CONTEXTUALITY AND NONCONTEXTUALITY IN HUMAN CHOICE BEHAVIOR

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To the people in my life.

Sandra, Gabriel y Sebastián.

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ABSTRACT

Cervantes Botero, Víctor Hernando. PhD, Purdue University, August 2020. Contextuality and Noncontextuality in Human Choice Behavior. Major Professor: Ehtibar N. Dzhafarov.

The Contextuality-by-Default theory describes the contextual effects on random variables: how the identity of random variables changes from one context to another. Direct influences and true contextuality constitute different types of effects of contexts upon sets of random variables. Changes in the distributions of random variables across contexts define direct influences. True contextuality is defined by the impossibility of sewing all the variables of a system of random variables into a particular overall joint distribution. In the absence of direct influences, the theory specializes to the theory of selective influences in psychology and the traditional treatment of contextuality in quantum mechanics. Consistently connected (i.e., with no direct influences) noncontextual systems are the systems with selective influences. However, observable systems of human behavior are seldom consistently connected. Contextuality-by-Default allows one to classify and measure the degree of deviation from or adherence to the pattern of selective influences, both for consistently and inconsistently connected systems.

The papers here included follow the development of the Contextuality-by-Default theory. The theory is presented for cyclic systems of binary random variables, for arbitrary systems of binary random variables, and for systems that include categorical random variables. Although contextuality has been searched for in human behavior since at least the 1990s, I report here the first experiments that have demonstrated contextuality in choice behavior without making the mistake of ignoring the direct influences present in the systems of random variables. A psychophysical experiment

was conducted and then analyzed using the theory for systems of binary random variables. Its results showed no contextuality in a double-detection paradigm, that is, in an experiment in which each participant was asked to make dual conjoint judgments of signal detection for two stimuli at a time. Several crowdsourcing experiments were conducted and analyzed using the theory for cyclic systems of binary random variables. These experiments demonstrate contextuality using a between-subjects experimental design. Among them, the Snow Queen experiment, in which each participant made two conjoint choices in accordance with a simple story line, provided a methodological template (used afterward to design the other crowdsourcing experiments) for systematically exploring contextuality. Lastly, another psychophysical experiment was conducted and then analyzed using the theory for systems with categorical random variables. This one is the first experiment that demonstrates contextuality in a within-subject design.

In addition to the experimental work reported in these papers, I also present the development of the Contextuality-by-Default theory from the theory for cyclic systems to the theory for systems with categorical random variables. The nominal dominance theorem, which states a necessary condition for noncontextuality of systems where all dichotomizations of categorical variables are considered, is the most relevant theoretical result of this development.

The role that the notion of contextuality can play in psychology is difficult to fully understand at our present stage of knowledge. Most obviously, contextuality analysis is a generalization of the traditional psychological problem of selective influences. It is, in fact, the only existing theoretical tool for classifying and quantifying patterns of deviations from the hypothesis of selective influences. It is less evident whether the degree of (non)contextuality correlates with specific aspects of behavior that may be of interest. Although some such correlations seem to suggest themselves, to be certain and precise in identifying them, we need to expand our knowledge of the degree of (non)contextuality to a broader class of behavioral systems.

1. INTRODUCTION

The Contextuality-by-Default theory, which is the subject of the papers collected in this dissertation, is an extension of the theory of *selective influences*. The latter was introduced by Saul Sternberg (1969) to characterize the situations when a set of factors are identified as influencing a set of responses. The notion of selectivity itself may be traced back at least as far as the work by Donders (1868/1969) on the use of the differences method for assessing the speed of mental processes. There, in the analysis of response times, selectivity of influences would be assumed when an additional mental process was hypothesised to be performed without affecting other mental processes under some experimental manipulations. Thus, differences of response times produced estimates of the duration of the interposed process. The theory of selective influences has been further developed by Townsend (1984), Townsend and Nozawa (1995), Schweickert and Xi (2011), Schweickert, Fisher, and Kyongje (2012), and Dzhafarov (1999, 2001, 2003), among others. Contextuality-by-Default is being developed by Dzhafarov and collaborators, myself among them (Dzhafarov, 2016; Dzhafarov, Cervantes, & Kujala, 2017; Dzhafarov & Kujala, 2013, 2014; Dzhafarov, Kujala, & Cervantes, 2016; Kujala & Dzhafarov, 2016a, 2016b; Kujala, Dzhafarov, & Larsson, 2015). In this dissertation, I include some of the published papers I have co-authored that have made theoretical or experimental contributions to this theory. In the remainder of this chapter, I will briefly recount the development of the Contextuality-by-Default theory during the period since I joined the research team. Along the way, I will mention the papers included as the subsequent chapters and will indicate my specific contributions to each of them. While I participated in all major aspects of these papers, I will only focus on describing the parts for which my contribution was critical.

1.1 Selective influences and contextuality

By 2014, the work on the Contextuality-by-Default theory, as a generalization of the theory of selective influences, was underway. A presentation of the theory of selective influences that already contains some of the core elements of Contextuality-by-Default is found in the chapter by Dzhafarov and Kujala (2017), originally prepared in 2013 and 2014. At its core, the theory develops from the explicit recognition that random variables recorded under mutually exclusive conditions are necessarily different random variables: the different contexts under which they are recorded form an inherent part of their identity. Moreover, any two variables recorded in different contexts do not possess a joint distribution, and they are said to be *stochastically unrelated*. These two characteristics may be easily noticed if one considers, for example, the prognosis of patients suffering of some disease depending on whether they receive a placebo or an effective drug. The distributions of outcomes for the patients are different for the two groups; clearly, they represent different variables. If a patient receives a placebo, then they were not administered the drug, and vice-versa. No joint observation can be made for a patient under both conditions; the variables are not jointly distributed.

The manner by which Contextuality-by-Default formalizes selective influences can be elaborated using the simple diagram of selective influences in Figure 1.1. According to the diagram, factors α and β selectively influence responses A and B, respectively. As a consequence, changes in the levels of α produce changes in the distribution of responses A but not do not affect the distribution for B, and symmetrically for β, B and A. If factors α and β can each be manipulated on two different levels, say α_1, α_2

$$\begin{tikzcd}
\alpha \arrow[d] & \beta \arrow[d] \\
A & B
\end{tikzcd}$$

Fig. 1.1. Diagram of selective influences. Factors α and β selectively influence responses A and B, respectively.

and β_1, β_2, the system of random variables obtained by recording responses A and B under the four treatments generated by the levels of α and β may be represented as in Figure 1.2. For each of the treatments, there are two random variables being recorded together. For instance, under treatment α_1, β_1, we record observations of A that respond to α_1 together with observations of B that respond to β_1. The respective random variables are represented by $R_{\alpha_1}^{\alpha_1,\beta_1}$ and $R_{\beta_1}^{\alpha_1,\beta_1}$. Here, we identify the random variables representing these observations both by what they are responses to (by a subscript) and under which treatment they are recorded (by superscripts). In the figure, the boxes indicate the contexts of random variables recorded under the same treatment; the dashed lines connect the random variables that respond to the same level of the factor that selectively influences them.

In Contextuality-by-Default, the system of random variables in Figure 1.2 is usually represented by a matrix as the one in Figure 1.3. In such a matrix, the set of variables in the same row are the jointly distributed variables recorded in the same *context*; the set of variables in the same column are the stochastically-unrelated-to-each-other random variables responding to the same property or quantity. Within the Contextuality-by-Default theory, the former are usually referred to as *bunches*, or sometimes simply as *contexts*, and the latter are usually referred to as *connections* of random variables. The property to which the variables in the same connection

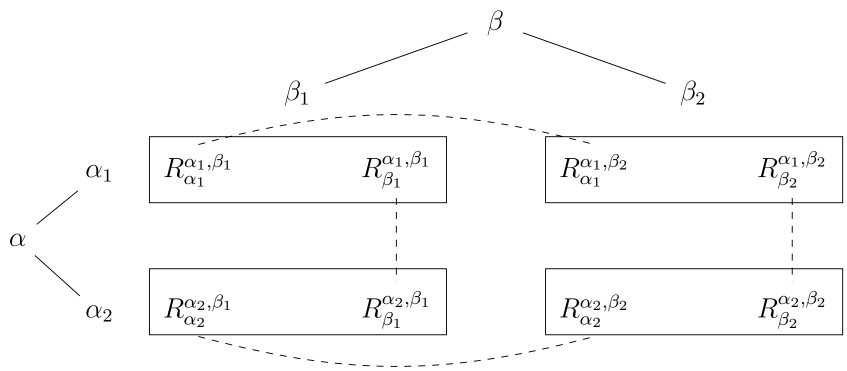

Fig. 1.2. Random variables associated with the diagram of selective influences in Figure 1.1. The boxes indicate the contexts of random variables recorded under the same treatment. The dashed lines represent the connections between random variables recorded at the same level of the factor that selectively influences them.

$R_{\alpha_1}^{\alpha_1,\beta_1}$	$R_{\beta_1}^{\alpha_1,\beta_1}$	$\cdot$	$\cdot$
$\cdot$	$R_{\beta_1}^{\alpha_2,\beta_1}$	$R_{\alpha_2}^{\alpha_2,\beta_1}$	$\cdot$
$\cdot$	$\cdot$	$R_{\alpha_2}^{\alpha_2,\beta_2}$	$R_{\beta_2}^{\alpha_2,\beta_2}$
$R_{\alpha_1}^{\alpha_1,\beta_2}$	$\cdot$	$\cdot$	$R_{\beta_2}^{\alpha_1,\beta_2}$

Fig. 1.3. System of random variables representing the situation from Figure 1.2.

respond to is referred to as their *content* (see for example, Cervantes & Dzhafarov, 2017b; Dzhafarov et al., 2016). In the example in Figure 1.3, the jointly distributed variables $R_{\alpha_1}^{\alpha_1,\beta_1}$ and $R_{\beta_1}^{\alpha_1,\beta_1}$ are recorded in the same context, and $(R_{\alpha_1}^{\alpha_1,\beta_1}, R_{\beta_1}^{\alpha_1,\beta_1})$ constitutes a bunch; the stochastically unrelated variables $R_{\alpha_1}^{\alpha_1,\beta_1}$ and $R_{\alpha_1}^{\alpha_1,\beta_2}$ both respond to the same level α_1 of factor α, and $\{\, R_{\alpha_1}^{\alpha_1,\beta_1}, R_{\alpha_1}^{\alpha_1,\beta_2} \,\}$ is the connection of random variables in the system that respond to content α_1.

Now, it follows from the definition of selective influences that the respective distributions of the random variables sharing a content should be equal to each other. Moreover, the notion of selective influence entails the counterfactual claim that for any given observation of, say, variable $R_{\alpha_1}^{\alpha_1,\beta_1}$, the observation would have been the same if it had been recorded in the context of treatment α_1, β_2 instead of α_1, β_1. An equivalent way to express this counterfactual is to state that if the variables $R_{\alpha_1}^{\alpha_1,\beta_1}$ and $R_{\alpha_1}^{\alpha_1,\beta_2}$ were jointly distributed they should not only share the same distribution, but should also be always equal to each other (Dzhafarov, 2019). This construction of a 'what-if' joint distribution is formalized by the probabilistic notion of *coupling*, and a coupling that always satisfies the equality of its random variables is called *identity coupling*. While this description suffices for the purposes of this introduction, a formal definition is given in Chapter 2 which reproduces the paper by Cervantes and Dzhafarov (2017b); additionally, a thorough treatment of probabilistic couplings can be found in Thorisson (2000). Finally, using the language of couplings and the contextual identification of the random variables in a system, selectivity of influences can be restated by saying that a coupling of all random variables in the system can be found in such a way that: a) marginalizing this coupling to the variables in a single context produces the same joint distribution originally recorded in that context; and b) marginalizing this coupling to any pair of variables in a connection produces their identity coupling (Dzhafarov & Kujala, 2017).

For the system in Figure 1.3, this means finding a set of eight jointly distributed random variables

$$\left(S_{\alpha_1}^{\alpha_1,\beta_1}, S_{\beta_1}^{\alpha_1,\beta_1}, S_{\beta_1}^{\alpha_2,\beta_1}, S_{\alpha_2}^{\alpha_2,\beta_1}, S_{\alpha_2}^{\alpha_2,\beta_2}, S_{\beta_2}^{\alpha_2,\beta_2}, S_{\beta_2}^{\alpha_1,\beta_2}, S_{\alpha_1}^{\alpha_1,\beta_2}\right)$$

such that the marginal $\left(S_{\alpha_1}^{\alpha_1,\beta_1}, S_{\beta_1}^{\alpha_1,\beta_1}\right)$ has the same joint distribution as the bunch $\left(R_{\alpha_1}^{\alpha_1,\beta_1}, R_{\beta_1}^{\alpha_1,\beta_1}\right)$ from the system in Figure 1.3; for the marginal $\left(S_{\alpha_1}^{\alpha_1,\beta_1}, S_{\alpha_1}^{\alpha_1,\beta_2}\right)$, corresponding to the connection $\{ R_{\alpha_1}^{\alpha_1,\beta_1}, R_{\alpha_1}^{\alpha_1,\beta_2} \}$, we have

$$\Pr(S_{\alpha_1}^{\alpha_1,\beta_1} = S_{\alpha_1}^{\alpha_1,\beta_2}) = 1;$$

and similarly for the other bunches and connections. Clearly, $\Pr(S_{\alpha_1}^{\alpha_1,\beta_1} = S_{\alpha_1}^{\alpha_1,\beta_2})$ can only be made equal to unity when the distributions of $R_{\alpha_1}^{\alpha_1,\beta_1}$ and $R_{\alpha_1}^{\alpha_1,\beta_1}$ are equal. When the distributions of all random variables sharing a content are the same for each of the connections in a system, the system is said to be *consistently connected*; otherwise, it is called *inconsistently connected*. An example of how the system in Figure 1.3 can be consistently connected is when all variables $R_{\alpha_i}^{\alpha_i,\beta_j}$ and $R_{\beta_j}^{\alpha_i,\beta_j}$ are binary, take values 0 or 1, and $\Pr(R_{\alpha_i}^{\alpha_i,\beta_j} = 1) = \Pr(R_{\beta_j}^{\alpha_i,\beta_j} = 1) = 1/2$ for $i, j = 1, 2$.

The traditional treatment of the notion of *noncontextuality* coincides with this informal description of selective influences. This treatment has developed from the analysis of some problems in logic and from the analysis of some predictions of quantum mechanics. According to this treatment, and using the language of Contextuality-by-Default, a system of random variables —as the one in Figure 1.3— is said to be noncontextual if the system is consistently connected, and the overall coupling containing the identity couplings, as described above, can be constructed. However, the notion of *noncontextuality* as used in the Contextuality-by-Default theory is more general. Whenever we consider random variables recorded under different conditions in a psychological research setting, we expect to observe changes in the distributions of these random variables. Thus, the respective system of random variables will be inconsistently connected and, consequently, an overall coupling that marginalizes to identity couplings for those pairs does not exist. In Contextuality-by-Default one takes a coupling that makes $\Pr(S_{\alpha_1}^{\alpha_1,\beta_1} = S_{\alpha_1}^{\alpha_1,\beta_2})$ as large as allowed by the distributions of $R_{\alpha_1}^{\alpha_1,\beta_1}$

	$\widetilde{Y}=1$	$\widetilde{Y}=0$	
$\widetilde{X}=1$	$1/8$	$5/24$	$1/3$
$\widetilde{X}=0$	0	$2/3$	$2/3$
	$1/8$	$7/8$	

Fig. 1.4. Maximal coupling of two binary random variables

and $R_{\alpha_1}^{\alpha_1,\beta_2}$. Such a coupling is called a *maximal coupling* and it coincides with the identity coupling whenever the respective distributions are the same. Unlike the identity coupling, a maximal coupling always exists. As an example of how to construct a maximal coupling consider two binary random variables, X and Y, such that both take values 0 or 1, and $\Pr(X=1)=1/3$ and $\Pr(Y=1)=1/8$. The joint distribution presented in Figure 1.4 is the maximal coupling of the variables X and Y constructed with the new variables $\widetilde{X}$ and $\widetilde{Y}$. The probability $\Pr(\widetilde{X}=\widetilde{Y})={}^{19}/_{24}$ and cannot be made larger while keeping $\Pr(\widetilde{X}=1)=\Pr(X=1)$ and $\Pr(\widetilde{Y}=1)=\Pr(Y=1)$.

A system is *noncontextual* when the overall coupling, with the maximal couplings as marginals corresponding to pairs of variables sharing a content, can be constructed. Otherwise, it is *contextual*. In this framework, selectiveness of influences is satisfied when the system of random variables is consistently connected and is noncontextual; and the traditional account of contextuality in quantum mechanics is obtained by restricting the analysis to consistently connected systems. With these definitions, the Contextuality-by-Default allows one to identify two kinds of contextual effects: a) *direct influences* which are defined by the lack of consistent connectedness and can be measured by the differences between the distributions of the random variables that share the same content; and b) *true contextuality* which is defined above by the nonexistence of the mentioned overall coupling. Defining measures of contextuality and of noncontextuality, as well as discovering the properties of such measures, is a work in progress (see e.g. Dzhafarov, Kujala, & Cervantes, 2020a, 2020b). The ubiquity of direct influences in psychological data prevents the possibility of selective influences, as well as the applicability of the traditional definition of noncontextuality, but it does not imply that the full behavior of a system is accounted for by the direct action of inputs upon the responses. An inconsistently connected system of random variables of behavioral data may still be contextual.

1.2 Contextuality in systems of binary random variables

In 2015, I designed and conducted a psychophysical experiment following an idea found in Dzhafarov and Kujala (2017, example 2.1 on p. 86). The idea consisted in the simultaneous presentation of two stimuli in each of which a certain signal should be detected by the participant: a double-detection experiment. This experimental paradigm provides a scheme in which both (in)consistent connectedness and (non)contextuality can be systematically studied. The experiment I designed is presented in Cervantes and Dzhafarov (2017b), and reproduced as Chapter 2 in this dissertation. For this paper, I designed, programmed (using Visual Basic), and conducted the experiment; I also carried out all data analyses with R, and prepared the first draft of the whole document.

The analysis of the double detection experiment presented in Chapter 2 is based on the theory of *cyclic systems* of random variables. These are systems in which all variables are binary, and where each connection and each context has precisely two random variables. The system presented in Figure 1.3 is an example of a cyclic system. The main reason for representing the results of this experiment by means of cyclic systems was that the original experimental idea consisted of exploring the detection of the presence (absence) of a signal in each of two stimuli. From this perspective, this experimental design provides the closest analogue to the most prominent example

	c-	-c	u-	-u	d-	-d
cc	$R^{cc}_{c\text{-}}$	$R^{cc}_{\text{-}c}$	·	·	·	·
uc	·	$R^{uc}_{\text{-}c}$	$R^{uc}_{u\text{-}}$	·	·	·
uu	·	·	$R^{uu}_{u\text{-}}$	$R^{uu}_{\text{-}u}$	·	·
du	·	·	·	$R^{du}_{\text{-}u}$	$R^{du}_{d\text{-}}$	·
dd	·	·	·	·	$R^{dd}_{d\text{-}}$	$R^{dd}_{\text{-}d}$
cu	$R^{cu}_{c\text{-}}$	·	·	$R^{cu}_{\text{-}u}$	·	·
ud	·	·	$R^{ud}_{u\text{-}}$	·	·	$R^{ud}_{\text{-}d}$
dc	·	$R^{dc}_{\text{-}c}$	·	·	$R^{dc}_{d\text{-}}$	·
cd	$R^{cd}_{c\text{-}}$	·	·	·	·	$R^{cd}_{\text{-}d}$

Fig. 1.5. Complete system of random variables for the double-detection experiment. The letters c, d, and u indicate that the dot is, respectively, in the center, below, or above. Thus, c- denotes the dot in the center of the left circle, -d denotes the dot shifted down in the right circle, etc. The resulting contexts are denoted cc, uc, uu, etc., the left (right) letter indicating the location of the dot in the left (respectively, right) circle.

of contextuality: that of the Alice-Bob EPR/Bohm paradigm in quantum physics, which is described by a cyclic system. A brief discussion of the Alice-Bob paradigm and how it is described as a cyclic system is presented in the Chapter 4.

For the experiment, the signal to be detected was the eccentricity of a dot inside a circle. The eccentricity of the dot was manipulated so that there were three levels: at the center, eccentric above, and eccentric below. A full representation of the experimental data considering all three levels for each factor (the location of the dot in the specific circle) produces a system of random variables that is not cyclic (see Figure 1.5). At the time, it was possible to perform contextuality analysis of cyclic systems, but the theory was not yet developed for arbitrary systems of random variables. The analysis of cyclic systems is based on a closed form criterion for

	c-	-c	u-	-u	d-	-d
cc	$R^{cc}_{c\text{-}}$	$R^{cc}_{\text{-}c}$	·	·	·	·
uc	·	$R^{uc}_{\text{-}c}$	$R^{uc}_{u\text{-}}$	·	·	·
uu	·	·	$R^{uu}_{u\text{-}}$	$R^{uu}_{\text{-}u}$	·	·
du	·	·	·	$R^{du}_{\text{-}u}$	$R^{du}_{d\text{-}}$	·
dd	·	·	·	·	$R^{dd}_{d\text{-}}$	$R^{dd}_{\text{-}d}$
cd	$R^{cd}_{c\text{-}}$	·	·	·	·	$R^{cd}_{\text{-}d}$

Fig. 1.6. Example of a system of random variables for the double-detection experiment, extracted from the complete system presented in Figure 1.5.

contextuality that was proved in 2015-2016 (Kujala & Dzhafarov, 2016b; Kujala et al., 2015). We applied this criterion to cyclic systems extracted from the complete data set. These systems were extracted either by taking subsystems of the system in Figure 1.5, or by clustering two of the factor levels together as a single content (for example, we could cluster the above and below forms of eccentricity, creating thereby a single category of an eccentric dot). Figure 1.6 shows an example of an extracted subsystem. Figure 1.7 shows an example of a redefined system obtained by clustering of contents. These analyses form the core of the paper reprinted as Chapter 2.

Later developments of the Contextuality-by-Default theory permitted the analysis of the contextuality of arbitrary systems of binary random variables (Dzhafarov & Kujala, 2016). The data from the double-detection experiment were then reanalysed using the linear programming task defined for such systems of random variables. The results of this analysis were published in Cervantes and Dzhafarov (2017a) which is reprinted here as Chapter 3. For this paper, I have implemented and run the linear programming task used for the analysis, and also prepared the first draft of most of the document.

	c-	*-c*	*e-*	*-e*
cc	$R^{cc}_{c\text{-}}$	$R^{cc}_{\text{-}c}$	·	·
ec	·	$R^{ec}_{\text{-}c}$	$R^{ec}_{e\text{-}}$	·
ee	·	·	$R^{ee}_{e\text{-}}$	$R^{ee}_{\text{-}e}$
ce	$R^{ce}_{c\text{-}}$	·	·	$R^{ce}_{\text{-}e}$

Fig. 1.7. Example of a system of random variables for the double-detection experiment, with clustered factor levels. The letters c and e indicate that the dot is in the center, or eccentric. The interpretation is otherwise as in Figure 1.5

The results from this double-detection experiment indicated that selective influences were not satisfied in any of the systems of random variables considered. However, the violation of selectivity was only of the direct influence type and not that of true contextuality. Up to the time when this experiment was conducted and analysed, most evidence indicated that true contextuality could not be found in systems of random variables that represented behavioral data. The analyses of this experiment, together with those of another psychophysical experiment conducted in our lab by Ru Zhang (Zhang & Dzhafarov, 2017), added to this evidence. The evidence collected hitherto included several attempts by different teams of researchers that had examined and searched for contextual effects in psychology corresponding to true contextuality. These teams had regularly employed a between-subjects design, and the data from their experiments were represented by cyclic systems of binary random variables. Several of these attempts at searching for contextuality were discussed and reanalysed under the framework of the Contextuality-by-Defaul theory in Dzhafarov, Zhang, and Kujala (2015) and in Dzhafarov, Kujala, Cervantes, Zhang, and Jones (2016). Similarly to the results reported in Chapters 2 and 3 for the double-detection experiment, the analyses of these data indicated no contextuality in human behavior.

We further explored the possibility of contextuality in human decision-making using a similar between-subjects design. We designed a crowdsourcing experiment (we refer to it as the Snow Queen experiment) where participants made a pair of conjoint choices depending on the context/treatment to which they were assigned. The goal of this experiment was to exploit the properties of the criterion of contextuality in order to investigate the possibility of contextuality in a setting similar to those that had been usually employed. This experiment was reported in Cervantes and Dzhafarov (2018c) which is reproduced in this dissertation as Chapter 4. The Snow Queen experiment provided the first experimental data where true contextuality could be established in human behavior. For this paper, I implemented the experiment using Purdue's Qualtrics platform for developing online surveys and recruited participants from Amazon's Mechanical Turk; I performed the contextuality analysis and statistical analysis in R; I also prepared the methods and results sections of the paper, and the first draft of the discussion.

After communicating the results from the Snow Queen experiment (Cervantes & Dzhafarov, 2018a, 2018b; Dzhafarov & Cervantes, 2018), we were approached by Irina Basieva (then of the City University London) and Andrei Khrennikov (Linnaeus University, Sweden) to conduct a series of additional experiments. Six crowdsourcing experiments were conducted with analogous formal structures and following the design strategy from the Snow Queen experiment. The results from these experiments were reported in Basieva, Cervantes, Dzhafarov, and Khrennikov (2019) and are reproduced here as Chapter 5. For this paper, I co-designed the experiments together with the coauthors of the paper, and I helped implementing them on the Qualtrics platform of City University of London. I also conducted the contextuality and statistical analyses of the experiments, and prepared a first skeletal draft of the paper. These experiments, together with the Snow Queen experiment, accomplish the following: a) they establish that true contextuality may be found in human choices using the usual between-subjects design; b) they present a methodological approach by which contextuality may be more systematically investigated in psychology; and c) they show that the methodology of the Snow Queen experiment does not force a system to be contextual: contextuality or its absence is an empirical property of a particular system.

Along with these experimental results, we have advanced the theory of contextuality for cyclic systems of binary random variables. We now have a complete geometric characterization of arbitrary cyclic systems and their (non)contextuality. This characterization is presented in Dzhafarov, Kujala, and Cervantes (2020a), where the polytope of noncontextual joint distributions and well defined measures of contextuality (for a system outside of the polytope) and noncontextuality (for a system within) are characterized. From this characterization, we also achieve some additional intuition of how likely it is to find contextuality for arbitrary cyclic systems. This intuition is formally laid out in Dzhafarov, Kujala, and Cervantes (2020b). With the description of the polytope and of the measures defined on it, the development of the theory of Contextuality-by-Default for contextuality in cyclic systems of binary random variables may be considered complete.

1.3 Contextuality in systems with categorical random variables

Besides the work on systems with binary random variables recounted in the previous section, I have also been involved in developing the theory for defining and analyzing contextuality of systems with random variables that take more than two values. In this development, we have been guided by the consideration that the theory of contextuality for binary random variables has some properties that are desirable to preserve in all generalizations of the theory. First and foremost, for systems where random variables sharing a content have the same distribution, the theory specializes to the traditional theory of contextuality. A second property is that the maximal coupling of a pair of binary random variables is unique. Lastly, for any noncontextual system, a subsystem obtained by dropping any subset of its random variables remains noncontextual.

In Dzhafarov, Cervantes, and Kujala (2017), we propose a possible way in which the theory may be extended to categorical random variables. It is based on the observation that any categorical variable may be represented by some set of jointly distributed binary random variables. In addition, for strictly categorical random variables, another property that may be deemed desirable is that a noncontextual system remains noncontextual under coarse-graining of categories. Consider a loaded die for which the random variable Z represents the outcome from a single roll. Let the distribution of Z be as depicted in Table 1.1. We clearly have the same information about the outcome of the die from the set of binary random variables

$$\{\, Z_j : Z_j = 1 \text{ if } Z = j, \text{ and } Z_j = 0 \text{ if } Z \neq j,\ \text{ where } j = 1, \ldots, 6 \,\}\,.$$

These six random variables are jointly distributed, and the joint outcomes that have nonzero probabilities are presented in Table 1.2. For this die, one may also be interested in coarse-grainings such as whether the outcome is odd or even, or the sets of two values of the die that would add up to seven. These coarse-grainings create two new random variables whose distributions are presented in Table 1.3.

The theory developed in Dzhafarov et al. (2017) presents a way to represent categorical variables by systems of binary random variables consisting of the set of all possible dichotomizations of the categorical variables. This is necessary to preserve noncontextuality under any coarse-graining. In this paper, we also derive a necessary condition for noncontextuality of the system thus constructed. This paper is

Table 1.1.
Distribution of the random variable Z

Z	$\Pr(Z = z)$
1	$6/21$
2	$5/21$
3	$4/21$
4	$3/21$
5	$2/21$
6	$1/21$

Table 1.2.
Probabilities of some joint events of random variables Z_j

Event	Pr(Event)
$\{ Z_1 = 1, Z_2 = 0, Z_3 = 0, Z_4 = 0, Z_5 = 0, Z_6 = 0 \}$	$6/21$
$\{ Z_1 = 0, Z_2 = 1, Z_3 = 0, Z_4 = 0, Z_5 = 0, Z_6 = 0 \}$	$5/21$
$\{ Z_1 = 0, Z_2 = 0, Z_3 = 1, Z_4 = 0, Z_5 = 0, Z_6 = 0 \}$	$4/21$
$\{ Z_1 = 0, Z_2 = 0, Z_3 = 0, Z_4 = 1, Z_5 = 0, Z_6 = 0 \}$	$3/21$
$\{ Z_1 = 0, Z_2 = 0, Z_3 = 0, Z_4 = 0, Z_5 = 1, Z_6 = 0 \}$	$2/21$
$\{ Z_1 = 0, Z_2 = 0, Z_3 = 0, Z_4 = 0, Z_5 = 0, Z_6 = 1 \}$	$1/21$

Table 1.3.
Distributions of two variables that coarse-grain the values of Z

Z_{odd}	$\Pr(Z_{\text{odd}} = z)$
odd	$12/21$
even	$9/21$

Z_{sum}	$\Pr(Z_{\text{sum}} = z)$
$\{ 1, 6 \}$	$7/21$
$\{ 2, 5 \}$	$7/21$
$\{ 3, 4 \}$	$7/21$

reproduced here as Chapter 6. For this paper, I formulated the initial conjecture and contributed to the discovery of the proof of the mentioned condition (nominal dominance theorem). I also contributed to the writing of the paper.

With this necessary condition at hand, we revisited the previous double-detection experiment. We designed a new psychophysical experiment where the task asked of the participants was to *identify* the location of two stimuli, instead of detecting two signals. As in the double-detection experiment, a dot could be located in the center or be eccentric for each of two circles. There were five levels (center, up, down, left, and right) for each of the two dot locations, and the responses were to identify these locations. The results from this experiment are reported in Cervantes and Dzhafarov (2019) which is reprinted here as Chapter 7. For this paper, I designed, programmed in PsychoPy/Python, and conducted the experiment; I also performed all data analyses with R, and prepared the first draft of the whole document. This experiment is the first one to establish contextuality in a within-subject experimental design.

1.4 Contextuality and scientific inquiry

The implications of the notion of contextuality for scientific research, and in particular, for psychology, are difficult to fully understand at our present stage of knowledge. Some considerations are made in the papers reproduced as Chapters 4, 5 and 6. I summarize them in this section.

The notion of contextuality and the results of contextuality analysis have clear substantive implications in quantum mechanics. The quantum-mechanical theory is used to determine a system's behavior, and for several quantum-mechanical experimental paradigms, such as the EPR/Bohm paradigm, the conditions under which the resulting systems are (non)contextual are well known. There, the notion of contextuality helps to separate quantum-mechanical models from those of classical mechanics. In addition, the results of contextuality analysis play a role as 'witnesses' of the desired quantum behavior (Abramsky, Barbosa, & Mansfield, 2017). Moreover, the degree of contextuality has been found to correlate with the computational advantage of quantum computing over conventional one (Abramsky et al., 2017; Frembs, Roberts, & Bartlett, 2018).

In psychology, the notion of noncontextuality specializes to the one of selective influences under certain circumstances: consistently connected noncontextual systems are the systems with selective influences. In these cases, the results of contextuality analysis determine the tenability of the hypothesis of selectivity of influences. The analysis is essentially a classification and measurement of the degree of deviation from or adherence to the pattern of selective influences. While selective influences is the traditional topic in psychology, observable systems of human behavior are seldom consistently connected, precluding thereby the possibility of selective influences. Contextuality analysis, and the Contextuality-by-Default theory in particular, allow us to perform the classification and measurement of contextual effects into direct influences and true contextuality for both consistently and inconsistently connected systems. An example of the application of contextuality analysis to the substantive issue of characterizing mental architectures is presented by Zhang and Dzhafarov (2015).

Unlike in quantum mechanics, we do not possess a psychological theory that helps determine the conditions when a system is (non)contextual. However, as is the case for measurements in quantum mechanics, the stochasticity of responses in most areas of psychology cannot be reduced by progressively greater control of stimuli and conditions. Thus, the status and role of contextuality can be expected to be similar. For now, by analogy with the contextuality advantage mentioned above, one may point out that the degree of similarity or unanimity of decisions in experiments with between-subjects design can correlate with the degree of (non)contextuality of the system of random variables that represent them. If, for example, across pools of respondents, respondents agreed among themselves on what options to choose in each context, the system of its responses would then become deterministic and noncontextual. However, it is not possible to replace contextuality with some measure of unanimity because deep noncontextuality can be achieved with nondeterministic systems. Whether the degree of (non)contextuality correlates with this or other aspects of behavior that may be of interest in their own right, requires that we gain more knowledge of the degree of (non)contextuality for a much larger class of behavioral systems.

More generally, the considerations of contextuality with respect to theoretical models lead also to identify classes of viable explanations. As pointed in Chapter 5, given a system of random variables representing some data, if contextuality or noncontextuality is established for them, a model is to be rejected if it fails to predict this property. An example of a simple decision model that should be rejected because it only predicts contextual systems for the experiments reported in the paper reprinted as Chapter 5 is also presented in that chapter. The issue of what classes of causal models are consistent with contextual systems is explored for some types of quantum paradigms by Cavalcanti (2018), and Pearl and Cavalcanti (2019). The relationship between contextuality, as defined in the Contextuality-by-Default theory, and causal models has been explored by Jones (2019).

Lastly, in addition to the previous considerations, the theory of contextuality brings to the fore the nature of random variables. This is debated especially in Chapter 4, where the relational nature of contextuality as a property of a system of random variables is discussed: that part of the identity of a random variable is given by its relation with the other random variables with which it is jointly distributed. The nature of the interrelations among variables, for different types of random variables, also comes into play for developing the theory and appropriately defining contextuality for different classes of systems of random variables, as discussed in Chapters 6 and 8.

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2. EXPLORATION OF CONTEXTUALITY IN A PSYCHOPHYSICAL DOUBLE-DETECTION EXPERIMENT

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Abstract The Contextuality-by-Default (CbD) theory allows one to separate contextuality from context-dependent errors and violations of selective influences (aka "no-signaling" or "no-disturbance" principles). This makes the theory especially applicable to behavioral systems, where violations of selective influences are ubiquitous. For cyclic systems with binary random variables, CbD provides necessary and sufficient conditions for noncontextuality, and these conditions are known to be breached in certain quantum systems. We apply the theory of cyclic systems to a psychophysical double-detection experiment, in which observers were asked to determine presence or absence of a signal property in each of two simultaneously presented stimuli. The results, as in all other behavioral and social systems previous analyzed, indicate lack of contextuality. The role of context in double-detection is confined to lack of selectiveness: the distribution of responses to one of the stimuli is influenced by the state of the other stimulus.

Keywords: Contextuality · Cyclic systems · Inconsistent connectedness · Psychophysics

The Contextuality-by-Default (CbD) theory [9, 10] describes systems of measurements with respect to the conditions under which they are recorded and determines the tenability of a non-contextual description of the system. In this paper, we study the double-detection paradigm suggested in Refs. [6, 8]. In this paradigm, two stimuli are presented to an observer simultaneously (left-right), each on one of several possible levels. The observer is asked to state (Yes/No), for each of the two observation areas, whether it contains a particular target property (signal). The signal is objectively present in a subset of levels of a stimulus. When such experimental situation includes only two levels for each stimulus (e.g., present/absent), the system of measurements is formally equivalent to that of the Einstein–Podolski–Rosen/Bohm (EPR/B) paradigm (see e.g, Ref. [6]).

2.1 Contextuality in CbD

We briefly recapitulate the concepts of the CbD, to make this paper self-sufficient. For detailed discussions see Refs. [9, 10]; the proofs may be found in Refs. [11, 14, 15].

Definition 1. (System of measurements) A *system of measurements* is a matrix $\mathfrak{R}_{n\times m}$, in which columns correspond to the properties $\{q_1,\dots,q_n\}$ and rows to the contexts $\{c_1,\dots,c_m\}$. A cell (i,j) contains the random variable R_i^j if q_i is measured in context c_j, and the cell is left empty otherwise.

When adopting the CbD framework, the first goal is to produce a matrix $\mathfrak{R}$ that formally represents the experiment and its results.

Definition 2. (Connections and bunches) The random variables in any column of a system of measurements form a *connection* for the corresponding property; denote the connection for property q_i by $\mathfrak{R}_i$. Those in any row form a *bunch* representing the corresponding context; denote the bunch for context c_j by R^j.

Note that elements of a connection are necessarily ("by default") pairwise distinct and pairwise stochastically unrelated, i.e., no R_i^j and R_i^k with $k \neq j$ have a joint distributions. Consequently, the system $\mathfrak{R}$ does not have a joint probability distribution including all of its elements. See Refs. [5, 10].

Definition 3. (Coupling) Let X_i, with $i \in \mathrm{I}$, an index set, be a random variable on a probability space $(\mathrm{X}_i, \Sigma_i, P_i)$. Let $\{ Y_i : i \in \mathrm{I}\}$ be a collection of jointly distributed random variables (i.e., a random variable in its own right) on a probability space (Y, Ω, p). The random variable $\{ Y_i : i \in \mathrm{I}\}$ is called a *coupling* of the collection $\{X_i : i \in \mathrm{I}\}$ if for all $i \in \mathrm{I}$, $Y_i \overset{d}{=} X_i$, where $\overset{d}{=}$ denotes identity in distribution.

Definition 4. (Maximal coupling) Let $Y = (Y_i : i \in \mathrm{I})$ be a coupling of a collection $\{X_i : i \in \mathrm{I}\}$. And let M be the event where $\{ Y_i = Y_j$ for all $i, j \in \mathrm{I}\}$. If Pr(M) is the largest possible among all couplings of $\{X_i : i \in \mathrm{I}\}$, then Y is a *maximal coupling* of $\{X_i : i \in \mathrm{I}\}$.

Definition 5. (Contextual system) Let $\mathfrak{R}$ be a system of measurements. Let S be a coupling of $\mathfrak{R}$ such that for each $c_j \in \{c_1, \dots, c_m\}$, S^j is a coupling of R^j contained in S. The system $\mathfrak{R}$ is said to be *non-contextual* if it has a coupling S such that for all $q_i \in \{q_1, \dots, q_n\}$, the coupling S_i is a maximal coupling.

Definition 6. (Cyclic system with binary variables) Let $\mathfrak{R}$ be a system of measurements such that (a) each context contains two properties; (b) each property is measured in two different contexts; (c) no two contexts share more than one property; and (d) each measurement is a binary random variable, with values ± 1. Then the system $\mathfrak{R}$ is a *cyclic system with binary variables* and in the following will be simply called *a cyclic system.*

Remark 1. Note that a cyclic system is composed of the same number n of connections and of bunches, and it contains $2n$ random variables. We shall say that a cyclic system has rank n or is of rank n to explicitly refer to this number.

Definition 7. (Consistent connections) Let $\mathfrak{R}_i$ be a connection in a system $\mathfrak{R}$. It is said that $\mathfrak{R}_i$ is a *consistent connection* if for all $c_j, c_k \in \{c_1, \ldots, c_m\}$ such that R_i^j and R_i^k are defined (i.e., both cells (i, j) and (i, k) of $\mathfrak{R}$ are not empty), $R_i^j \overset{d}{=} R_i^k$.

Definition 8. (Consistently connected system) A system of measurements $\mathfrak{R}$ is said to be *consistently connected* if for all $q_i \in \{q_1, \ldots, q_n\}$, the connection $\mathfrak{R}_i$ is a consistent connection. For a cyclic system, define[a]

$$\mathrm{ICC} = \sum_{i=1}^{n} \left| \left\langle R_i^j \right\rangle - \left\langle R_i^k \right\rangle \right| .$$

ICC provides a measure of how inconsistent the connections are in the system.

Definition 9. (Contextuality in cyclic systems) Let $\mathfrak{R}$ be a cyclic system with n binary variables. Let

$$\mathsf{s}_1(x_1, x_2, \ldots, x_n) = \max \left\{ \sum_{k=1}^{n} a_k x_k : a_k = \pm 1 \text{ and } \prod_{k=1}^{n} a_k = -1 \right\} .$$

Let

$$\Lambda C = \mathsf{s}_1 \left(\left\{ \left\langle R_i^j R_{i'}^j \right\rangle : q_i, q_{i'} \text{ measured in } c_j, \text{ and } c_j \in \{c_1, \ldots, c_m\} \right\} \right)$$

Let $\Delta C = \Lambda C - \mathrm{ICC} - (n - 2)$. The quantity ΔC is a *measure of contextuality* for cyclic systems.

Theorem 1. (Cyclic system contextuality criterion, [14]) A cyclic system $\mathfrak{R}$ is contextual if and only if $\Delta C > 0$.

Remark 2. ΔC for a consistently connected cyclic system with $n = 4$ reduces to the Bell/CHSH inequalities [3, 10].

[a](Explanatory note added in the dissertation.) Here, $\langle \cdot \rangle$ denotes expected value.

2.2 Contextuality in Behavioral and Social Data

In Ref. [13] many empirical studies of behavioral and social systems were reviewed. Most of those systems come from social data; that is, an observation for each measurement was the result of posing a question to a person, and the set of observations comes from questioning groups of people. For all the studies considered there, the CbD analyses showed that the systems, treated as cyclic systems ranging from rank 2 to 4, were non-contextual. Only one of the studies reviewed in Ref. [13] dealt with responses from a single person to multiple replications of stimuli.

Now, a key modeling problem in cognitive psychology has been determining whether a set of inputs selectively influences a set of response variables (Refs. [4, 16–18]). The formal theory of selective influences has been developed for the case of consistent connectedness, which has been treated as a necessary condition of selective influences; it follows from this formalism that selectiveness of influences in a consistently connected system is negated precisely in the case where it is contextual [6].

However, in most, if not all, behavioral systems, some form of influence upon a given random output is expected from most, if not all, of the system's inputs (Ref. [17]). This means that in the behavioral domain inconsistently connected systems are ubiquitous. While the presence of inconsistent connections rules out the possibility of selective influences, it does not imply that the full behavior of the system is accounted for by the direct action of inputs upon the outputs; an inconsistently connected behavioral system may still be contextual in the sense of CbD.

The double detection paradigm suggested in [6, 8] provides a framework where both (in)consistent connectedness and contextuality can be studied in a manner very similar to how they are studied in quantum-mechanical systems (or could be studied, because consistent connectedness in quantum physics is often assumed rather than documented).

