

ORTHOGONAL POLYNOMIALS ON S-CURVES ASSOCIATED WITH GENUS  
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To my parents  
Bassam and Lina

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## ABSTRACT

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We consider polynomials  $P_n(z)$  satisfying orthogonality relations

$$\int z^k P_n(z) \rho(z; N) d\mu(z) = 0 \quad \text{for} \quad k = 0, 1, \dots, n-1$$

where the measure  $\mu$  is, in general, a complex-valued Borel measure supported on subsets of the complex plane. In our considerations, we will focus on measures of the form  $d\mu(z) = \rho(z)dz$  where the function  $\rho$  may depend on other auxiliary parameters. Much of the asymptotic analysis is done via the Riemann-Hilbert problem and the Deift-Zhou nonlinear steepest descent method, and relies heavily on notions from logarithmic potential theory.

# 1. INTRODUCTION

The story of orthogonal polynomials is an extremely old one, and spans over many types of orthogonality (classical, Sobolev, non-Hermitian, discrete, etc.) and enjoys many applications in approximation theory, mathematical physics, and numerical methods, to name a very few. In this dissertation, we will study orthogonal polynomials satisfying non-Hermitian orthogonality conditions with respect to a variety of measures supported in the complex plane.

Unlike classical polynomials, the degree of polynomials orthogonal w.r.t a non-positive weight up to order  $n$  may not necessarily be  $n$ . This problem comes to the fore when the support of the measure of orthogonality is a so-called *S-contour* associated with a Riemann surface of genus  $g > 0$ , and can pose a challenge as far as asymptotic analysis is concerned. The main objective is to attain large-degree asymptotic formulas for the first nontrivial appearance of the aforementioned obstacles; polynomials on S-contours associated with genus one Riemann surfaces.

## Outline

The rest of this document is organized in the following fashion: in Chapter 2, we introduce orthogonal polynomials via Padé approximants, and discuss their most basic properties. Chapter 3 is a very brief survey of results regarding the convergence Padé approximants in a variety of senses. Chapter 4 is a bird's-eye-view introduction to the method we will be heavily relying on to achieve asymptotic results: Riemann-Hilbert analysis. In Chapter 5, we apply this method of analysis to a specific family of Jacobi-type orthogonal polynomials whose degrees exhibit a novel mechanism of degeneration, and state the relevant results. Proofs of these statements are provided in Chapter 6, and rely on multiple formulas listed in Appendix A. Chapter 7 is devoted

to a different type of orthogonality, so-called *varying* orthogonality. The analysis of these polynomials requires some notions from potential theory in external fields, which are introduced there. The analysis of the corresponding orthogonal polynomials is provided in Chapter 8. Finally, we consider polynomials corresponding to a cubic polynomial potential (compared to a degree 1 in Chapter 7) in Chapter 9, with analysis deferred to Chapter 10.

## 2. PADÉ APPROXIMANTS AND ORTHOGONALITY

Consider the following problem: given a function  $f(z)$ , the objective is to find a rational function with prescribed type that agrees with  $f$  “as much as possible.” Roughly speaking, one can view this as an attempt to analytically continue a function of form (2.1.1) into a larger domain and via these rational approximants. These objects have fallen in and out of vogue often; they were first considered by Georg Frobenius, who derived what are now known as Frobenius identities, connecting approximants of different orders and offering an effective method for explicit computation. Later on, Charles Hermite used a certain generalization of these rational functions (so-called Hermite-Padé approximants) to prove the transcendence of  $e$ . His student, Henri Padé, arranged these approximants, now known as *Padé approximants* in tables (Padé tables) and studied structural properties of these tables. Much of this and more on computation and application of Padé approximants can be found in Baker and Graves-Morris books [1, 2]. We will be mainly interested in questions of convergence of Padé approximants.

In this chapter, we introduce the precise formulation of the problem described above, and connect it with orthogonal polynomials. To move on with our study, we will need to recall some basic facts about orthogonal polynomials and their various properties and how they apply to Padé approximation. The theory of orthogonal polynomials is an old one with deep roots, and so we will confine the discussion to matters relevant in the forthcoming chapters. For a general reference on orthogonal polynomials, see the classic book of Szegő [3] and the more recent book by Stahl and Totik [4]. We then state and prove Markov’s theorem, the first (and one of the few) results regarding convergence of approximants.

## 2.1 Padé Approximants at Infinity

Consider the function

$$f(z) = \sum_{i=0}^{\infty} \frac{\mu_i}{z^{i+1}}. \quad (2.1.1)$$

We seek polynomials  $P_n, Q_n$  with  $\deg P_n \leq n$  so that

$$R_n(z) := (P_n f - Q_n)(z) = \mathcal{O}\left(\frac{1}{z^{n+1}}\right). \quad (2.1.2)$$

Equation (2.1.2) imposes  $2n + 1$  conditions on  $2n$  variables, and hence always has a nontrivial solution, and no solution is such that  $P_n \equiv 0$ . We let  $P_n$  denote the monic polynomial.

**Definition.** The  $n^{\text{th}}$  **diagonal Padé approximant**

$$[n/n]_f(z) := \frac{Q_n(z)}{P_n(z)} \quad (2.1.3)$$

While the pair  $(Q_n, P_n)$  that solves (2.1.2) may not be unique (for one, we can multiple both by constants, but more serious non-uniqueness can arise), the following still holds

**Proposition 2.1.1.** *The ratio  $Q_n/P_n$  is unique.*

*Proof.* Suppose there exists two pairs of solutions to (2.1.2),  $(Q_n, P_n)$  and  $(\tilde{Q}_n, \tilde{P}_n)$ , then

$$\begin{aligned} (P_n f - Q_n)(z) &= \frac{c}{z^{n+1}} + \dots, \\ (\tilde{P}_n f - \tilde{Q}_n)(z) &= \frac{\tilde{c}}{z^{n+1}} + \dots. \end{aligned}$$

where  $c, \tilde{c}$  may vanish (this corresponds to approximants *over-interpolating* at infinity). Eliminating  $f$  and noting that  $\deg P_n, \deg \tilde{P}_n \leq n$  yields

$$Q_n \tilde{P}_n - \tilde{Q}_n P_n = \frac{k}{z} + \dots$$

but, since the left hand side is polynomial, we conclude that  $Q_n \tilde{P}_n - \tilde{Q}_n P_n \equiv 0 \implies Q_n/P_n = \tilde{Q}_n/\tilde{P}_n$  □

While we will focus most of our attention on the diagonal Padé approximants defined above, It is still important to make the following definition:

**Definition.** Let  $P_n, Q_m$  be polynomials with  $\deg P_n \leq n$ ,  $\deg Q_m \leq m$  that satisfy

$$(P_n f - Q_m)(z) = \mathcal{O}\left(\frac{1}{z^{m+1}}\right). \quad (2.1.4)$$

Then the ratio

$$[m/n]_f(z) := \frac{Q_m(z)}{P_n(z)} \quad (2.1.5)$$

is the **Padé approximant of type (m, n)**.

Much like the diagonal case, equation (2.1.4) imposes  $n + m + 1$  conditions on  $n + m$  variables, and hence always has a nontrivial solution, and no solution is such that  $P_n \equiv 0$ . Arranging all Padé approximants into a table yields the **Padé Table**. For more on this and the structure of this table, see [1].

In the special case where  $\{\mu_i\}_{i=0}^\infty$  is a sequence of numbers that coincides with the set of moments of some *compactly supported* Borel measure  $\mu$ , i.e.

$$\mu_i = \int x^i d\mu(x) \quad \text{for } i = 0, 1, \dots$$

then we can write

$$f(z) = \int \frac{d\mu(x)}{z - x} \quad (2.1.6)$$

Properties of polynomials  $P_n$  will depend on the nature of this measure  $\mu$ , with the most important property (for us) being orthogonality. Suppose (2.1.1) converges in  $\{|z| > R\}$  for some  $R > 0$  and let  $\Gamma \subset \{|z| > R\}$  be any curve encircling infinity. Then, it follows from (2.1.6), an interchange of integrals, and an application of Cauchy's theorem that for  $k = 0, 1, \dots, n - 1$

$$0 = \int_{\Gamma} x^k (P_n f - Q_n)(x) dx = \int_{\Gamma} x^k (P_n f)(x) dx = \int z^k P_n(z) d\mu(z). \quad (2.1.7)$$

Hence, we arrive at **non-hermitian orthogonality relations**, dubbed so for the lack of a conjugation,

$$\int z^k P_n(z) d\mu(z) = 0 \quad \text{for } k = 0, 1, \dots, n - 1. \quad (2.1.8)$$



## 2.2 The Case of a Positive Measure on $\mathbb{R}$

The problem of existence of a measure as in (2.1.6) and its uniqueness goes by many names, but the case of a positive measure supported on  $\mathbb{R}$  goes by the **Hamburger moment problem**. We will not be concerned with this problem too much, but for more see [5, Chapter 2 Section 7] or [6, Chapter 2] (amongst many others). We note the resolution of the existence portion of the Hamburger problem.

**Theorem 2.2.1.** *Given a sequence of real numbers  $\{\mu_i\}_{i=1}^\infty$ , then a solution to the Hamburger problem exists if and only if the **Hankel matrices**  $[\mu_{i+j}]_{i,j=0}^n$  are positive definite for all  $n \in \mathbb{N}$ .*

With this in mind, it follows that

$$D_n := \det [\mu_{i+j}]_{i,j=0,1,\dots,n} = \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n+1} \end{vmatrix} > 0 \quad (2.2.1)$$

and we can construct

$$P_n(x) = \frac{1}{D_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix}, \quad P_0(x) = 1 \quad (2.2.2)$$

this formula makes orthogonality clear while showing that  $\deg P_n = n$ .

**Definition.** The **orthonormal polynomials**, denoted with the lowercase  $p_n(x)$ , are polynomials that satisfy

$$\int_{-\infty}^{\infty} p_m(x) p_n(x) d\mu(x) = \delta_{mn} \quad (2.2.3)$$

where  $\delta_{mn} = 0$  or  $1$  according to whether  $m \neq n$  or  $m = n$ , respectively.

These polynomials are related to the monic polynomials by a normalizing factor  $k_n P_n(x) = p_n(x)$ . Looking at (2.2.2) and (2.1.8), we see that

$$k_n = \sqrt{\frac{D_{n-1}}{D_n}} \quad \text{and} \quad \frac{1}{k_n^2} = \int_{-\infty}^{\infty} P_n^2(x) \, d\mu(x). \quad (2.2.4)$$

In the following subsections, we follow [3] to highlight the main properties of  $P_n$  when  $\mu$  is a positive measure.

### 2.2.1 Three-Term Recurrence Relation

A celebrated and well-studied property of orthogonal polynomials is the **three-term recurrence relation**

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \alpha_n^2 P_{n-1}(x) \quad (2.2.5)$$

where

$$\alpha_n = \frac{k_{n-1}}{k_n} \quad \text{and} \quad \beta_n = k_n^2 \cdot \int_{-\infty}^{\infty} x P_n(x) \, d\mu(x). \quad (2.2.6)$$

This can be verified by noting that the polynomial  $xP_n(x)$  can be written as a linear combination of  $\{P_k(x)\}_{k=0}^{n+1}$ , and applying (2.1.8) to solve for the coefficients yields (2.2.5). In fact, it was shown by Favard in [7] that the converse also holds: given  $\alpha_n, \beta_n \in \mathbb{R}$ , polynomials defined by (2.2.5) and initial conditions  $\tilde{P}_{-1}(x) \equiv 0$ ,  $\tilde{P}_0(x) \equiv 1$  form a family of polynomials orthogonal with respect to some positive measure  $\tilde{\mu}$ .

Applying the above result to the expression  $p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)$  yields the **Christoffel - Darboux formula**

$$\sum_{i=0}^n p_i(x)p_i(y) = \frac{k_n}{k_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x - y} \quad (2.2.7)$$

where, by considering the limit  $x \rightarrow y$ , we arrive at a special case

$$\sum_{i=0}^n p_i^2(x) = \frac{k_n}{k_{n+1}} (p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)) \quad (2.2.8)$$

### 2.2.2 Zeros

Observe that the zeros of  $P_n(x)$  all lie in the convex hull of  $\text{supp}(\mu)$ . Indeed, from the orthogonality condition (2.1.8) we have

$$\int_{\text{supp}(\mu)} P_n(x) \, d\mu(x) = 0$$

which implies that  $P_n(x)$  must change signs at least once within the interval. Label the zeros  $x_l$ ,  $l = 1, \dots, n$ . Then we have that  $x_l$  is in the convex hull of  $\text{supp}(\mu)$  for  $l \leq n$ . Now suppose that  $l < n$ , then this contradicts the relation

$$\int_{\text{supp}(\mu)} P_n(x)(x - x_1) \cdots (x - x_l) \, d\mu(x) = 0$$

since the integrand has constant sign.

In fact, when  $\text{supp}(\mu) = [a, b]$  (i.e.  $\text{supp}(\mu)$  is convex), the zeros of  $P_n(x)$ ,  $P_{n+1}(x)$  satisfy the following **interlacing property**:

**Theorem 2.2.2.** *Let  $a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = b$  be the zeros of  $P_n(x)$ , then in each interval  $[x_i, x_{i+1}]$  lies exactly one zero of  $P_{n+1}(x)$ .*

*Proof.* Consider two consecutive zeros of  $p_n(x)$ ,  $x_i$  and  $x_{i+1}$ ,  $i \in [1, n-1]$ . Then  $P'_n(x_i)P'_n(x_{i+1}) < 0$ . Furthermore, it follows from (2.2.8) that  $P'_n(x_i)P_{n+1}(x_i) < 0$  and  $P'_n(x_{i+1})P_{n+1}(x_{i+1}) < 0$ . Taking the product of the left hand side of the last two inequalities yields  $P_{n+1}(x_i)P_{n+1}(x_{i+1}) < 0$ , which implies the existence of an odd number of zeros of  $P_{n+1}$  in the interval  $[x_i, x_{i+1}]$ . Furthermore, observe that  $P'_n(x_n) > 0$  and so,  $P_{n+1}(x_n) < 0$ , but since  $P_n(x) \rightarrow +\infty$  as  $x \rightarrow +\infty$ , we have that  $P_{n+1}(x)$  must possess one zero between  $x_n$  and  $x_{n+1}$ . Similar argument yields at least one zero of  $P_{n+1}$  in  $[x_0, x_1]$ , and since  $P_{n+1}(x)$  has only  $n+1$  zeros, the result follows.  $\square$

### 2.2.3 Gauss-Jacobi Quadrature Formula

In fact, orthogonality with respect to a positive measure supported on an interval grants us the following quadrature formula:

**Theorem 2.2.3.** *Let  $a < x_1 < x_2 < \cdots < x_n < b$  be the zeros of  $P_n(x)$ . Then,  $\exists \lambda_i$ ,  $i = 1, 2, \dots, n$  so that*

$$\int_a^b Q(x) \, d\mu(x) = \lambda_1 Q(x_1) + \cdots + \lambda_n Q(x_n) \quad (2.2.9)$$

*for any polynomials  $Q$  with  $\deg Q \leq 2n - 1$ . In fact,  $\lambda_i$  are given by*

$$\lambda_i = \int_a^b \left( \frac{P_n(x)}{P'_n(x_i)(x - x_i)} \right)^2 \, d\mu(x) > 0 \quad (2.2.10)$$

*and satisfy*

$$\lambda_1 + \cdots + \lambda_n = \mu([a, b]). \quad (2.2.11)$$

*Proof.* We begin by constructing the Lagrange interpolating polynomial  $L(x)$  of degree  $n$ , which agrees with  $Q$  at nodes  $x_1, \dots, x_n$ . This can be written explicitly as

$$L(x) = \sum_{i=1}^n Q(x_i) \frac{P_n(x)}{P'_n(x_i)(x - x_i)} := \sum_{i=1}^n Q(x_i) l_i(x).$$

It follows then that  $Q - L$  is divisible by  $P_n$ , and hence  $(Q - L)(x) = (P_n \cdot r)(x)$  where  $\deg r \leq n - 1$ . Hence,

$$\begin{aligned} \int_a^b Q(x) \, d\mu(x) &= \int_a^b L(x) \, d\mu(x) + \int_a^b P_n(x) r(x) \, d\mu(x) \\ &= \int_a^b L(x) \, d\mu(x) = \sum_{i=1}^n Q(x_i) \lambda_i \end{aligned}$$

where we write

$$\lambda_i := \int_a^b \frac{P_n(x)}{P'_n(x_i)(x - x_i)} \, d\mu(x).$$

To see (2.2.10), (2.2.11), we simply need to apply (2.2.9) to the polynomials  $Q(x) = l_i^2(x)$ ,  $Q(x) \equiv 1$ , respectively.  $\square$

#### 2.2.4 Markov's Theorem

We now state and prove the first result regarding the convergence of Padé approximants

**Theorem 2.2.4.** *Let  $\mu$  be a positive Borel measure with  $\text{supp}(\mu) = [a, b] \subset \mathbb{R}$  and  $f$  be as in (2.1.6). Then*

$$[n/n]_f(z) \rightarrow f(z) \quad \text{as } n \rightarrow \infty$$

*locally uniformly<sup>1</sup> in  $\overline{\mathbb{C}} \setminus \text{supp}(\mu)$*

*Proof.* The first and main observation is that we can write

$$[n/n]_f(z) = \sum_{i=1}^n \frac{\lambda_i}{z - x_i} \quad (2.2.12)$$

where  $\lambda_i$ 's are as in Theorem 2.2.3. Indeed, we can write for some  $\lambda_i^*$

$$[n/n]_f(z) = \sum_{i=1}^n \frac{\lambda_i^*}{z - x_i} \implies \sum_{i=0}^{\infty} \frac{1}{z^{i+1}} \left( \sum_{j=1}^n \lambda_j x_j^i \right) \quad (2.2.13)$$

Since, by definition,

$$(f - [n/n]_f)(z) = \mathcal{O}\left(\frac{1}{z^{2n+1}}\right)$$

we conclude that

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \cdots & \ddots & \vdots \\ x_1^n & x_2^n & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} \lambda_1^* \\ \lambda_2^* \\ \vdots \\ \lambda_n^* \end{pmatrix} = \begin{pmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}.$$

However, by Theorem 2.2.3, the same equation holds with  $\lambda_i^*$  replaced with  $\lambda_i$ . Since the above matrix is a Vandermonde and  $x_i$ 's are distinct (zeros of  $P_n$  are simple, see Section 2.2.2), it is invertible and we must have  $\lambda_i^* = \lambda_i$  for  $i = 1, 2, \dots, n$ .

Let  $\mu_n := \sum_{i=1}^n \lambda_{n,i} \delta_{x_{n,i}}$  where we added  $n$  to the subscript to emphasize the dependence on  $n$ . WLOG, suppose  $\mu([a, b]) = 1$  (otherwise, all  $\epsilon$ 's below need to be adjusted by a factor of  $1/\mu([a, b])$ ). Then, it follows from the above computation that

$$[n/n]_f(z) = \int \frac{d\mu_n}{z - x}$$

---

<sup>1</sup>by “locally uniform convergence” on a domain  $D$  we mean uniform convergence on compact subsets of  $D$ .

and since for  $z \in \mathbb{C} \setminus [a, b]$  the function  $\frac{1}{z - x}$  is continuous (in  $x$ ) on  $[a, b]$ , showing convergence of  $[n/n]_f$  is equivalent to showing  $\mu_n \rightarrow \mu$  weakly. The later follows from the density of polynomials in  $C([a, b])$  (Stone - Weierstrass' theorem) and the identity

$$\int_a^b T(x) \, d\mu(x) = \int_a^b T(x) \, d\mu_n(x)$$

granted to us by Theorem 2.2.3. Uniformity follows by observing that, given any compact subset  $K \subset \mathbb{C} \setminus [a, b]$ , the family  $[n/n]_f$  is analytic and uniformly bounded on  $K$ , and hence normal.  $\square$

### 2.3 The Case of a Complex-Valued Measure

While orthogonal polynomials on  $\mathbb{R}$  will be our spiritual guides (and offer a good source of computable examples), when allowing for complex-valued measures we will concern ourselves with Borel measures supported on compact subsets of  $\mathbb{C}$ , not just  $\mathbb{R}$ . With the assumption of positivity of the measure dropped, definition (2.2.2) no longer guarantees as many properties as in the previous case. First and foremost, we no longer have  $\deg P_n = n$ , since the associated Hankel determinants are not necessarily positive-definite (in fact, the formula that appears in (2.2.2) must now be considered with extreme care). The best one can say is that  $\deg P_n \leq n$ .

*Example (Bad Example).* Consider polynomials orthogonal with respect to the measure  $d\mu(x) = w(x) \, dx$ ,  $x \in [0, \infty)$  where

$$w(x) = \sin(2\pi \log x) \exp(-\log^2(x)) .$$

Using the substitution  $u = \log x - \frac{n+1}{2}$ , one immediately arrives at the identity

$$\int_0^\infty x^n w(x) \, dx = 0 \quad \text{for } n = 0, 1, 2, \dots$$

In particular, this implies that the function  $P_0 \equiv 1$  satisfies (2.1.8) for any  $n \in \mathbb{N}$ , and so does any polynomial for that matter!

**Proposition 2.3.1.** *Let  $\mu$  be a complex-valued Borel measure supported on a subset of  $\mathbb{C}$  and for which all moments exist. Then, the minimal-degree monic polynomial  $P_n(z)$  satisfying*

$$\int z^k P_n(z) d\mu(z) = 0 \quad \text{for } k = 0, 1, \dots, n-1 \quad (2.3.1)$$

*is unique. Here and in all that follows, integral is taken over  $\text{supp}(\mu)$  unless it is stated otherwise.*

*Proof.* Suppose to the contrary that there exists another monic polynomial  $\tilde{P}_n(z)$  with  $\deg P_n = \deg \tilde{P}_n$  that satisfies (2.1.8). Then, the polynomial  $(P_n - \tilde{P}_n)(z)$  also satisfies (2.1.8), but  $\deg(P_n - \tilde{P}_n) < \deg P_n$ , contradicting minimality.  $\square$

For the rest of this document, we will denote by  $P_n$  this minimal degree polynomial. The loss of positivity costs us all the information we had about zeros, we cannot even count the number of zeros of  $P_n$ . However, Figure 2.1 suggests that zeros do have some peculiar structure. It is reasonable to think that in the case of algebraic functions,

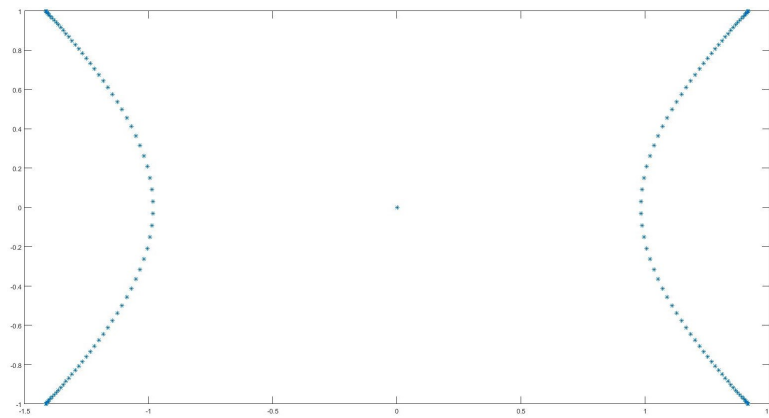


Fig. 2.1. Zeros of polynomials  $P_{150}$  associated with  $f_1(z) = \sqrt{1 - \frac{2}{z^2} + \frac{9}{z^4}}$

the structure of the zero-attracting curve depends on the location of branch points, however, it turns out that this not all, see Figure 3.1 in Chapter 3 for example.

