$ds^2 = ds_1^2 + f^2 ds_2^2 \,, \tag{5.1}$$

where f is a smooth function depending on the coordinates of M only; said a warping function. This notion, called by **warped product**, must be credited to Bishop and O'Neill [23]. However, years earlier, metrics in the form of (5.1) were being studied with different names; in [24], for instance, they were called semi-reducible Riemannian spaces. Moreover, metrics of such form with arbitrary signature can be easily considered in the realm of pseudo-Riemannian geometry. Particularly, if (M, ds_1^2), $(\mathbb{R}, dt^2)$ are Riemannian manifolds, then

$$ds^2 = f^2 dt^2 - ds_1^2 \,, \tag{5.2}$$

is a warped product metric with Lorentz signature. When M is 3-dimensional, (5.2) defines the line element of a **standard static spacetime** (see [25], p.360).

The class of warped product manifolds has shown itself to be rich, both wide and diverse, playing important roles in differential geometry as well as in physics. To illustrate, Bishop and O'Neill introduced warped products in [23] as means to construct a large class of complete Riemannian manifolds with negative curvature. For this reason, it seems valuable to study warped product metrics without the quadratic restriction, in the setting of Finsler geometry. Notably, progress in this direction has been stimulated by efforts to expand general relativity, such as the work of Asanov (e.g. [26], [27], [28]), which later motivated Kozma, Peter and Varga to study product manifolds $M \times N$ endowed with a Finsler metric

$$F = \sqrt{F_1^2 + f^2 F_2^2} \,, \tag{5.3}$$

called warped product, where (M, F_1), (N, F_2) are Finsler manifolds and f is a smooth function on M (see [29]). Following the definition of Beem [13], one may take $L = F^2$ to consider pseudo-Finsler metrics. For example, if (M, F_1) is a 3-dimensional Finsler manifold and $(\mathbb{R}, F_2)$ is a Minkowski space, then

$$L = f^2 F_2^2 - F_1^2 \tag{5.4}$$

is a Finsler metric with Lorentz signature, and $(\mathbb{R}\times M, L)$ may be regarded as a Finsler static spacetime. This is the case for [30], where Li and Chang studied metrics in the form of (5.4), given on coordinates $((t, r, \theta, \varphi), (y^t, y^r, y^\theta, y^\varphi))$ of the tangent bundle by

$$L = f^2(y^t)^2 - \left[g^2(y^r)^2 + r^2\overline{F}^2\right],$$

with $\overline{F}$ a Finsler metric on coordinates $(\theta, \varphi, y^\theta, y^\varphi)$ and f, g functions of r. They suggested the vacuum field equation for Finsler spacetime is equivalent to the vanishing of the Ricci scalar, and obtained a non-Riemannian exact solution similar to the Schwarzschild metric.

Recently, Chen, Shen and Zhao have considered product manifolds $\mathbb{R} \times M$ with Finsler metrics arising from warped products in the following way: if (M, α^2), $(\mathbb{R}, dt^2)$ are Riemannian manifolds, then $F^2 = dt^2 + f^2(t)\alpha^2$ is a warped product, which may be rewritten as $F = \alpha\sqrt{\left(\frac{dt}{\alpha}\right)^2 + f^2(t)}$. Letting $z = \frac{dt}{\alpha}$, they defined a class of Finsler metrics by

$$F = \alpha\sqrt{\phi(z,t)}\,, \tag{5.5}$$

which are also called warped product, where ϕ is a suitable function on $\mathbb{R}^2$ (see [1]). For $L = \alpha^2\phi(z, t)$, one may study pseudo-Finsler metrics with Lorentz signature, that can be thought of as Finsler Robertson-Walker spacetimes.

In this work, we wish to consider Finsler metrics of similar type as (5.5), with another "warping", one that is consistent with the form of metrics modeling static spacetimes and simplified by spherical symmetry over spatial coordinates, which emerged from the Schwarzschild metric in isotropic coordinates (as shown bellow).

5.1 Motivation

The Schwarzschild metric is very likely the most famous exact solution to the Einstein field equation; it was also the first to be derived, by Karl Schwarzschild, in a work [31] published only two months after Einstein's paper [32]. It describes the gravitational field around a static, spherically symmetric single body with no charge. In Schwarzschild coordinates (t, r, θ, φ), the solution is

$$ds^2 = \gamma c^2 dt^2 - \gamma^{-1} dr^2 - r^2 d\Omega^2 \,,$$

with $\gamma = 1 - \frac{m}{r}$, $m = \frac{2GM}{c^2}$ and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$, where G denotes the gravitational constant, M the mass of the central body and c the speed of light in vacuum.

In isotropic polar coordinates $(t, \rho, \theta, \varphi)$, given by

$$r = \left(1 + \frac{m}{4\rho}\right)^2 \rho$$

(see for example [33], p. 93), the Schwarzschild metric becomes

$$ds^2 = \frac{\left(1 - \frac{m}{4\rho}\right)^2}{\left(1 + \frac{m}{4\rho}\right)^2} c^2 dt^2 - \left(1 + \frac{m}{4\rho}\right)^4 \left[d\rho^2 + \rho^2 d\Omega^2\right] \,,$$

where $d\Omega^2$ and m are as before. Notice that the exterior region $r > m$ corresponds to $0 < \rho < \frac{m}{4}$ or $\rho > \frac{m}{4}$, because ρ doubly covers r:

$$\rho = \frac{1}{2}\left(r - \frac{m}{2} \pm \sqrt{r^2 - mr}\right)$$

Taking the spherical change of coordinates

$$\begin{cases} x^1 = \rho \sin\theta \cos\varphi \\ x^2 = \rho \sin\theta \sin\varphi \\ x^3 = \rho \cos\theta \end{cases}$$

the Schwarzschild metric is written in isotropic rectangular coordinates (t, x^1, x^2, x^3) as

$$ds^2 = \frac{\left(1 - \frac{m}{4\rho}\right)^2}{\left(1 + \frac{m}{4\rho}\right)^2} c^2 dt^2 - \left(1 + \frac{m}{4\rho}\right)^4 \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] \,, \tag{5.6}$$

where $\rho = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$.

In this system of coordinates, lightlike orbits (i.e. $ds^2 = 0$) are easily described by

$$\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2 = \frac{\left(1 - \frac{m}{4\rho}\right)^2}{\left(1 + \frac{m}{4\rho}\right)^6} c^2 \,,$$

which yields the same velocity in all spatial directions; hence the name, isotropic.

Letting $\alpha = \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}$ and $z = \frac{dt}{\alpha}$, the Schwarzschild metric (5.6) is written as

$$ds^2 = \alpha^2 \left[\frac{\left(1 - \frac{m}{4\rho}\right)^2}{\left(1 + \frac{m}{4\rho}\right)^2} c^2 z^2 - \left(1 + \frac{m}{4\rho}\right)^4 \right] ,$$

which has the form of a Finsler warped product metric $F^2 = \alpha^2 \phi(z, \rho)$.

