THE ERROR ESTIMATION IN FINITE ELEMENT METHODS FOR ELLIPTIC EQUATIONS WITH LOW REGULARITY

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This dissertation is dedicated to my family for the unwavering support and encouragement over the years.

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TABLE OF CONTENTS

Page

Page

LIST OF FIGURES

ABSTRACT

Yang, Jing Ph.D., Purdue University, May 2020. The error estimation in finite element methods for elliptic equations with low regularity. Major Professor: Professor Zhiqiang Cai .

This dissertation contains two parts: one part is about the error estimate for the finite element approximation to elliptic PDEs with discontinuous Dirichlet boundary data, the other is about the error estimate of the DG method for elliptic equations with low regularity.

Elliptic problems with low regularities arise in many applications, error estimate for sufficiently smooth solutions have been thoroughly studied but few results have been obtained for elliptic problems with low regularities. Part I provides an error estimate for finite element approximation to elliptic partial differential equations (PDEs) with discontinuous Dirichlet boundary data. Solutions of problems of this type are not in H^1 and, hence, the standard variational formulation is not valid. To circumvent this difficulty, an error estimate of a finite element approximation in the $W^{1,r}(\Omega)$ $(0 < r < 2)$ norm is obtained through a regularization by constructing a continuous approximation of the Dirichlet boundary data. With discontinuous boundary data, the variational form is not valid since the solution for the general elliptic equations is not in H^1. By using the $W^{1,r}$ $(1 < r < 2)$ regularity and constructing continuous approximation to the boundary data, here we present error estimates for general elliptic equations.

Part II presents a class of DG methods and proves the stability when the solution belong to $H^{1+\epsilon}$ where $\epsilon < 1/2$ could be very small. we derive a non-standard variational formulation for advection-diffusion-reaction problems. The formulation is defined in an appropriate function space that permits discontinuity across element

interfaces and does not require piece wise $H^s(\Omega)$, $s \geq 3/2$, smoothness. Hence, both continuous and discontinuous (including Crouzeix-Raviart) finite element spaces may be used and are conforming with respect to this variational formulation. Then it establishes the *a priori* error estimates of these methods when the underlying problem is not piece wise $H^{3/2}$ regular. The constant in the estimate is independent of the parameters of the underlying problem. Error analysis presented here is new. The analysis makes use of the discrete coercivity of the bilinear form, an error equation, and an efficiency bound of the continuous finite element approximation obtained in the *a posteriori* error estimation. Finally a new DG method is introduced i to overcome the difficulty in convergence analysis in the standard DG methods and also proves the stability.

1. INTRODUCTION

Partial differential equations with low regularity arise in many physical modelling problems. The low regularity usually comes from the non-smoothness of the modellings, for example, the non-smoothness of the domain, the non-smoothness of the boundary data and the non-smoothness of the coefficients. The finite element method (FEM) is the most widely used method for solving PDE problems of engineering and mathematical models. In the study of FEM, the error estimate plays an important role, which is one of the main topics of my Ph.D. research. There are two major types of error estimates for finite element methods, a priori and a posteriori error estimates. The main feature of a priori estimates is that they give us the order of convergence of a given method, that is, they tell us the finite element error $\|u - u_h\|$ in some norm $\|\cdot\|$ is $O(h^\lambda)$, where h is the maximum mesh size and λ is positive. And in adaptive mesh refinement, a posteriori error estimates are used to indicate where the error is large and a mesh refinement is then placed in those elements. The process is repeated until a satisfactory error tolerance is reached. And the low regularity may lead to the difficulty in the error analysis.

In this chapter, we briefly introduce some examples of the elliptic problems with low regularity and some preliminaries for finite element methods. In this thesis, boldface letters represent vectors, vector fields, or tensors and light- face letters represent scalar or scalar valued functions. The letter C with or without subscripts denotes a generic positive constant, possibly different at different occur- rences.

1.1 PDEs with non-smooth boundary data

The partial differential equations with discontinuous boundary data have arisen in many physical models. The difficulty of this kind of problems is that when the boundary data is not continuous, the solution will not belong to H^1 space, and hence, does not satisfy the standard variational formulation. Even though finite element approximations may be defined as usual by choosing a value of either $g_D(x^-)$ or $g_D(x^+)$ at discontinuous point $x \in \partial\Omega$, it is difficult to estimate error bound of finite element approximation due to lack of error equation. In the following, we introduce two examples of this kind of problems.

1.1.1 Poisson equations with non-smooth boundary data

In [9], Apel, Nicaise and Pfefferer first studied Poisson equations with L^2 boundary data.

Consider the boundary value problem

$$-\triangle u = f, \text{ in } \Omega, \qquad u = y, \text{ on } \Gamma := \partial\Omega,$$

with right hand side $f \in H^{-1}(\Omega)$ and boundary data $y \in L^2(\Gamma)$. We assume $\Omega \in \Re^2$ to be a bounded polygonal domain with boundary Γ. Such problems arise in optimal control when the Dirichlet boundary control is considered in $L^2(\Gamma)$ only, see for example the papers [11, 20].

This paper introduces the most popular method to solve this kind of problem, which is the transposition method. It is based on the use of some integration by parts and leads to the very weak formulation: Find $u \in U$ such that

$$(u, \triangle v)_\Omega = (u, \partial_n v)_\Gamma - (f, v)_\Omega, \quad \forall v \in V$$

with $(w, v)_G := \int_G wv$ denoting the L^2 (G) scalar product or an appropriate duality product. The main issue is to find the appropriate trial space U and test space V. The main drawback of the very weak formulation is the fact that a conforming discretization of the test space should be made by C^1-elements.

1.1.2 Stokes equations with non-smooth Dirichlet boundary data

In [3], it provides strict error estimates for different finite element approximations of the two-dimensional Stokes lid driven cavity flow.

Consider the two-dimensional Stokes driven cavity problem:

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in} \quad \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \quad \text{in} \quad \Omega, \\ \boldsymbol{u} &= \boldsymbol{g} \quad \text{on} \quad \partial\Omega \end{cases}$$

where Ω is a bounded polygonal domain, $\boldsymbol{u}$ is the velocity, p is the pressure, $\boldsymbol{f}$ is the external force and $\boldsymbol{g}$ is the velocity boundary data satisfying $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} = 0$.

The main approach is to regularize the problem by constructing a continuous approximation to the discontinuous boundary data.

1.2 PDEs with non-smooth coefficients

PDEs with non-smooth coefficients arise in a lot of modelling problems, such as molecular electrostatics [25], geophysics [26], ecology [27], astrophysics [28].

Example 1.2.1 *Consider the following interface problem (i.e., the diffusion problem with discontinuous coefficients):*

$$-\nabla \cdot (\alpha(x) \nabla u) = f \quad \text{in } \Omega$$

with homogeneous Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded polygonal domain in $\Re^d$ with $d = 2$ or 3; $f \in L^2(\Omega)$ is a given function; and diffusion coefficient $\alpha(x)$ is positive and piecewise constant with possible large jumps across subdomain boundaries (interfaces):

$$\alpha(x) = \alpha_i > 0 \quad \text{in } \Omega_i \quad \text{for } i = 1, ..., n.$$

Here, ${\Omega_i}_{i=1}^n$ is a partition of the domain Ω with Ω_i being an open polygonal domain.

In [20], Bernardi1 and Verfürth studied the error estimates for this kind of problem and proved the estimates to be robust with respect to jumps of the coefficients. The key technique is the using of a modification of Clement's quasi-interpolation operator which allows to obtain estimates for the interpolation error which are independent of the size of the jumps of α. But in the analysis, the Quasi-monotonicity assumption is needed, which is the following:

Quasi-monotonicity assumption. Assume that any two different subdomains Ω_i and Ω_j , which share at least one point, have a connected path passing from Ω_i to Ω_j through adjacent subdomains such that the diffusion coefficient $\alpha(x)$ is monotone along this path.

In [22], Cai, He and Zhang proved the robustness of estimations without QMA by using the efficiency bound of the a posteriori error estimation.

1.3 Preliminaries

The finite element method (FEM) is the most widely used numerical method for solving partial differential equations in two or three space variables (i.e., some boundary value problems). To solve a problem, the FEM subdivides a large system into smaller, simpler parts that are called finite elements. This is achieved by a particular space discretization in the space dimensions, which is implemented by the construction of a mesh of the object: the numerical domain for the solution, which has a finite number of points. The finite element method formulation of a boundary value problem finally results in a system of algebraic equations. The method approximates the unknown function over the domain. The simple equations that model these finite elements are then assembled into a larger system of equations that models the entire problem. The FEM then uses variational methods from the calculus of variations to approximate a solution by minimizing an associated error function.

1.3.1 Sobolev spaces and norms

Let Ω is a bounded open connected set in $\Re^d$, $k \in \mathbb{N} \cup \{0\}$ and $r \in [1, \infty]$. Define the Sobelov space $W^{k,p}$ is defined by

$$W^{k,r}(\Omega) := \{u \in L^r(\Omega) : D^\alpha u \in L^r(\Omega),\ \forall \alpha \text{ with } |\alpha| \leq k\},$$

where $D^\alpha u$ are the weak derivatives of u.

This space is equipped with the norm

$$\|u\|_{k,r,\Omega} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^r(\Omega)}.$$

When $r = 2$, the Sobolev space $W^{s,2}(\Omega)$ and $W^{s,2}(\partial\Omega)$ are denoted by $H^s(\Omega)$ and $H^s(\partial\Omega)$, and the associated inner product are denoted by $(\cdot,\cdot)_{s,\Omega}$ and $(\cdot,\cdot)_{s,\partial\Omega}$, respectively. (We omit the subscript Ω from the inner product and norm designation when there is no risk of confusion.)

And the fractional Sobolev norm is defined as follows(see, e.g., [4,6]). For $s = m+t$ with integer $m \geq 0$ and $0 < t < 1$, the norm $\|\cdot\|_{s,r,\Omega}$ is defined by:

$$\|v\|_{s,r,\Omega} = \left(\|v\|^r_{m,r,\Omega} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha(x) - D^\alpha(y)|^r}{|x-y|^{1+tr}} dxdy \right)^{1/r}$$

for all $v \in W^{s,r}(\Omega)$.

1.3.2 Variational formulations

Suppose that

$$\begin{cases} H \text{ is a Hilbert space }, \\ V \ \text{ is a closed subspace} \in H, \\ a(\cdot,\cdot) \ \text{ is a bounded, coercive bilinear form on } V. \end{cases}$$

In general, a variational formulation is posted as followed:

Given $F \in V'$, find $u \in V$ such that $a(u,v) = F(v), \forall v \in V$.

Example 1.3.1 *Let $\Omega \in \Re^2$ to be a bounded polygonal domain with boundary Γ. Consider the boundary value problem*

$$\triangle u = f, \ \ in\ \Omega \qquad u = g, \ \ on\ \Gamma.$$

Define the solution space

$$H_g^1(\Omega) = \{v \in H^1(\Omega) : v = g \text{ on } \Gamma\}$$

and

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\}.$$

The corresponding variational problem is to find $u \in H_g^1(\Omega)$ such that

$$a(u, v) = F(v), \ \forall v \in H_0^1(\Omega),$$

where $a(u, v) = (\nabla u, \nabla v)$ and $F(v) = (f, v)$.

2. GENERAL ELLIPTIC PROBLEMS WITH NON-SMOOTH DIRICHLET BOUNDARY DATA

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. Assume that Γ_D and Γ_N are connected open sets and that two internal angles at $\overline{\Gamma}_D \cap \overline{\Gamma}_N$ are less than or equal to $\pi/2$.

Consider the following general elliptic partial differential equation

$$\begin{cases} -\alpha\Delta u + \boldsymbol{\beta}\cdot\nabla u + cu &= f \qquad \text{in} \quad \Omega, \\ u &= g_{_D} \qquad \text{on} \quad \Gamma_D, \\ \alpha\nabla u\cdot\boldsymbol{n} &= g_{_N} \qquad \text{on} \quad \Gamma_N, \end{cases} \tag{2.0.1}$$

where $f \in L^2(\Omega)$ and $g_{_N} \in H^{1/2}(\Gamma_N)$. Let $\Gamma_D = \cup_{i=0}^{n}\Gamma_{D_i}$ and the Dirichlet boundary data $g_{_D}$ is piecewise $H^{3/2}$ on Γ_D, i.e., $g_{_D} \in H^{3/2}(\Gamma_{D_i})$ for $i = 1, 2, ..., n$. It is easy to check that $g_{_D} \in W^{1-1/r,r}(\Gamma_D)$ for all $1 < r < 2$ not in $H^{1/2}(\Gamma_D)$. Assume that there exists a positive constant ρ_0 such that

$$\rho := -\frac{1}{2}\nabla\cdot\boldsymbol{\beta} + c \geq \rho_0 > 0,$$

which guarantees the coercivity of the bilinear form of the problem.

Problem (2.0.1) has the following regularity property [5]:

Theorem 2.0.1 *Let Ω be a convex polygon. Assume that $f \in W^{-1,r}(\Omega)$, $g_{_N} \in W^{-1/r,r}(\Gamma_N)$ and $g_{_D} \in W^{1-1/r,r}(\Gamma_D)$ where $1 < r < 2$, then problem (2.0.1) has a unique solution $u \in W^{1,r}(\Omega)$ satisfying*

$$\|u\|_{1,r,\Omega} \leq C_r \left(\|f\|_{-1,r,\Omega} + \|g_{_D}\|_{1-1/r,r,\Gamma_D} + \|g_{_N}\|_{-1/r,r,\Gamma_N}\right),$$

where C_r is a positive constant independent of f, $g_{_D}$, and $g_{_N}$, but may depend on r.

2.1 Smooth approximation of Dirichlet boundary data $g_{_D}$

To regularize problem (2.0.1), we introduce a smooth approximation $g_\epsilon \in H^{3/2}(\Gamma_D)$ to the Dirichlet boundary data $g_{_D}$. To this end, let

$$\boldsymbol{r}(t) = \langle x(t), y(t) \rangle, \qquad \text{for } 0 < t < 1$$

be a parametrization of Γ_D. Without loss of generality, assume that the parametrized curve is oriented counterclockwise.

Let $\{t_i\}_{i=1}^n$ be a partition of interval $[0, 1]$ such that

$$0 = t_0 < t_1 < \cdots < t_{n+1} = 1 \tag{2.1.1}$$

and that $\boldsymbol{r}(t)$ for $t_i \le t \le t_{i+1}$ is a parametrization of Γ_{D_i} for $i = 0, 1, \cdots, n$. Let

$$g_{_D}(t) = g_{_D}(x(t), y(t)) \quad \text{and} \quad g_\epsilon(t) = g_\epsilon(x(t), y(t)),$$

where $g_\epsilon(t) = g_{_D}(t)$ on $t \in [0, 1] \backslash \cup_{i=1}^n [t_i, t_i + \epsilon]$ for a sufficiently small $\epsilon > 0$ such that $t_i + 2\epsilon < t_{i+1}$ for $i = 1, \cdots, n$.

On the interval $[t_i, t_i+\epsilon]$ for $i = 1, \cdots, n$, let $g_\epsilon(t)$ be the cubic Hermit interpolant of $g_{_D}$ using data: $\{g_{_D}(t_i^-),\, g_{_D}'(t_i^-),\, g_{_D}(t_i+\epsilon),\, g_{_D}'(t_i+\epsilon)\}$, where $g_{_D}(t_i^-) = \lim\limits_{t \to t_i^-} = g_{_D}(t)$. That is, $g_\epsilon(t)$ satisfies the following interpolation conditions:

$$g_\epsilon^{(k)}(t_i) = g_{_D}^{(k)}(t_i^-) \quad \text{and} \quad g_\epsilon^{(k)}(t_i + \epsilon) = g_{_D}^{(k)}(t_i + \epsilon) \quad \text{for } k = 0,\, 1. \tag{2.1.2}$$

Let $\phi_k(t)$ and $\psi_k(t)$ for $k = 0,\, 1$ be the basis function of the Hermit cubic polynomial on interval $[0,\, 1]$, then

$$\phi_0(t) = (t-1)^2(2t+1), \quad \psi_0(t) = t(t-1)^2,$$

$$\phi_1(t) = t^2(-2t+3), \quad \text{and} \quad \psi_1(t) = t^2(t-1).$$

On $[t_i,\, t_i + \epsilon]$, $g_\epsilon(t)$ has the form of

$$\begin{aligned} g_\epsilon(t) \;\; = \;\; & g_{_D}(t_i^-)\, \phi_0\left(\frac{t - t_i}{\epsilon}\right) + g_{_D}(t_i + \epsilon)\, \phi_1\left(\frac{t - t_i}{\epsilon}\right) \\ & + \epsilon\, g_{_D}'(t_i^-)\, \psi_0\left(\frac{t - t_i}{\epsilon}\right) + \epsilon\, g_{_D}'(t_i + \epsilon)\, \psi_1\left(\frac{t - t_i}{\epsilon}\right). \end{aligned}$$

Remark 2.1.1 *Since $g_\epsilon(t)$ is in $C^1(t_i,\, t_i{+}\epsilon)$ for $i = 1, 2,\ , ..., \, n$, hence $g_\epsilon \in H^{3/2}(\Gamma_D)$. It is easy to see that on $(t_i, t_i + \epsilon)$, there exist positive constants c_0, c_1, and c_2 independent of ϵ such that*

$$g_\epsilon(t) \le c_0, \quad g'_\epsilon(t) \le c_1\epsilon^{-1}, \quad \textit{and} \quad g''_\epsilon(t) \le c_2\epsilon^{-2}.$$

To derive the error estimates in Sobolev norms, the following inequalities will be needed.