In the next chapter, we will explore questions regarding the location of zeros, convergence of approximants, and degree of  $P_n$ .

### 3. CONVERGENCE OF PADÉ APPROXIMANTS

In this chapter, we will briefly mention some of the early results on convergence of Padé approximants, starting with the earliest result by Robert de Montessus de Ballore's theorem [8]

**Theorem 3.0.1.** *Let  $f(z)$  be a function meromorphic in the disk  $|z| \leq R$  with  $n$  poles at distinct points  $z_1, z_2, \dots, z_n$  with*

$$0 < |z_1| \leq |z_2| \leq \dots \leq |z_n| < R.$$

*Let  $m_k$  be the multiplicity of the pole at  $z_k$  and  $M := \sum_{k=1}^n m_k$ , then*

$$f(z) = \lim_{L \rightarrow \infty} [L/M]_f(z)$$

*locally uniformly in  $\{z \mid |z| \leq R, z \neq z_k, k = 0, 1, \dots, n\}$ .*

For a proof and discussion, see [1]. After this, results focused on weaker notions of convergence of approximants, with the first result being that of John Nuttall [9], with later refinements by Jean Zinn-Justin and Christian Pommerenke. To move any further into the modern theory will require some notions from potential theory, which we will recall in this chapter as well, with the main references being [10] and [11]. We use the language of potential theory to introduce the body of work on now commonly known as Gonchar-Rakhmanov-Stahl (GRS) theory, named after Andrei Gonchar, Evgenii Rakhmanov, and Herbert Stahl.



### 3.1 Convergence in Measure

The first theorem is due to Nuttall.

**Theorem 3.1.1** (see [9]). *Let  $f(z)$  be a function meromorphic in a compact region  $D \subset \mathbb{C}$ . Then, given any  $\epsilon, \delta > 0$  and  $j \in \mathbb{Z}$ , there is  $N_0$  so that  $\forall N > N_0$ ,*

$$|[n/n + j]_f(z) - f(z)| < \epsilon$$

*for all  $z \in D_\epsilon$ , where  $m(D \setminus D_\epsilon) < \delta$ .*

Here we already begin to see potential theory seeping in, as the original proof relies on Pólya and Szegő's work in [12]. Later, Zinn-Justin generalized the result above to sequences  $[L_k/M_k]_f, k = 1, 2, \dots$  with the property that for any  $0 < \lambda < 1$ ,

$$\lambda < \frac{L_k}{M_K} < \frac{1}{\lambda}$$

The next upgrade of this type came from Pommerenke in [13], who allows for essential singularities, and refines the size of the exceptional set. To state his result, we will need some notions from potential theory.

### 3.2 Convergence in Capacity

**Definition.** Let  $\mu$  be a Borel measure with compact support in  $\mathbb{C}$ , then the **logarithmic energy** of  $\mu$  is defined to be

$$I(\mu) := - \int \int \log |z - x| \, d\mu(z) d\mu(x) := \int U^\mu(z) \, d\mu(x)$$

where  $U^\mu(z)$  is the **logarithmic potential** of  $\mu$ .

**Definition.** A set  $K$  is said to be **polar** if for every  $\mu$  as above supported on  $K$ ,  $I(\mu) = -\infty$ .

**Definition.** The **capacity** of a set  $K$  is

$$\text{cp}(K) = e^{\inf I(\mu)}$$

where the infimum is taken over all Borel probability measures  $\mu$ .

It is clear from this definition that a set is polar if and only if its capacity is zero, and this notion helps us gauge what sets are “close” to being polar. We say a property holds **quasi-everywhere (q.e.)** if it holds off of a polar set. In fact, due to the inequality (see [10, Theorem 5.3.5] for a proof)

$$m(K) \leq \pi(\text{cp}(K))^2, \quad (3.2.1)$$

a property that holds q.e. holds a.e., while the opposite is not true. For example, the usual Cantor set has positive capacity, but measure zero (see [14, Chapter 5, Section 6.4] for a proof).

We are ready to state Pommerenke’s result:

**Theorem 3.2.1** (see [13]). *Let  $K \subset \mathbb{C}$  be a compact set with  $\text{cp}(K) = 0$  and let  $f(z)$  be (single-valued and) meromorphic in  $\mathbb{C} \setminus K$ . Then, for  $\epsilon, \delta > 0, r > 1, 0 < \lambda < 1$  there exists  $m_0$  such that for  $m > m_0$  and  $\lambda \leq \frac{m}{n} \leq \frac{1}{\lambda}$ ,*

$$|[m/n]_f - f(z)| < \epsilon^m$$

when  $|z| \leq r, z \notin K_{mn}$  where  $\text{cp}(K_{mn}) < \delta$ .

**Definition.** A sequence  $f_n, n = 1, 2, \dots$ , is said to **converge to  $f$  in capacity** in a domain  $D$  if for any  $\epsilon > 0$  and every compact  $K \subset D \cap \mathbb{C}$ , we have

$$\lim_{n \rightarrow \infty} \text{cp}(\{z \in K \mid |(f - f_n)(z)| > \epsilon\}) = 0$$

That is to say, Theorem 3.2.1 implies the convergence in capacity of  $[m/n]_f$  to  $f$ . The requirement that  $f$  be single-valued is important here, and given a function  $f$  defined by its series at infinity as in (2.1.1), it is not necessarily clear whether or not the function is single-valued as we proceed by analytic continuation into the finite plane. However, we can handle a subclass of such functions.

**Definition.** A function  $f$  is said to belong to the **Stahl class**, denoted  $f \in \mathcal{S}$ , if

1.  $f$  has a continuation along any arc originating at infinity that belongs to some set  $\mathbb{C} \setminus E_f$ ,

2.  $\text{cp}(E_f) = 0$ ,
3. there exists points in  $\mathbb{C} \setminus E_f$  at which  $f$  possesses at least two distinct continuations.

**Definition.** Given  $f \in \mathcal{S}$ , a compact set  $K \subset \mathbb{C}$  is said to be **admissible** if  $\overline{\mathbb{C}} \setminus K$  is connected and  $f$  is meromorphic and single-valued there.

The following theorem is due to Herbert Stahl, and summarizes work done in [15–18]

**Theorem 3.2.2.** *Given  $f \in \mathcal{S}$ , there exists a unique admissible compact  $\Delta_f$  such that  $\text{cp}(\Delta_f) \leq \text{cp}(K)$  for any admissible compact  $K$  and  $\Delta_f \subset K$  for any admissible  $K$  satisfying  $\text{cp}(K) = \text{cp}(\Delta_f)$ . Furthermore, Padé approximants  $[n/n]_f(z)$  converge to  $f$  in logarithmic capacity in  $D_f := \mathbb{C} \setminus \Delta_f$ .*

The problem of finding a continuum containing a finite set of points, which would be the problem of finding  $\Delta_f$  when  $f$  were algebraic, is also known as **Chebotarev’s Problem**. The curves traced out by zeros of the orthogonal polynomials in Figure 2.1 are exactly those minimal capacity branch cuts, and Figure 3.1 below further shows how the monodromy of the function  $f$  dictates the connectedness of the zero-attracting curve.

### 3.2.1 Anatomy of $\Delta_f$

In fact, Stahl provides many characterizations of this compact set  $\Delta_f$  in his works cited above.

**Theorem 3.2.3.** *Let  $\Delta_f$  be as in Theorem 3.2.2. Then,*

$$\Delta_f = E_0 \cup E_1 \cup \bigcup \Delta_j$$

where  $E_0 \subset E_f$ ,  $E_1$  consist of isolated points to which  $f$  has continuation from the point at infinity leading to at least two distinct function elements, and  $\Delta_j$  are open analytic arcs.

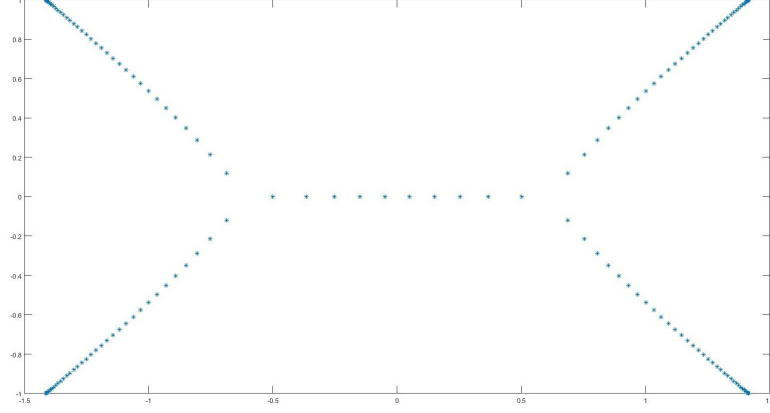


Fig. 3.1. Zeros of polynomials  $P_{150}$  associated with  $f_2(z) = \sqrt[4]{1 - \frac{2}{z^2} + \frac{9}{z^4}}$

**Definition.** The **Green function** of a domain  $D$ , denoted  $g_D(z; \infty)$  is defined by the following properties:

1.  $g_D(z; \infty)$  is non-negative and subharmonic in  $\mathbb{C} \setminus \{\infty\}$ , and harmonic in  $D \setminus \{\infty\}$
2.  $g_D(z; \infty) = \log |z| + \mathcal{O}(1)$  as  $z \rightarrow \infty$
3.  $g_D(z; \infty) = 0$  q.e. on  $\mathbb{C} \setminus D$

**Definition.** The (unique) **equilibrium measure** of a compact, nonpolar set  $K$  is a Borel measure satisfying

$$I(\mu_{eq}) = \inf I(\nu) \quad (3.2.2)$$

where the infimum is taken from the set of Borel probability measure on  $K$ . It is a matter of checking and applying Frostman's Theorem (see [10, Theorem 3.3.4] for example) to see that for a domain  $D \ni \infty$  we have

$$g_D(z; \infty) = \begin{cases} U^{\mu_{eq}}(z) - I(\mu_{eq}) & \text{for } z \in \mathbb{C} \\ \infty & \text{for } z = \infty \end{cases}$$

We this in mind, we characterize the set  $\Delta_f$  using the following **S-property**:

**Theorem 3.2.4.** *Let  $\partial/\partial \mathbf{n}^\pm$  be the one-sided normal derivatives on  $\bigcup \Delta_j$ . Then*

$$\frac{\partial g_{D_f}}{\partial \mathbf{n}^+} = \frac{\partial g_{D_f}}{\partial \mathbf{n}^-} \quad \text{on} \quad \bigcup \Delta_j \quad (3.2.3)$$

where the domain  $D_f$  was defined in Theorem 3.2.2. This is equivalent to

$$\frac{\partial U^\mu}{\partial \mathbf{n}^+} = \frac{\partial U^\mu}{\partial \mathbf{n}^-} \quad \text{on} \quad \bigcup \Delta_j \quad (3.2.4)$$

where  $\mu$  is the equilibrium measure of  $\partial D_f$

Observe that since  $\Delta_j$  are analytic arcs, it follows that all points in  $\Delta_j$  are regular (in the sense of Ransford, see [10, Theorem 4.2.2]) and so,  $g_{D_f}(z)$  is identically zero on  $\bigcup \Delta_j$ . In particular, combining this with the S-property implies that  $g_{D_f}(z)$  can be harmonically continued across each  $\Delta_j$  using the reflection principle.

The final characterization, and the one we will often resort to, requires the following definition.

**Definition.** Given a function  $Q(z)$  meromorphic on a domain  $D$ , a **trajectory** (respectively, **orthogonal trajectory**) of the quadratic differential  $Q(z)(dz)^2$  is a maximal arc along which

$$Q(z(t)) (z'(t))^2 > 0 \quad (\text{respectively, } Q(z(t)) (z'(t))^2 < 0)$$

for any smooth parametrization  $z(t) : [0, 1] \rightarrow \mathbb{C}$ . The **critical points** of the quadratic differential  $Q(z)dz^2$  are the zeros and poles of  $Q(z)$ . A trajectory is said to be **critical** if it is incident with a finite critical point, and said to be **short** if it is incident with only finite critical points. Finally, the **critical (orthogonal) graph** of the quadratic differential  $Q(z)dz^2$  is the union of all critical (orthogonal) trajectories.

**Theorem 3.2.5.** *Let*

$$h_{\Delta_f}(z) := 2\partial_z g_{D_f}(z),$$

where  $2\partial_z := \partial_x - i\partial_y$ . Then, function  $h_{\Delta_f}^2$  is holomorphic in  $D_f$ , has a zero of order 2 at infinity, and the arcs  $\Delta_j$  are critical trajectories of the quadratic differential  $-h_{\Delta_f}^2(z)dz^2$ .

Note that if the function  $f$  is algebraic, then the set of singularities  $E_0 \cup E_1$  is finite, and so  $h_{\Delta_f}^2$  is rational. In fact, the quadratic differential takes on the form

$$-\left(\frac{V}{A}\right)(dz)^2$$

where  $A = (z - a_1) \cdots (z - a_p)$  where  $a_i$ 's are some of the branch points of  $f$  and  $V$  is a uniquely determined polynomial.

### 3.3 Stronger Notions of Convergence

We saw that in the real case, Markov's theorem asserts convergence of  $[n/n]_f$  locally uniformly in some region of  $\mathbb{C}$ , and the natural question is: does a similar statement hold true in the general case? **The Baker-Gammel-Wills conjecture** asks the following:

Given a function  $f$  meromorphic in the unit disk, there exists an infinite sequence of natural numbers  $\Lambda(f)$  such that, along this subsequence,  $[n/n]_f$  converge to  $f$  locally uniformly on compact subsets of the disk omitting poles.

Unfortunately, this is not true. Doron Lubinsky observed in [19] that the Roger-Ramanujan continued fraction with a carefully chosen value of  $q$  is a counter-example. This continued fraction is meromorphic in the unit disk and is not algebraic. Later on, Viktor Buslaev in [20] found yet another counter-example, this time it was an algebraic function holomorphic in the unit disk. The main obstacle is the appearance of what are known in the approximation theory circles as “wandering” or “spurious” poles, see Figure 3.2. Wandering poles were observed earlier, see [21] for example. While a general convergence statement may not be attainable, restricting to Markov functions as in (2.1.6) allows us to not only prove convergence in certain cases, but even quantify the convergence.

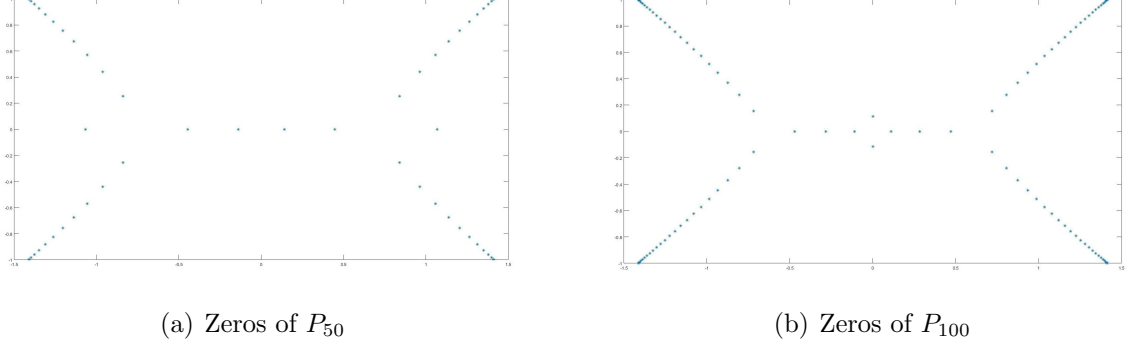


Fig. 3.2. Poles of approximants to  $f_2(z) = \sqrt[4]{1 - \frac{2}{z^2} + \frac{9}{z^4}}$ . Note the two zeros that appear to not lie on the zero-attracting curve.

### 3.4 Quantifying Markov's Theorem

The first result in this direction is due to Sergi Bernstein and Gabor Szegő.

**Theorem 3.4.1.** *Let  $q(x)$  be a positive polynomial on  $[-1, 1]$  and*

$$d\mu(x) = \frac{1}{\pi} \frac{dx}{q(x)\sqrt{1-x^2}}, \quad x \in [-1, 1],$$

*then*

$$p_n(z) = \gamma_n \left( \Psi_n^{(0)} + q\Psi_n^{(1)} \right) (z) \quad (3.4.1)$$

$$(\hat{\mu} - \pi_n)(z) = \frac{2}{\sqrt{z^2 - 1}} \frac{\Psi_n^{(1)}(z)}{\left( \Psi_n^{(0)} + q\Psi_n^{(1)} \right) (z)} \quad (3.4.2)$$

for all  $n > \frac{1}{2} \deg q$ , where  $p_n$  are the orthonormal polynomials associated with  $\mu$  and  $S_q$  is the unique holomorphic and non-vanishing function in  $\mathbb{C} \setminus [-1, 1]$  such that  $|S_q^\pm|^2 = q$  on  $[-1, 1]$ . Explicitly,

$$S_p^2(z) = \prod_{j=1}^{\deg q} \frac{z - z_j}{\Phi(z)} \cdot \frac{1 - \Phi(z)\overline{\Phi(z_j)}}{\Phi(z) - \Phi(z_j)} \quad (3.4.3)$$

where  $\Phi(z) = z + \sqrt{z^2 - 1}$ ,  $\sqrt{z^2 - 1} = z + \mathcal{O}(1)$  as  $z \rightarrow \infty$ ,  $q(z) = \prod_{j=1}^{\deg q} (z - z_j)$ , and

$$\begin{cases} \Psi_n^{(0)}(z) &:= \Phi^n(z) S_q(z), \\ \Psi_n^{(1)}(z) &:= \frac{1}{(\Phi^n S_q)(z)} \end{cases} \quad \text{for } z \in \mathbb{C} \setminus [-1, 1].$$

Observe that this is an exact formula for polynomials  $p_n$ , not an asymptotic one. Later, Nuttall-Singh generalized this. To state their result, we make a definition motivated by Stahl's work (see Theorems 3.2.2, 3.2.3, 3.2.4)

**Definition.** A compact set  $\Delta$  with a connected complement is called an **algebraic S-contour** if

$$\Delta = E_0 \cup E_1 \cup \bigcup \Delta_j$$

where  $\Delta_j$ 's are open analytic arcs,  $E_0 \cup E_1$  is a finite sets,  $E_0$  consists of points that are the endpoint of exactly one  $\Delta_j$ , while  $E_1$  consists of points that are endpoints of at least 3  $\Delta_j$ 's.

**Theorem 3.4.2** (see [22]). *Let  $\Delta$  be an algebraic S-contour and  $E_\Delta \subset (E_0 \cup E_1)$  be the set of points with odd valence. Define*

$$w_\Delta^2(z) := \prod_{e \in E_\Delta} (z - e) \quad \text{where } z^{-g-1} w_\Delta \rightarrow 1 \quad \text{as } z \rightarrow \infty$$

Furthermore, let  $q$  be a non-vanishing polynomial on  $\Delta$  and

$$f_q(z) := \frac{1}{\pi i} \int_\Delta \frac{1}{x - z} \frac{dx}{q(x) w_\Delta^+(x)}.$$

Then,

$$(f_q - \pi_n)(z) = \frac{2}{w_\Delta(z)} \frac{\Psi_n^{(1)}(z)}{\left( \Psi_n^{(0)} + q \Psi_n^{(1)} \right)(z)},$$

where functions  $\Psi^{(j)}(z)$  are known as Baker-Akhiezer functions, and will be defined in Chapter 4.

Maxim Yattselev replaced  $1/q(x)$  with a holomorphic weight in [23], and later on, along with Laurent Baratchart in [24] and Alexander Aptekarev in [25] extended the result in a different direction, replacing  $w_\Delta$  with any algebraic function with branch points in *generic position*, i.e. points in  $E_0 \cup E_1$  have valence at most 3. The main difference between these results and Nuttall-Singh is that formulas for the orthogonal polynomials and error of approximation are not exact, but asymptotic. These are the so-called *strong asymptotics* of orthogonal polynomials. In this work we show



that we can allow for points of valence 4 at the price of considering possibly sparser subsequences of indices. In the next chapter, we state the precise problem at hand and introduce a certain Riemann surface  $\mathfrak{R}$  on which functions describing the asymptotics of polynomials  $P_n$  are naturally defined.

#### 4. THE RIEMANN-HILBERT PROBLEM

To study the large degree asymptotics of orthogonal polynomials, we will employ the nonlinear steepest descent analysis of Riemann-Hilbert problems (RHPs). The following will drive our analysis: suppose we are given a sufficiently smooth contour  $\Delta$  with some orientation and a jump matrix  $\mathbf{J}(s, n)$  defined for  $s \in \Delta$ , where  $n$  is a parameter. By solving a RHP, we mean (roughly speaking) finding a matrix-valued function  $\mathbf{M}$  that is analytic on  $\mathbb{C} \setminus \Delta$  satisfying **RHP- $\mathbf{M}$**

1.  $\mathbf{M}(z, n) = \mathbf{I} + \mathcal{O}\left(\frac{1}{z}\right)$  as  $z \rightarrow \infty$ ,
2.  $\mathbf{M}_+(s, n) = \mathbf{M}_-(s, n)\mathbf{J}(s, n)$  for  $s \in \Delta \setminus \{\text{end points} \cup \text{points with valence} \geq 3\}$

One is immediately faced with the question of existence of a solution. The following theorem asserts such existence in a specific setting. For a proof, see [26, Corollary 7.108] or [27, Theorem 5.1.5].

**Theorem 4.0.1.** *Let **RHP- $\mathbf{M}$**  be as above, and assume that for some fixed  $N$  and  $\epsilon > 0$  we have*

$$\|\mathbf{J} - \mathbf{I}\|_{L^2(\Delta) \cap L^\infty(\Delta)} \leq \frac{C}{n^\epsilon} \quad \text{for } n \geq N. \quad (4.0.1)$$

*Then, for  $n$  large enough, **RHP- $\mathbf{M}$**  is uniquely solvable, and*

$$\|\mathbf{M} - \mathbf{I}\| \leq \frac{C}{n^\epsilon(1 + |z|^{1/2})} \quad (4.0.2)$$

*holds locally uniformly on  $\overline{\mathbb{C}} \setminus \Delta$ .*

#### 4.1 Riemann-Hilbert Problem for Orthogonal Polynomials

Henceforth, we will consider polynomials  $P_n$  satisfying (2.1.8) with measure  $d\mu = \rho(z, n)dz$ ,  $z \in \Delta$  where  $\Delta$  is a finite union of smooth arcs and <sup>1</sup>  $|\rho(z, n)| \sim |z - e|^{\alpha_e}$ ,  $\alpha_e > -1$ , where  $e$  is an endpoint or cusp of  $\Delta$ , and are otherwise holomorphic in a neighborhood of  $\Delta$ . In this setting, the associated Padé approximants are defined by the relation (2.1.2) with

$$f(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{\rho(s)ds}{s - z}, \quad z \in \overline{\mathbb{C}} \setminus \Delta$$

**Lemma 4.1.1.** *Let polynomial  $P_n$  be as above. Furthermore, suppose  $P_n$  and  $R_{n-1}$  (see (2.1.2)) are such that*

$$\deg(P_n) = n \quad \text{and} \quad R_{n-1}(z) \sim z^{-n} \quad \text{as} \quad z \rightarrow \infty. \quad (4.1.1)$$

Let  $k_{n-1}$  be a constant such that  $k_{n-1}R_{n-1}(z) = z^{-n}[1 + o(1)]$  near infinity. Then the matrix

$$\mathbf{Y} = \begin{pmatrix} P_n(z) & R_n(z) \\ k_{n-1}P_{n-1}(z) & k_{n-1}R_{n-1}(z) \end{pmatrix} \quad (4.1.2)$$

solves the following RHP (denoted **RHP- $\mathbf{Y}$** ):

(a)  $\mathbf{Y}$  is analytic in  $\overline{\mathbb{C}} \setminus \Delta$  and  $\lim_{z \rightarrow \infty} \mathbf{Y}(z)z^{-n\sigma_3} = \mathbf{I}^2$ ;

(b)  $\mathbf{Y}$  has continuous traces on  $\Delta$  that satisfy  $\mathbf{Y}_+ = \mathbf{Y}_- \begin{pmatrix} 1 & \rho(s, n) \\ 0 & 1 \end{pmatrix}$ ;

---

<sup>1</sup>In what follows we write  $|g_1(z)| \sim |g_2(z)|$  as  $z \rightarrow z_0$  if there exists a constant  $C > 1$  such that  $C^{-1}|g_1(z)| \leq |g_2(z)| \leq C|g_1(z)|$  for all  $z$  close to  $z_0$ .