5.2 Geometric Quantities

Set $M = \mathbb{R} \times \mathbb{R}^n$ with coordinates on TM

$$x = (x^0, \overline{x}),\ \overline{x} = (x^1, \ldots, x^n)\,,$$
$$y = (y^0, \overline{y}),\ \overline{y} = (y^1, \ldots, y^n)\,;$$

and consider a Finsler metric

$$F = \alpha \sqrt{\phi(z, \rho)}\,, \tag{5.7}$$

where $\alpha = |\overline{y}|$, $z = \frac{y^0}{|\overline{y}|}$ and $\rho = |\overline{x}|$. Throughout our work, the following convention for indices is adopted: A, B, ... range from 0 to n; i, j, ... range from 1 to n.

This construction is the same as [1] but for the “warping”. Consequently, any calculations involving F and its derivatives of any degree with respect to y^A only will be similar in form to the calculations in [1], e.g. the fundamental form. The effects of the warping only appear when derivatives of F with respect to x^A are involved, e.g. spray coefficients. So the Hessian matrix, $g_{AB} = \frac{1}{2}[F^2]_{y^A y^B}$, is

$$(g_{AB}) = \left(\begin{array}{c|c} \frac{1}{2}\phi_{zz} & \frac{1}{2}\Omega_z \frac{y^j}{\alpha} \\ \hline \frac{1}{2}\Omega_z \frac{y^i}{\alpha} & \frac{1}{2}\Omega\delta_{ij} - \frac{1}{2} z \Omega_z \frac{y^i y^j}{\alpha^2} \end{array} \right) , \tag{5.8}$$

where

$$\Omega := 2\phi - z\phi_z\,, \tag{5.9}$$

and the same argument as [1] to verify non-degeneracy of F applies. It actually simplifies, because α is the Euclidean metric here.

If $S^A_B = g_{BC}\delta^{CA}$, then $S = S^A_B dx^B \otimes \frac{\partial}{\partial x^A}$ can be considered an endomorphism on π^*T^*M. Fixed $(x,y) \in TM$, pick a linear basis of π^*T^*M

$$p^0 = dx^0\,, \quad p^1 = \frac{y^i}{\alpha}dx^i\,, \quad p^\gamma = p^\gamma_i dx^i\,, \quad \gamma = 2, \ldots, n\,,$$

such that $p^\gamma_i \frac{y^i}{\alpha} = 0$.

The matrix of S under the basis $\{p^A\}^n_{A=0}$ is

$$[S]_{\{p^A\}} = \left(\begin{array}{cc|ccc} \frac{1}{2}\phi_{zz} & \frac{1}{2}\Omega_z & 0 & \cdots & 0 \\ \frac{1}{2}\Omega_z & \frac{1}{2}(\Omega - z\Omega_z) & 0 & \cdots & 0 \\ \hline 0 & 0 & \frac{1}{2}\Omega & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & \frac{1}{2}\Omega \end{array}\right)$$

So the eigenvalues of (S^A_B) are given by:

$$\lambda_2 = \ldots = \lambda_n = \tfrac{1}{2}\Omega$$

$$\lambda_0 + \lambda_1 = \tfrac{1}{2}(\phi_{zz} + (\Omega - z\Omega_z))$$

$$\lambda_0\lambda_1 = \tfrac{1}{4}(\phi_{zz}(\Omega - z\Omega_z) - \Omega_z^2)$$

Hence,

$$\det(g_{AB}) = \frac{1}{2^{n+1}}\Omega^{n-1}\Lambda\,,$$

where

$$\Lambda := \phi_{zz}(\Omega - z\Omega_z) - \Omega_z^2 = 2\phi\phi_{zz} - \phi_z^2\,, \tag{5.10}$$

and:

Proposition 5.2.1 (Proposition 4.1, [1]) *$F = \alpha\sqrt{\phi(z,\rho)}$ is strongly convex if and only if $\Omega, \Lambda > 0$.*

Proof The eigenvalues of (g_{AB}) coincide with those of (S^A_B), so F is strongly convex if and only if $\lambda_A > 0$. Clearly, if the metric is strongly convex, then $\Omega, \Lambda > 0$. Conversely, assume $\Omega, \Lambda > 0$. So $\lambda_2 = \ldots = \lambda_n = \frac{1}{2}\Omega > 0$ and λ_0, λ_1 are solutions of the quadratic equation $\lambda^2 - (\lambda_0 + \lambda_1)\lambda + \lambda_0\lambda_1 = 0$, whose discriminant is

$$\Delta = (\lambda_0 + \lambda_1)^2 - 4\lambda_0\lambda_1 = \frac{1}{4}(\phi_{zz} - (\Omega - z\Omega_z))^2 + \Omega_z^2 \geq 0\,.$$

Hence, λ_0 and λ_1 are real numbers. Moreover, since $\lambda_0\lambda_1 = \frac{1}{4}\Lambda > 0$, they have the same sign. From $\Lambda = 2\phi\phi_{zz} - \phi_z^2 > 0$, we obtain $\phi_{zz} > 0$. At this moment, from $\Lambda = \phi_{zz}(\Omega - z\Omega_z) - \Omega_z^2 > 0$, we have $\Omega - z\Omega_z > 0$. Thus, $\lambda_0 + \lambda_1 > 0$ and we conclude λ_0, λ_1 are positive real numbers. ■

One may consider pseudo-Finsler metrics by letting $L = \alpha^2\phi(z,\rho)$. These metrics have Lorentz signature $(+,-,\ldots,-)$ if $\Omega, \Lambda < 0$, or $(-,+,\ldots,+)$ if $\Omega > 0$ and $\Lambda < 0$.

Henceforth, assume (g_{AB}) is non-degenerate. In this case, the inverse of $[S]_{\{p^A\}}$ is easily obtained:

$$[S]^{-1}_{\{p^A\}} = \left(\begin{array}{cc|ccc}
\frac{2}{\Lambda}(\Omega - z\Omega_z) & -\frac{2}{\Lambda}\Omega_z & 0 & \cdots & 0 \\
-\frac{2}{\Lambda}\Omega_z & \frac{2}{\Lambda}\phi_{zz} & 0 & \cdots & 0 \\
\hline
0 & 0 & 2\Omega^{-1} & & 0 \\
\vdots & \vdots & & \ddots & \\
0 & 0 & 0 & & 2\Omega^{-1}
\end{array}\right)$$

Therefore, the inverse of (g_{AB}) is

$$(g^{AB}) = \left(\begin{array}{c|c}
\frac{2}{\Lambda}(\Omega - z\Omega_z) & -\frac{2}{\Lambda}\Omega_z\frac{y^j}{\alpha} \\
\hline
-\frac{2}{\Lambda}\Omega_z\frac{y^i}{\alpha} & \frac{2}{\Omega}\delta^{ij} + \frac{2\phi_z\Omega_z}{\Omega\Lambda}\frac{y^iy^j}{\alpha^2}
\end{array}\right). \tag{5.11}$$