2.3 Method

2.3.1 Participants

Three volunteers, two females and one male, graduate students at Purdue University, served as participants for the experiment, including the first author of this paper. They were recruited and compensated in accordance to Purdue University's IRB protocol #1202011876, for the research study "Selective Probabilistic Causality As Interdisciplinary Methodology" under which this experiment was conducted. All participants reported normal or corrected to normal vision and were aged around 30. They are identified as $P1 - P3$ in the text and their experience with psychophysical experiments ranged from none to more than three previous participations.

2.3.2 Apparatus

The experiment was run using a personal computer with an Intel® Core™ processor running Windows XP, a 24-in. monitor with a resolution of 1920×1200 pixels (px), and a standard US 104-key keyboard. A chin-rest with forehead support was used so that the distance between subject and monitor was kept at 90 cm; this made each pixel on the screen to occupy about 62 s arc of the subjects' visual field.

2.3.3 Stimuli

The stimuli were similar to those from Refs. [1, 12]. They consisted of two circles drawn in solid grey (RGB 100, 100, 100) on a black background in a computer screen, with a dot drawn at or near their center. The circles radius was 135 px with their centers 320 px apart; the dots and circumference lines were 4 px wide. The offset of each dot with respect to the center of each circle, when they were not presented at

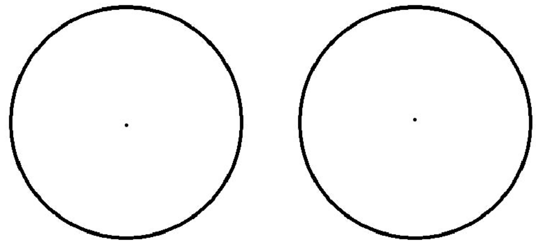

Fig. 2.1. Stimulus example

the center, was 4 px. An example of the stimuli (in reversed contrast) is shown in Fig. 2.1.

2.3.4 Procedure

Each participant performed nine experimental sessions. At the beginning of each experimental session, the chin-rest and chair heights were adjusted so that the subject could sit and use the keyboard comfortably. The time available for each session was 30 minutes, during which the participants responded in 560 (non-practice) trials (except for participant $P3$ in the sixth session, who only responded in 557 trials) preceded by up to 30 practice trials. The number of practice trials was set to 30 during the first two sessions and reduced to 15 during subsequent sessions. After each practice trial, the subject received feedback about whether their response for each circle was correct or not. The responses to practice trials were excluded from the analyses. Additionally, depending on their previous experience in psychophysical experiments the participants had up to three training sessions, also excluded from subsequent analyses.

Instructions for the experiment were presented to each participant verbally and written in the screen. In each trial the participant was required to judge for each circle whether the dot presented was displaced from the center or not. The stimuli were displayed until the subject produced their response. The responses were given by pressing and holding together two keys, one for each circle. Then, the dots in each circle were removed and a "Press the space bar to continue" message was flashed on top of the screen. After pressing the space bar, the message was removed and the next stimuli pair were presented after 400 ms. (Reaction times were measured from the onset of stimulus display until a valid response was recorded, but they were not used in the data analysis.)

2.3.5 Experimental Conditions

In each of two circles the dot presented could be located either at its center, or 4 px above, or else 4 px under the center. These locations produce a total of nine experimental conditions.

During each session, excepting the practice trials, the dot was presented at the center in a half of the trials; above the center in a quarter of them; and below the center in the remaining quarter, for each of the circles. Table 2.1 presents the proportions of allocations of trials to each of the 9 conditions.

For each session, each trial was randomly assigned to one of the conditions in accordance with Table 2.1. The number of experimental sessions was chosen so that the expected number of (non-practice) trials in the conditions with lowest probabilities was at least 300. This number of observations was chosen based on Refs. [2], whose results show that coverage errors with respect to nominal values are below 1% for almost all confidence intervals for proportions with $n > 300$.

Table 2.1.
Probabilities with which a trial was allocated to one of the 9 experimental conditions.

	Center	Up	Down
Center	1/4	1/8	1/8
Up	1/8	1/16	1/16
Down	1/8	1/16	1/16

2.4 Analyses

Based on the experimental design depicted in Table 2.1, we specify the following properties:

- l_c: a dot is presented in the center of the left circle;
- r_c: a dot is presented in the center of the right circle;
- l_u: a dot is presented above the center of the left circle;
- r_u: a dot is presented above the center of the right circle;
- l_d: a dot is presented below the center of the left circle; and
- r_d: a dot is presented below the center of the right circle.

The 9 experimental conditions (contexts) then are denoted $l_c r_c, l_c r_u$, etc. Thus, the system of measurements depicted by the matrix in Fig. 2.2 represents the complete 3×3 design given in Table 2.1.

We approach the exploration of this system through the theory of contextuality for cyclic systems in two ways. Firstly, note that from the system in Fig. 2.2 we can extract six different cyclic subsystems of rank 6 and nine of rank 4. One of the rank 4 subsystems is presented in the left matrix in Fig. 2.3. One of the rank 6 subsystems is shown in the right matrix in Fig. 2.3.

	l_c	r_c	l_u	r_u	l_d	r_d
$l_c r_c$	$R^{l_c r_c}_{l_c}$	$R^{l_c r_c}_{r_c}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$l_u r_c$	$\cdot$	$R^{l_u r_c}_{r_c}$	$R^{l_u r_c}_{l_u}$	$\cdot$	$\cdot$	$\cdot$
$l_u r_u$	$\cdot$	$\cdot$	$R^{l_u r_u}_{l_u}$	$R^{l_u r_u}_{r_u}$	$\cdot$	$\cdot$
$l_d r_u$	$\cdot$	$\cdot$	$\cdot$	$R^{l_d r_u}_{r_u}$	$R^{l_d r_u}_{l_d}$	$\cdot$
$l_d r_d$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$R^{l_d r_d}_{l_d}$	$R^{l_d r_d}_{r_d}$
$l_c r_u$	$R^{l_c r_u}_{l_c}$	$\cdot$	$\cdot$	$R^{l_c r_u}_{r_u}$	$\cdot$	$\cdot$
$l_u r_d$	$\cdot$	$\cdot$	$R^{l_u r_d}_{l_u}$	$\cdot$	$\cdot$	$R^{l_u r_d}_{r_d}$
$l_d r_c$	$\cdot$	$R^{l_d r_c}_{r_c}$	$\cdot$	$\cdot$	$R^{l_d r_c}_{l_d}$	$\cdot$
$l_c r_d$	$R^{l_c r_d}_{l_c}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$R^{l_c r_d}_{r_d}$

Fig. 2.2. System of measurements for double detection experiment.

	l_c	r_c	l_u	r_u
$l_c r_c$	$R^{l_c r_c}_{l_c}$	$R^{l_c r_c}_{r_c}$	$\cdot$	$\cdot$
$l_u r_c$	$\cdot$	$R^{l_u r_c}_{r_c}$	$R^{l_u r_c}_{l_u}$	$\cdot$
$l_u r_u$	$\cdot$	$\cdot$	$R^{l_u r_u}_{l_u}$	$R^{l_u r_u}_{r_u}$
$l_c r_u$	$R^{l_c r_u}_{l_c}$	$\cdot$	$\cdot$	$R^{l_c r_u}_{r_u}$

	l_c	r_c	l_u	r_u	l_d	r_d
$l_c r_c$	$R^{l_c r_c}_{l_c}$	$R^{l_c r_c}_{r_c}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$l_u r_c$	$\cdot$	$R^{l_u r_c}_{r_c}$	$R^{l_u r_c}_{l_u}$	$\cdot$	$\cdot$	$\cdot$
$l_u r_u$	$\cdot$	$\cdot$	$R^{l_u r_u}_{l_u}$	$R^{l_u r_u}_{r_u}$	$\cdot$	$\cdot$
$l_d r_u$	$\cdot$	$\cdot$	$\cdot$	$R^{l_d r_u}_{r_u}$	$R^{l_d r_u}_{l_d}$	$\cdot$
$l_d r_d$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$R^{l_d r_d}_{l_d}$	$R^{l_d r_d}_{r_d}$
$l_c r_d$	$R^{l_c r_d}_{l_c}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$R^{l_c r_d}_{r_d}$

Fig. 2.3. Examples of cyclic subsystems of rank 4 and 6.

Secondly, in addition to the definition of the quantities as presented above, there are several interesting systems produced by redefining these quantities.[1] From the description of the double-detection paradigm, one can argue, e.g., that the center location may be viewed as a signal to be detected, with either of the two off-center locations being treated as absence of the signal. This way of looking at the stimuli induces the following definition of the properties to be measured:

- l_c: a dot is presented in the center of the left circle;

[1]There are also several uninteresting ways to construct systems of measurements for the conditions and measurements in this experiment. Examples of how to construct them and why they are not interesting may be found in Ref. [7]

$$\begin{array}{|ccc|} & x & y \\ l_x r_y & R_x^{l_x r_y} & R_y^{l_x r_y} \\ l_y r_x & R_x^{l_y r_x} & R_y^{l_y r_x} \end{array}$$

Fig. 2.4. Rank 2 systems structure where (x, y) is any of $(c, ud), (cu, d), (cd, u), (c, u), (c, d), (u, d)$.

– r_c: a dot is presented in the center of the right circle;

– l_{ud}: a dot is presented off-center in the left circle;

– r_{ud}: a dot is presented off-center in the right circle.

Analogously one could also consider l_{cu}, l_{cd}, r_{cu}, r_{cd}, as properties to be measured in appropriately chosen contexts,

Another way of dealing with our data is to consider the locations of the dots as properties to be measured (by responses attributing to them to a left or to a right circle). For instance, a pair of properties can be chosen as

– c: a dot is presented in the center of a circle; and

– ud: a dot is presented off the center of a circle.

A systematic application of both of these redefinitions leads to also consider quantities l_{cu}, l_{cd}, r_{cu}, r_{cd}, u, cd, d, and cu with the analogous interpretations. In this way, six systems of rank 2 and 27 systems of rank 4 may be constructed. Thus, we shall consider systems with the structures depicted by the matrices in Figs. 2.4, 2.5, and 2.6.

	l_x	r_x	l_y	r_y
$l_x r_x$	$R_{l_x}^{l_x r_x}$	$R_{r_x}^{l_x r_x}$	$\cdot$	$\cdot$
$l_y r_x$	$\cdot$	$R_{r_x}^{l_y r_x}$	$R_{l_y}^{l_y r_x}$	$\cdot$
$l_y r_y$	$\cdot$	$\cdot$	$R_{l_y}^{l_y r_y}$	$R_{r_y}^{l_y r_y}$
$l_x r_y$	$R_{l_x}^{l_x r_y}$	$\cdot$	$\cdot$	$R_{r_y}^{l_x r_y}$

Fig. 2.5. Rank 4 systems structure where (l_x, l_y) is any of $(l_c, l_{ud}), (l_{cu}, l_d), (l_{cd}, l_u), (l_c, l_u), (l_c, l_d), (l_u, l_d)$, and (r_x, r_y) is any of $\{(r_c, r_{ud}), (r_{cu}, r_d), (r_{cd}, r_u), (r_c, r_u), (r_c, r_d), (r_u, r_d)\}$.

	l_x	r_x	l_y	r_y	l_z	r_z
$l_x r_x$	$R_{l_x}^{l_x r_x}$	$R_{r_x}^{l_x r_x}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$l_y r_x$	$\cdot$	$R_{r_x}^{l_y r_x}$	$R_{l_y}^{l_y r_x}$	$\cdot$	$\cdot$	$\cdot$
$l_y r_y$	$\cdot$	$\cdot$	$R_{l_y}^{l_y r_y}$	$R_{r_y}^{l_y r_y}$	$\cdot$	$\cdot$
$l_z r_y$	$\cdot$	$\cdot$	$\cdot$	$R_{r_y}^{l_z r_y}$	$R_{l_z}^{l_z r_y}$	$\cdot$
$l_z r_z$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$R_{l_z}^{l_z r_z}$	$R_{r_z}^{l_z r_z}$
$l_x r_z$	$R_{l_x}^{l_x r_z}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$R_{r_z}^{l_x r_z}$

Fig. 2.6. Rank 6 systems structure where (x, y, z) is any of $(c, u, d), (c, d, u), (u, c, d), (d, c, u), (u, d, c), (d, u, c)$.

2.5 Results

2.5.1 Results for Cyclic Subsystems

Table 2.2 presents the individual data for all of the expectations used in the calculations of all subsystems. Note that the statistics associated with the redefined quantities are obtained by an apropriate linear combination of those in Table 2.2 with weights proportional to the number of trials of the combined conditions.

Table 2.3 presents the values of ΛC, ICC, and ΔC calculated for each participant and each of the rank 6 cyclic subsystems. Table 2.4 presents the respective values

Table 2.2.
Individual level data

		P1			P2			P3		
l	r	$\langle R_l^{lr}\rangle$	$\langle R_r^{lr}\rangle$	$\langle R_l^{lr}R_r^{lr}\rangle$	$\langle R_l^{lr}\rangle$	$\langle R_r^{lr}\rangle$	$\langle R_l^{lr}R_r^{lr}\rangle$	$\langle R_l^{lr}\rangle$	$\langle R_r^{lr}\rangle$	$\langle R_l^{lr}R_r^{lr}\rangle$
l_c	r_c	0.4349	0.2730	0.4825	0.7317	0.5683	0.3984	0.3582	0.1946	−0.0913
l_c	r_u	0.6190	−0.5397	−0.2095	0.7016	−0.0825	−0.2413	0.6762	−0.8508	−0.6159
l_c	r_d	−0.1873	0.2698	0.4095	0.8857	−0.8635	−0.7937	0.3937	−0.3524	−0.3429
l_u	r_c	−0.5048	0.1175	0.2254	−0.2063	0.5238	−0.5302	−0.7302	0.6603	−0.5683
l_u	r_u	0.0476	−0.0286	0.4794	0.1111	0.1683	0.2190	−0.4904	−0.6624	0.4459
l_u	r_d	−0.8476	−0.0857	0.1619	0.2254	−0.7778	−0.4222	−0.7643	−0.2166	0.0446
l_d	r_c	0.5873	−0.3937	−0.0825	−0.6667	0.7810	−0.5238	−0.4159	0.3429	−0.4762
l_d	r_u	0.5619	−0.9111	−0.5365	−0.7333	0.2635	−0.4286	−0.2508	−0.7079	0.0095
l_d	r_d	0.5111	0.3016	0.4730	−0.5175	−0.5937	0.5810	−0.3079	−0.1746	0.0413

Table 2.3.
Contextuality cyclic subsystems of rank 6

System	P1			P2			P3		
$(l_x, l_y, l_z), (r_x, r_y, r_z)$	ΛC	ICC	ΔC	ΛC	ICC	ΔC	ΛC	ICC	ΔC
$(l_c, l_d, l_u), (r_d, r_u, r_c)$	1.6254	2.4127	-4.7873	2.4571	1.3714	-2.9143	2.0382	1.0779	-3.0397
$(l_d, l_c, l_u), (r_c, r_d, r_u)$	1.7143	2.4889	-4.7746	2.4508	1.4286	-2.9778	2.4078	1.3138	-2.9060
$(l_d, l_u, l_c), (r_c, r_u, r_d)$	1.9873	3.4476	-5.4603	2.3476	0.7286	-2.3810	1.4104	0.8040	-3.3936
$(l_c, l_u, l_d), (r_d, r_c, r_u)$	2.6063	2.2952	-3.6889	2.9508	1.0968	-2.1460	1.4991	1.0213	-3.5222
$(l_u, l_c, l_d), (r_d, r_u, r_c)$	1.7238	2.7206	-4.9968	2.3857	0.9413	-2.5556	1.7151	1.0784	-3.3633
$(l_u, l_d, l_c), (r_c, r_d, r_u)$	1.7651	1.4921	-3.7270	2.1190	1.2524	-3.1333	1.3708	1.0598	-3.6890

Table 2.4.
Contextuality cyclic subsystems of rank 4

System	P1			P2			P3		
$(l_x, l_y), (r_x, r_y)$	ΛC	ICC	ΔC	ΛC	ICC	ΔC	ΛC	ICC	ΔC
$(l_c, l_u), (r_c, r_u)$	1.3968	1.4032	-2.0063	0.9508	0.6429	-1.6921	1.7213	1.2118	-1.4904
$(l_c, l_u), (r_c, r_d)$	0.9556	1.4762	-2.5206	2.1444	0.7159	-0.5714	1.0470	0.6711	-1.6241
$(l_c, l_u), (r_u, r_d)$	1.2603	2.5683	-3.3079	1.6762	0.6349	-0.9587	1.3600	0.8806	-1.5206
$(l_c, l_d), (r_c, r_u)$	1.3111	1.2476	-1.9365	1.5921	0.6556	-1.0635	1.1929	0.7742	-1.5812
$(l_c, l_d), (r_c, r_d)$	1.4476	1.3968	-1.9492	1.5000	0.7857	-1.2857	0.9517	0.4694	-1.5177
$(l_c, l_d), (r_u, r_d)$	1.2095	1.2603	-2.0508	2.0444	1.0159	-0.9714	0.9905	0.6603	-1.6698
$(l_u, l_d), (r_c, r_u)$	1.1587	1.9714	-2.8127	1.7016	0.7365	-1.0349	1.4808	0.7678	-1.2870
$(l_u, l_d), (r_c, r_d)$	0.9429	1.3175	-2.3746	2.0571	1.0222	-0.9651	1.0478	0.5015	-1.4538
$(l_u, l_d), (r_u, r_d)$	1.6508	2.2159	-2.5651	1.2127	0.6095	-1.3968	0.5222	0.4185	-1.8963

for each of the rank 4 cyclic subsystems. For all participants, the subsystems are noncontextual.

Table 2.5.
Contextuality cyclic systems of rank 2

System	$P1$			$P2$			$P3$		
(x,y)	ΛC	ICC	ΔC	ΛC	ICC	ΔC	ΛC	ICC	ΔC
(c, ud)	0.0286	0.5302	-0.5016	0.0095	0.1778	-0.1683	0.0429	0.0619	-0.0190
(cd, u)	0.5228	0.5947	-0.0720	0.1905	0.2286	-0.0381	0.0430	0.0631	-0.0201
(cu, d)	0.5608	0.5862	-0.0254	0.1778	0.2032	-0.0254	0.1003	0.0695	0.0308
(c, u)	0.4349	0.5365	-0.1016	0.2889	0.3016	-0.0127	0.0476	0.1365	-0.0889
(c, d)	0.4921	0.5238	-0.0317	0.2698	0.3016	-0.0317	0.1333	0.1143	0.0190
(u, d)	0.6984	0.7111	-0.0127	0.0063	0.0825	-0.0762	0.0351	0.0906	-0.0556

2.5.2 Results for Cyclic Systems with Redefined Quantities

Table 2.5 presents the values of ΛC, ICC, and ΔC calculated for each participant for each of the rank 2 cyclic systems, and Table 2.6 shows those for the rank 4 cyclic systems. Note that for participant $P3$, two of the rank 2 systems, those with $(x, y) = (c, d)$ and $(x, y) = (cu, d)$, have a positive ΔC value, which might suggest that these two systems show contextuality. However, their respective confidence intervals, $\Delta C_{(cu,d)} \in (-0.267, 0.241)$ and $\Delta C_{(c,d)} \in (-0.233, 0.215)$,[2] indicate that the values are consistent with lack of contextuality.

2.6 Conclusions

The experiment presented in this paper illustrates the use of the double factorial paradigm in the search of contextuality in behavioral systems, namely in the responses of human observers in a double-detection task. This paradigm provides the closest analogue in psychophysical research to the Alice-Bob EPR/Bohm paradigm.

[2]95% confidence intervals corrected by Bonferroni for the number of tests for ΔC values in the experiment. However, it should be noted that even uncorrected intervals covered the value 0.

Table 2.6.
Contextuality cyclic systems of rank 4

System	$P1$			$P2$			$P3$		
$(l_x, l_y), (r_x, r_y)$	ΛC	ICC	ΔC	ΛC	ICC	ΔC	ΛC	ICC	ΔC
$(l_c, l_{ud}), (r_c, r_{ud})$	0.6556	0.7032	-2.0476	1.4556	0.5921	-1.1365	1.2281	0.7648	-1.5367
$(l_c, l_{ud}), (r_{cd}, r_u)$	0.7926	1.1228	-2.3302	0.6720	0.5238	-1.8519	1.3525	0.9192	-1.5667
$(l_c, l_{ud}), (r_{cu}, r_d)$	0.9407	1.3937	-2.4529	1.2857	0.7460	-1.4603	0.9247	0.5181	-1.5934
$(l_c, l_{ud}), (r_c, r_u)$	0.7349	0.9286	-2.1937	1.2714	0.5381	-1.2667	1.4568	0.9931	-1.5363
$(l_c, l_{ud}), (r_c, r_d)$	1.1381	1.4048	-2.2667	1.6397	0.7063	-1.0667	0.9993	0.5365	-1.5371
$(l_c, l_{ud}), (r_u, r_d)$	0.9079	1.5111	-2.6032	1.2190	0.8254	-1.6063	1.1431	0.7703	-1.6271
$(l_{cd}, l_u), (r_c, r_{ud})$	0.7841	0.4688	-1.6847	1.0423	0.6106	-1.5683	1.3443	0.7911	-1.4469
$(l_{cd}, l_u), (r_{cd}, r_u)$	1.3418	1.6402	-2.2984	1.0681	0.4804	-1.4123	1.4357	0.9428	-1.5070
$(l_{cd}, l_u), (r_{cu}, r_d)$	0.8127	1.6275	-2.8148	0.9975	0.5284	-1.5309	0.7726	0.5453	-1.7727
$(l_{cd}, l_u), (r_c, r_u)$	1.3175	1.3683	-2.0508	0.9619	0.6106	-1.6487	1.6412	1.0639	-1.4227
$(l_{cd}, l_u), (r_c, r_d)$	0.7884	1.2159	-2.4275	1.3788	0.7037	-1.3249	1.0473	0.5867	-1.5394
$(l_{cd}, l_u), (r_u, r_d)$	1.3905	2.4508	-3.0603	1.2804	0.4487	-1.1683	1.0235	0.6986	-1.6751
$(l_{cu}, l_d), (r_c, r_{ud})$	0.6212	0.9725	-2.3513	0.9153	0.6868	-1.7714	0.9903	0.4030	-1.4127
$(l_{cu}, l_d), (r_{cd}, r_u)$	1.0328	1.3848	-2.3520	0.6603	0.6145	-1.9541	0.8142	0.5372	-1.7230
$(l_{cu}, l_d), (r_{cu}, r_d)$	1.3051	1.4399	-2.1347	1.7129	0.8698	-1.1570	0.8240	0.2918	-1.4677
$(l_{cu}, l_d), (r_c, r_u)$	0.9958	1.4889	-2.4931	1.1291	0.6423	-1.5132	0.9988	0.5452	-1.5464
$(l_{cu}, l_d), (r_c, r_d)$	1.2794	1.3704	-2.0910	1.6857	0.8646	-1.1788	0.9818	0.2608	-1.2790
$(l_{cu}, l_d), (r_u, r_d)$	1.3566	1.5788	-2.2222	1.7672	0.8804	-1.1132	0.5084	0.5496	-2.0412
$(l_c, l_u), (r_c, r_{ud})$	0.9286	0.5571	-1.6286	1.5476	0.6492	-1.1016	1.3842	0.9073	-1.5231
$(l_c, l_u), (r_{cd}, r_u)$	1.3513	1.7915	-2.4402	0.9534	0.5069	-1.5534	1.6014	1.1021	-1.5007
$(l_c, l_u), (r_{cu}, r_d)$	0.8095	1.6328	-2.8233	1.6815	0.6296	-0.9481	0.8847	0.6947	-1.8101
$(l_c, l_d), (r_c, r_{ud})$	0.6333	1.1063	-2.4730	1.3635	0.6238	-1.2603	1.0723	0.6218	-1.5495
$(l_c, l_d), (r_{cd}, r_u)$	1.1016	1.1968	-2.0952	0.8265	0.7757	-1.9492	1.1043	0.7363	-1.6320
$(l_c, l_d), (r_{cu}, r_d)$	1.3683	1.3513	-1.9831	1.6815	0.8624	-1.1810	0.9647	0.4479	-1.4833
$(l_u, l_d), (r_c, r_{ud})$	0.5968	0.9143	-2.3175	1.2317	0.8127	-1.5810	1.2643	0.5585	-1.2942
$(l_u, l_d), (r_{cd}, r_u)$	1.3228	1.7608	-2.4381	1.2974	0.6180	-1.3206	1.1044	0.6240	-1.5195
$(l_u, l_d), (r_{cu}, r_d)$	1.1788	1.6169	-2.4381	1.7757	0.8847	-1.1090	0.5485	0.4365	-1.8880

We have found that for the participants in the study there was no evidence of contextuality in their responses. These results add to the existing evidence that points towards lack of contextuality in psychology (cf. Ref. [13].)

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3. ADVANCED ANALYSIS OF QUANTUM CONTEXTUALITY IN A PSYCHOPHYSICAL DOUBLE-DETECTION EXPERIMENT

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Abstract The results of behavioral experiments typically exhibit inconsistent connectedness, i.e., they violate the condition known as "no-signaling," "no-disturbance," or "marginal selectivity." This prevents one from evaluating these experiments in terms of quantum contextuality if the latter understood traditionally (as, e.g., in the Kochen-Specker theorem or Bell-type inequalities). The Contextuality-by-Default (CbD) theory separates contextuality from inconsistent connectedness. When applied to quantum physical experiments that exhibit inconsistent connectedness (due to context-dependent errors and/or signaling), the CbD computations reveal quantum contextuality in spite of this. When applied to a large body of published behavioral experiments, the CbD computations reveal no quantum contextuality: all context-dependence in these experiments is described by inconsistent connectedness alone. Until recently, however, experimental analysis of contextuality was confined to so-called cyclic systems of binary random variables. Here, we present the results of a psychophysical double-detection experiment that do not form a cyclic system: their analysis requires that we use a recent modification of CbD, one that makes the class of noncontextual systems more restricted. Nevertheless our results once again indicate

that when inconsistent connectedness is taken into account, the system exhibits no contextuality.

Keywords: contextuality, cyclic systems, double-detection, inconsistent connectedness, psychophysics.

In recent years there were many reports of behavioral experiments (Accardi, Khrennikov, Ohya, Tanaka, & Yamato, 2016; Aerts & Sozzo, 2014, 2015; Aerts, Sozzo, & Veloz, 2015; Asano, Hashimoto, Khrennikov, Ohya, & Tanaka, 2014; Bruza, Kitto, Ramm, & Sitbon, 2015; Cervantes & Dzhafarov, 2017; Dzhafarov, Zhang, & Kujala, 2015; Khrennikov, 2015; Sozzo, 2015; Wang, Solloway, Shiffrin, & Busemeyer, 2014; Zhang & Dzhafarov, 2017) aimed at (or interpretable as aimed at) revealing *contextuality* of the kind predicted by and experimentally confirmed in quantum physics (Bell, 1964; Clauser, Horne, Shimony, & Holt, 1969; Fine, 1982; Hensen et al., 2015; Klyachko, Can, Binicioğlu, & Shumovsky, 2008; Kochen & Specker, 1967; Kurzyński, Ramanathan, & Kaszlikowski, 2012; Lapkiewicz et al., 2011). All known to us behavioral data, however, violate a certain condition that makes a direct application of the traditional quantum contextuality analysis impossible. This condition is variously called "no-signaling" or "no-disturbance" in quantum physics (Bacciagaluppi, 2015, 2016; Cereceda, 2000; Kofler & Brukner, 2013; Kurzyński, Cabello, & Kaszlikowski, 2014; Popescu & Rohrlich, 1994; Ramanathan, Soeda, Kurzyński, & Kaszlikowski, 2012) and "marginal selectivity" in psychology (Dzhafarov, 2003; Townsend & Schweickert, 1989; Zhang & Dzhafarov, 2015). It is a required condition for the traditional quantum contextuality analysis, even though it is often violated in quantum mechanical experiments as well (this issue was first systematically discussed in Adenier & Khrennikov, 2007; see also Adenier & Khrennikov, 2016; Lapkiewicz et al., 2011, 2013). The Contextuality-by-Default (CbD) theory (de Barros, Dzhafarov, Kujala, &

Oas, 2015; Dzhafarov, 2016; Dzhafarov & Kujala, 2014a, 2014b, 2015, 2016a, 2016b, 2017a, in press; Dzhafarov, Kujala, & Cervantes, 2016; Dzhafarov, Kujala, & Larsson, 2015) overcomes this difficulty by proposing a principled way of separating contextuality proper from *inconsistent connectedness* (the CbD term for violations of the "no-signaling" or "marginal selectivity" condition). This theory was used to reanalyze the behavioral experiments aimed at contextuality, with the conclusion that they provide no evidence for contextuality (Cervantes & Dzhafarov, 2017; Dzhafarov, Kujala, Cervantes, Zhang, & Jones, 2016; Dzhafarov, Zhang, & Kujala, 2015; Zhang & Dzhafarov, 2017): inconsistent connectedness is the only form of context-dependence that we have in them. By contrast, when CbD is used to reanalyze a quantum-mechanical experiment that exhibits inconsistent connectedness (Lapkiewicz et al., 2011), contextuality proper (on top of inconsistent connectedness) is established beyond doubt (Kujala, Dzhafarov, & Larsson, 2015).

Virtually all experiments aimed at revealing contextuality, both in quantum physics and in behavioral sciences, deal with a special kind of systems of random variables, called *cyclic systems* in CbD (Kujala et al., 2015). In these systems each property is measured in precisely two different contexts, and each context contains two properties being measured together. If, in addition, all random variables in the system are binary (each indicating presence or absence of a certain property), then the system is amenable to complete and exhaustive contextuality analysis (Dzhafarov & Kujala, 2016a; Dzhafarov, Kujala, & Cervantes, 2016; Dzhafarov, Kujala, & Larsson, 2015; Kujala et al., 2015). In spite of their prominence in quantum theory, however, it is highly desirable to extend contextuality analysis beyond the class of cyclic systems. Many researchers (although not the present authors) find the lack of contextuality in behavioral data to be a disappointing negative result. What if this result is due to the fact that cyclic systems in human behavior are too simple? What if it is "too easy" for a cyclic system to be noncontextual? These are valid questions, and they will have no definite answers until we have a predictive theory of (at least certain types of) human behavior on a par with quantum mechanics.

In the absence of a predictive theory, the only, admittedly imperfect way of dealing with these considerations is to expand the experimentation and contextuality analysis to progressively broader classes of systems. In this paper we make a first step in this direction by analyzing a psychophysical experiment whose results form a non-cyclic system of random variables. This experiment was reported previously (Cervantes & Dzhafarov, 2017), but its analysis was confined to extracting from it a large number of cyclic subsystems and showing all of them to be noncontextual. It is mathematically possible, however, that a system is contextual with all its cyclic subsystems being noncontextual.

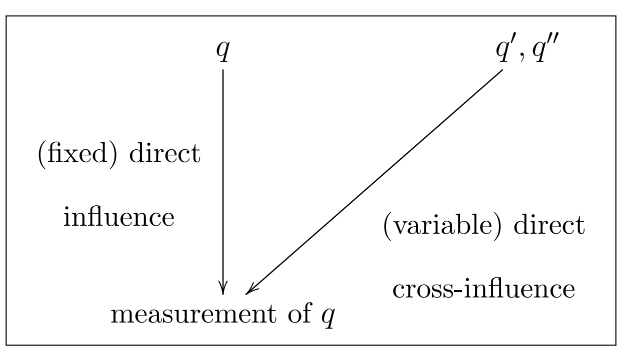
A satisfactory way to expand the contextuality analysis beyond cyclic systems was proposed in a recent modification of CbD, dubbed "CbD 2.0" (Dzhafarov & Kujala, 2017a, in press): it is essentially the original CbD in which the measurements of the same property (say, responses to the same stimulus) are analyzed in pairs only. This modification has compelling reasons behind it, The main one is that in the modified theory a subsystem of a noncontextual system is always noncontextual. Another reason is that contextuality analysis is reduced to the problem of compatibility of two *uniquely defined* sets of distributions: the empirically known distributions of context-sharing random variables and the distributions of the "multimaximal couplings" of the random variables measuring the same property in different contexts. All of this is clarified below (Section 3.2). The modification in question does not affect the theory of cyclic systems, so the results mentioned earlier remain unchanged. However, when it comes to non-cyclic systems, the modification makes the requirements that a system should satisfy to be noncontextual more stringent.

The plan of the paper is as follows. In Sections 3.1 and 3.2 we present the basics of the CbD theory, in the "CbD 2.0" version. The discussion is primarily confined to systems of binary random variables (dichotomic measurements), both for simplicity and because the double-detection experiment to be analyzed involves only dichotomic judgments. In Section 3.3 we apply this theory to the results of our double-detection experiment. Our conclusion is that in spite of the notion of noncontextuality we use being more restrictive than in the original version of the CbD theory, the double detection experiment does not exhibit any contextuality.

3.1 Introduction to contextuality

Every experiment results in a system of random variables. In most physics experiments these random variables are interpreted as measurements of properties, in most behavioral experiments they are interpreted as responses to stimuli, such as questions. For brevity we will use the term "measurement" in both meanings (because responding to a stimulus can always be viewed as a form of measurement). *What* is being measured therefore is part of the identity of a random variable representing a measurement. It is referred to as the *content* of the random variable. The content, however, does not specify a random variable uniquely, because one and the same content can be measured under different conditions, referred to as *contexts*. For instance, if a content q is measured simultaneously with measurements of other contents, in some cases q' and in other cases q'', then in the former cases the context is $c = (q, q')$ and in the latter ones it is $c' = (q, q'')$. As in Dzhafarov and Kujala (2016a, 2017a), we will write "conteXt" and "conteNt" to prevent their confusion in reading. The conteXt and conteNt of a random variable uniquely identify it within a given system of random variables. So each random variable in a system is double-indexed, R_q^c.

According to the CbD theory's main principle (Dzhafarov, 2016; Dzhafarov & Kujala, 2014a, 2016a, 2016b, in press; Dzhafarov, Kujala, & Cervantes, 2016), two random variables R_q^c and $R_{q'}^{c'}$ are jointly distributed if and only if $c = c'$, i.e., if and only if they are recorded in the same conteXt. Otherwise they are *stochastically unrelated*, i.e., joint probabilities for them are undefined. This means, in particular, that any two R_q^c and $R_q^{c'}$ with the same conteNt in different conteXts are stochastically unrelated (which implies, among other things, that they can never be considered to be one and the same random variable). Their individual distributions may be the same but they need not be. If these distributions are different, the system exhibits a form of context-dependence. However, in CbD, this context-dependence by itself does not say that the system is contextual in the sense related to how this term is used in quantum mechanics. Rather the difference in the distributions is treated as manifestation of information/energy flowing to the measurements of conteNt q from elements of the contexts c, c' other than q. We will refer to this transfer of information/energy as *direct cross-influences*. Thus, if $c = (q, q')$ and $c' = (q, q'')$, the conteNt q does, of course, directly influence its measurement, but, with q fixed, the second conteNt in the pair can also affect this measurement. This can sometimes be attributed to some physical action of q' or q'' upon the process measuring q, or (as another form of information transfer) it can be a form of contextual bias, a change in the procedure by which q is measured depending on what else is being measured.



The difference between the distributions of R_q^c and $R_q^{c'}$ (equivalently, the strength of the direct cross-influences responsible for this difference) is measured in CbD by the probability with which R_q^c and $R_q^{c'}$ could be made to coincide *if they were jointly distributed*. This means that we consider all *couplings* of $R_q^c, R_q^{c'}$, i.e., the jointly distributed pairs of random variables $T_q^c, T_q^{c'}$ whose respective individual distributions are the same as those of $R_q^c, R_q^{c'}$, and among these pairs we find the one(s) with the maximal possible probability of $T_q^c = T_q^{c'}$. The larger this maximal probability, the closer the two distributions to each other, and the weaker the direct cross-influences by conteXts c, c' upon the measurement of q. This maximal probability is 1 if and only if the two distributions are identical, and it is 0 if and only if the two distributions have disjoint supports.

Consider now an experiment represented by a system of random variables R_q^c with varying c and q, and suppose that we have computed the maximal probability just described for each pair of random variables that share a conteNt. And we know (or can empirically estimate) the joint distributions of all random variables that share a conteXt. Intuitively, quantum contextuality is about whether these computed maximal probabilities and these empirically defined joint distributions are mutually compatible. If they are not, then one can say that conteXts force the random variables sharing conteNts to be more dissimilar than they are made by direct cross-influences alone. The system then can be considered *contextual*.

To understand this without conceptual and technical complications, consider first a cyclic system of binary random variables (Dzhafarov, Kujala, & Larsson, 2015; Kujala & Dzhafarov, 2015; Kujala et al., 2015). It is depicted in Fig. 3.1. The conteXts and conteNts are such that, with appropriate enumeration, in conteXt c_i

one measures precisely two cyclically-successive conteNts $q_i, q_{i\oplus 1}$ (where $i = 1, \dots, n$; $i \oplus 1 = i + 1$ for $i < n$; and $n \oplus 1 = 1$):

$$
\begin{tikzcd}
q_1 \arrow[r, "c_1"] & q_2 \arrow[r, "c_2"] & \cdots \arrow[r, "c_{n-2}"] & q_{n-1} \arrow[r, "c_{n-1}"] & q_n, \arrow[llll, bend left, "c_n"]
\end{tikzcd}
$$

Each pair $R_i^i, R_{i\oplus 1}^i$ ($i = 1, \dots, n$) of random variables sharing a conteXt (within a row in Fig. 3.1) are jointly distributed. Since all the measurements in the system are binary (± 1), the joint distribution of R_j^i is uniquely determined by three probabilities,

$$
\begin{gathered}
p_i^i = \Pr\left[R_i^i = 1\right], \; p_{i\oplus 1}^i = \Pr\left[R_{i\oplus 1}^i = 1\right], \\
p^i = \Pr\left[R_i^i = R_{i\oplus 1}^i = 1\right].
\end{gathered}
\tag{3.1}
$$

Random variables $R_i^{i\ominus 1}, R_i^i$ within a column share a conteNt, and we compute for each such a pair the magnitude of direct cross-influences, $\max \Pr\left[T_i^i = T_i^{i\ominus 1}\right]$, across all couplings $\left(T_i^{i\ominus 1}, T_i^i\right)$ of $R_i^{i\ominus 1}, R_i^i$: in this case the couplings are the pairs $\left(T_i^{i\ominus 1}, T_i^i\right)$ with all possible values of $\Pr\left[T_i^i = T_i^{i\ominus 1} = 1\right]$ and with

$$
\Pr\left[T_i^i = 1\right] = p_i^i, \; \Pr\left[T_i^{i\ominus 1} = 1\right] = p_i^{i\ominus 1}. \tag{3.2}
$$

Here, $i = 1, \dots, n$; $i \ominus 1 = i - 1$ for $i > 1$; and $1 \ominus 1 = n$. The coupling $\left(T_i^{i\ominus 1}, T_i^i\right)$ with this property is called *maximal coupling*. It is easy to show (Thorisson, 2000) that this maximal coupling always exists and is defined by complementing (3.2) with

$$
p_i = \Pr\left[T_i^i = T_i^{i\ominus 1} = 1\right] = \min\left\{p_i^i, p_i^{i\ominus 1}\right\}. \tag{3.3}
$$

The probabilities (3.1) and (3.3) are shown in Fig. 3.2. Note that (3.2) and (3.3) uniquely define the joint distribution of the two random variables $T_i^{i\ominus 1}, T_i^i$ within each

⋆	⋆			⋯			c_1
	⋆	⋆		⋯			c_2
		⋆	⋆	⋯			c_3
⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮
				⋯	⋆	⋆	c_{n-1}
⋆				⋯		⋆	c_n
q_1	q_2	q_3	q_4	⋯	q_{n-1}	q_n	CYC

Fig. 3.1. A cyclic system (shown here for a sufficiently large n, although n can be as small as 2 or 3). The system involves n conteNts $q_1, \ldots, q_n$ and n conteXts $c_1, \ldots, c_n$. The star symbol in the (c_i, q_j)-cell indicates that conteNt q_j was measured in conteXt c_i, and the result of the measurement is random variable R_j^i; otherwise q_j was not measured in c_i and the cell is left empty. All R_j^i are binary random variables, with possible values denoted $+1$ and -1.

column of the matrix, in the same way as (3.1) uniquely define the joint distribution of $R_i^i, R_{i\oplus1}^i$ within each row of the matrix. The only difference is that the row-wise joint distributions are empirical reality, whereas the column-wise joint distributions are constructed artificially to depict the direct cross-influences. Contextuality in CbD is all about the compatibility of these column-wise and row-wise joint distributions: the system is considered noncontextual if all these probabilities can be achieved within a jointly distributed set of $2n$ random variables. In other words[a], we seek a set of jointly distributed random variables S_j^i replacing the star symbols in Fig. 3.1, such that

$$\begin{array}{ll} (i) & \Pr\left[S_i^i = 1\right] = p_i^i,\ \Pr\left[S_{i\oplus1}^i = 1\right] = p_{i\oplus1}^i, \\ \\ (ii) & \Pr\left[S_i^i = S_{i\oplus1}^i = 1\right] = p^i, \\ \\ (iii) & \Pr\left[S_i^i = S_i^{i\ominus1} = 1\right] = p_i = \min\left\{p_i^i, p_i^{i\ominus1}\right\} \end{array} \tag{3.4}$$

[a](Erratum note added in the dissertation.) Originally "word". Corrected in the text.

p_1^1	p_2^1			$\cdots$			p^1
	p_2^2	p_3^2		$\cdots$			p^2
		p_3^3	p_4^3	$\cdots$			p^3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
				$\cdots$	p_{n-1}^{n-1}	p_n^{n-1}	p^{n-1}
p_1^{n-1}				$\cdots$		p_n^n	p^n
p_1	p_2	p_3	p	$\cdots$	p_{n-1}	p_n	$\boxed{\text{CYC}}$

Fig. 3.2. The probability values that characterize the cyclic system in Fig. 3.1 in accordance with (3.1) and (3.3). The system is noncontextual if there is a set of $2n$ jointly distributed random variables $\left(S_j^i : i = 1, \ldots, n; j = i \text{ or } j = i \oplus 1\right)$ with $\Pr\left[S_j^i = 1\right] = p_j^i$, $\Pr\left[S_i^i = S_{i\oplus 1}^i = 1\right] = p^i$, and $\Pr\left[S_i^i = S_i^{i\ominus 1} = 1\right] = p_i = \min\left\{p_i^i, p_i^{i\ominus 1}\right\}$.

The equations (i) and (ii) in (3.4) tell us that the set of the S_j^i-variables we seek is a coupling of the original random variables R_j^i arranged row-wise in Fig. 3.1: in each row the variables R_j^i have a well-defined joint distribution, but different rows are stochastically unrelated, so the coupling "saws them together" in a single joint distribution. The equations (i) and (iii) in (3.4) tell us that the set of the S_j^i-variables is a coupling for the column-wise maximal couplings T_j^i: in each of the columns the variables T_j^i have a well-defined joint distribution, but different columns are stochastically unrelated because the maximal couplings were computed for each column separately; so the coupling "saws the columns together" in a single joint distribution. It is easy to see that each of these two couplings (of the rows and of the columns) exists, because the random variables in the different rows do not overlap, and the same is true for different columns. In a typical case, each of the two couplings can be constructed in an infinity of ways, and the question is whether a jointly distributed set of $2n$ random variables can be simultaneously a coupling for the rows and for the columns. If the answer to this question is negative, the conteXts intervene beyond the effect of the direct cross-influences.

	q_1	q_2	q_3	q_4	
	⋆	⋆			c_1
		⋆	⋆		c_2
			⋆	⋆	c_3
	⋆			⋆	c_4
					$\boxed{\mathsf{CYC}_4}$

	q_1	q_2	q_3	q_4	
	⋆	⋆		⋆	c_1
		⋆	⋆		c_2
			⋆	⋆	c_3
	⋆			⋆	c_4
					$\boxed{\mathsf{X}}$

Fig. 3.3. A cyclic system with $n = 4$ (CYC_4) and a system X obtained from CYC_4 by adding to it the random variable R_4^1.