Lemma 2.1.2 *For any $r \in (1,\, 2)$ and $t_i \in [0, 1],\ i = 1,\, ...,\, n$ defined in (2.1.1), we have the following estimates:*

$$\int_0^{t_i}\int_{t_i}^{t_i+\epsilon} \frac{(x - t_i)^r}{|x - y|^r}\, dxdy \le \epsilon^2 \ln \epsilon^{-1}, \tag{2.1.3}$$

$$\int_{t_i+\epsilon}^{1}\int_{t_i}^{t_i+\epsilon} \frac{(t_i + \epsilon - x)^r}{|x - y|^r}\, dxdy \le \epsilon^2 \ln \epsilon^{-1}. \tag{2.1.4}$$

Moreover, if $0 < \epsilon < 1/2$, then

$$\int_0^{t_i}\int_{t_i}^{t_i+\epsilon} \frac{(x - t_i)^2}{|x - y|^2} dxdy \le \epsilon^2/2. \tag{2.1.5}$$

Proof It suffices to prove (2.1.3) since (2.1.4) can be shown in a similar fashion. To this end, a direct integration gives

$$I \equiv \int_0^{t_i}\int_{t_i}^{t_i+\epsilon} \frac{(x - t_i)^r}{|x - y|^r} dxdy = \int_{t_i}^{t_i+\epsilon} (x - t_i)^r\, \frac{(x - t_i)^{1-r} - x^{1-r}}{r - 1}\, dx. \tag{2.1.6}$$

Let $h(x, r) = (x - t_i)^{1-r} - x^{1-r}$, since $h(x, 1) = 0$, there exists a $\xi \in (1,\, r)$ such that

$$\begin{aligned} \frac{h(x, r)}{r - 1} &= \frac{h(x, r) - h(x, 1)}{r - 1} = \frac{\partial h}{\partial r}(x, \xi) \\ &= x^{1-r} \ln x - (x - t_i)^{1-r} \ln(x - t_i) \le -(x - t_i)^{1-r} \ln(x - t_i), \end{aligned}$$

which, together with (2.1.6), gives

$$I \le -\int_{t_i}^{t_i+\epsilon} (x - t_i) \ln(x - t_i)\, dx = \frac{1}{2}\epsilon^2 \left(\frac{1}{2} - \ln \epsilon\right) \le \epsilon^2 \ln \epsilon^{-1}.$$

It follows from a direct integration and Tayler expansion of the function $\ln(1 + x)$ at $x = 0$ that

$$\int_0^{t_i}\int_{t_i}^{t_i+\epsilon} \frac{(x - t_i)^2}{|x - y|^2} dxdy = t_i\epsilon - t_i^2 \ln(t_i + \epsilon) + t_i^2 \ln(t_i) = t_i\epsilon - t_i^2 \ln(1 + \frac{\epsilon}{t_i}) \le \epsilon^2/2,$$

which implies (2.1.5). This completes the proof of the lemma. ∎

With Lemma 2.1.2, we are ready to estimate approximation property of g_ϵ and its upper bounds.

Theorem 2.1.3 *Let g_D and g_ϵ be the discontinuous Dirichlet boundary data and its continuous approximation as previously defined, respectively. Then for any $1 < r < 2$ and $1/2 < \alpha \le 1$, the following estimates hold:*

$$\begin{aligned} \|g_D - g_\epsilon\|^r_{1-\frac{1}{r},r,\Gamma_D} &\lesssim \epsilon^{2-r}\ln\epsilon^{-1}, \\ \textit{and} \quad \|g_\epsilon\|^2_{1/2+\alpha,\Gamma_D} &\lesssim \epsilon^{-2\alpha}\ln\epsilon^{-1}. \end{aligned}$$

Here and thereafter, we use the symbol $\lesssim$ for less than or equal to up to a constant independent of ϵ and r .

Proof Let

$$\delta g = g_D - g_\epsilon = \begin{cases} g_D - g_\epsilon, & t \in \cup_{i=1}^n (t_i, t_i+\epsilon), \\ 0, & \text{otherwise.} \end{cases}$$

From the definition of the fractional Sobolev norm, it follows that

$$\begin{aligned} & \|\delta g\|^r_{1-\frac{1}{r},r,\Gamma_D} \\ = \ & \|\delta g\|^r_{0,r,\Gamma_D} + \int_{\Gamma_D}\int_{\Gamma_D} \frac{|\delta g(x) - \delta g(y)|^r}{|x-y|^r} dxdy \\ = \ & \sum_{i=1}^n \int_{t_i}^{t_i+\epsilon} |\delta g|^r dx + \sum_{i=1}^n\sum_{j=1}^n \int_{t_j}^{t_j+\epsilon}\int_{t_i}^{t_i+\epsilon} \frac{|\delta g(x) - \delta g(y)|^r}{|x-y|^r} dxdy \\ & +2\sum_{i=1}^n \int_{t_i}^{t_i+\epsilon}\int_{[0,1]\setminus\cup_{i=1}^n[t_i,t_i+\epsilon]} \frac{|\delta g(y)|^r}{|x-y|^r} dxdy \\ \equiv \ & II + II_1 + II_2. \end{aligned}$$

Let $M = \max\{\max_i \max_{\Gamma_{D_i}} |\delta g|, \max_i \max_{\Gamma_{D_i}} |g'_D|, \max_i \max_{\Gamma_{D_i}} |g_D|\}$, by the triangle inequality, we have that

$$II \le nM^r\epsilon \lesssim \epsilon.$$

To bound II_1, it follows from the triangle inequality, the definition of δg, and Remark 2.1.1 that for $i = j$,

$$|\delta g(x) - \delta g(y)| \leq |g_{_D}(x) - g_{_D}(y)| + |g_\epsilon(x) - g_\epsilon(y)| \lesssim |x - y| + \epsilon^{-1}|x - y|.$$

For $i \neq j$, it implies that $|x - y| > \epsilon$ and

$$\begin{aligned} |\delta g(x) - \delta g(y)|^r &\leq |\delta g(x) - \delta g(t_i)|^r + |\delta g(y) - \delta g(t_j)|^r \\ &\lesssim \epsilon^{-r}|x - t_i|^r + \epsilon^{-r}|y - t_j|^r. \end{aligned}$$

Together with Lemma 2.1.2, we have that

$$\begin{aligned} II_1 &\lesssim \sum_{i=1}^{n} \int_{t_i}^{t_i+\epsilon} \int_{t_i}^{t_i+\epsilon} \frac{|\delta g(x) - \delta g(y)|^r}{|x - y|^r} dxdy \\ &\quad + \sum_{i=1}^{n} \sum_{j \neq i} \int_{t_j}^{t_j+\epsilon} \int_{t_i}^{t_i+\epsilon} \frac{|\delta g(x) - \delta g(y)|^r}{|x - y|^r} dxdy \\ &\lesssim \sum_{i=1}^{n} \int_{t_i}^{t_i+\epsilon} \int_{t_i}^{t_i+\epsilon} \frac{\epsilon^{-r}|x - y|^r}{|x - y|^r} dxdy \\ &\quad + \sum_{i=1}^{n} \sum_{j \neq i} \int_{t_j}^{t_j+\epsilon} \int_{t_i}^{t_i+\epsilon} \epsilon^{-r} \frac{|x - t_i|^r + |y - t_j|^r}{|x - y|^r} dxdy \\ &\lesssim \epsilon^2 \cdot \epsilon^{-r} + \epsilon^{-r} \cdot \epsilon^2 \ln \epsilon^{-1} \lesssim \epsilon^{2-r} \ln \epsilon^{-1} \end{aligned}$$

Arguing in a similar way, for $y \in [t_i, t_i + \epsilon]$, it follows that

$$|\delta g(y)|^r \leq |\delta g(y) - \delta g(t_i)|^r \lesssim \epsilon^{-r}|y - t_i|^r.$$

Together with Lemma 2.1.2, it implies that

$$II_2 \lesssim \sum_{i=1}^{n} \int_{t_i}^{t_i+\epsilon} \int_{[0,1]\setminus[t_i, t_i+\epsilon]} \frac{\epsilon^{-r}|y - t_i|^r}{|x - y|^r} dxdy \lesssim \epsilon^{2-r} \ln \epsilon^{-1}.$$

Combining II, II_1, and II_2 implies that

$$\|\delta g\|^r_{1-\frac{1}{r}, r, \Gamma_D} \lesssim \epsilon^{2-r} \ln \epsilon^{-1},$$

which completes the proof of the first inequality.

To prove the second inequality, since $g_{_D} \in H^{3/2}(\Gamma_{D_i})$, the embedding theorem implies that $g_{_D} \in H^{1/2+\alpha}(\Gamma_{D_i})$. With $g_\epsilon = g_{_D} - \delta g$, it suffices to estimate $\|\delta g_\epsilon\|_{1/2+\alpha,\Gamma_D}$. To this end, it follows from the fractional Sobelov norm and the construction of δg that

$$\begin{aligned}
& \|\delta g\|^2_{1/2+\alpha,\Gamma_D} \\
= \; & \|\delta g\|^2_{1,\Gamma_D} + \int_{\Gamma_D}\int_{\Gamma_D} \frac{|\delta g'(x)-\delta g'(y)|^2}{|x-y|^{2\alpha}} dxdy. \\
\leq \; & \sum_{i=1}^n \int_{t_i}^{t_i+\epsilon} \left(|\delta g|^2 + |\delta g'|^2\right) + \sum_{j=1}^n \sum_{i=1}^n \int_{t_j}^{t_j+\epsilon} \int_{t_i}^{t_i+\epsilon} \frac{|\delta g'(x)-\delta g'(y)|^2}{|x-y|^{2\alpha}} dxdy \\
& +2\sum_{i=1}^n \int_{[0,1]\setminus\cup_{i=1}^n [t_i,t_i+\epsilon]} \int_{t_i}^{t_i+\epsilon} \frac{|\delta g'(x)|^2}{|x-y|^{2\alpha}} dxdy.
\end{aligned}$$

It follows from Remark 2.1.1 and Lemma 2.1.2 that

$$\sum_{i=0}^n \int_{t_i}^{t_i+\epsilon} \left(|\delta g|^2 + |\delta g'|^2\right) \lesssim \epsilon(M^2+\epsilon^{-2}) \lesssim \epsilon^{-1}.$$

Arguing as before, consider the integral over $x \in [t_i, t_i+\epsilon]$ and $y \in [t_j, t_j+\epsilon]$. For $i=j$, it follows from the triangle inequality, the definition of δg and Remark 2.1.1 that

$$|\delta g'(x) - \delta g'(y)|^2 \leq |g'_{_D}(x) - g'_{_D}(y)|^2 + |g'_\epsilon(x) - g'_\epsilon(y)|^2 \lesssim M^2 + \epsilon^{-4}|x-y|^2.$$

And for $i \neq j$, it implies that

$$\begin{aligned}
|\delta g'(x) - \delta g'(y)|^2 \; & \leq \; |\delta g'(x) - \delta g'(t_i) - \delta g'(y) + \delta g'(t_j)|^2 \\
& \lesssim \; \epsilon^{-4}|x-t_i|^2 + \epsilon^{-4}|y-t_j|^2 + M^2.
\end{aligned}$$

Together with Lemma 2.1.2, it implies that

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\int_{t_j}^{t_j+\epsilon}\int_{t_i}^{t_i+\epsilon}\frac{|\delta g'(x)-\delta g'(y)|^2}{|x-y|^{2\alpha}}dxdy$$

$$\lesssim \sum_{i=1}^{n}\int_{t_i}^{t_i+\epsilon}\int_{t_i}^{t_i+\epsilon}\epsilon^{-4}|x-y|^{2-2\alpha}dxdy+\sum_{i=1}^{n}\sum_{j\neq i}\int_{t_j}^{t_j+\epsilon}\int_{t_i}^{t_i+\epsilon}\epsilon^{-4}\frac{|x-t_i|^2}{|x-y|^{2\alpha}}dxdy$$

$$+\sum_{i=1}^{n}\sum_{j\neq i}\int_{t_j}^{t_j+\epsilon}\int_{t_i}^{t_i+\epsilon}\epsilon^{-4}\frac{|y-t_j|^2}{|x-y|^{2\alpha}}dxdy+\sum_{i=1}^{n}\sum_{j\neq i}\int_{t_j}^{t_j+\epsilon}\int_{t_i}^{t_i+\epsilon}\frac{1}{|x-y|^{2\alpha}}dxdy$$

$$\lesssim \epsilon^{-4}\cdot\epsilon^{2-2\alpha}\epsilon^2+\epsilon^{-4}\cdot\epsilon^2\ln\epsilon^{-1}\cdot\epsilon^{2-2\alpha}+\epsilon^2\cdot\epsilon^{-2\alpha}\lesssim\epsilon^{-2\alpha}\ln\epsilon^{-1}$$

Similarly, it can be proved that

$$\sum_{i=1}^{n}\int_{[0,1]\setminus\cup_{i=1}^{n}[t_i,t_i+\epsilon]}\int_{t_i}^{t_i+\epsilon}\frac{|\delta g'(x)|^2}{|x-y|^{2\alpha}}dxdy\lesssim\epsilon^{-2\alpha}\ln\epsilon^{-1}.$$

Combining all the parts gives that

$$\|\delta g\|^2_{1/2+\alpha,\Gamma_D}\lesssim\epsilon^{-2\alpha}\ln\epsilon^{-1},$$

which, in turn, proves the result. ■

2.2 *A priori* error estimate

With the continuous approximation g_ϵ of the Dirichlet data g_D, consider the following regularized problem:

$$\left\{\begin{array}{rcll} -\alpha\Delta u_\epsilon+\boldsymbol{\beta}\cdot\nabla u_\epsilon+cu_\epsilon &=& f & \text{in}\quad \Omega,\\ u_\epsilon &=& g_\epsilon & \text{on}\quad \Gamma_D,\\ \alpha\nabla u_\epsilon\cdot\boldsymbol{n} &=& g_N & \text{on}\quad \Gamma_N,\end{array}\right. \tag{2.2.1}$$

Let $H^1(\Omega)=W^{1,2}(\Omega)$ and let

$$H^1_{g_\epsilon,D}(\Omega):=\{v\in H^1(\Omega): v=g_\epsilon \text{ on } \Gamma_D\}\quad\text{and}\quad H^1_D(\Omega):=H^1_{0,D}(\Omega).$$

The corresponding variational formulation of the problem in (2.2.1) is to find $u_\epsilon\in H^1_{g_\epsilon,D}(\Omega)$ such that

$$a(u_\epsilon,v)=f(v),\quad\forall\, v\in H^1_{0,D}(\Omega), \tag{2.2.2}$$

where the bilinear and linear forms are defined by

$$a(u_\epsilon, v) = (\alpha \nabla u_\epsilon,\, \nabla v) + (\boldsymbol{\beta} \cdot \nabla u_\epsilon, v) + (c\, u_\epsilon, v) \text{ and } f(v) = (f, v) + (g_{_N}, v)_{\Gamma_N},$$

respectively.

To discretize problem (2.2.2), let $\mathcal{T}_h = \{K\}$ be a finite element triangulation of the domain Ω. Denote by h_K the diameter of the element K and let $h = \max_{K \in \mathcal{T}_h} h_K$. For each element $K \in \mathcal{T}_h$, let $P_k(K)$ be the space of polynomials of degree less than or equal to k.