The spray coefficients $G^C = \frac{1}{4} g^{CA} \left([F^2]_{y^A x^B} y^B - [F^2]_{x^A} \right)$ are:

$$G^0 = (U + zV)(x^m y^m)\alpha \,, \tag{5.12a}$$

$$G^i = (V + W) y^i (x^m y^m) - W x^i \alpha^2 \,, \tag{5.12b}$$

where

$$U := \frac{1}{2\rho\Lambda}(2\phi\phi_{z\rho} - \phi_z\phi_\rho) \,, \tag{5.13a}$$

$$V := \frac{1}{2\rho\Lambda}(\phi_\rho\phi_{zz} - \phi_z\phi_{z\rho}) \,, \tag{5.13b}$$

$$W := \frac{1}{2\rho\Omega}\phi_\rho \,. \tag{5.13c}$$

The Riemann curvature by Berwald's formula

$$R^C_B = 2[G^C]_{x^B} - [G^C]_{x^A y^B} y^A + 2G^A [G^C]_{y^A y^B} - [G^C]_{y^A} [G^A]_{y^B}$$

gives

$$\begin{aligned} R^0_0 = & \left[\rho^2(U + zV)W_z - (2\rho^2 W + 1)(U_z + V + zV_z)\right] \alpha^2 \\ & + \Bigg[2(V + W)(U_z + V + zV_z) - (V_z + W_z)(U + zV) + 2U(U_{zz} + 2V_z + zV_{zz}) \\ & \quad - \frac{1}{\rho}(U_{z\rho} + V_\rho + zV_{z\rho}) - (U_z + V + zV_z)^2 - (U - zU_z - z^2 V_z)V_z \Bigg] (x^m y^m)^2 \end{aligned} \tag{5.14a}$$

$$\begin{aligned} R^i_j = & - \left[2W + (2\rho^2 W + 1)(V + W)\right] \alpha^2 \delta^i_j \\ & + \left[(V + W)^2 + 2U(V_z + W_z) - \frac{1}{\rho}(V_\rho + W_\rho) \right] (x^m y^m)^2 \delta^i_j \\ & + \left[2W(2W - zW_z) + W_z(U - zW) - \frac{2}{\rho} W_\rho \right] \alpha^2 x^i x^j + \big[(V + W) \\ & \qquad + z(V_z + W_z)(2\rho^2 W + 1) + (\rho^2(V + W) + 1)(2W - zW_z) \big] y^i y^j \\ & - \Bigg[2zU(V_{zz} + W_{zz}) + (3U - zU_z - zV + 5zW)(V_z + W_z) \\ & \quad - \frac{z}{\rho}(V_{z\rho} + W_{z\rho}) \Bigg] (x^m y^m)^2 \frac{y^i y^j}{\alpha^2} + \Bigg[-(2W - zW_z)^2 - 2U(W_z - zW_{zz}) \\ & \qquad + \frac{1}{\rho}(2W_\rho - zW_{z\rho}) + W_z(U - zU_z + z^2 W_z) \Bigg] (x^m y^m) x^i y^j \\ & + \left[-(V + W)^2 + (V_z + W_z)(U + 3zW) + \frac{1}{\rho}(V_\rho + W_\rho) \right] (x^m y^m) x^j y^i \end{aligned} \tag{5.14b}$$

$$\begin{aligned}
R_j^0 = z & \left[(2\rho^2 W + 1)(V + U_z + zV_z) - \rho^2 W_z (U + zV)\right] \alpha y^j \\
& + \Bigg[z(U + zV)(V_z + W_z) - 2zU(U_{zz} + 2V_z + zV_{zz}) \\
& + (U - zU_z - z^2 V_z)(5W - U_z) - \frac{1}{\rho}(U_\rho - zU_{z\rho} - z^2 V_{z\rho}) \Bigg] (x^m y^m)^2 \frac{y^j}{\alpha} \\
& + \Bigg[(U + zV)(U_z - V + zV_z - 2W) + (V - 3W)(U - zU_z - z^2 V_z) \\
& \qquad + \frac{1}{\rho}(U_\rho + zV_\rho) \Bigg] (x^m y^m) \alpha x^j
\end{aligned} \tag{5.14c}$$

$$\begin{aligned}
R_0^i = & \left[\rho^2 W_z (V - W) - (2\rho^2 W + 1) V_z\right] \alpha y^i + \Bigg[(2W - V - U_z)(V_z + W_z) \\
& \qquad + 2U(V_{zz} + W_{zz}) - \frac{1}{\rho}(V_{z\rho} + W_{z\rho}) \Bigg] (x^m y^m)^2 \frac{y^i}{\alpha} \\
& + \left[(U_z - W) W_z - 2U W_{zz} + \frac{1}{\rho} W_{z\rho} \right] (x^m y^m) \alpha x^i
\end{aligned} \tag{5.14d}$$

After simplification, the Ricci curvature is:

$$\begin{aligned}
\mathrm{Ric} &= \sum R_A^A \\
&= \left[-(2\rho^2 W + 1)(U_z + nV + (n-3)W) - 2(nW + \rho^2 W_z (U - zW))\right] \alpha^2 \\
& \quad + \Bigg[2U(U_{zz} + nV_z + (n-2)W_z) - \frac{1}{\rho}(U_{z\rho} + nV_\rho + (n-3)W_\rho) + nV(V + 2W) \\
& \qquad + W((n-5)W + 2zW_z) + U_z(2W - U_z) \Bigg] (x^m y^m)^2
\end{aligned}$$

Let the Ricci curvature components be

$$P(z, \rho) := -(2\rho^2 W + 1)(U_z + nV + (n-3)W) - 2(nW + \rho^2 W_z (U - zW)) \tag{5.15a}$$

$$\begin{aligned}
Q(z, \rho) := & \, 2U(U_{zz} + nV_z + (n-2)W_z) - \frac{1}{\rho}(U_{z\rho} + nV_\rho + (n-3)W_\rho) \\
& + nV(V + 2W) + W((n-5)W + 2zW_z) + U_z(2W - U_z)
\end{aligned} \tag{5.15b}$$

So

$$\begin{aligned}
\mathrm{Ric} &= P\left(\frac{y^0}{|\overline{y}|}, |\overline{x}|\right) \langle \overline{y}, \overline{y} \rangle + Q\left(\frac{y^0}{|\overline{y}|}, |\overline{x}|\right) \langle \overline{x}, \overline{y} \rangle^2 \\
&= \left\langle P\left(\frac{y^0}{|\overline{y}|}, |\overline{x}|\right) \overline{y} + Q\left(\frac{y^0}{|\overline{y}|}, |\overline{x}|\right) \langle \overline{x}, \overline{y} \rangle \overline{x}, \overline{y} \right\rangle
\end{aligned} \tag{5.16}$$

Theorem 5.2.1 (Theorem 1 of [2]) *For $n \geq 2$, $F = \alpha\sqrt{\phi(z,\rho)}$ is Ricci-flat if and only if $P(z,\rho) = Q(z,\rho) = 0$. Furthermore, the Ricci-flat condition is weaker when $n = 1$; namely, $P(z,\rho) + \rho^2 Q(z,\rho) = 0$.*

Proof Suppose Ric $= 0$. Let e_i denote the n-dimensional vector with 1 in the i^{th} entry and zeros elsewhere. Take $\overline{y} = e_i$ and $\overline{x} = \rho e_j$ for $\rho \geq 0$. By equation (5.16),

$$P\left(y^0, \rho\right) + Q\left(y^0, \rho\right)\rho^2 \delta^{ij} = 0\,,\ \forall i, j\,.$$

For $n \geq 2$, pick $i \neq j$ to get $P(y^0, \rho) = 0$. Now set $i = j$ to conclude $Q(y^0, \rho) = 0$ for $\rho \neq 0$. Finally, $Q(y^0, 0) = 0$ by continuity. The remaining assertions are clear. ■

The above proof suggests metrics F that are singular on $(x^0, 0)$ or metrics F defined on $\mathbb{R} \times \mathbb{R}^n \setminus \{0\}$ should also be considered. This becomes evident on the examples bellow.