3.2 Contextuality in arbitrary systems of binary measurements

Let us discuss now how the analysis just presented extends beyond cyclic systems. We will continue to assume that all the random variables in play are binary.

Consider Fig. 3.3. The system X is not cyclic, as it has three random variables in the first row (conteXt c_1) and three random variables in the fourth column (conteNt q_4). The number and arrangement of the random variables in a row, however, is immaterial for the logic of the contextuality analysis. The joint distribution of R_1^1, R_2^1, R_4^1 in the first row of X is uniquely defined empirically. It simply requires more probabilities than in (3.1) to be described:

$$
\begin{gathered}
p_1^1 = \Pr\left[R_1^1 = 1\right],\ p_2^1 = \Pr\left[R_2^1 = 1\right],\\
p_4^1 = \Pr\left[R_4^1 = 1\right],\\
\\
p_{12}^1 = \Pr\left[R_1^1 = R_2^1 = 1\right],\\
p_{24}^1 = \Pr\left[R_2^1 = R_4^1 = 1\right],\\
p_{14}^1 = \Pr\left[R_1^1 = R_4^1 = 1\right],\\
\\
p_{124}^1 = \Pr\left[R_1^1 = R_2^1 = R_4^1 = 1\right].
\end{gathered}
\tag{3.5}
$$

Nor does anything change in how one treats the pairs of the conteNt-sharing random variables in the first three columns: one computes the maximal coupling for each of these columns. One faces choices, however, when dealing with the three random variables in the fourth column. What is the right way of generalizing the maximal coupling in this case? There is a compelling reason (Dzhafarov & Kujala, 2017a, in press) to consider the three conteNt-sharing random variables one pair at a time, and to compute maximal couplings for them separately. This means finding a jointly distributed triple (T_4^1, T_4^3, T_4^4) whose elements are distributional copies of, respectively, R_4^1, R_4^3, R_4^4, i.e.,

$$\begin{gathered} \Pr\left[T_4^1 = 1\right] = p_4^1,\ \Pr\left[T_4^3 = 1\right] = p_4^3, \\ \Pr\left[T_4^4 = 1\right] = p_4^4, \end{gathered} \tag{3.6}$$

such that (T_4^1, T_4^3) is the maximal coupling of R_4^1, R_4^3, (T_4^3, T_4^4) is the maximal coupling of R_4^3, R_4^4, and (T_4^1, T_4^4) is the maximal coupling of R_4^1, R_4^4. In terms of probability values,

$$\begin{aligned} \Pr\left[T_4^1 = T_4^3 = 1\right] &= \min\left\{p_4^1, p_4^3\right\}, \\ \Pr\left[T_4^3 = T_4^4 = 1\right] &= \min\left\{p_4^3, p_4^4\right\}, \\ \Pr\left[T_4^1 = T_4^4 = 1\right] &= \min\left\{p_4^1, p_4^4\right\}. \end{aligned} \tag{3.7}$$

As shown in Dzhafarov and Kujala (2017a, in press), such a coupling (called *multi-maximal* in CbD) always exists, and it is unique (as all the random variables here are binary). The above-mentioned compelling reason for maximizing the couplings pairwise is that then, if the system is noncontextual, it will remain noncontextual after one deletes from it one or more random variables. In other words, any subsystem of a noncontextual system is noncontextual. This would not be true, for instance, if we only maximized the value of $\Pr\left[T_1^1 = T_2^1 = T_4^1 = 1\right]$. At the same time, the maximization of $\Pr\left[T_1^1 = T_2^1 = T_4^1 = 1\right]$ is achieved "automatically" if (3.7) is satisfied. Moreover, one of the equalities in (3.7) is redundant as it can be derived from the other two: if, e.g., $p_4^3 \leq p_4^1 \leq p_4^4$,[b] then the redundant equality in (3.7) is the second one. Generalizing, we have the following theorem.

Theorem 1 (Dzhafarov & Kujala, 2017a, in press)**.** Let $R_q^1, \ldots, R_q^k$, $k > 1$, be binary (± 1) random variables with conteXts enumerated so that

$$p_q^1 = \Pr\left[R_q^1 = 1\right] \leq \ldots \leq \Pr\left[R_q^k = 1\right] = p_q^k.$$

Then there is a unique set of jointly distributed $\left(T_q^1, \ldots, T_q^k\right)$ such that $\left(T_q^i, T_q^{i+1}\right)$ is the maximal coupling of R_q^i, R_q^{i+1}, for $i = 1, \ldots, k-1$. The coupling $\left(T_q^1, \ldots, T_q^k\right)$ has the following properties.

(i) For any subset $\{i_1, \ldots, i_m\} \subseteq (1, \ldots, k)$ with $m \leq k$, $\left(T_q^{i_1}, \ldots, T_q^{i_m}\right)$ is the maximal coupling of $R_q^{i_1}, \ldots, R_q^{i_m}$, i.e., $\Pr\left[T_q^{i_1} = \ldots = T_q^{i_m}\right]$ has the maximal possible value among all couplings of $R_q^{i_1}, \ldots, R_q^{i_m}$. In particular, for any $i, j \in (1, \ldots, k)$, $\left(T_q^i, T_q^j\right)$ is the maximal coupling of $R_q^i, \ldots, R_q^j$.

(ii) The distribution of $\left(T_q^1, \ldots, T_q^k\right)$[c] is defined by

$$\Pr\left[T_q^1 = \ldots = T_q^k = 1\right] = p_1,$$

$$\begin{gathered}\Pr\left[T_q^1 = \ldots = T_q^l = -1\ ;\ T_q^{l+1} = \ldots = T_q^k = 1\right] = p_{l+1} - p_l, \\ (\text{for } l = 1, \ldots, k-1)\end{gathered} \tag{3.8}$$

$$\Pr\left[T_q^1 = \ldots = T_q^k = -1\right] = 1 - p_k,$$

with all other combinations of values having probability zero.

[b](Erratum note added in the dissertation.) Originally $p_4^3 \leq p_4^1 \leq p_{44}$. Corrected in the text.

[c](Erratum note added in the dissertation.) Originally $\left(T_q^{i_1}, \ldots, T_q^{i_m}\right)$. Corrected in the text.

Now we can formulate the generalization of the definition of contextuality given in the previous section.

Definition 2. A system of binary random variables R_q^c is noncontextual if there exists a jointly distributed set of (correspondingly labeled) random variables S_q^c such that (i) for every conteXt c, the joint distribution of all S_q^c with this value of c is identical to the joint distribution of the corresponding R_q^c; and (ii) for every conteNt q, the joint distribution of all S_q^c with this value of q forms the (unique) multimaximal coupling of the corresponding R_q^c.

The notion of contextuality is, once again, about compatibility of the uniquely determined row-wise and column-wise distributions. The row distributions are empirically given, the column distributions are computed as multimaximal couplings, and the question is whether it is possible to find a single coupling for both the rows and the columns. Once again, the logic of the approach is that if the coupling in question does not exist, it means that the conteXts force some pairs of the random variables measuring the same conteNt to be more dissimilar than they are made by direct cross-influences alone — and the system is therefore contextual.

If a system of random variables turns out to be contextual, one can compute the degree of its contextuality as the smallest possible total variation of *quasi-couplings* of this system. A quasi-coupling differs from a coupling in that the probabilities for its values are replaced with arbitrary real numbers (not necessarily nonnegative) that sum to 1. The existence of quasi-couplings for any system and the uniqueness of the minimum total variation are proved in Dzhafarov and Kujala (2016a). We need not discuss this otherwise important topic further because the experimental results reported below reveal no contextuality.

3.3 Double-detection experiment

We now apply the theory just described to the results of a double-detection experiment. We remind the reader that this experiment was previously described in Cervantes and Dzhafarov (2017), but to keep this paper self-sufficient we recapitulate the procedural details below. In Cervantes and Dzhafarov (2017) the system formed by the data was analyzed by extracting from it a multitude of cyclic subsystems. In this paper we analyze the system in its entirety.

The double-detection experiment is one of only two contextuality-aimed experiments known to us that uses a within-subject design, i.e., with probabilities estimated from the responses of a single person to multiple replications of stimuli. (The other such experiment is the psychophysical matching one described in Dzhafarov, Zhang, & Kujala, 2015; Zhang & Dzhafarov, 2017.) Most experiments use aggregation of responses obtained from many persons. The double detection paradigm suggested in Dzhafarov and Kujala (2012, 2017b) provides a framework where both (in)consistent connectedness and contextuality can be studied in a manner very similar to how they are studied in quantum-mechanical systems (or could be studied, because consistent connectedness in quantum physics is often assumed rather than documented).

3.3.1 Method

3.3.1.1 Participants

The participants were three volunteers, graduate students at Purdue University, two females and one male (the first author of this paper), aged around 30, with normal or corrected to normal vision. The experimental program was regulated by the Purdue University's IRB protocol #1202011876. The participants are identified as $P1 - P3$ in the text below.

3.3.1.2 Equipment

A personal computer was used with an Intel® Core™ processor running Windows XP, a 24-in. monitor with a resolution of 1920×1200 pixels (px), and a standard US 104-key keyboard. The participant's head was steadied in a chin-rest with forehead support at 90 cm distance from the monitor; at this distance a pixel on the screen subtended 62 s arc.

3.3.1.3 Stimuli

The stimuli presented on the computer screen consisted of two brightly grey colored circles (RGB 100-100-100) on a black background, with their centers 320 px apart horizontally, each circle having the radius of 135 px and circumference 4 px wide. Each circle contained a dot of 4 px in diameter in its center or 4 px away from it, in the upward or downward direction. An example of the stimuli (in reversed contrast and scaled) is shown in Fig. 3.4.

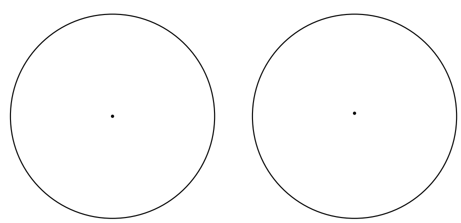


Fig. 3.4. An example of the stimulus in the double-detection experiment. In the left circle the dot is in the center, in the right one it is shifted 4 px upwards. The participant's task was to say, for each of the two circles, whether the dot was in the center (the answer coded 1) or off-center (the answer coded -1), irrespective of whether it was shifted up or down.

3.3.1.4 Procedure

In each trial the participant was asked to indicate, for each circle, whether the dot was in its center or not in the center (irrespective of in what direction). The responses were given by pressing in any order and holding together two designated keys, one for each circle, and the stimuli were displayed until both keys were pressed. Then, the dots in each circle disappeared, and a “Press the space bar to continue” message appeared above the circles. Pressing the space bar removed the message, and the next pair of dots appeared 400 ms later. (Response times were recorded but not used in the data analysis.)

Each participant completed nine experimental sessions, each lasting 30 minutes and containing about 560 trials recorded and used for the analysis, preceded by several practice trials. In each practice trial the participants received feedback as to whether their response for each of the two circles was correct or not. No feedback was given in the non-practice trials. The experimental sessions were preceded by one to three training sessions, excluded from the analysis.

3.3.2 Experimental ConteXts and ConteNts

In each of two circles the dot presented could be located either at its center, or 4 px above the center, or else 4 px under the center. These pairs of locations produce a total of nine conteXts. During each session, excepting the practice trials, the dot was presented at the center in a half of the trials, above the center in a quarter of them, and below the center in the remaining quarter, for each of the circles. Fig. 3.5 presents the proportions of allocations of trials to each of the 9 conditions.

For each session, each trial was randomly assigned to one of the conditions in accordance with Table 3.5. The number of experimental sessions was chosen so that

		Right			
		Center (-c)	Up (-u)	Down (-d)	
Left	Center (c-)	$1/4$ (cc)	$1/8$ (cu)	$1/8$ (cd)	$1/2$
	Up (u-)	$1/8$ (uc)	$1/16$ (uu)	$1/16$ (ud)	$1/4$
	Down (d-)	$1/8$ (dc)	$1/16$ (du)	$1/16$ (dd)	$1/4$
		$1/2$	$1/4$	$1/4$	

Fig. 3.5. Probabilities with which a trial was allocated to one of the 9 conteXts, with the notation used for the conteXts and the conteNts: c, u, and d denote that the dot is, respectively, in the center, shifted up, or shifted down. The 9 conteXts are denoted cc, cu, du, etc., the left (right) symbol indicating the location of the dot in the left (respectively, right) circle. To denote conteNts, the location of a dot is shown on the left or on the right with a dash filling the other side: thus, c- denotes the dot in the center of the left circle, -d denotes the dot shifted down in the right circle, etc.

the expected number of (non-practice) trials in the conditions with lowest probabilities was at least 300. This number of observations was chosen based on Cepeda Cuervo et al. (2008), whose results show that coverage errors with respect to nominal values are below 1% for almost all confidence intervals for proportions with $n > 300$.

The system of random variables representing the data is shown in Fig. 3.6.

3.3.3 Results

The results are shown in Figs. 3.7– 3.9, one for each of the three participants. Each row, together with its margins, specifies an empirical estimate of the joint distribution of the two random variables sharing the corresponding conteXt. This distribution is shown in the format

$$\Pr[X=1], \Pr[Y=1], \Pr[X=Y=1],$$

	c-	-c	u-	-u	d-	-d
cc	⋆	⋆				
uc		⋆	⋆			
uu			⋆	⋆		
du				⋆	⋆	
dd					⋆	⋆
cu	⋆			⋆		
ud			⋆			⋆
dc		⋆			⋆	
cd	⋆					⋆

Fig. 3.6. The conteNt-conteXt system of measurements for the double detection experiment. The cell corresponding to context xy and content z (with z being x- or -y), if it contains a star, represents the random variable R_z^{xy}; the absence of a star means that content z was not measured in context xy. For instance, $xy = cc$ and $z = c$- define a random variable $R_{c\text{-}}^{cc}$. The random variables within a given row (in the same conteXt) are jointly distributed. In our design there are two random variables, $R_{x\text{-}}^{xy}$ and $R_{\text{-}y}^{xy}$ in each conteXt xy, and their joint distribution is uniquely defined by three probabilities: $\Pr\left[R_{x\text{-}}^{xy} = 1\right]$, $\Pr\left[R_{\text{-}y}^{xy} = 1\right]$, and $\Pr\left[R_{x\text{-}}^{xy} = R_{\text{-}y}^{xy} = 1\right]$.

where X and Y are the two variables in the same row. Each column, together with its margins, shows an empirical estimate of the multimaximal coupling of the three random variables sharing the corresponding conteNt. The distribution of the coupling is shown in the format

$$\begin{array}{c} \Pr\left[A = 1\right] \\ \Pr\left[B = 1\right] \\ \Pr\left[C = 1\right] \\ \Pr\left[A = B = 1\right] \\ \Pr\left[B = C = 1\right] \\ \Pr\left[A = C = 1\right] \end{array},$$

where A, B, C are the three random variables in the same column listed from top down. The analysis of contextuality consists in considering a set of jointly distributed 18 binary random variables (corresponding to the star symbols in Fig. 3.6), and determining whether the 2^{18} values of this set can be assigned probabilities that sum to the probabilities whose empirical estimates are shown in the data matrices (Figs. 3.7–3.9). This is a standard linear programing task,

$$\underset{46\times 2^{18}}{\mathbf{M}}\overset{2^{18}\times 1}{\mathbf{Q}} = \underset{46\times 1}{\mathbf{P}},\ \mathbf{Q} > 0 \text{ (componentwise)}.$$

The number of the rows in $\mathbf{M}$ and $\mathbf{P}$ (i.e., the number of linear constraints imposed on $\mathbf{Q}$) is the number of the probability estimates shown in each of the data matrices (45) plus the constraint that ensures that all the 2^{18} probabilities in $\mathbf{Q}$ sum to 1. (The number of the probability estimates could be reduced from 45 to 39, because one of the three marginal probabilities for each column could be eliminated. We did not, however, make use of this small reduction in our computations.) The linear programing was performed by using the GLPK (GNU Linear Programming Kit) package (version 4.6; Makhorin, 2012) and the R interface to the package (Rglpk, version 0.6-1; Theussl & Hornik, 2015).

The outcome of the analysis is that, for all three participants, the system of linear equations has a solution in nonnegative values —that is, the data matrices in Figs. 3.7–3.9 describe noncontextual systems of random variables. Note that in this case the empirical estimates were fit by the solution precisely, eliminating the need for statistical analysis.

3.4 Conclusion

The experiment presented in this paper illustrates the use of the double factorial paradigm in the search of contextuality in behavioral systems, namely in the responses of human observers in a double-detection task. This paradigm provides the closest analogue in psychophysical research to the Alice-Bob EPR/Bohm paradigm (Bell, 1964; Clauser et al., 1969; Fine, 1982). We have found that for the participants in

P1	*c*-	-*c*	*u*-	-*u*	*d*-	-*d*	$\Pr[X = Y = 1]$	# of trials
cc	.7175	.6365					.5476	1260
uc		.5587	.2476				.2095	630
uu			.5238	.4857			.3746	315
du				.0444	.7810		.0286	315
dd					.7556	.6508	.5714	315
cu	.8095			.2302			.2175	630
ud			.0762			.4571	.0571	315
dc		.3032			.7937		.2778	630
cd	.4063					.6349	.3730	630
$\Pr[A = B = 1]$	.7175	.5587	.2476	.0444	.7556	.4571		
$\Pr[B = C = 1]$	.4063	.3032	.0762	.0444	.7556	.4571		
$\Pr[A = C = 1]$	.4063	.3032	.0762	.2302	.7810	.6349		

Fig. 3.7. Empirical data (relative frequencies) for the conteNt-conteXt system in Fig. 3.6 for participant P1. For every conteXt xy and every conteNt z measured in xy (either x- or -y), the cell for R_z^{xy} contains the frequency estimate of $\Pr[R_z^{xy} = 1]$; the right margins of the row for xy show the frequency estimate of $\Pr\left[R_{x\text{-}}^{xy} = R_{\text{-}y}^{xy} = 1\right]$ and the total number of measurements in this conteXt. Since xy and z vary, the column for joint probabilities denotes the two random variables by $X = R_{x\text{-}}^{xy}$ and $Y = R_{\text{-}y}^{xy}$. The bottom margins in the column for conteNt z show the three frequency estimates of the maximal values of $\Pr[R_z^{xy} = R_z^{uv} = 1]$, $\Pr[R_z^{uv} = R_z^{st} = 1]$, and $\Pr[R_z^{xy} = R_z^{st} = 1]$ (where xy, uv, st are three conteXts in which z was measured). To make notation compact, the three random variables in each column are labeled A, B, C (from top down), and the three probabilities are shown as $\Pr[A = B = 1]$, $\Pr[B = C = 1]$, and $\Pr[A = C = 1]$ (one of which is always redundant but shown for completeness).

the study there was no evidence of contextuality in their responses. These results add to the existing evidence that points towards lack of contextuality in behavioral data (Cervantes & Dzhafarov, 2017; Dzhafarov, Kujala, Cervantes, Zhang, & Jones, 2016; Dzhafarov, Zhang, & Kujala, 2015; Zhang & Dzhafarov, 2017). The present result is in fact stronger than the previous ones, as it uses a more stringent than before criterion of noncontextuality. This criterion is based on multimaximality rather than on the simple maximality of the couplings in cyclic systems. However, we

P2	c-	-c	u-	-u	d-	-d	$\Pr[X = Y = 1]$	# of trials
cc	.8659	.7841					.6746	1260
uc		.7619	.3968				.1968	630
uu			.5556	.5841			.3746	315
du				.6317	.1333		.0254	315
dd					.2413	.2032	.1175	315
cu	.8508			.4587			.3444	630
ud			.6127			.1111	.0063	315
dc		.8905			.1667		.1476	630
cd	.9429					.0683	.0571	630
$\Pr[A = B = 1]$	.8508	.7619	.3968	.5556	.1333	.1111		
$\Pr[B = C = 1]$	.8508	.7619	.5556	.4587	.1667	.0683		
$\Pr[A = C = 1]$	.8659	.7841	.3968	.4587	.1333	.0683		

Fig. 3.8. Empirical data (frequencies) for the conteNt-conteXt system in Fig. 3.6 for participant P2. The rest is as in Fig. 3.7.

P3	c-	-c	u-	-u	d-	-d	$\Pr[X = Y = 1]$	# of trials
cc	.6791	.5973					.3654	1259
uc		.8302	.1349				.0905	630
uu			.2548	.1688			.0732	314
du				.1460	.3746		.0127	315
dd					.3460	.4127	.1397	315
cu	.8381			.0746			.0524	630
ud			.1178			.3917	.0159	314
dc		.6714			.2921		.1127	630
cd	.6968					.3238	.1746	630
$\Pr[A = B = 1]$	.6791	.5973	.1349	.1460	.3460	.3917		
$\Pr[B = C = 1]$	.6968	.6714	.1178	.0746	.2921	.3238		
$\Pr[A = C = 1]$	.6791	.5973	.1178	.0746	.2921	.3238		

Fig. 3.9. Empirical data (frequencies) for the conteNt-conteXt system in Fig. 3.6 for participant P3. The rest is as in Fig. 3.7.

should emphasize that in the absence of a predictive theory on a par with quantum mechanics, no failure to find contextuality in even a large number of experiments can be safely generalized: contextuality may very well be found under as yet unexplored modifications of experimental conditions. Consider, e.g., the Alice-Bob EPR/Bohm paradigm, and imagine that we have no theory that could guide us in choosing the specific axes along which Alice and Bob are to measure the spins in their respective particles. It would be rather unlikely to hit at a "right" combination of the angles by pure chance, and after numerous failures one could very well conclude, in this case wrongly, that contextuality is absent in this paradigm. More work is needed.

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4. SNOW QUEEN IS EVIL AND BEAUTIFUL: EXPERIMENTAL EVIDENCE FOR PROBABILISTIC CONTEXTUALITY IN HUMAN CHOICES

 Cervantes, V. H., & Dzhafarov, E. N. (2018). Snow queen is evil and beautiful: Experimental evidence for probabilistic contextuality in human choices. *Decision, 5*(3), 193—204. https://doi.org/10.1037/dec0000095

Abstract We present unambiguous experimental evidence for (quantum-like) probabilistic contextuality in psychology. All previous attempts to find contextuality in a psychological experiment were unsuccessful because of the gross violations of marginal selectivity in behavioral data, making the traditional mathematical tests developed in quantum mechanics inapplicable. In our crowdsourcing experiment respondents were making two simple choices: of one of two characters in a story (The Snow Queen by Hans Christian Andersen), and of one of two characteristics, such as Kind and Evil, so that the character and the characteristic chosen matched the story line. The formal structure of the experiment imitated that of the Einstein-Podolsky-Rosen paradigm in the Bohm-Bell version. Marginal selectivity was violated, indicating that the two choices were directly influencing each other, but the application of a mathematical test developed in the Contextuality-by-Default theory, extending the traditional quantum-mechanical test, indicated a strong presence of contextuality proper, not reducible to direct influences.

Keywords: concept combinations, context-dependence, contextuality, direct influences, marginal selectivity.

It is commonplace to say that human behavior is *context-dependent*. What is usually meant by this is that one's response to stimulus S (performance in task S) depends on other stimuli (tasks) S'. Asked to explain the meaning of LINE, one's answer will depend on whether the word is preceded by CHORUS or OPENING. Visual size perception, if interpreted as a response to retinal size, is influenced by distance cues. In all such cases one can avoid speaking of context-dependence by simply including the relevant elements of S' into S: visual size is a response to both retinal size and distance cues, the meaning of LINE is a response to the word LINE and to the words preceding it. J. J. Gibson's (1950, 1960) psychophysics was, essentially, a change from understanding a percept as a response to a target stimulus modified by context stimuli (as, e.g., in H. von Helmholtz's, 1867, theory of unconscious inference) to a "direct" response to all relevant aspects of the optical flow.

This form of context-dependence is depicted in Figure 4.1, with the acknowledgement of the obvious fact that all psychological responses are *random variables*, generally varying from one presentation to another or from one person to another (Thurstonian cases I and II, respectively). Figure 4.1 therefore presents a *probabilistic response* R to S, such that its distribution is influenced not only by S but also by S'. This means, of course, that the *identity* of the response R as a random variable

$$\begin{tikzcd}
S \arrow[d, "directly"'] & S' \arrow[dl, bend right, "directly"'] \\
R &
\end{tikzcd}$$

Fig. 4.1. R is a random variable interpreted as a response to S: As S changes, the distribution of R generally changes. It also changes as the "context stimuli" S' change. The influences of S and S' upon R are both direct.

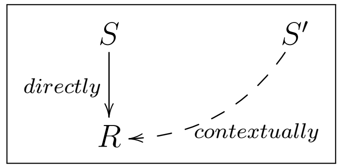

Fig. 4.2. The situation when R, interpreted as a response to S, changes its identity but not its distribution as "context stimuli" S' change. This can be revealed by looking at how R is codistributed with other random variables as S' changes. The influence of S on R is direct, while influence of S' on R is "purely contextual."

is different for different S', at a fixed S: One and the same random variable cannot have two different distributions.

One might think that all context-dependence is of this nature: we simply have some "secondary" factors influencing the distribution of one's response to a "primary" one. Quantum mechanics, however, provides striking examples of another form of context-dependence, when the distribution of R at a fixed S does not change with S', but R nevertheless is not one and the same random variable at different values of S'. This type of context-dependence is schematically depicted in Figure 4.2, and can be called "purely contextual." To make sure there is no logical problem here, different random variables R' and R'' may very well have the same distribution (as in the case of two different fair coins). One can distinguish them if, e.g., R' is positively correlated with some random variable A, R'' is negatively correlated with some B, and A always equals B. Obviously then, R' and R'' cannot be one and the same random variable, even if identically distributed.

An example of pure contextuality in quantum mechanics that is especially relevant for us (because our behavioral experiment follows its formal structure) is the Einstein-Podolsky-Rosen paradigm in the Bohm-Bell version (EPR/BB, Figure 4.3). Let us denote the spins along axes α_i and β_j by, respectively, A_i and B_j. As it turns out, the axes can be chosen so that it is impossible for the identity of A_i not to depend on the choice of β_j and for the identity of B_j not to depend on the choice of α_i. This is established by the following reasoning. If we assume that A_i is one and the same random variable under β_1 and β_2 (and analogously for B_j under α_1 and α_2), then we should have four jointly distributed random variables A_1, A_2, B_1, B_2, and the observed pairs of measurements by Alice and Bob then should be derivable from this distribution as its marginals (A_1, B_1), (A_1, B_2), (A_2, B_1), and (A_2, B_2). If so, these pairwise joint distributions should satisfy the following inequality, abbreviated CHSH (Clauser, Horne, Shimony, & Holt, 1969; Fine, 1982):

$$\max_{k,l\in\{1,2\}} \left| \sum_{i,j\in\{1,2\}} \mathsf{E}\left[A_i B_j\right] - 2\mathsf{E}\left[A_k B_l\right] \right| - 2 \leq 0, \tag{4.1}$$

where E is expected value. Now, the expected values in the CHSH inequality can be computed for any axes $\alpha_1, \alpha_2, \beta_1, \beta_2$ by using the principles of quantum mechanics, and it turns out that for certain choices of these axes these expected values violate the inequality. By reductio ad absurdum, therefore, we have to reject the initial assumption that A_i is the same for both choices of β_j and B_j is the same for both choices of α_i.

In other words, A_i and B_j measured together are in fact A_i^j and B_j^i, so that, e.g., A_1^1 (Alice's measurement along axis α_1 when Bob has chosen axis β_1) is different from A_1^2 (Alice's measurement along the same axis when Bob has chosen β_2). However, we do not have the same situation as in Figure 4.1: Bob's choice of an axis cannot directly influence Alice's measurement because this choice and the measurement are simultaneous (in some inertial frame of reference). They cannot be causally related.[1] In the past, this situation was often presented as paradoxical, with Einstein famously

[1]Note that the "pure contextuality" we have here is not a characteristic of the physical system comprised of the two particles in Figure 4.3. Rather it is a characteristic of the system of random variables representing a particular choice of two axes by Alice and two axes by Bob. For the same

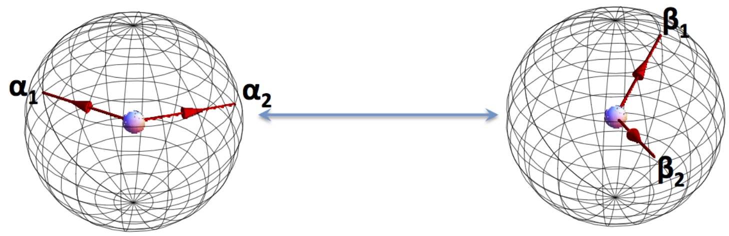

Fig. 4.3. Einstein-Podolsky-Rosen paradigm adapted to spins by Bohm (Bohm & Aharonov, 1957) and famously investigated by Bell (1964). Two spin-1/2 particles (e.g., electrons) are created in what is called a "singlet state" and move away from each other. Alice measures the spin of the left particle along one of the two axes denoted α_1 and α_2, Bob simultaneously does the same for the right particle along one of the two axes denoted β_1 and β_2. Spins are binary random variables, with values $+1$ or -1. Adapted from Dzhafarov & Kujala (2016a). See the online article for the color version of this figure.

referring to it as "a spooky action at a distance." In fact, contextual influences involve no "actions" (i.e., no transfer of energy or information). They simply reflect a fundamental fact of probability theory, that part of the identity of a random variable is what other random variables it is jointly distributed with (see Dzhafarov & Kujala, 2014a, 2016a, 2017b, for probabilistic foundations of contextuality). A simple analogy would be the property of being or not being "the brightest star in the sky" considered part of each star's identity: The identity of a given star then can change depending on the brightness of stars that do not influence it directly. It is a basic but fascinating aspect of reality, fundamentally different from direct influences in being non-causal (see Dzhafarov & Kujala, 2016a, for a detailed discussion).[2]

two entangled particles but a different choice of the four axes, the system of random variables representing them may very well exhibit no contextuality.

[2]A formal definition of a random variable in probability theory is that it is a measurable function mapping one probability space into another, and it is jointly distributed with any other measurable function defined on the same domain probability space. Conversely, the set of all random variables with which it is jointly distributed define the domain space of this random variable, which obviously is part of this variable's identity.

With contextuality (or lack thereof) understood as a property of a system of random variables describing an aspect of a physical system (see Footnote 1), there are no known principles, in physics or elsewhere, that would confine all contextual systems to quantum mechanics. Quantum mechanical computations may establish certain properties of a set of particles, and then by means of classical probability theory one may establish that a certain system of random variables describing these properties forms a contextual system. No quantum mechanical computation, however, is based on contextuality as a physical property. It is not surprising therefore that numerous attempts were made to reveal probabilistic contextuality analogous to the EPR/BB one outside quantum physics, in particular, in human cognition and decision making (Aerts, 2014; Aerts et al., 2017 ; Aerts, Gabora, & Sozzo, 2013; Asano, Hashimoto, Khrennikov, Ohya, & Tanaka, 2014; Bruza, Kitto, Nelson, & McEvoy, 2009; Bruza, Kitto, Ramm, & Sitbon, 2015; Bruza, Wang, & Busemeyer, 2015). The idea of constructing a behavioral analogue of a quantum-mechanical experiment is simple: each experimental setting (e.g., an axis chosen by Alice) is replaced with a task of responding to a stimulus or question, and the measurement outcome (e.g., the spin along this axis) is replaced with a response given to this stimulus or question. With these correspondences, the design of a behavioral experiment can be made formally identical to that of the quantum one. For instance, in the experiment described in Aerts et al. (2013), the axis α_i corresponded to the task of choosing between two animals (one pair for $i = 1$, another for $i = 2$), and β_j corresponded to the task of choosing between two animal sounds (again, different pairs for $j = 1$ and $j = 2$). The respondent was asked to choose an animal in response to α_i and to choose the best matching animal sound in response to β_j. The expectation in this experiment was that the responses to α_i and β_j could be treated as random variables A_i and B_j, respectively, and the CHSH inequality (4.1) could then be used to reveal the presence or absence of contextuality.

Here, however, the study in question, as well as all other studies mentioned above, faced a serious difficulty (Dzhafarov & Kujala 2014b; Dzhafarov, Kujala, Cervantes, Zhang, & Jones, 2016; Dzhafarov, Zhang, & Kujala, 2015). The CHSH inequality (4.1) and other traditional contextuality tests in quantum mechanics are derived under the assumption of "no-signaling" (Abramsky & Brandenburger, 2011; Adenier & Khrennikov, 2017) or "marginal selectivity" (Dzhafarov & Kujala, 2014b), which is the condition ensuring that the context does not influence random variables directly. Thus, in the classical version of EPR/BB, the distribution of A_i does not depend on whether it is measured together with B_1 or B_2. Without this condition the expression in (4.1) would be hopelessly confused, as the symbols it contains for random variables then would change their meaning within the expression. In human behavior, however, this condition is almost never satisfied: As response to a stimulus S is typically directly influenced by any stimulus S' in the temporal-spatial vicinity of S. For instance, in Aerts et al. (2013), when choosing between Tiger and Cat (task α_2), Tiger was chosen with probability 0.86 when combined with the choice between Growls and Winnies (task β_1), but Tiger was only chosen with probability 0.23 when combined with the choice between Snorts and Meows (β_2). There is no way therefore one can denote the response to α_2 by A_2 and use Inequality 4.1. The change in the distribution of the response to α_2 indicates that it is directly influenced by the choice of the sound, while the CHSH inequality expressly excludes this possibility (Dzhafarov & Kujala 2014b).

However, the presence of direct influences from S' to R does not automatically exclude the presence of pure contextuality: it is possible, as schematically shown in Figure 4.4, that contextual influences coexist with direct ones. The situation depicted

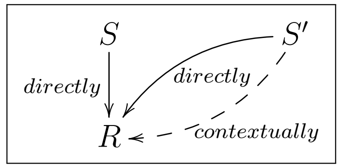

Fig. 4.4. A combination of the situations depicted in Figures 4.1 and 4.2. As S' changes, the distribution of R changes (i.e., S' directly influences R), but the identity of R (revealed by looking at how R is codistributed with other random variables) changes more than the change in its distribution can explain. Here, influence of S' on R is in part direct and in part contextual.

in Figure 4.2 is merely a special case, when the change in the distribution of R with S' is nil, so whatever change in the identity of R is observed in response to changes in S', it is purely contextual. More generally, however, one can consider the possibility that the distribution of R does change with S', but the extent of this change is not sufficient to account for the extent of the changes in R's identity, as revealed by its joint distribution with other random variables. This combined form of context-dependence has been studied in the mathematical theory called Contextuality-by-Default (CbD; Dzhafarov, Cervantes, & Kujala, 2017; Dzhafarov & Kujala, 2014a, 2016a, 2016b, 2017a; Dzhafarov, Kujala, & Cervantes, 2016; Dzhafarov, Kujala, & Larsson, 2015; Kujala, Dzhafarov, & Larsson, 2015).

When applied to the EPR/BB system, the logic of CbD is as follows. One determines the maximal probability with which A_1^1 could equal A_1^2 if the two were jointly distributed. This probability is a measure of difference between the two distributions (the smaller the probability the larger the difference). Analogously one determines the maximal probabilities of $A_2^1 = A_2^2$, $B_1^1 = B_1^2$, and $B_2^1 = B_2^2$. If this measure of difference between the distributions is sufficient to account for the entire difference between the random variables A_1^1 and A_1^2, B_1^1 and B_1^2, and so forth, then these maximal probabilities should be compatible with the observed joint distributions of (A_1^1, B_1^1), (A_1^2, B_2^1), (A_2^1, B_1^2), and (A_2^2, B_2^2). If they are, the system in noncontextual. If they are not, then A_1^1 and A_1^2, or B_1^1 and B_1^2, and so forth, have to be more dissimilar as random variables than they are because of the difference between their distributions. Such a system exhibits contextuality proper ("on top of" direct influences).[3]

It is proved (Dzhafarov, Kujala, & Larsson, 2015; Kujala & Dzhafarov, 2016) that the EPR/BB system is noncontextual if and only if

$$\begin{aligned}\max_{k,l\in\{1,2\}} \left|\sum_{i,j\in\{1,2\}} \mathsf{E}\left[A_i^j B_j^i\right] - 2\mathsf{E}\left[A_k^l B_l^k\right]\right| \\ - \sum_{i\in\{1,2\}} \left|\mathsf{E}\left[A_i^1\right] - \mathsf{E}\left[A_i^2\right]\right| \\ - \sum_{j\in\{1,2\}} \left|\mathsf{E}\left[B_j^1\right] - \mathsf{E}\left[B_j^2\right]\right| - 2 \le 0.\end{aligned} \tag{4.2}$$

The formula generalizes the CHSH Inequality (4.1), which obtains if the second and third sums in the expression are zero (no-signaling or marginal selectivity condition). When this formula was applied to behavioral experiments imitating the EPR/BB design, all available data (Aerts, 2014; Aerts et al., 2013, 2017; Bruza, Kitto, Ramm, & Sitbon, 2015; Cervantes & Dzhafarov, 2017a; Zhang & Dzhafarov, 2017) were in compliance with lack of contextuality. The same conclusion (lack of contextuality) was reached regarding behavioral experiments with other designs (Asano et al., 2014; Cervantes & Dzhafarov, 2017b; Wang & Busemeyer, 2013; Wang, Solloway, Shiffrin, & Busemeyer, 2014). This series of negative results led Dzhafarov et al. (2015) and Dzhafarov et al. (2016) to hypothesize that all context-dependence in behavioral and social data may be due to direct influences, with no contextuality proper.

[3]To avoid technicalities, the formulation given is far from being general and is less than rigorous. A rigorous formulation for the EPR/BB system involves considering *maximal couplings* of (A_1^1, B_1^1), (A_1^2, B_2^1), (A_2^1, B_1^2), and (A_2^2, B_2^2). More complex systems require *dichotomizations* of the random variables and *multimaximal couplings* (Dzhafarov, Cervantes, & Kujala, 2017; Dzhafarov & Kujala, 2017a, 2017b).

Inspection of Inequality 4.2, however, suggests another possibility: perhaps the correlations between A and B variables in the previous attempts imitating the formal structure of the EPR/BB experiment were not strong enough. The maximum of the first sum in (4.2) is large if the four expectations $\mathsf{E}\left[A_i^j B_j^i\right]$ are large in absolute value, and one of them has the sign opposite to the sign of the remaining three. What if this maximum were large enough to offset the terms reflecting violations of marginal selectivity and to make the left-hand side of the expression positive? Here, we report an experiment in which contextuality proper is definitely established by achieving the desired pattern of sufficiently large correlations between A and B variables.

The design of the experiment is similar to other behavioral imitations of the EPR/BB paradigm: the choice of an axis is replaced by a choice between two options, the options corresponding to each α-axis being two characters from a story, and the options corresponding to each β-axis being two characteristics which characters from the story may possess. The story was The Snow Queen by Hans Christian Andersen, and, for example, the pair (α_1, β_1) was the offer to choose between Gerda and the Troll (the result being A_1^1) and also to choose between Beautiful and Unattractive (B_1^1), so that the two choices match the story line (in which Gerda is Beautiful and the Troll is Unattractive). The choices are offered to many people in a crowdsourcing experiment, and the probabilities are estimated by the proportions of people making this or that pair of choices. The expectation is that a respondent who understands the story line would choose a “correct” combination of a character and a characteristic (e.g., either Gerda and Beautiful, or the Troll and Unattractive). If so, the max of the first sum in Inequality 4.2 should equal 4 (its maximal possible value), and the presence or absence of contextuality would depend only on the the relative proportions of people preferring one correct choice to another. We will see, however, that a fraction of respondents, more than 8%, chose “incorrect” options.

4.1 Method

4.1.1 Participants

A total of 1,989 participants signed up for the study on Amazon's Mechanical Turk (Barr, J., 2005) and indicated their agreement with a standard informed consent page in exchange for financial compensation ($0.10). No demographics were required nor recorded. A total of 1,799 of the participants completed the experiment by answering the two questions posed to them. They will be referred to below as respondents, and their responses were used for the analysis. The number of respondents was planned to exceed 1,600, estimated to be more than sufficient for construction of 99.99% bootstrap confidence intervals (as explained in Results). The data were collected in January 12-14, 2017.

4.1.2 Materials and Procedure

The experiment was set up as a "survey" on Purdue University's Qualtrics platform (Purdue University, 2015). Each participant was randomly assigned to one of four conditions, referred to as contexts (Table 4.1). The experiment consisted in the participant being presented with the instructions ("story line") and, on the same computer screen, offered to make two choices forming the context assigned to this participant: of a character from a given pair of characters, and of a suitable characteristic of this character from a given pair of characteristics. For example, in Context 3 (Table 4.1), the computer screen looked as shown in Figure 4.5, asking to choose between Snow Queen and Old Finn Woman and to choose between Beautiful and

Table 4.1.
Each Context Consisted of Two Choices, Between Two Characters and Between Two Characteristics

	Character choice	Characteristic choice	N total (correct)
Context 1	Gerda	Beautiful	447 (425)
	Troll	Unattractive	
Context 2	Gerda	Kind	453 (429)
	Troll	Evil	
Context 3	Snow Queen	Beautiful	446 (410)
	Old Finn Woman	Unattractive	
Context 4	Snow Queen	Kind	453 (388)
	Old Finn Woman	Evil	

Note. N total is the number of respondents assigned to each context (the number in parentheses shows the subset of respondents whose answers were correct, in accordance with the story line).

Unattractive, with the instruction that the two choices had to be true to the story line (which says that Snow Queen is Beautiful and Old Finn Woman is Unattractive).[4]

4.2 Results

We present the results first for correct responses only, and then for all responses, with the numbers of respondents shown in Table 4.1. In Tables 4.2 and 4.3,[a] we present the observed proportions for each combination of choices in the first and second group, respectively. We refer to these tables of proportions (or probabilities they estimate) as "systems," in accordance with the terminology of "context-content systems" introduced in Dzhafarov & Kujala (2016a).

[4] As pointed out at the end of the Discussion section, the logic of CbD dictates that only one context (one pair of choices) be presented to a given respondent, dividing thereby the pool of respondents into four groups, one responding to Context 1, another to Context 2, etc.

[a] (Erratum note added in the dissertation.) In Tables 4.2–4.4, the columns with the heading "Mar. character" appear with the heading "Mar. characteristic". Corrected in text.