Denote the continuous linear finite element space associated with the triangulation by

$$V_h = \{v \in H^1(\Omega) : v|_K \in P_1(K) \quad \forall K \in \mathcal{T}\}.$$

Let $\tilde{g}_\epsilon$ be the linear interpolation of g_ϵ and let

$$V_{h,\tilde{g}_\epsilon} = \{v \in V_h : v = \tilde{g}_\epsilon \text{ on } \Gamma_D\} \quad \text{and} \quad V_{h,0} = V_h \cap H^1_D(\Omega).$$

Then the finite element approximation is to find $u_{\epsilon,\,h} \in V_{h,\tilde{g}_\epsilon}$ such that

$$a(u_{\epsilon,\,h}, v_h) = f(v_h), \quad \forall v_h \in V_{h,0}. \tag{2.2.3}$$

The following theorem gives the detailed error estimates.

Theorem 2.2.1 *Let u, u_ϵ, and $u_{\epsilon,\,h}$ be the solutions of* (2.0.1), (2.2.1), *and* (2.2.3), *respectively. Assume that $u_\epsilon \in H^{1+\alpha}(\Omega)$ for $1/2 < \alpha \le 1$, then for $1 < r < 2$ and $\epsilon = O(h)$, the following error estimate holds:*

$$\|u - u_{\epsilon,\,h}\|_{2-2/r,\Omega} \;\lesssim\; h^{2/r+\alpha-2} \ln h^{-1}.$$

Proof Let I_h be the nodal interpolation operator from $H^s(\Omega)$ $(s > 1)$ into V_h. Then by the triangle inequality, the embedding theorem and Theorem 2.0.1, we have

$$\begin{aligned}
&\|u - u_{\epsilon,\,h}\|_{2-2/r,\Omega} \\
\le\;& \|u - u_\epsilon\|_{2-2/r,\Omega} + \|u_\epsilon - I_h u_\epsilon\|_{2-2/r,\Omega} + \|I_h u_\epsilon - u_{\epsilon,\,h}\|_{2-2/r,\Omega} \\
\lesssim\;& \|u - u_\epsilon\|_{1,r,\Omega} + \|u_\epsilon - I_h u_\epsilon\|_{2-2/r,\Omega} + \|I_h u_\epsilon - u_{\epsilon,\,h}\|_{2-2/r,\Omega} \\
\lesssim\;& C_r \|g_{_D} - g_\epsilon\|_{1-1/r,r,\Gamma_D} + \|u_\epsilon - I_h u_\epsilon\|_{2-2/r,\Omega} + \|I_h u_\epsilon - u_{\epsilon,\,h}\|_{2-2/r,\Omega}.
\end{aligned}$$

It follows from the approximation property of I_h (see, e.g., [1]), the inverse inequality and the fact that u_ϵ is in $H^{1+\alpha}(\Omega)$ that

$$
\begin{aligned}
\|u_\epsilon - I_h u_\epsilon\|_{2-2/r,\Omega} &\lesssim h^{2/r+\alpha-1}\|u_\epsilon\|_{1+\alpha,\Omega} \\
&\lesssim h^{2/r+\alpha-1}\left(\|g_\epsilon\|_{\alpha+1/2,\Gamma_D}+\|g_N\|_{\alpha-1/2,\Gamma_N}+\|f\|_{\alpha-1,\Omega}\right)
\end{aligned}
$$

and that

$$
\begin{aligned}
\|I_h u_\epsilon - u_{\epsilon,h}\|_{2-2/r,\Omega} &\lesssim h^{2/r-2}\|I_h u_\epsilon - u_{\epsilon,h}\|_{0,\Omega} \\
&\lesssim h^{2/r-2}(\|I_h u_\epsilon - u_\epsilon\|_{0,\Omega}+\|u_\epsilon - u_{\epsilon,h}\|_{0,\Omega}) \\
&\lesssim h^{2/r-2}(h^{1+\alpha}+h^{2\alpha})\|u_\epsilon\|_{1+\alpha,\Omega} \\
&\lesssim h^{2/r+2\alpha-2}\left(\|g_\epsilon\|_{1/2+\alpha,\Gamma_D}+\|g_N\|_{\alpha-1/2,\Gamma_N}+\|f\|_{\alpha-1,\Omega}\right).
\end{aligned}
$$

Finally for $\epsilon = O(h)$ and $1/2 < \alpha \leq 1$, it follows from Theorem 2.1.3 that

$$
\begin{aligned}
\|u - u_{\epsilon,h}\|_{2-2/r,\Omega} &\lesssim C_r\|g_D - g_\epsilon\|_{1-\frac{1}{r},r,\Gamma_D}+h^{2/r+2\alpha-2}\|g_\epsilon\|_{1/2+\alpha,\Gamma_D} \\
&\lesssim C_r\left(\epsilon^{2-r}\ln\epsilon^{-1}\right)^{1/r}+h^{2/r+2\alpha-2}\left(\epsilon^{-2\alpha}\ln\epsilon\right)^{1/2} \\
&\lesssim C_r h^{2/r-1}\ln h^{-1}+h^{2/r+\alpha-2}\ln h^{-1} \\
&\lesssim h^{2/r+\alpha-2}\ln h^{-1}.
\end{aligned}
$$

This completes the proof of the theorem. ■

Remark 2.2.2 *The order of the error estimate in* Theorem 2.2.1 *is not optimal when $\alpha < 1$ due to the use of the triangle inequality and the L^2 norm estimate of the finite element approximation of $u_\epsilon \in H^{1+\alpha}(\Omega)$,*

$$
\|u_\epsilon - u_{\epsilon,h}\|_{0,\Omega}\lesssim h^{2\alpha}\|u_\epsilon\|_{1+\alpha,\Omega},
$$

in the estimate of the difference between the interpolation and the finite element approximation of u_ϵ, $\|I_h u_\epsilon - u_{\epsilon,h}\|_{2-2/r,\Omega}$. In general, the quantity $I_h u_\epsilon - u_{\epsilon,h}$ should be much smaller than the quantity $u_\epsilon - u_{\epsilon,h}$. For example, the former is equal to zero *for the Poisson equation in one dimension, and one has (see, e.g., [7]) the following expansion:*

$$I_h u_\epsilon - u_{\epsilon,\,h} = C(x)\, h^2 + o(h^2)$$

for smooth u_ϵ and for two dimensions. With the assumption

$$\|I_h u_\epsilon - u_{\epsilon,\,h}\|_{0,\Omega} \lesssim h^{1+\alpha} \|u_\epsilon\|_{1+\alpha,\Omega},$$

the estimate in Theorem 2.2.1 *may be improved to be the order of* $h^{2/r-1} \ln h^{-1}$.

2.3 Numerical results

In this section, we report numerical results of solving the elliptic problems with discontinuous Dirichlet boundary condition. Let $\Omega = (0,\,1)^2$ be the unit square, consider the following problems

$$-\triangle u + u = 1, \quad \text{in } \Omega,$$

with the discontinuous boundary conditions:

$$\text{either} \quad \Pi_1 : \begin{cases} u(x,\,1) = 1, & x \in [0,1] \\ u(x,\,y) = 0, & (x,\,y) \in \partial\Omega \setminus [0,\,1] \times \{1\} \end{cases}$$

$$\text{or} \quad \Pi_2 : \begin{cases} u(x,\,1) = 1, & x \in (0,1) \\ u(x,\,y) = 0 & (x,\,y) \in \partial\Omega \setminus (0,\,1) \times \{1\}. \end{cases}$$

The domain Ω is partitioned by a uniform triangulation with triangle elements. Continuous linear finite elements are used for all numerical experiments. Numerical solutions with boundary conditions Π_1 and Π_2 are respectively depicted in Figures 2.1 and 2.2 with 225 degrees of freedom and in Figures 2.3 and 2.4 with 16129 degrees of freedom.

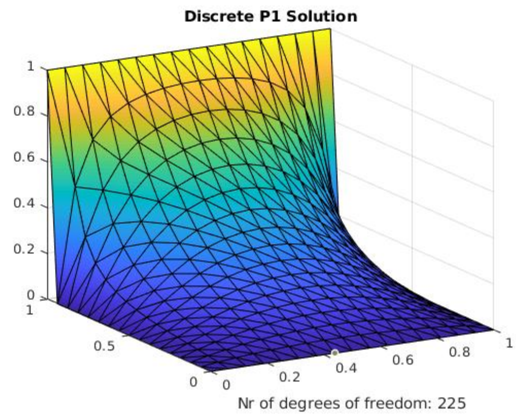

Figure 2.1. Π_1

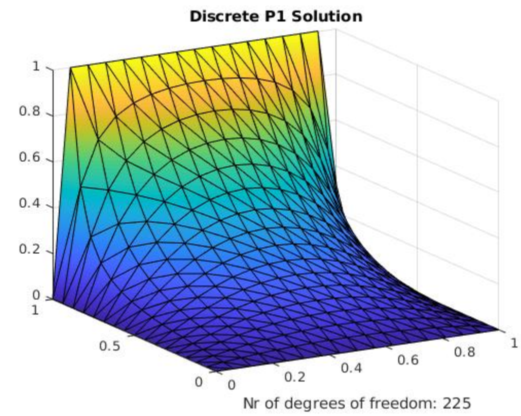

Figure 2.2. Π_2

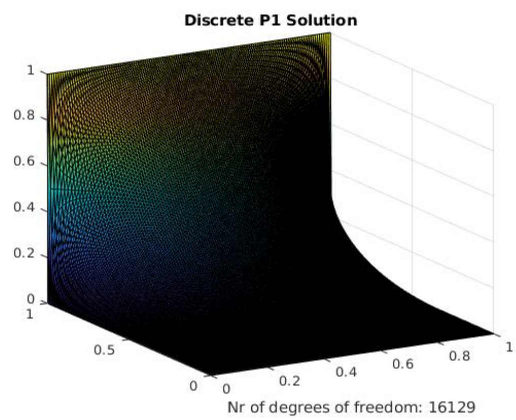

Figure 2.3. Π_1

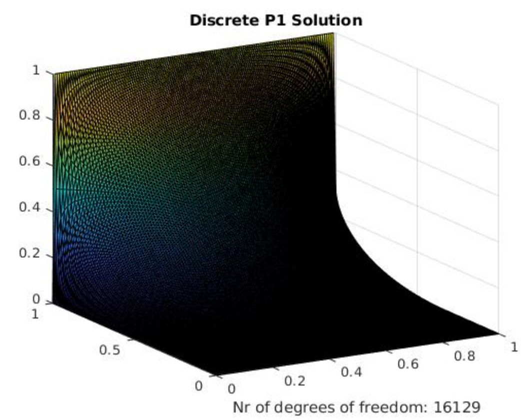

Figure 2.4. Π_2

Since those two boundary conditions differs only at two points $(0, 1)$ and $(1, 1)$, the H^1 norm of the difference of two solutions with the boundary conditions Π_1 and Π_2 on the domain Ω excluding three elements with nodes $(0, 1)$ and $(1, 1)$ is depicted in Figure 2.5. It shows that two solutions corresponding to the two boundary conditions are super close.

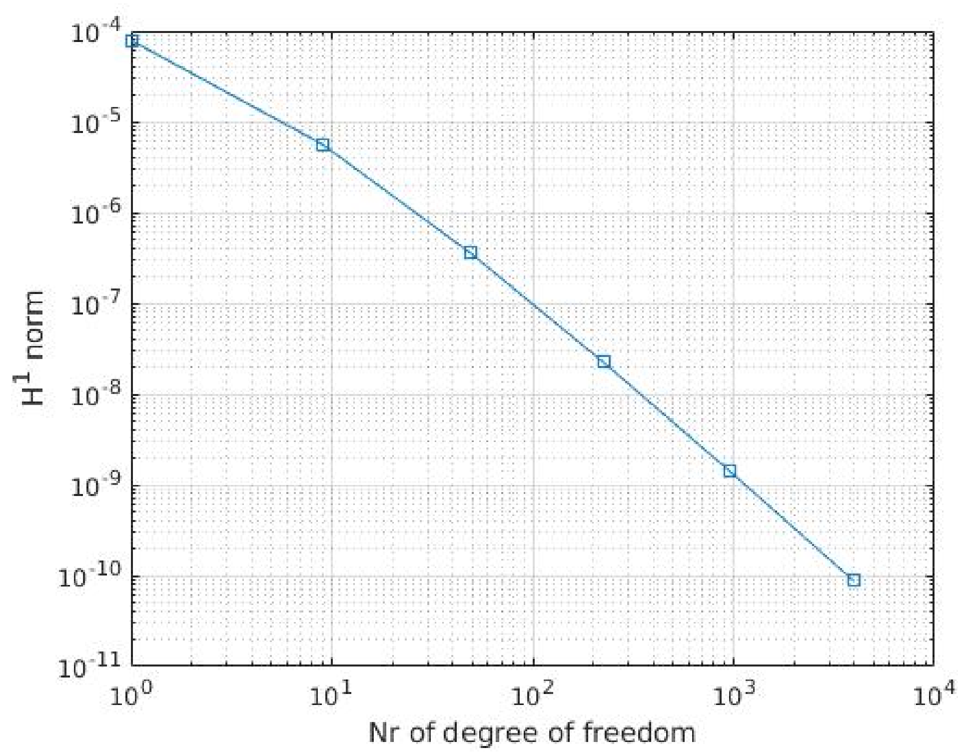

Figure 2.5. H1 norm of the difference of two solutions

3. DISCONTINUOUS GALERKIN METHODS

Discontinuous Galerkin methods (DG methods) are a class of numerical methods for solving partial differential equations. They combine the features of the finite element and the finite volume framework and have been successfully applied to hyperbolic, elliptic, parabolic and mixed form problems arising from a wide range of applications. DG methods have in particular received considerable interest for problems with a dominant first-order part, e.g. in electrodynamics, fluid mechanics and plasma physics.

Discontinuous Galerkin methods were first proposed and analyzed in the early 1970s as a technique to numerically solve partial differential equations. In 1973 Reed and Hill introduced a DG method to solve the hyperbolic neutron transport equation. Recently, Ayuso and Marini in [12] and Ern, Stephansen, and Zunino in [13] studied discontinuous Galerkin (DG) finite element methods for advection-diffusion-reaction problems. Optimal *a priori* error estimates in suitable norms were established provided that the exact solution is at least in $H^{3/2+\epsilon}$, for any $\epsilon > 0$. For comments and remarks on various DG methods studied by researchers, we refer readers to [12, 13] and references therein.

3.1 Notations

Throughout the paper, we will use the standard notations for the norms and seminorms in Sobolev Space. For a domain Ω, denote the Sobolev space by $W^{s,r}(\Omega)$ equipped with the standard Sobolev norm $\|\cdot\|_{s,r,\Omega}$ and seminorm $|\cdot|_{s,r,\Omega}$, where s is a real number and $1 \leq r \leq \infty$. When $r = 2$, $W^{s,2}(\Omega)$ is a Hilbert space and is denoted by $H^s(\Omega)$ with the norm $\|\cdot\|_{s,\Omega}$ and seminorm $|\cdot|_{s,\Omega}$. (We omit the subscript Ω from the inner product and norm designation when there is no risk of confusion.) To keep the homogeneity of dimensions, on a domain Ω with diameter L we define

$$\|v\|_{k,\Omega}^2 := \sum_{s=0}^{k} L^{2s} |v|_{s,\Omega}^2 \quad \text{for} \quad v \in H^k(\Omega),\ k \geq 0 \tag{3.1.1}$$

and

$$\|v\|_{k,\infty,\Omega} := \sum_{s=0}^{k} L^{s} |v|_{s,\infty,\Omega} \quad \text{for} \quad v \in W^{k,\infty}(\Omega),\ k \geq 0. \tag{3.1.2}$$

3.2 Jumps and Averages

Let $\mathcal{T}_h = \{K\}$ be a finite element triangulation of the domain Ω. Let h_K be the diameter of the element $K \in \mathcal{T}_h$ and $h = \max_{K \in \mathcal{T}_h} h_K$. Assume that the triangulation $\mathcal{T}_h$ is regular and also the interfaces $F = \{\partial\Omega_i \cap \partial\Omega_j \ :\ i, j = 1, ..., n\}$ do not cut through any element $K \in \mathcal{T}_h$.