5.3 Examples

5.3.1 Riemannian

Suppose $\phi(z,\rho) = \mathrm{e}^{f(\rho)}\, z^2 + \mathrm{e}^{g(\rho)}$. So $\Omega = 2\,\mathrm{e}^g$, $\Lambda = 4\,\mathrm{e}^{f+g}$ and $F = \alpha\sqrt{\phi}$ gives a positive-definite Riemannian metric.

The Ricci curvature components are:

$$P = -\frac{1}{4\rho}\left[p_2\, \mathrm{e}^{f-g}\, z^2 + p_0\right]$$
$$Q = -\frac{1}{4\rho^3} q_0$$

where

$$p_2 = 2\rho f'' + \rho(f')^2 + (n-2)\rho f' g' + 2(n-1) f'$$
$$p_0 = 2\rho g'' + (n-2)\rho(g')^2 + \rho f' g' + 2f' + 2(2n-3) g'$$
$$q_0 = 2\rho f'' + 2(n-2)\rho g'' + \rho(f')^2 - 2\rho f' g' - (n-2)\rho(g')^2 - 2f' - 2(n-2) g'$$

By independence of z and ρ, the Ricci-flat equations for $n \geq 2$ become $p_2 = p_0 = q_0 = 0$. Taking $q_0 - p_2 + n p_0 = 0$ yields:

$$4(n-1)\rho g'' + (n-2)(n-1)\rho(g')^2 + 4(n-1)^2 g' = 0$$

For $n \geq 3$:

$$4\rho g'' + (n-2)\rho(g')^2 + 4(n-1)g' = 0 \tag{5.17}$$

If $g' = 0$, then (5.17) is trivially satisfied. So $g(\rho) = \overline{B}$ constant is a solution. Otherwise, (5.17) is a Bernoulli differential equation in g', which can be transformed to a linear ODE by letting $u := (g')^{-1}$. The equation reduces to:

$$4\rho u' - (n-2)\rho - 4(n-1)u = 0$$

Its solution gives $g(\rho) = \ln(B|\rho^{2-n} + C|^{\frac{4}{n-2}})$, for $B, C \in \mathbb{R}$ constants with $B > 0$.

To find f, substitute g in $p_0 = 0$. When $g(\rho) = \overline{B}$, $f' = 0$ and f is constant also, say $f(\rho) = \overline{A}$. For $g(\rho) = \ln(B|\rho^{2-n} + C|^{\frac{4}{n-2}})$, equation $p_0 = 0$ gives:

$$f' = \frac{4(n-2)C\rho^{1-n}}{(C - \rho^{2-n})(C + \rho^{2-n})}$$

So $f(\rho) = \ln\left[A\left(\frac{C-\rho^{2-n}}{C+\rho^{2-n}}\right)^2\right]$, for some constant $A > 0$.

Therefore, when $n \geq 3$, solutions are:

$$\phi(z, \rho) = Az^2 + B\,,\ A, B > 0 \tag{5.18a}$$

$$\phi(z, \rho) = A\left(\frac{C - \rho^{2-n}}{C + \rho^{2-n}}\right)^2 z^2 + B|\rho^{2-n} + C|^{\frac{4}{n-2}}\,,\ A, B > 0,\ C \in \mathbb{R} \tag{5.18b}$$

For $n = 2$, equation (5.17) still holds, but it is already linear:

$$\rho g'' + g' = 0$$

So $g(\rho) = \ln(B|\rho|^C)$, for $B, C \in \mathbb{R}$ constants with $B > 0$. Substitute g in $p_0 = 0$ to get:

$$(C + 2)f' = 0$$

If $C \neq -2$, then $f' = 0$. So $f(\rho) = \overline{A}$. When $C = -2$, equation $p_2 = 0$ yields:

$$2\rho f'' + \rho(f')^2 + 2f' = 0$$

If $f' = 0$, the above equation is trivially satisfied and then $f(\rho) = \overline{A}$. Else, it is a Bernoulli equation in f'. As before, let $u := (f')^{-1}$ to get a linear ODE:

$$2\rho u' - \rho - 2u = 0$$

It gives $f(\rho) = \ln(A_1 + A_2 \ln|\rho|)^2$, for real constants A_1, A_2.

Thus, for $n = 2$, solutions are:

$$\phi(z,\rho) = Az^2 + B|\rho|^C\,,\ A, B > 0,\, C \in \mathbb{R} \setminus \{-2\} \tag{5.19a}$$

$$\phi(z,\rho) = (A_1 + A_2 \ln|\rho|)^2 z^2 + B\rho^{-2}\,,\ A_1, A_2 \in \mathbb{R},\, B > 0 \tag{5.19b}$$

For $n = 1$, the Ricci-flat condition gives $p_2 = p_0 + q_0 = 0$, by independence of z and ρ. This gives:

$$2f'' + (f')^2 - f'g' = 0$$

So either $f(\rho) = \overline{A}$ and g is an arbitrary smooth function of ρ, or $g = \ln(f')^2 + f + \overline{B}$ for any smooth function f of ρ.

Hence, solutions for $n = 1$ are:

$$\phi(z,\rho) = Az^2 + \mathrm{e}^{g(\rho)}\,,\ A > 0\,,\ g \in C^\infty \tag{5.20a}$$

$$\phi(z,\rho) = \mathrm{e}^{f(\rho)}\left(z^2 + B[f'(\rho)]^2\right),\ f \in C^\infty,\, B > 0 \tag{5.20b}$$

Finally, if $\phi(z,\rho) = \mathrm{e}^{f(\rho)}\, z^2 - \mathrm{e}^{g(\rho)}$, then $\Omega = -2\,\mathrm{e}^g$ and $\Lambda = -4\,\mathrm{e}^{f+g}$. So the associated metric $L = \alpha^2\phi$ has Lorentz signature $(+, -, \ldots, -)$. In this case, the Ricci curvature components are:

$$P = \frac{1}{4\rho}\left[p_2\,\mathrm{e}^{f-g}\, z^2 - p_0\right]$$
$$Q = -\frac{1}{4\rho^3} q_0$$

where p_2, p_0 and q_0 are as before. Thus, by independence of z and ρ, the Ricci-flat equations reduce to the same system as the positive-definite case.