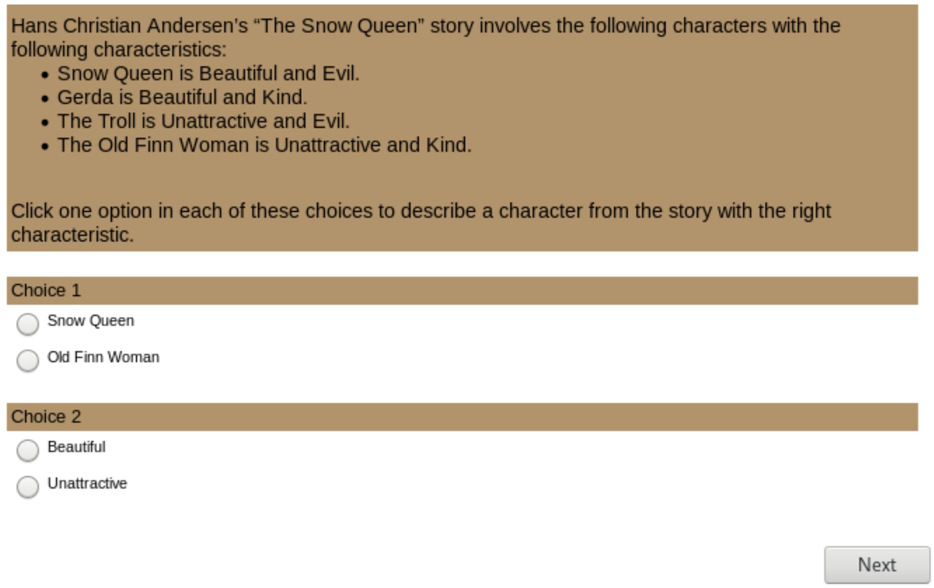

Fig. 4.5. The appearance of the computer screen for participants assigned to Context 3.

4.2.1 System of Correct Choices

In this system, the max of the first sum in Inequality 4.2 equals 4 (its maximal possible value), and the presence or absence of contextuality depends only on the the relative proportions of two correct pairs of choices. The system is contextual on the sample level: The left-hand side of Inequality 4.2 equals 0.452. To evaluate how reliable this figure is, a bootstrap confidence interval (Davison & Hinkley, 1997) was calculated by generating $n = 400,000$ resamples from each of the contexts, computing the lefthand side of Inequality 4.2 for each of them, choosing a confidence level C, and finding the $\frac{1-C}{2}$ and $1 - \frac{1-C}{2}$ quantiles of their distribution. The histogram of the distribution is shown in the upper panel of Figure 4.6. For this system, the 99.99% bootstrap confidence interval for the lefthand side of Inequality 4.2 is $[0.226, 0.668]$. The confidence needed for the bootstrap interval to cover zero exceeds 99.999% because none of the $400,000$ resamples produced a nonpositive value.

Table 4.2.
Observed Proportions of Correct Choices for Each of the Four Contexts

	B_1^1				B_2^1		
Context 1	Beautiful	Unattractive	Mar. character	Context 2	Kind	Evil	Mar. character
A_1^1				A_1^2			
Gerda	0.887	0.000	0.887	Gerda	0.841	0.000	0.841
Troll	0.000	0.113	0.113	Troll	0.000	0.159	0.159
Mar. characteristic	0.887	0.113		Mar. characteristic	0.841	0.159	

	B_1^2				B_2^2		
Context 3	Beautiful	Unattractive	Mar. character	Context 4	Kind	Evil	Mar. character
A_2^1				A_2^2			
Snow Queen	0.837	0.000	0.837	Snow Queen	0.000	0.626	0.626
Old Finn woman	0.000	0.163	0.163	Old Finn woman	0.374	0.000	0.374
Mar. characteristic	0.837	0.163		Mar. characteristic	0.374	0.627	

Note. Mar. = marginal observed proportions. To apply Inequality 4.2, one of the two options (no matter which) in each choice is encoded by +1, the other by −1.

4.2.2 System of All Responses

This system is contextual on the sample level: The left-hand side of Inequality 4.2 equals 0.279. A bootstrap confidence interval was calculated by generating $n = 400,000$ resamples and analyzing them in the same way as for the system of correct responses. The histogram of the distribution of values of the lefthand side of Inequality 4.2 is shown in the lower panel of Figure 4.6. For this system, the 99.99% bootstrap confidence interval for the lefthand side of Inequality 4.2 is $[0.008, 0.506]$.

4.3 Discussion

We have demonstrated that a contextual system of random variables formally analogous to the EPR/BB system in quantum mechanics can be observed in human behavior. It has been done without making the mistake of ignoring lack of marginal selectivity in psychological data. Marginal selectivity (or no-signaling condition), in application to the EPR/BB system, means that the second and third sums in Inequality 4.2 are zero. If this were the case in our experiments (e.g., if the two correct

Table 4.3.
Observed Proportions of All Choices, for Each of the Four Contexts

Context 1	B_1^1 Beautiful	B_1^1 Unattractive	Mar. character	Context 2	B_2^1 Kind	B_2^1 Evil	Mar. character
A_1^1				A_1^2			
Gerda	0.843	0.020	0.864	Gerda	0.797	0.035	0.832
Troll	0.029	0.107	0.136	Troll	0.018	0.150	0.168
Mar. characteristic	0.872	0.128		Mar. characteristic	0.815	0.185	
Context 3	B_1^2 **Beautiful**	B_1^2 **Unattractive**	**Mar. character**	**Context 4**	B_2^2 **Kind**	B_2^2 **Evil**	**Mar. character**
A_2^1				A_2^2			
Snow Queen	0.769	0.011	0.780	Snow Queen	0.135	0.536	0.671
Old Finn woman	0.070	0.150	0.220	Old Finn woman	0.320	0.009	0.329
Mar. characteristic	0.839	0.161		Mar. characteristic	0.455	0.545	

Note. Mar. = marginal observed proportions.

choices of the character-characteristic pairs were made with equal probability), the lefthand side of Inequality 4.2 for the system with correct choices would have the maximal theoretically possible value, 2. This would make the system a so-called PR box (Popescu & Rohrlich, 1994), a system forbidden by laws of both classical and quantum mechanics. There is no a priori reason why a behavioral system could not violate boundaries established by quantum mechanics, but the sample level contextuality value of 0.452 obtained in our experiment for correct responses is quite moderate, well below the quantum boundary (so-called Tsirelson bound) of $2\left(\sqrt{2}-1\right)$.[5] Recall that application of Inequality 4.2 and similar formulas to all previously reported experimental data showed no contextuality at all, leading Dzhafarov et al. (2015) and Dzhafarov et al. (2016) to consider the possibility that all context-dependence in psychology is because of direct influences only. This hypothesis is now falsified.[6]

[5]Note, however, that the derivability of the Tsirelson bound without assuming non-signaling is not obvious and requires special investigation.

[6]It is worth mentioning that violations of marginal selectivity or no-signaling condition (the general CbD term being "consistent connectedness") are also common in quantum physical experiments (Adenier & Khrennikov, 2007; Khrennikov, 2017, pp. 25-28). Compared with behavioral data, however, inconsistent connectedness in quantum mechanics is relatively small, even when statistically significant, and with the use of CbD theory pure contextuality can usually be established at extremely

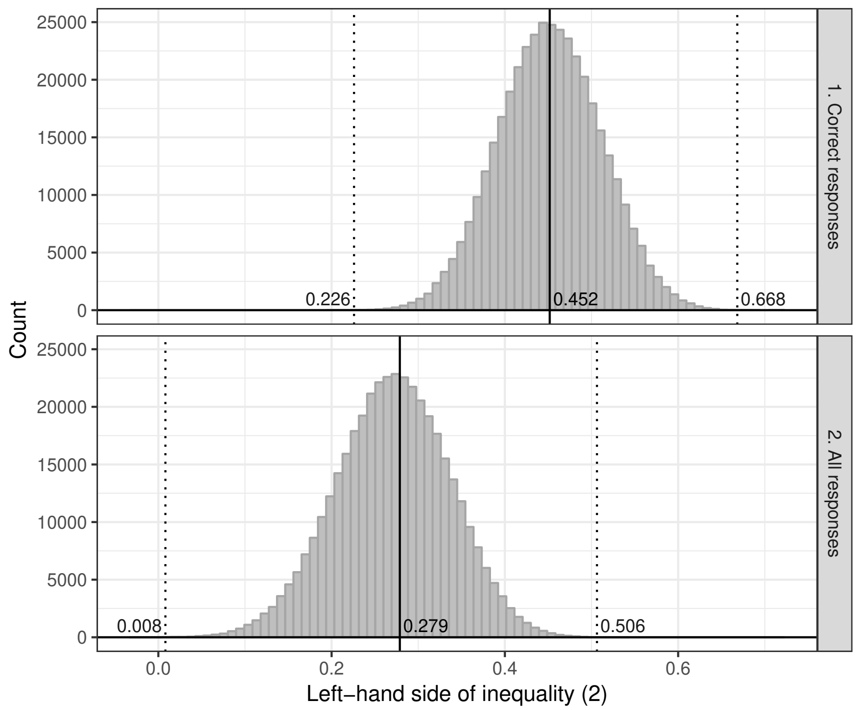

Fig. 4.6. Histograms of the bootstrap values of the lefthand side of Inequality 4.2, for correct responses (upper panel) and for all responses (lower panel). The solid vertical line indicates the location of the observed sample value. The vertical dotted lines indicate the locations of the 99.99% bootstrap confidence intervals.

Contextuality in our experiment was exhibited by both the system of correct responses and the system of all responses, correct and incorrect. It is not clear, however, why some respondents made incorrect choices to begin with. The possibilities range from misunderstanding of the instructions to deliberate non-compliance. This makes no difference for the formal contextuality analysis, but one might consider the legitimacy of excluding incorrect choices as outliers.

Focusing on the system of correct responses, one might wonder if a story line that makes one of the two choices in each context rigidly determined by the other choice

high level of confidence (see, e.g., the analysis of experimental data in Kujala, Dzhafarov, & Larsson, 2015).

Table 4.4.
Hypothetical Proportions of Correct Choices for Each of the Four Contexts

Context 1	B_1^1 Beautiful	Unattractive	Mar. character	Context 2	B_2^1 Kind	Evil	Mar. character
A_1^1				A_1^2			
Gerda	0.817	0.000	0.817	Gerda	0.911	0.000	0.911
Troll	0.000	0.183	0.183	Troll	0.000	0.089	0.089
Mar. characteristic	0.817	0.183		Mar. characteristic	0.911	0.089	
Context 3	Beautiful	Unattractive	Mar. character	Context 4	Kind	Evil	Mar. character
A_2^1				A_2^2			
Snow Queen	0.907	0.000	0.907	Snow Queen	0.000	0.696	0.696
Old Finn woman	0.000	0.093	0.093	Old Finn woman	0.304	0.000	0.304
Mar. characteristic	0.907	0.093		Mar. characteristic	0.304	0.696	

Note. This system is obtained by adding or subtracting .07 to/from each of the nonzero probabilities in Table 4.2.

(Table 4.1) may somehow predetermine the contextuality of the system. Could the results reported in this paper be essentially forced by the experiment's design? It is easy to see that this is not the case. For example, in Table 4.4 all responses are correct but the system is noncontextual, with the left-hand side of Inequality 4.2 equal to -0.004. No superficial inspection of this system would reveal a qualitative difference from the one in Table 4.2. The question should not be therefore whether noncontextuality is compatible with the story line, but whether the latter makes it "rare."

One way of making the meaning of "rare" precise is as follows. The experimental design we use (considering only correct responses) makes the value

$$\max_{k,l\in\{1,2\}} \left| \sum_{i,j\in\{1,2\}} \mathsf{E}\left[A_i^j B_j^i\right] - 2\mathsf{E}\left[A_k^l B_l^k\right] \right| \tag{4.3}$$

equal to 4, its maximal possible value. The system's (non)contextuality therefore is determined entirely by the value of

$$\sum_{i\in\{1,2\}} \left|\mathsf{E}\left[A_i^1\right] - \mathsf{E}\left[A_i^2\right]\right| + \sum_{j\in\{1,2\}} \left|\mathsf{E}\left[B_j^1\right] - \mathsf{E}\left[B_j^2\right]\right|. \tag{4.4}$$

The system is contextual if and only if this expression's value is less than 2. In the system of correct responses

$$\begin{aligned} a &= \mathsf{E}\left[A_1^1\right] = \mathsf{E}\left[B_1^1\right], \\ b &= \mathsf{E}\left[A_1^2\right] = \mathsf{E}\left[B_2^1\right], \\ c &= \mathsf{E}\left[A_2^1\right] = \mathsf{E}\left[B_1^2\right], \\ d &= \mathsf{E}\left[A_2^2\right] = -\mathsf{E}\left[B_2^2\right]. \end{aligned} \tag{4.5}$$

Each of the four values a, b, c, d ranges between -1 and 1. It is reasonable now to ask how probable it is that these four values chosen "randomly and independently" (meaning that the quadruple of the expected values is uniformly distributed within the 4-dimensional cube) would yield (4.4) equal to or exceeding 2. The answer is easily obtained by Monte Carlo simulation, and the probability in question, that is, the probability that a randomly created system is noncontextual, turns out to be about 0.6667. This is hardly a "rare" event.

In psychological terms, the interpretation of context-dependence in our experiment is straightforward: the meaning of such characteristics as Kind versus Evil or Beautiful versus Unattractive is different depending on what choice of characters is offered to ascribe these concepts to. This difference, however, cannot be fully explained by assuming that the impact of the character choice upon the characteristic choice is "direct" (analogous to a signal propagating from Bob's measurement to Alice's measurement). The direct influence is there without doubt, manifested in the lack of marginal selectivity in our data, but the context-dependence contains a component of pure contextuality. We have no psychological terms to discern the two parts of context-dependence. The value of contextuality analysis here is in that it provides rigorous analytic discernments where "ordinary" psychological analysis is underdeveloped or moot.

Note that the term "direct influences" in CbD refers to mathematical properties of a specific system of random variables rather than to physical or psychological mechanisms. Although the initial intuition of direct influences involves conventional schemes with forces and energy transfer, in the mathematical theory direct influences are *defined* by the differences between the distributions of random variables measuring (responding to) the same property in different contexts. In the EPR/BB system, the difference between the distributions A_1^1 and A_1^2 is, by definition, the difference between the direct influences exerted by β_1 and β_2 (or, simply, the direct influence of the β) upon the measurement of α_1. If the two distributions are identical, β exerts no direct influence, because we only think of particular random variables and of differences in their distributions. It is perfectly possible that the two identical distributions would differ from the distribution of some A_1^3, had there been a third context in which α_1 were measured alone or together with some β_3. Moreover, it is possible that a physical theory could establish that the influences exerted by β_1 and β_2 are physically different despite affecting the distributions of A_1^1 and A_1^2 identically (see, e.g., Filk, 2015). Our analysis, however, does not depend on this or that physical or psychological theory. Even in the case of the classical EPR/BB system with two particles, the Bohmian version of quantum mechanics allows for the possibility of direct influences being responsible for the entire picture, albeit defying special relativity. However, the EPR/BB system with a specific choice of axes would remain contextual even if the Bohmian mechanics became universally accepted. As everything else in CbD, "direct influence" is not a physical term (although it may be assigned a physical interpretation in many cases), it is a mathematical term that is relative to the system of random variables in play.[7]

[7]Of course, if the systems with physically certified direct influences that are not reflected in the differences between the distributions were ubiquitous, the CbD analysis would be less interesting to physicists. This is too complex an issue to discuss in a paper focusing on a single experiment. We believe in the "no-conspiracy" principle reflected in Einstein's famous "Subtle is the Lord, but

Our experiment establishes a clear template for designing analogous experiments aimed at pure contextuality, whether in the EPR/BB or similar format. In the terminology of CbD, the EPR/BB system is a cyclic system of rank 4 (Dzhafarov, Kujala, & Larsson, 2015; Kujala, Dzhafarov, & Larsson, 2015). This system involves eight binary random variables, A_j^i, B_i^j $(i, j \in \{1, 2\})$, and the design maximizing the chances of this system exhibiting contextuality ("on top of" direct influences) is as follows. Label the values of all the random variables $+1$ and -1 and create a "story line" in which $+1$ of A_j^i and $+1$ of B_i^j are associated with a very high probability in three out of four pairs $\left(A_j^i, B_i^j\right)$, and with a very low probability in the fourth pair (or vice versa). For other cyclic systems (say, of ranks 3 or 5) the criteria of contextuality are similar to (4.2), and the design can be constructed similarly.

An important feature of the design is that each respondent should be assigned to a single context only, instead of asking each of them to make (in the case of the EPR/BB system) all four pairs of choices $(\alpha_1, \beta_1), \ldots, (\alpha_2, \beta_2)$, whether presented simultaneously, in a fixed order, or a variable order. The reason for this is that making all four pairs of choices would have created an empirical joint distribution of the eight random variables in play,

$$A_1^1, B_1^1, A_1^2, B_2^1, A_2^1, B_1^2, A_2^2, B_2^2,$$

contravening the logic of CbD in which different contexts are mutually exclusive, and different pairs $\left(A_i^j, B_j^i\right)$ are not jointly distributed (are *stochastically unrelated* to each other). Contextuality analysis consists in finding out whether a joint distribution can be *imposed* on these eight random variables, subject to certain constraints (maximality of the probabilities of $A_1^1 = A_1^2$, $B_1^1 = B_1^2$, etc.). For this analysis an empirical

malicious He is not." All known to us examples of hidden direct influences are artificially constructed on paper, with even slight modifications revealing them.

joint distribution involving, say, A_1^1 and A_1^2 would be a nuisance relation. It would have to be ignored, and an additional theory would be required to know how the ignored relations affect the results of the analysis. Consider, for example, the fact that every given choice (e.g., between Gerda and Troll) in the EPR/BB system enters in two different contexts. The respondent would normally remember her previous choice when facing it the second time, albeit in combination with another pair of characteristics (which in turn, will appear once again, in combination with another pair of characters). It is clear that the respondent's choice would depend on the previously made one in some complex way (e.g., the strategy may be adopted to always repeat it, or to always choose a new option). This would affect the marginal distributions of the choices in some unknown way. Note that our design is not different from how the measurements are made in the quantum-mechanical EPR/BB system, where only one pair of measurements can be performed on a given pair of entangled particles.

Jerome Busemeyer (personal communication, November 2017) mentioned to us that the "respondents" in our design need not be people, they can be any entities to which A and B properties can be probabilistically assigned. This observation points at ways of searching for contextual systems outside both quantum mechanics and psychology.

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5. TRUE CONTEXTUALITY BEATS DIRECT INFLUENCES IN HUMAN DECISION MAKING

 Basieva, I., Cervantes, V. H., Dzhafarov, E. N., & Khrennikov, A. (2019). True contextuality beats direct influences in human decision making. *Journal of Experimental Psychology: General, 148*(11), 1925–1937. https://doi.org/10.1037/xge0000585

Abstract In quantum physics there are well-known situations when measurements of the same property in different contexts (under different conditions) have the same probability distribution but cannot be represented by one and the same random variable. Such systems of random variables are called contextual. More generally, true contextuality is observed when different contexts force measurements of the same property (in psychology, responses to the same question) to be more dissimilar random variables than warranted by the difference of their distributions. The difference in distributions is itself a form of context-dependence, but of another nature: it is attributable to direct causal influences exerted by contexts upon the random variables. The Contextuality-by-Default theory allows one to separate true contextuality from direct influences in the overall context-dependence. The Contextuality-by-Default analysis of numerous previous attempts to demonstrate contextuality in human judgments shows that all context-dependence in them can be accounted for by direct influences, with no true contextuality present. However, contextual systems in human behavior can be found. In this paper we present a series of crowd-sourcing experiments that exhibit true contextuality in simple decision making. The design of these experiments is an elaboration of one introduced in the Snow Queen experiment (*De-*

cision 5, 193-204, 2018), in which contextuality was for the first time demonstrated unequivocally.

Keywords: concept combinations, context-dependence, contextuality, direct influences.

A response to a stimulus (say, a question) is generally a *random variable* that can take on different values (say, Yes or No) with certain probabilities. The *identity* of a random variable, in nontechnical terms, is what uniquely distinguishes this random variable from other random variables.[1] The *distribution* of this random variable (probabilities with which it takes on different values) is part of this identity, but clearly not the entire identity: Think of a handful of fair coins—a set of distinct random variables with the same distribution. Other stimuli (e.g., other questions posed together or prior to a given one) may *directly influence* the identity of the response to the given stimulus by changing its distribution. In fact, this change in the distribution, mathematically, is how the directness of the influence is defined. True *contextuality* is such dependence of the identity of a response to a stimulus on other stimuli that cannot be wholly explained by such direct influences. We will elaborate this definition below.

Contextuality is at the very heart of quantum mechanics (see, e.g., Liang, Spekkens, & Wiseman, 2011), where it can be observed by eliminating (or at least greatly reducing) all direct influences by experimental design. (In quantum physics "response to a stimulus" has to be replaced with "measurement of a property," but this is in essence the same input-output relation.) This paper addresses a question that ever since the 1990's interested researchers in physics, computer science, and psychology,

[1]In rigorous mathematical terms, a random variable is defined as a (measurable) function mapping a domain probability space into another (measurable) space. Its distribution is just one property of this function, the probability measure it induces on the codomain space.

the question of whether true contextuality can be observed outside quantum mechanics, with special interest (largely for philosophical reasons we will not be discussing) in whether it is present in human behavior. Many previous behavioral experiments designed to answer this question (e.g., Aerts, 2014; Aerts, Gabora, & Sozzo, 2013; Asano, Hashimoto, Khrennikov, Ohya, & Tanaka, 2014; Bruza, Kitto, Nelson, & McEvoy, 2009; Bruza, Kitto, Ramm, & Sitbon, 2015; Bruza, Wang, & Busemeyer, 2015; Cervantes & Dzhafarov, 2017a, 2017b; Dzhafarov & Kujala, 2014b; Dzhafarov, Kujala, Cervantes, Zhang, & Jones, 2016; Dzhafarov, Zhang, & Kujala, 2015; Zhang & Dzhafarov, 2017) have been shown to result in systems of random variables that are noncontextual. This prompted Dzhafarov, Zhang, and Kujala (2015) to consider the possibility that human behavior may never exhibit true contextuality. It turns out, however, that contextual systems in human behavior can be found. In this paper we describe a series of experiments that, added to one previously conducted (Cervantes & Dzhafarov, 2018), demonstrate this unequivocally.

It should be emphasized at the outset that it would be incorrect to think of contextuality as being surprising and strange while noncontextuality is trivial and expected. In the absence of constraints imposed by a general psychological theory, comparable to quantum mechanics, we have no justification for such judgements. One might argue in fact that it is most surprising that so many experiments in psychology are described by noncontextual systems of random variables. Nor would it be correct to assume that typical psychological models, even very simple ones, can only predict noncontextual systems: thus, in the concluding section of this paper we mention a simple model that, on the contrary, predicts only contextual systems (and has to be dismissed because of this). Contextuality analysis is not a predictive model of behavior, and both contextual and noncontextual systems are compatible with ordinary psychological models. In that, as we point out in the Discussion, psychology is not different from quantum physics, in which (non)contextuality of a system is established based on the laws of quantum physics but is not used to derive or revise them. What contextuality analysis elucidates is the nature and structure of random variables—arguably, the most basic and mandatory construct in the scientific analysis of empirical systems, whether in psychology or elsewhere. In a well-defined and mathematically rigorous sense, in a contextual system random variables form true "wholes" that cannot be reduced to sets of distinct random variables measuring or responding to specific elements of contexts while being also cross-influenced by other elements of contexts. This makes contextuality analysis inherently interesting, but we need much greater knowledge of which behavioral systems are contextual and which are not in order to determine to what other properties of behavior these characteristics are related. We will return to the role and meaning of contextuality after we introduce necessary definitions, theoretical results, and empirical evidence.

5.1 Direct Influences and True Contextuality

We introduce the basic notions related to contextuality analysis using a simple example—responses to three Yes/No questions asked two at a time. Most of the experiments reported below are of this kind. Let, for example, the three questions be:

q_1: Do you like chocolate?

q_2: Are you afraid of pain?

q_3: Do you see your dentist regularly?

Let a very large group of people be divided into three subgroups: In the first subgroup each respondent is asked questions q_1 and q_2; in the second subgroup, each respondent is asked questions q_2 and q_3; and in the third subgroup, the questions are q_3 and q_1. We call these pairwise arrangements of questions *contexts*, and we denote them

c_1, c_2, and c_3, respectively. It does not matter for the example whether the questions are asked in a fixed order, randomized order, or (if in writing) simultaneously. A response to question q_i asked in context c_j is a random variable that we denote R_i^j: Some of the people in the subgroup corresponding to context c_j will answer question q_i with Yes, others with No. Assuming the subgroups are so large that statistical issues can be ignored, by counting the numbers of responses we can get a good estimate of the probability distribution for our random variable:

$$R_i^j : \begin{array}{|c|c||c|} \hline Yes & No & \text{response} \\ \hline p_i^j & 1 - p_i^j & \text{probability} \\ \hline \end{array} . \tag{5.1}$$

All in all, we have six random variables in play, and they can be arranged in the form of the following *content-context* matrix (Dzhafarov & Kujala, 2016):

$$\begin{array}{|c|c|c||c|} \hline R_1^1 & R_2^1 & & c_1 \\ \hline & R_2^2 & R_3^2 & c_2 \\ \hline R_1^3 & & R_3^3 & c_3 \\ \hline \hline q_1 & q_2 & q_3 & \text{system } \mathcal{R}_3 \\ \hline \end{array} . \tag{5.2}$$

Now the distributions of the responses to question q_i should be expected to differ, depending on the context in which it is asked. For instance, when q_1 (Do you like chocolate?) is asked in combination with q_2 (Are you afraid of pain?), the probability of $R_1^1 =$"Yes, I like chocolate" may be relatively high because chocolate is usually liked, and the mentioning of pain in q_2 may make it sound especially comforting. However, when the same question q_1 is asked in context c_3, in combination with mentioning a dentist, the probability of $R_1^3 =$"Yes, I like chocolate" may very well be lower. The same reasoning applies to the two other questions: the responses to each of them will generally be distributed differently, depending on its context. This type of influence exerted by a context on the responses to questions within this context can be called *direct influence*. Indeed, the dependence of R_1^1 (responding to q_1) on q_2 (another question in the same context) is essentially of the same nature as the dependence of R_1^1 on q_1: A response to q_1 is based on the information contained in q_1 and (even if to a lesser extent) on the information contained in q_2. The other question in the same context can be viewed as part of the question to which a response is given.

Is all context-dependence of this direct influence variety? As it turns out, the answer is negative. Imagine, for example, that all direct influences are eliminated by some procedural trick, and each question in each context is answered Yes with probability 1/2. This means, in particular, that R_1^1 and R_1^3 have one and the same distribution,

$$R_1^1 : \begin{array}{|c|c|}\hline Yes & No \\ \hline 1/2 & 1/2 \\ \hline \end{array}, \quad R_1^3 : \begin{array}{|c|c|}\hline Yes & No \\ \hline 1/2 & 1/2 \\ \hline \end{array}, \tag{5.3}$$

and if one does not take into account their relations to R_2^1 (in context c_1) and to R_3^3 (in context c_3), one could consider R_1^1 and R_1^3 as if they were always equal to each other—essentially one and the same random variable;[2] and similarly for R_2^1 and R_2^2, and for R_3^2 and R_3^3. If one looks at each column of matrix (5.2) separately, ignoring the row-wise joint distributions, then one can write

$$\begin{aligned} R_1^1 &= R_1^3 \\ R_2^1 &= R_2^2 \quad . \\ R_3^2 &= R_3^3 \end{aligned} \tag{5.4}$$

Consider, however, the possibility that no respondent ever gives the same answer to both questions posed to her. Thus, if she answers Yes to q_1 in context c_1 (which can

[2]The "as if" here serves to circumvent the technicalities associated with the fact that, strictly speaking, we are dealing here not with R_1^1 and R_1^3 themselves but with their probabilistic copies (*couplings*) that are jointly distributed. See Dzhafarov and Kujala (2014a, 2017b) for details.

happen with probability 1/2), she always answers No to q_2, and vice versa. Denoting Yes and No by $+1$ and -1, respectively, we have a chain of equalities

$$\begin{aligned} R_1^1 &= -R_2^1 \\ R_2^2 &= -R_3^2 \ , \\ R_1^3 &= -R_3^3 \end{aligned} \tag{5.5}$$

and it is clear that (5.4) and (5.5) cannot be satisfied together: combining them would lead to a numerical contradiction. We should conclude therefore that when the joint distributions within contexts are taken into account, R_1^1 and R_1^3 or R_2^1 and R_2^2 or R_3^2 and R_3^3 cannot be considered always equal to each other. In at least one of these pairs, the two random variables should be more different than it is warranted by their individual distributions (which are, in this example, identical). This is a situation in which we can say that the system exhibits *true contextuality*, the kind of context-dependence that is not reducible to direct influences (in this example, absent).

Empirical data, especially outside quantum physics, almost always involve some direct influences, but the logic of finding out whether they also involve true contextuality remains the same. Continuing to use matrix (5.2) as a demonstration tool, we first look at the columns of the matrix one by one, ignoring the contexts. For each pair of random variables in a column (responses to the same question), we find out how close to each other they could be made if they were jointly distributed. In other words, we find the maximal probabilities with which each of the equalities in (5.4) can be satisfied. Then we investigate whether all the variables in our system can be made jointly distributed while preserving these maximal probabilities. If the answer is negative, we conclude that the contexts force the random variables sharing a column to be more dissimilar than warranted by direct influences (differences in their individual distributions). We then call such a system *contextual*. Otherwise it is *noncontextual*. This is the gist of the approach to contextuality called Contextuality-by-Default (CbD), and we illustrate it in the next section by a detailed numerical example.

CbD forms the theoretical basis for the design and analysis of our experiments. For completeness, however, another approach to the notion of contextuality should be mentioned, one treating context-dependent probabilities as a generalization of conditional probabilities defined through Bayes's formula (Khrennikov, 2009). With some additional assumptions these contextual probabilities can be represented by quantum-theoretical formalisms—state vectors in complex Hilbert space and Hermitian operators or their generalizations. Applications of such approach to cognitive psychology can be found in Khrennikov (2010) and Busemeyer and Bruza (2012), among other monographs and papers. CbD, by contrast, is squarely within classical probability theory. Although contextuality in CbD can be called quantum-like because of the origins of the concept in quantum physics, CbD uses no quantum formalisms.

5.1.1 Numerical Example and Interpretation

The following numerical example illustrates how CbD works. Let there be just two dichotomous questions, q_1 and q_2, answered in two contexts, c_1 and c_2 (e.g., in two different orders, as in Wang & Busemeyer, 2013). The content-context matrix here is

$$\begin{array}{|c|c||c|}
\hline
R_1^1 & R_2^1 & c_1 \\
\hline
R_1^2 & R_2^2 & c_2 \\
\hline
\hline
q_1 & q_2 & \text{system } \mathcal{R}_2 \\
\hline
\end{array}. \tag{5.6}$$

Assume that the joint distributions along the rows of the matrix are as shown:

c_1	$R_2^1 = Yes$	$R_2^1 = No$	
$R_1^1 = Yes$	1/2	0	1/2
$R_1^1 = No$	0	1/2	1/2
	1/2	1/2	

c_2	$R_1^2 = Yes$	$R_1^2 = No$	
$R_2^2 = Yes$	a	$1/2 - a$	1/2
$R_2^2 = No$	$3/4 - a$	$a - 1/4$	1/2
	3/4	1/4	

, (5.7)

where a is some value between 1/4 and 1/2. Knowing these distributions means that, for any filling of the matrix (5.6) with values of the random variables $R_1^1, R_2^1, R_1^2, R_2^2$ (Yes or No, for a total of 16 combinations), we know the row-wise probabilities: e.g.,

$R_1^1 = Yes$	$R_2^1 = Yes$	$p_1\,(Yes, Yes) = 1/2$
$R_1^2 = Yes$	$R_2^2 = No$	$p_2\,(Yes, No) = 3/4 - a$

. (5.8)

We see from (5.7) that R_2^1 and R_2^2 (the responses to question q_2) are distributed identically. Because of this, if they were jointly distributed (see footnote 1), the maximal probability with which they could be equal to each other would be 1:

q_2	$R_2^2 = Yes$	$R_2^2 = No$	
$R_2^1 = Yes$	1/2	0	1/2
$R_2^1 = No$	0	1/2	1/2
	1/2	1/2	

. (5.9)

The responses to question q_1, however, are distributed differently, and in the imaginary matrix of their joint distribution,

q_1	$R_1^2 = Yes$	$R_1^2 = No$	
$R_1^1 = Yes$	1/2	0	1/2
$R_1^1 = No$	1/4	1/4	1/2
	3/4	1/4	

, (5.10)

the maximal possible probability of $R_1^1 = R_1^2 = Yes$ is 1/2, and the maximal possible value of $R_1^1 = R_1^2 = No$ is 1/4. Therefore, if they were jointly distributed, the maximal probability with which $R_1^1 = R_1^2$ would be 3/4. Now, with these imaginary distributions, for any filling of the matrix (5.6) with Yes-No values of the random variables $R_1^1, R_2^1, R_1^2, R_2^2$, we also have the column-wise probabilities: e.g.,

$R_1^1 = Yes$	$R_2^1 = Yes$
$R_1^2 = Yes$	$R_2^2 = No$
$p_1'(Yes, Yes) = 1/2$	$p_2'(Yes, No) = 0$

. (5.11)

The problem we have to solve now is: Are these column-wise probabilities compatible with the row-wise probabilities in (5.8)? The compatibility means that, to any of the 16 filling of the matrix (5.6) with values of the random variables $R_1^1, R_2^1, R_1^2, R_2^2$, we can assign a probability, e.g., $p(R_1^1 = Yes, R_2^1 = Yes, R_1^2 = Yes, R_2^2 = No)$, such that the row-wise sums of these probabilities agree with (5.8) and the column-wise sums agree with (5.11). This is a classical linear programming problem: For any given value of a, it is guaranteed that either such an assignment of probabilities will be found (so that the system is noncontextual) or the determination will be made that such an assignment does not exist (the system is contextual). In our case, however, one need not resort to linear programing to see that no such assignment of probabilities is possible for any value of a other than 1/2. Indeed, we see from the c_1 distribution in (5.7) and from (5.9) that, with probability 1,

$$R_1^1 = R_2^1 = R_2^2. \tag{5.12}$$

So, R_1^1 and R_2^2 are essentially the same random variable, say X. But, from (5.10), this X equals R_1^2 with probability 3/4, whereas from the c_2 distribution in (5.7), this X equals R_1^2 with probability $2a - 1/4$, which is not 3/4 if $a \neq 1/2$. The conclusion is that the joint distributions along the two rows of the content-context matrix (5.6) prevent the responses to the same questions in the two columns of the matrix to be as close to each other as they can be if the two columns are viewed separately. The system therefore is contextual for any $a \neq 1/2$.

Why is this interesting? In psychological terms, the interpretation of the question order effect seems straightforward: The first question reminds something or draws one's attention to something that is relevant to the second question. What is shown by the contextuality analysis of our hypothetical question-order system is that this interpretation is only sufficient for $a = 1/2$, being incomplete in all other cases. The responses to two questions posed in a particular order form a "whole" that cannot be reduced to an action of the first question upon the second response: the identity of the two random variables changes beyond the effect of this action on their distributions. We will return to this issue in the concluding section of the paper.

The reader should not forget that we are discussing a numerical example rather than experimental data. The large body of experimental data on the question-order effect collected by Wang and Busemeyer (2013) has been subjected to contextual analysis in Dzhafarov, Zhang, and Kujala (2015), the result being that the responses to any of the many pairs of questions studied exhibit no contextuality. In fact, almost all question pairs are in a good agreement with the QQ law discovered by Wang and Busemeyer (2013),

$$\Pr\left[R_1^1 = R_2^1\right] = \Pr\left[R_1^2 = R_2^2\right], \tag{5.13}$$

and, as shown in Dzhafarov, Zhang, and Kujala (2015), this law implies no contextuality: This system of random variables is entirely describable in terms of each response being dependent on its own question, plus the second response[a] being also influenced by the first question. The idea of a whole being irreducible to interacting

[a](Erratum note added in the dissertation.) Originally "respond". Corrected in the text.

parts is not therefore an automatically applicable formula. To see if it is applicable at all, in psychology, one should look for empirical evidence elsewhere. Such evidence is presented below.

5.1.2 Contextuality-by-Default

CbD was developed (Dzhafarov, Cervantes, Kujala, 2017; Dzhafarov & Kujala 2014a, 2016, 2017a, 2017b; Kujala, Dzhafarov, & Larsson, 2015) as a generalization of the quantum-mechanical notion of contextuality (Abramsky & Brandenburger, 2011; Fine, 1982; Kochen & Specker, 1967; Kurzynski, Ramanathan, & Kaszlikowski, 2012). The latter only applies to *consistently connected* systems, those in which direct influences are absent, that is, responses to the same stimulus (or measurements of the same property) in different contexts are distributed identically. In physics this requirement is known by such names as no-signaling, no-disturbance, and so forth; in psychology it is known as *marginal selectivity* (Dzhafarov, 2003; Townsend & Schweickert, 1989). This requirement is never satisfied in behavioral experiments (Dzhafarov & Kujala, 2014b; Dzhafarov et al., 2016; Dzhafarov, Zhang, & Kujala, 2015), and it is often violated in quantum physical experiments too (Adenier & Khrennikov, 2017; Kujala et al., 2015). The main difficulty faced by many previous attempts to reveal contextuality in human behavior was that they could not apply mathematical tests predicated on the assumption of consistent connectedness to systems in which this requirement does not hold. As mentioned in the introduction, a CbD-based analysis of these experiments (Dzhafarov & Kujala, 2014b; Dzhafarov et al., 2016; Dzhafarov, Zhang, & Kujala, 2015) showed that all context-dependence in them was attributable to direct influences. The first unequivocal evidence of the existence of contextual systems in human behavior was provided by Cervantes and Dzhafarov's (2018) Snow Queen experiment.

The idea underlying the design of the Snow Queen experiment (and all the experiments reported below) is suggested by the criterion (necessary and sufficient condition) of contextuality when CbD is applied to cyclic systems with dichotomous random variables (Dzhafarov, Kujala, & Larsson, 2015; Kujala & Dzhafarov, 2016; Kujala et al., 2015). In such a system n questions and n contexts can be arranged as

$$\begin{tikzcd} q_1 \arrow[r, dash, "c_1"] \arrow[rrrr, dash, bend right=15, "c_n"'] & q_2 \arrow[r, dash, "c_2"] & \cdots \arrow[r, dash, "c_{n-2}"] & q_{n-1} \arrow[r, dash, "c_{n-1}"] & q_n \end{tikzcd} \tag{5.14}$$

The number n is referred to as the rank of the system. The question-order system (5.6) considered in the *Numerical Example and Interpretation* section is the smallest possible cyclic system, of rank 2,

$$\begin{tikzcd} q_1 \arrow[r, dash, "c_1"] \arrow[r, dash, bend right, "c_2"'] & q_2. \end{tikzcd} \tag{5.15}$$

The system (5.2) in the *Direct Influences and True Contextuality* section is a cyclic system of rank 3,

$$\begin{tikzcd} q_1 \arrow[r, dash, "c_1"] \arrow[rr, dash, bend right=20, "c_3"'] & q_2 \arrow[r, dash, "c_2"] & q_3, \end{tikzcd} \tag{5.16}$$

and it is used in four of the six experiments reported below. The remaining two are analyzed as cyclic systems of rank 4,

$$\begin{tikzcd} q_1 \arrow[r, dash, "c_1"] \arrow[rrr, dash, bend right=15, "c_n"'] & q_2 \arrow[r, dash, "c_2"] & q_3 \arrow[r, dash, "c_{n-1}"] & q_4 \end{tikzcd}, \tag{5.17}$$

with the content-context matrix

$$
\begin{array}{|c|c|c|c||c|}
\hline
R_1^1 & R_2^1 & & & c_1 \\
\hline
 & R_2^2 & R_3^2 & & c_2 \\
\hline
 & & R_3^3 & R_4^3 & c_3 \\
\hline
R_1^4 & & & R_4^4 & c_4 \\
\hline
\hline
q_1 & q_2 & q_3 & q_4 & \text{system } \mathcal{R}_4 \\
\hline
\end{array}
\tag{5.18}
$$

To formulate the criterion of contextuality in cyclic systems, we encode the values of our random variables by $+1$ and -1. Then the products of the random variables in the same context, such as $R_1^1 R_2^1$, are well-defined, and so are the expected values $\mathsf{E}[R_1^1 R_2^1]$, $\mathsf{E}[R_2^2 R_3^2]$, and so forth. For instance, if the joint distribution of R_1^1 and R_2^1 (responses to questions q_1 and q_2 in context c_1) is

$$
\begin{array}{c|c|c|c}
c_1 & R_2^1 = +1 & R_2^1 = -1 & \\
\hline
R_1^1 = +1 & a & b & a+b \\
\hline
R_1^1 = -1 & c & d & c+d \\
\hline
 & a+c & b+d &
\end{array},
\tag{5.19}
$$

then $R_1^1 R_2^1$ has the distribution

$$
\begin{array}{|c|c|}
R_1^1 R_2^1 = +1 & R_1^1 R_2^1 = -1 \\
\hline
a+d & b+c \\
\hline
\end{array},
\tag{5.20}
$$

and the distribution of R_1^1 and R_2^1 is described by the expected values

$$
\begin{aligned}
\mathsf{E}[R_1^1] &= (a+b) - (c+d), \\
\mathsf{E}[R_2^1] &= (a+c) - (b+d), \\
\mathsf{E}[R_1^1 R_2^1] &= (a+d) - (b+c).
\end{aligned}
\tag{5.21}
$$

We will also need a special function, s_{odd}: given some real numbers $x_1, \ldots, x_n$,

$$s_{odd}(x_1, \ldots, x_n) = \max(\pm x_1 \pm \ldots \pm x_n), \tag{5.22}$$

where each $\pm$ is to be replaced with $+$ or $-$, and the maximum is taken over all choices that contain an odd number of minus signs. Thus,

$$\begin{gathered} s_{odd}(x, y) = \max(-x + y, x - y), \\ s_{odd}(x, y, z) = \max(-x + y + z, x - y + z, x + y - z, -x - y - z), \\ \text{etc.} \end{gathered} \tag{5.23}$$

The theorem proved by Kujala and Dzhafarov (2016) says that a cyclic system of rank n is contextual (exhibits true contextuality) if and only if

$$D = s_{odd}\left(\mathsf{E}\left[R_1^1 R_2^1\right], \mathsf{E}\left[R_2^2 R_3^2\right], \ldots, \mathsf{E}\left[R_n^n R_1^n\right]\right) - (n-2) - \Delta > 0, \tag{5.24}$$

where

$$\Delta = \left|\mathsf{E}\left[R_1^1\right] - \mathsf{E}\left[R_1^n\right]\right| + \left|\mathsf{E}\left[R_2^1\right] - \mathsf{E}\left[R_2^2\right]\right| + \ldots + \left|\mathsf{E}\left[R_n^{n-1}\right] - \mathsf{E}\left[R_n^n\right]\right|. \tag{5.25}$$

The value of Δ is a measure of direct influences, or of inconsistent connectedness. It shows how much, overall, the distributions of responses to one and the same question differ in different contexts. If $\Delta = 0$, the system is consistently connected: the response to a given question is not influenced by the other questions with which it co-occurs in the same context.[3] One can loosely interpret s_{odd} as a measure of the potential true contextuality: It shows how much, overall, the identities of the random variables responding to the same question differ in different contexts. The contex-

[3]The special case of (5.24) for $\Delta = 0$ was proved, by very different mathematical means, in Araújo, Quintino, Budroni, Cunha, & Cabello (2013).

tuality test for a cyclic system therefore can be viewed as a test of whether these differences exceed those due to direct influences alone. The failure of the previous attempts to find contextuality in behavioral data may be described by saying that the empirical situations chosen for investigation had too strong direct influences for the amount of potential true contextuality they contained.

The idea of the Snow Queen experiment was to make the value of s_{odd} as large as possible, increasing its chances of beating Δ, a quantity that cannot be controlled by experimental design.[4] The formal structure of the experiment was a cyclic system of rank 4, with q_1 and q_3 being two choices of characters from a story (Snow Queen, by H.C. Andersen), and q_2 and q_4 being two choices of attributes of these characters.