Let $\mathcal{E}_K$ be the set of three edges of element $K \in \mathcal{T}_h$. Denote the set of all edges of the triangulation $\mathcal{T}_h$ by

$$\mathcal{E} := \mathcal{E}_{_I} \cup \mathcal{E}_{_D} \cup \mathcal{E}_{_N},$$

where $\mathcal{E}_{_I}$ is the set of all interior element edges, and $\mathcal{E}_{_D}$ and $\mathcal{E}_{_N}$ are the sets of all boundary edges belonging to the respective Γ_D and Γ_N. And define

$$\mathcal{E}_{\Gamma^{\pm}} := \mathcal{E} \cap \Gamma^{\pm}.$$

For each $e \in \mathcal{E}$, let h_e be the length of the edge e and $\boldsymbol{n}_e$ be a unit normal vector to e. For each interior edge $e \in \mathcal{E}_I$, choose $\boldsymbol{n}_e$ such that $\boldsymbol{\beta} \cdot \boldsymbol{n}_e > 0$ and let K_e^- and K_e^+ be the two elements sharing the common edge e such that the unit outward normal vector of K_e^- coincides with $\boldsymbol{n}_e$. When $e \in \mathcal{E}_{\Gamma^{\pm}}$, $\boldsymbol{n}_e$ is the unit outward normal vector and denote the element by $K_e^{\pm}$. For any $e \in \mathcal{E}$, denote by $v|_e^-$ and $v|_e^+$, respectively, the traces of a function v over e.

Define jumps over edges by

$$[\![v]\!]_e := \begin{cases} v|_e^- - v|_e^+ & e \in \mathcal{E}_I, \\ v|_e^- & e \in \mathcal{E}_{\Gamma^-}, \\ v|_e^+ & e \in \mathcal{E}_{\Gamma^+}. \end{cases}$$

Let w_e^+ and w_e^- be weights defined on e satisfying

$$w_e^+(x) + w_e^-(x) = 1, \tag{3.2.1}$$

and define the following weighted averages by

$$\{v(x)\}_w^e = \begin{cases} w_e^- v_e^- + w_e^+ v_e^+ & e \in \mathcal{E}_I, \\ v|_e^- & e \in \mathcal{E}_{\Gamma^-}, \\ v|_e^+ & e \in \mathcal{E}_{\Gamma^+}, \end{cases} \text{ and } \{v(x)\}_e^w = \begin{cases} w_e^+ v_e^- + w_e^- v_e^+ & e \in \mathcal{E}_I, \\ v|_e^+ & e \in \mathcal{E}_{\Gamma^-}, \\ v|_e^- & e \in \mathcal{E}_{\Gamma^+} \end{cases}$$

for all $e \in \mathcal{E}$. Denote by $\{v(x)\}_e$ the weighted average of v with $w_e^+ = w_e^- = \frac{1}{2}$. When there is no ambiguity, the subscript or superscript e in the designation of the jump and the weighted averages will be dropped. A simple calculation leads to the following identity:

$$[\![uv]\!]_e = \{v\}_e^w [\![u]\!]_e + \{u\}_w^e [\![v]\!]_e. \tag{3.2.2}$$

Let e be the interface of elements K_e^+ and K_e^-, i.e., $e = \partial K_e^+ \cap \partial K_e^-$, and denote by α_e^+ and α_e^- the diffusion coefficients on K_e^+ and K_e^-, respectively. Denote by

$$W_e = \{\alpha\}_w^e$$

the weighted average of α on edge e. For boundary edges, set

$$w_e^- = 1, \quad W_e = k_e^- \quad \text{if} \quad e \in \Gamma^- \quad \text{and} \quad w_e^+ = 1, \quad W_e = k_e^+ \quad \text{if} \quad e \in \Gamma^+.$$

In this paper, in order to guarantee the robust convergence, we take harmonic weights $w_e^{\pm} = \frac{\alpha_e^{\mp}}{\alpha_e^- + \alpha_e^+}$. Let $\alpha_{e,\min} = \min\{\alpha_e^+, \alpha_e^-\}$ and $\alpha_{e,\max} = \max\{\alpha_e^+, \alpha_e^-\}$, thus

$$W_e = \frac{2\alpha_e^+ \alpha_e^-}{\alpha_e^+ + \alpha_e^-} \quad \text{and} \quad \alpha_{e,\min} \leq W_e \leq 2\alpha_{e,\min}. \tag{3.2.3}$$

4. ADVECTION-DIFFUSION-REACTION PROBLEMS WITH NON-SMOOTH COEFFICIENTS

Let Ω be a bounded polygonal domain in $\Re^2$ with boundary $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$ and let $\boldsymbol{n} = (n_1, n_2)$ be the outward unit vector normal to the boundary. Let $\boldsymbol{\beta} = (\beta_1, \beta_2)^t \in W^{1,\infty}(\Omega)^2$ be the velocity vector field defined on $\overline{\Omega}$. Define inflow and outflow boundaries of $\partial\Omega$ by

$$\Gamma^- = \{x \in \partial\Omega : \boldsymbol{\beta}(x) \cdot \boldsymbol{n}(x) < 0\} \quad \text{and} \quad \Gamma^+ = \{x \in \partial\Omega : \boldsymbol{\beta}(x) \cdot \boldsymbol{n}(x) > 0$$

respectively, and let

$$\Gamma_D^{\pm} = \Gamma_D \cap \Gamma^{\pm} \quad \text{and} \quad \Gamma_N^{\pm} = \Gamma_N \cap \Gamma^{\pm}.$$

Consider the following advection-diffusion-reaction problem with discontinuous diffusion coefficients:

$$-\nabla \cdot (\alpha(x)\nabla u - \boldsymbol{\beta} u) + \gamma u = f \quad \text{in } \Omega \tag{4.0.1}$$

with boundary conditions

$$u = g_{\scriptscriptstyle D} \quad \text{on } \Gamma_D \quad \text{and} \quad \boldsymbol{n} \cdot \left(\boldsymbol{\beta} u \chi_{\Gamma_N^-} - \alpha \nabla u\right) = g_{\scriptscriptstyle N} \quad \text{on } \Gamma_N, \tag{4.0.2}$$

where $f \in L^2(\Omega)$, $g_{\scriptscriptstyle D} \in H^{1/2}(\Gamma_D)$, and $g_{\scriptscriptstyle N} \in H^{-1/2}(\Gamma_N)$ are given functions; $\chi_{\Gamma_N^-}$ is the characteristic function of the set Γ_N^-; and the diffusion coefficient $\alpha(x)$ is non-negative and piecewise constant on polygonal subdomains of Ω with possible large jumps across subdomain boundaries (interfaces):

$$\alpha(x) = \alpha_i \geq 0 \quad \text{in } \Omega_i \quad \text{for } i = 1, ..., n.$$

Here, $\{\Omega_i\}_{i=1}^n$ is a partition of the domain Ω with Ω_i being an open polygonal domain.

For the stability and error analysis, the assumptions on the coefficients introduced in [12, 24] are adopted in this paper:

(1) There exists a constant $\rho_0 \geq 0$ such that

$$\rho(x) = \frac{1}{2}\nabla \cdot \boldsymbol{\beta} + \gamma \geq \rho_0 \geq 0, \quad \text{in } \Omega; \tag{4.0.3}$$

(2) The advection field has no closed curves and stationary points. This implies that there exists a $\eta \in W^{2,\infty}(\Omega)$ such that

$$\boldsymbol{\beta} \cdot \nabla\eta \geq 2b_0 := 2\frac{\|\boldsymbol{\beta}\|_{0,\infty,\Omega}}{L}, \quad \text{in } \Omega; \tag{4.0.4}$$

(3) There exists a constant $c_{\boldsymbol{\beta}} > 0$ such that

$$|\boldsymbol{\beta}(x)| \geq c_{\boldsymbol{\beta}}\|\boldsymbol{\beta}\|_{1,\infty,\Omega}, \quad \text{in } \Omega; \tag{4.0.5}$$

(4) There exists a constant $c_\rho > 0$ such that

$$\|\rho\|_{0,\infty,K} \leq c_\rho(\min_K \rho(x) + b_0), \quad \forall K \in \mathcal{T}_h, \tag{4.0.6}$$

where $\mathcal{T}_h = \{K\}$ is a given shape-regular triangulation of Ω.

Remark 4.0.1 *Assumption* (5.3a) *guarantees the stability of the advection-reaction part. Also, the following useful inequality is deduced from* (3.1.2) *and* (4.0.5) *:*

$$|\boldsymbol{\beta}|_{1,\infty,\Omega} \leq \frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{L} \leq \frac{1}{c_\beta}\frac{\|\boldsymbol{\beta}\|_{0,\infty,\Omega}}{L} = \frac{b_0}{c_\beta}. \tag{4.0.7}$$

4.1 Variational formulations

Following [21], we derive a variational formulation of problem (4.0.1) - (4.0.2) held for piecewise smooth test functions. The key of this derivation is the introduction of a proper solution space in which integrals over inter-edges are well-defined. Moreover, the proper solution space is crucial for *a priori* error estimates of the underlying problem with low regularity.

Let u be the solution of problem (4.0.1) - (4.0.2), then it is well known from the regularity estimate [14] that u is in $H^{1+s}(\Omega)$ for some positive s which could be very small. Since $f \in L^2(\Omega)$, it is then easy to see that divergences of the diffusion and advection fluxes, $\alpha\nabla u$ and $\boldsymbol{\beta} u$, are square integrable, i.e.,

$$\alpha\nabla u,\ \ u\boldsymbol{\beta} \in H(\mathrm{div};\Omega) \equiv \{\boldsymbol{\tau} \in L^2(\Omega)^2 \,:\, \nabla\cdot\boldsymbol{\tau} \in L^2(\Omega)\}. \tag{4.1.1}$$

Consider the following solution space

$$V^{1+\epsilon}(\mathcal{T}_h) = \{v \in H^{1+\epsilon}(\mathcal{T}_h) \,:\, \nabla\cdot(\alpha\nabla v) \in L^2(K),\ \forall\, K \in \mathcal{T}_h\}$$

for $0 < \epsilon \ll 1$, where $H^s(\mathcal{T}_h)$ is the broken Sobolev space of degree $s > 0$ with respect to $\mathcal{T}_h$:

$$H^s(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K \in H^s(K),\ \forall\, K \in \mathcal{T}_h\}.$$

Denote the discrete gradient and divergence operators by

$$(\nabla_h v)|_K = \nabla(v|_K) \quad \text{and} \quad (\nabla_h\cdot\boldsymbol{\tau})|_K = \nabla\cdot(\boldsymbol{\tau}|_K),$$

for all $K \in \mathcal{T}_h$, respectively.

Multiplying equation (4.0.1) by a test function $v \in V^{1+\epsilon}(\mathcal{T}_h)$, integrating by parts, and using boundary conditions (4.0.2), we have the following :

$$\begin{aligned}(f,\,v) \;=\;& (\alpha\nabla_h u, \nabla_h v) - \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e [\![\alpha\nabla u\cdot\boldsymbol{n}_e v]\!] + \sum_{e\in\mathcal{E}_N}\int_e g_{\scriptscriptstyle N} v \\ & + (u,\, -\boldsymbol{\beta}\cdot\nabla_h v + \gamma v) + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_{\Gamma+}}\int_e [\![\beta_e u v]\!] + \sum_{e\in\mathcal{E}_{D-}}\int_e \beta_e g_{\scriptscriptstyle D} v,\end{aligned}$$

where $\mathcal{E}_{D^-} = \mathcal{E}_D \cap \Gamma^-$ and $\beta_e = \boldsymbol{\beta}\cdot\boldsymbol{n}_e$. Note that the Dirichlet boundary condition is used on the inflow boundary. By (4.1.1), it is easy to see that the normal components of the diffusion and advection fluxes are continuous across the internal edges. Then for any $e \in \mathcal{E}_I$ and $v \in V^{1+\epsilon}(\mathcal{T}_h)$,

$$\int_e [\![\alpha\nabla u\cdot\boldsymbol{n}_e]\!]\,\{v\}^w\,ds = 0 \quad \text{and} \quad \int_e [\![u\boldsymbol{\beta}\cdot\boldsymbol{n}_e]\!]\,\{v\}^w\,ds = 0.$$

By identity (3.2.2) and the Dirichlet boundary condition in (4.0.2), we have that for all $v \in V^{1+\epsilon}(\mathcal{T}_h)$,

$$(\alpha\nabla_h u, \nabla_h v) + (u,\, -\boldsymbol{\beta}\cdot\nabla_h v + \gamma v) - \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \{\alpha\nabla u\cdot\boldsymbol{n}_e\}_w[\![v]\!]$$
$$+ \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_{\Gamma^+}}\int_e \{\beta_e u\}_w[\![v]\!] = (f,v) - \sum_{e\in\Gamma_N}\int_e g_{_N} v - \sum_{e\in\mathcal{E}_{D^-}}\int_e \beta_e g_{_D} v. \tag{4.1.2}$$

Since the derivation does not make use of the continuity of the solution, one needs to impose such a continuity in order to achieve stability. To do so, it is natural and well-known to stabilize the diffusion and the advection operators by adding proper jump terms of the solution. Following the idea of [13] (also see [21]), we stabilize the diffusion operator by adding the following equation :

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \gamma_{_\theta} h_e^{-1} W_e[\![u]\!][\![v]\!]\,ds = \sum_{e\in\mathcal{E}_D}\gamma_{_\theta} h_e^{-1} W_e\int_e g_{_D} v\,ds,\ \forall\, v\in V^{1+\epsilon}(\mathcal{T}_h). \tag{4.1.3}$$

Since the diffusion operator is self-adjoint, it is then natural to symmetrize the diffusion part by adding the following equation:

$$\theta\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \{\alpha\nabla v\cdot\boldsymbol{n}_e\}_w[\![u]\!]\,ds = \theta\sum_{e\in\mathcal{E}_D}\int_e g_{_D}(\alpha\nabla v\cdot\boldsymbol{n}_e)\,ds,\ \forall\, v\in V^{1+\epsilon}(\mathcal{T}_h) \tag{4.1.4}$$

with $\theta = \{-1, 0, 1\}$. Both (4.1.3) and (4.1.4) follow from the continuity of $u \in H^{1+s}(\Omega)$ and the Dirichlet boundary condition. When $\theta = 1$, (4.1.4) plays a role of stabilization and, hence, (4.1.3) is not needed. For the advection-reaction term, introduce the following general upwind average:

$$\{\beta_e u\}^e_{up} = \beta_e\xi_e^- u^- + \beta_e\xi_e^+ u^+, \quad \text{where } \xi_e^- + \xi_e^+ = 1 \text{ and } \xi_e^- > 1/2, \tag{4.1.5}$$

which is more general than that in [12] since ξ_e^+ could be negative. When $\xi_e^- = 1$, (4.1.5) is the classic upwind. As pointed out in [23], the jump-stabilization is more general than the classic upwind. But it is easy to see that the jump-stabilization is equivalent to (4.1.5).

Now, define bilinear forms for $u, v \in V^{1+\epsilon}(\mathcal{T}_h)$ by

$$\begin{aligned} a_{d,\theta}(u,v) &= (\alpha\nabla_h u, \nabla_h v) + \theta \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{\alpha\nabla v \cdot \boldsymbol{n}_e\}_w [\![u]\!]\, ds \\ &\quad - \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \{\alpha\nabla u \cdot \boldsymbol{n}_e\}_w [\![v]\!]\, ds + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \gamma_\theta h_e^{-1} W_e [\![u]\!][\![v]\!]\, ds \end{aligned} \tag{4.1.6}$$

for $\theta \in \{-1, 0, 1\}$ and

$$a_c(u,v) = (u,\, -\boldsymbol{\beta}\cdot\nabla_h v + \gamma v) + \sum_{e\in\mathcal{E}_I} \int_e \{\beta_e u\}_{up} [\![v]\!]\, ds + \sum_{e\in\mathcal{E}_{\Gamma+}} \int_e \beta_e uv\, ds. \tag{4.1.7}$$

Define the linear form for $v \in V^{1+\epsilon}(\mathcal{T}_h)$ by

$$\begin{aligned} f_\theta(v) &= (f,\, v) + \sum_{e\in\mathcal{E}_D} \gamma_\theta h_e^{-1} W_e \int_e g_D v\, ds + \sum_{e\in\mathcal{E}_N} \int_e g_N v\, ds \\ &\quad + \theta \sum_{e\in\mathcal{E}_D} \int_e g_D (k\nabla v \cdot \boldsymbol{n}_e)\, ds - \sum_{e\in\mathcal{E}_{D-}} \int_e (\boldsymbol{\beta}\cdot\boldsymbol{n}_e) g_D\, v\, ds. \end{aligned}$$

The weak solution of (4.0.1) - (4.0.2) satisfies the following variational problem: find $u \in V^{1+\epsilon}(\mathcal{T}_h)$ such that

$$a_\theta(u,\, v) \equiv a_{d,\theta}(u,\, v) + a_c(u,\, v) = f_\theta(v), \quad \forall v \in V^{1+\epsilon}(\mathcal{T}_h). \tag{4.1.8}$$

4.2 Discontinuous finite element approximation

Let $P_k(K)$ be the space of polynomials of degree at most k on element $K \in \mathcal{T}_h$. Denote the discontinuous finite element space associated with the triangulation $\mathcal{T}_h$ by

$$\mathcal{U}_h^k = \{v \in L^2(\Omega) \,:\, v|_K \in P_k(K),\ \forall K \in \mathcal{T}_h\}.$$

Discontinuous Galerkin (DG) finite element method is to find $u_h \in \mathcal{U}_h^k \subset V^{1+\epsilon}(\mathcal{T}_h)$ such that

$$a_\theta(u_h,\, v) = f_\theta(v), \quad \forall\ v \in \mathcal{U}_h^k. \tag{4.2.1}$$

The method corresponding to $\theta = -1$ and the classic upwind was introduced and analyzed recently in [13] for different boundary conditions. When $\alpha(x) = \varepsilon$, the methods corresponding to $\theta = 0, 1$ and the classic upwind reproduce the first two methods in [12]; the third (introduced in [17]) and fourth methods in [12] are corresponding to (4.2.1) with the respective classic and general upwind averages for both the diffusion and advection terms. *A priori* error bounds for DG methods had been established by various researchers (see [12, 13] and references therein) provided that the solution is at least piecewise $H^{3/2+\epsilon}$ smooth and that γ_θ is large enough.