5.3.2 m^{th}-root

If $\phi(z,\rho) = \left(\mathrm{e}^{f(\rho)}\, z^m + \mathrm{e}^{g(\rho)}\right)^{\frac{2}{m}}$ for an even integer $m > 2$, then $\Omega = \frac{2\,\mathrm{e}^g}{(\mathrm{e}^f\, z^m + \mathrm{e}^g)^{1-\frac{2}{m}}}$ and $\Lambda = \frac{4(m-1)\,\mathrm{e}^{f+g}\, z^{m-2}}{(\mathrm{e}^f\, z^m + \mathrm{e}^g)^{2(1-\frac{2}{m})}}$. So $F = \alpha\sqrt{\phi}$ is a positive-definite m^{th}-root metric.

The Ricci curvature components are:

$$P = -\frac{1}{2m^2(m-1)\rho}\left[p_{2m}\,\mathrm{e}^{2(f-g)}\,z^{2m} + p_m\,\mathrm{e}^{f-g}\,z^m + p_0\right]$$
$$Q = \frac{1}{4m^2(m-1)^2\rho^3}\left[q_{2m}\,\mathrm{e}^{2(f-g)}\,z^{2m} - q_m\,\mathrm{e}^{f-g}\,z^m - q_0\right]$$

where

$$\begin{aligned}
p_{2m} &= (m-2)(m+n-2)\rho(f')^2\\
p_m &= 2m(m-1)\rho f'' + m(m-1)\rho(f')^2 + (n-2)(3m-4)\rho f'g'\\
&\quad + m[(n-2)(3m-4)+2(m-1)]f'\\
p_0 &= 2m(m-1)\rho g'' + 2(m-1)(n-2)\rho(g')^2 + m\rho f'g' + m^2 f' + 2m(m-1)(2n-3)g'\\
q_{2m} &= (m-2)[2m^2+(n-2)(3m-2)]\rho(f')^2\\
q_m &= 2(m-2)[m(m-1)(n-2)\rho f'' - m(m+n-1)\rho(f')^2 + 2(n-2)(m-1)\rho f'g'\\
&\qquad\qquad + m(m-1)(n-2)f']\\
q_0 &= 2m^2(m-1)\rho f'' + 4m(m-1)^2(n-2)\rho g'' + m^2\rho(f')^2 - 4m(m-1)\rho f'g'\\
&\quad - 4(m-1)^2(n-2)\rho(g')^2 - 2m^2(m-1)f' - 4m(m-1)^2(n-2)g'
\end{aligned}$$

By independence of z and ρ, the Ricci-flat equations for $n \geq 2$ are $p_{2m} = p_m = p_0 = q_{2m} = q_m = q_0 = 0$. Since $m > 2$, $p_{2m} = q_{2m} = 0$ imply $f' = 0$, and equations $p_m = q_m = 0$ are automatically satisfied. The remaining equations reduce to:

$$m\rho g'' + (n-2)\rho(g')^2 + (2n-3)mg' = 0 \tag{5.21}$$

$$(n-2)[m\rho g'' - \rho(g')^2 - mg'] = 0 \tag{5.22}$$

So $f(\rho) = \overline{A}$ and $g(\rho)$ must be determined from the above equations.

For $n \geq 3$, combine equations (5.21) and (5.22) to eliminate g''. This gives:

$$g'(\rho g' + 2m) = 0$$

If $g' = 0$, then $g(\rho) = \overline{B}$. Otherwise, $\rho g' + 2m = 0$ and so $g(\rho) = \ln(B\rho^{-2m})$ for some constant $B > 0$.

Therefore, when $n \geq 3$, solutions are:

$$\phi(z,\rho) = (Az^m + B)^{\frac{2}{m}}\,,\ A, B > 0 \tag{5.23a}$$

$$\phi(z,\rho) = (Az^m + B\rho^{-2m})^{\frac{2}{m}}\,,\ A, B > 0 \tag{5.23b}$$

For $n = 2$, (5.22) is vacuous and (5.21) gives a linear ODE:

$$\rho g'' + g' = 0$$

So $g(\rho) = \ln(B|\rho|^C)$, for constants $B > 0$ and $C \in \mathbb{R}$.

Hence, solutions for $n = 2$ are:

$$\phi(z, \rho) = (Az^m + B|\rho|^C)^{\frac{2}{m}}\,,\ A, B > 0\,,\ C \in \mathbb{R} \tag{5.24}$$

For $n = 1$, the Ricci-flat equations are $2(m-1)p_{2m} - q_{2m} = 2(m-1)p_m + q_m = p_0 + q_0 = 0$, by independence of z and ρ. As before, since $m > 2$, $2(m-1)p_{2m} - q_{2m} = 0$ implies $f' = 0$, and equation $2(m-1)p_m + q_m = 0$ is automatically satisfied. The remaining equation gives:

$$m\rho g'' - \rho(g')^2 - mg' = 0$$

If $g' = 0$, the above equation is trivially satisfied; then $g(\rho) = \overline{B}$. Otherwise, this is yet again a Bernoulli equation in g'. Let $u := (g')^{-1}$ to obtain a linear ODE:

$$m\rho u' + \rho + mu = 0$$

Its solution gives $g(\rho) = \ln(B|\rho^2 + C|^{-m})$ for constants $B > 0$ and $C \in \mathbb{R}$.

So, for $n = 1$, solutions are:

$$\phi(z, \rho) = (Az^m + B)^{\frac{2}{m}}\,,\ A, B > 0 \tag{5.25a}$$

$$\phi(z, \rho) = (Az^m + B|\rho^2 + C|^{-m})^{\frac{2}{m}}\,,\ A, B > 0\,,\ C \in \mathbb{R} \tag{5.25b}$$

For an odd integer $m > 2$, all formulas still hold, but the metric generated changes signature according to the sign of z, because it determines the sign of Λ. For $m = 2$, ϕ simplifies to give a Riemannian metric; in this case, the non-trivial Ricci-flat equations are multiples of the previously found equations for Riemannian metrics.

Finally, taking $\phi(z, \rho) = \left(\mathrm{e}^{f(\rho)}\, z^m - \mathrm{e}^{g(\rho)}\right)^{\frac{2}{m}}$ for some integer $m > 2$ with $m \equiv 2 (\mathrm{mod}\ 4)$ gives a well-defined metric $L = \alpha^2\phi$ with Lorentz signature $(+, -, \ldots, -)$, since $\Omega = -\frac{2\,\mathrm{e}^g}{(\mathrm{e}^f\, z^m - \mathrm{e}^g)^{1-\frac{2}{m}}}$ and $\Lambda = -\frac{4(m-1)\,\mathrm{e}^{f+g}\, z^{m-2}}{(\mathrm{e}^f\, z^m - \mathrm{e}^g)^{2(1-\frac{2}{m})}}$.

In this setting, the Ricci components are:

$$P = -\frac{1}{2m^2(m-1)\rho}\left[p_{2m}\,\mathrm{e}^{2(f-g)}\,z^{2m} - p_2\,\mathrm{e}^{f-g}\,z^2 + p_0\right]$$
$$Q = \frac{1}{4m^2(m-1)^2\rho^3}\left[q_{2m}\,\mathrm{e}^{2(f-g)}\,z^{2m} + q_m\,\mathrm{e}^{f-g}\,z^m - q_0\right]$$

where p_i, q_j are as before. Thus, the Ricci-flat equations coincide with the positive-definite case.