R_1^1	R_2^1			c_1
	R_2^2	R_3^2		c_2
		R_3^3	R_4^3	c_3
R_1^4			R_4^4	c_4
q_1 : Gerda / Troll	q_2 : beautiful / unattractive	q_3 : Snow Queen / old Finn woman	q_4 : kind / evil	system $\mathcal{SQ}_4$

. (5.26)

For instance, in context c_3, a respondent could choose either Snow Queen or old Finn woman, and also choose either kind or evil. The instruction said the choices had to match the story line. The respondents knew, for example, that Snow Queen is beautiful and evil, and that the old Finn woman is unattractive and kind.[5] It is easy to show that if all respondents followed the instruction correctly, s_{odd} in this experiment had to have the maximal possible value of 4. The amount of direct influences measured by Δ was considerable, but the left-hand side expression in (5.24) was well above

[4]In physics the situation is different: One can eliminate or greatly reduce direct influences by, e.g., separating two entangled particles by a space-time interval that prevents transmission of a signal between them.

[5]This instruction is an analogue of the quantum-mechanical preparation, an empirical procedure preceding an experiment with the aim of creating a specific pattern of high correlations between measurements.

zero, with very high statistical reliability (evaluated by 99.99% bootstrap confidence intervals).

One possible criticism of the Snow Queen experiment can be that the paired choices were too asymmetric: Choice of a character, such as Gerda, and choice of a characteristic, such as beautiful, seem too different in nature. In the experiments reported below the paired choices were on a par. Otherwise, the experiments followed the same logic, ensuring the highest possible value for s_{odd}. This value equals n, the rank of the cyclic system. In quantum physics, the systems with this property (if, additionally, they are consistently connected, i.e., $\Delta = 0$), are called PR boxes, after Popescu and Rohrlich (1994). In our experiments n was 3 or 4.

5.2 Method

5.2.1 Participants

We recruited $6,192$ participants on CrowdFlower (2018) between February 7 and 12, 2018. They agreed to participate in this study by accepting a standard consent from. The consent form and the interactive experimental procedure were provided via a Qualtrics survey hosted by City University London. The study was approved by City University London Research Ethics Committee, PSYETH (S/L) 17/18 09. (The number of participants was chosen so that we could construct reliable 99.99% bootstrap confidence intervals for each context in each experiment, as described below.)

5.2.2 Materials and Procedure

Each respondent participated in all six experiments, in a random order. For each of the experiments, each participant was randomly and independently assigned to one

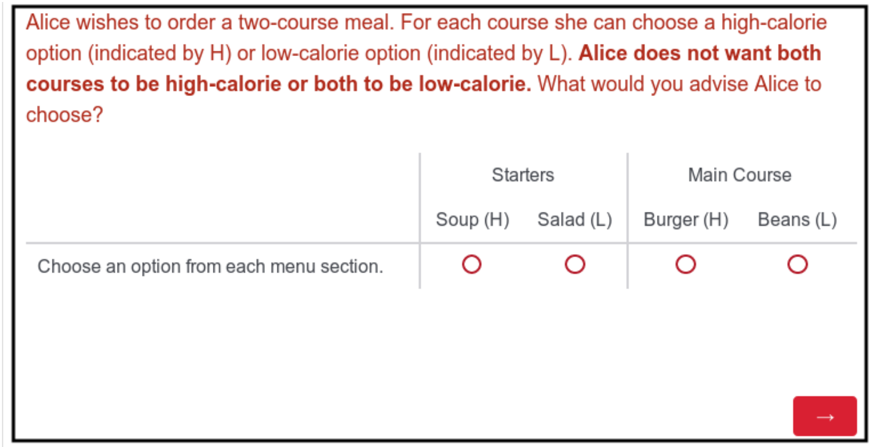


Fig. 5.1. The appearance of the computer screen to the participant if assigned to context c_1 in Experiment 1. The participant was required to choose an option for each question, in this case each menu section; the next experiment or the end of the survey would be reached by clicking the Next arrow. If the participant had made both choices in accordance with the instructions, in this case having chosen Soup (H) with Beans (L) or Salad (L) with Burger (H), clicking the 'Next' arrow allowed the survey to continue; otherwise the participants were prompted to revise or complete their responses. See the online article for the color version of this figure.

of the conditions (contexts). In each context, a participant was introduced to a pair of choices to be made by a fictional Alice; each choice was between two alternatives. There were three contexts in Experiments 1-4, and four contexts in Experiments 5 and 6. Figure 5.1 shows the way the instruction and choices were presented to respondents in one context of Experiment 1.

5.3 Experiments 1–4

In Experiments 1–4, in each context, the character Alice was faced with two choices of a set of three dichotomous choices. The participant was asked to select a pair of responses that respected Alice's preferences as stated in the instructions (see Figure 5.1). The system would not allow the respondent to make only one choice or two choices contradicting the instructions. The following depicts the situations presented, whereas Table 5.1 summarizes the sets of dichotomous choices.

5.3.1 Experiment "Meals."

Alice wishes to order a two-course meal. For each course she can choose a high-calorie option (indicated by H) or a low-calorie option (indicated by L). Alice does not want both courses to be high calorie nor does she want both of them to be low calorie.

5.3.2 Experiment "Clothes."

Alice is dressing for work, and chooses two pieces of clothing. She does not want both of them to be plain, nor does she want both of them to be fancy.

5.3.3 Experiment "Presents."

Alice wishes to buy two presents for her nephew's birthday. She can choose either a more expensive option (indicated by E) or a cheaper option (indicated by C). Alice does not want both presents to be expensive or both presents to be cheap.

5.3.4 Experiment "Exercises."

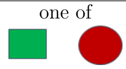
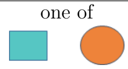
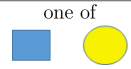
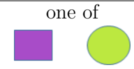
Alice is doing two physical exercises. Alice does not want both exercises to be hard or both to be easy.

Table 5.1.
Dichotomous Choices in Experiments 1–4

Experiment	q_1	q_2	q_3
1. Meals	Starters: Soup (H)* or Salad (L)	Main course: Burger (H)* or Beans (L)	Dessert: Cake (H)* or Coffee (L)
2. Clothes	Skirt: Plain* or Fancy	Blouse: Plain* or Fancy	Jacket: Plain* or Fancy
3. Presents	Book: Big expensive book (E)* or Smaller book (C)	Soft toy (bear): (E)* or (C)	Construction set: (E)* or (C)
4. Exercises	Arms: Hard* or Easy	Back: Hard* or Easy	Legs: Hard* or Easy

*Note.*Each respondent was asked to make two choices (q_1&q_2 or q_2&q_3 or q_3&q_1), randomly and independently assigned to this respondent in each experiment.
* Denotes the response encoded with +1

Table 5.2.
Dichotomous Choices in Experiments 5 and 6

Experiment	q_1	q_2	q_3	q_4
5. Directions	West—East fork $\leftarrow$ or $\rightarrow$	NorthWest—SouthEast fork $\nwarrow$ or $\searrow$	North—South fork $\uparrow$ or $\downarrow$	NorthEast—SouthWest fork $\nearrow$ or $\swarrow$
6. Colored figures	one of 	one of 	one of 	one of 

Note. Each respondent was asked to make two choices (q_1 and q_2, or q_2 and q_3, or q_3 and q_4, or q_4 and q_1), randomly and independently assigned to this respondent in each experiment. For each choice q_i, the response encoded by +1 is the one on the left: e.g., for q_1 in Experiment 5, the response $\leftarrow$ was encoded by +1.

5.4 Experiments 5 and 6

In experiments 5 and 6, in each context, the character Alice was faced with two choices out of a set of four. In all other respects the procedure was similar to that in Experiments 1–4. The participant was asked to select a pair of responses that respected the character's preferences as stated in the instructions. The following depicts the situations presented, whereas Table 5.2 summarizes the sets of dichotomous choices.

5.4.1 Experiment "Directions."

Alice goes for a walk, and has to choose path directions at forks. Alice wants the two directions to be as similar as possible (i.e., the angle between them to be as small as possible).

5.4.2 Experiment "Colored figures."

Alice is taking a drawing lesson, and is presented with two pairs consisting of a square and a circle (the pairs being labeled as Section 1 and Section 2). Alice needs to choose one figure from each section, and she wants the two figures chosen to be of similar color.

5.5 Results

In Experiments 1–4, irrespective of the specific content of the questions, there were three dichotomous choices, q_1, q_2, q_3, offered to the respondents two at a time. Denoting, for each of the choices, one of the response options $+1$ and the other -1, the results have the following form:

$$
\begin{array}{c|c|c|c}
c_1 & R_2^1 = 1 & R_2^1 = -1 & \\
\hline
R_1^1 = 1 & 0 & p_1 & p_1 \\
\hline
R_1^1 = -1 & 1-p_1 & 0 & 1-p_1 \\
\hline
 & 1-p_1 & p_1 &
\end{array}
\quad
\begin{array}{c|c|c|c}
c_2 & R_3^2 = 1 & R_3^2 = -1 & \\
\hline
R_2^2 = 1 & 0 & p_2 & p_2 \\
\hline
R_2^2 = -1 & 1-p_2 & 0 & 1-p_2 \\
\hline
 & 1-p_2 & p_2 &
\end{array}
$$

$$
\begin{array}{c|c|c|c}
c_3 & R_1^3 = 1 & R_1^3 = -1 & \\
\hline
R_3^3 = 1 & 0 & p_3 & p_3 \\
\hline
R_3^3 = -1 & 1-p_3 & 0 & 1-p_3 \\
\hline
 & 1-p_3 & p_3 &
\end{array}
\tag{5.27}
$$

Table 5.3.
Probability Estimates $\hat{p}_1, \hat{p}_2, \hat{p}_3$ That Determine the Outcomes of Experiments 1–4 in Accordance With (5.27), and the Sizes N_1, N_2, N_3 of the Samples From Which These Estimates Were Computed.

	c_1		c_2		c_3	
Experiment	$\hat{p}_1$	N_1	$\hat{p}_2$	N_2	$\hat{p}_3$	N_3
1. Meals	.349	2,090	.658	2,052	.653	2,050
2. Clothes	.639	1,996	.566	2,086	.435	2,110
3. Presents	.547	2,081	.387	2,052	.515	2,059
4. Exercises	.590	2,058	.306	2,024	.580	2,110

In reference to the CbD criterion (5.24)–(5.25), it follows that in these experiments

$$s_{odd}\left(\mathsf{E}\left[R_1^1 R_2^1\right], \mathsf{E}\left[R_2^2 R_3^2\right], \mathsf{E}\left[R_3^3 R_1^3\right]\right) = s_{odd}\left(-1,-1,-1\right) = 3, \tag{5.28}$$

so that D in (5.24) is

$$D = 2 - \Delta, \tag{5.29}$$

where

$$\begin{aligned}\Delta &= \left|\mathsf{E}\left[R_1^1\right] - \mathsf{E}\left[R_1^3\right]\right| + \left|\mathsf{E}\left[R_2^2\right] - \mathsf{E}\left[R_2^1\right]\right| + \left|\mathsf{E}\left[R_3^3\right] - \mathsf{E}\left[R_3^2\right]\right| \\ &= 2\left|p_1 + p_3 - 1\right| + 2\left|p_2 + p_1 - 1\right| + 2\left|p_3 + p_2 - 1\right|.\end{aligned}, \tag{5.30}$$

Table 5.3 presents the observed values of $\hat{p}_1, \hat{p}_2$ and $\hat{p}_3$ for each context of each of Experiments 1–4 and the corresponding numbers of participants from which these probabilities were estimated.

In Experiments 5 and 6, there were four dichotomous choices, q_1, q_2, q_3, q_4, and each respondent was offered two of them, forming one of four possible contexts. Denoting, again, for each of the choices, one of the response options $+1$ and another -1, the results have the following form:

c_1	$R_2^1 = 1$	$R_2^1 = -1$	
$R_1^1 = 1$	p_1	0	p_1
$R_1^1 = -1$	0	$1 - p_1$	$1 - p_1$
	p_1	$1 - p_1$	

c_2	$R_3^2 = 1$	$R_3^2 = -1$	
$R_2^2 = 1$	p_2	0	p_2
$R_2^2 = -1$	0	$1 - p_2$	$1 - p_2$
	p_2	$1 - p_2$	

(5.31)

c_3	$R_4^3 = 1$	$R_4^3 = -1$	
$R_3^3 = 1$	p_3	0	p_3
$R_3^3 = -1$	0	$1 - p_3$	$1 - p_3$
	p_3	$1 - p_3$	

c_4	$R_1^4 = 1$	$R_1^4 = -1$	
$R_4^4 = 1$	0	p_4	p_4
$R_4^4 = -1$	$1 - p_4$	0	$1 - p_4$
	$1 - p_4$	p_4	

In reference to the CbD criterion (5.24)–(5.25), it follows that in these experiments

$$s_{odd}\left(\mathsf{E}\left[R_1^1 R_2^1\right], \mathsf{E}\left[R_2^2 R_3^2\right], \mathsf{E}\left[R_3^3 R_4^3\right], \mathsf{E}\left[R_4^4 R_1^4\right]\right) = s_{odd}\left(1, 1, 1, -1\right) = 4, \tag{5.32}$$

where, once again,

$$D = 2 - \Delta, \tag{5.33}$$

where

$$\begin{aligned}\Delta = &\left|\mathsf{E}\left[R_1^1\right] - \mathsf{E}\left[R_1^4\right]\right| + \left|\mathsf{E}\left[R_2^1\right] - \mathsf{E}\left[R_2^2\right]\right| + \left|\mathsf{E}\left[R_3^2\right] - \mathsf{E}\left[R_3^3\right]\right| + \left|\mathsf{E}\left[R_4^3\right] - \mathsf{E}\left[R_4^4\right]\right| \\ &= 2\left|p_1 + p_4 - 1\right| + 2\left|p_2 - p_1\right| + 2\left|p_3 - p_2\right| + 2\left|p_4 - p_3\right|.\end{aligned} \tag{5.34}$$

Table 5.4 presents the observed values of $\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4$ in Experiment 5 and 6, and the corresponding numbers of participants from which these probabilities were estimated.

Table 5.5 shows the estimated values of $D = 2 - \Delta$ in all our experiments. We see that contextuality is observed in Experiments 1–4 and 5. Experiment 6, however,

Table 5.4.
Probability Estimates $\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4$ That Determine the Outcomes of Experiments 5 and 6 in Accordance With (5.31), and the Sizes N_1, N_2, N_3, N_4 of the Samples From Which These Estimates Were Computed.

	c_1		c_2		c_3		c_4	
Experiment	$\hat{p}_1$	N_1	$\hat{p}_2$	N_2	$\hat{p}_3$	N_3	$\hat{p}_4$	N_4
5. Directions	.471	1,549	.706	1,504	.645	1,537	.750	1,602
6. Colored figures	.419	1,603	.819	1,589	.360	1,482	.154	1,517*

* One participant assigned to context c_4 was excluded from Experiment 6 because she or he did not complete the responses in accordance with the instructions.

Table 5.5.
Estimated Values of $D = 2 - \Delta$ in Experiments 1–4 ($n = 3$) and 5-6 ($n = 4$)

Experiment	1. Meals	2. Clothes	3. Presents	4. Exercises	5. Directions	6. Colored figures
$\hat{D} = 2 - \hat{\Delta}$	1.361	1.440	1.548	1.223	0.758	$-.984$

Note. Positive (negative) values of D indicate contextuality (resp., noncontextuality).

shows no contextuality: The negative value in the last column indicates that direct influences here are all one needs to account for the results.

We evaluate statistical reliability of these results in two ways. The first way is to compute an upper bound for the standard deviation of $\hat{D}$ and use it to conservatively test the null-hypothesis $D = 0$ (the maximal noncontextual value) against $D > 0$ (contextuality). In Experiment 6 the alternative hypotheses changes to $D < 0$ (noncontextuality), with $D = 0$ in the null hypothesis interpreted as the infimum of contextual values. We begin by observing that each $\hat{p}_i$ has variance $\frac{p_i(1-p_i)}{N_i} \leq \frac{1}{4N_i} \leq \frac{1}{4N_{min}}$, where N_{min} is the smallest among N_i for a given experiment, as shown in Tables 5.3 and 5.4. Using the independent coupling of stochastically unrelated $\hat{p}_i$'s, commonly adopted in statistics, each summand in (5.30) and (5.34) has a variance bounded by $\frac{2}{N_{min}}$. The different summands are not independent, but the standard

Table 5.6.
Statistical Significance of Contextuality in Experiment 1-5 and of Non-contextuality in Experiment 6.

Experiment	1. Meals	2. Clothes	3. Presents	4. Exercises	5. Directions	6. Colored
$\hat{D} = 2 - \hat{\Delta}$	1.361	1.440	1.548	1.223	.758	$-.984$
N_{min}	$2,050$	$1,996$	$2,052$	$2,024$	$1,504$	$1,482$
Upper bound for SD of $\hat{D}$	.094	.095	.094	.095	.146	.147
Number of st. dev. from zero	> 14.5	> 15.1	> 16.5	> 12.9	> 5.1	> 6.6
t-distribution p-value	$< 10^{-45}$	$< 10^{-48}$	$< 10^{-57}$	$< 10^{-36}$	$< 10^{-6}$	$< 10^{-10}$
Chebyshev p-value	$< .005$	$< .005$	$< .004$	$< .006$	$< .038$	$< .023$

deviation of the sum cannot exceed the sum of their standard deviations. This means that $3\sqrt{\frac{2}{N_{min}}}$ for Experiments 1–4 and $4\sqrt{\frac{2}{N_{min}}}$ for Experiments 5–6 are upper bounds for the standard deviation of $\hat{D}$. These values are reported in Table 5.6. If we assume applicability of the central limit theorem, given the very large sample sizes, the t-distribution-based p-values are essentially zero. If we make no assumptions, the maximally conservative p-values based on Chebyshev's inequality are still below the conventional significance levels.

In our second statistical analysis, we computed bootstrap distributions and constructed the 99.99% bootstrap confidence intervals for D from $500,000$ independent resamples for each context of each experiment (Davison & Hinkley, 1997). These are presented in Figure 5.2. As we see, the left end points of the confidence intervals for Experiments 1–5 are well above zero. For Experiment 6, the 99.99% bootstrap confidence interval (see Figure 5.2) has the right endpoint well below zero, indicating reliable lack of contextuality.

5.6 Discussion

Our results confirm beyond doubt the presence of true contextuality, separated from direct influences, in simple decision making. Compared to the Snow Queen

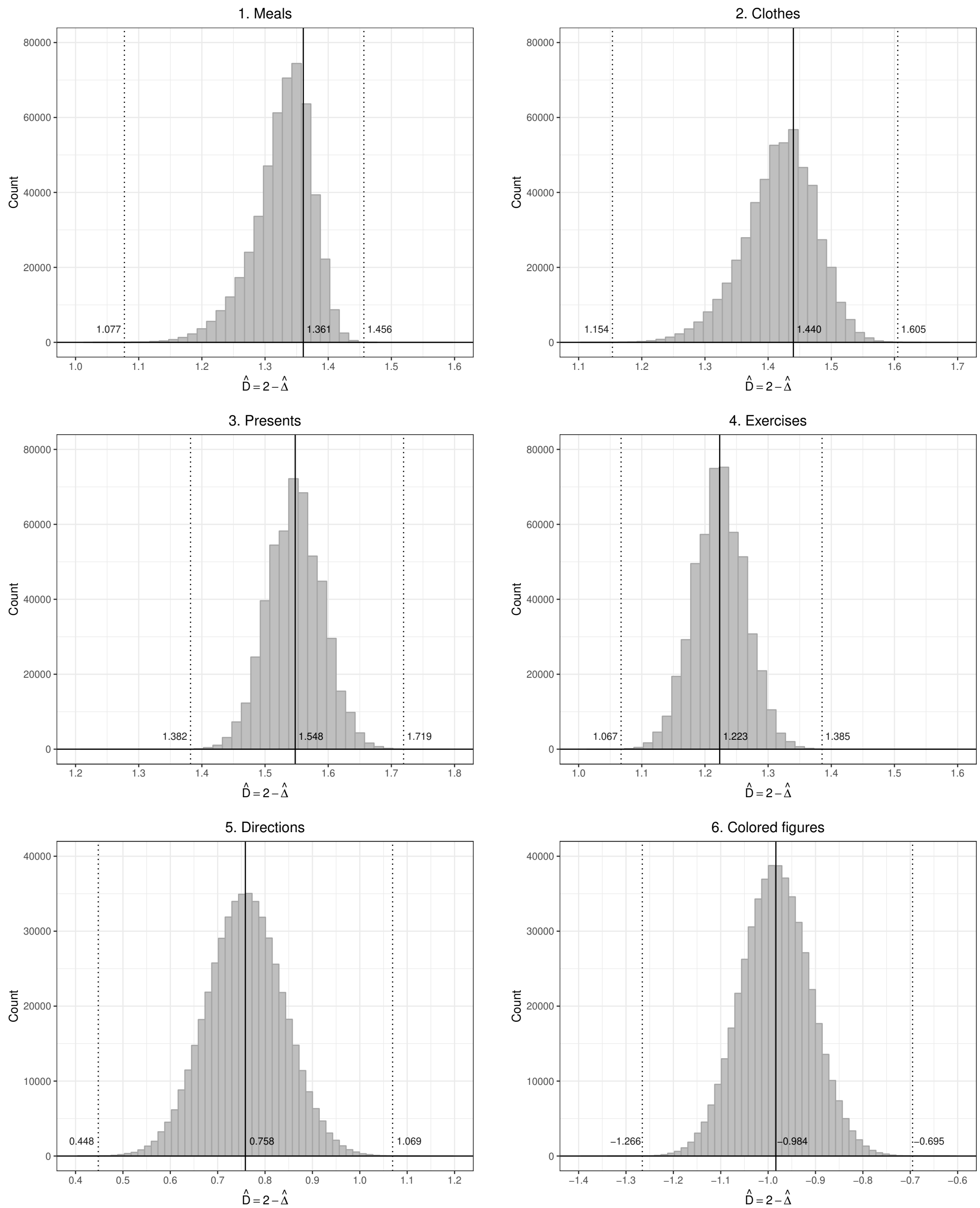


Fig. 5.2. Histograms of the bootstrap values of $\hat{D} = 2 - \hat{\Delta}$ for Experiments 1–6. The solid vertical line indicates the location of the observed sample value. The vertical dotted lines indicate the locations of the 99.99% bootstrap confidence intervals.

experiment (Cervantes & Dzhafarov, 2018), in which the paired choices belonged to different categories (choice of characters, such as Gerda or Troll, was paired with the choice of characteristics, such as kind or evil), in our experiments the paired choices belonged to the same category (e.g., two levels of arm exercises were paired with two levels of leg exercises). The fact that our results are similar to those of the Snow Queen experiment shows that this difference is immaterial. What is material is the design that ensures a very large value of s_{odd} in the contextuality criterion (5.24). In our experiments it was in fact the largest possible value, one equal to the rank of the cyclic system, n. This value in all but one of our experiments was sufficient to beat direct influences, measured by Δ (in the sense that their difference exceeded $n-2$). The one exception we got, with "Colored figures," is also valuable, as it shows that the presence of true contextuality in our experiment is an empirical finding rather than mathematical consequence of the design: Even with s_{odd} maximal in value, direct influences may very well exceed the value of $s_{odd} - (n-2)$, making the the value of D in (5.24) negative.

As explained in Cervantes and Dzhafarov (2018), in much greater detail than in the present brief recap, it is important that the design we used was between subjects, that is, each respondent in each experiment was assigned to a single context only. The reason for this is that if a single respondent were asked to make pairs of choices in all three contexts (in Experiments 1–4) or in all four contexts (in Experiments 5 and 6), it would have created an empirical joint distribution of all the random variables in the respective systems. This would contravene the logic of CbD, in which different contexts are mutually exclusive, and the random variables in different rows of content-context matrices are stochastically unrelated (have no joint distribution).

One might question another aspect of our experimental design: the fact that the respondents were not allowed to contravene their instructions and make incorrect choices (e.g., choose two "high" options or two "low" options in Experiments 1–4). The main reason for this is that in a crowdsourcing experiment, with no additional information about the respondents, it is difficult to understand what could lead a person not to follow the simple instructions. Ideally, one would want to separate data due to deliberate non-compliance or disregard from honest mistakes, and this is impossible. In fact, it is hard to fathom what an honest mistake in a situation as simple as ours might be. In the Snow Queen experiment (Cervantes and Dzhafarov, 2018), in which the choices were, arguably, less simple than in the present experiments, incorrect responses were allowed, and their percentage was just over 8%. Their inclusion or exclusion did not make any difference for analysis and conclusions.

In the opening of the paper and at the end of the *Numerical Example and Interpretation* section, we alluded to the interpretation of true contextuality in terms of the wholes irreducible to interacting parts. One must not mistake this interpretation for the old adage that "the whole is something besides the parts" (Aristotle) or, as reformulated by Kurt Koffka (1935), "the whole is something else than the sum of its parts" (p. 176). These and similar statements are not only vague, they have also been rendered essentially meaningless by their indiscriminate application to all kinds of situations. In most of cases one has a justifiable suspicion that what is meant is that parts interact, or that someone can discern a pattern in them. This is probably always true when the parts are deterministic entities. In the case of random variables, however, there is a rigorous analytic meaning of saying that the whole is different from, and indeed greater than a system of parts with all their interactions. Random variables measuring or responding to one and the same "part" (property or stimulus) have different identities in different "wholes" (contexts), with the difference being greater than warranted by the mere distributional differences caused by their interactions with other elements of the wholes. If this sounds too philosophical to be of importance in scientific practice, we have an example of quantum mechanics to counter this view.

Contextuality in quantum mechanics is not a predictive theory, and it is never used to derive any parts of quantum-mechanical theory. Rather the other way around: Quantum-mechanical theory is used to determine a system's behavior, from which it is possible to establish if the system is contextual. Thus, in the most famous example of quantum contextuality, involving spins of entangled particles (Bell, 1964, 1966), the correlations between spins are computed by standard quantum-theoretic formulas, and the results are used to establish that, for certain choices of axes along which the spins are measured, the system is contextual. The computations themselves make no use of contextuality, nor are they being amended in any way as a result of establishing contextuality or lack thereof. Nevertheless, the contextuality analysis of spins of entangled particles (Bell, 1964, 1966; Clauser, Horne, Shimony, & Holt, 1969; Fine 1982), mathematically related to a special case of our contextuality criterion (5.24), with $n = 4$ and $\Delta = 0$, is considered highly significant. A prominent experimental physicist, Alain Aspect, called it "one of the profound discoveries of the [20th] century" (Aspect, 1999), and teams of experimentalists have put much effort into verifying that the quantum-mechanical predictions used to derive it are correct (Handsteiner et al., 2017). The reason for this is, of course, that contextuality reveals something about one of the most fundamental aspects of quantum theory: the nature of random variables used to describe quantum phenomena. Thus, it is significant that typical systems of random variables describing classical mechanics happen to be noncontextual, wheras some quantum-mechanical systems are contextual. In time it has also become clear that, in addition to its foundational significance, quantum contextuality correlates with physical properties that can be used for practical purposes. Physicists and computer scientists at present are beginning to pose the question of "contextuality advantage" or "contextuality as a resource," which is the question of whether contextuality or noncontextuality of a system can be utilized for practical purposes. It is argued, for example, that the degree of contextuality (a notion we have not discussed in this paper; see Dzhafarov et al., 2017; Kujala & Dzhafarov, 2016) is directly related to computational advantage of quantum computing over conventional one (Abramsky, Barbosa, & Mansfield, 2017; Frembs, Roberts, & Bartlett, 2018).

Psychology shares the mandatory use of random variables with quantum physics: stochasticity of responses in most areas of psychology is inherent, it cannot be reduced by progressively greater control of stimuli and conditions. The status and role of contextuality therefore can be expected to be similar. The same as in quantum physics, contextuality analysis is not a predictive model competing with other models. Thus, in constructing a model to fit our data, contextuality analysis can help only in the trivial sense: as with any other property of the data, if contextuality or noncontextuality of them is established, a model is to be rejected if it fails to predict this property. As an example, one could attempt to fit our data by a model with responses being chosen from some covertly evoked initial responses actualized with the aid of some conflict resolution scheme. Assume that each question q has a probability h of being covertly answered $+1$ (standing here for one of the two options), and that in a context $c = (q, q')$ these covert responses occur independently, so that $(+1, +1)$ occurs with probability hh', $(+1, -1)$ with probability $h(1 - h')$ and so forth. If the combination of covert responses is allowed by the instructions (e.g., West and North-West in Experiment 5, or Red and Orange in Experiment 6), they turn into observed responses; if the combination is prohibited (say, West and South-East, or Red and Blue), the respondent randomly flips one of the two responses, say, with probability 1/2. Then the observed probability of choosing an allowed combination $(+1, -1)$ is computed as $h(1 - h') + hh'/2 + (1 - h)(1 - h')/2$. This model can be shown to predict that a system in our experiments is contextual, but it is incompatible with the noncontextuality in Experiment 6. This was only one example, however. Simple models that can predict both contextual and noncontextual outcomes in our experiments can be readily constructed, because all one has to predict are three probabilities (p_1, p_2, p_3) in (5.27) for Experiments 1–4, and four probabilities (p_1, p_2, p_3, p_4) in (5.31) for Experiments 5 and 6. Consider, for example, a model with eight triples $(+1, +1, +1), (+1, +1, -1), \ldots, (-1, -1, -1)$, mental states evoked with certain probabilities, with the following decision rule: if the context is (q_i, q_j), $i, j = 1, 2, 3$ and the mental state contains r_i ($+1$ or -1) and r_j ($+1$ or -1) in the ith and jth positions, respectively, then respond (r_i, r_j) if this response combination is allowable; if the combination is forbidden, choose one of the allowable combinations with probability 1/2. The model has 7 free parameters, and it can fit (p_1, p_2, p_3) in Experiments 1–4 precisely. For Experiments 5 and 6, the eight triples have to be replaced with 16 quadruples. We need not get into discussing such models here: It was not a purpose of our experiments to achieve a deeper understanding of how someone chooses to eat soup and beans over burger and salad. Rather our aim was to capitalize on the psychological transparency and modeling simplicity of such choices to firmly establish that quantum-like contextuality can be observed outside quantum physics, in human behavior. Recall that many previous attempts to demonstrate behavioral contextuality have failed, so our paper is only one of the first two steps (the other one being the Snow Queen experiment in Cervantes & Dzhafarov, 2018) on the path of identifying contextual systems in human behavior.

Thinking by analogy with the contextuality advantage mentioned above, can we, at this early stage of exploration, point out any properties of human behavior as correlating with or being indicated by contextuality? One obvious fact is that in our experiments contextuality is negatively related to the value of Δ, the amount of direct influences. Lack of direct influences means that the probability of choosing a particular option, say, burger, is the same irrespective of what context this option is included in (e.g., whether the plain skirt is chosen in the skirt-blouse combination or in the jacket-skirt one). The lack of direct influences would result in the maximal possible value of $D = 2$. This simplicity, however, is specific to our design, in which s_{odd} function does not vary. For a more general class of systems of random variables, one cannot simply replace contextuality with a measure inversely related to the amount of direct influences (we even have examples when the two are synergistic rather than antagonistic). Another dimension of human behavior that can be related to contextuality can be called the degree of similarity or unanimity of decisions across pools of respondents, or across repeated responses by the same person when a within-subject design is possible (as in Cervantes & Dzhafarov, 2017a, 2017b, and Zhang & Dzhafarov, 2017). Consider, for example, one of our Experiments 1–4 and assume that the respondents agreed among themselves on what option to choose in each context. The system then would become deterministic and noncontextual, with $D = -4$ or $D = 0$, depending on the pattern of choices agreed upon. Small deviations from an agreed-on pattern would result in small deviations from the corresponding values of D. On the other extreme, we have maximal diversity, when in each context the opposite options are chosen with equal probabilities. In this case the system would reach the maximal possible degree of contextuality. Again, it is not possible to simply replace contextuality with some measure of unanimity, such as variance: The maximal value of contextuality can also be achieved without maximal diversity of responses, and deep noncontextuality, with D between -4 and 0, can be achieved with nondeterministic systems. With due caution, one can conjecture that the degree of (non)contextuality, for a given format of the content-context matrix, may reflect a combination of the two dimensions mentioned: (in)consistency of choices across contexts (reflecting the amount of direct influences) and unanimity/diversity of choices made in each context across a pool of respondents or repeated in a within-subject design (reflecting the amount of determinism/stochasticity). We will not know if this or other relations of contextuality to various aspects of behavior can be established until we broaden our knowledge of the degree of (non)contextuality to a much larger class of behavioral systems.

5.7 References

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6. CONTEXTUALITY IN CANONICAL SYSTEMS OF RANDOM VARIABLES

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Abstract Random variables representing measurements, broadly understood to include any responses to any inputs, form a system in which each of them is uniquely identified by its content (that which it measures) and its context (the conditions under which it is recorded). Two random variables are jointly distributed if and only if they share a context. In a canonical representation of a system, all random variables are binary, and every content-sharing pair of random variables has a unique maximal coupling (the joint distribution imposed on them so that they coincide with maximal possible probability). The system is contextual if these maximal couplings are incompatible with the joint distributions of the context-sharing random variables. We propose to represent any system of measurements in a canonical form and to consider the system contextual if and only if its canonical representation is contextual. As an illustration, we establish a criterion for contextuality of the canonical system consisting of all dichotomizations of a single pair of content-sharing categorical random variables.

This article is part of the themed issue 'Second quantum revolution: foundational questions'.

Keywords: canonical systems, contextuality, dichotomization, direct influences, measurements

6.1 Introduction

We begin by recapitulating the basics of our theory of 'quantum-like' contextuality, and then explain how this theory is developed in this paper. The name of the theory is *Contextuality-by-Default* (CbD), and its recent accounts can be found in [1–3].

Remark 6.1.1. We use the following two notation conventions throughout the paper: (1) owing to its frequent occurrence, we abbreviate the term *random variable* as *rv* (*rvs* in plural); and (2) we unconventionally capitalize the words *conteNt* and *conteXt* to prevent their confusion in reading.

The matrix below represents the smallest possible version of what we call a *cyclic system* [4–7]:

$$\begin{array}{|c|c|c}
\hline
R_1^1 & R_2^1 & c=1 \\
\hline
R_1^2 & R_2^2 & c=2 \\
\hline
\multicolumn{1}{c}{q=1} & \multicolumn{1}{c}{q=2} & \boxed{\mathcal{R}}
\end{array}\ .$$

Each of the rvs R_q^c represents measurements of one of two properties, $q = 1$ or $q = 2$, under one of two conditions, $c = 1$ or $c = 2$. The 'properties' q can also be called 'objects', 'inputs', 'stimuli', etc. depending on the application, and we refer to q generically as the *conteNt* of the measurement R_q^c. The superscript c in R_q^c describes how and under what circumstances q is measured, including what other conteNts are measured together with q. We refer to c generically (and traditionally) as the *conteXt* of the measurement R_q^c. The conteNt-conteXt pair (q, c) provides a *unique identification* of R_q^c within the system of measurements $\mathcal{R}$. In addition, being an rv, R_q^c is characterized by its *distribution*. In this paper, consideration is confined to *categorical rvs*, those with a finite number of values. The term 'measurement' is understood very broadly, to include any response to any input or stimulus.

Let us begin with the simplest case of the system $\mathcal{R}$, when all four rvs R_q^c are *binary*. In quantum physics, R_q^c may describe a measurement of spin along one of two fixed axes, $q = 1$ or $q = 2$, in a spin-$1/2$ particle. In psychology, R_q^c may describe a response to one of two Yes–No questions, $q = 1$ or $q = 2$. In both applications, in conteXt $c = 1$ one measures first $q = 1$ and then $q = 2$; in conteXt $c = 2$ the measurements are made in the opposite order. The rvs sharing a conteXt c are recorded in pairs, (R_1^c, R_2^c), which means that they are *jointly distributed* and can be viewed as a single (here, four-valued) rv. No such joint distribution is defined for rvs in different conteXts, such as R_2^1 and R_1^2. They are *stochastically unrelated* (to each other): one cannot ask about the probability of an 'event' $[R_2^1 = x, R_1^2 = y]$, as no such 'event' is defined. In particular, two conteNt-sharing rvs, R_q^1 and R_q^2, are always stochastically unrelated, hence they can never be considered one and the same rv, even if they are identically distributed (see [1] for a detailed probabilistic analysis).

In both applications mentioned, the distributions of R_q^1 and R_q^2 are de facto different. In the quantum-mechanical example, the first spin measurement generally changes the state of the particle [8]. Assuming identical preparations in both conteXts c, therefore, the state of the particle when a q-spin is measured first will be different from that when it is measured second. In the behavioural example, one's response to a question asked second will generally be influenced by the question asked first [9, 10]. This creates obvious *conteXt-dependence* of the measurements, but this is not what we call *contextuality* in our theory. The original meaning of the term in quantum mechanics, when translated into the language of probability theory (as in [1, 3, 11] and, with caveats, [6, 12–17]), is that measurements of one and the same physical property q have to be represented by different rvs depending on what other properties are being measured together with q—even when the laws of physics exclude all *direct interactions* (energy/information transfer) between the measurements. By extension, when such direct interactions are present, as they are in our two applications of the system $\mathcal{R}$, we speak of contextuality only if the dependence of R_q^c on c is greater, in a well-defined sense, than just the changes in its distribution. Contextuality is a non-causal aspect of conteXt-dependence, revealed in the probabilistic relations between different measurements rather than in their individual distributions.

This is how this understanding is implemented in CbD. We characterize the conteXt-induced changes in the individual distributions, i.e., the difference between those of R_q^1 and R_q^2, by *maximally coupling* them. This means that we replace R_q^1 and R_q^2 with jointly distributed T_q^1 and T_q^2 that have the same respective individual distributions, and among all such couplings we find one with the maximal value of $\Pr\left[T_q^1 = T_q^2\right]$. This maximal coupling $\left(T_q^1, T_q^2\right)$ always exists and is unique. The next step is to see if there exists an *overall coupling* S of $\mathcal{R}$, a jointly distributed quadruple with elements corresponding to those of $\mathcal{R}$,

$$\begin{array}{|c|c|l}
\hline
S_1^1 & S_2^1 & c=1 \\
\hline
S_1^2 & S_2^2 & c=2 \\
\hline
\multicolumn{1}{c}{q=1} & \multicolumn{1}{c}{q=2} & \boxed{S}
\end{array} \quad ,$$

such that its rows (S_1^c, S_2^c) are distributed as the rows of $\mathcal{R}$ and its columns $\left(S_q^1, S_q^2\right)$ are distributed as the maximal couplings $\left(T_q^1, T_q^2\right)$ of the columns of $\mathcal{R}$. If such a *maximally connected* coupling S does not exist, one can say that the within-conteXt (row-wise) relations prevent different measurements of the same conteNt (column-wise) from being as close to each other as this is allowed by the direct influences alone. Put differently, the relations of R_q^1 and R_q^2 with their same-conteXt counterparts force them, if imposed a joint distribution on, to coincide less frequently than if these relations are ignored. The system then is deemed *contextual.* Conversely, if the coupling S above exists, the within-conteXt relations do not make the measurements of R_q^1 and R_q^2 any more dissimilar than required by the direct influences: the system is *non-contextual.*

The (non)existence of S is determined by a simple linear programing procedure [3, 4]: in our example, S has 2^4 possible values, and we find out if they can be assigned nonnegative numbers (probability masses) that sum to the given row-wise probabilities $\Pr\left[R_1^c = x, R_2^c = y\right]$ and the computed column-wise probabilities $\Pr\left[T_q^1 = x, T_q^2 = y\right]$. There is also a simple criterion (inequality) for the existence of a solution for this system of equations [4–6]. Using it one can show, e.g., that in our quantum-mechanical application the system $\mathcal{R}$ is always non-contextual, and this is also true for the behavioural application if one adopts the model proposed in [9] (see [18] for details). Mathematically, however, the system $\mathcal{R}$ can be contextual, and if it is, CbD provides a simple way of computing the *degree* of its contextuality [3]: one replaces the probability masses in the above linear programing task with *quasi-probabilities*, allowed to be negative, and finds among the solutions the minimum sum of their absolute values (see §6.2.3).

Although most of these principles and procedures of CbD have been formulated for arbitrary systems of measurements [3, 11], they only work without complications with systems that satisfy the following two constraints: (A) they contain only binary rvs, and (B) there are no more than two rvs sharing a conteNt (i.e., occupying the same column). What we propose in this paper is to always present a system of measurements in a *canonical form*, which is in essence one with the properties A and B. The cyclic systems form a subclass of canonical systems, rich enough to cover most experimental paradigms of traditional interest in quantum-mechanical and behavioural contextuality studies [3, 4, 6, 11, 18, 19], but far from satisfactory generality.

What are the complications one faces if a system does not satisfy the properties A and B? Consider the system below, with all its rvs binary but with three rather than two of them in each column:

R_1^1	R_2^1	$c = 1$
R_1^2	R_2^2	$c = 2$
R_1^3	R_2^3	$c = 3$
$q = 1$	$q = 2$	$\boxed{\mathcal{R}'}$

How does CbD apply here? In the earlier version of the theory (summarized in [3, 11]), we computed the couplings $\left(T_q^1, T_q^2, T_q^3\right)$ of each column that maximize $\Pr\left[T_q^1 = T_q^2 = T_q^3\right]$. One problem with this approach is that the maximal coupling $\left(T_q^1, T_q^2, T_q^3\right)$, while it always exists, is not defined uniquely. What should be the contextuality analysis of $\mathcal{R}'$ if the within-conteXt (row-wise) distributions are compatible with some but not all combinations of the maximal couplings for the two columns? Shall one then speak of a partial (non)contextuality? Originally, we proposed to consider a system non-contextual if it is compatible with at least one of these pairs of maximal couplings, but in addition to being arbitrary, this leads to another complication: it may then very well happen that the system $\mathcal{R}'$ is non-contextual but one of its subsystems, e.g. $\mathcal{R}$, is contextual. This is contrary to one's intuition of non-contextuality.

In the most recent publications therefore [1, 2], we modified our approach into 'CbD 2.0', by positing that a coupling for conteNt-sharing measurements should be computed so that it maximizes the probability of coincidence for every *pair* (equivalently, every subset) of them. In our case, this means maximization of $\Pr\left[T_q^1 = T_q^2\right]$, $\Pr\left[T_q^2 = T_q^3\right]$, and $\Pr\left[T_q^1 = T_q^3\right]$ (it is in fact sufficient to maximize only certain pairs rather than all of them, but this is not critical here). Such a coupling $\left(T_q^1, T_q^2, T_q^3\right)$ is called *multimaximal*. With only binary rvs involved, a multimaximal coupling always exists and is unique; and a subsystem of a non-contextual system then is always non-contextual.

Returning to system $\mathcal{R}$, consider now the situation when the measurements involved are not dichotomous. For example, let the two successive spin measurements along axes $q = 1$ and $q = 2$ be made on a hypothetical spin-2 particle, with the measurement outcomes denoted by $\{-2, -1, 0, 1, 2\}$. In the behavioural application, let the questions asked allow five answers each, labeled in the same way. A maximal coupling in this situation exists for each column of $\mathcal{R}$, but not uniquely. This takes us back to the problem of what one should do if the row-wise distributions are compatible with some but not all pairs of these maximal couplings. Another problem is even harder. If the system is deemed non-contextual, one may consider it desirable that it remain non-contextual after some of the measurement outcomes are 'lumped together.' Thus, one may wish to consider $\{-2, -1, 0, 1, 2\}$ in terms of 'negative-zero-positive', lumping together -2 with -1 and 2 with 1. Or one may wish to look at the outcomes in terms of 'zero-non-zero.' As it turns out, a non-contextual system may become contextual after such coarsening of some of its measurements.