In the remainder of this section, we prove the stability that implies the well-posedness of (4.2.1). To this end, for any $v \in \mathcal{U}_h^k$, define the DG norms for the diffusion and advection-reaction parts by

$$\|v\|_d^2 \;:=\; \|\alpha^{1/2}\nabla_h v\|_{0,\Omega}^2 + \|v\|_j^2 \tag{4.2.2}$$

with

$$\|v\|_j^2 \;:=\; \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} h_e^{-1} W_e \|[\![v]\!]\|_{0,e}^2$$

and

$$\|v\|_c^2 \;:=\; \|(\overline{\rho}+b_0)^{1/2} v\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}} \|c_e^{1/2}[\![v]\!]\|_{0,e}^2 \tag{4.2.3}$$

respectively, where $b_0 = \|\boldsymbol{\beta}\|_{0,\infty}/L$, $\overline{\rho}$ is a piece wise constant function defined as

$$\overline{\rho}_K(x) = \min_{x\in K} \rho_K(x) = \min\left(\frac{1}{2}\nabla\cdot\boldsymbol{\beta} + \gamma\right)_K, \quad \forall K \in \mathcal{T}_h \tag{5.3a}$$

and

$$c_e = \begin{cases} \left(\xi_e^- - \frac{1}{2}\right)\beta_e, & \text{on } e \in \mathcal{E}_I, \\ \frac{1}{2}\beta_e, & \text{on } e \in \mathcal{E}_{\Gamma^+}, \\ -\frac{1}{2}\beta_e, & \text{on } e \in \mathcal{E}_{\Gamma^-}. \end{cases} \tag{5.3b}$$

The DG norm is defined as

$$\|v\|_{DG} \;=\; \left(\|v\|_d^2 + \|v\|_c^2\right)^{1/2} \tag{4.2.4}$$

4.3 Stability

To prove the stability, we introduce the two useful lemmas as following.

Lemma 4.3.1 *For any $u \in \mathcal{U}_h^k$ and $v \in V^{1+\epsilon}(\mathcal{T}_h)$, there exists a positive constant C_g, depending only on the degree of the polynomial and the triangulation, such that*

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \Big|\{\alpha\nabla_h u \cdot \boldsymbol{n}_e\}_w [\![v]\!]\Big| ds \le C_g \|\alpha^{1/2}\nabla u\|_{0,\Omega}\|v\|_j \tag{4.3.1}$$

and

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \Big|\{\alpha u\}_w [\![v]\!]\Big| ds \le C_g \|\alpha^{1/2} u\|_{0,\Omega}\|v\|_j. \tag{4.3.2}$$

Proof It follows from the definition of W_e and the harmonic averages that

$$w_e^\omega \sqrt{\alpha_e^\omega} \le \frac{\sqrt{2}}{2}\sqrt{W_e}, \quad \text{where} \quad \omega = -, +.$$

Together with the inverse and the Cauchy-Schwarz inequalities, it gives that

$$\begin{aligned}
\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \Big|\{\alpha\nabla u\cdot\boldsymbol{n}_e\}_w [\![v]\!]\Big| ds &= \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \Big|(w_e^+\alpha_e^+\nabla u\cdot n_e^+ + w_e^-\alpha_e^-\nabla u\cdot n_e^-)[\![v]\!]\Big| ds \\
&\le c_1 \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} h_e^{-1/2} W_e^{1/2} \|[\![v]\!]\|_{0,e} \sum_{\omega=+,-} \|\alpha^{1/2}\nabla u\|_{0,K^\omega} \\
&\le C_1 \|\alpha^{1/2}\nabla u\|_{0,\Omega}\|v\|_j,
\end{aligned}$$

where C_1 may depend on the polynomial degree k and the triangulation $\mathcal{T}_h$, is independent of α and h.

In a similar way, we obtain that

$$\begin{aligned}
\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \Big|\{\alpha u\}_w [\![v]\!]\Big| ds &= \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \Big|(w_e^+\alpha_e^+ u^+ + w_e^-\alpha_e^- u^-)[\![v]\!]\Big| ds \\
&\le c_2 \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} h_e^{-1/2} W_e^{1/2} \|[\![v]\!]\|_{0,e} \sum_{\omega=+,-} \|\alpha^{1/2} u\|_{0,K^\omega} \\
&\le C_2 \|\alpha^{1/2} u\|_{0,\Omega}\|v\|_j,
\end{aligned}$$

Where C_2 may depend on the triangulation $\mathcal{T}_h$ and the polynomial degree k. Let $C_g = \max\{C_1, C_2\}$ and this completes the proof of the lemma. ■

Lemma 4.3.2 *For any function $v \in \mathcal{U}_h^k$, there exists a positive constant C_p, depending on the minimum angel of the triangulation $\mathcal{T}_h$ of Ω, such that*

$$\|\alpha^{1/2} v\|_{0,\Omega} \leq C_p L \left(\|\alpha^{1/2}\nabla_h v\|_{0,\Omega}^2 + \|v\|_j^2\right)^{1/2}, \tag{4.3.3}$$

where L is the diameter of the domain Ω.

Proof For any piece wise H^1 function v, the following Poincaré-Friedrichs inequality is proved in [16]:

$$\|v\|_{0,\Omega} \leq CL \left(\|\nabla_h v\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} h_e^{-1}\|[\![v]\!]\|_{0,e}^2\right)^{1/2} \tag{4.3.4}$$

where C is a positive constant depending on the minimum angle of the triangulation $\mathcal{T}_h$ of Ω.

Since the diffusion coefficient α is piece wise constant, (4.3.4) implies that

$$\|\alpha^{1/2} v\|_{0,\Omega} \leq CL \left(\|\alpha^{1/2}\nabla_h v\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} h_e^{-1}\|[\![\alpha^{1/2} v]\!]\|_{0,e}^2\right)^{1/2},$$

for any $v \in \mathcal{U}_h^k$.

To show the validity of (4.3.3), it suffices to prove that

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} h_e^{-1}\|[\![\alpha^{1/2} v]\!]\|_{0,e}^2 \leq C\left(\|\alpha^{1/2}\nabla_h v\|_{0,\Omega}^2 + \|v\|_j^2\right), \tag{4.3.5}$$

for any $v \in \mathcal{U}_h^k$.

To this end, let $\alpha_{e,min} = \alpha_e^- < \alpha_e^+$. It follows from the trace inequality and (3.2.3) that for each $e \in \mathcal{E}_I \cup \mathcal{E}_D$,

$$\begin{aligned}
\|[\![\alpha^{1/2} v]\!]\|_{0,e}^2 &= \|\sqrt{\alpha_e^-}v^- - \sqrt{\alpha_e^+}v^+\|_{0,e}^2 \\
&= \|\sqrt{\alpha_e^-}(v^- - v^+) + (\sqrt{\alpha_e^-} - \sqrt{\alpha_e^+})v^+\|_{0,e}^2 \\
&\leq 2\left(\|\alpha_{e,min}^{1/2}[\![v]\!]\|_{0,e}^2 + \|\sqrt{\alpha_e^+}v^+\|_{0,e}^2\right) \\
&\leq C\left(W_e\|[\![v]\!]\|_{0,e}^2 + h_{K_e^+}\|\sqrt{\alpha}\nabla_h v\|_{0,K^+}^2\right).
\end{aligned}$$

Multiplying by h_e^{-1} and summing up over $e \in \mathcal{E}_I \cup \mathcal{E}_D$ imply (4.3.5). This completes the proof of the lemma. ∎

To establish the stability of the bilinear form $a_\theta(\cdot,\cdot)$ in the DG norm, we follow the idea in [12]. To this end, introduce the weight function

$$\varphi = e^{-\eta} + \mathcal{K} := \chi + \mathcal{K}, \tag{4.3.6}$$

where η is defined in (4.0.4) and $\mathcal{K}$ is a positive constant.

Since $\eta \in W^{1,\infty}(\Omega)$, there exist positive constants χ_1, χ_2, and χ_3 such that

$$\chi_1 \le \chi \le \chi_2 \quad \text{and} \quad \|\nabla\chi\|_\infty \le \chi_3. \tag{4.3.7}$$

Choose the constant $\mathcal{K}$ such that

$$\chi_1 + \mathcal{K} > 6(1 + C_g)C_p L\chi_3 \quad \text{and} \quad 2(\chi_1 + \mathcal{K}) > \chi_2 + \mathcal{K}. \tag{4.3.8}$$

with C_g and C_p defined in Lemma 4.3.1 and Lemma 4.3.2, respectively.

Lemma 4.3.3 *Let $a_{d,\theta}(\cdot,\cdot)$ and $a_c(\cdot,\cdot)$ be the bilinear forms defined in (4.1.6) and (4.1.7), respectively, with $\gamma_\theta \ge \gamma_0 > \max\{9C_g^2, 1\}$. For any $v_h \in \mathcal{U}_h^k$, the following inequalities hold:*

$$a_{d,\theta}(v_h, \varphi v_h) \ge \frac{\chi_1 + \mathcal{K}}{6} \|\!|v_h|\!\|_d^2, \qquad a_c(v_h, \varphi v_h) \ge \chi_1 \|\!|v_h|\!\|_c^2 \tag{4.3.9}$$

and

$$\|\!|\varphi v_h|\!\|_{DG} \le \sqrt{5}(\chi_1 + \mathcal{K})\|\!|v_h|\!\|_{DG}. \tag{4.3.10}$$

Proof By the definition of the bilinear form $a_{d,\theta}$ and the continuity of φ, we have

$$\begin{aligned}
& a_{d,\theta}(v_h, \varphi v_h) \\
= \; & (\alpha\nabla_h v_h, \varphi\nabla_h v_h) + (\alpha\nabla_h v_h, v_h\nabla\varphi) + \theta \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e (\nabla\varphi\cdot\boldsymbol{n}_e)\{\alpha v_h\}_w [\![v_h]\!] \\
& + (\theta - 1)\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \varphi\{\alpha\nabla v_h\cdot\boldsymbol{n}_e\}_w [\![v_h]\!] + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} \int_e \gamma_\theta h_e^{-1} W_e \varphi [\![v_h]\!]^2.
\end{aligned}$$

It follows from the Cauchy-Schwarz inequality, (4.3.7), and Lemma 4.3.1 and 4.3.2 that

$$(\alpha\nabla_h v_h, v_h\nabla\varphi) \le \chi_3 \|\alpha^{1/2}\nabla_h v_h\|_{0,\Omega} \|\alpha^{1/2} v_h\|_{0,\Omega} \le \chi_3 C_p L \|\!|v_h|\!\|_d^2,$$

and that

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e(\nabla\varphi\cdot\boldsymbol{n}_e)\{\alpha v_h\}_w[\![v_h]\!] \le \chi_3 C_g\|\alpha^{1/2}v_h\|_{0,\Omega}|\!|\!|v_h|\!|\!|_d \le \chi_3 C_g C_p L|\!|\!|v_h|\!|\!|_d^2.$$

By Lemma 4.3.1, (4.3.8), and the assumption that $\gamma_\theta \ge \gamma_0 > \max\{9C_g^2, 1\}$, we have

$$\begin{aligned}\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\varphi\{\alpha\nabla v_h\cdot\boldsymbol{n}_e\}_w[\![v_h]\!] &\le (\chi_2+\mathcal{K})C_g\|\alpha^{1/2}\nabla_h v_h\|_{0,\Omega}\|v_h\|_j\\ &\le \frac{(\chi_1+\mathcal{K})}{3}\left(\|\alpha^{1/2}\nabla_h v_h\|_{0,\Omega}^2+\gamma_0\|v_h\|_j^2\right).\end{aligned}$$

For $\theta\in\{-1,0,1\}$, combining the above equality and inequalities gives that

$$\begin{aligned}a_{d,\theta}(v_h,\varphi v_h) &\ge (\chi_1+\mathcal{K})\left(\|\alpha^{1/2}\nabla_h v_h\|_{0,\Omega}^2+\gamma_0\|v_h\|_j^2\right)-\chi_3 C_p L|\!|\!|v_h|\!|\!|_d^2\\ &\quad -\chi_3 C_g C_p L|\!|\!|v_h|\!|\!|_d^2-\frac{2(\chi_1+\mathcal{K})}{3}\left(\|\alpha^{1/2}\nabla_h v_h\|_{0,\Omega}^2+\gamma_0\|v_h\|_j^2\right)\\ &\ge \left(\frac{\chi_1+\mathcal{K}}{3}-(1+C_g)\chi_3 C_p L\right)|\!|\!|v_h|\!|\!|_d^2\\ &\ge \frac{\chi_1+\mathcal{K}}{6}|\!|\!|v_h|\!|\!|_d^2.\end{aligned}$$

The last inequality used (4.3.8). And this proves the first inequality in (4.3.9).