<sup>2</sup>Hereafter, we set  $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and  $\mathbf{I}$  to be the identity matrix.

(c)  $\mathbf{Y}$  behaves like

$$\mathbf{Y}(z) = \begin{cases} \mathcal{O} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } \alpha_e > 0, \\ \mathcal{O} \begin{pmatrix} 1 & \log |z - e| \\ 1 & \log |z - e| \end{pmatrix} & \text{if } \alpha_e = 0, \\ \mathcal{O} \begin{pmatrix} 1 & |z - e|^{\alpha_e} \\ 1 & |z - e|^{\alpha_e} \end{pmatrix} & \text{if } -1 < \alpha_e < 0, \end{cases} \quad \text{as } z \rightarrow e.$$

Conversely, if **RHP- $\mathbf{Y}$**  is solvable, then its solution necessarily has the form (4.1.2) and the polynomial  $P_n$  and the function  $R_{n-1}$  satisfy (4.1.1).

*Sketch of Proof.* In the forward direction, **RHP- $\mathbf{Y}$** (a) follows from (4.1.1), **RHP- $\mathbf{Y}$** (b) follows from the Plemelj-Sokhotski formulas (see [28]), while **RHP- $\mathbf{Y}$** (c) follows from the integral representation

$$R_n(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{(\rho \cdot P_n)(x)}{x - z} dx \quad (4.1.3)$$

and properties of Cauchy integrals (see [28, Section 8]). The converse direction follows from similar considerations and is shown in, for example, [25].  $\square$

*Remark 4.1.2.* Observe that the orthogonality relations we have described in Chapter 2, the weight of orthogonality  $\rho$  was assumed to be independent of  $n$ . However, such *varying orthogonality* appears naturally, and will be discussed extensively in Chapter 7.

This while this is not a RHP of the type discussed in the beginning of this chapter, once the initial RHP is established, the analysis follows a series of transformations to arrive at a “small norm problem.” The following are the highlights.

## 4.2 First Transformation

To begin solving this problem, one normalizes the behavior of  $\mathbf{Y}$  at infinity in the following fashion

$$\mathbf{T}(z) := \mathbf{Y}(z) \cdot \exp(-ng(z)\sigma_3), \quad (4.2.1)$$

where the function  $g$ , hereafter the **g-function** is chosen so that  $\mathbf{T} = \mathbf{I} + \mathcal{O}\left(\frac{1}{z}\right)$ . For this to happen,  $g$  must

1. be analytic on  $\mathbb{C} \setminus \Delta$ ,
2. be bounded on  $\Delta$ ,
3. satisfy  $g(z) \sim \ln z$  as  $z \rightarrow \infty$ .

It follows from **RHP- $\mathbf{Y}$** (b) that,

$$\mathbf{T}_+(s) = \mathbf{T}_-(s) \begin{pmatrix} e^{-n(g_+ - g_-)(s)} & \rho(s, n)e^{n(g_+ + g_-)(s)} \\ 0 & e^{n(g_+ - g_-)(s)} \end{pmatrix} \quad (4.2.2)$$

Since the eventual goal is to arrive at a RHP whose jumps is close to  $\mathbf{I}$  for large  $n$ , (4.2.2) forces the requirements

4.  $(g_+ - g_-)(s) \in i\mathbb{R}$  for  $s \in \Delta$
5. There is a constant  $\ell$  (known as the **Robin** or **modified Robin Constant**) such that  $\rho(s, n)e^{n(g_+ + g_- - \ell)(s)}$  is bounded as  $n \rightarrow \infty$ .

Requirement 4 can be viewed as a compromise: we cannot make both diagonal entries small simultaneously, and we will settle for oscillatory entries, and enforce requirement 5 to handle the off-diagonal term. Observe that the freedom of introducing  $\ell$  is allowed to us by simply modifying the above transformation to  $\mathbf{T} = e^{(n\ell\sigma_3/2)}\mathbf{Y}e^{(-n(g(z)-\ell/2))}$ .

### 4.3 Opening the Lenses

To take care of the oscillatory diagonal terms, the strategy is to consider contours encircling each arc of  $\Delta$  (see Figure 6.1, for example) and transfer the jump onto those via factorizations in the spirit of

$$\begin{pmatrix} a \cdot b & b \\ 0 & b/a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1/a & 1 \end{pmatrix} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \quad (4.3.1)$$

One chooses the lenses above judiciously so as to guarantee that  $a$  is decaying on one “lip” while  $1/a$  is decaying on the other. This of course heavily relies on the  $g$  function and its properties, and this type of identity will be highlighted in Chapter 8.

### 4.4 Solving a Riemann-Hilbert Problem

Since the first and last matrices on the right hand side of (4.3.1) are exponentially small on the lenses (away from  $\Delta$ ), we can focus on solving a global RHP, where only the jump on  $\Delta$  is considered. These can be explicitly solved via the Szegő function (cf. Section 5.3) and Theta functions.

To arrive at a RHP whose jumps are close to identity, one also needs to exactly solve (or prove existence of a solution at least) local RHP’s in neighborhoods of points where the lenses from above intersect  $\Delta$ . These are often solved with the help of special functions (cf. Section 6.3.2, for example).

### 4.5 Extracting Asymptotics

Once at this stage, one can apply Theorem 4.0.1, and reversing all the transformations above yields an asymptotic expression for  $\mathbf{Y}$  and in turn  $P_n(z)$  as  $n \rightarrow \infty$ .

Large  $n$  asymptotics of a solution of a RHP were first achieved by Its in [29, 30], where he reduced the initial RHP to solving a global problem along with “local” RHPs. This is an interesting feature of this method of analysis: applying the method yields asymptotics for all  $z \in \mathbb{C}$ , but requires solving RHP in the whole plane as

well. The method was generalized and standardized in the work of Deift and Zhou in [31], and this method (highlighted above) is now known as the nonlinear steepest descent method, an ode to the integral steepest descent method. In the context of orthogonal polynomials the early RHP appearances include [32–34], but the first connection between RHPs and orthogonal polynomials was made in [35, 36].

In the next chapter we will return to our original problem of asymptotics of orthogonal polynomials and introduce all the key players.

## 5. MODEL PROBLEM: ORTHOGONAL POLYNOMIALS ON A CROSS

A version of this chapter appeared in [37].

To understand the effect of a point of valence 4 in an algebraic S-contour, we consider the following model problem: the asymptotic behavior of polynomials  $P_n(z)$  satisfying non-Hermitian orthogonality relations

$$\int_{\Delta} x^k P_n(x) \rho(x) ds = 0, \quad k = 0, \dots, n-1, \quad (5.0.1)$$

where  $\Delta := [-a, a] \cup [-ib, ib]$ ,  $a, b > 0$ , and  $\rho(s)$  is a Jacobi-type weight. These polynomials correspond to Padé approximants of

$$f(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{\rho(s) ds}{s - z}, \quad z \in \overline{\mathbb{C}} \setminus \Delta \quad (5.0.2)$$

where we require the weight  $\rho(x)$  belong to the following class.

**Definition.** Let  $\ell$  be a positive integer or infinity. We shall say that a function  $\rho(s)$  on  $\Delta$  belongs to the class  $\mathcal{W}_{\ell}$  if

- (i)  $\rho_i(s) := \rho|_{\Delta_i}(s)$  factors as a product  $\rho_i(s) = \rho_i^*(s)(s - a_i)^{\alpha_i}$ , where the function  $\rho_i^*(z)$  is non-vanishing and holomorphic in some neighborhood of  $\Delta_i$ ,  $\alpha_i > -1$ , and  $(z - a_i)^{\alpha_i}$  is a branch holomorphic across  $\Delta \setminus \{a_i\}$ ,  $i \in \{1, 2, 3, 4\}$ ;
- (ii) the ratio  $(\rho_1 \rho_3)(z)/(\rho_2 \rho_4)(z)$  is constant in some neighborhood of the origin;
- (iii) it holds that  $\rho_1(0) + \rho_2(0) + \rho_3(0) + \rho_4(0) = 0$ ;
- (iv) the quantities  $\rho_i^{(l)}(0)/\rho_i(0)$ ,  $0 \leq l < \ell$ , do not depend on  $i \in \{1, 2, 3, 4\}$ .

Observe that conditions (ii) and (iii) say that one of the functions  $\rho_i(z)$  is fully determined by the other three. In particular, it must hold that

$$\rho_4(z) = -(\rho_1 + \rho_2 + \rho_3)(0)(\rho_2/\rho_1 \rho_3)(0)(\rho_1 \rho_3/\rho_2)(z).$$



Notice also that  $\mathcal{W}_{\ell_1} \subset \mathcal{W}_{\ell_2}$  whenever  $\ell_2 < \ell_1$  and that  $\rho(s) \in \mathcal{W}_\infty$  if and only if there exists a function  $F(z)$ , holomorphic in some neighborhood of  $\Delta \setminus \{a_1, a_2, a_3, a_4\}$ , such that  $\rho_i(s) = c_i F|_{\Delta_i}(s)$  for some constants  $c_i$  that add up to zero.

In particular, this class includes functions

$$\sum_{i=1}^4 C_i \log(z - a_i) \quad \text{and} \quad \prod_{i=1}^4 (z - a_i)^{\alpha_i},$$

where the constants  $C_i$  add up to zero and the exponents  $-1 < \alpha_i \notin \mathbb{Z}$  add up to an integer, possess branches holomorphic off  $\Delta$  that can be represented by (5.0.2) for certain weight functions in  $\mathcal{W}_\infty$  (the second function can be represented by (5.0.2) up to an addition of a polynomial of degree  $\sum_{i=1}^4 \alpha_i$ ).

Holomorphy of the weights  $\rho_i(z)$  allows one to deform  $\Delta$  in (5.0.1) to any cross-like contour consisting of four arcs connecting the points  $a_i$  to the origin (some central point if the weight add up to zero in a neighborhood of the origin). However, Theorem 3.2.2 suggests that the attracting contour is essentially characterized by having the smallest logarithmic capacity among all continua containing  $\{a_1, a_2, a_3, a_4\}$ . It is also known from Theorem 3.2.5 (see also [38, 39] for the appearance of the same problem in the geometric function theory literature) that this contour must consist of the orthogonal critical trajectories of the quadratic differential

$$\frac{(z - b_1)(z - b_2)dz^2}{(z^2 - a^2)(z^2 + b^2)} \tag{5.0.3}$$

for some uniquely determined constants  $b_1, b_2$ . It can be readily verified that  $\Delta$  is the desired contour and  $b_1 = b_2 = 0$ .

The functions describing the asymptotic behavior of the polynomials  $P_n(z)$  are constructed in three steps, carried out in Sections 5.2-5.4, and naturally defined on a Riemann surface corresponding to  $\Delta$  that is introduced in Section 5.1. The main results are stated in Sections 5.5 and 5.6.

### 5.1 Riemann Surface

Let  $\Delta = \cup_{i=1}^4 \Delta_i$  be as above. Set

$$w(z) := \sqrt{(z^2 - a^2)(z^2 + b^2)}, \quad z \in \mathbb{C} \setminus \Delta, \quad (5.1.1)$$

to be the branch normalized so that  $w(z) = z^2 + \mathcal{O}(z)$  as  $z \rightarrow \infty$ . Denote by  $\mathfrak{R}$  the Riemann surface of  $w(z)$  realized as a two-sheeted ramified cover of  $\overline{\mathbb{C}}$  constructed in the following manner. Two copies of  $\overline{\mathbb{C}}$  are cut along each arc  $\Delta_i$ . These copies are glued together along the cuts in such a manner that the right (resp. left) side of the arc  $\Delta_i$  belonging to the first copy, say  $\mathfrak{R}^{(0)}$ , is joined with the left (resp. right) side of the same arc  $\Delta_i$  only belonging to the second copy,  $\mathfrak{R}^{(1)}$ .

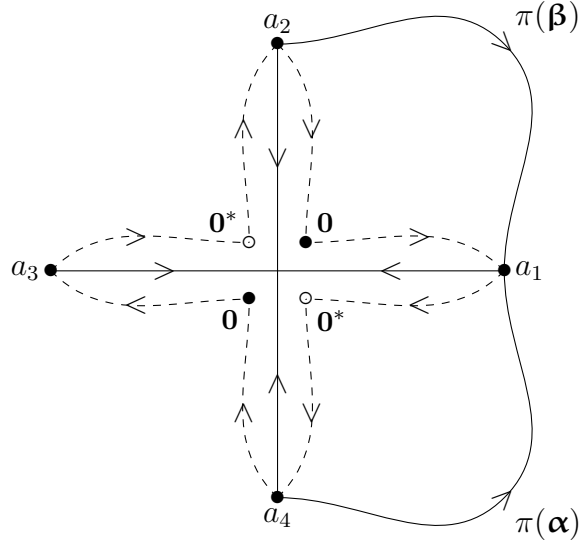


Fig. 5.1. The arcs  $\Delta_i$  together with their orientation (solid lines), a schematic representation of the arcs  $\Delta_i = \pi^{-1}(\Delta_i)$  (dashed lines) as viewed from  $\mathfrak{R}^{(0)}$ , and the chosen homology basis  $\{\alpha, \beta\}$  projected down from  $\mathfrak{R}^{(0)}$ .

We denote by  $\pi$  the canonical projection  $\pi : \mathfrak{R} \rightarrow \overline{\mathbb{C}}$  and define  $\Delta := \pi^{-1}(\Delta)$ ,  $\Delta_i := \pi^{-1}(\Delta_i)$ ,  $i \in \{1, 2, 3, 4\}$ . Then  $\Delta$  is a curve on  $\mathfrak{R}$  that intersects itself exactly twice (once at each point on top of the origin), see Figures 5.1 and 5.2. We orient  $\Delta$  so that  $\mathfrak{R}^{(0)}$  remains on the left when  $\Delta$  is traversed in the positive direction.

We shall denote by  $z^{(k)}$ ,  $k \in \{0, 1\}$ , the point on  $\mathfrak{R}^{(k)}$  with canonical projection  $z$  and designate the symbol  $\cdot^*$  to stand for the conformal involution that sends  $z^{(k)}$  into  $z^{(1-k)}$ ,  $k \in \{0, 1\}$ . We use bold lower case letters such as  $\mathbf{z}, \mathbf{t}, \mathbf{s}$  to indicate points on  $\mathfrak{R}$  with canonical projections  $z, t, s$ . Since  $\mathfrak{R}$  is elliptic (genus 1), any homology basis on  $\mathfrak{R}$  consists of only two cycles. In what follows, we choose cycles  $\alpha, \beta$  to be involution-symmetric and such that  $\pi(\alpha), \pi(\beta)$  are rectifiable Jordan arcs joining  $a_1, a_2$  and  $a_4, a_1$ , respectively, oriented as on Figures 5.1 and 5.2.

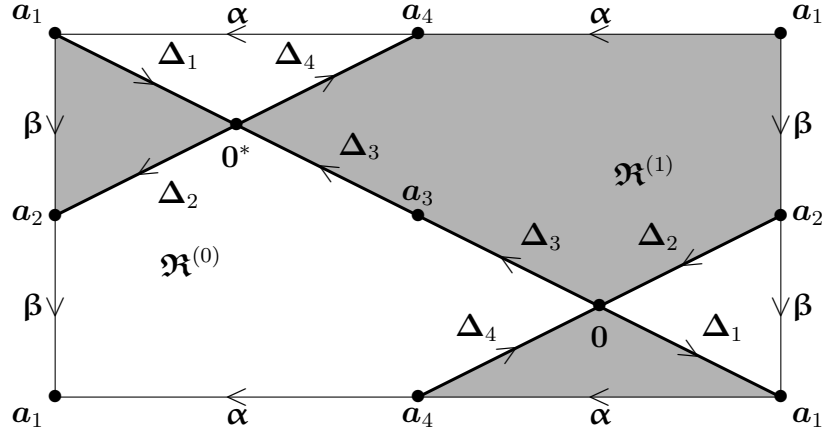


Fig. 5.2. Schematic representation of the surface  $\mathfrak{R}$  (shaded region represents  $\mathfrak{R}^{(1)}$ ), which topologically is a torus, the arcs  $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ , and the homology basis  $\alpha, \beta$ .

## 5.2 Geometric Term

The main goal of this subsection is to define the function  $\Phi(\mathbf{z})$ , see (5.2.5), that will be responsible for the rate of growth of the polynomials  $Q_n(z)$  and is determined solely by the contour of orthogonality  $\Delta$ .

With a slight abuse of notation, let us set

$$w(\mathbf{z}) := (-1)^k w(z), \quad \mathbf{z} \in \mathfrak{R}^{(k)} \setminus \Delta, \quad k \in \{0, 1\},$$

which we then extend by continuity to  $\Delta$ . Clearly,  $w(z)$  is a meromorphic function on  $\mathfrak{R}$  with simple zeros at the ramification points of  $\mathfrak{R}$ , double poles at  $\infty^{(0)}$  and  $\infty^{(1)}$ , and otherwise non-vanishing and finite. Thus,

$$\Omega(z) := \left( \oint_{\alpha} \frac{ds}{w(s)} \right)^{-1} \frac{dz}{w(z)} \quad (5.2.1)$$

is the holomorphic differential on  $\mathfrak{R}$  normalized to have unit period on  $\alpha$ . In this case it was shown by Riemann that the constant

$$B := \oint_{\beta} \Omega \quad (5.2.2)$$

has positive purely imaginary part. Further, since  $z/w(z)$  has simple poles at the ramification point of  $\mathfrak{R}$ , simple zeros at  $\infty^{(0)}$  and  $\infty^{(1)}$ , and behaves like  $1/z$  around  $\infty^{(0)}$ , the differential

$$G(z) := \frac{zdz}{w(z)}$$

is meromorphic on  $\mathfrak{R}$  having two simple poles at  $\infty^{(1)}$  and  $\infty^{(0)}$  with respective residues 1 and  $-1$ .  $G(z)$  is also distinguished by having a purely imaginary period on any cycle on  $\mathfrak{R}$ . Indeed, it is enough to verify this claim on the cycles of any homology basis. To this end, define

$$\omega := -\frac{1}{2\pi i} \oint_{\beta} G \quad \text{and} \quad \tau := \frac{1}{2\pi i} \oint_{\alpha} G. \quad (5.2.3)$$

By deforming  $\alpha$  (resp.  $\beta$ ) into  $-\Delta_1 - \Delta_4$  (resp.  $\Delta_1 + \Delta_2$ ) and using the symmetry  $G(z^*) = -G(z)$ , one gets that

$$\omega = \tau = \frac{1}{4\pi i} \oint_{\Gamma} \frac{zdz}{w(z)} = \frac{1}{2}, \quad (5.2.4)$$

where  $\Gamma$  is any positively oriented rectifiable Jordan curve encircling  $\Delta$ , which does verify the claim about  $G(z)$  having purely imaginary periods. Let

$$\Phi(z) := \exp \left\{ \int_{a_3}^z G \right\}, \quad z \in \mathfrak{R}_{\alpha,\beta} \setminus \{\infty^{(0)}, \infty^{(1)}\}, \quad (5.2.5)$$

where  $\mathfrak{R}_{\alpha,\beta} := \mathfrak{R} \setminus \{\alpha, \beta\}$  and the path of integration lies entirely in  $\mathfrak{R}_{\alpha,\beta} \setminus \{\infty^{(0)}, \infty^{(1)}\}$ .

The function  $\Phi(z)$  is holomorphic and non-vanishing on  $\mathfrak{R}_{\alpha,\beta}$  except for a simple pole

at  $\infty^{(0)}$  and a simple zero at  $\infty^{(1)}$ . Furthermore, it possesses continuous traces on both sides of each cycle of the canonical basis that satisfy<sup>1</sup>

$$\Phi_+(s) = -\Phi_-(s), \quad s \in \alpha \cup \beta, \quad (5.2.6)$$

by (5.2.3)–(5.2.4). It is not a difficult computation to check that  $\Phi(z)\Phi(z^*) \equiv 1$  and

$$|\Phi(z)| = \exp \{(-1)^k g_\Delta(z; \infty)\}, \quad z \in \mathfrak{R}^{(k)}, \quad (5.2.7)$$

$k \in \{0, 1\}$ , where  $g_\Delta(z; \infty)$  is the Green function for  $\overline{\mathbb{C}} \setminus \Delta$  with pole at  $\infty$ .<sup>2</sup> In fact, the above properties allow us to verify that

$$\Phi^2(z^{(k)}) = \frac{2}{a^2 + b^2} \left( z^2 + \frac{b^2 - a^2}{2} + (-1)^k w(z) \right), \quad (5.2.8)$$

$k \in \{0, 1\}$ . In particular, this implies that the logarithmic capacity of  $\Delta$  is equal to  $\sqrt{a^2 + b^2}/2$  since

$$\Phi(z^{(0)}) = \frac{-2z}{\sqrt{a^2 + b^2}} + \mathcal{O}(1) \quad \text{as } z \rightarrow \infty \quad (5.2.9)$$

(the sign in (5.2.9) is determined by the fact that  $\Phi(\mathbf{a}_3) = 1$  and  $\Phi(z)$  is non-vanishing on  $\pi^{-1}((-\infty, -a))$ ). Observe also that a calculus level computation tells us that

$$\Phi(\mathbf{0}) = \overline{\Phi(\mathbf{0}^*)} = \exp \left\{ i \arctan \left( \frac{a}{b} \right) \right\}, \quad (5.2.10)$$

where the point  $\mathbf{0}$  and  $\mathbf{0}^*$  are defined as on Figure 5.1.

### 5.3 Szegő Function

It is known since the work of Szegő that the finer details of the asymptotics of  $Q_n(z)$  are captured by the so-called Szegő function, which depends only on the weight of orthogonality. Below, we construct this function for  $\rho(s) \in \mathcal{W}_1$ .

Given  $i \in \{1, 2, 3, 4\}$ , fix  $\log \rho_i(s)$  to be a branch continuous on  $\Delta_i \setminus \{a_i\}$ , selected so that

$$\nu := \frac{1}{2\pi i} \sum_{i=1}^4 (-1)^i \log \rho_i(0) \quad \text{satisfies} \quad \operatorname{Re}(\nu) \in \left( -\frac{1}{2}, \frac{1}{2} \right]. \quad (5.3.1)$$

<sup>1</sup>Here and in what follows we state jump relations understanding that they hold outside the points of self-intersection of the considered arcs.

<sup>2</sup> $g_\Delta(z; \infty)$  is equal to zero on  $\Delta$ , is positive and harmonic in  $\mathbb{C} \setminus \Delta$ , and satisfies  $g(z; \infty) = \log |z| + \mathcal{O}(1)$  as  $z \rightarrow \infty$ .