With some thought, one might consider these equations for other values of m. When $m > 2$ is divisible by 4, one may take $L = \alpha^m\left(\mathrm{e}^{f(\rho)}\,z^m - \mathrm{e}^{g(\rho)}\right)$ to consider Finsler spacetimes in the sense of Pfeifer and Wohlfarth [14]. When $m > 2$ is odd, $F = \alpha\left(\mathrm{e}^{f(\rho)}\,z^m - \mathrm{e}^{g(\rho)}\right)^{\frac{1}{m}}$ already makes sense. However, in both cases, one needs to become concerned with the domain of z and ρ to ensure Ω, Λ are defined and their sign give the appropriate signature.

5.3.3 Randers

Assume $\phi(z,\rho) = (\sqrt{\mathrm{e}^{f(\rho)}\,z^2 + \mathrm{e}^{g(\rho)}} + \varepsilon\,\mathrm{e}^{\frac{f(\rho)}{2}}\,z)^2$ with $0 < |\varepsilon| < 1$, so $F = \alpha\sqrt{\phi}$ gives a positive-definite Randers metric.

Indeed, $\Omega = 2\left(\frac{\sqrt{\mathrm{e}^f\,z^2+\mathrm{e}^g}+\varepsilon\,\mathrm{e}^{\frac{f}{2}}\,z}{\sqrt{\mathrm{e}^f\,z^2+\mathrm{e}^g}}\right)\mathrm{e}^g$ and $\Lambda = 4\left(\frac{\sqrt{\mathrm{e}^f\,z^2+\mathrm{e}^g}+\varepsilon\,\mathrm{e}^{\frac{f}{2}}\,z}{\sqrt{\mathrm{e}^f\,z^2+\mathrm{e}^g}}\right)^3\mathrm{e}^{f+g}$.

The Ricci curvature components are:

$$P = -\frac{1}{4\rho\sqrt{\mathrm{e}^f\,z^2+\mathrm{e}^g}(\sqrt{\mathrm{e}^f\,z^2+\mathrm{e}^g} + \varepsilon\,\mathrm{e}^{\frac{f}{2}}\,z)}\Big[p_4\,\mathrm{e}^{2f-g}\,z^4 + 2\varepsilon p_3\,\mathrm{e}^{\frac{3f}{2}-g}\sqrt{\mathrm{e}^f\,z^2+\mathrm{e}^g}z^3$$
$$+ p_2\,\mathrm{e}^f\,z^2 + \varepsilon p_1\,\mathrm{e}^{\frac{f}{2}}\sqrt{\mathrm{e}^f\,z^2+\mathrm{e}^g}z + p_0\,\mathrm{e}^g\Big]$$
$$Q = \frac{1}{4\rho^3(\mathrm{e}^f\,z^2+\mathrm{e}^g)^2(\sqrt{\mathrm{e}^f\,z^2+\mathrm{e}^g} + \varepsilon\,\mathrm{e}^{\frac{f}{2}}\,z)^2}\Big[q_6\,\mathrm{e}^{3f}\,z^6 + 2\varepsilon q_5\,\mathrm{e}^{\frac{5f}{2}}\sqrt{\mathrm{e}^f\,z^2+\mathrm{e}^g}z^5$$
$$+ 2q_4\,\mathrm{e}^{2f+g}\,z^4 + 4\varepsilon q_3 e^{\frac{3f}{2}+g}\sqrt{\mathrm{e}^f\,z^2+\mathrm{e}^g}z^3 + q_2\,\mathrm{e}^{f+2g}\,z^2 + 2\varepsilon q_1\,\mathrm{e}^{\frac{f}{2}+2g}\sqrt{\mathrm{e}^f\,z^2+\mathrm{e}^g}z$$
$$+ q_0\,\mathrm{e}^{3g}\Big]$$

where p_i, q_j are functions of ρ, f, g and its derivatives of order up to two. Particularly,

$$p_4 = 2(\varepsilon^2+1)\rho f'' - ((n+1)\varepsilon^2 - 1)\rho(f')^2 + (n-2)(\varepsilon^2+1)\rho f'g' + 2(n-1)(\varepsilon^2+1)f'$$
$$p_3 = 2\rho f'' - \frac{1}{4}((n+2)\varepsilon^2 + (n-2))\rho(f')^2 + (n-2)\rho f'g' + 2(n-1)f'$$

For $n \geq 2$, the Ricci-flat equations reduce to $p_i = q_j = 0$, by independence of z and ρ. Taking $p_4 - (\varepsilon^2 + 1)p_3 = 0$ reads

$$\frac{(n+2)}{4}(\varepsilon^2 - 1)^2 \rho (f')^2 = 0\,.$$

So $f' = 0$, and the remaining equations simplify to:

$$2\rho g'' + (n-2)\rho(g')^2 + 2(2n-3)g' = 0 \tag{5.26}$$

$$(n-2)[2\rho g'' - \rho(g')^2 - 2g'] = 0 \tag{5.27}$$

Hence, $f(\rho) = \overline{A}$ and $g(\rho)$ must be determined from the above equations.

When $n \geq 3$, one may combine equations (5.26) and (5.27) to eliminate g'', obtaining:

$$(n-1)(\rho g' + 4)g' = 0$$

If $g' = 0$, then $g(\rho) = \overline{B}$. Otherwise, $\rho g' + 4 = 0$ and so $g(\rho) = \ln(B\rho^{-4})$ for some constant $B > 0$.

Thus, solutions for $n \geq 3$ are:

$$\phi(z,\rho) = (\sqrt{Az^2 + B} + \varepsilon\sqrt{A}z)^2\,,\ A, B > 0\,,\ 0 < |\varepsilon| < 1 \tag{5.28a}$$

$$\phi(z,\rho) = (\sqrt{Az^2 + B\rho^{-4}} + \varepsilon\sqrt{A}z)^2\,,\ A, B > 0\,,\ 0 < |\varepsilon| < 1\,,\ C \in \mathbb{R} \tag{5.28b}$$

For $n = 2$, (5.27) is vacuous and (5.26) becomes a linear ODE:

$$\rho g'' + g' = 0$$

So $g(\rho) = \ln(B|\rho|^C)$, for constants $B > 0$ and $C \in \mathbb{R}$.

Hence, when $n = 2$, solutions are:

$$\phi(z,\rho) = \left(\sqrt{Az^2 + B|\rho|^C} + \varepsilon\sqrt{A}z\right)^2\,,\ A, B > 0\,,\ C \in \mathbb{R},\ 0 < |\varepsilon| < 1 \tag{5.29}$$

Finally, for $n = 1$, the Ricci-flat condition once again implies $f' = 0$, although the computation is lengthier and will be omitted. All remaining equations are automatically satisfied.