For the advection-reaction part, it follows from the identity that $v_h\nabla v_h=\frac{1}{2}\nabla_h(v_h^2)$, integration by parts, and the continuity of ϕ and $\boldsymbol{\beta}$ that

$$\begin{aligned}(v_h,-\boldsymbol{\beta}\cdot\nabla_h(\varphi v_h)) &= -\frac{1}{2}\int_\Omega\varphi\boldsymbol{\beta}\cdot\nabla_h(v_h^2)-\int_\Omega(\boldsymbol{\beta}\cdot\nabla\varphi)v_h^2\\ &= \frac{1}{2}\int_\Omega v_h^2\nabla\cdot(\varphi\boldsymbol{\beta})-\frac{1}{2}\sum_{K\in\mathcal{T}_h}\int_{\partial K}\varphi v_h^2\boldsymbol{\beta}\cdot\boldsymbol{n}-\int_\Omega(\boldsymbol{\beta}\cdot\nabla\varphi)v_h^2\\ &= \frac{1}{2}\int_\Omega(\nabla\cdot\boldsymbol{\beta})\varphi v_h^2-\frac{1}{2}\int_\Omega(\boldsymbol{\beta}\cdot\nabla\varphi)v_h^2-\frac{1}{2}\sum_{e\in\mathcal{E}}\int_e\beta_e\varphi[\![v_h^2]\!].\end{aligned}$$

With the definition of c_e in (5.3b), a simple computation gives that

$$\begin{aligned}
&-\frac{1}{2}\sum_{e\in\mathcal{E}}\int_e \beta_e\varphi[\![v_h^2]\!] + \sum_{e\in\mathcal{E}_I}\int_e \{\beta_e v_h\}_{up}[\![\varphi v_h]\!] + \sum_{e\in\mathcal{E}_{\Gamma+}}\int_e \beta_e\varphi v_h^2 \\
= \; & -\frac{1}{2}\sum_{e\in\mathcal{E}_I}\int_e \beta_e\varphi(v_h^+ + v_h^-)[\![v_h]\!] - \frac{1}{2}\sum_{e\in\mathcal{E}_{\Gamma-}}\int_e \beta_e\varphi v_h^2 + \frac{1}{2}\sum_{e\in\mathcal{E}_{\Gamma+}}\int_e \beta_e\varphi v_h^2 \\
&+\sum_{e\in\mathcal{E}_I}\int_e \beta_e\varphi(\xi_e^+ v_h^+ + \xi_e^- v_h^-)[\![v_h]\!] \; = \sum_{e\in\mathcal{E}}\int_e c_e\varphi[\![v_h]\!]^2.
\end{aligned}$$

Combining these two identities gives that

$$\begin{aligned}
a_c(v_h,\varphi v_h) \;&=\; (v_h, -\boldsymbol{\beta}\cdot\nabla_h(\varphi v_h) + \gamma\varphi v_h) + \sum_{e\in\mathcal{E}_I}\int_e \{\beta_e v_h\}_{up}[\![\varphi v_h]\!] + \sum_{e\in\mathcal{E}_{\Gamma+}}\int_e \beta_e\varphi v_h^2 \\
&=\; \int_\Omega (\gamma + \frac{1}{2}\nabla\cdot\boldsymbol{\beta})\varphi v_h^2 - \frac{1}{2}\int_\Omega (\boldsymbol{\beta}\cdot\nabla\varphi)v_h^2 + \sum_{e\in\mathcal{E}}\int_e c_e\varphi[\![v_h]\!]^2.
\end{aligned}$$

From (4.0.4) and (4.3.7), we have

$$-\boldsymbol{\beta}\cdot\nabla\varphi = (\boldsymbol{\beta}\cdot\nabla\eta)e^{-\eta} \ge 2b_0 e^{-\eta} \ge 2b_0\chi_1.$$

Together with the definition of $\overline{\rho}$ in (5.3a), we obtain that

$$\begin{aligned}
a_c(v_h,\varphi v_h) \;&\ge\; (\chi_1+\mathcal{K})\int_\Omega \overline{\rho}v_h^2 + \chi_1\int_\Omega b_0 v_h^2 + (\chi_1+\mathcal{K})\sum_{e\in\mathcal{E}}\int_e c_e[\![v_h]\!]^2 \\
&\ge\; \chi_1\|(\overline{\rho}+b_0)^{1/2}v_h\|_{0,\Omega}^2 + \chi_1\sum_{e\in\mathcal{E}}\|c_e^{1/2}[\![v_h]\!]\|_{0,e}^2 \\
&\ge\; \chi_1|\!|\!|v_h|\!|\!|_c^2,
\end{aligned}$$

which proves the second inequality in (4.3.9).

To estimate the upper bound of the DG norm of φv_h, Lemma 4.3.2, (4.3.7) and (4.3.8) give that

$$\begin{aligned}
|\!|\!|\varphi v_h|\!|\!|_d^2 \;&=\; \|\alpha^{1/2}\varphi\nabla_h v_h\|_{0,\Omega}^2 + \|\alpha^{1/2}v_h\nabla\varphi\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e h_e^{-1}W_e\varphi^2[\![v_h]\!]^2 \\
&\le\; ((\chi_2+\mathcal{K})^2 + \chi_3^2C_p^2L^2)|\!|\!|v_h|\!|\!|_d^2 \\
&\le\; 5(\chi_1+\mathcal{K})^2|\!|\!|v_h|\!|\!|_d^2,
\end{aligned}$$

and that

$$\|\varphi v_h\|_c^2 = \|(\overline{\rho}+b_0)^{1/2}\varphi v_h\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}} \|c_e^{1/2}\varphi[\![v_h]\!]\|_{0,e}^2$$
$$\leq (\chi_2+\mathcal{K})^2 \|v_h\|_c^2,$$

which implies that

$$\|\varphi v_h\|_{DG} \leq (5(\chi_1+K)^2\|v_h\|_d^2 + (\chi_2+\mathcal{K})^2\|v_h\|_c^2)^{1/2}$$
$$\leq \sqrt{5}(\chi_1+\mathcal{K})\|v_h\|_{DG},$$

which proves (4.3.10) and, hence, completes the proof of the lemma. ■

The following lemma is about the approximation results of the L_2-projection in the DG space, which have been proved in [18] and [19].

Lemma 4.3.4 *Let $\varphi \in W^{1,\infty}(\Omega)$ be the function defined in (4.3). For any $v_h \in \mathcal{U}_h^k$, let $\widetilde{\varphi v_h}$ be the L_2-projection of φv_h into $\mathcal{U}_h^k$, then the following estimates hold:*

$$\|\varphi v_h - \widetilde{\varphi v_h}\|_{p,2,\Omega} \leq Ch^{1-p}\|\chi\|_{1,\infty,\Omega}\|v_h\|_{0,\Omega}/L, \quad p=0,1$$

and

$$\left(\sum_{e\in\mathcal{E}} \|\varphi v_h - \widetilde{\varphi v_h}\|_{0,e}^2\right)^{1/2} \leq Ch^{1/2}\|\chi\|_{1,\infty,\Omega}\|v_h\|_{0,\Omega}/L,$$

where C is a positive constant independent of $\mathcal{K}$ and L is the diameter of Ω.

With Lemma 4.3.4, we estimate the upper bounds of the norms $\|\varphi v_h - \widetilde{\varphi v_h}\|_d$ and $\|\varphi v_h - \widetilde{\varphi v_h}\|_c$ in the following lemma.

Lemma 4.3.5 *For any $v_h \in \mathcal{U}_h^k$, then the following estimates hold:*

$$\|\widetilde{\varphi v_h} - \varphi v_h\|_d \leq CC_p\|\chi\|_{1,\infty}\|v_h\|_d$$

and

$$\|\widetilde{\varphi v_h} - \varphi v_h\|_c \leq C\left(\frac{h}{L}\right)^{1/2}\|\chi\|_{1,\infty}\|(\overline{\rho}+b_0)^{1/2}v_h\|_{0,\Omega}.$$

Proof For any function $v_h \in \mathcal{U}_h^k$, since α is a piece wise constant function, then $\alpha^{1/2} v_h \in \mathcal{U}_h^k$ and $\alpha^{1/2}\widetilde{\varphi v_h}$ is the L^2 projection of $\alpha^{1/2}\varphi v_h$ into $\mathcal{U}_h^k$.

Lemma 4.3.4 gives that

$$\|\alpha^{1/2}\varphi v_h - \alpha^{1/2}\widetilde{\varphi v_h}\|_{p,2,\Omega} \le Ch^{1-p}\|\chi\|_{1,\infty}\|\alpha^{1/2}v_h\|_{0,\Omega}/L, \quad p = 0, 1$$

and

$$\left(\sum_{e\in\mathcal{E}} \|\alpha^{1/2}\varphi v_h - \alpha^{1/2}\widetilde{\varphi v_h}\|_{0,e}^2\right)^{1/2} \le Ch^{1/2}\|\chi\|_{1,\infty,\Omega}\|\alpha^{1/2}v_h\|_{0,\Omega}/L.$$

Together with the definition of d-norm in (4.2.2), the fact that $\alpha_{e,\min} \le W_e \le 2\alpha_{e,\min}$ and Lemma 4.3.2, we have

$$\begin{aligned}
|||\varphi v_h - \widetilde{\varphi v_h}|||_d^2 &= \|\alpha^{1/2}\nabla_h(\varphi v_h - \widetilde{\varphi v_h})\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} h_e^{-1}W_e\|[\![\varphi v_h - \widetilde{\varphi v_h}]\!]\|_{0,e}^2 \\
&\le C^2\|\chi\|_{1,\infty}^2\|\alpha^{1/2}v_h\|_{0,\Omega}^2/L^2 \\
&\le C^2C_p^2\|\chi\|_{1,\infty}^2|||v_h|||_d^2,
\end{aligned}$$

which proves the first inequality.

In a similar way, by the fact that $\overline{\rho} + b_0$ is a piece wise constant function and Lemma 4.3.4, we have that

$$\|(\overline{\rho}+b_0)^{1/2}(\varphi v_h - \widetilde{\varphi v_h}\|_{p,2,\Omega} \le Ch^{1-p}\|\chi\|_{1,\infty}\|(\overline{\rho}+b_0)^{1/2}v_h\|_{0,\Omega}/L, \quad p = 0, 1.$$

Together with the inequality that

$$|c_e| \le \|\boldsymbol{\beta}\|_{0,\infty} \le= b_0 L, \ \forall e \in \mathcal{E}$$

and the fact that $h/L \le 1$, we obtain that

$$\begin{aligned}
|||\widetilde{\varphi v_h} - \varphi v_h|||_c &= \left(\|(\overline{\rho}+b_0)^{1/2}(\widetilde{\varphi v_h} - \varphi v_h)\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}}\|c_e^{1/2}[\![\widetilde{\varphi v_h} - \varphi v_h]\!]\|_{0,e}^2\right)^{1/2} \\
&\le \left(C^2\frac{h^2}{L^2}\|\chi\|_{1,\infty}^2\|(\overline{\rho}+b_0)^{1/2}v_h\|_{0,\Omega}^2 + b_0LC^2\frac{h}{L^2}\|\chi\|_{1,\infty}^2\|v_h\|_{0,\Omega}^2\right)^{1/2} \\
&\le C\left(\frac{h}{L}\right)^{1/2}\|\chi\|_{1,\infty}\|(\overline{\rho}+b_0)^{1/2}v_h\|_{0,\Omega},
\end{aligned}$$

which proves the second inequality and, hence, completes the proof of the lemma. ■

Lemma 4.3.6 *Under the same hypotheses of Lemma 4.3.3, for any* $v_h \in \mathcal{U}_h^k$*, there exist constants* χ_4 *and* χ_5 *independent of* $\mathcal{K}$*, such that*

$$a_d(v_h, \varphi v_h - \widetilde{\varphi v_h}) \leq \chi_4 |||v_h|||_d^2 \tag{5.15a}$$

and that

$$a_c(v_h, \varphi v_h - \widetilde{\varphi v_h}) \leq \chi_5 (h/L)^{1/2} |||v_h|||_c^2. \tag{5.15b}$$

Proof By the definition of $a_{d,\theta}$ in (4.1.6), the Cauchy-Schwarz inequality, the assumption that $\gamma_\theta \geq \gamma_0 > \max 9C_g^2, 1$, Lemma 4.3.1, and Lemma 4.3.4, we have that

$$\begin{aligned}
& a_{d,\theta}(v_h, \widetilde{\varphi v_h} - \varphi v_h) \\
= \;& (\alpha \nabla_h v_h, \nabla_h(\widetilde{\varphi v_h} - \varphi v_h)) + \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \gamma_\theta h_e^{-1} W_e [\![v_h]\!] [\![\widetilde{\varphi v_h} - \varphi v_h]\!] ds \\
& - \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \{\alpha \nabla_h v_h \cdot n_e\}_w [\![\widetilde{\varphi v_h} - \varphi v_h]\!] ds + \theta \sum_{e \in \mathcal{E}_I \cup \mathcal{E}_D} \int_e \{\alpha \nabla_h(\widetilde{\varphi v_h} - \varphi v_h) \cdot n_e\}_w [\![v_h]\!] ds \\
\leq \;& \gamma_\theta |||v_h|||_d |||\widetilde{\varphi v_h} - \varphi v_h|||_d + C_g \|\alpha^{1/2} \nabla v_h\|_{0,\Omega} \|\widetilde{\varphi v_h} - \varphi v_h\|_j + C \frac{\|\chi\|_{1,\infty}}{L} \|\alpha^{1/2} v_h\|_{0,\Omega} \|v_h\|_j \\
\leq \;& (\gamma_\theta + C_g + C C_p \|\chi\|_{1,\infty,\Omega}) \, |||v_h|||_d |||\widetilde{\varphi v_h} - \varphi v_h|||_d.
\end{aligned}$$

This proves the validity of (5.15a) with $\chi_4 = \gamma_\theta + C_g + C C_p \|\chi\|_{1,\infty,\Omega}$, independent of $\mathcal{K}$.

Rewriting the advection - reaction part by integration by parts and using (3.2.2) give that, for any $u, v \in V^{1+\epsilon}(\mathcal{T}_h)$,

$$\begin{aligned}
a_c(u, v) &= (u, \gamma v) + (\nabla_h(u\boldsymbol{\beta}), v) - \sum_{e \in \mathcal{E}} \int_e \beta_e [\![uv]\!] + \sum_{e \in \mathcal{E}_I} \int_e \beta_e \{u\}_{up} [\![v]\!] + \sum_{e \in \Gamma^+} \int_e \beta_e uv \\
&= (u, (\gamma + \nabla \cdot \boldsymbol{\beta}) v) + (\boldsymbol{\beta} \cdot \nabla_h u, v) - \sum_{e \in \mathcal{E}_I} \int_e \beta_e \{v\}^{up} [\![u]\!] - \sum_{e \in \Gamma^-} \int_e \beta_e uv \\
&= (u, (\gamma + \nabla \cdot \boldsymbol{\beta}) v) + (\boldsymbol{\beta} \cdot \nabla_h u, v) + \sum_{e \in \mathcal{E}_I} \int_e c_e [\![u]\!] [\![v]\!] - \sum_{e \in \Gamma^- \cup \mathcal{E}_I} \int_e \beta_e [\![u]\!] \{v\}.
\end{aligned}$$

Let $P\boldsymbol{\beta}$ be the L_2 projection of $\boldsymbol{\beta}$ onto $\mathcal{U}_h^0$, i.e., the space of piece wise constant with respect to $\mathcal{T}_h$ with the following approximation property holds:

$$\|\boldsymbol{\beta} - P\boldsymbol{\beta}\|_{0,\infty,\Omega} \leq C h |\boldsymbol{\beta}|_{1,\infty,\Omega}. \tag{4.3.11}$$

Since $P\boldsymbol{\beta}\cdot\nabla_h v_h \in \mathcal{U}_h^k$, the definition of $\widetilde{\varphi v_h}$ gives that

$$\int_\Omega P\boldsymbol{\beta}\cdot\nabla_h v_h(\varphi v_h - \widetilde{\varphi v_h}) = 0.$$

Combining the identities gives that

$$\begin{aligned}
&a_c(v_h, \widetilde{\varphi v_h} - \varphi v_h)\\
= \;& \int_\Omega (\gamma+\nabla\cdot\boldsymbol{\beta}) v_h(\widetilde{\varphi v_h} - \varphi v_h) + \int_\Omega (\widetilde{\varphi v_h} - \varphi v_h)(\boldsymbol{\beta} - P\boldsymbol{\beta})\cdot\nabla_h v_h\\
&+\sum_{e\in\mathcal{E}_I}\int_e c_e[\![v_h]\!][\![\widetilde{\varphi v_h} - \varphi v_h]\!] - \sum_{e\in\mathcal{E}_{\Gamma^-}\cup\mathcal{E}_I}\int_e \beta_e[\![v_h]\!]\{\widetilde{\varphi v_h} - \varphi v_h\}\\
:= \;& I + II + III + IV.
\end{aligned}$$

It follows from (4.0.6), (4.0.7) and Lemma 4.3.4 that

$$\begin{aligned}
I &= \int_\Omega \rho v_h(\widetilde{\varphi v_h} - \varphi v_h) + \frac{1}{2}\int_\Omega \nabla\cdot\boldsymbol{\beta} v_h(\widetilde{\varphi v_h} - \varphi v_h)\\
&\le c_\rho\|(\overline{\rho}+b_0)^{1/2}v_h\|_\Omega\|(\overline{\rho}+b_0)^{1/2}(\widetilde{\varphi v_h} - \varphi v_h)\|_\Omega + \frac{b_0}{2c_{\boldsymbol{\beta}}}\|v_h\|_\Omega\|\widetilde{\varphi v_h} - \varphi v_h\|_\Omega\\
&\le (c_\rho + \frac{1}{2c_\beta})C\frac{h}{L}\|\chi\|_{1,\infty}\|(\overline{\rho}+b_0)^{1/2}v_h\|_{0,\Omega}^2.
\end{aligned}$$

Using (4.3.11), (4.0.7), Lemma 4.3.4 and the inverse inequality gives that

$$II \le Ch|\boldsymbol{\beta}|_{1,\infty}\|\nabla_h v_h\|\frac{h}{L}\|\chi\|_{1,\infty}\|v_h\| \le C\frac{h}{L}\frac{b_0}{c_{\boldsymbol{\beta}}}\|\chi\|_{1,\infty,\Omega}\|v_h\|_{0,\Omega}^2.$$

By (4.0.4), Lemma 4.3.4 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
III + IV &\le C\left(\sum_{e\in\mathcal{E}}\|c_e^{1/2}[\![v_h]\!]\|_{0,e}\right)\left(\frac{h^{1/2}}{L}\|\boldsymbol{\beta}\|_{0,\infty}^{1/2}\|\chi\|_{1,\infty}\|v_h\|\right)\\
&\le C\left(\frac{h}{L}\right)^{1/2}\|\chi\|_{1,\infty}\left(\sum_{e\in\mathcal{E}}\|c_e^{1/2}[\![v_h]\!]\|_{0,e}^2 + b_0\|v_h\|_{0,\Omega}^2\right).
\end{aligned}$$

Together with the fact that $h/L < 1$, we obtain that

$$a_c(v_h, \widetilde{\varphi v_h} - \varphi v_h) \le (1 + c_\rho + \frac{2}{c_\beta})C\|\chi\|_{k+1,\infty,\Omega}\left(\frac{h}{L}\right)^{1/2}|\!|\!|v_h|\!|\!|_c^2,$$

which completes the proof with $\chi_5 = (1 + c_\rho + \frac{2}{c_\beta})C\|\chi\|_{k+1,\infty,\Omega}$. ■

Next theorem gives the stability of the variational form.