Further, it can be readily verified that we can set

$$\log w_+(s) = \log |w_+(s)| + (-1)^i \frac{\pi i}{2}, \quad s \in \Delta_i^\circ, \quad (5.3.2)$$

where, as usual,  $w_+(s)$  is the trace of (5.1.1) on the positive side of  $\Delta_i^\circ$  according to the chosen orientation. We also let  $\log(\rho_i w_+)(s)$  to stand for  $\log \rho_i(s) + \log w_+(s)$  with the just selected branches. Put

$$S_\rho(\mathbf{z}) := \exp \left\{ -\frac{1}{4\pi i} \oint_{\Delta} \log(\rho w_+)(s) \Omega_{\mathbf{z}, \mathbf{z}^*}(\mathbf{s}) \right\}, \quad (5.3.3)$$

where  $\Omega_{\mathbf{z}, \mathbf{z}^*}(\mathbf{s})$  is the meromorphic differential with two simple poles at  $\mathbf{z}$  and  $\mathbf{z}^*$  with respective residues 1 and  $-1$  normalized to have zero period on  $\alpha$ . When  $\mathbf{z}$  does not lie on top of the point at infinity, it can be readily verified that

$$\Omega_{\mathbf{z}, \mathbf{z}^*}(\mathbf{s}) = \frac{w(\mathbf{z})}{s - z} \frac{ds}{w(\mathbf{s})} - \left( \oint_{\alpha} \frac{w(\mathbf{z})}{t - z} \frac{dt}{w(\mathbf{t})} \right) \Omega(\mathbf{s}), \quad (5.3.4)$$

where  $\Omega(\mathbf{s})$  is the holomorphic differential (5.2.1).

**Proposition 5.3.1.** *Let  $\rho(s) \in \mathcal{W}_1$  and  $S_\rho(\mathbf{z})$  be given by (5.3.3). Define*

$$c_\rho := \frac{1}{2\pi i} \oint_{\Delta} \log(\rho w_+) \Omega. \quad (5.3.5)$$

*Then  $S_\rho(\mathbf{z})$  is a holomorphic and non-vanishing function in  $\Re \setminus \{\Delta \cup \alpha\}$  with continuous traces on  $(\Delta \cup \alpha) \setminus \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{0}, \mathbf{0}^*\}$  that satisfy*

$$S_{\rho+}(\mathbf{s}) = S_{\rho-}(\mathbf{s}) \begin{cases} \exp \{2\pi i c_\rho\}, & \mathbf{s} \in \alpha, \\ 1/(\rho w_+)(s), & \mathbf{s} \in \Delta. \end{cases} \quad (5.3.6)$$

*It also holds that  $S_\rho(\mathbf{z})S_\rho(\mathbf{z}^*) \equiv 1$  and*<sup>3</sup>

$$|S_\rho(z^{(0)})| \sim \begin{cases} |z - a_i|^{-(2\alpha_i+1)/4} & \text{as } z \rightarrow a_i, \\ |z|^{(-1)^j \operatorname{Re}(\nu)} & \text{as } \mathcal{Q}_j \ni z \rightarrow 0, \end{cases} \quad (5.3.7)$$

*for  $i, j \in \{1, 2, 3, 4\}$ , where  $\mathcal{Q}_j$  is the  $j$ -th quadrant and  $\nu$  is given by (5.3.1).*

Proposition 5.3.1 is proved in Appendix B.

<sup>3</sup>In what follows we write  $|g_1(z)| \sim |g_2(z)|$  as  $z \rightarrow z_0$  if there exists a constant  $C > 1$  such that  $C^{-1}|g_1(z)| \leq |g_2(z)| \leq C|g_1(z)|$  for all  $z$  close to  $z_0$ .

## 5.4 Theta Function

Let  $\text{Jac}(\mathfrak{R}) := \mathbb{C}/\{\mathbb{Z} + \mathbf{B}\mathbb{Z}\}$  be the Jacobi variety of  $\mathfrak{R}$ , where  $\mathbf{B}$  is given by (5.2.2). We shall represent elements of  $\text{Jac}(\mathfrak{R})$  as equivalence classes  $[s] = \{s + l + \mathbf{B}m : l, m \in \mathbb{Z}\}$ , where  $s \in \mathbb{C}$ . Since  $\mathfrak{R}$  is elliptic, Abel's map

$$z \in \mathfrak{R} \mapsto \left[ \int_{a_3}^z \Omega \right] \in \text{Jac}(\mathfrak{R})$$

is a holomorphic bijection. Hence, given any  $s \in \mathbb{C}$ , there exists a unique  $z_{[s]} \in \mathfrak{R}$  such that  $\left[ \int_{a_3}^{z_{[s]}} \Omega \right] = [s]$ .

Denote by  $\theta(\zeta)$  the Riemann theta function associated to  $\mathbf{B}$ , i.e.,

$$\theta(\zeta) := \sum_{n \in \mathbb{Z}} \exp \{ \pi i \mathbf{B} n^2 + 2\pi i n \zeta \}.$$

As shown by Riemann,  $\theta(\zeta)$  is an entire, even function that satisfies

$$\theta(\zeta + l + m\mathbf{B}) = \theta(\zeta) \exp \{ -\pi i m^2 \mathbf{B} - 2\pi i m \zeta \} \quad (5.4.1)$$

for any integers  $l, m$ . Moreover, its zeros are simple and  $\theta(\zeta) = 0$  if and only if  $[\zeta] = [(1 + \mathbf{B})/2]$ . The constant  $(1 + \mathbf{B})/2$ , known as the Riemann constant, will appear often in our computations. So, we choose to abbreviate the representatives of its “half”-classes by

$$\mathbf{K}_+ := (1 + \mathbf{B})/4 \quad \text{and} \quad \mathbf{K}_- := (1 - \mathbf{B})/4, \quad (5.4.2)$$

i.e.,  $[2\mathbf{K}_+] = [2\mathbf{K}_-]$ . The symmetries of  $\Omega(z)$  ( $\Omega(-z) = -\Omega(z) = \Omega(z^*)$ ) yield that

$$\int_{\infty^{(1)}}^{\infty^{(0)}} \Omega = \frac{1}{2} \int_{\delta} \Omega = 2\mathbf{K}_+ \quad \Rightarrow \quad \int_{a_3}^{\infty^{(k)}} \Omega = (-1)^k \mathbf{K}_+, \quad (5.4.3)$$

$k \in \{0, 1\}$ , where  $\delta = \pi^{-1}((-\infty, -a] \cup [a, \infty))$  is a cycle on  $\mathfrak{R}$  oriented from  $\infty^{(1)}$  to  $\infty^{(0)}$  (on Figure 5.2,  $\delta$  would be represented by the anti-diagonal), which is clearly homologous to  $\alpha + \beta$ .

With  $c_\rho$  as in Proposition 5.3.1, define

$$T_k(z) := \exp \left\{ \pi i k \int_{a_3}^z \Omega \right\} \frac{\theta \left( \int_{a_3}^z \Omega - c_\rho - (-1)^k \mathbf{K}_+ \right)}{\theta \left( \int_{a_3}^z \Omega - \mathbf{K}_+ \right)} \quad (5.4.4)$$

for  $k \in \{0, 1\}$  and  $z \in \mathfrak{R}_{\alpha, \beta}$ , where the path of integration lies entirely within  $\mathfrak{R}_{\alpha, \beta}$ .  $T_k(z)$  is a meromorphic function that is finite and non-vanishing except for a simple pole at  $\infty^{(1)}$ , see (5.4.3), and a simple zero at  $z_k := z_{[c_\rho - (-1)^k \mathbf{K}_+]}$ , where  $z_k \in \mathfrak{R}$  is uniquely characterized by

$$\int_{\mathbf{a}_3}^{z_k} \Omega = c_\rho - (-1)^k \mathbf{K}_+ + l_k + m_k \mathbf{B}, \quad (5.4.5)$$

$k \in \{0, 1\}$ , for some  $l_0, m_0, l_1, m_1 \in \mathbb{Z}$ . Furthermore, it follows from the normalization in (5.2.1), the definition of  $\mathbf{B}$  in (5.2.2), and (5.4.1) that

$$T_{k+}(\mathbf{s}) = T_{k-}(\mathbf{s}) \begin{cases} \exp \{2\pi i(k/2 - c_\rho)\}, & \mathbf{s} \in \alpha, \\ \exp \{\pi i k\}, & \mathbf{s} \in \beta. \end{cases} \quad (5.4.6)$$

Now we are ready to define the function that will be responsible for the asymptotic behavior of the polynomials  $Q_n(z)$ . Given  $\rho(s) \in \mathcal{W}_1$ , let  $c_\rho$  be defined by (5.3.5). Set

$$\{0, 1\} \ni \iota(n) := n \mod 2, \quad n \in \mathbb{Z},$$

to be the parity function. Then it follows from (5.2.6), (5.3.6), and (5.4.6) that the function

$$\Psi_n(z) := (\Phi^n S_\rho T_{\iota(n)})(z), \quad z \in \mathfrak{R} \setminus \Delta, \quad (5.4.7)$$

is meromorphic in  $\mathfrak{R} \setminus \Delta$  with a pole of order  $n$  at  $\infty^{(0)}$ , a zero of multiplicity  $n - 1$  at  $\infty^{(1)}$ , a simple zero at  $z_{\iota(n)}$ , and otherwise non-vanishing and finite, whose traces on  $\Delta$  satisfy

$$\Psi_{n+}(\mathbf{s}) = \Psi_{n-}(\mathbf{s})/(\rho w_+)(s), \quad \mathbf{s} \in \Delta, \quad (5.4.8)$$

and whose behavior around the ramification points of  $\mathfrak{R}$  as well as  $\mathbf{0}^*, \mathbf{0}$  is governed by (5.3.7).

## 5.5 Asymptotics

In this section we formulate the main theorem on the behavior of the polynomials  $Q_n(z)$ . As was alluded to in the introduction, we do not expect to be able to handle



all the possible indices  $n$  as  $Q_n(s)$  might have degree smaller than  $n$ . One source of this degeneration already can be seen from (5.4.7) since this function can have a pole of order  $n - 1$  at  $\infty^{(0)}$  when  $\mathbf{z}_{i(n)} = \infty^{(0)}$ . In fact, this is the only reason for the degeneration in the generic cases described in [25]. However, this is no longer the case for the considered model.

To restrict the indices we need the following, unfortunately very technical, definition. Let us set

$$\varsigma_\nu := \begin{cases} 1, & \operatorname{Re}(\nu) > 0, \\ -1, & \operatorname{Re}(\nu) < 0, \end{cases} \quad \text{and} \quad \mathbf{o} := \begin{cases} \mathbf{0}, & \operatorname{Re}(\nu) > 0, \\ \mathbf{0}^*, & \operatorname{Re}(\nu) < 0. \end{cases} \quad (5.5.1)$$

We do not make any choice for  $\varsigma_\nu$  and  $\mathbf{o}$  when  $\operatorname{Re}(\nu) = 0$ . Given  $\rho(s) \in \mathcal{W}_1$  and the constant  $c_\rho$  from (5.3.5), define

$$A_{\rho,n} := \begin{cases} \sigma_{i(n)} A'_{\rho,n} \Phi(\mathbf{z}_{i(n)}) \Phi^{2(n-1)}(\mathbf{o}), & \operatorname{Re}(\nu) \neq 0, \\ 0, & \operatorname{Re}(\nu) = 0, \end{cases} \quad (5.5.2)$$

where  $\sigma_k := (-1)^{l_k + m_k + k}$ ,  $k \in \{0, 1\}$ , see (5.4.5), and

$$A'_{\rho,n} := A_\rho e^{\pi i \varsigma_\nu (c_\rho + 1/4)} \frac{\sqrt{a^2 + b^2}}{2} \frac{\Gamma(1 - \varsigma_\nu \nu)}{\sqrt{2\pi}} \times \left[ \lim_{z \rightarrow 0, \arg(z) = 5\pi/4} |z|^{2\nu} S_\rho^2(z^{(0)}) \right]^{\varsigma_\nu} \left( \frac{ab}{2n} \right)^{1/2 - \varsigma_\nu \nu},$$

and

$$A_\rho := e^{\pi i \nu} \rho_3(0) \frac{(\rho_2 + \rho_3)(0)}{\rho_2(0)} \quad \text{or} \quad A_\rho := \frac{1}{(ab)^2} \frac{(\rho_3 + \rho_4)(0)}{(\rho_3 \rho_4)(0)}$$

depending on whether  $\operatorname{Re}(\nu) > 0$  or  $\operatorname{Re}(\nu) < 0$  (it follows from the last display in Section B.1, devoted to the proof of Proposition 5.3.1, that the limit in the definition of the constant  $A'_{\rho,n}$  is indeed well defined).

Given the above constants  $A_{\rho,n}$  and  $\epsilon \in (0, 1/2)$ , we define subsequences of allowable indices  $n$  for the weight  $\rho(s)$  by

$$\mathbb{N}_{\rho,\epsilon} := \{n \in \mathbb{N} : \mathbf{z}_{i(n)} \neq \infty^{(0)} \text{ and } |1 - A_{\rho,n}| \geq \epsilon\}. \quad (5.5.3)$$

The following proposition states that such sequences are non-empty.

**Proposition 5.5.1.** *Let  $\mathbb{N}_{\rho,\epsilon}$  be given by (5.5.3). If  $[c_\rho] = [0]$  or  $[c_\rho] = [(1+B)/2]$ , then it holds that*

$$\mathbb{N}_{\rho,\epsilon} = \mathbb{N}_\rho := \begin{cases} 2\mathbb{N} & \text{when } [c_\rho] = [0], \\ \mathbb{N} \setminus 2\mathbb{N} & \text{when } [c_\rho] = [(1+B)/2]. \end{cases} \quad (5.5.4)$$

*If  $[c_\rho] \neq [0]$  and  $[c_\rho] \neq [(1+B)/2]$  while  $\operatorname{Re}(\nu) \in (-1/2, 1/2)$ , it holds that*

$$\mathbb{N}_{\rho,\epsilon} = \mathbb{N}_\rho := \mathbb{N}. \quad (5.5.5)$$

*If  $[c_\rho] \neq [0]$  and  $[c_\rho] \neq [(1+B)/2]$ , and  $\operatorname{Re}(\nu) = 1/2$ , then  $\mathbb{N}_{\rho,\epsilon}$  is an infinite subsequence with gaps of size at most 2 (clearly, this is the only case when  $\mathbb{N}_{\rho,\epsilon}$  might depend on  $\epsilon$ ).*

The proof of Proposition 5.5.1 is delegated to Appendix B.

When  $\operatorname{Re}(\nu) < 1/2$ , the sequence  $\mathbb{N}_{\rho,\epsilon} = \mathbb{N}_\rho$  is equal to the whole set of the natural numbers or consists of every other one. This is consistent with the explanation given at the beginning of the subsection and is supported by the examples in Sections C.1 and C.2 where two weights  $\rho(s)$  are provided for which  $Q_{2n}(z) = Q_{2n+1}(z)$ . As mentioned before, this is a generic behavior observed in [25]. On the technical level this degeneration manifests itself as our inability to construct the “global parametrix”, see Section 6.2, since we are no longer able to properly renormalize  $Q_n(z)$  by  $\Psi_n(z^{(0)})$  when  $z_{\iota(n)} = \infty^{(0)}$ .

When  $\operatorname{Re}(\nu) = 1/2$ , new phenomenon occurs. The sequence  $\mathbb{N}_{\rho,\epsilon}$  can have gaps of size 2 depending on the behavior of the constants  $A_{n,\rho}$ . This suggests that there might be indices  $n$  such that  $Q_n(z) = Q_{n+1}(z) = Q_{n+2}(z)$ . Such a possibility can in fact occur, see Section C.3 for an example. On the technical level, the second condition in (5.5.3) appears in an attempt to match the behavior of  $Q_n(z)$  at the origin, that is, during the construction of the so-called “local parametrix”, see Sections 6.3.2 and 6.3.2, and manifests itself through the constants  $L_{ni}$ , see (5.5.11).

Recall that the weight  $\rho(s)$  defines two constants:  $\ell$ , which says how well the restrictions of  $\rho(s)$  to different segments  $\Delta_i$  match each other at the origin, and  $\nu$ ,

defined in (5.3.1). Our analysis does not allow us to handle all possible combinations of these constants. In what follows we assume that

$$|\operatorname{Re}(\nu)| \in \begin{cases} [0, \sqrt{7}/2 - 1) & \text{when } \ell = 1, \\ [0, 1/2) & \text{when } \ell = 2, \\ [0, 1/2] & \text{when } \ell > 3. \end{cases} \quad (5.5.6)$$

This technical condition appears in the rate of decay of the error, which we quantify by the following exponent:

$$d_{\nu, \ell} := \begin{cases} \frac{(\frac{1}{2} + |\operatorname{Re}(\nu)|)(\ell - 2|\operatorname{Re}(\nu)|)}{\ell + 1 + 2|\operatorname{Re}(\nu)|}, & \ell \geq \frac{4|\operatorname{Re}(\nu)|(1 + |\operatorname{Re}(\nu)|)}{1 - 2|\operatorname{Re}(\nu)|}, \\ \frac{\ell(3 - 2|\operatorname{Re}(\nu)|) - 2|\operatorname{Re}(\nu)|(3 + 2|\operatorname{Re}(\nu)|)}{2(\ell + 3 + 2|\operatorname{Re}(\nu)|)}, & \text{otherwise,} \end{cases} \quad (5.5.7)$$

where we understand that  $d_{\nu, \infty} = 1/2 + |\operatorname{Re}(\nu)|$ . It is a straightforward computation to check that requiring positivity of the numerator of  $d_{\nu, \ell}$  in the second line of (5.5.7) produces restriction (5.5.6). Observe also that  $d_{1/2, \ell} = \frac{\ell - 2}{\ell + 4}$ .

**Theorem 5.5.2.** *Let  $\rho(s) \in \mathcal{W}_\ell$ , where  $\ell$  is a positive integer or infinity. Define  $\nu$  by (5.3.1) and assume that (5.5.6) is satisfied. Let  $\Psi_n(z)$  be given by (5.4.7) and  $\mathbb{N}_{\rho, \epsilon}$  be as in (5.5.3) for some  $\epsilon \in (0, 1/2)$  fixed. Then it holds for all  $n \in \mathbb{N}_{\rho, \epsilon}$  large enough that*

$$Q_n(z) = \gamma_n(1 + v_{n1}(z))\Psi_n(z^{(0)}) + \gamma_n v_{n2}(z)\Psi_{n-1}(z^{(0)}) \quad (5.5.8)$$

for  $z \in \mathbb{C} \setminus \Delta$ , where  $\gamma_n := \lim_{z \rightarrow \infty} z^n \Psi_n^{-1}(z^{(0)})$  is the normalizing constant;

$$Q_n(s) = \gamma_n(1 + v_{n1}(s)) \left( \Psi_{n+}^{(0)}(s) + \Psi_{n-}^{(0)}(s) \right) + \gamma_n v_{n2}(s) \left( \Psi_{n-1+}^{(0)}(s) + \Psi_{n-1-}^{(0)}(s) \right) \quad (5.5.9)$$

for  $s \in \Delta^\circ$ , where  $\Psi_{n\pm}^{(0)}(s)$  are the traces of  $\Psi_n(z^{(0)})$  on the positive and negative sides of  $\Delta$ . The functions  $v_{ni}(z)$  are such that

$$v_{ni}(\infty) = 0 \quad \text{and} \quad v_{ni}(z) = L_{ni}z^{-1} + \mathcal{O}(n^{-d_{\nu, \ell}}) \quad (5.5.10)$$

where  $\mathcal{O}(\cdot)$  holds locally uniformly on  $\overline{\mathbb{C}} \setminus \Delta$  in (5.5.8) and on  $\Delta^\circ$  in (5.5.9),  $d_{\nu, \ell}$  was defined in (5.5.7), and  $L_{ni}$  are constants given by

$$L_{ni} = (-1)^{i(n)} \frac{A_{\rho, n}}{1 - A_{\rho, n}} \left( -\frac{\Phi T_{i(n)}}{T_{i(n-1)}} \right)^{i-1} (\mathbf{o}) \frac{(T_0/T_1)(\mathbf{o})}{(T_0/T_1)'(\mathbf{o})} \quad (5.5.11)$$

when  $|\pi(\mathbf{z}_k)| < \infty$ ,  $i \in \{1, 2\}$ , where  $\mathbf{o}$  was defined in (5.5.1) (when  $|\pi(\mathbf{z}_k)| = \infty$ , the expressions for  $L_{ni}$  are even more cumbersome and therefore are omitted here).

Notice that the behavior of the polynomials  $Q_n(z)$  is qualitatively different for  $\operatorname{Re}(\nu) < 1/2$  and  $\operatorname{Re}(\nu) = 1/2$  as the first summand in (5.5.10) is decaying in the former case by (B.2.1), but does not decay in the latter.

Recall that the traces of  $\Phi(\mathbf{z})$  are unimodular on  $\Delta$ , see (5.2.7). Since  $\Psi_n(\mathbf{z}) = (S_\rho T_{i(n)})(\mathbf{z})\Phi^n(\mathbf{z})$ , it is exactly the sum of the terms  $(\Phi_+^{(0)}(s))^n$  and  $(\Phi_-^{(0)}(s))^n$  that creates oscillations describing the zeros of  $Q_n(z)$ . Of course, since the traces of  $(S_\rho T_{i(n)})_\pm^{(0)}(s)$  are in general complex-valued, the zeros of  $Q_n(z)$  do not lie exactly on  $\Delta$ . However, we do prove that (5.5.9) holds on compact subsets “close” to  $\Delta^\circ$ , where  $\Psi_{n\pm}^{(0)}(s)$  are analytically continued from  $\Delta^\circ$  into the complex plane with the help of (5.4.8).

When  $\ell < \infty$ , we cannot control the error functions  $v_{ni}(z)$  around the origin and therefore cannot describe the polynomials  $Q_n(z)$  there (however, we can extend (5.5.9) to hold on a sequence of compact subsets of  $\Delta^\circ$  that are allowed to approach the origin with a certain speed at the expense of worsening the rate of decay in the error estimates). When  $\ell = \infty$ , we can provide an asymptotic formula for  $Q_n(z)$  around the origin, but due to its technical nature we placed it at the very end of the paper in Section 6.6.

Theorem 5.5.2, as well as Theorem 5.6.1 further below, is proved in Chapter 6 with the derivation of some technical identities relegated to Appendix A.

## 5.6 Padé Approximation

Given  $\widehat{\rho}(z)$  as in (5.0.2), it follows from the orthogonality relations (5.0.1) that there exists a polynomial  $P_n(z)$  of degree at most  $n - 1$  such that

$$R_n(z) := (Q_n \widehat{\rho})(z) - P_n(z) = \mathcal{O}(z^{-n-1}) \quad \text{as } z \rightarrow \infty. \quad (5.6.1)$$

The rational function  $[n/n]_{\widehat{\rho}}(z) := P_n(z)/Q_n(z)$  is called the  $n$ -th diagonal Padé approximant to  $\widehat{\rho}(z)$ .