Both these problems can be resolved if we agree that *every measurement included in the system, empirically recorded or computed from those empirically recorded, should be represented by a set of binary rvs.* Let us denote by D^c_{qW} the Bernoulli rv that equals 1 if the value of R^c_q is within the subset W of its possible values. We call D^c_{qW} a *split* (of the original rv). We posit that a measurement with k distinct values should always be represented by k 'detectors' of these values, i.e. the splits with one-element subsets W. Thus, in our system $\mathcal{R}$, each measurement R^c_q should be replaced with the jointly distributed splits

$$\left(D^c_{q\{-2\}}, D^c_{q\{-1\}}, D^c_{q\{0\}}, D^c_{q\{1\}}, D^c_{q\{2\}}\right).$$

If one is also interested in the coarsening of $R^c_{q'}$ into values 'negative-zero-positive', then the list should be expanded into

$$\left(D^c_{q\{-2\}}, D^c_{q\{-1\}}, D^c_{q\{0\}}, D^c_{q\{1\}}, D^c_{q\{2\}}, D^c_{q\{-2,-1\}}, D^c_{q\{1,2\}}\right).$$

If one wishes to include *all* possible coarsenings of the original rvs in $\mathcal{R}$, then the set of binary rvs should consist of *all* possible splits. As every dichotomization creating a split should be applied to all rvs sharing a conteNt, one ends up replacing the system $\mathcal{R}$ with

$D^1_{1\{-2\}}$	$\cdots$	$D^1_{1\{2\}}$	$D^1_{1\{-2,-1\}}$	$\cdots$	$D^1_{1\{1,2\}}$	$\cdots$	$D^1_{2\{1,2\}}$	$c=1$
$D^2_{1\{-2\}}$	$\cdots$	$D^2_{1\{2\}}$	$D^2_{1\{-2,-1\}}$	$\cdots$	$D^2_{1\{1,2\}}$	$\cdots$	$D^2_{2\{1,2\}}$	$c=2$
$q=1\{-2\}$	$\cdots$	$q=1\{2\}$	$q=1\{-2,-1\}$	$\cdots$	$q=1\{1,2\}$	$\cdots$	$q=2\{1,2\}$	$\boxed{\mathcal{D}}$

There are $(2^5-2)/2 = 15$ distinct dichotomizations of the set $\{-2,-1,0,1,2\}$, and the 15 subsets W in D^c_{qW} should be chosen to avoid duplication, such as in $D^c_{q\{0,1\}}$ and $D^c_{q\{-2,-1,2\}}$. Once duplication is prevented, however, all splits of all rvs one is interested in should be included. It is irrelevant that some of them can be presented as functions of the others. In fact, any split of our R^c_q can be presented as a function of just three splits, chosen, e.g., as

$$D^c_{q'} = D^c_{q\{-1,1\}}, D^c_{q''} = D^c_{q\{0,1\}}, D^c_{q'''} = D^c_{q\{2\}}.$$

It is easy to show, however, that in the subsystem

$D^1_{1'}$	$D^1_{1''}$	$D^1_{1'''}$	$f\left(D^1_{1'}, D^1_{1''}, D^1_{1'''}\right)$	$c=1$
$D^2_{1'}$	$D^2_{1''}$	$D^2_{1'''}$	$f\left(D^2_{1'}, D^2_{1''}, D^2_{1'''}\right)$	$c=2$
$q=1'$	$q=1''$	$q=1'''$	q^*	$\boxed{\mathcal{D}'}$

of the system $\mathcal{D}$, the f-transformation of the maximal couplings of the first three columns, because these couplings are not jointly distributed, would not determine the coupling of the fourth column, let alone ensure that this coupling is maximal.

There is no general prescription as to which rvs should or should not be included in the system representing an empirical set of measurements: what one includes (e.g. what coarsenings of the rvs already in play one considers) reflects what aspects of the empirical situation one is interested in. Once a set of rvs is chosen, however, we uniquely form their splits and place them in a canonical system.

The remainder of the paper is organized as follows. In §6.2, we present the abstract version of CbD applicable to all possible systems of categorical (and not only categorical) rvs. In §6.3, we formalize the idea of representing any system of rvs by their splits and applying contextuality analysis to these representations only. In §6.4, we investigate the representation of all coarsenings of a single pair of conteNt-sharing rvs by all possible splits. In the concluding section, we explain why one might wish to consider only some rather than all possible splits.

Remark 6.1.2. The proofs of the formal propositions in the paper, unless obvious or referenced as presented elsewhere, are given in electronic supplementary material, file S[a], together with additional theorems and examples.

6.2 Formal theory of contextuality

6.2.1 Basic notions

The definition of a system of rvs requires two non-empty finite sets, a set of *conteNts* Q and a set of *conteXts* C. There is a relation

$$\prec \subseteq Q \times C, \tag{6.2.1}$$

[a](Explanatory note added in the dissertation.) The supplementary material is reproduced as Section 6.7

such that the projections of $\prec$ into Q and C equal Q and C, respectively (this means that for every $q \in Q$, there is a $c \in C$, and vice versa, such that $q \prec c$). We read both $q \prec c$ and $c \succ q$ as '*q is measured in c*'.

A categorical rv is one with a finite set of values and its power set as the codomain sigma-algebra. A system of (categorical) rvs is a double-indexed set (we use calligraphic letters for sets of random variables)

$$\mathcal{R} = \left\{ R_q^c : q \in Q, c \in C, q \prec c \right\}, \tag{6.2.2}$$

such that (i) any R_q^c and $R_q^{c'}$ have the same set of possible values; (ii) R_q^c and $R_{q'}^{c'}$ are jointly distributed if $c = c'$; and (iii) if $c \neq c'$, R_q^c and $R_{q'}^{c'}$ are *stochastically unrelated* (possess no joint distribution). For any $c \in C$, the subset

$$\mathcal{R}^c = \left\{ R_q^c : q \in Q, q \prec c \right\} = R^c \tag{6.2.3}$$

of $\mathcal{R}$ is called a *bunch* (of rvs) corresponding to c. As the elements of a bunch are jointly distributed, the bunch is a (categorical) rv in its own right, so it can be also written as R^c. Note that we do not distinguish the representations of $\mathcal{R}$ as (6.2.2) and as

$$\mathcal{R} = \left\{ R^c : c \in C \right\}. \tag{6.2.4}$$

(See [1, 3] for a detailed probabilisitic analysis.)

For any $q \in Q$, the subset

$$\mathcal{R}_q = \left\{ R_q^c : c \in C, q \prec c \right\} \tag{6.2.5}$$

of $\mathcal{R}$ is called a *connection* (between the bunches of rvs) corresponding to q. Any two elements of a connection are stochastically unrelated, so it is not an rv.

6.2.2 General definition of (non)contextuality

A (probabilistic) *coupling* Y of a set of rvs $\{X_1, \ldots, X_n\}$ is a set of jointly distributed $\{Y_1, \ldots, Y_n\}$ such that $Y_i \sim X_i$ for $i = 1, \ldots, n$. The tilde $\sim$ stands for 'has the same distribution as'.

An (overall) coupling S of a system $\mathcal{R}$ in (6.2.2) is a coupling of its bunches. That is, it is an rv

$$S = \{S^c : c \in C\} \tag{6.2.6}$$

(with jointly distributed components) such that $S^c \sim R^c$ for any $c \in C$. This implies that

$$S^c = \left\{S_q^c : q \in Q, q \prec c\right\} \tag{6.2.7}$$

is a set of jointly distributed rvs in a one-to-one correspondence with the identically labeled elements of $\mathcal{R}$.

For a given $q \in Q$, a coupling T_q of a connection $\mathcal{R}_q$ is an rv

$$T_q = \left\{T_q^c : c \in C, q \prec c\right\} \tag{6.2.8}$$

such that $T_q^c \sim R_q^c$. In particular, if S is a coupling of $\mathcal{R}$, then

$$S_q = \left\{S_q^c : c \in C, q \prec c\right\} \tag{6.2.9}$$

is a coupling of $\mathcal{R}_q$, for any $q \in Q$.

Definition 6.2.1. Given a set $\mathcal{T} = \{T^c : c \in C\}$ of couplings for all connections in a system $\mathcal{R}$, the system is said to be *non-contextual with respect to* $\mathcal{T}$ if $\mathcal{R}$ has a coupling S with $S_q \sim T_q$ for any $q \in Q$. Otherwise $\mathcal{R}$ is said to be *contextual with respect to* $\mathcal{T}$.

Put differently, $\mathcal{R}$ is non-contextual with respect to $\mathcal{T}$ if and only if there is a jointly distributed set

$$S = \left\{ S_q^c : q \in Q, c \in C, q \prec c \right\}, \tag{6.2.10}$$

such that, for every $c \in C$, $S^c \sim R^c$, and for every $q \in Q$, $S_q \sim T_q$. A coupling S with this property is called $\mathcal{T}$-*connected*.

If the couplings T_q are characterized by some property C such that one and only one coupling T_q satisfies this property for any given connection $\mathcal{R}_q$, then the definition can be rephrased as follows:

Definition 6.2.2. $\mathcal{R}$ is said to be *non-contextual with respect to property* C if it has a C-*connected* coupling S, defined as one with S_q satisfying C for any $q \in Q$. Otherwise $\mathcal{R}$ is said to be *contextual with respect to* C.

Remark 6.2.3. In §6.3.3, we will use the property of *(multi)maximality* to play the role of C, and the couplings in question then are referred to as *(multi)maximally-connected*.

6.2.3 Degree of contextuality

A *quasi-distribution* on a finite set V is a function $V \to \mathbb{R}$ (real numbers) such that the numbers assigned to the elements of V sum to 1. We will refer to these numbers as *quasi-probability masses*. A *quasi-rv* X is defined analogously to an rv but with a quasi-distribution instead of a distribution.

A *quasi-coupling* X of $\mathcal{R}$ is defined as a quasi-rv

$$X = \left\{ X_q^c : q \in Q, c \in C, q \prec c \right\}, \tag{6.2.11}$$

such that $X^c \sim R^c$ for every $c \in C$. We have the following results.

Theorem 6.2.4. **([3, Theorem 6.1])** *For any system $\mathcal{R}$ and any set $\mathcal{T}$ of couplings for the connections of $\mathcal{R}$, there is a quasi-coupling X of $\mathcal{R}$ such that $X_q = \left\{X_q^c : c \in C, q \prec c\right\} \sim T_q$ for any $q \in Q$.*

The *total variation* of X is denoted by $\|X\|$ and defined as the sum of the absolute values of the quasi-probability masses assigned to all values of X.

Theorem 6.2.5. **([3, Section 6.3])** *The total variation $\|X\|$ reaches its minimum in the class of all quasi-couplings X satisfying the conditions of theorem 6.2.4.*

If $\min \|X\|$ is 1, then all quasi-probability masses are non-negative, and the system $\mathcal{R}$ is non-contextual with respect to $\mathcal{T}$. If $\min \|X\| > 1$, then the system is contextual with respect to $\mathcal{T}$, and $\min \|X\| - 1$ can be taken as a (universally applicable) measure of the *degree of contextuality.*

6.3 Splits and canonical representations

6.3.1 Expansions of the original system

One is often interested not only in a system of empirically measured rvs $\mathcal{R}$ but also in some transformations thereof. Each such a transformation $F_{q_1,\ldots,q_k}$ is labeled by a set of conteNts, $q_1, \ldots, q_k$, and it takes as its arguments the rvs $R_{q_1}^c, \ldots, R_{q_k}^c$ in each conteXt c such that $c \succ q_1, \ldots, q_k$. The outcome,

$$R_{q^*}^c = F_{q_1,\ldots,q_k}\left(R_{q_1}^c, \ldots, R_{q_k}^c\right), \tag{6.3.1}$$

is an rv interpreted as measuring a new conteNt q^* in the conteXt c. One is free to choose any such transformations and form the corresponding new conteNts, as there can be no rules mandating what one should be interested in measuring.

Using various transformations to add new conteNts and new rvs to the original system *expands* it into a larger system. Two types of expansions that are of particular interest are *expansion-through-joining* and *expansion-through-coarsening*. Joining is defined as

$$R^c_{q_1}, \ldots, R^c_{q_k} \longmapsto \left(R^c_{q_1}, \ldots, R^c_{q_k}\right) = R^c_{q'}, \tag{6.3.2}$$

whereas coarsening is transformation

$$R^c_q \longmapsto F_q\left(R^c_q\right) = R^c_{q''}. \tag{6.3.3}$$

In fact any other transformation $F_{q_1,\ldots,q_k}\left(R^c_{q_1}, \ldots, R^c_{q_k}\right)$ can be presented as joining followed by coarsening.

Example 6.3.1 (Joining)**.** Consider the system

$$\begin{array}{|c|c|c|l}
\hline
R^1_1 & R^1_2 & \cdot & c=1 \\
\hline
R^2_1 & R^2_2 & \cdot & c=2 \\
\hline
R^3_1 & \cdot & R^3_3 & c=3 \\
\hline
\cdot & R^4_2 & R^4_3 & c=4 \\
\hline
\multicolumn{1}{c}{q=1} & \multicolumn{1}{c}{q=2} & \multicolumn{1}{c}{q=3} & \boxed{\mathcal{R}}
\end{array} \; .$$

It contains the jointly distributed R^1_1, R^1_2 and also the jointly distributed R^2_1, R^2_2, but in determining the maximal couplings of R^1_1, R^2_1 and of R^1_2, R^2_2 in the first and second columns these row-wise joints are not used. In some applications, this would be unacceptable (e.g., in the theory of selective influences [20, 21] and in the approach advocated by Abramsky and colleagues [22, 23] this is never acceptable), and then the following expansion has to be used:

$$
\begin{array}{|c|c|c|c|c}
\hline
R_1^1 & R_2^1 & \cdot & (R_1^1, R_2^1) & c=1 \\
\hline
R_1^2 & R_2^2 & \cdot & (R_1^2, R_2^2) & 2 \\
\hline
R_1^3 & \cdot & R_3^3 & \cdot & 3 \\
\hline
\cdot & R_2^4 & R_3^4 & \cdot & 4 \\
\hline
\multicolumn{1}{c}{q=1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{12} & \boxed{\mathcal{R}^*}
\end{array}
\;.
$$

Example 6.3.2 (Coarsening)**.** If V is a set of possible values of R_q^c, then $U = F_q(V)$ is the set of possible values of the rv $R_{q^*}^c = F_q(R_q^c)$. This rv is a coarsening of R_q^c. Note that any rv is its own coarsening. As the way one labels the values of U is usually irrelevant, each such function F_q can be presented as a partition of V. Consider, e.g. the 'mini'-system

$$
\begin{array}{|c|c}
\hline
R_q^1 & c=1 \\
\hline
R_q^2 & c=2 \\
\hline
\multicolumn{1}{c}{q} & \boxed{\mathcal{R}}
\end{array}
\;,
$$

and let the two rvs take values on $\{1,2,3,4,5\}$. If these values are considered ordered, $1 < \ldots < 5$, one may be interested in all possible partitions of $\{1,2,3,4,5\}$ into subsets of consecutive numbers, such as $\{12\,|\,34\,|\,5\}$, $\{1\,|\,2345\}$, etc. There are 15 such partitions (counting $\{1\,|\,2\,|\,3\,|\,4\,|\,5\}$ that defines the original rvs R_q^c, but excluding the trivial partition $\{12345\}$). If the values $1,2,3,4,5$ are treated as unordered labels, one might consider all possible non-trivial partitions, such as $\{\{14\},\{25\},\{3\}\}$, $\{\{145\},\{23\}\}$, etc. There are 51 such partitions. In either of these two coarsening schemes the partitions can be ordered in some way, and the respective expanded systems then become

$$
\begin{array}{|c|c|c|c|c}
\hline
R_q^1 & R_{q1'}^1 & \cdots & R_{q14'}^1 & c=1 \\
\hline
R_q^2 & R_{q1'}^2 & \cdots & R_{q14'}^2 & c=2 \\
\hline
\multicolumn{1}{c}{q} & \multicolumn{1}{c}{q1'} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{q14'} & \boxed{\mathcal{R}'}
\end{array}
\quad \text{and} \quad
\begin{array}{|c|c|c|c|c}
\hline
R_q^1 & R_{q1''}^1 & \cdots & R_{q50''}^1 & c=1 \\
\hline
R_q^2 & R_{q1''}^2 & \cdots & R_{q50''}^2 & c=2 \\
\hline
\multicolumn{1}{c}{q} & \multicolumn{1}{c}{q1''} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{q50''} & \boxed{\mathcal{R}''}
\end{array}
$$

Remark 6.3.3. Although the number of the states (combinations of the values of the elements) of the bunch R^c in $\mathcal{R}'$ and especially in $\mathcal{R}''$ is very large, the support of each bunch (the set of the states with non-zero probabilities) has the same size as that of the initial random variable R_q^c in $\mathcal{R}$ (i.e. in our example, it cannot exceed 5). This follows from the facts that each event $R_q^c = x$ uniquely defines the state of R^c in $\mathcal{R}'$ and in $\mathcal{R}''$, and that $\sum_x \Pr\left[R_q^c = x\right] = 1$.

6.3.2 Dichotomizations and canonical/split representations

Definition 6.3.4. A *dichotomization* of a set V is a function $f : V \to \{0,1\}$. Applying such an f to an rv R with the set of possible values V, we get a binary rv $f(R)$. We call this $f(R)$ a *split* of the original R.

If R_q^c is an element of a system $\mathcal{R}$, let us agree to identify $f\left(R_q^c\right)$ as D_{qW}^c, where $W = f^{-1}(1)$, with the understanding that D_{qW}^c and $D_{q(V-W)}^c$ are indistinguishable. To make the choice definitive, we always choose W as the smaller of W and $V - W$; in the case they have the same number of elements, we order the elements of V, say $1 < 2 < \ldots < k$, and then choose W as lexicographically preceding $V - W$.

With $V = \{1, 2, \ldots, k\}$, the jointly distributed set of splits

$$\left\{D_{q\{1\}}^c, D_{q\{2\}}^c, \ldots, D_{q\{k\}}^c\right\} \tag{6.3.4}$$

is called the *split representation* of R_q^c. If $k = 2$, then R_q^c is its own split representation, because $D_{q\{1\}}^c$ and $D_{q\{2\}}^c$ are indistinguishable.

Definition 6.3.5. The system $\mathcal{D}$ obtained from a system $\mathcal{R}$ by replacing each of its elements by its split representations is called the *canonical* (or *split*) *representation* of $\mathcal{R}$.

Example 6.3.6 (continuing example 6.3.1). Let all rvs in $\mathcal{R}$ be binary, 0/1, whence (R_1^1, R_2^1) and (R_1^2, R_2^2) in $\mathcal{R}^*$ have four values each: $00, 01, 10$ and 11. Replacing them with the split representations and observing that the first three columns do not change, we get the following canonical representation of $\mathcal{R}^*$:

$D_1^1 = R_1^1$	$D_2^1 = R_2^1$	$\cdot$	$D^1_{12\{00\}}$	$D^1_{12\{01\}}$	$D^1_{12\{10\}}$	$D^1_{12\{11\}}$	$c = 1$
$D_1^2 = R_1^2$	$D_2^2 = R_2^2$	$\cdot$	$D^2_{12\{00\}}$	$D^2_{12\{01\}}$	$D^2_{12\{10\}}$	$D^2_{12\{11\}}$	2
$D_1^3 = R_1^3$	$\cdot$	$D_3^3 = R_3^3$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	3 .
	$D_2^4 = R_2^4$	$D_3^4 = R_3^4$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	4
$q = 1$	2	3	$12\,\{00\}$	$12\,\{01\}$	$12\,\{10\}$	$12\,\{11\}$	$\boxed{\mathcal{D}^*}$

Example 6.3.7 (continuing example 6.3.2). For the system $\mathcal{R}'$, it is clear that the split representations of the 15 coarsenings of R_q^c variously overlap: e.g. $D^1_{q\{3\}}$ belongs to the split representations of R_q^1 and of the coarsenings defined by the partitions $\{12\,|\,3\,|\,45\}$, $\{1\,|\,2\,|\,3\,|\,45\}$, and $\{12\,|\,3\,|\,4\,|\,5\}$. Following our rules, W in the splits D^c_{qW} comprising the split representation of $\mathcal{R}'$ are (when written as strings) $1, 2, 3, 4, 5, 12, 23, 34, 45$ and 15 (note that, e.g., the split of the coarsening $\{1|23|4|5\}$ with $W = \{1, 23\}$ should be denoted $D^1_{q\{1,23\}}$ according to our definitions, but this is the same random variable as $D^1_{q\{45\}}$ which we have included in the list). For the system $\mathcal{R}''$ the canonical representation, obviously, consists of all possible splits of R_q^c. It will be the target of the analysis presented in §6.4.

6.3.3 Multimaximality for canonical representations

If each connection in a canonical representation $\mathcal{D}$ contains just two rvs, one can compute unique maximal couplings for all of these connections. The determination of whether $\mathcal{D}^*$ is (non)contextual then can proceed in compliance with the general theory presented in §6.2.2, and amounts to determining if $\mathcal{D}^*$ has a *maximally connected* coupling S (see remark 6.2.3). If no such coupling exists, the computation of the degree of contextuality in $\mathcal{D}^*$ can be done in compliance with §6.2.3.

In a more general case, however, with an arbitrary number of rvs in each connection, maximal couplings should be replaced with computing what we call *multimaximal couplings* [1, 2].

Definition 6.3.8. A coupling T_q of a connection $\mathcal{D}_q$ of a split representation $\mathcal{D}$ is called *multimaximal* if, for any $c, c' \in C$ such that $c, c' \succ q$, $\Pr\left[T_q^c = T_q^{c'}\right]$ is maximal over all possible couplings of $\mathcal{D}_q$. (If the connection contains two rvs, its multimaximal coupling is simply maximal.)

A multimaximal coupling is known to have the following properties.

Multimax1: The multimaximal coupling exists and is unique for any connection $\mathcal{D}_q$ ([2] Corollary 1).

Multmax2: T_q is a multimaximal coupling of $\mathcal{D}_q$ if and only if any subset of T_q is a maximal coupling for the corresponding subset of $\mathcal{D}_q$ ([2, Theorem 5]; [1, Theorem 2.3]).

Multimax3: In a connection $\mathcal{D}_q$, if $\{c_1, \ldots, c_n\}$ is the set of all $c \succ q$ enumerated so that

$$\Pr\left[D_q^{c_1} = 1\right] \leq \ldots \leq \Pr\left[D_q^{c_n} = 1\right],$$

then T_q is a multimaximal coupling of $\mathcal{D}_q$ if and only if $\Pr\left[T_q^{c_i} = T_q^{c_{i+1}}\right]$ is maximal for $i = 1, \ldots, n-1$, over all possible couplings of $\mathcal{D}_q$ ([1, Theorem 2.3]).

6.4 The largest canonical representation of a two-element connection

We consider here the case when one is interested in all possible coarsenings of the rvs in a system. The canonical/split representation of the system then contains all splits of all rvs. We will investigate in detail a fragment of the original (expanded) system involving just two k-valued rvs within a single connection:

$$\begin{array}{|c|l} \hline R_1^1 & c=1 \\ \hline R_1^2 & c=2 \\ \hline \multicolumn{1}{c}{q=1} & \boxed{\mathcal{R}} \end{array}$$

The canonical system with all splits of these k-valued rvs is

$$\begin{array}{r|c|c|c|c|l} \cline{2-5} D^1: & D^1_{W1} & D^1_{W2} & & D^1_{W\left(2^{k-1}-1\right)} & c=1 \\ \cline{2-5} D^2: & D^2_{W1} & D^2_{W2} & \cdots & D^2_{W\left(2^{k-1}-1\right)} & c=2 \\ \cline{2-5} \multicolumn{1}{c}{} & \multicolumn{1}{c}{q=W1} & \multicolumn{1}{c}{W2} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{W\left(2^{k-1}-1\right)} & \boxed{\mathcal{D}} \end{array}$$

where $W1$, $W2$, etc. are the subsets $f^{-1}(1)$ chosen as explained in §6.3.2 from the $2^{k-1}-1$ distinct dichotomizations f of $\{1,\ldots,k\}$. The number $2^{k-1}-1$ is arrived at by taking the number of all subsets, subtracting 2 improper subsets, and dividing by 2 because one chooses only one of W and $\{1,2,\ldots,k\}-W$. The goal is to determine whether $\mathcal{D}$ is contextual. If it is, then any canonical system that includes $\mathcal{D}$ as its subsystem (i.e., represents an original system with $\mathcal{R}$ as part of its connection) is contextual.

The two original rvs have distributions

$$\Pr\left[R_1^1=i\right]=p_i \text{ and } \Pr\left[R_1^2=i\right]=q_i, \quad i=1,2,\ldots,k. \tag{6.4.1}$$

A state (or value) of a bunch in the system $\mathcal{D}$ is a vector of $2^{k-1}-1$ zeroes and ones. However, the support of each of the bunches in system $\mathcal{D}$ consists of at most k

corresponding states, and we can enumerate them by any k symbols, say, $1, 2, \ldots, k$, as in the original variable:

$$\Pr\left[D^1 = i\right] = p_i,\ \Pr\left[D^2 = i\right] = q_i \quad , i = 1, 2, \ldots, k, \tag{6.4.2}$$

As a result, $\mathcal{D} = \{D^1, D^2\}$ has k^2 possible states that we can denote ij, with $i, j \in \{1, 2, \ldots, k\}$. A coupling $S = \left(S_q^1, S_q^2\right)$ of $\mathcal{D}$ assigns probabilities

$$r_{ij} = \Pr\left[S_q^1 = i, S_q^2 = j\right], \quad i, j \in \{1, \ldots, k\}, \tag{6.4.3}$$

to these k^2 states so that they satisfy $2k$ linear constraints imposed by (6.4.1),

$$\sum_{j=1}^{k} r_{ij} = p_i \text{ and } \sum_{i=1}^{k} r_{ij} = q_j, \quad i, j \in \{1, \ldots, k\}. \tag{6.4.4}$$

If S is maximally connected, then it should also satisfy $2^{k-1} - 1$ linear constraints imposed by the maximal couplings of the corresponding connections. Specifically, if $W = \{i_1, \ldots, i_m\} \subset \{1, \ldots, k\}$, then the maximal coupling (S_W^1, S_W^2) of (D_W^1, D_W^2) is distributed as

$$\left.\begin{array}{rl} & \Pr\left[S_W^1 = 1\right] = \Pr\left[D_W^1 = 1\right] = p_{i_1} + p_{i_2} + \ldots + p_{i_m} \\ & \Pr\left[S_W^2 = 1\right] = \Pr\left[D_W^2 = 1\right] = q_{i_1} + q_{i_2} + \ldots + q_{i_m} \\ \text{and} & \Pr\left[S_W^1 = S_W^2 = 1\right] = \min\left(p_{i_1} + p_{i_2} + \ldots + p_{i_m}, q_{i_1} + q_{i_2} + \ldots + q_{i_m}\right) \end{array}\right\}. \tag{6.4.5}$$

Let us use the term m-split to designate any split D_W with an m-element set W ($m \leq k/2$). Thus, D_W with $W = \{i\}$ is a 1-split, with $W = \{i, j\}$ it is a 2-split, and the higher-order splits appear beginning with $k > 5$. Theorem 6.4.3 and its corollaries below show that in determining whether the system $\mathcal{D}$ is contextual, one needs to consider only the 1-splits and 2-splits. Let us use the term *1–2 system* for this subsystem of $\mathcal{D}$. An overall coupling S of $\mathcal{D}$ contains as its part a maximally connected coupling of the 1–2 system if and only if the probabilities r_{ij} in (6.4.3) satisfy (6.4.5) for $m = 1$ and $m = 2$:

$$r_{ii} = \min\left(p_i, q_i\right), \quad i \in \{1, \ldots, k\} \tag{6.4.6}$$

and

$$r_{ii} + r_{ij} + r_{ji} + r_{jj} = \min\left(p_i + p_j, q_i + q_j\right), \quad i, j \in \{1, \ldots, k\}, \quad i < j. \tag{6.4.7}$$

That is, a maximally connected coupling of the 1–2 system is described by the $3k+\binom{k}{2}$ linear equations (6.4.4), (6.4.6) and (6.4.7). We have therefore the following necessary condition for non-contextuality of $\mathcal{D}$.

Theorem 6.4.1. *If the system $\mathcal{D}$ is non-contextual, then the $3k+\binom{k}{2}$ linear equations (6.4.4), (6.4.6) and (6.4.7) are satisfied.*

Remark 6.4.2. Note that $3k + \binom{k}{2} < k^2$ for $k > 5$. (For completeness only, theorem 6.7.1 in electronic supplementary material, file S, shows that the rank of this system of equations is $2k - 1 + \binom{k}{2}$.)

Theorem 6.4.3. *In a maximally connected coupling S of $\mathcal{D}$ with $k > 5$, the distributions of the 1-splits and 2-splits uniquely determine the probabilities of all higher-order splits. Specifically, for any $2 < m \leq k/2$, and any $W = \{i_1, \ldots, i_m\} \subset \{1, \ldots, k\}$, the probability that the corresponding m-split equals 1 is*

$$\begin{aligned}&\min\left(p_{i_1} + p_{i_2} + \ldots + p_{i_m}, q_{i_1} + q_{i_2} + \ldots + q_{i_m}\right) = \textstyle\sum_{j=1}^{m} \min\left(p_{i_j}, q_{i_j}\right)\\ &+ \textstyle\sum_{j=1}^{m-1} \sum_{j'=j+1}^{m} \left[\min\left(p_{i_j} + p_{i_{j'}}, q_{i_j} + q_{i_{j'}}\right) - \min\left(p_{i_j}, q_{i_j}\right) - \min\left(p_{i_{j'}}, q_{i_{j'}}\right)\right].\end{aligned} \tag{6.4.8}$$

It is easy to find numerical examples of the distributions of R_1^1 and R_1^2 for which (6.4.8) is violated (see example 6.7.2 in electronic supplementary material, file S). As shown below, however, (6.4.8) cannot be violated if a maximally connected coupling for the 1–2 system exists. It follows from the fact that the statement of theorem 6.4.1 can be reversed: (6.4.4), (6.4.6) and (6.4.7) imply that $\mathcal{D}$ is non-contextual. We establish this fact by first characterizing the distributions of R_1^1 and R_1^2 for a non-contextual 1–2 system (theorem 6.4.4 with corollary 6.4.5), and then showing that (6.4.8) always holds for such distributions (theorem 6.4.6).

Theorem 6.4.4. *A maximally connected coupling for a 1–2 system is unique if it exists. In this coupling, the only pairs of ij in (6.4.3) that may have non-zero probabilities assigned to them are the diagonal states $\{11, 22, \ldots, kk\}$ and either the states $\{i1, i2, \ldots, ik\}$ for a single fixed i or the states $\{1j, 2j, \ldots, kj\}$ for a single fixed j $(i, j = 1, \ldots, k)$.*

Assuming, with no loss of generality, that the single fixed i or the single fixed j in the formulation above is 2, the theorem says that the non-zero probabilities assigned to the states of the maximally connected coupling (shown below for $k = 4$) could only occupy the cells marked with asterisks:

$$
\begin{array}{c|c|c|c|c|}
 & 1 & 2 & 3 & 4 \\
\hline
1 & * & * & 0 & 0 \\
\hline
2 & 0 & * & 0 & 0 \\
\hline
3 & 0 & * & * & 0 \\
\hline
4 & 0 & * & 0 & * \\
\hline
\end{array}
\quad \text{or} \quad
\begin{array}{c|c|c|c|c|}
 & 1 & 2 & 3 & 4 \\
\hline
1 & * & 0 & 0 & 0 \\
\hline
2 & * & * & * & * \\
\hline
3 & 0 & 0 & * & 0 \\
\hline
4 & 0 & 0 & 0 & * \\
\hline
\end{array}.
$$

Corollary 6.4.5. *The 1–2 system for the original rvs R_1^1, R_1^2 has a maximally connected coupling if and only if either $p_i > q_i$ for no more than one i (this single possible i being the single fixed i in the formulation of the theorem), or $p_j < q_j$ for no more* *than one j (this single possible j being the single fixed j in the formulation of the theorem), $i, j \in \{1, \ldots, k\}$.*

The relationship between $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_k)$ described in this corollary is some form of stochastic dominance for categorical rvs, but it does not seem to have been previously identified. We propose to say that R_1^1 *nominally dominates* R_1^2 if $p_i < q_i$ for no more than one value of $i = 1, \ldots, k$ (i.e., $p_i \geq q_i$ for at least $k-1$ of them). Two categorical rvs nominally dominate each other if and only if either they are identically distributed or $k = 2$.[a] Using this notion, and combining corollary 6.4.5 with theorems 6.4.1 and 6.4.4, we get the main result of this section.

Theorem 6.4.6. *The system $\mathcal{D}$ is non-contextual if and only if its 1–2 subsystem is non-contextual, i.e., if and only if one of the* R_1^1 and R_1^2 *nominally dominates the other.*

6.5 Concluding remarks

Contextuality analysis of an empirical situation involves the following sequence of steps:

$$\begin{array}{ccccccc} \text{empirical} & & \text{initial} & & \text{expanded} & & \text{canonical/split} \\ & \longrightarrow & & \longrightarrow & & \longrightarrow & \\ \text{measurements} & & \text{system of rvs} & & \text{system of rvs} & & \text{representation} \end{array}$$

In the initial system, measurements are represented by rvs each of which generally has multiple values. Expansion means adding to the system new conteNts with corresponding connections (conteNt-sharing rvs) computed as functions of the existing connections. In a canonical representation of the system all rvs are binary, and the

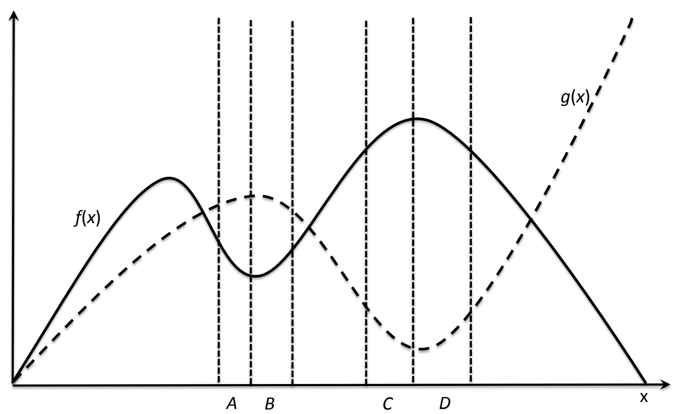
[a](Erratum note added in the dissertation.) This statement is incorrect. A correct description would be "Two categorical rvs nominally dominate each other if they are identically distributed or $k = 2, 3$". Noting that two rvs may still nominally dominate each other when they are not identically distributed and $k \geq 4$.

connections are coupled multimaximally, meaning essentially that one deals with their elements pair-wise. The issue of contextuality is reduced to that of compatibility of the unique couplings for pairs of conteNt-sharing rvs with the known distributions of the conteXt-sharing bunches of rvs. Coupling the connections multimaximally ensures that a non-contextual system has all its subsystems non-contextual too.

The canonical system of rvs is uniquely determined by the expanded system, but the latter is inherently non-unique, it depends on what aspects of the empirical situation one wishes to include in the system. Thus, it is one's choice rather than a general rule whether one considers a multi-valued measurement as representable by *all* or only *some* of its possible coarsenings. If one chooses all coarsenings, the split/canonical representation involves all dichotomizations, and then theorem 6.4.6 says that the canonical system is non-contextual only if, for any pair of rvs $R_q^c, R_q^{c'}$ in the expanded system, one of them, say R_q^c, 'nominally dominates' the other. This domination means that $\Pr\left[R_q^c = x\right] < \Pr\left[R_q^{c'} = x\right]$ holds for no more than one value x of these rvs: a stringent necessary condition for non-contextuality, likely to be violated in many empirical systems.

This is of special interest for contextuality studies outside quantum physics. Historically, the search for non-quantum contextual systems was motivated by the possibility of applying quantum-theoretic formalisms in such fields as biology [24], psychology [9, 25, 26], economics [26, 27] and political science [28]. In CbD, the notion of contextuality is not tied to quantum formalisms in any special way. The possibility of non-quantum contextual systems here is motivated by treating contextuality as an abstract probabilistic issue: there are no a priori reasons why a system of rvs describing, say, human behaviour could not be contextual if it is qualitatively (i.e. up to specific probability values) the same as a contextual one describing particle spins. Nevertheless, all known to us systems with dichotomous responses investigated for potential contextuality (with the exception of one, very recent experiment) have been found to be non-contextual [18, 19, 29]. The use of canonical representations with dichotomizations of multiple-choice responses offers new possibilities.

In some cases, however, the use of all possible dichotomizations is not justifiable. Notably, if the values of an rv are linearly ordered, $x_1 < x_2 < \ldots, x_N$, it may be natural to only allow dichotomizations f with $f^{-1}(1)$ containing several successive values, $\{x_l, x_{l+1}, \ldots, x_L\}$, for some $l, L \in \{1, \ldots, N\}$. An even stronger restriction would be to only allow 'cuts', with $f^{-1}(1) = \{x_l, x_{l+1}, \ldots, x_N\}$ or $\{x_1, x_2, \ldots, x_{l-1}\}$.



Stronger restrictions on possible dichotomizations translate into stronger restrictions on the pairs $R_q^c, R_{q'}^c$ whose canonical representation is contextual. This fact is especially important if one considers expanding CbD beyond categorical rvs. Thus, it is easy to see that if one considers all possible dichotomizations of two conteNt-sharing rvs with continuous densities on the set of real numbers, then the system will be contextual whenever the two distributions are not identical. Let the densities of these rvs be $f(x)$ and $g(x)$ shown in the graphic above. If the set of all splits of these rvs forms a non-contextual system, then any discretization of these rvs should satisfy corollary 6.4.5 to theorem 6.4.4. That is, for any $k > 2$ and any partition $H_1, \ldots, H_k$ of the set of reals into intervals, we should have either

$$\begin{array}{c} \int_{H_i} f(x)\,dx < \int_{H_i} g(x)\,dx \text{ for no more than one of } i = 1, \ldots, k, \\ \text{or} \\ \int_{H_i} f(x)\,dx > \int_{H_i} g(x)\,dx \text{ for no more than one of } i = 1, \ldots, k. \end{array} \quad (6.5.1)$$

This is, however, impossible unless $f(x) = g(x)$. If they are different, then f exceeds g on some interval, and g exceeds f on some other interval. If we take any two subintervals within each of these intervals (in the graphic they are denoted by A, B and C, D), any partition $H_1, \ldots, H_k$ that includes A, B, C, D will violate (6.5.1). The development of the theory of canonical representations with variously restricted sets of splits is a task for future work.

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6.7 Supplementary Text to "Contextuality in Canonical Systems of Random Variables" by Ehtibar N. Dzhafarov, Víctor H. Cervantes, and Janne V. Kujala

Theorem 6.7.1 (Section 6.4, Remark 6.4.2)**.** *The rank of the system of linear equations (6.4.4)-(6.4.6)-(6.4.7) is* $2k-1+\binom{k}{2}$.

Proof of Theorem 6.7.1. This system of linear equations can be written as

$$\mathbf{M} \times \mathbf{X} = \mathbf{P},$$

where

$$\mathbf{P}^T = \begin{pmatrix} \overbrace{p_1, \ldots, p_k}^{k}, \overbrace{q_1, \ldots, q_k}^{k}, \overbrace{\min\left(p_1, q_1\right), \ldots, \min\left(p_k, q_k\right)}^{k}, \\ \overbrace{\min\left(p_1 + p_2, q_1 + q_2\right), \ldots, \min\left(p_{k-1} + p_k, q_{k-1} + q_k\right)}^{\binom{k}{2}} \end{pmatrix},$$

$$\mathbf{X}^T = \left\{x_{ij} : i, j \in \{1, \ldots, k\}\right\},$$

and $\mathbf{M}$ is a Boolean matrix. The $\left(k + k + k + \binom{k}{2}\right)$ rows of matrix $\mathbf{M}$ correspond to the elements of $\mathbf{P}$ and can be labeled as

$$\left(\overbrace{\mathbf{r}_{1\cdot}, \ldots, \mathbf{r}_{k\cdot}}^{k}, \overbrace{\mathbf{r}_{\cdot 1}, \ldots, \mathbf{r}_{\cdot k}}^{k}, \overbrace{\mathbf{r}_{11}, \ldots, \mathbf{r}_{kk}}^{k}, \overbrace{\mathbf{r}_{12}, \ldots, \mathbf{r}_{k-1,k}}^{\binom{k}{2}} \right),$$

whereas the k^2 columns of $\mathbf{M}$ correspond to the elements of $\mathbf{X}$ and can be labeled as

$$\left\{\mathbf{c}_{ij} : i, j \in \{1, \ldots, k\}\right\}.$$

Thus, if $k = 4$, the matrix $\mathbf{M}$ is

$\mathbf{r} \backslash \mathbf{c}$	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
1·	1	1	1	1												
2·					1	1	1	1								
3·									1	1	1	1				
4·													1	1	1	1
·1	1				1				1				1			
·2		1				1				1				1		
·3			1				1				1				1	
·4				1				1				1				1
11	1															
22						1										
33											1					
44																1
12	1	1			1	1										
13	1		1						1		1					
14	1			1									1			1
23						1	1			1	1					
24						1		1						1		1
34											1	1			1	1

We will continue to illustrate the steps of the proof using this matrix. We begin by adding to $\mathbf{M}$ the row $\mathbf{r}_{all}$ with all cells equal to 1, and denote the new matrix $\mathbf{M}'$.