Theorem 4.3.7 *Under the hypotheses of* Lemma 4.3.3*, there exist positive constants* a_0 *and* h_0 *such that for all* $h < h_0$ *and* $v_h \in \mathcal{U}_h^k$*,*

$$\sup_{w_h \in \mathcal{U}_h^k} \frac{a_\theta(v_h, w_h)}{\|w_h\|_{DG}} \geq a_0 \|v_h\|_{DG}. \tag{4.3.12}$$

Proof For any $v_h \in \mathcal{U}_h^k$, let $w_h = \widetilde{\varphi v_h} \in \mathcal{U}_h^k$ be the L_2 projection of φv_h onto $\mathcal{U}_h^k$.

First it follows from the triangle inequality, Lemma 4.3.3 and Lemma 4.3.5 that

$$\|\widetilde{\varphi v_h}\|_{DG} \leq (\|\widetilde{\varphi v_h} - \varphi v_h\|_{DG} + \|\varphi v_h\|_{DG}) \leq C\|v_h\|_{DG}.$$

To show the validity of (4.3.12), it suffices to show that

$$a_\theta(v_h, w_h) \geq C\|v_h\|_{DG}^2. \tag{4.3.13}$$

To this end, by Lemma 4.3.3 and Lemma 4.3.6, we have that

$$\begin{aligned} a_{d,\theta}(v_h, \widetilde{\varphi v_h}) &= a_{d,\theta}(v_h, \widetilde{\varphi v_h} - \varphi v_h) + a_{d,\theta}(v_h, \varphi v_h) \\ &\geq \left(\frac{\chi_1 + \mathcal{K}}{6} - \chi_4\right) \|v_h\|_d^2. \end{aligned}$$

Note that in Lemma 4.3.6, the constant χ_4 is independent of $\mathcal{K}$, so we can choose $\mathcal{K}$ such that $\chi_1 + \mathcal{K}$ is bigger that $12\chi_4$. Then it follows that

$$a_{d,\theta}(v_h, \widetilde{\varphi v_h}) \geq \chi_4 \|v_h\|_d^2.$$

And in a similar way, then for $h < h_0$ we have that

$$a_c(v_h, \widetilde{\varphi v_h}) \geq c\|v_h\|_c^2,$$

with c only depending on χ_1 and χ_5.

Combining the two inequalities gives (4.3.13) and, hence, completes the proof of the theorem. ∎

4.4 A priori error estimate

In this section, we establish the *a priori* error estimate in the norm (4.2.4) for the discontinuous finite element methods presented.

Let P be the $L2$-projection in $\mathcal{U}_h^k$. The standard approximation argument in [20, 21] gives that: for $u \in V^{1+\epsilon}(\mathcal{T}_h) \cap H^{1+s}(\mathcal{T}_h)$ with $\epsilon \leq s \leq 1$,

$$\|\alpha^{1/2}\nabla(u - Pu)\|_{\epsilon,\Omega} \leq C \left(\sum_{K\in\mathcal{T}_h} h_K^{2(s-\epsilon)} \|\alpha^{1/2}\nabla u\|_{s,K}^2 \right)^{1/2}, \tag{4.4.1}$$

$$\|u - Pu\|_{r,p,K} \leq Ch^{s+1-r}|u|_{s+1,p,K}, \quad r = 0, 1,\ 1 \leq p \leq \infty,\ K \in \mathcal{T}_h. \tag{4.4.2}$$

Together with the trace inequality, the following estimate holds:

$$\|u - Pu\|_{0,e} \ \leq \ Ch_{K_e}^{s+1/2}|u|_{s+1,K_e},\ \forall e \in \mathcal{E}. \tag{4.4.3}$$

Let f_k be the L_2 projection of f onto $\mathcal{U}_h^k$, define

$$osc(f, K) \ := \ \frac{h_K}{\sqrt{\alpha_K}}\|f - f_{k-1}\|_{0,K}$$

and

$$asc(f) \ := \ \left(\sum_{K\in\mathcal{T}_h} osc(f, K)^2 \right)^{1/2}.$$

Remark 4.4.1 *The symbol $\lesssim$ used in this section denotes lower than or equal, up to a positive constant depending only on the triangulation $\mathcal{T}_h$, the domain Ω , the polynomial degree k, independent of the coefficients of the problem and h.*

The next lemma proved in [22] gives a trace inequality of functions with low regularities.

Lemma 4.4.2 *For any $K \in \mathcal{T}_h$, assume that $v \in V^{1+s}(K)$ and $w_h \in P_k(K)$, then the following trace inequality holds:*

$$\int_e (\nabla v \cdot \boldsymbol{n}) w_h ds \ \lesssim \ h_e^{-1/2}\|w_h\|_{0,e}(\|\nabla v\|_{0,K} + h_K\|\triangle v\|_{0,K}).$$

Lemma 4.4.3 *Let $u \in V^{1+s}(\mathcal{T}_h) \bigcap H^{1+\epsilon}(\Omega)$ be the solution of* (4.0.1) *with boundary conditions* (4.0.2). *Let $v \in \mathcal{U}_h^k$, and set $\xi = u - v$. Then on any $K \in \mathcal{T}_h$, the following estimate holds:*

$$h_K \|\alpha^{1/2} \triangle \xi\|_{0,K} \ \lesssim \ \|\alpha^{1/2} \nabla \xi\|_{0,K} + \frac{h_K}{\sqrt{\alpha}} \|\nabla \cdot (\boldsymbol{\beta} \xi) + \gamma \xi\|_{0,K} + osc(f, K).$$

Proof For any $K \in \mathcal{T}_h$, define

$$r_K \ = \ -\nabla \cdot (\alpha \nabla v) + \nabla \cdot (\boldsymbol{\beta} v) + \gamma v - f_{k-1}.$$

It follows that

$$\begin{aligned}
h_K \|\alpha^{1/2} \triangle \xi\|_{0,K} &= h_K \alpha^{-1/2} \|\nabla \cdot (\alpha \nabla u) - \nabla \cdot (\alpha \nabla v)\|_{0,K} \\
&= h_K \alpha^{-1/2} \|\nabla \cdot (\boldsymbol{\beta} u) + \gamma u - f - \nabla \cdot (\alpha \nabla v)\|_{0,K} \\
&= h_K \alpha^{-1/2} \|r_K + f_{k-1} - f + \nabla \cdot (\boldsymbol{\beta} \xi) + \gamma \xi\|_{0,K} \\
&\leq h_K \alpha^{-1/2} \left(\|r_K\|_{0,K} + \|\nabla \cdot (\boldsymbol{\beta} \xi) + \gamma \xi\|_{0,K}\right) + osc(f, K).
\end{aligned}$$

Let ψ_K be the local interior bubble function on K, then we have

$$\begin{aligned}
\|r_K\|_{0,K}^2 &\lesssim \int_K \left(-\nabla \cdot (\alpha \nabla v) + \nabla \cdot (\boldsymbol{\beta} v) + \gamma v - f_{k-1}\right) r_K \psi_K \\
&= \int_K (\nabla \cdot (\alpha \nabla \xi) - \nabla \cdot (\boldsymbol{\beta} \xi) - \gamma \xi + f - f_{k-1}) r_K \psi_K \\
&= \left(-\int_K \alpha \nabla \xi \nabla (r_K \psi_K) + \int_K (f - f_{k-1} - \nabla \cdot (\boldsymbol{\beta} \xi) - \gamma \xi) r_K \psi_K\right) \\
&\lesssim \left(\|\alpha \nabla \xi\| |r_K \psi_K|_1 + (\|\nabla \cdot (\boldsymbol{\beta} \xi) + \gamma \xi\| + \|f - f_{k-1}\|\right) \|r_K \psi_K\| \\
&\lesssim \left(h_K^{-1} \|\alpha \nabla \xi\| + \|\nabla \cdot (\boldsymbol{\beta} \xi) + \gamma \xi\| + \|f - f_{k-1}\|\right) \|r_K\|_{0,K}.
\end{aligned}$$

It follows that

$$\|r_K\|_{0,K} \ \lesssim \ h_K^{-1} \|\alpha \nabla \xi\|_{0,K} + \|\nabla \cdot (\boldsymbol{\beta} \xi) + \gamma \xi\|_{0,K} + \|f - f_{k-1}\|_{0,K}.$$

Finally we obtain

$$h_K\|\alpha^{1/2}\triangle\xi\|_{0,K} \lesssim \|\alpha^{1/2}\nabla\xi\|_{0,K}+\frac{h_K}{\sqrt{\alpha}}\|\nabla\cdot(\boldsymbol{\beta}\xi)+\gamma\xi\|_{0,K}+osc(f,K).$$

■

Theorem 4.4.4 *Let $u \in V^{1+s}(\mathcal{T}_h)\bigcap H^{1+\epsilon}(\Omega)$ be the solution of (4.0.1) with boundary conditions (4.0.2), and $u|_K\in H^{1+s_K}$ be the restriction on $K \in \mathcal{T}_h$. Let u_h be the solution of discrete problem* (4.2.1)*. There exists a positive constant C, depending on the domain, the triangulation $\mathcal{T}_h$ and the polynomial degree (but independent of mesh size h and the coefficients of the problem), such that*

$$\begin{aligned}\|\!|u-u_h|\!\|_{DG} \leq\ & C\sum_{K\in\mathcal{T}_h} h_K^{s_K}|u|_{1+s_K,K}\Big(\alpha_K^{1/2}+h_K^{1/2}\|\boldsymbol{\beta}\|_{0,\infty,\Omega}^{1/2}+h_K\|\rho\|_{0,\infty,\Omega}^{1/2}\\ & +h_K^2\alpha_K^{-1/2}\|\rho\|_{0,\infty,\Omega}+h_K\alpha_K^{-1/2}\|\boldsymbol{\beta}\|_{0,\infty,\Omega}\Big)+osc(f).\end{aligned}$$

Proof Define

$$E=u-Pu \quad \text{and} \quad E_h=u_h-Pu.$$

It follows from Theorem 4.3.7 and the error equation that

$$a_0\|\!|E_h|\!\|_{DG} \leq \frac{a_\theta(E_h,v_h)}{\|\!|v_h|\!\|_{DG}}=\frac{a_\theta(E,v_h)}{\|\!|v_h|\!\|_{DG}}.$$

First consider the diffusion part. The definition of $a_{d,\theta}$ in (4.1.6)gives that

$$\begin{aligned}a_{d,\theta}(E,v_h) =\ & (\alpha\nabla_h E,\nabla_h v_h)+\theta\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\{\alpha\nabla v_h\cdot\boldsymbol{n}_e\}_w[\![E]\!]\\ & -\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\{\alpha\nabla E\cdot\boldsymbol{n}_e\}_w[\![v_h]\!]+\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\gamma_\theta h_e^{-1}W_e[\![E]\!][\![v_h]\!]\\ :=\ & I1+I2+I3+I4\end{aligned}$$

It follows easily from Lemma 4.3.1 and the Cauchy-Schwarz inequality that

$$I1+I2+I4\lesssim \|\!|E|\!\|_d\|\!|v_h|\!\|_d.$$

Using Lemma 4.4.2, Lemma 4.4.3 and the Cauchy-Schwarz inequality gives that

$$\begin{aligned} I3 &\le \sum_{e\in\mathcal{E}_I\bigcup\mathcal{E}_D} h_e^{-1/2}W_e^{1/2}\|[\![v_h]\!]\|_{0,e}\sum_{\omega=+,-}\Big(\|\alpha^{1/2}\nabla E\|_{0,K^\omega}+h_{K^\omega}\|\alpha^{1/2}\triangle E\|_{0,K^\omega}\Big)\\ &\le \|v_h\|_j\Big(\|\alpha^{1/2}\nabla_h E\|_{0,\Omega}+\sum_{K\in\mathcal{T}_h} h_K\|\alpha_K^{1/2}\triangle E\|_{0,K}\Big)\\ &\le \|v_h\|_d\Big(|\!|\!|E|\!|\!|_d+\sum_{K\in\mathcal{T}_h}\frac{h_K}{\sqrt{\alpha_K}}\|\nabla\cdot(\boldsymbol{\beta}E)+\gamma E\|_{0,K}+osc(f)\Big). \end{aligned}$$

Summing up all the terms gives that

$$a_{d,\theta}(E,v_h)\lesssim |\!|\!|v_h|\!|\!|\Big(|\!|\!|E|\!|\!|_d+\sum_{K\in\mathcal{T}_h}\frac{h_K}{\sqrt{\alpha_K}}\|\nabla\cdot(\boldsymbol{\beta}E)+\gamma E\|_{0,K}+osc(f)\Big).$$

It follows from (4.0.5), (4.0.7), (4.4.1)-(4.4.3) and the fact that $h/L<1$ that

$$\begin{aligned} \|\nabla\cdot(\boldsymbol{\beta}E)+\gamma E\|_{0,K} &= \|\rho E+E\nabla\cdot\boldsymbol{\beta}/2+\boldsymbol{\beta}\cdot\nabla E\|_{0,K}\\ &\lesssim (\|\rho\|_{0,\infty,\Omega}+|\boldsymbol{\beta}|_{1,\infty})\|e\|_{0,K}+\|\boldsymbol{\beta}\|_{0,\infty,\Omega}|e|_{1,K}\\ &\lesssim h^{1+s_K}\|\rho\|_{0,\infty,\Omega}|u|_{1+s_K,K}+h^{s_K}\|\boldsymbol{\beta}\|_{0,\infty,\Omega}|u|_{1+s_K,K} \end{aligned}$$

and that

$$|\!|\!|E|\!|\!|_d \lesssim \sum_{K\in\mathcal{T}_h} h_K^{s_K}\alpha_K^{1/2}|u|_{1+s_K,K}.$$

Hence, we obtain that

$$a_{d,\theta}(E,v_h) \lesssim \|v_h\|_d\left(\sum_{K\in\mathcal{T}_h} h_K^{s_K}\alpha_K^{1/2}\left(1+\frac{h_K^2\|\rho\|_{0,\infty,\Omega}}{\alpha_K}+\frac{h_K\|\boldsymbol{\beta}\|_{0,\infty,\Omega}}{\alpha_K}\right)|u|_{1+s_K,K}+osc(f)\right).$$

Next consider the convection-reaction part. It follows from the definition of a_c that

$$a_c(E,v_h) = (E,\,-\boldsymbol{\beta}\cdot\nabla_h v_h+\gamma v_h)+\sum_{e\in\mathcal{E}_I}\int_e\{\beta_e E\}_{up}[\![v_h]\!]+\sum_{e\in\mathcal{E}_{\Gamma+}}\int_e\beta_e E\,v_h.$$

It follows from the definition of the projection and $P\boldsymbol{\beta}\cdot\nabla_h v_h\in\mathcal{U}_h^k$ that

$$\int_\Omega P\boldsymbol{\beta}\cdot\nabla_h v_h E=\int_\Omega P\boldsymbol{\beta}\cdot\nabla_h v_h(u-Pu)=0.$$