**Theorem 5.6.1.** *Let  $\widehat{\rho}(z)$  be given by (5.0.2) and  $R_n(z)$  be defined by (5.6.1). In the setting of Theorem 5.5.2, it holds for all  $n \in \mathbb{N}_{\rho, \epsilon}$  large enough that*

$$(wR_n)(z) = \gamma_n(1 + v_{n1}(z))\Psi_n(z^{(1)}) + \gamma_n v_{n2}(z)\Psi_{n-1}(z^{(1)}) \quad (5.6.2)$$

*locally uniformly in  $\overline{\mathbb{C}} \setminus \Delta$ , where  $v_{ni}(z)$  are the same as in Theorem 5.5.2.*

## 6. RIEMANN-HILBERT ANALYSIS: CASE OF THE CROSS

The starting point of the analysis is **RHP-Y** as stated in Chapter 4 with the weight  $\rho \in \mathcal{W}_\ell$

### 6.1 Opening of the Lenses

Let  $\delta_0 > 0$  be small enough so that all the functions  $\rho_i(z)$  are holomorphic in some neighborhood of  $\{|z| \leq \delta_0\}$ . Define  $\tilde{\Delta}_i$  and  $\tilde{\Delta}_i^\circ$  to be the closed and open segments connecting the origin and  $\delta_0 e^{(2i-1)\pi i/4}$ ,  $i \in \{1, 2, 3, 4\}$ , that are oriented towards the origin. Further, let  $\Gamma_{i-}, \Gamma_{i+}$  be open smooth arcs that lie within the

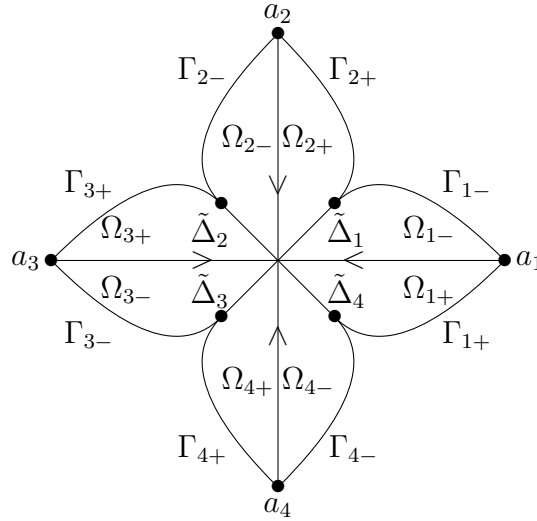


Fig. 6.1. The arcs  $\Delta_i$ ,  $\tilde{\Delta}_i$  and  $\Gamma_{i\pm}$ , and domains  $\Omega_{i\pm}$ .

domain of holomorphy of  $\rho_i(z)$  and connect  $a_i$  to  $\delta_0 e^{(2i-1)\pi i/4}$ ,  $\delta_0 e^{(2i-3)\pi i/4}$ , respectively.

We orient  $\Gamma_{i\pm}$  away from  $a_i$  and assume that no open arcs  $\Delta_i^\circ, \tilde{\Delta}_i^\circ, \Gamma_{i\pm}$  intersect, see Figure 6.1. We denote by  $\Omega_{i\pm}$  the domain partially bounded by  $\Delta_i$  and  $\Gamma_{i\pm}$ . Let

$$\mathbf{X}(z) := \mathbf{Y}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ \mp 1/\rho_i(z) & 1 \end{pmatrix}, & z \in \Omega_{i\pm}, \\ \mathbf{I}, & z \notin \bar{\Omega}_{i+} \cup \bar{\Omega}_{i-}. \end{cases} \quad (6.1.1)$$

Then  $\mathbf{X}(z)$  satisfies the following Riemann-Hilbert problem (RHP- $\mathbf{X}$ ):

- (a)  $\mathbf{X}(z)$  is analytic in  $\mathbb{C} \setminus \cup_i(\Delta_i \cup \tilde{\Delta}_i \cup \Gamma_{i\pm})$  and  $\lim_{z \rightarrow \infty} \mathbf{X}(z)z^{-n\sigma_3} = \mathbf{I}$ ;
- (b)  $\mathbf{X}(z)$  has continuous traces on each  $\Delta_i^\circ$ ,  $\tilde{\Delta}_i^\circ$ , and  $\Gamma_{i\pm}$  that satisfy

$$\mathbf{X}_+(s) = \mathbf{X}_-(s) \begin{cases} \begin{pmatrix} 1 & 0 \\ 1/\rho_i(s) & 1 \end{pmatrix}, & s \in \Gamma_{i+} \cup \Gamma_{i-}, \\ \begin{pmatrix} 0 & \rho_i(s) \\ -1/\rho_i(s) & 0 \end{pmatrix}, & s \in \Delta_i^\circ, \\ \begin{pmatrix} 1 & 0 \\ \frac{1}{\rho_i(s)} + \frac{1}{\rho_{i+1}(s)} & 1 \end{pmatrix}, & s \in \tilde{\Delta}_i^\circ, \end{cases}$$

where  $i \in \{1, 2, 3, 4\}$  and  $\rho_5 := \rho_1$ .

- (c)  $\mathbf{X}(z)$  is bounded around the origin and behaves like

$$\mathbf{X}(z) = \begin{cases} \mathcal{O} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{if } \alpha_i > 0, \\ \mathcal{O} \begin{pmatrix} 1 & \log |z - a_i| \\ 1 & \log |z - a_i| \end{pmatrix} & \text{if } \alpha_i = 0, \\ \mathcal{O} \begin{pmatrix} 1 & |z - a_i|^{\alpha_i} \\ 1 & |z - a_i|^{\alpha_i} \end{pmatrix} & \text{if } -1 < \alpha_i < 0, \end{cases}$$

as  $z \rightarrow a_i$  from outside the lens while from inside the lens,

$$\mathbf{X}(z) = \begin{cases} \mathcal{O} \begin{pmatrix} |z - a_i|^{-\alpha_i} & 1 \\ |z - a_i|^{-\alpha_i} & 1 \end{pmatrix} & \text{if } \alpha_i > 0, \\ \mathcal{O} \begin{pmatrix} 1 & \log |z - a_i| \\ 1 & \log |z - a_i| \end{pmatrix} & \text{if } \alpha_i = 0, \\ \mathcal{O} \begin{pmatrix} 1 & |z - a_i|^{\alpha_i} \\ 1 & |z - a_i|^{\alpha_i} \end{pmatrix} & \text{if } -1 < \alpha_i < 0. \end{cases}$$

The following observation can be easily checked: **RHP- $\mathbf{X}$**  is solvable if and only if **RHP- $\mathbf{Y}$**  is solvable. When solutions of **RHP- $\mathbf{X}$**  and **RHP- $\mathbf{Y}$**  exist, they are unique and connected by (6.1.1).

## 6.2 Global Parametrix

Let  $\Psi_n(z)$  be given by (5.4.7). For each  $n \in \mathbb{N}_{\rho, \epsilon}$ , define

$$\mathbf{N}(z) := \begin{pmatrix} \gamma_n & 0 \\ 0 & \gamma_{n-1}^* \end{pmatrix} \begin{pmatrix} \Psi_n(z^{(0)}) & \Psi_n(z^{(1)})/w(z) \\ \Psi_{n-1}(z^{(0)}) & \Psi_{n-1}(z^{(1)})/w(z) \end{pmatrix}, \quad (6.2.1)$$

where the constants  $\gamma_n$  and  $\gamma_{n-1}^*$  are defined by the relations

$$\lim_{z \rightarrow \infty} \gamma_n z^{-n} \Psi_n(z^{(0)}) = 1 \quad \text{and} \quad \lim_{z \rightarrow \infty} \gamma_{n-1}^* z^n \Psi_{n-1}(z^{(1)})/w(z) = 1. \quad (6.2.2)$$

Such constants do exist, see the explanation after Proposition 5.5.1. The product  $\gamma_n \gamma_{n-1}^*$  assumes only two necessarily finite and non-zero values depending on the parity of  $n$  (when  $|\pi(\mathbf{z}_k)| < \infty$ , it is equal to  $X_n^{-1}$ , see (A.0.6)). The matrix  $\mathbf{N}(z)$  solves the following Riemann-Hilbert problem (**RHP- $\mathbf{N}$** ):

- (a)  $\mathbf{N}(z)$  is analytic in  $\mathbb{C} \setminus \Delta$  and  $\lim_{z \rightarrow \infty} \mathbf{N}(z) z^{-n\sigma_3} = \mathbf{I}$ ;
- (b)  $\mathbf{N}(z)$  has continuous traces on  $\Delta^\circ$  that satisfy

$$\mathbf{N}_+(s) = \mathbf{N}_-(s) \begin{pmatrix} 0 & \rho(s) \\ -1/\rho(s) & 0 \end{pmatrix}, \quad s \in \Delta^\circ;$$



(c)  $\mathbf{N}(z)$  satisfies

$$\mathbf{N}(z) = \mathcal{O} \begin{pmatrix} |z - a_i|^{-(2\alpha_i+1)/4} & |z - a_i|^{(2\alpha_i-1)/4} \\ |z - a_i|^{-(2\alpha_i+1)/4} & |z - a_i|^{(2\alpha_i-1)/4} \end{pmatrix} \quad \text{as } z \rightarrow a_i,$$

$i \in \{1, 2, 3, 4\}$ , and

$$\mathbf{N}(z) = \mathcal{O} \begin{pmatrix} |z|^{(-1)^j \text{Re}(\nu)} & |z|^{(-1)^{j+1} \text{Re}(\nu)} \\ |z|^{(-1)^j \text{Re}(\nu)} & |z|^{(-1)^{j+1} \text{Re}(\nu)} \end{pmatrix} \quad \text{as } z \rightarrow 0,$$

where  $j \in \{1, 2, 3, 4\}$  is the number of the quadrant from which  $z \rightarrow 0$  and  $\nu$  is given by (5.3.1).

Indeed, **RHP- $\mathbf{N}$** (a) holds by construction, while **RHP- $\mathbf{N}$** (b,c) follow from (5.4.8) and (5.3.7), respectively (notice that the actual rate of behavior in **RHP- $\mathbf{N}$** (c) can be different if the considered point happens to coincide with  $z_{i(n)}$  or  $z_{i(n-1)}$ ). Notice also that  $\det(\mathbf{N}(z)) \equiv 1$  since this is an entire function (it clearly has no jumps and it can have at most square root singularities at the points  $a_i$ ) that converges to 1 at infinity.

For later calculations it will be convenient to set

$$\mathbf{M}^*(z) := \begin{pmatrix} (S_\rho T_{i(n)})(z^{(0)}) & (S_\rho T_{i(n)})(z^{(1)})/w(z) \\ (S_\rho T_{i(n-1)}/\Phi)(z^{(0)}) & (S_\rho T_{i(n-1)}/\Phi)(z^{(1)})/w(z) \end{pmatrix}, \quad (6.2.3)$$

and  $\mathbf{M}(z) := (\mathbf{I} + \mathbf{L}_\nu/z) \mathbf{M}^*(z)$ , where  $\mathbf{L}_\nu$  is a certain constant matrix with zero trace and determinant defined further below in (6.3.19). Observe that  $\mathbf{N}(z) = \mathbf{C} \mathbf{M}^*(z) \mathbf{D}(z)$ , where

$$\mathbf{C} := \begin{pmatrix} \gamma_n & 0 \\ 0 & \gamma_{n-1}^* \end{pmatrix} \quad \text{and} \quad \mathbf{D}(z) := \Phi^{n\sigma_3}(z^{(0)}). \quad (6.2.4)$$

When  $\text{Re}(\nu) \in (-1/2, 1/2)$ , it is possible to take  $\mathbf{L}_\nu$  to be the zero matrix, but this would worsen the error rates in (5.5.8) and (5.6.2). When  $\text{Re}(\nu) = 1/2$ , our analysis necessitates introduction of  $\mathbf{L}_\nu$ . Notice that neither the normalization of  $\mathbf{M}(z)$  at infinity nor its determinant do not depend on  $\mathbf{L}_\nu$ . In fact, it holds that  $\det(\mathbf{M}(z)) = \det(\mathbf{M}^*(z)) = (\gamma_n \gamma_{n-1}^*)^{-1}$ .

### 6.3 Local Analysis

#### 6.3.1 Local Parametrix around $a_i$

Let  $U_i$  be a disk around  $a_i$  of small enough radius so that  $\rho_i(z)$  is holomorphic around  $\overline{U}_i$ ,  $i \in \{1, 2, 3, 4\}$ . In this section we construct solution of **RHP- $\mathbf{X}$**  locally in each  $U_i$ . More precisely, we seeking a solution of the following local Riemann-Hilbert problem (**RHP- $\mathbf{P}_{a_i}$** ):

(a,b,c)  $\mathbf{P}_{a_i}(z)$  satisfies **RHP- $\mathbf{X}$** (a,b,c) within  $U_i$ ;

(d)  $\mathbf{P}_{a_i}(s) = \mathbf{M}(s)(\mathbf{I} + \mathcal{O}(1/n))\mathbf{D}(s)$  uniformly for  $s \in \partial U_i$ .

We shall only construct a solution of **RHP- $\mathbf{P}_{a_1}$**  as other constructions are almost identical.

#### Model Problem

Below, we always assume that the real line as well as its subintervals are oriented from left to right. Further, we set

$$I_{\pm} := \{z : \arg(\zeta) = \pm 2\pi/3\},$$

where the rays  $I_{\pm}$  are oriented towards the origin. Given  $\alpha > -1$ , let  $\Psi_{\alpha}(\zeta)$  be a matrix-valued function such that

(a)  $\Psi_{\alpha}(\zeta)$  is analytic in  $\mathbb{C} \setminus (I_+ \cup I_- \cup (-\infty, 0])$ ;

(b)  $\Psi_{\alpha}(\zeta)$  has continuous traces on  $I_+ \cup I_- \cup (-\infty, 0)$  that satisfy

$$\Psi_{\alpha+} = \Psi_{\alpha-} \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } (-\infty, 0), \\ \begin{pmatrix} 1 & 0 \\ e^{\pm \pi i \alpha} & 1 \end{pmatrix} & \text{on } I_{\pm}; \end{cases}$$

(c) as  $\zeta \rightarrow 0$  it holds that

$$\Psi_\alpha(\zeta) = \mathcal{O} \begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{\alpha/2} \end{pmatrix} \quad \text{and} \quad \Psi_\alpha(\zeta) = \mathcal{O} \begin{pmatrix} \log |\zeta| & \log |\zeta| \\ \log |\zeta| & \log |\zeta| \end{pmatrix}$$

when  $\alpha < 0$  and  $\alpha = 0$ , respectively, and

$$\Psi_\alpha(\zeta) = \mathcal{O} \begin{pmatrix} |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{\alpha/2} & |\zeta|^{-\alpha/2} \end{pmatrix} \quad \text{and} \quad \Psi_\alpha(\zeta) = \mathcal{O} \begin{pmatrix} |\zeta|^{-\alpha/2} & |\zeta|^{-\alpha/2} \\ |\zeta|^{-\alpha/2} & |\zeta|^{-\alpha/2} \end{pmatrix}$$

when  $\alpha > 0$ , for  $|\arg(\zeta)| < 2\pi/3$  and  $2\pi/3 < |\arg(\zeta)| < \pi$ , respectively;

(d) it holds uniformly in  $\mathbb{C} \setminus (I_+ \cup I_- \cup (-\infty, 0])$  that

$$\Psi_\alpha(\zeta) = \mathbf{S}(\zeta) (\mathbf{I} + \mathcal{O}(\zeta^{-1/2})) \exp \{2\zeta^{1/2}\sigma_3\},$$

where  $\mathbf{S}(\zeta) := \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \begin{pmatrix} 1 & \mathbf{i} \\ \mathbf{i} & 1 \end{pmatrix}$  and we take the principal branch of  $\zeta^{1/4}$ .

Explicit construction of this matrix can be found in [40] (it uses modified Bessel and Hankel functions). Observe that

$$\mathbf{S}_+(\zeta) = \mathbf{S}_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (6.3.1)$$

since the principal branch of  $\zeta^{1/4}$  satisfies  $\zeta_+^{1/4} = \mathbf{i}\zeta_-^{1/4}$ . Also notice that the matrix  $\sigma_3 \Psi_\alpha(\zeta) \sigma_3$  satisfies **RHP- $\Psi_\alpha$**  only with the reversed orientation of  $(-\infty, 0]$  and  $I_\pm$ .

## Conformal Map

Since  $w(z)$  has a square root singularity at  $a_1$  and satisfies  $w_+(s) = -w_-(s)$ ,  $s \in \Delta$ , the function

$$\zeta_{a_1}(z) := \left( \frac{1}{2} \int_{a_1}^z \frac{s ds}{w(s)} \right)^2, \quad z \in U_1, \quad (6.3.2)$$

is holomorphic in  $U_1$  with a simple zero at  $a_1$ . Thus, the radius of  $U_1$  can be made small enough so that  $\zeta_{a_1}(z)$  is conformal on  $\overline{U}_1$ . Observe that  $s ds/w_\pm(s)$  is purely imaginary on  $\Delta_1^\circ$  and therefore  $\zeta_{a_1}(z)$  maps  $\Delta_1 \cap U_1$  into the negative reals. It is also

rather obvious that  $\zeta_{a_1}(z)$  maps the interval  $(a_1, \infty) \cap U_1$  into the positive reals. As we have had some freedom in choosing the arcs  $\Gamma_{1\pm}$ , we shall choose them within  $U_1$  so that  $\Gamma_{1-}$  is mapped into  $I_+$  and  $\Gamma_{1+}$  is mapped into  $I_-$ . Notice that the orientation of the images of  $\Delta_1, \Gamma_{1+}, \Gamma_{1-}$  under  $\zeta_{a_1}(z)$  are opposite from the ones of  $(-\infty, 0], I_-, I_+$ .

In what follows, we understand that  $\zeta_{a_1}^{1/2}(z)$  stands for the branch given by the expression in the parenthesis in (6.3.2).

### Matrix $\mathbf{P}_{a_1}$

According to the definition of the class  $\mathcal{W}_1$ , it holds that

$$\rho(z) = \rho_1^*(z)(a_1 - z)^{\alpha_1}, \quad z \in U_1,$$

where  $\rho_1^*(z)$  is non-vanishing and holomorphic in  $U_1$  and  $(a_1 - z)^{\alpha_1}$  is the branch holomorphic in  $U_1 \setminus [a_1, \infty)$  and positive on  $\Delta_1$ . Define

$$r_{a_1}(z) := \sqrt{\rho_1^*(z)}(z - a_1)^{\alpha_1/2}, \quad z \in U_1 \setminus \Delta_1,$$

where  $(z - a_1)^{\alpha_1/2}$  is the principle branch. It clearly holds that

$$(z - a_1)^{\alpha_1} = e^{\pm \pi i \alpha_1} (a_1 - z)^{\alpha_1}, \quad z \in U_1^\pm,$$

where  $U_1^\pm := U_1 \cap \{\pm \text{Im}(z) > 0\}$ . Then

$$\begin{cases} r_{a_1+}(s)r_{a_1-}(s) = \rho(s), & s \in \Delta_1 \cap U_1, \\ r_{a_1}^2(z) = \rho(z)e^{\pm \pi i \alpha_1}, & z \in U_1^\pm. \end{cases}$$

The above relations and **RHP- $\Psi_\alpha$** (a,b,c) imply that

$$\mathbf{P}_{a_1}(z) := \mathbf{E}_{a_1}(z)\sigma_3\mathbf{\Psi}_{\alpha_1}\left(n^2\zeta_{a_1}(z)\right)\sigma_3r_{a_1}^{-\sigma_3}(z) \quad (6.3.3)$$

satisfies **RHP- $\mathbf{P}_{a_1}$** (a,b,c) for any holomorphic matrix  $\mathbf{E}_{a_1}(z)$ .

### Matrix $\mathbf{E}_{a_1}$

Now we choose  $\mathbf{E}_{a_1}(z)$  so that **RHP- $\mathbf{P}_{a_1}$** (d) is fulfilled. To this end, denote by  $V_1, V_2, V_3$  the sectors within  $U_1$  delimited by  $\pi(\alpha) \cup \pi(\beta)$ ,  $\pi(\beta) \cup \Delta_1$ , and  $\Delta_1 \cup \pi(\alpha)$ ,

respectively, see Figure 5.1. Let  $\gamma \subset \mathbb{C} \setminus \Delta$  be a path from  $a_3$  to  $a_1$  that does not intersect  $\pi(\alpha), \pi(\beta)$ . Further, let  $\gamma := \pi^{-1}(\gamma)$  be a cycle oriented so that  $\gamma^{(0)} := \gamma \cap \mathfrak{R}^{(0)}$  proceeds from  $a_3$  to  $a_1$ . Define

$$K_{a_1}(z) := \begin{cases} \exp \left\{ \int_{\gamma^{(0)}} G \right\} = \exp \{ \pi i (\tau - \omega) \} = 1, & z \in V_1, \\ \exp \left\{ \int_{\gamma^{(0)} - \alpha} G \right\} = \exp \{ -\pi i (\tau + \omega) \} = -1, & z \in V_2, \\ \exp \left\{ \int_{\gamma^{(0)} - \beta} G \right\} = \exp \{ \pi i (\tau + \omega) \} = -1, & z \in V_3, \end{cases}$$

where we used the symmetry  $G(z^*) = -G(z)$ , the fact that  $\gamma$  is homologous to  $\alpha + \beta$ , see Figure 5.2, and (5.2.3)–(5.2.4). Recalling the definition of  $\Phi(z)$  in (5.2.5) (the path of integration must lie in  $\mathfrak{R}_{\alpha, \beta}$ ), one can see that

$$\Phi(z^{(0)}) = K_{a_1}(z) \exp \{ 2\zeta_{a_1}^{1/2}(z) \}, \quad z \in V_1 \cup V_2 \cup V_3.$$

Clearly,  $|K_{a_1}(z)| = 1$ . It now follows from **RHP- $\Psi_\alpha$** (d) that

$$\mathbf{P}_{a_1}(s) = \mathbf{E}_{a_1}(s) \sigma_3 \mathbf{S}(n^2 \zeta_{a_1}(s)) \sigma_3 r_{a_1}^{-\sigma_3}(s) K_{a_1}^{-n\sigma_3}(s) (\mathbf{I} + \mathcal{O}(1/n)) \mathbf{D}(s)$$

for  $s \in \partial U_1$ . Thus, if the matrix

$$\mathbf{E}_{a_1}(z) := \mathbf{M}(z) K_{a_1}^{n\sigma_3}(z) r_{a_1}^{\sigma_3}(z) \sigma_3 \mathbf{S}^{-1}(n^2 \zeta_{a_1}(z)) \sigma_3$$

is holomorphic in  $U_1$ , **RHP- $\mathbf{P}_{a_1}$** (d) is clearly fulfilled. The fact that it has no jumps on  $\Delta_1, \pi(\alpha), \pi(\beta)$  follows from **RHP- $\mathbf{N}$** (b), (6.3.1), (5.2.6), and the definition of  $K_{a_1}(z)$ . Thus, it is holomorphic in  $U_1 \setminus \{a_1\}$ . Since  $|r_{a_1}(z)| \sim |z - a_1|^{\alpha_1/2}$ ,  $\mathbf{S}^{-1}(n^2 \zeta_{a_1}(z)) \sim |z - a_1|^{\sigma_3/4}$ , and  $\mathbf{M}(z)$  satisfies **RHP- $\mathbf{N}$** (c) around  $a_1$ , the desired claim follows.

### 6.3.2 Approximate Local Parametrix around the Origin

Let  $0 < \delta \leq \delta_0$ , see Section 6.1. We can assume that the closure of  $U_\delta := \{|z| < \delta\}$  is disjoint from  $\pi(\alpha), \pi(\beta)$ . In this section we construct an approximate solution of **RHP- $\mathbf{X}$**  in  $U_\delta$  when  $\ell < \infty$  and an exact solution of **RHP- $\mathbf{X}$**  in  $U_\delta$  when  $\ell = \infty$ .