$\mathbf{r}$ \ $\mathbf{c}$	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
1·	1	1	1	1												
2·					1	1	1	1								
3·									1	1	1	1				
4·													1	1	1	1
·1	1				1				1				1			
·2		1				1				1				1		
·3			1				1				1				1	
·4				1				1				1				1
11	1															
22						1										
33											1					
44																1
12	1	1			1	1										
13	1		1						1		1					
14	1			1									1			1
23						1	1			1	1					
24						1		1						1		1
34											1	1			1	1
all	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

This does not change the rank of the matrix since $\mathbf{r}_{all}$ is the sum of all $\mathbf{r}_{\cdot i}$. Then we observe that the rows $\mathbf{r}_{k\cdot}$, $\mathbf{r}_{\cdot k}$, and all $\mathbf{r}_{ik}$ with $i < k$ can be deleted as they are linear combinations of the remaining rows of $\mathbf{M}'$. Indeed, it can be checked directly that

$$\mathbf{r}_{k\cdot} = \mathbf{r}_{all} - \sum_{i=1}^{k-1} \mathbf{r}_{i\cdot},$$

$$\mathbf{r}_{\cdot k} = \mathbf{r}_{all} - \sum_{i=1}^{k-1} \mathbf{r}_{\cdot i},$$

$$\left(\mathbf{r}_{ik} - \mathbf{r}_{ii} - \mathbf{r}_{kk}\right) = \left(\mathbf{r}_{i\cdot} - \mathbf{r}_{ii}\right) + \left(\mathbf{r}_{\cdot i} - \mathbf{r}_{ii}\right) - \sum_{l<i} \left(\mathbf{r}_{li} - \mathbf{r}_{ll} - \mathbf{r}_{ii}\right) - \sum_{l>i}^{l<k} \left(\mathbf{r}_{il} - \mathbf{r}_{ii} - \mathbf{r}_{ll}\right),$$

for all $i < k$. Moreover, one can also delete $\mathbf{r}_{kk}$, because

$$\sum_{i<j<k} \left(\mathbf{r}_{ij} - \mathbf{r}_{ii} - \mathbf{r}_{jj}\right) + \sum_{i<k} \left(\mathbf{r}_{ik} - \mathbf{r}_{ii} - \mathbf{r}_{kk}\right) + \sum_{i<k} \mathbf{r}_{ii} + \mathbf{r}_{kk} = \mathbf{r}_{all}.$$

Let the resulting matrix be $\mathbf{M}''$:

$\mathbf{r}$ \ $\mathbf{c}$	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
1·	1	1	1	1												
2·					1	1	1	1								
3·									1	1	1	1				
·1	1				1				1				1			
·2		1				1				1				1		
·3			1				1				1				1	
11	1															
22						1										
33											1					
12	1	1			1	1										
13	1		1						1		1					
23						1	1			1	1					
all	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

This matrix contains

$$\overbrace{3k + \binom{k}{2}}^{initial} - \underbrace{3}_{\mathbf{r}_{k\cdot},\mathbf{r}_{\cdot k},\mathbf{r}_{kk}} - \overbrace{(k-1)}^{all\ \mathbf{r}_{ik}, i<k} + \underbrace{1}_{\mathbf{r}_{all}} = 2k - 1 + \binom{k}{2}$$

rows. We prove that this matrix is of full row rank. Consider equation

$$\sum_{all\ \mathbf{r}\text{ in }\mathbf{M}''} \alpha_{\mathbf{r}} \mathbf{r} = 0.$$

We use the following principle: if a row $\mathbf{r}$ intersects a columns whose only nonzero entry is in the row $\mathbf{r}$, then $\alpha_{\mathbf{r}} = 0$, and we can delete the row $\mathbf{r}$ from the matrix, decreasing the row rank of the matrix by 1. The following statements can be directly verified.

$\mathbf{r}_{all}$ can be deleted because column $\mathbf{c}_{kk}$ has its only 1 in $\mathbf{r}_{all}$.

$\mathbf{r}$ \ $\mathbf{c}$	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
1·	1	1	1	1												
2·					1	1	1	1								
3·									1	1	1	1				
·1	1				1				1				1			
·2		1				1				1				1		
·3			1				1				1				1	
11	1															
22						1										
33											1					
12	1	1			1	1										
13	1		1						1		1					
23						1	1			1	1					

Then each of $\mathbf{r}_{\cdot i}$ can be deleted because the column $\mathbf{c}_{ki}$ has its only 1 in $\mathbf{r}_{\cdot i}$ ($i = 1, \ldots, k-1$).

$\mathbf{r}$ \ $\mathbf{c}$	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
1·	1	1	1	1												
2·					1	1	1	1								
3·									1	1	1	1				
11	1															
22						1										
33											1					
12	1	1			1	1										
13	1		1						1		1					
23						1	1			1	1					

Then each of $\mathbf{r}_{i\cdot}$ can be deleted because the column $\mathbf{c}_{ik}$ has its only 1 in $\mathbf{r}_{i\cdot}$ $(i = 1, \ldots, k-1)$.

$\mathbf{r}$ \ $\mathbf{c}$	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
11	1															
22						1										
33											1					
12	1	1			1	1										
13	1		1						1		1					
23						1	1			1	1					

Then each of $\mathbf{r}_{ij}$ can be deleted because the column $\mathbf{c}_{ji}$ has its only 1 in $\mathbf{r}_{ij}$ $(i, j \in \{1, \ldots, k-1\}, i < j)$.

$\mathbf{r}$ \ $\mathbf{c}$	11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44
11	1															
22						1										
33											1					

This leaves only $\mathbf{r}_{11}, \ldots, \mathbf{r}_{(k-1)(k-1)}$ that are obviously linearly independent. $\square$

Theorem (Section 6.4, Theorem 6.4.3)**.** *In a maximally-connected coupling S of $\mathcal{D}$ with $k > 5$, the distributions of the 1-splits and 2-splits uniquely determine the probabilities of all higher-order splits. Specifically, for any $2 < m \leq k/2$, and any $W = \{i_1, \ldots, i_m\} \subset \{1, \ldots, k\}$, the probability that the corresponding m-split equals 1 is*

$$\begin{aligned} &\min\left(p_{i_1} + p_{i_2} + \ldots + p_{i_m}, q_{i_1} + q_{i_2} + \ldots + q_{i_m}\right) = \textstyle\sum_{j=1}^{m} \min\left(p_{i_j}, q_{i_j}\right) \\ &+ \textstyle\sum_{j=1}^{m-1} \sum_{j'=j+1}^{m} \left[\min\left(p_{i_j} + p_{i_{j'}}, q_{i_j} + q_{i_{j'}}\right) - \min\left(p_{i_j}, q_{i_j}\right) - \min\left(p_{i_{j'}}, q_{i_{j'}}\right)\right]. \end{aligned} \tag{6.7.1}$$

Proof of Theorem 6.4.3. From (6.4.6) and (6.4.7),

$$\begin{array}{rcl} r_{12}+r_{21} & = & \min(p_1+p_2, q_1+q_2) - \min(p_1,q_1) - \min(p_2,q_2) \\ \vdots & \vdots & \vdots \\ r_{ij}+r_{ji} & = & \min(p_i+p_j, q_i+q_j) - \min(p_i,q_i) - \min(p_j,q_j) \qquad (i<j). \\ \vdots & \vdots & \vdots \\ r_{(k-1)k}+r_{k(k-1)} & = & \min(p_{k-1}+p_k, q_{k-1}+q_k) - \min(p_{k-1},q_{k-1}) - \min(p_k,q_k) \end{array}$$

Consider an m-split with $2 < m \leq k/2$, and assume without loss of generality that $W = (1, \dots, m)$. We have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} r_{ij} = \min(p_1+\ldots+p_m, q_1+\ldots+q_m)\,. \tag{6.7.2}$$

The left-hand-side sum can be presented as

$$\sum_{i=1}^{m} r_{ii} + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(r_{ij}+r_{ji})$$

$$= \sum_{i=1}^{m}\min(p_i,q_i) + \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\left[\min(p_i+p_j, q_i+q_j) - \min(p_i,q_i) - \min(p_j,q_j)\right],$$

whence we get (6.4.8). □

Example 6.7.2. (showing that the relation (6.4.8) may be violated, see Section 6.4.) If

$R_1^1=$	1	2	3	4	0	0
prob. mass $p=$	.6	.1	.1	.2	0	0

,

$R_1^2=$	1	2	3	4	0	0
prob. mass $q=$	.2	.3	.4	.1	0	0

,

then

$$\overbrace{\min\left(p_1+p_2+p_3, q_1+q_2+q_3\right)}^{.8}$$

$$\neq \left.\begin{array}{rl} \min\left(p_1,q_1\right) & .2 \\ +\min\left(p_2,q_2\right) & .1 \\ +\min\left(p_3,q_3\right) & .1 \\ +\min\left(p_1+p_2,q_1+q_2\right)-\min\left(p_1,q_1\right)-\min\left(p_2,q_2\right) & .5-.2-.1 \\ +\min\left(p_1+p_3,q_1+q_3\right)-\min\left(p_1,q_1\right)-\min\left(p_3,q_3\right) & .6-.2-.1 \\ +\min\left(p_2+p_3,q_2+q_3\right)-\min\left(p_2,q_2\right)-\min\left(p_3,q_3\right) & .2-.1-.1 \end{array}\right\} = .5$$

Theorem (Section 6.4, Theorem 6.4.4)**.** *A maximally-connected coupling for a 1-2 system is unique if it exists. In this coupling, the only pairs of ij in (6.4.3) that may have nonzero probabilities assigned to them are the diagonal states $\{11, 22, \ldots, kk\}$ and either the states $\{i1, i2, \ldots, ik\}$ for a single fixed i or the states $\{1j, 2j, \ldots, kj\}$ for a single fixed j $(i, j = 1, \ldots, k)$.*

Proof of Theorem 6.4.4. (The matrices illustrating the proof are shown for $k > 6$ but the theorem is valid for all $k > 1$.) If the only nonzero entries in the matrix are in the main diagonal, the theorem is trivially true. Assume therefore that $r_{ij} > 0$ for some $i \neq j$. Without loss of generality, we can assume that $r_{12} > 0$ and $p_1 + p_2 \leq q_1 + q_2$. Indeed, if some $r_{ij} > 0$, we can always rename the values so that $i = 1$ and $j = 2$; and if $p_1 + p_2 > q_1 + q_2$, then we can simply rename all ps into qs and vice versa. In the following we will use the expression "r_{ij} is p-minimized" if $p_i + p_j \leq q_i + q_j$, and "r_{ij} is q-minimized" if $p_i + p_j \geq q_i + q_j$ (in both cases, $i \neq j$).

We have (the empty cells are those whose value is to be determined later)

	1	2	3	4	5	6	...	
1	r_{11}	$r_{12} > 0$						p_1
2	r_{21}	r_{22}						p_2
3			r_{33}					
4				r_{44}				
5					r_{55}			
6						r_{66}		
$\vdots$							$\vdots\vdots\vdots$	$\vdots$
	q_1	q_2					...	

..

From (6.4.6)-(6.4.7), $r_{11} + r_{12} + r_{21} + r_{22} = \min\{p_1 + p_2 q_1 + q_2\}$, and since r_{12} is p-minimized, $r_{11} + r_{12} + r_{21} + r_{22} = p_1 + p_2$. This means

	1	2	3	4	5	6	...	
1	r_{11}	$r_{12} > 0$	0	0	0	0	**0**	$p_1 = r_{11} + r_{12}$
2	r_{21}	r_{22}	0	0	0	0	**0**	$p_2 = r_{21} + r_{22}$
3			r_{33}					
4				r_{44}				
5					r_{55}			
6						r_{66}		
$\vdots$							$\vdots\vdots\vdots$	$\vdots$
	$q_1 \geq r_{11} + r_{21}$	$q_2 \geq r_{12} + r_{22}$					...	

.

We also should have

	1	2	3	4	5	6	$\dots$	
1	r_{11}	$r_{12} > 0$	0	0	0	0	$\mathbf{0}$	$p_1 = r_{11} + r_{12}$
2	0	r_{22}	0	0	0	0	$\mathbf{0}$	$p_2 = r_{22}$
3	0		r_{33}					
4	0			r_{44}				
5	0				r_{55}			
6	0					r_{66}		
$\vdots$	$\mathbf{0}$						$\vdots\vdots\vdots$	$\vdots$
	$q_1 = r_{11}$	$q_2 \geq r_{12} + r_{22}$					$\dots$	

because $r_{11} = \min\{p_1, q_1\}$ and $r_{11} < p_1$.

Generalizing, we have established the following rules:

(R1) If $r_{ij} > 0$ and it is p-minimized, then all non-diagonal elements in the rows i and j are zero except for r_{ij}, and all non-diagonal elements in the column i are zero.

(R2) (By symmetry, on exchanging ps and qs) If $r_{ij} > 0$ and it is q-minimized, then all non-diagonal elements in the columns i and j are zero except for r_{ij}, and all non-diagonal elements in the row j are zero.

Returning to our special arrangement of the rows and columns, let us prove now that all r_{1j} with $j > 2$ are q-minimized. Assume the contrary, and with no loss of generality, let $r_{15} = 0$ be p-minimized. This would mean that

$$r_{15} + r_{51} = p_1 + p_5 - r_{11} - r_{55} = r_{12} + p_5 - r_{55} = 0,$$

which could only be true if $r_{12} = 0$, which it is not.

	1	2	3	4	5	6	$\dots$	
1	r_{11}	$r_{12} > 0$	0 $_{q-min}$	0 $_{q-min}$	0 $_{q-min}$	0 $_{q-min}$	**0** $_{q-min}$	$p_1 = r_{11} + r_{12}$
2	0	r_{22}	0	0	0	0	**0**	$p_2 = r_{22}$
3	0		r_{33}					
4	0			r_{44}				
5	0				r_{55}			p_5
6	0					r_{66}		
$\vdots$	**0**						$\vdots\vdots\vdots$	$\vdots$
	$q_1 = r_{11}$	$q_2 \geq r_{12} + r_{22}$					$\dots$	

Generalizing, we have established two additional rules:

(R3) If r_{ij} and $r_{ij'}$ are both p-minimized (for pairwise distinct i, j, j'), then they are both zero (because if one of them is not, say $r_{ij} > 0$, then $r_{ij'} = 0$ and it must be q-minimized).

(R4) (By symmetry, on exchanging ps and qs) If r_{ij} and $r_{i'j}$ are both q-minimized (for pairwise distinct i, i', j), then they are both zero.

Returning to our special arrangement of the rows and columns, it follows that nowhere in the matrix can we have $r_{ij} > 0$ ($i > 2$) which is q-minimized. Indeed, if $j > 2$, then this would have contradicted R4 (because the zeros in the first row are all q-minimized), and if $j = 2$, it would have contradicted R2 (because $r_{12} > 0$).

Let us prove now that if $j > 2$ and $i > 2$ and $i \neq j$, then there is no $r_{ij} > 0$ that is p-minimized. Assume the contrary: $r_{ij} > 0$ and q-minimized, and consider r_{2i}, r_{i2}. With no loss of generality, let (i, j)=$(4, 6)$. In accordance with R1, we fill in the 4th and the 6th rows with zeros, and we fill in the 4th column with zeros too:

	1	2	3	4	5	6	$\dots$	
1	r_{11}	$r_{12} > 0$	0	0	0	0	$\mathbf{0}$	$p_1 = r_{11} + r_{12}$
2	0	r_{22}	0	0	0	0	$\mathbf{0}$	$p_2 = r_{22}$
3	0		r_{33}	0				
4	0	0	0	r_{44}	0	$r_{46} > 0$	$\mathbf{0}$	$p_4 = r_{44} + r_{46}$
5	0			0	r_{55}			
6	0	0	0	$r_{64} = 0$	0	r_{66}	$\mathbf{0}$	$p_6 = r_{66}$
$\vdots$	$\mathbf{0}$			$\mathbf{0}$			$\vdots\vdots\vdots$	$\vdots$
	$q_1 = r_{11}$	$q_2 \geq r_{12} + r_{22}$		$q_4 = r_{44}$		$q_6 \geq r_{46} + r_{66}$	$\dots$	

Then r_{24}, r_{42} are both zero, whence $\min(p_2 + p_4, q_2 + q_4)$ must equal $r_{22} + r_{44}$ to be a maximal coupling. But

$$\min(p_2 + p_4, q_2 + q_4) = \min(r_{22} + r_{44} + r_{46}, r_{12} + r_{22} + r_{44} + x) > r_{22} + r_{44},$$

since both r_{12} and r_{46} are positive, a contradiction.

We come to the conclusion that the only positive non-diagonal elements in the matrix can be in the column 2 (and they are all p-minimized).

	1	2	3	4	5	6	$\dots$	
1	r_{11}	$r_{12} > 0$	0	0	0	0	$\mathbf{0}$	$p_1 = r_{11} + r_{12}$
2	0	r_{22}	0	0	0	0	$\mathbf{0}$	$p_2 = r_{22}$
3	0	$r_{32} \geq 0$	r_{33}	0	0	0	$\mathbf{0}$	$p_3 = r_{32} + r_{33}$
4	0	$r_{42} \geq 0$	0	r_{44}	0	0	$\mathbf{0}$	$p_4 = r_{42} + r_{44}$
5	0	$r_{52} \geq 0$	0	0	r_{55}	0	$\mathbf{0}$	$p_5 = r_{52} + r_{55}$
6	0	$r_{62} \geq 0$	0	0	0	r_{66}	$\mathbf{0}$	$p_6 = r_{62} + r_{66}$
$\vdots$	$\mathbf{0}$	$\vdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\vdots\vdots\vdots$	$\vdots$
	$q_1 = r_{11}$	$q_2 \geq r_{12} + r_{22}$	$q_3 = r_{33}$	$q_4 = r_{44}$	$q_5 = r_{55}$	$q_6 = r_{66}$	$\dots$	

Generalizing, let $r_{ij} > 0$ and $i \neq j$. Then, if r_{ij} is p-minimized, all non-diagonal elements of the matrix outside column j are zero (and the non-diagonal elements in the jth column are p-minimized); if r_{ij} is q-minimized, then all non-diagonal elements of the matrix outside row i are zero (and the non-diagonal elements in the ith row are q-minimized).

It is easy to check that such a construction is always internally consistent. $\square$

Corollary (Section 6.4, Corollary 6.4.5)**.** *The 1-2 system for the original rvs* R_1^1, R_1^2 *has a maximally-connected coupling if and only if either* $p_i > q_i$ *for no more than one* i *(this single possible* i *being the single fixed* i *in the formulation of the theorem), or* $p_j < q_j$ *for no more than one* j *(this single possible* j *being the single fixed* j *in the formulation of the theorem),* $i, j \in \{1, \ldots, k\}$*.*

Proof of Corollary 6.4.5. The "only if" part is obvious. To demonstrate the "if" part, consider (without loss of generality) the arrangement

	1	2	3	4	5	6	$\ldots$	
1							$\ldots$	$p_1 \geq q_1$
2							$\ldots$	p_2
3							$\ldots$	$p_3 \geq q_3$
4							$\ldots$	$p_4 \geq q_4$
5							$\ldots$	$p_5 \geq q_5$
6							$\ldots$	$p_6 \geq q_6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots\vdots\vdots$	$\vdots$
	q_1	$q_2 \geq p_2$	q_3	q_4	q_5	q_6	$\ldots$	

and fill it in as

	1	2	3	4	5	6	$\dots$	
1	q_1	$p_1 - q_1$	0	0	0	0	$\mathbf{0}$	$p_1 \geq q_1$
2	0	p_2	0	0	0	0	$\mathbf{0}$	p_2
3	0	$p_3 - q_3$	q_3	0	0	0	$\mathbf{0}$	$p_3 \geq q_3$
4	0	$p_4 - q_4$	0	q_4	0	0	$\mathbf{0}$	$p_4 \geq q_4$
5	0	$p_5 - q_5$	0	0	q_5	0	$\mathbf{0}$	$p_5 \geq q_5$
6	0	$p_6 - q_6$	0	0	0	q_6	$\mathbf{0}$	$p_6 \geq q_6$
$\vdots$	$\mathbf{0}$	$\vdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\vdots\vdots\vdots$	$\vdots$
	q_1	$q_2 \geq p_2$	q_3	q_4	q_5	q_6	$\dots$	

with the empty cells filled in with zeros. Check that (a) all rows sum to the marginals; (b) the second column sums to

$$\sum_{i=1}^{k} p_i - \left(\sum_{i=1}^{k} q_i - q_2 \right) = q_2;$$

(c) the rest of the columns sum to the marginals; (d) all r_{ii} are $\min(p_i, q_i)$; and (e) for all pairs r_{ij} $(i \neq j)$ the sums $r_{ii} + r_{ij} + r_{ji} + r_{jj}$ equal $\min(p_i + p_j, q_i + q_j)$. The latter is proved by considering first all $j \neq 2$, where it is obvious, and then $j = 2$ where the computation is, for $i \neq 2$,

$$r_{ii} + r_{i2} + r_{2i} + r_{22} = q_i + (p_i - q_i) + 0 + p_2 = p_i + p_2,$$

as it should be because the values in the second column are to be p-minimized. $\square$

Theorem (Section 6.4, Theorem 6.4.6)**.** *The system $\mathcal{D}$ is noncontextual if and only if its 1-2 subsystem is noncontextual, i.e., if and only if one of the* R_1^1 and R_1^2 *nominally dominates the other.*

Proof of Theorem 6.4.6. The "only if" part is Theorem 6.4.1. All we need to proof the "if " part is to check that the relation (6.4.8) holds. Assume the arrangement is as in the previous corollary. Consider first any set $i_1, \ldots, i_m$ that does not include 2:

$$\min\left(p_{i_1} + p_{i_2} + \ldots + p_{i_m}, q_{i_1} + q_{i_2} + \ldots + q_{i_m}\right) = q_{i_1} + q_{i_2} + \ldots + q_{i_m},$$

$$\sum_{j=1}^{m} \min\left(p_{i_j}, q_{i_j}\right) = q_{i_1} + q_{i_2} + \ldots + q_{i_m},$$

$$\min\left(p_{i_j} + p_{i_{j'}}, q_{i_j} + q_{i_{j'}}\right) - \min\left(p_{i_j}, q_{i_j}\right) - \min\left(p_{i_{j'}}, q_{i_{j'}}\right) = 0.$$

So, (6.4.8) holds. If one of the indices (let it be i_1) is 2, then

$$q_2 + q_{i_2} + \ldots + q_{i_m} = \left(p_2 + \sum_{x \neq 2} (p_x - q_x)\right) + q_{i_2} + \ldots + q_{i_m} > p_2 + p_{i_2} + \ldots + p_{i_m},$$

so

$$\min\left(p_2 + p_{i_2} + \ldots + p_{i_m}, q_2 + q_{i_2} + \ldots + q_{i_m}\right) = p_2 + p_{i_2} + \ldots + p_{i_m}.$$

We also have

$$\sum_{j=1}^{m} \min\left(p_{i_j}, q_{i_j}\right) = p_2 + q_{i_2} + \ldots + q_{i_m},$$

and for any $j \neq 2, j' \neq 2$,

$$\min\left(p_{i_j} + p_{i_{j'}}, q_{i_j} + q_{i_{j'}}\right) - \min\left(p_{i_j}, q_{i_j}\right) - \min\left(p_{i_{j'}}, q_{i_{j'}}\right) = 0,$$

$$\min\left(p_2 + p_{i_j}, q_2 + q_{i_j}\right) - \min\left(p_2, q_2\right) - \min\left(p_{i_j}, q_{i_j}\right) = p_{i_j} - q_{i_j}.$$

Since index $i_1 = 2$ is paired with each of $i_2, \ldots, i_m$ only once, the right-hand side in (6.4.8) is

$$p_2 + q_{i_2} + (p_{i_2} - q_{i_2}) + \ldots + q_{i_m} + (p_{i_m} - q_{i_m}) = p_2 + p_{i_2} + \ldots + p_{i_m}.$$

$\square$

7. TRUE CONTEXTUALITY IN A PSYCHOPHYSICAL EXPERIMENT

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Abstract Recent crowdsourcing experiments have shown that true contextuality of the kind found in quantum mechanics can also be present in human behavior. In these experiments simple human choices were aggregated over large numbers of respondents, with each respondent dealing with a single context (set of questions asked). In this paper we present experimental evidence of contextuality in individual human behavior, in a psychophysical experiment with repeated presentations of visual stimuli in randomly varying conteXts (arrangements of stimuli). The analysis is based on the Contextuality-by-Default (CbD) theory whose relevant aspects are reviewed in the paper. CbD allows one to detect contextuality in the presence of direct influences, i.e., when responses to the same stimuli have different distributions in different contexts. The experiment presented is also the first one in which contextuality is demonstrated for responses that are not dichotomous, with five options to choose among. CbD requires that random variables representing such responses be dichotomized before they are subjected to contextuality analysis. A theorem says that a system consisting of all possible dichotomizations of responses has to be contextual if these responses violate a certain condition, called nominal dominance. In our experiment nominal dominance was violated in all data sets, with very high statistical reliability established by bootstrapping.

Keywords: Contextuality, Inconsistent connectedness, Nominal dominance, Psychophysics.

Contextuality (or lack thereof) is a characteristic of a *system of random variables.* A set of random variables forms a system if each random variable R_q^c in it is uniquely identified by its *content* q and its *context* c. The content q is that which the random variable measures or responds to, while the context c is a complex of recorded conditions under which this random variable is observed. As an example, the following set of random variables,

$$\begin{array}{|c|c|c|c||c}
\hline
R_1^1 & R_2^1 & & R_4^1 & c=1 \\
\hline
 & R_2^2 & R_3^2 & & c=2 \\
\hline
R_1^3 & R_2^3 & R_3^3 & & c=3 \\
\hline\hline
q=1 & q=2 & q=3 & q=4 & \text{system } \mathcal{E}
\end{array}, \tag{7.1}$$

forms a system with three contexts and four contents.

To prevent possible misreadings, we will follow the convention adopted in Dzhafarov and Kujala (2016a) and capitalize the distinguishing letters in the words "conteNt" and "conteXt."

The conteNts could be, e.g., four stimuli (say, questions or light flashes), and conteXts be defined by which two or three of them are presented in a single trial, say, in a fixed succession. Thus, in conteXt $c = 1$, three stimuli ($q = 1$, $q = 2$, and $q = 4$) are presented, and each of them is being responded to in accordance with some instructions. Depending on the arrangements, a response to a given stimulus can be given immediately after it is presented or after all three of them are presented — such experimental details are immaterial for contextuality analysis insofar as responses and stimuli are in a one-to-one correspondence. The responses in the conteXt $c = 1$ are the random variables R_1^1, R_2^1, R_4^1 shown in the first row of (7.1). They may be binary (e.g., Yes/No, or I saw it/I did not see it), or they can be multi-valued ones (e.g., each stimulus may have a name, and the task may be to identify which stimulus was shown). The difference between binary and more-than-binary responses plays a central role in the present paper.

Let us explain the intuition behind the notion of contextuality using (7.1). The random variables within a given conteXt are jointly distributed, and the marginal distribution of a given-conteNt variable may depend on the conteXt in which it is recorded. Thus, the distributions of R_2^2 and R_2^3 may be different, so by knowing the distribution one can guess in which of the two conteXts, $c = 2$ or $c = 3$, the conteNt $q = 2$ is being responded to. This means that the effect of a conteXt upon a distribution is *information-carrying*, i.e., it is a causal influence. We call such influences *direct*. The terminology used in physics for direct influences is "signaling," "disturbance," "invasiveness," etc. (Cereceda, 2000; Leggett & Garg, 1985). In psychology we usually speak of "violations of marginal selectivity" (Dzhafarov, 2003; Dzhafarov & Kujala, 2016b). If, e.g., the conteNts in (7.1) are questions, and in each conteXt they are posed in a succession, in the order of their values ($q = 1, 2, 3, 4$), then the response R_2^2 to $q = 2$ in conteXt $c = 2$ may very well differ in distribution from the response R_2^3 to the same $q = 2$ in conteXt $c = 3$, because in the later case the respondent could have been affected by the previously asked $q = 1$. The (dis)similarity of two conteNt-sharing variables, such as R_2^2 and R_2^3, can be measured by how often their values could coincide *had they been jointly distributed* (de facto, they are not, because they occur in mutually exclusive conteXts). In other words, the similarity of R_2^2 and R_2^3 is measured by the maximal value of $\Pr[T_2^2 = T_2^3]$ among all jointly distributed pairs $\{T_2^2, T_2^3\}$ such that T_2^2 is distributed as R_2^2, and T_2^3 as R_2^3. Any such a pair $\{T_2^2, T_2^3\}$ is called a *coupling* of R_2^2 and R_2^3, and the couplings with the maximal value of $\Pr[T_2^2 = T_2^3]$ are called *maximal*. We can find maximal couplings for all other conteNt-sharing pairs $\left\{R_q^c, R_q^{c'}\right\}$. For some of them we may expect no distributional differences (in our example with questions it could be, e.g., R_1^1, R_1^3, as in both these cases $q = 1$ is asked first), and then the maximal value of $\Pr\left[T_1^1 = T_1^3\right]$ will be 1. This is the case of traditional interest in quantum physics. However, generally, both in physics and psychology, differences in distributions of conteNt-sharing random variables should be expected and taken into account. Direct influence is, of course, a form of conteXt-dependence, but it is very different from what is considered *contextuality* in the proper sense of the word. The latter is detected in the system by showing that the just mentioned maximal couplings of the conteNt-sharing pairs are not compatible with the joint distributions of the random variables within conteXts. In other words, a system is contextual if the joint distributions within conteXts force the conteNt-sharing pairs across conteNts to be more dissimilar than they could be if taken without the conteXts. While direct influences exerted by conteNts are causal (information-carrying), true contextuality is of a correlational, non-causal nature.[1] More rigorous definitions are given below, in Section 7.1.

To provide historical perspective, contextuality (without using this term at first) was introduced in quantum physics by Bell (1964, 1966) and Kochen and Specker (1967). They demonstrated that one could meaningfully address, using only observable measurements, the question famously discussed in Bohr's (1935) critique of Einstein, Podolsky, and Rosen (1935). The question is whether all measurement outcomes in a system of measurements can be presented as being determined by

[1]To prevent objections, direct influences are *defined* in our theory as the differences in distributions, so one cannot speak of "hidden" influences (Filk, 2015, 2016). Thus, if the variables in system $\mathcal{E}$ are binary, $+1/-1$, and $\Pr\left[R_1^1 = 1\right] = \Pr\left[R_1^3 = 1\right] = 0.5$, one can imagine that "in reality" conteXt $c = 3$ somehow acts upon the "potential values" of R_1^3 reversing their signs, $R_1^3 \to -R_1^3$, without changing the distribution. However, this is not considered a "direct influence," because in the given system of random variables these unnoticeable changes do not carry information. If one can actually observe the changes $R_1^3 \to -R_1^3$, the system of random variables one deals with changes dramatically, and the CbD analysis then changes accordingly (Dzhafarov, Cervantes, & Kujala, 2017; Dzhafarov & Kon, 2018; Dzhafarov & Kujala, 2018).

some "hidden" random variable in a conteXt-independent way, i.e., using conteXt-independent mappings from the values of this hidden variable into the values of the observed measurement outcomes. With the work of Fine (1982a, 1982b) and Suppes and Zanotti (1981), it became clear that contextuality can also be formulated in terms of the (non)existence of certain *joint distributions involving random variables recorded in different conteXts.* Although some researchers disagree (Griffiths, 2017), this seems to have become a common way of understanding contextuality (Abramsky, Barbosa, Kishida, Lal, & Mansfield, 2015; Abramsky, & Brandenburger, 2011; Araújo, Quintino, Budroni, Cunha, & Cabello, 2013; Budroni, 2016; Budroni & Emary, 2014; Cabello, 2013; Khrennikov, 2008; Klyachko, Can, Binicioglu, & Shumovsky, 2008; Kurzynski, Ramanathan, & Kaszlikowski, 2012; Liang, Spekkens, & Wiseman, 2011; Ramanathan, Soeda, Kurzynski, & Kaszlikowski, 2012). Probabilistic underpinnings of this understanding have been critically examined by Khrennikov (2000a, 2000b, 2001) and Dzhafarov and Kujala (2016a, 2017a). Irrespective of the debated issues and disagreements, however, contextuality analysis has been moved from physics to probability theory, making it apparent that random variables in contextuality analysis need not represent quantum measurements, they can also be, e.g., responses of biological organisms to stimuli. However, the search for contextuality in psychology was frustrated by the fact that all behavioral systems of random variables exhibit strong direct influences, whereas the theory of contextuality in quantum mechanics, until recently, was only developed for *consistently connected systems*, those in which conteNt-sharing random variables have identical distributions. When direct influences are taken into account, a large body of experimental data collected in search of contextuality can be shown to exhibit no contextuality (Dzhafarov, & Kujala, 2014; Dzhafarov, Kujala, Cervantes, Zhang, & Jones, 2016; Dzhafarov, Zhang & Kujala, 2015). Nevertheless two very recent series of experiments unequivocally demonstrate that behavioral data (simple conjoint choices made by people) can be represented by contextual systems of random variables (Basieva, Cervantes, Dzhafarov, & Khrennikov, 2019; Cervantes & Dzhafarov, 2018). These experiments dealt with responses aggregated over large pools of people, with each person making choices within a single conteXt.

This paper presents the first experimental evidence of contextuality in *individual human behavior*. In the experiment presented below, each of the three participants made repeated choices in a series of randomized conteXts. A similar experiment, with essentially the same stimuli and similar instructions, has been conducted before, and analyzed in two different ways (Cervantes & Dzhafarov, 2017a, 2017b): both these analyses revealed no contextuality in the data. The main difference of that experiment from the present one is that in the former all choices were binary, whereas in the present experiment each choice was made among five options. This is an important difference in the theory presented below.

7.1 Contextuality-by-default theory

7.1.1 Generalities

A system of random variables is defined as a set of double-indexed random variables

$$\mathcal{R} = \left\{ R_q^c : c \in C, q \in Q, q \prec c \right\}, \tag{7.2}$$

where C is a set of *conteXts*, Q is a set of *conteNts*, and $q \prec c$ (or $c \succ q$) is read "conteNt q is recorded in conteXt c".[2] Examples of a conteNt q (the "thing" being measured or responded to) are particle's spin in a given direction in a Hilbert space,

[2]Here and throughout, we conveniently confuse R_q^c and $\left(R_q^c, c, q\right)$, so that, e.g., $\left\{R_q^c, R_{q'}^c\right\}$ consists of two random variables even if $R_q^c \equiv R_{q'}^c$, the same measurable function. Also, we follow the common tradition of conveniently confusing functions R_q^c with their values.

or a question asked of a person. Examples of a conteXt c may be subsets of conteNts measured "together" (simultaneously or sequentially), or different conditions associated with a given subset of conteNts (e.g., the order in which two fixed questions are asked). The corresponding R_q^c would then be the spin value (say, "up" or "down") along axis q in a given set c of measured properties, or the response (say, "yes" or "no") to question q asked before or after another question, q', with $c = (q', q)$. As a *random variable*, R_q^c is a measurable function from a probability space $(X^c, \varXi^c, \pi^c)$ to a measurable space $(Y_q, \varUpsilon_q)$, with the usual meaning of the components. The probability space $\left(Y_q, \varUpsilon_q, p_q^c\right)$ induced by this function is the *distribution* of R_q^c. The indices show that $(X^c, \varXi^c, \pi^c)$ is common to all R_q^c within a conteXt c, i.e., all such R_q^c are jointly distributed, reflecting the fact that their realizations are empirically linked. Put differently, for any $c \in C$, the set

$$R^c = \left\{R_q^c : q \in Q, q \prec c\right\} \tag{7.3}$$

can be viewed as a random variable. It is a principle of CbD that any $R_q^c, R_{q'}^{c'}$ with $c \neq c'$ are *stochastically unrelated*, i.e., $(X^c, \varXi^c, \pi^c) \neq (X^{c'}, \varXi^{c'}, \pi^{c'})$, reflecting the fact that conteXts are mutually exclusive, so no pairing of the values of R_q^c and $R_{q'}^{c'}$ is defined. In particular, the variables in

$$\mathcal{R}_q = \left\{R_q^c : c \in C, q \prec c\right\} \tag{7.4}$$

for a given q are not jointly distributed. However, the distributions of any $R_q^c, R_q^{c'}$ in $\mathcal{R}_q$ always share the same measurable space, $(Y_q, \varUpsilon_q)$, reflecting the fact that R_q^c and $R_q^{c'}$ have the same conteNt (i.e., they measure or respond to the same "thing").

The next definition is a modification of the usual one (Thorisson, 2000), to better suit our purposes. A (probabilistic) *coupling* of an indexed set of random variables

$\{V_i\}_{i\in I}$ is an identically indexed set of jointly distributed random variables $\{W_i\}_{i\in I}$ such that, for any subset $I' \subseteq I$, if the elements of $\{V_i\}_{i\in I'}$ are jointly distributed, then $\{W_i\}_{i\in I'} \overset{dist}{=} \{V_i\}_{i\in I'}$ (the same distribution). In particular, a coupling of a system $\mathcal{R}$ in (7.2) is a set

$$S = \left\{S_q^c : c \in C, q \in Q, q \prec c\right\} \tag{7.5}$$

of jointly distributed random variables, such that, for all $c \in C$,

$$R^c = \left\{R_q^c : q \in Q, q \prec c\right\} \overset{dist}{=} \left\{S_q^c : q \in Q, q \prec c\right\} = S^c. \tag{7.6}$$

Returning to our example (7.1), the following matrix of jointly distributed random variables (or simply, the following random variable) E,

S_1^1	S_2^1		S_4^1	$c=1$
	S_2^2	S_3^2		$c=2$
S_1^3	S_2^3	S_3^3		$c=3$
$q=1$	$q=2$	$q=3$	$q=4$	coupling E

, (7.7)

is a coupling of $\mathcal{E}$ if $S^c \overset{dist}{=} R^c$ for $c = 1, 2, 3$.

Let $\mathsf{P}_{\max}$ be the following statement, well-defined (in the sense of being true or false) for any two jointly distributed random variables A, B:

$$\mathsf{P}_{\max}(A, B) = \text{“}\Pr[A = B] \text{ is maximal possible, given the distributions of } A \text{ and } B.\text{”} \tag{7.8}$$

If a coupling $\left\{S_q^c, S_q^{c'}\right\}$ of two conteNt-sharing random variables R_q^c and $R_q^{c'}$ satisfies this statement, it is called a maximal coupling of R_q^c and $R_q^{c'}$. The system $\mathcal{R}$ is *noncontextual* if $\mathcal{R}$ has a coupling S in which any $\left\{S_q^c, S_q^{c'}\right\}$ is a *maximal coupling* of R_q^c and $R_q^{c'}$. Otherwise, if such a coupling S does not exist, the system is *contextual*.

Using our example in (7.7), system $\mathcal{E}$ is noncontextual if and only if among all its couplings E one can find at least one in which all equalities $S_1^1 = S_1^3$, $S_2^1 = S_2^2$, $S_2^2 = S_2^3$, and $S_3^2 = S_3^3$ occur with the maximal probability allowed by their individual distributions. Thus, if R_1^1 and R_1^3 are dichotomous, $+1/-1$, with $\Pr\left[R_1^1 = 1\right] = p$ and $\Pr\left[R_1^3 = 1\right] = q$, then the maximal possible probability of $S_1^1 = S_1^3$ is $1 - |p - q|$. Obviously, any subsystem of a noncontextual system (obtained by deleting some of the random variables) is noncontextual, or, equivalently, any system with a contextual subsystem is contextual.

7.1.2 Dichotomous random variables

Most systems of traditional interest consist of *dichotomous* random variables. Among basic properties of such systems one should mention the following (Dzhafarov, 2017; Dzhafarov et al., 2017; Dzhafarov & Kujala, 2017a, 2017b).

(P1) Adding to or removing from a system a deterministic random variable (attaining a single value with probability 1), or a variable that does not share its conteXt or its conteNt with other variables, does not change the system's (non)contextuality (in fact, does not change the degree of contextuality, but we do not discuss this notion here).

(P2) A set of conteNt-sharing random variables $\mathcal{R}_q = \left\{R_q^c : c \in C, q \prec c\right\}$ always has a *unique* coupling such that any two of its elements satisfy $\mathsf{P}_{\max}$. (Such a coupling is referred to as a *multimaximal coupling*).

(P3) $T_q = \left\{T_q^c : c \in C, q \prec c\right\}$ is a multimaximal coupling of $\mathcal{R}_q$ if and only if, for any $\{c_1, \ldots, c_k\} \subseteq C$, the probability of $T_q^{c_1} = \ldots = T_q^{c_k}$ is maximal among all couplings of $\left\{R_q^{c_1}, \ldots, R_q^{c_k}\right\}$.

(P4) If $\mathcal{R}_q = \left\{R_q^{c_1}, \ldots, R_q^{c_l}\right\}$ is enumerated so that $\Pr\left[R_q^{c_1} = 1\right] \leq \ldots \leq \Pr\left[R_q^{c_l} = 1\right]$, then T_q is a multimaximal coupling of $\mathcal{R}_q$ if and only if $\Pr\left[T_q^{c_i} = T_q^{c_{i+1}}\right]$ is maximal for $i = 1, \ldots, l-1$ among all possible couplings of $\mathcal{R}_q$.

Especially important in quantum-mechanical applications are cyclic systems of ranks $n = 2, 3, \ldots$. Denoting by $\oplus 1$ cyclic clockwise shift $1 \mapsto 2, \ldots, n-1 \mapsto n, n \mapsto 1$ (and by $\ominus 1$ the opposite shift), a cyclic system of rank n has conteXts $c = 1, \ldots, n$, conteNts $q = 1, \ldots, n$, and consists of dichotomous $(+1/-1)$ random variables $\left\{R_i^i, R_{i\oplus 1}^i : i = 1, \ldots, n\right\}$. Some examples of such systems are: for $n = 2$, question order effects (Wang & Busemeyer, 2013; Wang, Solloway, Shiffrin, & Busemeyer, 2014); for $n = 3$, the Suppes and Zanotti (1981), original Bell (1964), and Leggett and Garg (1985) systems in quantum mechanics, and simple decision making systems in cognition (Asano, Hashimoto, Khrennikov, Ohya, & Tanaka, 2014; Basieva et al., 2019); for $n = 4$, the EPR/Bohm-Bell-CHSH systems (Bell, 1966; Bohm & Aharonov, 1957; Clauser & Horne, 1974; Clauser, Horne, Shimony, & Holt, 1969; Fine, 1982a, 1982b), and decision making and psychophysical systems (Bruza, Kitto, Nelson, & McEvoy, 2009; Bruza, Kitto, Ramm, & Sitbon, 2015; Cervantes & Dzhafarov, 2017a, 2017b, 2018); for $n = 5$, the KCBS system (Klyachko et al., 2008; Lapkiewicz et al., 2011); for $n > 5$, some psychophysical systems (Zhang & Dzhafarov, 2016). The main theoretical result here is

Theorem 7.1.1 (Kujala & Dzhafarov, 2016)**.** A cyclic system of rank n is contextual if and only if (denoting expected value by $\langle \cdot \rangle$)

$$\max_{\iota_1, \ldots, \iota_k \in \{-1,1\}, \prod_{i=1}^{n} \iota_i = -1} \sum_{i=1}^{n} \iota_i \left\langle R_i^i R_{i\oplus 1}^i \right\rangle - (n-2) - \sum_{i=1}^{n} \left| \left\langle R_i^i \right\rangle - \left\langle R_i^{i\ominus 1} \right\rangle \right| > 0. \tag{7.9}$$

Prior to Kujala and Dzhafarov (2016), this general result was conjectured and proved for small values of n (Dzhafarov, Kujala, & Larsson, 2015; Kujala & Dzhafarov, 2015). The special case of this result for consistently connected systems had been proved, by very different means, in Araújo et al. (2013).

We do not have analogous closed-form criteria for non-cyclic systems, but the theory here is well-developed. There is a general linear programming method for establishing contextuality or lack thereof in any given system with finite sets C and Q and dichotomous random variables (Dzhafarov et al., 2017; Dzhafarov & Kujala, 2016a) (in fact, the method would work for any categorical random variables, but the CbD approach does not require this, see Section 7.1.3). The problem is reduced to a certain underdetermined system of linear equations,

$$\mathbf{MQ} = \mathbf{P}. \tag{7.10}$$

Here, $\mathbf{P} = \left(1, \overset{\#1}{\ldots}, \overset{\#2}{\ldots}\right)$, where #1 denotes all probabilities characterizing the distributions within the conteXts (e.g., $\Pr\left[R_1^1 = 1, R_2^1 = 1, R_3^1 = -1\right]$), and #2 denotes all probabilities characterizing the maximal couplings $\left\{T_q^c, T_q^{c'}\right\}$ of the separate conteNt-sharing pairs (e.g., $\Pr\left[T_2^1 = 1, T_2^2 = 1\right]$); $\mathbf{Q}$ is a vector of probabilities (summing to 1) for all possible values of the hypothetical coupling S; and $\mathbf{M}$ is a Boolean matrix with 1's in each row corresponding to values of S comprising the events whose probabilities are given in $\mathbf{P}$. The system is noncontextual if and only if these linear equations have a solution for $\mathbf{Q}$ with nonnegative components. The linear programming representation of CbD naturally leads to its geometric representations by polytopes and graph-theoretic renderings. A detailed version of the latter was recently proposed by Amaral, Duarte, and Oliveira (2018).