Together with (4.0.7), the inverse inequality and (4.4.1) - (4.4.2), it implies that

$$\begin{aligned}
\int_\Omega -\boldsymbol{\beta}\cdot\nabla_h v_h E &= \int_\Omega (P\boldsymbol{\beta}-\boldsymbol{\beta})\cdot\nabla_h v_h E \\
&\lesssim h|\boldsymbol{\beta}|_{1,\infty,\Omega}\|\nabla_h v_h\|_{0,\Omega}\|E\|_{0,\Omega} \\
&\lesssim \|b_0^{1/2}v_h\|_{0,\Omega}\|b_0^{1/2}E\|_{0,\Omega} \\
&\lesssim |\!|\!|v_h|\!|\!|_c \sum_{K\in\mathcal{T}_h} h_K^{1+s_K}\|\boldsymbol{\beta}\|_{0,\infty,\Omega}^{1/2}|u|_{1+s_K,K}.
\end{aligned}$$

Applying $\gamma=\rho-\frac{1}{2}\nabla\cdot\boldsymbol{\beta}$, (4.0.6), (4.0.7) and (4.4.1)-(4.4.3) gives that

$$\begin{aligned}
(E,\gamma v_h) &= \int_\Omega (\rho-\frac{1}{2}\nabla\cdot\boldsymbol{\beta})\,E\,v_h \\
&\lesssim c_\rho\|E\|_{0,\Omega}\|(\overline{\rho}+b_0)v_h\|_{0,\Omega}+\frac{b_0}{c_\beta}\|E\|_{0,\Omega}\|v_h\|_{0,\Omega} \\
&\lesssim |\!|\!|v_h|\!|\!|_c \sum_{K\in\mathcal{T}_h} (\|\overline{\rho}\|_{0,\Omega}+\|\boldsymbol{\beta}\|_{0,\infty,\Omega})^{1/2}\, h_K^{1+s_K}|u|_{1+s_K,K}
\end{aligned}$$

and

$$\sum_{e\in\mathcal{E}_I}\int_e \{\beta_e E\}_{up}[\![v_h]\!] + \sum_{e\in\mathcal{E}_{\Gamma+}}\int_e \beta_e E v_h \lesssim |\!|\!|v_h|\!|\!|_c \sum_{K\in\mathcal{T}_h} h_K^{1/2+s_K}\|\boldsymbol{\beta}\|_{0,\infty,\Omega}^{1/2}|u|_{1+s_K,K}.$$

Summing up the three parts gives that

$$a_c(E,v_h) \lesssim |\!|\!|v_h|\!|\!|_c \sum_{K\in\mathcal{T}_h} h_K^{1/2+s_K}\left(\|\boldsymbol{\beta}\|_{0,\infty,\Omega}^{1/2}+h_K^{1/2}\|\rho\|_{0,\infty,\Omega}^{1/2}\right)|u|_{1+s_K,K}.$$

Collecting the diffusion and convection-reaction parts implies that

$$\begin{aligned}
a_\theta(E,v_h) \lesssim |\!|\!|v_h|\!|\!|_{DG}\Bigg(&\sum_{K\in\mathcal{T}_h} h_K^{s_K}|u|_{1+s_K,K}\Big(\alpha_K^{1/2}+h_K^{1/2}\|\boldsymbol{\beta}\|_{0,\infty,\Omega}^{1/2}+h_K\|\rho\|_{0,\infty,\Omega}^{1/2} \\
&+h_K^2\alpha_K^{-1/2}\|\rho\|_{0,\infty,\Omega}+h_K\alpha_K^{-1/2}\|\boldsymbol{\beta}\|_{0,\infty,\Omega}\Big)+osc(f)\Bigg).
\end{aligned}$$

Together with (4.4.4) and the triangle inequality, it implies that

$$\begin{aligned}
|\!|\!|u-u_h|\!|\!|_{DG} \lesssim &\sum_{K\in\mathcal{T}_h} h_K^{s_K}|u|_{1+s_K,K}\Big(\alpha_K^{1/2}+h_K^{1/2}\|\boldsymbol{\beta}\|_{0,\infty,\Omega}^{1/2}+h_K\|\rho\|_{0,\infty,\Omega}^{1/2} \\
&+h_K^2\alpha_K^{-1/2}\|\rho\|_{0,\infty,\Omega}+h_K\alpha_K^{-1/2}\|\boldsymbol{\beta}\|_{0,\infty,\Omega}\Big)+osc(f).
\end{aligned}$$

■

4.5 A new discontinuous Galerkin method

In Chapter 3, we stabilize the diffusion operator by adding the following equation :

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \gamma_\theta h_e^{-1}W_e[\![u]\!][\![v]\!]\,ds=\sum_{e\in\mathcal{E}_D}\gamma_\theta h_e^{-1}W_e\int_e g_D v\,ds,\ \forall\, v\in V^{1+\epsilon}(\mathcal{T}_h).$$

The order h_e^{-1} may lead to the difficulty in the convergence analysis.

Considering this, for any $v\in V^{1+\epsilon}(\mathcal{T}_h)$, denote the tangential derivative along edge e by

$$\gamma_e(\nabla v)=\frac{\partial v}{\partial \boldsymbol{t}}.$$

And for any $v\in V^{1+\epsilon}(\mathcal{T}_h)$, we add the following term to stabilize :

$$\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e \gamma_\theta h_e W_e[\![\gamma_e(\nabla u)]\!][\![\gamma_e(\nabla v)]\!]\,ds=\sum_{e\in\mathcal{E}_D}\gamma_\theta h_e W_e\int_e \gamma_e(\nabla g_D)\gamma_e(\nabla v)\,ds.$$

Now, define the new bilinear form for $u,v\in V^{1+\epsilon}(\mathcal{T}_h)$ by

$$\begin{aligned}\widehat{a}_{d,\theta}(u,v) &= (\alpha\nabla_h u,\nabla_h v)+\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\gamma_\theta h_e W_e[\![\gamma_e(\nabla u)]\!][\![\gamma_e(\nabla v)]\!]\,ds\\ &\quad+\theta\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\{\alpha\nabla v\cdot\boldsymbol{n}_e\}_w[\![u]\!]\,ds-\sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D}\int_e\{\alpha\nabla u\cdot\boldsymbol{n}_e\}_w[\![v]\!]\,ds\end{aligned}$$

for $\theta\in\{-1,\,0,\,1\}$.

And define the new linear form for $v\in V^{1+\epsilon}(\mathcal{T}_h)$ by

$$\begin{aligned}\widehat{f}_\theta(v) &= (f,\,v)+\sum_{e\in\mathcal{E}_D}\gamma_\theta h_e W_e\int_e\gamma_e(\nabla g_D)\gamma_e(\nabla v)\,ds+\sum_{e\in\mathcal{E}_N}\int_e g_N v\,ds\\ &\quad+\theta\sum_{e\in\mathcal{E}_D}\int_e g_D(k\nabla v\cdot\boldsymbol{n}_e)\,ds-\sum_{e\in\mathcal{E}_{D-}}\int_e(\boldsymbol{\beta}\cdot\boldsymbol{n}_e)g_D\,v\,ds.\end{aligned}$$

The new variational formulation is to find $\widehat{u}\in V^{1+\epsilon}(\mathcal{T}_h)$ such that

$$\widehat{a}_\theta(\widehat{u},v)\equiv\widehat{a}_{d,\theta}(\widehat{u},\,v)+a_c(\widehat{u},v)=\widehat{f}_\theta(v),\quad\forall v\in V^{1+\epsilon}(\mathcal{T}_h).$$

To discretilize the problem, modify the DG finite element space associated with the triangulation $\mathcal{T}_h$ as

$$\widehat{\mathcal{U}}_h^k=\{v\in L^2(\Omega)\,:\,v|_K\in P_k(K),\ \forall K\in\mathcal{T}_h \text{ and } [\![\overline{v}]\!]_e=0,\ \forall e\in\mathcal{E}_I\},$$

where $\overline{v}_e = \frac{1}{|e|}\int_e v\,ds$ is the average of v on e.

The new DG finite element method is to find $\widehat{u}_h \in \widehat{\mathcal{U}}_h^k$ such that

$$\widehat{a}_\theta(\widehat{u}_h,\, v) = \widehat{f}_\theta(v), \quad \forall\; v \in \widehat{\mathcal{U}}_h^k.$$

For any $v \in \widehat{\mathcal{U}}_h^k$, define the norm for the modified DG space by

$$|\!|\!| v |\!|\!|_{dg}^2 \;\; = \;\; \|\alpha^{1/2}\nabla_h v\|_{0,\Omega}^2 + \|v\|_{dj}^2 + |\!|\!| v |\!|\!|_c^2,$$

where

$$\|v\|_{dj}^2 \;\; := \sum_{e\in\mathcal{E}_I\cup\mathcal{E}_D} h_e W_e \|[\![\gamma_e(\nabla v)]\!]\|_{0,e}^2.$$

The following lemma implies the equivalence between $\|[\![u]\!]\|$ and $h_e\|[\![\gamma_e(\nabla u)]\!]\|$ in the DG finite element space.

Lemma 4.5.1 *For any $v \in \widehat{\mathcal{U}_h^k}$ and any $e \in \mathcal{E}_I$, $\|[\![v]\!]\|_{0,e}$ and $h_e\|[\![\gamma_e(\nabla u)]\!]\|$ are equivalent, i.e, there exist positive constants c_m and c_M such that*

$$c_m\|[\![v]\!]\|_{0,e} \le h_e\|[\![\gamma_e(\nabla u)]\!]\| \le c_M\|[\![v]\!]\|_{0,e}.$$

Proof By a scaling argument, it suffices to prove that $\|[\![\gamma_e(\nabla v)]\!]\|= 0$ implies that $v \equiv 0$ on e. It follows that

$$[\![\gamma_e(\nabla v]\!]_e = [\![\frac{\partial v}{\partial \boldsymbol{t}_e}]\!]_e = \frac{\partial}{\partial \boldsymbol{t}_e}[\![v]\!]_e = 0.$$

Hence, $[\![v]\!]_e$ is a constant, which implies that

$$[\![v]\!]_e = \overline{[\![v]\!]_e} = \frac{1}{|e|}\int_e [\![v]\!]_e\,ds = [\![\overline{v}]\!]_e = 0.$$

This completes the proof of the lemma. ■

Corollary 4.5.2 *For any $v \in \widehat{\mathcal{U}}_h^k$, $a_{d,\theta}(v,v)$ and $\widehat{a}_{d,\theta}(v,v)$ are equivalent.*

5. CONCLUSION

In conclusion, this thesis discussed the error estimates in finite element methods for two typical kinds of non-smooth elliptic problems. Chapter 1 introduced some problems of low regularity. Chapter 2 discussed the a priori error estimate for elliptic equations with non-smooth boundary data. Chapter 3 introduced the discontinuous Galerkin methods, and Chapter 4 discussed the stability of the discontinuous Galerkin methods, and also the a priori error estimates for this kinds of problems.

The main part of the thesis is about the a priori error estimates. The a posterior error estimate also plays an important role in the adaptive finite element methods. For the a posterior error estimate, the low regularity may lead the difficulty in the analysis of the robustness of the the error estimates. For non-smooth boundary data problem, if we consider the adaptive finite element methods, we need a local indicator and a global error estimate. The indicator and error estimate may depend on the regularization process, which means they depend on ϵ. When ϵ is very small, the problem may have boundary layers like the singularly perturbed problems. So the robustness analysis of the error estimates is essential and also may be the main challenge. The non-smooth coefficients problem may face the similar situation. Since the coefficients are piece-wise constant, it may have the interior layers. And in the thesis, we only consider about the diffusion coefficients to be non-smooth. In some applications, the advection coefficients may be also non-smooth, which is also an interesting topic.

REFERENCES

REFERENCES

[1] S. Brenner and R. Scott, *The mathematical theory of finite element methods*, Springer-Verlag, New York, 2008.

[2] P. G. Ciarlet, *The finite element method for elliptic problems*, SIAM, Philadelphia, 2002.

[3] Z. Cai and Y. Wang, *An error estimate for two-dimensional Stokes driven cavity flow*, Math. Comp., 78 (2009), 771–787.

[4] R. Adams, and J. Fournier, *Sobolev Spaces*, Elsevier, Amsterdam, 2003.

[5] C. Kenig, *Harmonic analysis techniques for second order Elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics No. 83, AMS, Providence, R.I., 1994.

[6] E. Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136 (2012), 521–573.

[7] Q. Lin and J. Lin, *Finite Element Methods: Accuracy and Improvement*, Science Press, Beijing, 2006.

[8] T. Apel, S. Nicaise and J. Pfefferer, *Adapted Numerical Methods for the Poisson Equation with L^2 Boundary Data in NonConvex Domains*, SIAM J. Numer. Anal., 55 (2017), 1937–1957.

[9] T. Apel, S. Nicaise and J. Pfefferer, *Discretization of the Poisson equation with non-smooth data and emphasis on non-convex domains*, Numer. Methods Partial Differential Eq., 32(2016), 1433–1454.

[10] K. Deckelnick, A. Günther, and M. Hinze, *Finite element approximation of Dirichlet boundary control for elliptic PDEs on two- and three-dimensional curved domains*, SIAM J. Control Optim., 48(2009), 2798–2819.

[11] S. May, R. Rannacher, and B. Vexler, *Error analysis for a finite element approximation of elliptic Dirichlet boundary control problems*, SIAM J. Control Optim., 51(2013), 2585–2611.

[12] B. Ayuso and L.D. Marini , *Discontinuous Galerkin methods for advection-diffusion-reaction problems*, SIAM J. Numer. Anal., 47(2009), 1391–1420.

[13] A. Ern, A.F. Stephansen and P. Zunino, *A discontinuous Galerkin method with weighted averages for advection-diffusion equations with locally small and anisotropic diffusivity*, IMA J. Numer. Anal., 29(2009), 235–256.

[14] P. Grisvard, *Elliptic problems in nonsmooth domains*, SIAM, Philadelphia, 2011.

[15] F. Brezzi, B. Cockburn, L.D. Marini and E. Söli, *Stabilization mechanisms in discontinuous galerkin finite element methods*, Comut. Meth. Appl. Mech. Eng., 195 (25–28) (2006), 3293–3310.

[16] S.C. Brenner, *Poincarë–Friedrichs Inequalities for Piecewise H^1 Functions*, SIAM J. Numer. Anal., 41(2003), 306–324.

[17] P. Bochev, T. J. R. Hughes and G. Scovazzi, *A multiscale discontinuous Galerkin method*, Lect. Note Comput. Sci., 3743(2006), 84–93.

[18] J. A. Nitsche and A. H. Schatz, *On local approximation properties of L^2 projection on spline subspaces*, Appl. Anal. 2(1972), 161–168.

[19] L. Wahlbin, *Superconvergence in Galerkin finite element methods*, Springer-Verlag, New York, 2006.

[20] C. Bernardi, R. Verfürth, *Adaptive finite element methods for elliptic equations with non-smooth coefficients*, Numer. Math., 85(2000), 579–608.

[21] Z. Cai, X. Ye and S. Zhang, *Discontinuous Galerkin finite element methods for interface problems: a priori and a posteriori error estimations*, SIAM J. Numer. Anal., 49(2011), 1761–1787.

[22] Z. Cai, C. He and S. Zhang, *Discontinuous finite element methods for interface problems: Robust A Priori and A Posteriori error estimates*, SIAM J. Numer. Anal., 55(2017), 400–418.

[23] F. Brezzi, L.D. Marini and E. Süli,*Discontinuous Galerkin methods for first-order hyperbolic problems*, Math. Models Methods Appl. Sci., 14(2004), 1893–1903.

[24] A. Devinatz, R. Ellis and A. Friedman,*The asymptotic behavior of the first real eigenvalue of second order elliptic operators with a small parameter in the highest derivatives, II*, Indiana Univ. Math. J., 23(1974), 991–1011.

[25] M. Mascagni and N. A. Simonov, *Monte Carlo methods for calculating some physical properties of large molecules*, SIAM J. Sci. Comput., 26(1):339–357(electronic), 2004.

[26] J.M. Ramirez, E.A. Thomann, E.C. Waymire, Haggerty R., and Wood B. A, *generalized Taylor-Aris formula and Skew Diffusion*, Multiscale Model. Simul., 5(3):786–801, 2006.

[27] O. Ovaskainen and S. J. Cornell, *Biased movement at a boundary and conditional occupancy times for diffusion processes*, J. Appl. Probab., 40(3):557–580, 2003.

[28] M. Zhang, *Calculation of diffusive shock acceleration of charged particles by skew Brownian motion*, Astrophys. J., 541:428–435, 2000.

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- 3. Z. Cai and J. Yang, *The a priori error estimates of discontinuous Galerkin methods for advection-diffusion-reaction problems with low regularity*, to be submitted.
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