To this end, let functions  $b_i(z)$ ,  $i \in \{1, 2, 3, 4\}$ , be defined in  $\overline{U}_{\delta_0}$  by

$$b_1 := \frac{\rho_1 + \rho_2}{\rho_2}, \quad b_2 := -\frac{\rho_2 + \rho_3}{\rho_4}, \quad b_3 := -\frac{\rho_3 + \rho_4}{\rho_2}, \quad \text{and} \quad b_4 := \frac{\rho_1 + \rho_4}{\rho_4}, \quad (6.3.4)$$

which are holomorphic and non-vanishing on  $\overline{U}_\delta$ . It follows from item (iv) in the definition of class  $\mathcal{W}_l$  that

$$\frac{b_i(0)}{b_i(z)} - 1 = \mathcal{O}(z^\ell) \quad \text{as } z \rightarrow 0, \quad i \in \{1, 2, 3, 4\}. \quad (6.3.5)$$

Notice that  $b_i(z) \equiv b_i(0)$  when  $\ell = \infty$ . Observe also that  $b_1(0) = b_3(0)$  and  $b_2(0) = b_4(0)$  by item (ii) in the definition of class  $\mathcal{W}_l$ . We are seeking a solution of the following local Riemann-Hilbert problem (RHP- $\mathbf{P}_0$ ):

- (a)  $\mathbf{P}_0(z)$  satisfies **RHP-X**(a) within  $U_\delta$ ;
- (b)  $\mathbf{P}_0(z)$  satisfies **RHP-X**(b) within  $U_\delta$ , where the jump matrix on each  $\tilde{\Delta}_i^\circ$  needs to be replaced by

$$\begin{pmatrix} 1 & 0 \\ \frac{b_i(0)}{b_i(s)} \left( \frac{1}{\rho_i(s)} + \frac{1}{\rho_{i+1}(s)} \right) & 1 \end{pmatrix};$$

- (c)  $\mathbf{P}_0(s) = \mathbf{M}(s) \left( \mathbf{I} + \mathcal{O}((n\delta^2)^{-1/2 - |\operatorname{Re}(\nu)|}) \right) \mathbf{D}(s)$  uniformly for  $s \in \partial U_\delta$  and  $\delta \leq \delta_0$ .

### Model Problem

A construction, similar the one below, has been introduced in [41], see also [42] and the book [43, Chapter 2], in the context of integrable systems. Unfortunately, the local problem is not stated in the form and generality we need in any of these references. Thus, for the convenience of the reader, we provide an explicit expression for the local parametrix.

Let  $s_1, s_2 \in \mathbb{C}$  be independent parameters and let  $\nu \in \mathbb{C}$ ,  $\operatorname{Re}(\nu) \in (-\frac{1}{2}, \frac{1}{2}]$  be given by

$$e^{-2\pi i \nu} := 1 - s_1 s_2 \quad (6.3.6)$$

(we slightly abuse the notation here as the parameter  $\nu$  has already been fixed in (5.3.1); however, we shall use the construction below with parameters  $s_1, s_2$  such that (6.3.6) holds with  $\nu$  from (5.3.1)). Define constants  $d_1, d_2$  by

$$d_1 := -s_1 \frac{\Gamma(1 + \nu)}{\sqrt{2\pi}} \quad \text{and} \quad d_2 := -s_2 e^{\pi i \nu} \frac{\Gamma(1 - \nu)}{\sqrt{2\pi}}, \quad (6.3.7)$$

where  $\Gamma(z)$  is the standard Gamma function. It follows from the well-known Gamma function identities that

$$d_1 d_2 = i\nu. \quad (6.3.8)$$

Denote by  $D_\mu(\zeta)$  the parabolic cylinder function in Whittaker's notations, see [44, Section 12.2]. It is an entire function with the asymptotic expansion

$$D_\mu(\zeta) \sim e^{-\zeta^2/4} \zeta^\mu \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)} \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-2k)} \frac{1}{(2\zeta^2)^k} \quad (6.3.9)$$

valid uniformly in each  $|\arg(\zeta)| \leq 3\pi/4 - \epsilon$ ,  $\epsilon > 0$ , see [44, Equation (12.9.1)].

Let the matrix function  $\Psi_{s_1, s_2}(\zeta)$  be given by

$$\begin{aligned} & \begin{pmatrix} D_\nu(2\zeta) & d_1 D_{-\nu-1}(-2i\zeta) \\ d_2 D_{\nu-1}(2\zeta) & D_{-\nu}(-2i\zeta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-\pi i \nu/2} \end{pmatrix}, \quad \arg(\zeta) \in (0, \frac{\pi}{2}), \\ & \begin{pmatrix} D_\nu(-2\zeta) & d_1 D_{-\nu-1}(-2i\zeta) \\ -d_2 D_{\nu-1}(-2\zeta) & D_{-\nu}(-2i\zeta) \end{pmatrix} \begin{pmatrix} e^{\pi i \nu} & 0 \\ 0 & e^{-\pi i \nu/2} \end{pmatrix}, \quad \arg(\zeta) \in (\frac{\pi}{2}, \pi), \\ & \begin{pmatrix} D_\nu(-2\zeta) & -d_1 D_{-\nu-1}(2i\zeta) \\ -d_2 D_{\nu-1}(-2\zeta) & D_{-\nu}(2i\zeta) \end{pmatrix} \begin{pmatrix} e^{-\pi i \nu} & 0 \\ 0 & e^{\pi i \nu/2} \end{pmatrix}, \quad \arg(\zeta) \in (-\frac{\pi}{2}, -\pi), \\ & \begin{pmatrix} D_\nu(2\zeta) & -d_1 D_{-\nu-1}(2i\zeta) \\ d_2 D_{\nu-1}(2\zeta) & D_{-\nu}(2i\zeta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i \nu/2} \end{pmatrix}, \quad \arg(\zeta) \in (0, -\frac{\pi}{2}). \end{aligned}$$

Then,  $\Psi_{s_1, s_2}(\zeta)$  satisfies the following RH problem (RHP- $\Psi_{s_1, s_2}$ ):

- (a)  $\Psi_{s_1, s_2}(\zeta)$  is analytic in  $\mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R})$ ;
- (b)  $\Psi_{s_1, s_2}(\zeta)$  has continuous traces on  $\mathbb{R} \cup i\mathbb{R}$  outside of the origin that satisfy the jump relations shown in Figure 6.2;
- (c)  $\Psi_{s_1, s_2}(\zeta)$  has the following asymptotic expansion as  $\zeta \rightarrow \infty$ :

$$\left( \mathbf{I} + \frac{1}{2\zeta} \begin{pmatrix} 0 & id_1 \\ d_2 & 0 \end{pmatrix} + \frac{\nu(\nu-1)}{8\zeta^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}\left(\frac{1}{\zeta^3}\right) \right) (2\zeta)^{\nu\sigma_3} e^{-\zeta^2\sigma_3},$$

which holds uniformly in  $\mathbb{C}$ .

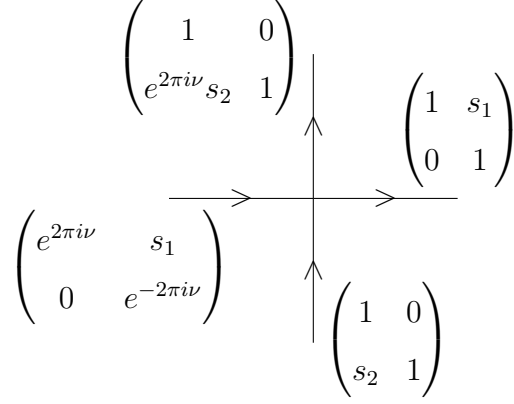


Fig. 6.2. Matrices  $\Psi_{s_1, s_2-}^{-1} \Psi_{s_1, s_2+}$  on the corresponding rays.

Indeed, **RHP- $\Psi_{s_1, s_2}$** (a) follows from the fact that  $D_\nu(\zeta)$  is entire, while **RHP- $\Psi_{s_1, s_2}$** (c) is a consequence of (6.3.9). The jump relations **RHP- $\Psi_{s_1, s_2}$** (b) can be verified using the identities  $\Gamma(-\nu)\Gamma(1+\nu) = -\pi/\sin(\pi\nu)$ , (6.3.6), and

$$D_\mu(2\xi) = e^{-\mu\pi i} D_\mu(-2\xi) + \frac{\sqrt{2\pi}}{\Gamma(-\mu)} e^{-(\mu+1)\pi i/2} D_{-\mu-1}(2i\xi),$$

suitably applied with parameter values  $\mu = -\nu, \nu - 1$  and  $\xi = \zeta, -\zeta, i\zeta$ . For later, it will be important for us to make the following observation. Define

$$d_\nu := \begin{cases} d_2, & \operatorname{Re}(\nu) > 0, \\ 0, & \operatorname{Re}(\nu) = 0, \\ id_1, & \operatorname{Re}(\nu) < 0 \end{cases} \quad \text{and} \quad \mathbf{A}_\nu := \begin{cases} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & \operatorname{Re}(\nu) \geq 0, \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \operatorname{Re}(\nu) < 0, \end{cases} \quad (6.3.10)$$

Recall that we set  $\varsigma_\nu = 1, 0, -1$  depending on whether  $\operatorname{Re}(\nu) > 0$ ,  $\operatorname{Re}(\nu) = 0$ , or  $\operatorname{Re}(\nu) < 0$ . Observe that

$$\begin{aligned} & (\mathbf{I} - (2\zeta)^{-1} d_\nu \mathbf{A}_\nu) \Psi_{s_1, s_2}(\zeta) \\ &= (2\zeta)^{\nu\sigma_3} \left( \mathbf{I} + (2\zeta)^{-1-2\varsigma_\nu\nu} d_{-\nu} \mathbf{A}_{-\nu} + \mathcal{O}(\zeta^{-1-|\varsigma_\nu|}) \right) e^{-\zeta^2\sigma_3}. \end{aligned} \quad (6.3.11)$$



## Conformal Map

Let, as before,  $\mathcal{Q}_j$  stand for the  $j$ -th quadrant,  $j \in \{1, 2, 3, 4\}$ . Set

$$\zeta_0(z) := \left( (-1)^{j-1} \int_0^z \frac{s ds}{w(s)} \right)^{1/2}, \quad z \in U_\delta \cap \mathcal{Q}_j. \quad (6.3.12)$$

Since  $w(z)$  is bounded at 0 and satisfies  $w_+(s) = -w_-(s)$ ,  $s \in \Delta$ , the branch of the square root can be chosen so that the function  $\zeta_0(z)$  is in fact holomorphic in  $U_\delta$  with a simple zero at the origin. Without loss of generality we can assume that  $\delta$  is small enough for  $\zeta_0(z)$  to be conformal on  $\bar{U}_\delta$ .

Since the integrand  $(-1)^{j-1} s ds / w(s)$  becomes negative purely imaginary on  $\Delta_1 \cup \Delta_3$ , the square root in (6.3.12) can be chosen so that  $\arg(\zeta_0(z)) = -\pi/4$ ,  $z \in \Delta_3^\circ$ . As we have had some freedom in selecting the arcs  $\tilde{\Delta}_i$ , we shall choose them so that  $\tilde{\Delta}_3^\circ$  and  $\tilde{\Delta}_1^\circ$  are mapped by  $\zeta_0(z)$  into positive and negative reals, respectively, while  $\tilde{\Delta}_4^\circ$  and  $\tilde{\Delta}_2^\circ$  are mapped into positive and negative purely imaginary numbers.

## Matrix $P_0$

Define the function  $r(z) := r_j(z)$ ,  $z \in \mathcal{Q}_j$ , where we let

$$r_1 := ie^{\pi i \nu} \sqrt{\rho_1}, \quad r_2 := ie^{-\pi i \nu} \frac{\rho_2}{\sqrt{\rho_1}}, \quad r_3 := -ie^{-\pi i \nu} \frac{\rho_4}{\sqrt{\rho_1}}, \quad r_4 := -ie^{-\pi i \nu} \sqrt{\rho_1} \quad (6.3.13)$$

for a fixed determination of  $\sqrt{\rho_1(z)}$ . Furthermore, let

$$\mathbf{J}(z) := \begin{cases} e^{2\pi i \nu \sigma_3}, & \arg z \in \left(-\frac{\pi}{2}, 0\right), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{2\pi i \nu \sigma_3}, & \arg z \in \left(0, \frac{\pi}{4}\right), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \arg z \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, -\pi\right), \\ \mathbf{I}, & \arg z \in \left(\frac{\pi}{2}, \pi\right). \end{cases} \quad (6.3.14)$$

Finally, recalling (6.3.4), put

$$s_1 := b_1(0) = b_3(0) \quad \text{and} \quad s_2 := b_2(0) = b_4(0). \quad (6.3.15)$$

Notice that since  $(\rho_1 + \rho_2 + \rho_3 + \rho_4)(0) = 0$ , the parameters  $s_1, s_2$  satisfy (6.3.6) with  $\nu$  given by (5.3.1). Then

$$\mathbf{P}_0(z) := \mathbf{E}_0(z) \mathbf{\Psi}_{s_1, s_2}(n^{1/2} \zeta_0(z)) \mathbf{J}^{-1}(z) r^{-\sigma_3}(z) \quad (6.3.16)$$

satisfies **RHP- $\mathbf{P}_0$** (a,b) for any matrix  $\mathbf{E}_0(z)$  holomorphic in  $U_\delta$ . Indeed, **RHP- $\mathbf{P}_0$** (a) is an immediate consequence of **RHP- $\mathbf{\Psi}_{s_1, s_2}$** (a). It further follows from **RHP- $\mathbf{\Psi}_{s_1, s_2}$** (b) that the jumps of  $\mathbf{P}_0(z)$  are as on Figure 6.3. To verify **RHP- $\mathbf{P}_0$** (b), it remains to

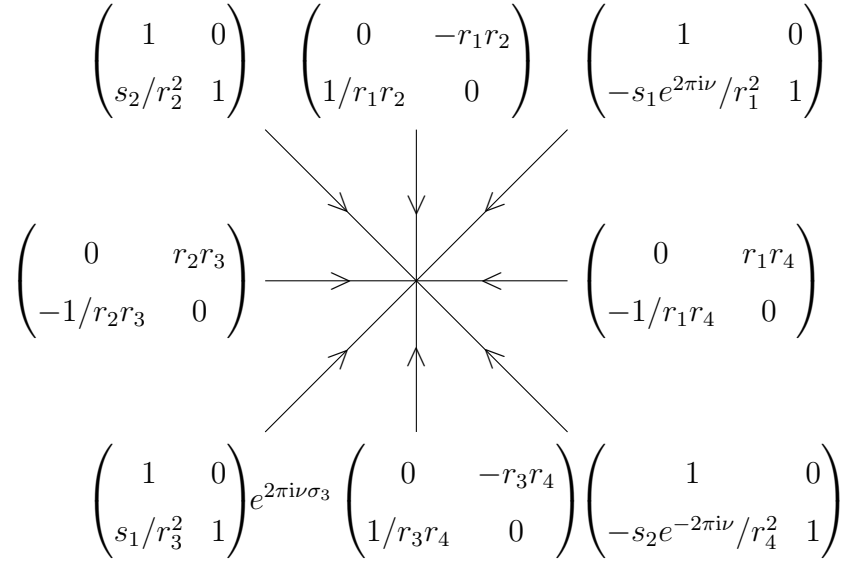


Fig. 6.3. The jump matrices of  $\mathbf{P}_0(z)$ .

observe that

$$r_1 r_4 = \rho_1, \quad -r_1 r_2 = \rho_2, \quad r_2 r_3 = e^{-2\pi i \nu} \rho_2 \rho_4 / \rho_1 = \rho_3, \quad -r_3 r_4 e^{2\pi i \nu} = \rho_4,$$

since  $e^{-2\pi i\nu} = (\rho_1\rho_3)/(\rho_2\rho_4)$ , and that

$$\begin{aligned} -e^{2\pi i\nu} \frac{s_1}{r_1^2} &= \frac{b_1(0)}{\rho_1} = \frac{b_1(0)}{b_1} \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right), \\ \frac{s_2}{r_2^2} &= -e^{2\pi i\nu} b_2(0) \frac{\rho_1}{\rho_2^2} = -b_2(0) \frac{\rho_4}{\rho_2\rho_3} = \frac{b_2(0)}{b_2} \left( \frac{1}{\rho_2} + \frac{1}{\rho_3} \right), \\ \frac{s_1}{r_3^2} &= -e^{2\pi i\nu} b_3(0) \frac{\rho_1}{\rho_4^2} = -b_3(0) \frac{\rho_2}{\rho_3\rho_4} = \frac{b_3(0)}{b_3} \left( \frac{1}{\rho_3} + \frac{1}{\rho_4} \right), \\ -e^{-2\pi i\nu} \frac{s_2}{r_4^2} &= \frac{b_4(0)}{\rho_1} = \frac{b_4(0)}{b_4} \left( \frac{1}{\rho_1} + \frac{1}{\rho_4} \right). \end{aligned}$$

Thus, it remains to choose  $\mathbf{E}_0(z)$  so that **RHP- $\mathbf{P}_0$** (c) is fulfilled.

### Matrix $\mathbf{E}_0$

Let  $\gamma$  be the part of  $\Delta_3$  that proceeds from  $\mathbf{a}_3$  to  $\mathbf{0}^*$ , see Figures 5.1 and 5.2. Define

$$K_0(z) := \begin{cases} \exp \left\{ -\int_{\gamma} G \right\} = \Phi(\mathbf{0}), & z \in \mathcal{Q}_1 \cup \mathcal{Q}_3, \\ \exp \left\{ \int_{\gamma} G \right\} = \Phi(\mathbf{0}^*), & z \in \mathcal{Q}_2 \cup \mathcal{Q}_4. \end{cases} \quad (6.3.17)$$

(5.2.10) immediately yields that  $|K_0(z)| \equiv 1$ . Define

$$\mathbf{E}_0^*(z) := \mathbf{M}^*(z) r^{\sigma_3}(z) K_0^{n\sigma_3}(z) \mathbf{J}(z) \zeta_0^{-\nu\sigma_3}(z), \quad (6.3.18)$$

see (6.2.3). From **RHP- $\mathbf{N}$** (b), the definition of  $\mathbf{J}(z)$ , and the fact that  $\zeta_0(z)$  maps  $\tilde{\Delta}_1^\circ$  into the negative reals, it follows that  $\mathbf{E}_0^*(z)$  is holomorphic in  $U_\delta \setminus \{0\}$ . Furthermore, **RHP- $\mathbf{N}$** (c) combined with the fact that  $\zeta_0(z)$  possesses a simple zero at  $z = 0$  imply that  $\mathbf{E}_0^*(z)$  is holomorphic in  $U_\delta$ . Observe also that the moduli of the entries of  $\mathbf{E}_0^*(z)$  depend only on the parity of  $n$ .

Put for brevity  $\epsilon_{\nu,n} := (4n)^{\varsigma_{\nu}\nu-1/2}$ , where, as before,  $\varsigma_{\nu}$  is equal to 1, 0,  $-1$  depending on whether  $\text{Re}(\nu)$  is positive, zero, or negative. Set

$$\mathbf{L}_{\nu} := \frac{d_{\nu}\epsilon_{\nu,n}}{\zeta_0'(0)D_n} \mathbf{E}_0^*(0) \mathbf{A}_{\nu} \mathbf{E}_0^{*-1}(0), \quad (6.3.19)$$

where  $d_{\nu}, \mathbf{A}_{\nu}$  were defined in (6.3.10) and we assume that

$$0 \neq D_n := 1 - d_{\nu}\epsilon_{\nu,n} (\zeta_0'(0))^{-1} E_{\nu} \quad (6.3.20)$$

with

$$E_\nu := \begin{cases} [\mathbf{E}_0^{\star-1}(0)\mathbf{E}_0^{\star'}(0)]_{12} & \text{if } \operatorname{Re}(\nu) \geq 0, \\ [\mathbf{E}_0^{\star-1}(0)\mathbf{E}_0^{\star'}(0)]_{21} & \text{if } \operatorname{Re}(\nu) < 0. \end{cases}$$

Notice that  $\mathbf{L}_\nu$  is the zero matrix when  $\operatorname{Re}(\nu) = 0$  as  $d_\nu = 0$  by (6.3.10). Let

$$\mathbf{E}_0(z) := (\mathbf{I} + \mathbf{L}_\nu/z)\mathbf{E}_0^\star(z)(4n)^{-\nu\sigma_3/2}(\mathbf{I} - d_\nu(2n^{1/2}\zeta_0(z))^{-1}\mathbf{A}_\nu). \quad (6.3.21)$$

Let us show that thus defined matrix  $\mathbf{E}_0(z)$  is holomorphic at the origin. Indeed, it has at most double pole there. It is quite simple to see that the coefficient next to  $z^{-2}$  is equal to

$$-d_\nu\epsilon_{\nu,n}(4n)^{-\varsigma_\nu\nu/2}(\zeta_0'(0))^{-1}\mathbf{L}_\nu\mathbf{E}_0^\star(0)\mathbf{A}_\nu,$$

which is equal to the zero matrix since  $\mathbf{A}_\nu^2$  is equal to the zero matrix. Using this observation we also get that the coefficient next to  $z^{-1}$  is equal to

$$\mathbf{L}_\nu\mathbf{E}_0^\star(0)(4n)^{-\nu\sigma_3/2} - d_\nu\epsilon_{\nu,n}(4n)^{-\varsigma_\nu\nu/2}(\zeta_0'(0))^{-1}(\mathbf{E}_0^\star(0) + \mathbf{L}_\nu\mathbf{E}_0^{\star'}(0))\mathbf{A}_\nu,$$

which simplifies to

$$\frac{d_\nu\epsilon_{\nu,n}(4n)^{-\varsigma_\nu\nu/2}}{\zeta_0'(0)D_n} \left(1 - \frac{d_\nu\epsilon_{\nu,n}}{\zeta_0'(0)}E_\nu - D_n\right) \mathbf{E}_0^\star(0)\mathbf{A}_\nu,$$

that is equal to the zero matrix by the very definition of  $D_n$ .