Therefore, for $n = 1$, solutions are:

$$\phi(z,\rho) = \left(\sqrt{Az^2 + \mathrm{e}^{g(\rho)}} + \varepsilon\sqrt{A}z\right)^2 \ , \ A > 0\,, \ g \in C^\infty\,, \ 0 < |\varepsilon| < 1 \tag{5.30}$$

Clearly, one may rewrite solutions as $\phi(z,\rho) = \left(\sqrt{Az^2 + \mathrm{e}^{g(\rho)}} + Dz\right)^2$ for any constant D satisfying $D^2A^{-1} < 1$. More generally, it is possible to look for solutions in the form $\phi(z,\rho) = \left(\sqrt{\mathrm{e}^{f(\rho)}\,z^2 + \mathrm{e}^{g(\rho)}} \pm h(\rho)z\right)^2$ with $h^2(\rho) < \mathrm{e}^{f(\rho)}$, but the calculations quickly become cumbersome. In either case, it is uncertain how to consider Lorentz signature (if possible).

6. CONCLUSIONS AND FUTURE WORK

The Hessian of the Ricci curvature

$$\mathrm{Ric}_{AB} = \frac{1}{2}\left[\mathrm{Ric}\right]_{y^A y^B}$$

was the first notion for Ricci curvature tensor of Finsler metrics introduced by Akbar-Zadeh [34]. Evidently, $\mathrm{Ric}_{AB} = 0$ if and only if $\mathrm{Ric} = 0$, and they imply the vanishing of the scalar curvature $R = g^{AB}\,\mathrm{Ric}_{AB}$. By defining the modified Einstein tensor

$$G_{AB} = \mathrm{Ric}_{AB} - \frac{1}{2} g_{AB} R$$

in [30], Li an Chang established the equivalence between the vacuum field equation for Finsler spacetime and the vanishing of the Ricci curvature. However, the notion of Ricci curvature tensor for Finsler metrics is not unique. If $R_{B\ CD}^{\ A}$ is the Riemann curvature tensor for Finsler metrics, then

$$\widetilde{\mathrm{Ric}}_{AB} = \frac{1}{2}\left(R_{A\ CB}^{\ C} + R_{B\ CA}^{\ C}\right)$$

is another notion of Ricci curvature tensor introduced by Li and Shen in [35]. Moreover, these Ricci tensors differ by a non-Riemannian quantity; namely,

$$\widetilde{\mathrm{Ric}}_{AB} - \mathrm{Ric}_{AB} = H_{AB} = \frac{1}{2}\left([\chi_B]_{y^A} + [\chi_A]_{y^B}\right)\,,$$

where the χ-curvature tensor is given by

$$\chi_A = \frac{1}{2}\left[\Pi_{x^B y^A} y^B - \Pi_{x^A} - 2\Pi_{y^A y^B} G^B\right]$$

with $\Pi = \frac{\partial G^C}{\partial y^C}$. So $\widetilde{\mathrm{Ric}}_{AB} = 0$ if and only if $\mathrm{Ric}_{AB} = 0$ and $H_{AB} = 0$; in words, the vanishing of $\widetilde{\mathrm{Ric}}_{AB}$ is a stronger condition than the vanishing of Ric_{AB}. In particular, if $\mathrm{Ric} = 0$ and $\chi_A = 0$, then $\widetilde{\mathrm{Ric}}_{AB} = 0$.

For the proposed metrics $F = \alpha\sqrt{\phi(z,\rho)}$, we have $\Pi = \Psi(x^m y^m)$, where

$$\Psi := U_z + (n+2)V + (n-1)W\,, \tag{6.1}$$

and the χ-curvature is

$$\chi_0 = \left[\frac{1}{2\rho}\Psi_{z\rho} - U\Psi_{zz} - W\Psi_z\right]\frac{(x^m y^m)^2}{\alpha} + \frac{1}{2}(2\rho^2 W + 1)\Psi_z\alpha \tag{6.2a}$$

$$\begin{aligned}\chi_i = &\left[zU\Psi_{zz} - \frac{z}{2\rho}\Psi_{z\rho} + (U + 2zW)\Psi_z\right]\frac{(x^m y^m)^2}{\alpha^2}y^i \\ &- \frac{z}{2}(2\rho^2 W + 1)\Psi_z y^i - (U + zW)\Psi_z(x^m y^m)x^i\end{aligned} \tag{6.2b}$$

Clearly, $\Psi_z = 0$ is a sufficient condition for the vanishing of the χ-curvature. By direct verification, all solutions in previous section satisfy $\Psi_z = 0$. Thus, they are strongly Ricci-flat metrics: $\widetilde{\mathrm{Ric}}_{AB} = \mathrm{Ric}_{AB} = 0$.

In addition to the examples presented here, it seems to be feasible (although lengthy) to construct other types of (strongly) Ricci-flat metrics in the proposed form; particularly, one may look for series expansions. The same type of construction also seems to work well for Ricci-isotropic metrics, $\mathrm{Ric} = [(n+1)-1]k(x)F^2$. At the very least the PDE characterization is similar to describe; namely, for $n \geq 2$, $F = \alpha\sqrt{\phi(z,\rho)}$ is Ricci isotropic if and only if $P = nk\phi$ and $Q = 0$. It might be wise, however, to spend such efforts with a wider class of warped product Finsler metrics, which may allow for global solutions on $\mathbb{R} \times M$; for instance, a class of Finsler metrics defined by

$$F = \alpha\sqrt{\phi(z,\overline{x})}\,, \tag{6.3}$$

for α any Riemannian metric on M, z as before and ϕ some appropriate function on $\mathbb{R} \times M$.

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APPENDIX

Derivatives

In this appendix, a collection of all derivatives needed to carryout the computations of Chapters 5 and 6.

Derivatives of F^2

$$\begin{aligned}
\left[F^2\right]_{y^0} &= \alpha\phi_z \\
\left[F^2\right]_{y^i} &= \Omega y^i \\
\left[F^2\right]_{x^0} &= 0 \\
\left[F^2\right]_{x^i} &= \frac{1}{\rho}\phi_\rho\alpha^2 x^i \\
\left[F^2\right]_{y^A x^0} &= 0 \\
\left[F^2\right]_{y^0 x^i} &= \frac{1}{\rho}\phi_{z\rho}\alpha x^i \\
\left[F^2\right]_{y^i x^j} &= \frac{1}{\rho}\Omega_\rho x^j y^i
\end{aligned}$$