7.1.3 Arbitrary random variables

The current version of CbD (Dzhafarov et al., 2017; Dzhafarov & Kujala, 2017a, 2017b) posits that all random variables in a system should be dichotomized before they are submitted to contextuality analysis. One reason for this is that the property **P2** in the previous section does not hold for non-dichotomous variables: a multimaximal coupling need not exist, and when it does, need not be unique. The other reason is that one expects a noncontextual system[a] to remain noncontextual if some values of a random variable are "lumped together" (e.g., if in $\{1, 2, 3, 4\}$ one ceases to distinguish 1 and 2) (Dzhafarov et al., 2017). Dichotomizations are easy if in the initial description of an empirical domain all random variables are categorical (i.e., have unordered finite sets of values). One then is interested in all possible dichotomizations: an n-valued random variable is replaced with $2^{n-1}-1$ distinct dichotomizations (with unordered pairs of values). For instance, if an initial R has values $\{1, 2, 3, 4\}$, in contextual analysis it is replaced with 7 jointly distributed values (see Box I). Assume, e.g., that in system $\mathcal{E}$ of (7.1) the variables for $q = 1$ have 4 values, variables for $q = 3$ have 3 values, and the other two variables are binary. Dichotomization of the system then transforms it into one shown in Box II where the numbers in parentheses encode different dichotomizations. The procedure effectively splits old conteNts into new conteNts. The size of the system increases only in visual appearance, because in each row of $\mathcal{E}^*$ the support of the joint distribution is precisely the same as in system $\mathcal{E}$. The original system is considered contextual if its dichotomization is contextual. The main result here is

Theorem 7.1.2 (Dzhafarov et al., 2017)**.** *A system of categorical random variables (before dichotomization) is contextual if, for some (q, c, c'), neither of $R_q^c, R_q^{c'}$ nominally dominates the other.*

[a](Erratum note added in the dissertation.) Originally "systems". Corrected in the text.

$$\begin{array}{r|c|c|c|c|c|c|c|}
 & R_{(1)} & R_{(2)} & R_{(3)} & R_{(4)} & R_{(5)} & R_{(6)} & R_{(7)} \\
\text{values:} & 1 \,\|\, 2,3,4 & 2 \,\|\, 1,3,4 & 3 \,\|\, 1,2,4 & 4 \,\|\, 1,2,3 & 1,2 \,\|\, 3,4 & 1,3 \,\|\, 2,4 & 1,4 \,\|\, 2,3
\end{array} \tag{7.11}$$

Box I.

$R^1_{1(1)}$	$R^1_{1(2)}$	$R^1_{1(3)}$	$R^1_{1(4)}$	$R^1_{1(5)}$	$R^1_{1(6)}$	$R^1_{1(7)}$	R^1_2				R^1_4	$c = 1$
							R^2_2	$R^2_{3(1)}$	$R^2_{3(2)}$	$R^2_{3(3)}$		$c = 2$
$R^3_{1(1)}$	$R^3_{1(2)}$	$R^3_{1(3)}$	$R^3_{1(4)}$	$R^3_{1(5)}$	$R^3_{1(6)}$	$R^3_{1(7)}$	R^3_2	$R^3_{3(1)}$	$R^3_{3(2)}$	$R^3_{3(3)}$		$c = 3$
$q = 1(1)$	1(2)	1(3)	1(4)	1(5)	1(6)	1(7)	2	3(1)	3(2)	3(3)	4	system $\mathcal{E}^*$

(7.12)

Box II.

(i) Values	1	2	3	4	5
probabilities for A:	0.1	0.2	0.2	0.5	0
probabilities for B:	0.1	0.2	0.2	0.5	0

,

(ii)	1	2	3	4	5
probabilities for A:	0.1	0.2	0.2	0.5	0
probabilities for B:	0.2	0.1	0.2	0.5	0

,

(iii) Values	1	2	3	4	5
probabilities for A:	0.1	0.2	0.2	0.5	0
probabilities for B:	0.5	0	0.1	0.4	0

,

(iv) Values	1	2	3	4	5
probabilities for A:	0.1	0.2	0.2	0.5	0
probabilities for B:	0.3	0.3	0.1	0.2	0.1

,

(7.13)

Box III.

It is this theorem that we use to analyze the experiment below. The meaning of nominal dominance is as follows: given A and B with the same set of values $\{1, \ldots, k\}$, A *nominally dominates* B if the inequality $\Pr[A = i] < \Pr[B = i]$ holds for no more than one value of $i = 1, \ldots, k$ (i.e., if $\Pr[A = i] \geq \Pr[B = i]$ for at least $k - 1$ of them). Thus, among the pairs of probability distributions shown in Box III, in (i) and (ii) A and B nominally dominate each other, in (iii) A nominally dominates B, and in (iv) neither of the two random variables nominally dominates the other.

The theorem above tells us that if we are interested in all possible dichotomizations, we may not need to actually create them to determine that the system is contextual. It suffices instead to find at least one instance when neither of two original (as observed, before dichotomization) conteNt-sharing random variables nominally dominates the other, as in (iv) above. The condition is only sufficient but not necessary for contextuality: if nominal dominance is found in all pairs of conteNt-sharing random variables, the system may or may not be contextual.

7.2 Double-identification experiment

7.2.1 Method

7.2.1.1 Participants

Three volunteers, graduate students at Purdue University, one female and two males (including the first author of this paper), with normal or corrected to normal vision, participated in this study. The experimental program was regulated by Purdue University's IRB protocol #1202011876. The participants are identified as P1, P2, and P3 in the text below.

7.2.1.2 Equipment

A personal computer was used with an Intel® Core™ processor running Windows XP, and with a 24-in. monitor with a resolution of 1920×1200 pixels (px). The participant's head was steadied in a chin-rest with forehead support at 90 cm distance from the monitor; at this distance a pixel on the screen subtended 62 sec arc. The response keys on a US 104-key keyboard were indicated by stickers with the corresponding response labels (see Figure 7.1).

7.2.1.3 Stimuli

The stimuli presented on the computer screen consisted of two brightly grey colored circles (RGB 100-100-100) on a black background, with their centers 320 px apart horizontally, each circle having the radius of 135 px and 4 px wide circumfer-

Fig. 7.1. Layout of the keyboard with the response keys stickers for left and right stimuli.

ence. Each circle contained within it a dot of 4 px in diameter, that could be located in the circle's center or 4 px away from it, in the left, right, upward or downward direction. An example of the stimuli is shown in Figure 7.2.

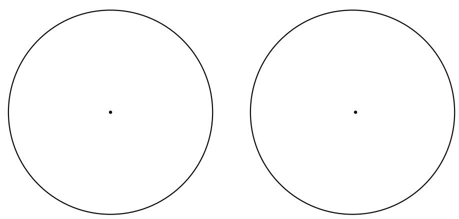

Fig. 7.2. An example of the stimuli in experiment (in reversed contrast and not to scale). In the left circle the dot is in the center, in the right one it is shifted to the right by 4 px ($\simeq$ 4.1 min arc). The participant's task was to identify the location of the dot in each of the two circles by pressing corresponding keys on a keyboard.

7.2.1.4 Procedure

In each trial the participant was asked to indicate, for each circle, whether the dot was in its center or shifted in one of the four directions (up, down, left, or right). The responses were given by pressing in any order and holding together two designated keys, one for each location in each circle, as shown in Fig. 7.2. The stimuli were displayed until both keys were pressed. Then, the dots in each circle disappeared, and the next pair of dots appeared 600 ms later. The circles, with or without the dots, remained displayed continuously throughout the experiment. (Response times were recorded but not used in the data analysis.)

Each participant completed between 20 and 23 experimental sessions, each lasting 30 minutes and consisting of about 380 trials recorded and used for subsequent analysis. The experimental sessions were preceded by two training sessions, excluded from the analysis. The first 75 trials of each training session were practice trials in which the participants received feedback as to whether their response for each of the two circles was correct or not. No feedback was given in the experimental trials.

7.2.2 Experimental conteXts and conteNts

In each of two circles the dot presented could be in one of 5 locations: at the center, or shifted to the left, right, up, or down. These locations formed conteNts of the random variables in the probabilistic description of the experiment, denoted as shown in Table 7.1. The same table shows that the 5×5 pairs of locations of the two dots formed 25 conteXts. In each experimental session, all conteXts were presented [close-to-]equal numbers of times (about 15).

For each session, each trial was randomly assigned to one of the conditions in Fig. 7.1. The number of experimental sessions was chosen so that the expected number of experimental trials in each of the conteXts was at least 300. This number of observations was chosen based on Cepeda Cuervo, Aguilar, Cervantes, Corrales, Díaz, and Rodríguez (2008), whose results show that coverage errors with respect to nominal values are below 1% for most confidence intervals for proportions with $n > 300$.

Table 7.1.
Notation used for the conteXts and the conteNts: c, l, r, u, and d denote that the dot is, respectively, in the center, shifted to the left, to the right, up, or down. The 25 conteXts are denoted cc, cu, du, etc., the left (right) symbol indicating the location of the dot in the left (respectively, right) circle. To denote conteNts, the location of a dot is shown on the left (for the left circle) or on the right (for the right circle) of a dash: thus, c-denotes the dot in the center of the left circle, -l denotes the dot shifted to the left in the right circle, etc.

		Right circle conteNts				
		(-c)	(-l)	(-r)	(-u)	(-d)
Left circle conteNts	Center (c-)	cc	cl	cr	cu	cd
	Left (l-)	lc	ll	lr	lu	ld
	Right (r-)	rc	rl	rr	ru	rd
	Up (u-)	uc	ul	ur	uu	ud
	Down (d-)	dc	dl	dr	du	dd

The system of random variables describing the experiment is shown in Fig. 7.3.

7.2.3 Results

The complete set of results obtained in the experiment (excluding training sessions) is stored in "Contextuality in a psychophysical double-identification experiment" , `https://doi.org/10.7910/DVN/FCT9VO`. The data used in the analysis of the nominal dominance condition are shown in Tables 7.A.1–7.A.3, placed in Appendix.[a] These tables show the estimated probabilities with which each of the three participants responded in each of five possible ways (center, left, right, up, and down) to the left stimulus and to the right stimulus, in each of the 25 conteXts. For all participants, the nominal dominance condition fails for at least one pair of random variables for each of the conteNts. This means that, for all three participants, the pattern of the results indicates contextuality.

[a](Explanatory note added in the dissertation.) Reproduced here as Section 7.5.

	c-	-c	l-	-l	r-	-r	u-	-u	d-	-d
cc	$\star$	$\star$								
cl	$\star$			$\star$						
cr	$\star$					$\star$				
cu	$\star$							$\star$		
cd	$\star$									$\star$
lc		$\star$	$\star$							
ll			$\star$	$\star$						
lr			$\star$			$\star$				
lu			$\star$					$\star$		
ld			$\star$							$\star$
rc		$\star$			$\star$					
rl				$\star$	$\star$					
rr					$\star$	$\star$				
ru					$\star$			$\star$		
rd					$\star$					$\star$
uc		$\star$					$\star$			
ul				$\star$			$\star$			
ur						$\star$	$\star$			
uu							$\star$	$\star$		
ud							$\star$			$\star$
dc		$\star$							$\star$	
dl				$\star$					$\star$	
dr						$\star$			$\star$	
du								$\star$	$\star$	
dd									$\star$	$\star$

Fig. 7.3. The conteNt-conteXt system of measurements for the double detection experiment. The cell corresponding to conteXt xy and conteNt z (with z being x- or -y), if it contains a star, represents the random variable R_z^{xy}; the absence of a star means that conteNt z was not measured in conteXt xy. For instance, $xy = cc$ and $z = c$- define a random variable $R_{c\text{-}}^{cc}$. There are two random jointly distributed variables, $R_{x\text{-}}^{xy}$ and $R_{\text{-}y}^{xy}$, in each conteXt xy, and their joint distribution is defined by the probabilities: $\Pr\left[R_{x\text{-}}^{xy} = j, R_{\text{-}y}^{xy} = k\right]$ where $j, k \in$\{center, left, right, up, down\}.

To assess the reliability of these results, we generated 100,000 bootstrap resamples for each participant: each bootstrap resample was generated by independently selecting, with replacement, a random sample from (and of the same size as) the responses given in the experiment to each of the two circles in each conteXt. The proportions of resamples in which nominal dominance was observed are presented in Table 7.2, for each conteNt separately, and (in the bottom row of the table) for all conteNts simultaneously. Note that it is the latter that matters for our analysis: the system may be noncontextual only if nominal dominance is satisfied for all pairs of conteNt-sharing random variables. This was observed for none of the resamples and none of the participants. We can model this situation, for each participant, as a sequence of 100,000 binomial trials with zero successes. If p denotes the probability of this happening (let us label this as a "success"), we can model the results, for each participant, as a Bernoulli sequence of length 100,000, with probability of a "success" (overall compliance with nominal dominance) being p, and the observed number of successes being zero. The exact 99.999% Clopper and Pearson (1934) confidence interval for p is $[0, 0.00012]$. We can clearly dismiss the possibility that our data result from random perturbations of a pattern that satisfies nominal dominance.

7.3 Discussion

Based on the CbD analysis of many published experiments in none of which contextuality was found, it was tempting to hypothesize that all behavioral systems were noncontextual (Dzhafarov et al., 2016; Dzhafarov, Zhang, & Kujala, 2015; Zhang & Dzhafarov, 2017).[b] This hypothesis was rejected by recent crowdsourcing experiments (Basieva et al., 2019; Cervantes & Dzhafarov, 2018), but the question remained open

[b](Erratum note added in the dissertation.) Reference "Dzhafarov, Zhang, & Kujala, 2015" appeared as "Dzhafarov, Kujala et al., 2015", and reference "Zhang & Dzhafarov, 2017" appeared as "Zhang & Dzhafarov, 2016". Corrected in the text.

Table 7.2.
Bootstrap estimates of the probabilities for the systems to satisfy the nominal dominance condition.

ConteNt	P1	P2	P3
c-	0.038	0.000	0.000
l-	0.000	0.000	0.224
r-	0.000	0.000	0.003
u-	0.429	0.000	0.023
d-	0.002	0.000	0.001
-*c*	0.412	0.000	0.000
-*l*	0.019	0.000	0.385
-*r*	0.000	0.000	0.015
-*u*	0.566	0.001	0.034
-*d*	0.001	0.000	0.000
Overall all contents	0.000	0.000	0.000

as to whether contextuality can also be observed in individual human behavior. In the crowdsourcing experiments the stimuli were questions to be answered in one of two ways. In such an experiment a repeated presentation of a question to the same person cannot be viewed as a repeated recording of the same random variable, because the person would most likely remember her previous answers and repeat them not to contradict herself, or would deliberately vary them due to the phenomenon of satiation. Therefore, to investigate contextuality in a within-subject paradigm, one has to use stimuli that do not have any distinguishing characteristics by which they can be remembered. Thus, if a variety of weak flashes varying in intensity are judged in terms of "I have seen it" or "I have not seen it," there is no way the observer may remember seeing a particular flash before, unless this flash was seen with probability 1. Analogously, in our experiment, there was no way a participant could remember seeing a specific dot position in one of the circles, as no position was identified perfectly.

A previously conducted experiment (Cervantes & Dzhafarov, 2017a, 2017b), similar to the one presented in this paper, revealed no contextuality, i.e., all conteXt-dependence in it could be attributed to direct influences. In that experiment the dots within two circles could vary on three levels (center, up, down) and the responses were dichotomous: "in the center" or "not in the center." As it turns out, switching to questions with five possible answers (and increasing the number of conteNts to five to match them) changed the system from noncontextual to contextual.

The overall conteXt-dependence in our experiment means that a given location q of the dot in a circle is judged differently for different locations q' of the dot in the other circle. This direct influence of q' on responses to q manifests itself in the changing distribution of the responses to q as q' changes. The contextuality of the system, however, shows that these direct influences cannot account for the entire situation: the changes in the identity of the random variable representing the responses to q in different conteXts are greater than warranted by their distributional differences. This is another way of stating the definition of a contextual system, according to which the joint distributions of the random variables within conteXts force conteNt-sharing random variables (responses to the same q at different q') to be more dissimilar than warranted by the difference in their distributions.

The relationship between the two forms of conteXt-dependence in a contextual system, direct influences and contextuality proper, is a complex issue of which we have very little knowledge at present. A remarkable fact is that this relationship seems to be different in systems of binary random variables (at least in cyclic systems, mentioned in Section 7.1.2) and in systems of multivalued random variables. As is

responses to q in context qq':	center	left	right	up	down
probabilities:	p_1	p_2	p_3	p_4	p_5

and

responses to q in context qq'':	center	left	right	up	down
probabilities:	p'_1	p'_2	p'_3	p'_4	p'_5

.

Box IV.

evident from (7.9), the direct influences and contextuality in a cyclic system are antagonistic. Direct influences in (7.9) are represented by

$$\sum_{i=1}^{n} \left| \left\langle R_i^i \right\rangle - \left\langle R_i^{i\ominus 1} \right\rangle \right| , \tag{7.14}$$

and as this quantity increases, the value of the left-hand-side expression in (7.9) decreases, making the system less likely to be contextual. In our present experiment the situation is more complex. Direct influences here are responsible for the differences between the distributions shown in Box IV;

In the absence of all direct influences, i.e., with $p_i = p'_i$ for all i, the nominal dominance is trivially satisfied. This does not mean that the system in noncontextual, but its contextuality will have to be established by other means, generally, by solving the linear programming task (7.10). Direct influences must be present to break the nominal dominance relation and thereby allow us to establish contextuality "easily." More work is needed to understand this relationship better.

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7.5 Appendix. Data tables

See Table 7.A.1–7.A.3.

Table 7.A.1.
Empirical estimates of marginal distributions for the conteNt-conteXt system in Fig. 7.3 for participant P1.

P1		Left response					Right response				
Context	Trials	Center	Left	Right	Up	Down	Center	Left	Right	Up	Down
cc	336	.318	.521	.000	.155	.006	.235	.455	.000	.310	.000
cl	334	.213	.656	.000	.132	.000	.015	.871	.000	.108	.006
cr	336	.390	.435	.000	.155	.021	.601	.060	.018	.318	.003
cu	336	.298	.554	.000	.143	.006	.021	.149	.000	.830	.000
cd	336	.265	.574	.000	.152	.009	.271	.613	.000	.030	.086
lc	334	.036	.931	.000	.027	.006	.195	.527	.000	.278	.000
ll	335	.024	.928	.000	.042	.006	.021	.860	.000	.119	.000
lr	335	.051	.913	.003	.030	.003	.558	.122	.018	.299	.003
lu	335	.054	.904	.000	.033	.009	.042	.176	.003	.779	.000
ld	334	.042	.910	.003	.042	.003	.314	.605	.000	.024	.057
rc	333	.763	.081	.033	.117	.006	.246	.483	.000	.270	.000
rl	334	.605	.159	.051	.183	.003	.018	.859	.000	.120	.003
rr	335	.782	.048	.024	.137	.009	.591	.042	.021	.346	.000
ru	336	.685	.077	.027	.202	.009	.045	.083	.000	.872	.000
rd	335	.701	.075	.036	.179	.009	.322	.555	.000	.033	.090
uc	335	.116	.269	.003	.612	.000	.200	.457	.000	.343	.000
ul	336	.062	.345	.000	.592	.000	.021	.872	.003	.101	.003
ur	334	.156	.216	.000	.629	.000	.581	.051	.027	.335	.006
uu	334	.084	.260	.000	.656	.000	.033	.108	.000	.859	.000
ud	335	.096	.191	.000	.713	.000	.343	.558	.000	.033	.066
dc	335	.337	.478	.000	.006	.179	.242	.460	.000	.296	.003
dl	334	.237	.599	.000	.006	.159	.012	.880	.000	.108	.000
dr	336	.312	.449	.000	.009	.229	.589	.054	.027	.330	.000
du	335	.310	.504	.000	.015	.170	.030	.116	.000	.854	.000
dd	335	.346	.451	.000	.006	.197	.370	.549	.000	.012	.069

Table 7.A.2.
Empirical estimates of marginal distributions for the conteNt-conteXt system in Fig. 7.3 for participant P2.

P2		Left response					Right response				
Context	Trials	Center	Left	Right	Up	Down	Center	Left	Right	Up	Down
cc	336	.616	.062	.039	.226	.057	.560	.062	.164	.202	.012
cl	336	.586	.080	.033	.268	.033	.265	.604	.015	.107	.009
cr	336	.586	.045	.062	.259	.048	.185	.000	.720	.071	.024
cu	336	.607	.083	.033	.220	.057	.131	.062	.089	.717	.000
cd	336	.580	.054	.024	.304	.039	.348	.033	.086	.024	.509
lc	336	.223	.604	.000	.134	.039	.610	.092	.119	.152	.027
ll	336	.214	.583	.003	.164	.036	.274	.548	.021	.134	.024
lr	336	.223	.586	.009	.158	.024	.220	.006	.682	.065	.027
lu	336	.310	.527	.000	.128	.036	.149	.042	.089	.720	.000
ld	336	.226	.557	.003	.179	.036	.333	.039	.098	.021	.509
rc	336	.339	.003	.443	.176	.039	.548	.086	.158	.173	.036
rl	336	.318	.012	.432	.205	.033	.247	.631	.018	.080	.024
rr	336	.310	.003	.429	.229	.030	.140	.018	.696	.116	.030
ru	336	.336	.000	.467	.158	.039	.170	.033	.074	.720	.003
rd	336	.351	.000	.405	.211	.033	.381	.054	.095	.030	.440
uc	336	.146	.018	.015	.818	.003	.646	.048	.134	.137	.036
ul	336	.131	.030	.015	.821	.003	.345	.545	.012	.068	.030
ur	336	.146	.030	.006	.815	.003	.235	.000	.688	.057	.021
uu	336	.167	.021	.018	.795	.000	.196	.036	.128	.637	.003
ud	336	.137	.015	.009	.836	.003	.390	.024	.068	.015	.503
dc	336	.354	.030	.036	.021	.560	.539	.057	.143	.229	.033
dl	336	.366	.039	.024	.018	.554	.220	.583	.006	.158	.033
dr	336	.375	.039	.006	.009	.571	.199	.003	.661	.119	.018
du	336	.393	.018	.033	.021	.536	.122	.027	.065	.786	.000
dd	336	.360	.039	.024	.036	.542	.393	.048	.116	.021	.423

Table 7.A.3.
Empirical estimates of marginal distributions for the conteNt-conteXt system in Fig. 7.3 for participant P3.

P3		Left response					Right response				
Context	Trials	Center	Left	Right	Up	Down	Center	Left	Right	Up	Down
cc	336	.738	.092	.012	.143	.015	.634	.149	.015	.188	.015
cl	337	.801	.053	.030	.110	.006	.027	.955	.000	.018	.000
cr	336	.762	.098	.021	.107	.012	.321	.003	.631	.036	.009
cu	336	.768	.086	.033	.098	.015	.128	.060	.000	.812	.000
cd	335	.785	.081	.021	.101	.012	.648	.081	.009	.003	.260
lc	337	.056	.935	.000	.009	.000	.700	.104	.039	.151	.006
ll	336	.060	.929	.000	.012	.000	.045	.929	.000	.027	.000
lr	337	.053	.929	.000	.015	.003	.288	.000	.680	.033	.000
lu	337	.059	.917	.000	.021	.003	.148	.059	.006	.786	.000
ld	336	.051	.938	.000	.012	.000	.676	.054	.015	.006	.250
rc	336	.336	.000	.649	.012	.003	.658	.125	.024	.185	.009
rl	337	.335	.009	.635	.021	.000	.027	.935	.000	.039	.000
rr	336	.312	.000	.670	.012	.006	.298	.000	.667	.036	.000
ru	337	.332	.000	.653	.015	.000	.142	.071	.000	.783	.003
rd	336	.280	.000	.699	.015	.006	.658	.074	.021	.012	.235
uc	336	.164	.033	.003	.801	.000	.699	.134	.021	.137	.009
ul	336	.143	.065	.003	.789	.000	.042	.943	.000	.012	.003
ur	336	.134	.033	.003	.830	.000	.327	.000	.631	.033	.009
uu	336	.202	.033	.003	.762	.000	.164	.062	.000	.774	.000
ud	337	.172	.021	.003	.804	.000	.668	.080	.015	.000	.237
dc	335	.603	.021	.012	.000	.364	.618	.137	.009	.230	.006
dl	337	.626	.030	.027	.000	.318	.030	.950	.000	.021	.000
dr	337	.644	.030	.021	.000	.306	.329	.000	.635	.033	.003
du	337	.638	.030	.015	.000	.318	.151	.080	.003	.766	.000
dd	336	.619	.039	.012	.003	.327	.708	.068	.009	.006	.208

8. DISCUSSION

Random variables representing measurements form systems in which each variable is identified both by its content (what it measures) and its context (the conditions under which it is recorded). Any two random variables are jointly distributed if and only if they are measured in the same context. The Contextuality-by-Default theory permits to separate *true contextuality* from *direct influences* (or, equivalently, from *inconsistent connectedness*) in the analysis of such systems. This makes the theory especially relevant to behavioral systems because the results of behavioral experiments normally exhibit inconsistent connectedness.

In Chapter 2 (Cervantes & Dzhafarov, 2017b), we applied the theory of cyclic systems to a psychophysical double-detection experiment, in which observers were asked to determine the presence or absence of a signal property in each of two simultaneously presented stimuli. In Chapter 3 (Cervantes & Dzhafarov, 2017a), we applied the theory for arbitrary systems of binary random variables to the same data. The experiment provides the closest analogue in psychophysical research to the Alice-Bob EPR/Bohm paradigm, the most prominent example of a contextual system in quantum mechanics. We have found that for the participants in the study there was no evidence of contextuality in their responses.

Now, the progress in the theory from Chapter 2 to Chapter 3, highlights one of the desirable properties of the theory for binary random variables (Dzhafarov & Kujala, 2017). This property is that a subsystem of a noncontextual system, obtained by dropping one or more of its random variables, remains noncontextual. In other words, noncontextuality of all subsystems of a system of random variables is necessary for the system to be noncontextual; however, it is not sufficient. Consider for instance

$c=1$	R^1_α	R^1_β
$c=2$	R^2_α	R^2_β
	α	β

Fig. 8.1. Example of a system whose all subsystems are noncontextual.

$c=1$	R^1_α	R^1_β
$c=2$	R^2_α	$\cdot$
	α	β

Fig. 8.2. Example of a subsystem of the system in Figure 8.1. This system is necessarily noncontextual.

the system of binary random variables in Figure 8.1. Regardless of the contextuality of the system, all subsystems that can be extracted from it are noncontextual. If, for example, we drop variable R^2_β, the resulting system, shown in Figure 8.2, is necessarilly noncontextual. This can be seen in the construction of an overall coupling of the system regardless of the actual probabilities involved. One such coupling is shown in Table 8.1. An analogous result would result if we had dropped any of the other three random variables in the system in Figure 8.1. It is also clear that dropping any additional variables from the system in Figure 8.2 will also result in a noncontextual system: a) if we drop R^1_α, any coupling of $\{R^1_\beta, R^2_\alpha\}$ will satisfy the definition; b) if we drop R^1_β, the maximal coupling of $\{R^1_\alpha, R^2_\alpha\}$ is the required coupling; c) if we drop R^2_α, then a copy of the jointly distributed (R^1_α, R^1_β) gives the required coupling; d) if we drop any two variables, we are left with a system of a single random variable which is trivially noncontextual. This example helps us to approach a less obvious example that shows a contextual system without contextual subsystems.[1]

[1]The following example is due to Janne Kujala (personal communication, 2019).

Table 8.1.
Coupling of the system in Figure 8.2. Without loss of generality, here we assumed $\Pr(R_\alpha^1 = 1, R_\beta^1 = 0) \le \Pr(R_\alpha^2 = 1) \le \Pr(R_\alpha^1 = 1)$.

Event	Pr(Event)
$\{S_\alpha^1 = 0, S_\beta^1 = 0, S_\alpha^2 = 0\}$	$\Pr(R_\alpha^1 = 0, R_\beta^1 = 0)$
$\{S_\alpha^1 = 0, S_\beta^1 = 0, S_\alpha^2 = 1\}$	0
$\{S_\alpha^1 = 0, S_\beta^1 = 1, S_\alpha^2 = 0\}$	$\Pr(R_\alpha^1 = 0, R_\beta^1 = 1)$
$\{S_\alpha^1 = 0, S_\beta^1 = 1, S_\alpha^2 = 1\}$	0
$\{S_\alpha^1 = 1, S_\beta^1 = 0, S_\alpha^2 = 0\}$	$\Pr(R_\alpha^1 = 1, R_\beta^1 = 0)$
$\{S_\alpha^1 = 1, S_\beta^1 = 0, S_\alpha^2 = 1\}$	0
$\{S_\alpha^1 = 1, S_\beta^1 = 1, S_\alpha^2 = 0\}$	$\Pr(R_\alpha^2 = 1) - \Pr(R_\alpha^1 = 1, R_\beta^1 = 0)$
$\{S_\alpha^1 = 1, S_\beta^1 = 1, S_\alpha^2 = 1\}$	$\Pr(R_\alpha^1 = 1, R_\beta^1 = 1)$

Consider the system of binary random variables in Figure 8.3. Let the variables in context 1 be independent, with

$$\Pr(R_\alpha^1 = i, R_\beta^1 = j, R_\gamma^1 = k) = 1/8, \text{ for } i, j, k = 0, 1.$$

And let the variables in context 2 be distributed as shown in Table 8.2. In this case there are three nontrivially noncontextual subsystems. These are found by dropping the two random variables in a single connection. If we drop, for example, the variables responding to content γ, we obtain the system in Figure 8.1. The distributions are known in this case; all random variables in the system have the same distribution,

$c = 1$	R_α^1	R_β^1	R_γ^1
$c = 2$	R_α^2	R_β^2	R_γ^2
	α	β	γ

Fig. 8.3. Example of a system whose all subsystems are noncontextual. The distributions of the random variables are given in-text and in Table 8.2.

Table 8.2.
Joint distribution of variables in context 2 from the system in Figure 8.3.

Event	Pr(Event)
$\{ R^2_\alpha = 0, R^2_\beta = 0, R^2_\gamma = 0 \}$	0
$\{ R^2_\alpha = 0, R^2_\beta = 0, R^2_\gamma = 1 \}$	1/4
$\{ R^2_\alpha = 0, R^2_\beta = 1, R^2_\gamma = 0 \}$	1/4
$\{ R^2_\alpha = 0, R^2_\beta = 1, R^2_\gamma = 1 \}$	0
$\{ R^2_\alpha = 1, R^2_\beta = 0, R^2_\gamma = 0 \}$	1/4
$\{ R^2_\alpha = 1, R^2_\beta = 0, R^2_\gamma = 1 \}$	0
$\{ R^2_\alpha = 1, R^2_\beta = 1, R^2_\gamma = 0 \}$	0
$\{ R^2_\alpha = 1, R^2_\beta = 1, R^2_\gamma = 1 \}$	1/4

and both bunches have the same joint distribution with its random variables being independent. We see that the system is noncontextual because the identity coupling of the two bunches satisfies the requirements for noncontextuality. The same result holds if we drop the random variables responding to either α or β. If we drop any additional variable, we have the same situation as in Figure 8.1. The remaining subsystems, those obtained by dropping a single random variable can easily be shown to be noncontextual. Whenever there is only one variable in a context or in a connection, contextuality of the system does not change by dropping that variable (Dzhafarov & Kujala, 2016). Hence, we use the results above to conclude that these subsystems are noncontextual. Lastly, we can see that the whole system is contextual by noting that in a coupling

$$\left(S^1_\alpha, S^1_\beta, S^1_\gamma, S^2_\alpha, S^2_\beta, S^2_\gamma\right)$$

the maximal sub-couplings for each connection are identity couplings. Consequently, any event of that coupling that includes $S^j_\iota = 0$ and $S^k_\iota = 1$, for $\iota = \alpha, \beta, \gamma$ and $j, k = 1, 2$, must have zero probability mass. If we assume that a coupling that shows the system to be noncontextual exists, then we reach the following contradiction

$$0 = \Pr(S^2_\alpha = 0, S^2_\beta = 0, S^2_\gamma = 0) = \Pr(S^1_\alpha = 0, S^1_\beta = 0, S^1_\gamma = 0) = {}^1\!/_8.$$

That is, the system presented in Figure 8.3 is contextual, and all of it subsystems are noncontextual.

Therefore, analyses based on subsystems of the system of interest, such as those in Cervantes and Dzhafarov (2017b) or Zhang and Dzhafarov (2017), are inconclusive when all the subsystems are found to be noncontextual. This is what makes the results in Chapter 3 stronger than those in Chapter 2. According with the results of the contextuality analysis in Chapter 3, the role of context in double-detection is attributed to direct influences: the distribution of responses to one of the stimuli is influenced by the state of the other stimulus.

Nonetheless, as discussed in Chapters 3 and 4 (Cervantes & Dzhafarov, 2017a, 2018), without a predictive theory, even if a large number of experiments fail to find contextuality, contextuality could be found under as yet unexplored experimental conditions. The likelihood of 'randomly' (or better, the epistemic probability one could assign to the event of) finding a noncontextual system varies with the structure of the system. For a system of random variables such as the one of the Snow Queen experiment, it was estimated as $2/3$. A relationship between these epistemic probabilities and the shape of the noncontextuality polytopes that describe cyclic systems of binary random variables has been established in Dzhafarov, Kujala, and Cervantes (2020, in press). These results show that, absent a guiding theory that would predict under which conditions contextuality can be found, it is rather unlikely that one will find a contextual system. The methodological procedure proposed in Chapter 4 (Cervantes & Dzhafarov, 2018) has provided a guide to explore contextuality more systematically. This procedure is predicated on the relatively simple structure of contextuality in cyclic systems of binary random variables. Using this procedure, we found unambiguous experimental evidence for contextuality in decision making. The experiments reported in Chapters 4 and 5, the Snow Queen experiment (Cervantes & Dzhafarov, 2018) and the 'Directions' experiment (experiment 5 in Basieva, Cervantes, Dzhafarov, & Khrennikov, 2019) demonstrated that a contextual system of random variables formally analogous to the EPR/Bohm system in quantum mechanics can be observed in human behavior. Experiments 1-4 in Basieva et al. (2019), demonstrated contextuality in cyclic systems of rank 3. Unlike the results reported in the literature preceding our work, these demostrations have been done without making the mistake of ignoring the inconsistently connectedness of the systems. Lastly, Experiment 6 ('Colored figures') shows that contextuality in our experiments is an empirical finding rather than mathematical consequence of the experimental design.

The issue of measuring the degree of contextuality is mentioned in several of the included papers. In Chapters 2, 4, and 5, the quantity

$$D = s_{odd}\left(\mathsf{E}\left[R_1^1 R_2^1\right], \mathsf{E}\left[R_2^2 R_3^2\right], \ldots, \mathsf{E}\left[R_n^n R_1^n\right]\right) - (n-2) - \Delta,$$

where

$$\Delta = \left|\mathsf{E}\left[R_1^1\right] - \mathsf{E}\left[R_1^n\right]\right| + \left|\mathsf{E}\left[R_2^1\right] - \mathsf{E}\left[R_2^2\right]\right| + \ldots + \left|\mathsf{E}\left[R_n^{n-1}\right] - \mathsf{E}\left[R_n^n\right]\right|,$$

is referred to as a measure of contextuality for cyclic systems of random variables. In Chapters 3 and 6, the minimum total variation of a quasi-coupling is used to define a measure of contextuality. These measures are two among several that have been proposed to measure contextuality (Abramsky, Barbosa, & Mansfield, 2017; Amaral, Duarte, & Oliveira, 2018; Kujala & Dzhafarov, 2019). A limitation of the total variation measure of contextuality is that it does not extend to a measure of noncontextuality. Kujala and Dzhafarov (2019) examined several proposed measures of contextuality, and found one (there called CNT_2) that does extend to a measure of noncontextuality (NCNT_2). Dzhafarov, Kujala, and Cervantes (in press) have shown that these measures, CNT_2 and NCNT_2, are closely related to quantity D for, respectively, contextual and noncontextual cyclic systems with binary random variables. More work is needed to describe the behavior of these measures for more general systems of random variables.

In the crowdsourcing experiments, and more generally in the between-subjects experiments that have searched for contextuality (Dzhafarov, Kujala, Cervantes, Zhang, & Jones, 2016; Dzhafarov, Zhang, & Kujala, 2015), the stimuli were questions to be answered in one of two contexts. In these experiments a repeated presentation of a question to the same person cannot be viewed as a repeated recording of the same random variable. However, if the same question is repeatedly posed to someone, the person would most likely remember them and their previous answers. Thus, to appropriately use a within-subject paradigm to study contextuality, the stimuli should not have any distinguishing characteristics by which they can be remembered. This was the case with the psychophysical experiment considered in Chapters 2 and 3; hence, we revisited its design for the empirical exploration of the theory of contextuality for systems of random variables that includes categorical variables (Dzhafarov, Cervantes, & Kujala, 2017). The experiment, presented in Chapter 7 (Cervantes & Dzhafarov, 2019), is also the first one that demonstrates contextuality for responses that are not dichotomous. This was shown by finding that the nominal dominance condition (see Theorem 6.4.6) was violated in the data of each participant, with very high statistical reliability established by bootstrapping. In this experiment, the change of the questions from detecting eccentricty to identifying it with five possible answers led to a contextual system rather than a noncontextual one. As noted in the discussion of Chapter 7, direct influences must be present to break the nominal dominance relation and allow us to establish contextuality "easily."

The generalization of the Contextuality-by-Default theory to arbitrary systems of categorical random variables satisfies the three desiderata for any generalization of a theory of contextuality that were mentioned in the Introduction, and in Chapter 5. First, the theory specializes to the traditional theory of contextuality when applied to consistently connected systems. Second, the maximal couplings of each pair of variables for the same content is unique. And third, any subsystem of a noncontextual system remains noncontextual. In addition to these properties, with the proposed generalization, we identify the characteristics that a system of random variables needs to have for the coarse-graining property to be satisfied: that a system obtained from coarse-graining variables of a noncontextual system is also noncontextual. In this paper, we also find a simple condition (the nominal dominance theorem) that may be used to detect contextuality in systems with categorical random variables when all coarse-grainings are deemed of interest.

A broader issue that is highlighted in this paper and, to a lesser extent, in the paper reproduced in Chapter 2 (Cervantes & Dzhafarov, 2017a; Dzhafarov et al., 2017) is that a system of random variables represents an empirical situation, and there is a one-to-many, rather than a on-to-one, mapping from the empirical situation to systems of random variables. Recall the sequence of steps that constitute the contextuality analysis of an empirical situation (Chapter 6):

$$\begin{array}{c} \text{empirical} \\ \text{measurements} \end{array} \longrightarrow \begin{array}{c} \text{initial} \\ \text{system of rvs} \end{array} \longrightarrow \begin{array}{c} \text{expanded} \\ \text{system of rvs} \end{array} \longrightarrow \begin{array}{c} \text{canonical/split} \\ \text{representation} \end{array}$$

The final canonical/split representation of the system of random variables is uniquely determined by the expanded system. The other two steps depend on one's choices regarding what aspects of the empirical situation one wishes to include in the system.

The expanded system of random variables generally has new contents added to the system via functions applied to the connections in the initial system. What contents are added to the system is not forced by the initial system. For instance, the expanded system may include *all* the coarsenings of some variables, if one considers that no coarsening of those variables should turn a noncontextual system into a contextual one. The expanded system may include functions of several variables, such as all the random vectors of variables responding to the same subset of contents,

$c = 1$	R^1_α	R^1_β	(R^1_α, R^1_β)
$c = 2$	R^2_α	R^2_β	(R^2_α, R^2_β)
	α	β	$\alpha\beta$

Fig. 8.4. Expanded system of random variables obtained by including the joining of contents α and β for the system in Figure 8.1.

as in the example in Figure 8.4. Expanded systems such as this one would reflect the desideratum that whenever a set of contents are measured together, the joint distributions of the corresponding random variables should also be maximally coupled.

In the initial empirical system, one also faces a non-unique choice as to what contents one decides to measure, and how the measurements will be recorded as random variables. It also includes what aspects of the measurement conditions one decides or is able to record as contexts. For instance, in the double-detection experiment described in Chapter 2, the choice of what defines a content may lead to the system that was analysed in Chapter 3 or to one of the 'redefined systems' analysed in Chapter 2. In the identification experiment, there was the choice to ask the participants to report the location of each dot rather than simply detect each dot's eccentricity, as well as the decision to record the location they reported instead of just recording, say, if their identification was correct or incorrect. We have proposed to represent any system of measurements in a canonical form and to consider the system contextual if and only if its canonical representation is contextual. However, by the above considerations, starting with the same set of empirical measurements, there are many different canonical representations. Each of them will reflect the aspects that led to that specific representation, and contextuality or lack thereof is a property of that canonical system.

With the above caveats, psychology shares mandatory use of random variables with quantum physics: stochasticity of responses in most areas of psychology is inherent, it cannot be significantly reduced, let alone eliminated, by progressively greater control of stimuli and conditions. The role of contextuality therefore can be expected to be similar. In quantum physics, contextuality analysis is not a predictive model competing with other models. When one considers a model to explain some data, contextuality analysis can help only in the trivial sense: as with any other property of the data, if contextuality or noncontextuality of them is established, a model is to be rejected if it fails to predict this property.

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Research areas and interests

- Mathematical psychology
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- Theory of measurement
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- Philosophy of science
- Behavioral decision theory
- Reproducible Science
- Psychometrics
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Affiliations

2016-present	Society for Mathematical Psychology
2009-present	Colegio Colombiano de Psicólogos
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Service

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2019	Reviewer of applications to doctoral and research visit scholarships for Fulbright Colombia
2019	Journal manuscript reviewer for Journal of Mathematical Psychology (ISSN 0022-2496).
2019	Journal manuscript reviewer for Philosophical Transactions of the Royal Society A (ISSN 1471-2962).
2018	Organization, along Ehtibar Dzhafarov and Maria Kon, of the 2018 Purdue Winer Memorial Lectures - Probability and Contextuality held November 9-12.

2017	Journal manuscript reviewer for Avances en Psicología Latinoamericana (ISSN 1794-4724).
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2013-2014	Interinstitutional task force for defining the instrument and protocol for assessing the quality of tests in current use in Colombia.
2013	Interinstitutional (Ministry of Education, National Administrative Department of Statistics –DANE– and National Institute for Education Assessment –ICFES–) task force for conforming the national schools directory in Colombia.
2006-2011	Journal manuscript reviewer for Avances en Medición (ISSN 1692-0023).

Skills

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Spanish	Native	**English**	Excellent (C2)
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Programming languages and specialized software

R	Excellent	**LaTeX**	Very good
SPSS	Good	**perl**	Good
Linux	Very good	**python**	Fair

MOOCs

2015	Programming for Everybody (Python). University of Michigan.
2015	An Introduction to Interactive Programming in Python (Part 1). Rice University.
2015	Reproducible Research. Johns Hopkins University.
2013	Game theory. Stanford University.

Honors & Awards

2017, 2018, 2020	**Graduate School Summer Research Grant Award**, Purdue University, USA	
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Cervantes, V. H. & Dzhafarov, E. N. (2019, July). Contextuality in human decision making. *Frontiers of Quantum and Mesoscopic Thermodynamics,* Prague, Czech Republic

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Cervantes, V. H. & Dzhafarov, E. N. (2016, August). Exploration of contextuality in a psychophysical double-detection experiment. *Society for Mathematical Psychology Meeting,* New Brunswick, NJ, USA

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