Now, recalling the definition of  $\Phi(z)$  in (5.2.5) and of  $\zeta_0(z)$  in (6.3.12), one can see that

$$\exp\{-\zeta_0^2(z)\} = e^{-\int_\gamma G} \begin{cases} \Phi(z^{(1)}), & z \in \mathcal{Q}_1 \cup \mathcal{Q}_3, \\ \Phi(z^{(0)}), & z \in \mathcal{Q}_2 \cup \mathcal{Q}_4. \end{cases} \quad (6.3.22)$$

In particular, since  $\mathbf{D}(z) = \Phi^{n\sigma_3}(z^{(0)})$  and  $\Phi(z^{(0)})\Phi(z^{(1)}) \equiv 1$ , it follows from (6.3.17) that

$$\exp\{-n\zeta_0^2(z)\sigma_3\}\mathbf{J}^{-1}(z) = \mathbf{J}^{-1}(z)K_0^{-n\sigma_3}(z)\mathbf{D}(z).$$

For brevity, let  $\mathbf{H}(z) := r^{\sigma_3}(z)K_0^{n\sigma_3}(z)\mathbf{J}(z)$ . Then we get from (6.3.11) and the previous identity that

$$\begin{aligned} \mathbf{E}_0(s)\Psi_{s_1,s_2}(n^{1/2}\zeta_0(s))\mathbf{J}^{-1}(s)r^{-\sigma_3}(s) = \\ \mathbf{M}(s)\mathbf{H}(s)\left(\mathbf{I} + \mathcal{O}\left((n\zeta_0^2(s))^{-1/2-|\operatorname{Re}(\nu)|}\right)\right)\mathbf{H}^{-1}(s)\mathbf{D}(s) = \\ \mathbf{M}(s)\left(\mathbf{I} + \mathcal{O}\left((n\delta^2)^{-1/2-|\operatorname{Re}(\nu)|}\right)\right)\mathbf{D}(s). \end{aligned}$$

It remains to show that (6.3.20) holds for all  $n \in \mathbb{N}_{\rho,\epsilon}$ . It follows from (5.5.2) that it is enough to show that

$$A_{\rho,n} = d_\nu \epsilon_{\nu,n} (\zeta'_0(0))^{-1} E_\nu. \quad (6.3.23)$$

### Existence of $\mathbf{L}_\nu$

Assume that  $\operatorname{Re}(\nu) > 0$ . It can be readily verified that

$$E_\nu = \gamma_n \gamma_{n-1}^* ([\mathbf{E}_0^{\star'}(0)]_{12} [\mathbf{E}_0^\star(0)]_{22} - [\mathbf{E}_0^{\star'}(0)]_{22} [\mathbf{E}_0^\star(0)]_{12}),$$

where we used the fact that  $\det(\mathbf{E}_0^\star(z)) = \det(\mathbf{M}^\star(z)) = (\gamma_n \gamma_{n-1}^*)^{-1}$ . Notice that  $d_2 \neq 0$  by (6.3.8). Using (6.3.18), (6.3.14), and (6.3.17) gives us that  $[\mathbf{E}_0^\star(z)]_{i2}$  is equal to

$$\zeta_0^\nu(z)\Phi^n(\mathbf{0}) \left\{ \begin{array}{ll} e^{-2\pi i\nu} r_1(z) [\mathbf{M}^\star(z)]_{i1}, & \arg(z) \in (0, \pi/4), \\ r_1(z) [\mathbf{M}^\star(z)]_{i1}, & \arg(z) \in (\pi/4, \pi/2), \\ [\mathbf{M}^\star(z)]_{i2}/r_2(z), & \arg(z) \in (\pi/2, \pi), \\ r_3(z) [\mathbf{M}^\star(z)]_{i1}, & \arg(z) \in (\pi, 3\pi/2), \\ e^{-2\pi i\nu} [\mathbf{M}^\star(z)]_{i2}/r_4(z), & \arg(z) \in (3\pi/2, 2\pi). \end{array} \right.$$

Define

$$S(z) := \zeta_0^\nu(z) \begin{cases} e^{-2\pi i \nu} r_1(z) S_\rho(z^{(0)}), & \arg(z) \in (0, \pi/4), \\ r_1(z) S_\rho(z^{(0)}), & \arg(z) \in (\pi/4, \pi/2), \\ S_\rho(z^{(1)})/(r_2 w)(z), & \arg(z) \in (\pi/2, \pi), \\ r_3(z) S_\rho(z^{(0)}), & \arg(z) \in (\pi, 3\pi/2), \\ e^{-2\pi i \nu} S_\rho(z^{(1)})/(r_4 w)(z), & \arg(z) \in (3\pi/2, 2\pi), \end{cases}$$

which is a holomorphic and non-vanishing function around the origin. Then we obtain from (6.2.3), (A.0.6), and (A.0.12) that

$$E_\nu = S^2(0) \Phi^{2n}(\mathbf{0}) Y_n X_n^{-1}. \quad (6.3.24)$$

When  $|\pi(\mathbf{z}_k)| = \infty$ , the first condition in the definition of  $\mathbb{N}_{\rho, \epsilon}$  implies that we are looking only at those indices  $n$  for which  $\mathbf{z}_{i(n)} = \infty^{(1)}$ . In this case  $A_{\rho, n} = 0$  by its very definition in (5.5.2) and it also follows from Lemma A.0.9 that  $Y_n = 0$  in this case. Hence, (6.3.23) does hold in this case.

Let now  $|\pi(\mathbf{z}_k)| < \infty$  and therefore the first condition in the definition of  $\mathbb{N}_{\rho, \epsilon}$  is void. It follows from (6.3.12) and (5.1.1) as well as the fact that  $\zeta_0(z)$  maps  $\{\arg(z) = 5\pi/4\}$  into the positive reals that

$$1/\zeta_0'(0) = e^{5\pi i/4} \sqrt{2ab}. \quad (6.3.25)$$

Since  $e^{-2\pi i \nu} = (\rho_1 \rho_3)(0)/(\rho_2 \rho_4)(0)$  by (5.3.1), we get from (6.3.13) that

$$S^2(0) = -(\rho_3 \rho_4 / \rho_2)(0) (2ab)^{-\nu} \lim_{z \rightarrow 0, \arg(z) = 5\pi/4} |z|^{2\nu} S_\rho^2(z^{(0)}). \quad (6.3.26)$$

Observe also that

$$d_2 = e^{\pi i \nu} \frac{(\rho_2 + \rho_3)(0)}{\rho_4(0)} \frac{\Gamma(1 - \nu)}{\sqrt{2\pi}} \quad (6.3.27)$$

by (6.3.7), (6.3.15), and (6.3.4). Then it follows from (A.0.16) and the very definitions of  $A_{\rho, n}$  in (5.5.2) that (6.3.24)–(6.3.27) yield (6.3.23). The proof of (6.3.23) in the case  $\operatorname{Re}(\nu) < 0$  is similar.

Since  $|\pi(\mathbf{z}_k)| < \infty$ , the quantities  $Y_n$  and  $Z_n$  in (A.0.12) and (A.0.14) are non-zero and equal to

$$W'_{i(n)}(\mathbf{o}) \frac{T_{i(n-1)}^2(\mathbf{o})}{\Phi(\mathbf{o})}, \quad W_{i(n)}(\mathbf{z}) := \frac{T_{i(n)}(\mathbf{z})}{T_{i(n-1)}(\mathbf{z})},$$

where  $\mathbf{o}$  was defined in (5.5.1). Hence, it follows from (6.3.19), (6.3.23), (6.3.24), and a computation similar to the one carried out at the beginning of this subsection that

$$\mathbf{L}_\nu = \frac{A_{\rho,n}}{1 - A_{\rho,n}} \frac{1}{W'_{i(n)}(\mathbf{o})} \begin{pmatrix} W_{i(n)}(\mathbf{o}) & -\Phi(\mathbf{o})W_{i(n)}^2(\mathbf{o}) \\ 1/\Phi(\mathbf{o}) & -W_{i(n)}(\mathbf{o}) \end{pmatrix}.$$

Moreover, since  $W_1(\mathbf{z}) = 1/W_0(\mathbf{z})$  we can rewrite the first row of  $\mathbf{L}_\nu$  as

$$(1 \ 0)\mathbf{L}_\nu = (-1)^{i(n)} \frac{A_{\rho,n}}{1 - A_{\rho,n}} \frac{W_0(\mathbf{o})}{W'_0(\mathbf{o})} \begin{pmatrix} 1 & -\Phi(\mathbf{o})W_{i(n)}(\mathbf{o}) \end{pmatrix}. \quad (6.3.28)$$

#### 6.4 Final Riemann-Hilbert Problem

In what follows, we assume that  $\delta = \delta_n \leq \delta_0$  in Section 6.3.2 when  $\ell < \infty$  and shall specify the exact dependence on  $n$  later on in this section. When  $\ell = \infty$ , we simply take  $\delta = \delta_0$ . Set  $U := \cup_{i=1}^4 U_{a_i}$  and define

$$\Sigma_n := (\partial U \cup \partial U_{\delta_n}) \cup \left( \cup_{i=1}^4 (\Gamma_{i-} \cup \Gamma_{i+} \cup \tilde{\Delta}_i) \setminus \bar{U} \right),$$

see Figure 6.4. We are looking for a solution of the following Riemann-Hilbert problem (RHP- $\mathbf{Z}$ ):

- (a)  $\mathbf{Z}(z)$  is analytic in  $\bar{\mathbb{C}} \setminus \Sigma_n$  and  $\lim_{z \rightarrow \infty} \mathbf{Z}(z) = \mathbf{I}$ ;
- (b)  $\mathbf{Z}(z)$  has continuous traces outside of non-smooth points of  $\Sigma_n$  that satisfy

$$\mathbf{Z}_+ = \mathbf{Z}_- \begin{cases} \mathbf{P}_{a_i}(\mathbf{M}\mathbf{D})^{-1}, & \text{on } \partial U_{a_i}, \\ \mathbf{P}_0(\mathbf{M}\mathbf{D})^{-1}, & \text{on } \partial U_\delta, \\ \mathbf{M}\mathbf{D} \begin{pmatrix} 1 & 0 \\ 1/\rho_i & 1 \end{pmatrix} (\mathbf{M}\mathbf{D})^{-1}, & \text{on } (\Gamma_{i+}^\circ \cup \Gamma_{i-}^\circ) \setminus \bar{U}, \end{cases}$$

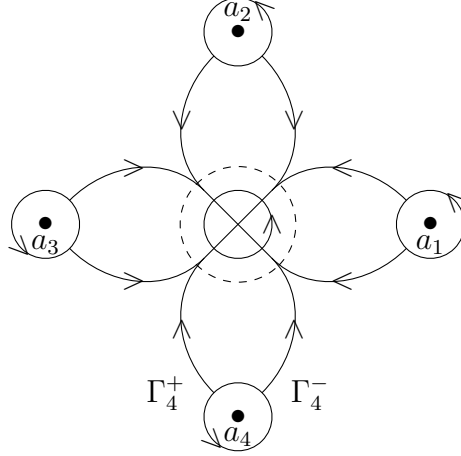


Fig. 6.4. Contour  $\Sigma_n$  for **RHP- $\mathbf{Z}$**  (dashed circle represents  $\{|z| = \delta_0\}$ ).

and

$$\mathbf{Z}_+ = \mathbf{Z}_- \begin{cases} \mathbf{M}\mathbf{D} \begin{pmatrix} 1 & 0 \\ \frac{\rho_i + \rho_{i+1}}{\rho_i \rho_{i+1}} & 1 \end{pmatrix} (\mathbf{M}\mathbf{D})^{-1}, & \text{on } \tilde{\Delta}_i^\circ \setminus \overline{U}_{\delta_n}, \\ \mathbf{P}_{0-} \begin{pmatrix} 1 & 0 \\ \frac{\rho_i + \rho_{i+1}}{\rho_i \rho_{i+1}} & 1 \end{pmatrix} \mathbf{P}_{0+}^{-1}, & \text{on } \tilde{\Delta}_i^\circ \cap U_{\delta_n} \end{cases}$$

(notice that the second set of jumps is not present when  $\ell = \infty$  as  $\delta_n = \delta_0$  and  $\mathbf{P}_0(z)$  is the exact parametrix).

It follows from **RHP- $\mathbf{P}_{a_i}$** (d) that the jump of  $\mathbf{Z}$  on  $\partial U_{a_i}$  can be written as

$$\mathbf{M}(s)(\mathbf{I} + \mathcal{O}(1/n))\mathbf{M}^{-1}(s) = \mathbf{I} + \mathcal{O}_\epsilon(1/n)$$

since the matrix  $\mathbf{M}(z)$  is invertible (its determinant is equal to the reciprocal of  $\gamma_n \gamma_{n-1}^*$ ), the matrix  $\mathbf{M}^*(z)$  depends only on the parity of  $n$ , see (6.2.3), and the matrix  $\mathbf{L}_\nu$  has trace and determinant zero as well as bounded entries for all  $n \in \mathbb{N}_{\rho, \epsilon}$



and each fixed  $\epsilon > 0$ , see (6.3.19). Similarly, we get from RHP- $\mathbf{P}_0$ (c) that the jump of  $\mathbf{Z}$  on  $\partial U_{\delta_n}$  can be written as

$$\begin{aligned} \mathbf{M}(s) \left( \mathbf{I} + \mathcal{O} \left( (n\delta_n^2)^{-1/2-|\operatorname{Re}(\nu)|} \right) \right) \mathbf{M}^{-1}(s) \\ = \mathbf{I} + (\mathbf{I} + \mathbf{L}_\nu/s) \mathcal{O} \left( (n\delta_n^2)^{-1/2-|\operatorname{Re}(\nu)|} \right) (\mathbf{I} - \mathbf{L}_\nu/s), \end{aligned}$$

where  $\mathcal{O}(\cdot)$  does not depend on  $n$ . Since  $\mathbf{L}_\nu = \mathcal{O}_\epsilon(n^{|\operatorname{Re}(\nu)|-1/2})$  by its very definition in (6.3.19), we get that the jump of  $\mathbf{Z}$  on  $\partial U_{\delta_n}$  can further be written as

$$\mathbf{I} + \mathcal{O}_\epsilon \left( (n\delta_n^2)^{-1/2-|\operatorname{Re}(\nu)|} \max \{1, n^{2|\operatorname{Re}(\nu)|}/(n\delta_n^2)\} \right).$$

One can easily check with the help of (6.2.1) and (6.2.3) that the jump of  $\mathbf{Z}$  on  $(\Gamma_{i+}^\circ \cup \Gamma_{i-}^\circ) \setminus \bar{U}$  is equal to

$$\begin{aligned} \mathbf{I} + \frac{\gamma_n \gamma_{n-1}^*}{(w^2 \rho_i)(s)} (\mathbf{I} + \mathbf{L}_\nu/s) \begin{pmatrix} (\Psi_n \Psi_{n-1})(s^{(1)}) & -\Psi_n^2(s^{(1)}) \\ \Psi_{n-1}^2(s^{(1)}) & -(\Psi_n \Psi_{n-1})(s^{(1)}) \end{pmatrix} (\mathbf{I} - \mathbf{L}_\nu/s) \\ = \mathbf{I} + \mathcal{O}_\epsilon(e^{-cn}) \end{aligned}$$

for some constant  $c > 0$  by (5.4.7) and since the maximum of  $|\Phi(s^{(1)})|$  on  $\Gamma_{i\pm} \setminus U$  is less than 1. The estimate of the jump of  $\mathbf{Z}$  on  $\tilde{\Delta}_i^\circ \setminus \bar{U}_{\delta_n}$  is analogous and yields

$$\mathbf{I} + \mathcal{O}_\epsilon \left( e^{-cn\delta_n^2} \max \{1, n^{2|\operatorname{Re}(\nu)|}/(n\delta_n^2)\} \right)$$

for an adjusted constant  $c > 0$ , where the rate estimate follows from (6.3.22) as

$$|\Phi(s^{(1)})| = \exp \{(-1)^i \operatorname{Re}(\zeta_0^2(s))\} = \mathcal{O}(e^{-c\delta_n^2}), \quad s \in \tilde{\Delta}_i \setminus U_{\delta_n},$$

since  $\zeta_0(z)$  is real on  $\tilde{\Delta}_1 \cup \tilde{\Delta}_3$  and is purely imaginary on  $\tilde{\Delta}_2 \cup \tilde{\Delta}_4$ .

Finally, it holds on  $\tilde{\Delta}_i^\circ \cap U_{\delta_n}$  that the jump of  $\mathbf{Z}$  is equal to

$$\begin{aligned} \mathbf{I} + \left( 1 - \frac{b_i(0)}{b_i(z)} \right) \frac{(\rho_i + \rho_{i+1})(s)}{(\rho_i \rho_{i+1})(s)} \mathbf{P}_{0+}(s) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{P}_{0+}^{-1}(s) = \\ \mathbf{I} + \mathcal{O}(\delta_n^\ell) \mathbf{E}_0(s) \begin{pmatrix} [\Psi_+(s)]_{1j} [\Psi_+(s)]_{2j} & -[\Psi_+(s)]_{1j}^2 \\ [\Psi_+(s)]_{2j}^2 & -[\Psi_+(s)]_{1j} [\Psi_+(s)]_{2j} \end{pmatrix} \mathbf{E}_0^{-1}(s) \end{aligned}$$

by (6.3.5) and (6.3.16), where  $j = 1$  for  $s \in \tilde{\Delta}_1 \cup \tilde{\Delta}_3$  and  $j = 2$  for  $s \in \tilde{\Delta}_2 \cup \tilde{\Delta}_4$ , and we set for brevity  $\Psi(z) := \Psi_{s_1, s_2}(n^{1/2}\zeta_0(z))$  (observe also that  $\det(\Psi(z)) \equiv 1$ ). It follows from the asymptotic expansion (6.3.9) that  $D_\mu(x)$  is bounded for  $x \geq 0$ . Thus, we deduce from the definition of  $\Psi(z)$  that the above jump matrix can be estimated as

$$\mathbf{I} + \mathcal{O}(\delta_n^\ell) \mathbf{E}_0(s) \mathcal{O}(1) \mathbf{E}_0^{-1}(s) = \mathbf{I} + \mathcal{O}_\epsilon(n^{|\operatorname{Re}(\nu)|} \delta_n^\ell),$$

where the last equality follows from (6.3.18) and (6.3.21) as  $\mathbf{E}_0(z)$  is equal to a bounded matrix that depends only on  $\epsilon_{\nu, n}$  multiplied by  $(4n)^{\nu\sigma_3/2}$  on the right.

When  $\ell \geq 4|\operatorname{Re}(\nu)|(1 + |\operatorname{Re}(\nu)|)/(1 - 2|\operatorname{Re}(\nu)|)$ , choose

$$\delta_n = \delta_0 \exp \left\{ -\frac{1}{2} \frac{1 + 4|\operatorname{Re}(\nu)|}{\ell + 1 + 2|\operatorname{Re}(\nu)|} \ln n \right\}. \quad (6.4.1)$$

Then it holds that  $n^{2|\operatorname{Re}(\nu)|}/(n\delta_n^2) = \mathcal{O}(1)$  and

$$n^{|\operatorname{Re}(\nu)|}(\delta_n/\delta_0)^\ell = (n(\delta_n/\delta_0)^2)^{-|\operatorname{Re}(\nu)|-1/2} = n^{-d_{\nu, \ell}}$$

with  $d_{\nu, \ell}$  defined in (5.5.7). Otherwise, take

$$\delta_n = \delta_0 \exp \left\{ -\frac{1}{2} \frac{3}{\ell + 3 + 2|\operatorname{Re}(\nu)|} \ln n \right\}.$$

In this case  $n^{2|\operatorname{Re}(\nu)|}/(n\delta_n^2) \rightarrow \infty$  as  $n \rightarrow \infty$  and

$$n^{|\operatorname{Re}(\nu)|}(\delta_n/\delta_0)^\ell = n^{2|\operatorname{Re}(\nu)|}(n(\delta_n/\delta_0)^2)^{-|\operatorname{Re}(\nu)|-3/2} = n^{-d_{\nu, \ell}}.$$

Since  $d_{\nu, \ell} < 1$ , it holds that the jumps of  $\mathbf{Z}$  on  $\Sigma_n$  are of order  $\mathbf{I} + \mathcal{O}_\epsilon(n^{-d_{\nu, \ell}})$ , where  $\mathcal{O}_\epsilon(\cdot)$  does not depend on  $n$ . Then, by arguing as in [26, Theorem 7.103 and Corollary 7.108] we obtain that the matrix  $\mathbf{Z}$  exists for all  $n \in \mathbb{N}_{\rho, \epsilon}$  large enough and that

$$\|\mathbf{Z}_\pm - \mathbf{I}\|_{2, \Sigma_n} = \mathcal{O}_\epsilon(n^{-d_{\nu, \ell}}).$$

Since the jumps of  $\mathbf{Z}$  on  $\Sigma_n$  are restrictions of holomorphic matrix functions, the standard deformation of the contour technique and the above estimate yield that

$$\mathbf{Z} = \mathbf{I} + \mathcal{O}_\epsilon(n^{-d_{\nu, \ell}}) \quad (6.4.2)$$

locally uniformly in  $\overline{\mathbb{C}} \setminus \{0\}$ .

### 6.5 Proofs of Theorems 5.5.2 and 5.6.1

Given  $\mathbf{Z}(z)$ , a solution of **RHP- $\mathbf{Z}$** ,  $\mathbf{P}_{a_i}(z)$  and  $\mathbf{P}_0(z)$ , defined in (6.3.3) and (6.3.16), respectively, and  $\mathbf{C}(\mathbf{MD})(z)$  from (6.2.3) and (6.2.4), it can be readily verified that

$$\mathbf{X}(z) := \mathbf{C}\mathbf{Z}(z) \begin{cases} \mathbf{P}_{a_i}(z), & z \in U_i, \ i \in \{1, 2, 3, 4\}, \\ \mathbf{P}_0(z), & z \in U_\delta, \\ (\mathbf{MD})(z), & \text{otherwise,} \end{cases} \quad (6.5.1)$$

solves **RHP- $\mathbf{X}$** . Given a closed set  $K \subset \overline{\mathbb{C}} \setminus \Delta$ , the contour  $\Sigma_n$  can always be adjusted so that  $K$  lies in the exterior domain of  $\Sigma_n$ . Then it follows from (6.1.1) that  $\mathbf{Y}(z) = \mathbf{X}(z)$  on  $K$ . Formulae (5.5.8) and (5.6.2) now follow immediately from (4.1.2), (6.1.1), (6.2.3), (6.2.4), and (5.4.7) since

$$w^{i-1}(z)[(\mathbf{ZMD})(z)]_{1i} = (1 + v_{n1}(z))\Psi_n(z^{(i-1)}) + v_{n2}(z)\Psi_{n-1}(z^{(i-1)}),$$

where  $1 + v_{n1}(z), v_{n2}(z)$  are the first row entries of  $\mathbf{Z}(z)(\mathbf{I} + \mathbf{L}_\nu/z)$ . Estimates (5.5.10) are direct consequence of (6.3.19) and (6.4.2). Relations (5.5.11) follow from (6.3.28). Similarly, if  $K$  is a compact subset of  $\Delta^\circ$ , the lens  $\Sigma_n$  can be arranged so that  $K$  does not intersect  $\overline{U} \cup \overline{U}_{\delta_n}$ . As before, we get that

$$\begin{aligned} [(\mathbf{ZMD})(z)]_{11} &= ((1 + v_{n1}(z))\Psi_n(z^{(0)}) + v_{n2}(z)\Psi_{n-1}(z^{(0)})) \pm \\ &\quad (\rho_i w)^{-1}(z) ((1 + v_{n1}(z))\Psi_n(z^{(1)}) + v_{n2}(z)\Psi_{n-1}(z^{(1)})) \end{aligned}$$

for  $z \in \Omega_{i\pm} \setminus (\overline{U} \cup \overline{U}_{\delta_n})$ . Formula (5.5.9) now follows by taking the trace of  $[(\mathbf{ZMD})(z)]_{11}$  on  $\Delta_{i\pm} \setminus (\overline{U} \cup \overline{U}_{\delta_n})$  and using (5.4.8).