Derivatives of G^A

$$\begin{aligned}
\left[G^A\right]_{x^0} &= 0 \\
\left[G^0\right]_{x^i} &= \frac{1}{\rho}(U_\rho + zV_\rho)x^i(x^m y^m)\alpha + (U + zV)y^i\alpha \\
\left[G^j\right]_{x^i} &= \frac{1}{\rho}(V_\rho + W_\rho)x^i y^j(x^m y^m) + (V + W)y^i y^j - \frac{1}{\rho}W_\rho x^i x^j\alpha^2 - W\delta_i^j\alpha^2 \\
\left[G^0\right]_{y^0} &= (U_z + V + zV_z)(x^m y^m) \\
\left[G^j\right]_{y^0} &= (V_z + W_z)(x^m y^m)\frac{y^j}{\alpha} - W_z x^j\alpha \\
\left[G^0\right]_{y^i} &= (U - zU_z - z^2V_z)(x^m y^m)\frac{y^i}{\alpha} + (U + zV)x^i\alpha \\
\left[G^j\right]_{y^i} &= (V+W)(x^m y^m)\delta_i^j - z(V_z+W_z)(x^m y^m)\frac{y^i y^j}{\alpha^2} + (V+W)x^i y^j + (zW_z - 2W)x^j y^i \\
\left[G^B\right]_{x^0 y^A} &= 0
\end{aligned}$$

$$
\begin{aligned}
\left[G^0\right]_{x^i y^0} &= \frac{1}{\rho}(U_{z\rho} + V_\rho + zV_{z\rho})(x^m y^m)x^i + (U_z + V + zV_z)y^i \\
\left[G^j\right]_{x^i y^0} &= \frac{1}{\rho}(V_{z\rho} + W_{z\rho})(x^m y^m)\frac{x^i y^j}{\alpha} + (V_z + W_z)\frac{y^i y^j}{\alpha} - \frac{1}{\rho}W_{z\rho}x^i x^j \alpha - W_z \delta_i^j \alpha \\
\left[G^0\right]_{x^i y^j} &= (U + zV)\delta_j^i \alpha + (U - zU_z - z^2 V_z)\frac{y^i y^j}{\alpha} \\
&\quad + \frac{1}{\rho}(U_\rho - zU_{z\rho} - z^2 V_{z\rho})(x^m y^m)\frac{x^i y^j}{\alpha} + \frac{1}{\rho}(U_\rho + zV_\rho)x^i x^j \alpha \\
\left[G^k\right]_{x^i y^j} &= (V + W)(\delta_j^i y^k + \delta_j^k y^i) + (zW_z - 2W)\delta_i^k y^j + \frac{1}{\rho}(V_\rho + W_\rho)(x^m y^m)x^i \delta_j^k \\
&\quad + \frac{1}{\rho}(V_\rho + W_\rho)x^i x^j y^k + \frac{1}{\rho}(zW_{z\rho} - 2W_\rho)x^i x^k y^j \\
&\quad - \frac{z}{\rho}(V_{z\rho} + W_{z\rho})(x^m y^m)\frac{x^i y^j y^k}{\alpha^2} - z(V_z + W_z)\frac{y^i y^j y^k}{\alpha^2} \\
\left[G^0\right]_{y^0 y^0} &= (U_{zz} + 2V_z + zV_{zz})\frac{(x^m y^m)}{\alpha} \\
\left[G^k\right]_{y^0 y^0} &= (V_{zz} + W_{zz})(x^m y^m)\frac{y^k}{\alpha^2} - W_{zz}x^k \\
\left[G^0\right]_{y^0 y^i} &= (U_z + V + zV_z)x^i - z(U_{zz} + 2\psi_z + zV_{zz})(x^m y^m)\frac{y^i}{\alpha^2} \\
\left[G^k\right]_{y^0 y^i} &= (V_z + W_z)\frac{(x^m y^m)}{\alpha}\delta_i^k + (V_z + W_z)\frac{x^i y^k}{\alpha} + (zW_{zz} - W_z)\frac{x^k y^i}{\alpha} \\
&\quad - (V_z + zV_{zz} + W_z + zW_{zz})\frac{(x^m y^m)}{\alpha}\frac{y^i y^k}{\alpha^2} \\
\left[G^0\right]_{y^i y^j} &= (U - zU_z - z^2 V_z)\frac{(x^m y^m)}{\alpha}\delta_j^i + (U - zU_z - z^2 V_z)\left[\frac{x^i y^j}{\alpha} + \frac{x^j y^i}{\alpha}\right] \\
&\quad + (-U + zU_z + z^2 U_{zz} + 3z^2 V_z + z^3 V_{zz})\frac{(x^m y^m)}{\alpha}\frac{y^i y^j}{\alpha^2} \\
\left[G^k\right]_{y^i y^j} &= z(3(V_z + W_z) + z(V_{zz} + W_{zz}))(x^m y^m)\frac{y^i y^j y^k}{\alpha^4} - z(V_z + W_z)\left[\frac{x^i y^j y^k}{\alpha^2} + \frac{x^j y^i y^k}{\alpha^2}\right] \\
&\quad - z(zW_{zz} - W_z)\frac{x^k y^i y^j}{\alpha^2} + (V + W)(x^i \delta_k^j + x^j \delta_i^k) \\
&\quad + (zW_z - 2W)x^k \delta_j^i - z(V_z + W_z)(x^m y^m)\left[\frac{\delta_i^j y^k}{\alpha^2} + \frac{y^i \delta_k^j}{\alpha^2} + \frac{\delta_i^k y^j}{\alpha^2}\right]
\end{aligned}
$$

Derivatives of Π

$$
\begin{aligned}
\Pi_{x^0} &= 0 \\
\Pi_{x^i} &= \Psi y^i + \frac{1}{\rho}\Psi_\rho (x^m y^m)x^i
\end{aligned}
$$

$$
\begin{aligned}
\Pi_{y^0} &= \Psi_z \frac{(x^m y^m)}{\alpha} \\
\Pi_{y^j} &= -z\Psi_z \frac{(x^m y^m)}{\alpha} \frac{y^j}{\alpha} + \Psi x^j \\
\Pi_{x^0 y^A} &= 0 \\
\Pi_{x^i y^0} &= \Psi_z \frac{y^i}{\alpha} + \frac{1}{\rho} \Psi_{z\rho} \frac{(x^m y^m)}{\alpha} x^i \\
\Pi_{x^i y^j} &= \Psi \delta_i^j - z\Psi_z \frac{y^i y^j}{\alpha^2} - \frac{z}{\rho} \Psi_{z\rho} \frac{(x^m y^m)}{\alpha} x^i \frac{y^j}{\alpha} + \frac{1}{\rho} \Psi_\rho x^i x^j \\
\Pi_{y^0 y^0} &= \Psi_{zz} \frac{(x^m y^m)}{\alpha^2} \\
\Pi_{y^0 y^j} &= \Psi_z \frac{x^j}{\alpha} - \left(\Psi_z + z\Psi_{zz}\right) \frac{(x^m y^m)}{\alpha^2} \frac{y^j}{\alpha} \\
\Pi_{y^i y^j} &= -z\Psi_z \left[\frac{(x^m y^m)}{\alpha^2} \delta_i^j + \frac{x^i y^j}{\alpha^2} + \frac{x^j y^i}{\alpha^2} \right] + z\left(3\Psi_z + z\Psi_{zz}\right) \frac{(x^m y^m)}{\alpha^2} \frac{y^i y^j}{\alpha^2}
\end{aligned}
$$

VITA

VITA

Patricia Marcal has been “math curious” since a young age, but only “came out” when she entered the University of Campinas – Unicamp (Brazil) in 2010. She was an exchange student at the University of Warwick (UK) for a year and graduated in pure mathematics from Unicamp in 2014. As an undergraduate, she studied Riemannian Geometry, which led to her interest in Finsler Geometry, the desire to become a student of Professor Z. Shen and the decision to enter the Ph.D. Program of the Department of Mathematical Sciences of IUPUI in 2015.