### 6.6 Behavior of $Q_n(z)$ around the Origin when $\ell = \infty$ and $|\operatorname{Re}(\nu)| < 1/2$

Assume that  $\ell = \infty$ . In this case  $\delta = \delta_n = \delta_0$  in (6.4.1) is independent of  $n$  and  $\mathbf{P}_0(z)$  is the exact parametrix (that is, the second group of jumps in **RHP- $\mathbf{Z}$** (b) is not present). Assume further that  $|\operatorname{Re}(\nu)| < 1/2$ . The definition of the matrix

function  $\mathbf{M}(z)$  as  $(\mathbf{I} + \mathbf{L}_\nu/z) \mathbf{M}^*(z)$  is absolutely necessary when  $|\operatorname{Re}(\nu)| = 1/2$ , see (6.2.3), but can be simplified to  $\mathbf{M}(z) = \mathbf{M}^*(z)$  when  $|\operatorname{Re}(\nu)| < 1/2$ . That is, we can take  $\mathbf{L}_\nu$  to be the zero matrix. In this case the error rate in  $\mathbf{RHP}\text{-}\mathbf{P}_0(\mathbf{c})$  will become  $\mathcal{O}(n^{|\operatorname{Re}(\nu)|-1/2})$  and the matrix  $\mathbf{E}_0(z)$  will simplify to

$$\mathbf{E}_0(z) = \mathbf{M}(z) K_0^{n\sigma_3}(z) r^{\sigma_3}(z) \mathbf{J}(z) (2\xi_n)^{-\nu\sigma_3}, \quad \xi_n := \sqrt{n}\zeta_0(z),$$

see (6.3.18) and (6.3.21). Assume now that  $z$  is in the second quadrant, in which case  $\mathbf{J} = \mathbf{I}$ . It then follows from (6.3.17) and (6.3.22) that  $K_0^n(z) = \Phi^n(z^{(0)}) e^{\xi_n^2}$ . Thus, we get from (6.3.16) as well as (6.2.1) and (6.2.4) that

$$\mathbf{P}_0(z) = \mathbf{E}_0(z) \Psi(\xi_n) r_2^{-\sigma_3}(z), \quad \mathbf{E}_0(z) = \mathbf{C}^{-1} \mathbf{N}(z) \left( r_2(z) e^{\xi_n^2} / (2\xi_n)^\nu \right)^{\sigma_3},$$

where we write  $\Psi(\zeta)$  for  $\Psi_{s_1, s_2}(\zeta)$ . Now, (4.1.2) and (6.1.1) yield that  $Q_n(z) = [\mathbf{X}(z)]_{11} + \rho_3^{-1}(z) [\mathbf{X}(z)]_{12}$  for  $z \in \Omega_{3+}$ . Therefore, we get from (6.5.1) that

$$\gamma_n^{-1} Q_n(s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{Z}(s) ([\mathbf{P}_0(s)]_{1+} + \rho_3^{-1}(s) [\mathbf{P}_0(s)]_{2+})$$

for  $s \in \Delta_3 \cap U_\delta$ , where  $[\mathbf{P}_0(z)]_i$  stands for the  $i$ -th column of  $\mathbf{P}_0(z)$ . It follows from the analyticity of  $\mathbf{E}_0(z)$  in  $U_\delta$  that

$$\gamma_n^{-1} Q_n(s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{Z}(s) \mathbf{E}_0(s) (r_2^{-1}(s) [\Psi(\xi_n)]_1 + r_3^{-1}(s) [\Psi(\xi_n)]_2)$$

since  $r_2(s)r_3(s) = \rho_3(s)$ . Using the expression for  $\mathbf{E}_0(z)$  from above as well as (6.2.1) and (5.4.8) we get that

$$\gamma_n^{-1} Q_n(s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{Z}(s) \begin{pmatrix} \Psi_{n+}^{(0)}(s) & \Psi_{n-}^{(0)}(s) \\ \Psi_{n-1+}^{(0)}(s) & \Psi_{n-1-}^{(0)}(s) \end{pmatrix} \begin{pmatrix} (2\xi_n)^{-\nu} A_\rho(\xi_n) \\ (2i\xi_n)^\nu B_\rho(\xi_n) \end{pmatrix} \quad (6.6.1)$$

for  $s \in \Delta_3 \cap U_\delta$ , where, since  $\zeta_0(s)$  has argument  $-\pi/4$  for  $s \in \Delta_3$ , we set

$$\begin{cases} A_\rho(\zeta) := e^{\zeta^2} (D_\nu(2\zeta) + \alpha_\rho D_{-\nu-1}(2i\zeta)) \\ B_\rho(\zeta) := e^{-\zeta^2} (D_{-\nu}(2i\zeta) + \beta_\rho D_{\nu-1}(2\zeta)) \end{cases}$$

with  $\alpha_\rho := -e^{\pi i \nu/2} d_1(r_2/r_3)(s) = d_1(\rho_4/\rho_2)(s)$ ,  $\beta_\rho := -e^{-\pi i \nu/2} d_2(\rho_2/\rho_4)(s)$  and  $d_1, d_2$  given by (6.3.7), which are constants by the definition of  $\mathcal{W}_\infty$ . Recall that  $\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{Z}(s)$ ,

the first row of  $\mathbf{Z}(s)$ , behaves like  $\begin{pmatrix} 1 + o(1) & o(1) \end{pmatrix}$ , where  $o(1) = \mathcal{O}(n^{|\operatorname{Re}(\nu)|-1/2})$ , in the considered case. Therefore, by multiplying (6.6.1) out, we can get an asymptotic expression for  $Q_n(s)$  around the origin on  $\Delta_3$ . Clearly, we can get similar expressions on the remaining arcs  $\Delta_1, \Delta_2$  and  $\Delta_4$ .

A computation along these lines can be performed in the case  $\operatorname{Re}(\nu) = 1/2$ , but the resulting formula is even more involved than (6.6.1).

## 6.7 Concluding Remarks

It is important to note that Jacobi-type functions  $f \in \mathcal{W}_\infty$  have been extensively studied for any number of branch points  $p$  in this connection. In the situation  $p = 2$ ,  $Q_n$ 's are, up to a change of variables, the usual Jacobi polynomials whose strong asymptotics can be found in [3], for example (also, see [40]). Strong asymptotics of polynomials orthogonal with respect to the weight  $h(x)(f_+ - f_-)(x)$ , where  $h(x)$  is holomorphic, non-vanishing on a neighborhood of  $\Delta$  and  $\alpha_i = -1/2$  for all  $i = 1, 2, \dots, p$  were studied in [23] for any  $p \in \mathbb{N}$ . If  $p$  is arbitrary, but the points at which arcs of  $\Delta$  meet are univalent or trivalent, and no  $a_i$  is a trivalent point, then strong asymptotics were obtained in [25]. This work, along with [45], completely resolves the situation  $p = 3$  (Vanlessen's work applies for any  $p$  when all  $a_i$ 's are colinear). However, in the case  $p = 4$ , there remain some non-trivial situations, shown in Figure 6.5. With the result above, the final case in need of analysis is the one depicted in Figure 6.5(d).

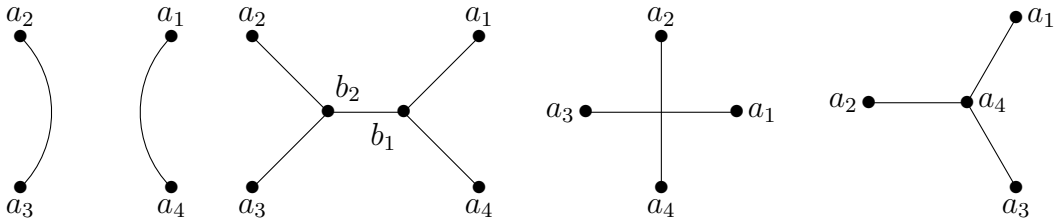


Fig. 6.5. Possible non-colinear arrangements of  $a_i$ 's

## 7. VARYING ORTHOGONALITY

A version of this chapter will appear in [46].

All polynomials considered thus far were orthogonal with respect to a measure  $d\mu(z) = \rho(z) dz$ ,  $z \in \Delta \subset \mathbb{C}$  where the density  $\rho(z)$  depended on  $z$  alone. However, polynomials satisfying a **varying orthogonality condition**:

$$\int_{\Delta} z^k P_n(z) \rho(z; N) dz = 0 \quad \text{for } k = 0, 1, \dots, n-1 \quad (7.0.1)$$

where  $N$  is a parameter, usually depending on  $n$ , crop in applications. We highlight here the so-called “kissing polynomials” that appear in numerical computation of oscillatory integrals [47–49], and polynomials that appear in connection with random matrix theory [50, 51], see also [52, 53].

### 7.1 Choice of Contour

As was discussed in Chapter 4, one can still form a matrix  $\mathbf{Y}$  and consider its associated RHP. However, we need to make the “correct” choice of the contour  $\Delta$ . In the non-varying case, existence of  $\Delta$  was given to us by Theorem 3.2.2, and characterizations followed in Theorems 3.2.3, 3.2.4, and 3.2.5. Unfortunately, the existence of an S-contour in the varying case is not so clear, and general results are rare. A general method of finding S-contours by way of solving a min-max problem is described in [54]. In the context of varying orthogonality, one must include a non-trivial external field to all potential-theoretic objects and in the definition of the S-property. We rely on [11] as a general reference on potential theory with external fields.

**Definition.** A **weight function**  $w(z) := \exp(-W(z))$  on a set  $K$  is said to be admissible if

1.  $W(z) : K \rightarrow (-\infty, \infty]$  is lower semi-continuous,

2.  $W(z) < \infty$  on a set of positive capacity,
3. If  $\infty \in K$ , then  $\lim_{\substack{|z| \rightarrow \infty \\ z \in K}} (W(z) - \log |z|) = \infty$ .

**Definition.** The **weighted logarithmic energy** of a measure  $\mu$  is defined as the integral

$$\begin{aligned} I_w(\mu) &:= - \int \int \log(|z - x|w(z)w(x)) \, d\mu(x)d\mu(z) \\ &= \int \int \log\left(\frac{1}{|z - x|}\right) \, d\mu(x)d\mu(z) + 2 \int W(z)d\mu(z). \end{aligned}$$

The **equilibrium measure** of a compact set  $K$  associated with weight  $w$  is the unique minimizer of the weighted energy

$$I_w(\mu_{K,w}) = \inf I_w(\nu)$$

where infimum is taken over all probability Borel measures supported on  $K$ .

Moreover, the measure  $\mu_{K,w}$  is also characterized as the unique measure  $\mu$  satisfying the variational condition (see [11] for more on this)

$$2U^\mu(z) + W(z) \begin{cases} = \ell & \text{for } z \in \text{supp}(\mu), \\ \geq \ell & \text{for } z \in K. \end{cases} \quad (7.1.1)$$

**Definition.** A set  $K$  comprised of a finite union of analytic arcs is said to have the **S-property with respect to the external field  $W$**  if it holds that (compare with Theorem 3.2.4) for a.e.  $z \in K$

$$\frac{\partial (U^{\mu_{K,w}} + W)}{\partial \mathbf{n}^+}(z) = \frac{\partial (U^{\mu_{K,w}} + W)}{\partial \mathbf{n}^-}(z) \quad (7.1.2)$$

The following is due to Gonchar and Rakhmanov [55]. Although their result is more general, this version establishes the importance of S-contours

**Theorem 7.1.1.** *Let  $D$  be a domain and  $\Delta \subset D$  be a system of a.e. smooth arcs of positive capacity. Suppose polynomials  $P_n$  satisfy an orthogonality relation*

$$\int_{\Delta} z^k P_n(z) e^{-nV(z)} f(z) \, dz = 0 \quad \text{for } k = 0, 1, \dots, n-1$$

where  $f$  is holomorphic off  $\Delta$ ,  $V$  holomorphic on  $D$ . If  $\Delta$  has the S-property in  $W(z) = \operatorname{Re} V(z)$  and if the complement of the support of equilibrium measure  $\mu_{\Delta,w}$  is connected, then

$$\frac{1}{n} \sum_{k=1}^n \delta_{x_k} \xrightarrow{*} \mu_{\Delta,w}$$

where  $\frac{1}{n} \sum_{k=1}^n \delta_{x_k}$  is the probability counting measure of the zeroes of  $P_n$ .

Note that this theorem makes no claim regarding the existence of such S-contours, but simply suggests that the S-property is what we should look for to ever hope to start our RH analysis of varying orthogonal polynomials. In this chapter we will consider the external field  $V_\lambda(z) = -i\lambda z$ ,  $\lambda > 0$  to demonstrate some of the difficulties that arise.

## 7.2 Kissing Polynomials: One-Cut Case

Consider polynomials  $P_n^\lambda(z)$  that satisfy the orthogonality condition

$$\int_{-1}^1 z^k P_n^\lambda(z) h(z) e^{i\lambda n z} dz = 0, \quad \text{for } k = 0, 1, \dots, n-1, \quad (7.2.1)$$

where

$$h(x) = (1-x)^\alpha (1+x)^\beta h^*(x) \quad (7.2.2)$$

and  $h^*(z)$  is holomorphic in a neighborhood containing the compact set delimited by  $\gamma_\lambda$  and  $[-1, 1]$  (see Theorem 7.2.1 below). Because of the analyticity of the integrand, one could deform the contour of integration  $[-1, 1]$  to any smooth arc connecting  $z = -1$  to  $z = 1$ . Then, the question becomes about finding such an arc  $\gamma_\lambda$  that has the S-property.



### 7.2.1 Geometry

In [54], Rakhmanov provides yet another characterization of S-curves as a set of trajectories of quadratic differentials. For the external field  $\operatorname{Re}(V_\lambda(z))$ , one needs to look for a measure  $\mu_\lambda$ ,  $\gamma_\lambda = \operatorname{supp}(\mu_\lambda)$ , such that  $Q_\lambda(z)$  defined by

$$Q_\lambda(z) = \left( \int \frac{d\mu_\lambda(s)}{s-z} + \frac{V'(z)}{2} \right)^2 = \left( \int \frac{d\mu_\lambda(s)}{s-z} - \frac{\lambda i}{2} \right)^2. \quad (7.2.3)$$

is a rational function, in which case  $\gamma_\lambda$  is a subset of short critical trajectories of  $-Q_\lambda(z)(dz)^2$ . Deaño showed in [48] that  $\mu_\lambda$  is such that  $\gamma_\lambda$  is a single arc for all  $\lambda_{cr} > \lambda > 0$ , where  $\lambda_{cr}$  is the unique solution of

$$2 \log \left( \frac{2 + \sqrt{\lambda_{cr}^2 + 4}}{\lambda_{cr}} \right) - \sqrt{\lambda_{cr}^2 + 4} = 0 \quad (\lambda_{cr} \approx 1.325...). \quad (7.2.4)$$

The following Theorem appears in [48].

**Theorem 7.2.1.** *Let  $V_\lambda(z) = -i\lambda z$  and  $\lambda \in [0, \lambda_{cr})$ . Then,*

1. *there exists a smooth curve  $\gamma_\lambda$  connecting  $z = 1$  and  $z = -1$  that is a part of the level set*

$$\operatorname{Re}(\phi(z)) = 0,$$

*where*

$$\phi(z) = 2 \log \varphi(z) + i\lambda w(z), \quad \varphi(z) := z + w(z), \quad (7.2.5)$$

*and  $w(z) := (z^2 - 1)^{1/2} = z + \mathcal{O}(z)$  and is analytic outside of  $\gamma_\lambda$ ;*

2. *the measure*

$$d\mu_\lambda(z) = -\frac{1}{2\pi i} \frac{2 + i\lambda z}{w(z)} dz$$

*is the equilibrium measure on  $\gamma_\lambda$  in the external field  $\operatorname{Re}(V_\lambda(z))$ .*

3.  *$\gamma_\lambda$  has the S-property in the field  $\operatorname{Re}(V_\lambda(z))$ .*

*Remark 7.2.2.* In fact, Deaño's proof shows that for  $\lambda = \lambda_{cr}$ ,  $\gamma_{\lambda_{cr}}$  is a union of two smooth curves that meet at  $2i/\lambda_{cr}$ .

In fact, Theorem 7.2.1(1) can be equivalently stated as follows:  $\gamma_\lambda$  is the unique short critical trajectory of  $Q_\lambda(z)(dz)^2$ , where

$$Q_\lambda(z)(dz)^2 = \frac{1}{4} \frac{(2 + i\lambda z)^2}{z^2 - 1} (dz)^2. \quad (7.2.6)$$

## 7.2.2 Asymptotics of Orthogonal Polynomials

Let  $\lambda_{cr}$  be as in (7.2.4). In the non-critical case ( $\lambda < \lambda_{cr}$ ), the situation was described completely for  $h(x) \equiv 1$  in [48]. To extend this result to  $h(x)$  as in (7.2.2), we need the following Szegő function

$$S_h(z) := \exp \left\{ \frac{w(z)}{2\pi i} \int_{\gamma_\lambda} \frac{\log[(w+h)(x)]}{z-x} \frac{dx}{w_+(x)} \right\}, \quad (7.2.7)$$

where  $w$  is as in (7.2.5) and  $h^*(z)$  is analytic in a neighborhood containing the compact set delimited by  $\gamma_\lambda \cup [-1, 1]$ . Properties of  $S$  will be discussed in Section 8.1.

**Theorem 7.2.3 (Subcritical Case  $\lambda < \lambda_{cr}$ ).** *Let  $0 \leq \lambda < \lambda_{cr}$  and  $h(z)$  be as above. Then for  $n$  large enough, polynomials  $P_n^\lambda$  have degree exactly  $n$  and locally uniformly for  $z \in \mathbb{C} \setminus \gamma_\lambda$*

$$P_n^\lambda(z) = \left( \frac{\varphi(z)}{2} \right)^n \exp \left( -\frac{i n \lambda}{2\varphi(z)} \right) \left( \frac{S_h(\infty)}{S_h(z)} + \mathcal{O} \left( \frac{1}{n} \right) \right) \quad \text{as } n \rightarrow \infty. \quad (7.2.8)$$

When  $\lambda = \lambda_{cr}$ , the geometry of  $\gamma_\lambda$  changes. More precisely,  $\gamma_{\lambda_{cr}}$  is no longer an analytic arc, but rather a union of two analytic arcs meeting at the angle  $\pi/2$  at the point  $2i/\lambda_{cr}$ , see [48]. However, by slightly changing the analysis, we may still write an asymptotic formula for  $P_n^\lambda$ .

**Theorem 7.2.4 (Critical Case  $\lambda = \lambda_{cr}$ ).** *Let  $\lambda = \lambda_{cr}$  and  $h(z)$  be as above. Then for  $n$  large enough, polynomials  $P_n^\lambda$  have degree exactly  $n$  and locally uniformly for  $z \in \mathbb{C} \setminus \gamma_\lambda$  and satisfy*

$$P_n^{\lambda_{cr}}(z) = \left( \frac{\varphi(z)}{2} \right)^n \exp \left( -\frac{i n \lambda_{cr}}{2\varphi(z)} \right) \left( \frac{S_h(\infty)}{S_h(z)} + \mathcal{O} \left( \frac{1}{\sqrt{n}} \right) \right) \quad \text{as } n \rightarrow \infty. \quad (7.2.9)$$

### 7.3 Kissing Polynomials: Two-Cut Case

#### 7.3.1 Geometry

When  $\lambda > \lambda_{cr}$  the quadratic differential in (7.2.6) ceases to possess a trajectory connecting  $z = -1, z = 1$ , and we must look for a new differential whose critical graph is such that  $[-1, 1]$  can be deformed to align with trajectories of  $-Q(z) (dz)^2$  or pass through regions of “exponential decay,” where  $\operatorname{Re}(\int^z Q^{1/2}(z)dz) < 0$ . This becomes important for the RH analysis (carries out in Chapter 8). To this end, we rely on Celsus and Silva’s work [56], where they consider a quadratic differential of the form

$$Q_\lambda(z; x) := -\frac{\lambda^2}{4} \frac{(z - z_\lambda(x))(z + \overline{z_\lambda(x)})}{z^2 - 1}, \quad \text{and} \quad z_\lambda(x) = x + \frac{2i}{\lambda}. \quad (7.3.1)$$

The following results appear in their work

**Theorem 7.3.1.** *Let  $\lambda > \lambda_{cr}$ . Then, there exists  $x_*(\lambda) \in (0, 1)$  for which*

$$\operatorname{Re} \left( \int_{z_\lambda(x_*)}^1 Q_\lambda(s) ds \right) = 0, \quad Q_\lambda(z) := Q_\lambda(z; x_*) \quad (7.3.2)$$

and  $\lim_{\lambda \rightarrow \infty} x_*(\lambda) = 1$ . In fact, there exist analytic arcs  $\gamma_1, \gamma_2$  such that  $\gamma_2$  is the reflection of  $\gamma_1$  across the imaginary axis and

(a) the arc  $\gamma_1$  starts at  $z = -1$ , ends at  $-\overline{z_\lambda(x_*)}$ , and satisfies

$$\operatorname{Re} \left( \int_{-1}^z Q_\lambda^{1/2}(s) ds \right) = 0 \quad \forall z \in \gamma_1; \quad (7.3.3)$$

(b) the arc  $\gamma_2$ , being the reflection of  $\gamma_1$  satisfies

$$\operatorname{Re} \left( \int_{z_\lambda(x_*)}^z Q_\lambda^{1/2}(s) ds \right) = 0 \quad \forall z \in \gamma_2.$$

The equilibrium measure in the external field  $\operatorname{Re}(V_\lambda(z))$  on  $\gamma_1 \cup \gamma_2$  is given by

$$d\mu_\lambda(s) = -\frac{1}{\pi i} Q_\lambda^{1/2}(s) ds, \quad s \in \gamma_1 \cup \gamma_2 \quad (7.3.4)$$

where (similar to the one-cut case) we take the branch of  $Q_\lambda^{1/2}$  holomorphic off  $\gamma_1 \cup \gamma_2$  and that satisfies

$$Q_\lambda^{1/2}(z) = \frac{\lambda i}{2} + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as } z \rightarrow \infty. \quad (7.3.5)$$

Moreover, Celsus and Silva show that the critical graph of  $-Q_\lambda(z)(dz)^2$  is as in Figure 7.1 below.

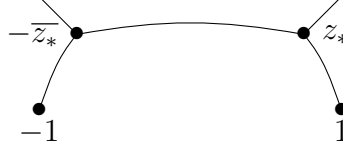


Fig. 7.1. Schematic representation of critical graph of  $-Q_\lambda(z)(dz)^2$  in the supercritical regime near  $z = -1, z = 1$ , with  $z_* := z_\lambda(x_*)$ .

### 7.3.2 Asymptotics of Orthogonal Polynomials

To present the results when  $\lambda > \lambda_{cr}$ , we construct the main term of the asymptotics using the approach of [25] relying on Theta functions, instead of the meromorphic differential approach taken in [56]. We begin by defining

$$\gamma(z) := \left( \frac{z + \bar{z}_*}{z - z_*} \frac{z - 1}{z + 1} \right)^{1/4}, \quad z \in \overline{\mathbb{C}} \setminus (\gamma_1 \cup \gamma_2), \quad z_* = z_\lambda(x_*) \quad (7.3.6)$$

where  $\gamma(z)$  is holomorphic off  $\gamma_1 \cup \gamma_2$  and the branch is chosen so that  $\gamma(\infty) = 1$ . Further, set

$$A(z) := \frac{\gamma(z) + \gamma^{-1}(z)}{2} \quad \text{and} \quad B(z) := \frac{\gamma(z) - \gamma^{-1}(z)}{-2i}. \quad (7.3.7)$$

The functions  $A(z)$  and  $B(z)$  are holomorphic in  $\overline{\mathbb{C}} \setminus (\gamma_1 \cup \gamma_2)$  and satisfy

$$A(\infty) = 1, \quad B(\infty) = 0, \quad \text{and} \quad A_\pm(s) = \pm B_\mp(s), \quad s \in (\gamma_1 \cup \gamma_2)^\circ := (\gamma_1 \cup \gamma_2) \setminus \{\pm 1, z_*, -\bar{z}_*\}. \quad (7.3.8)$$

The rest of our functions live on a Riemann surface, denoted  $\mathfrak{R}_\lambda$ , and so we define it here.

## Riemann Surface

Let  $\mathfrak{R} \equiv \mathfrak{R}_\lambda$  be the Riemann surface associated with the algebraic equation  $y^2 = Q_\lambda(z)$ , with  $Q_\lambda$  as in Theorem 7.3.1. This surface is realized as two copies of  $\mathbb{C}$  cut along  $\gamma_{1,2}$  and glued together in such a way that the right side of  $\gamma_i$  on  $\mathfrak{R}^{(0)}$ , the first sheet, is connected with the left side of the same arc on the second sheet,  $\mathfrak{R}^{(1)}$ . Furthermore,  $\pi : \mathfrak{R} \rightarrow \overline{\mathbb{C}}$  be the natural projection. We will denote points on the surface with boldface symbols  $\mathbf{z}, \mathbf{t}, \mathbf{s}$  and their projections by regular script  $z, s, t$  and  $F^{(i)}(z)$ ,  $i \in \{0, 1\}$ , stands for the pull-back under  $\pi(\mathbf{z})$  of a function  $F(z)$  from  $\mathfrak{R}^{(i)}$  into  $\overline{\mathbb{C}} \setminus (\gamma_1 \cup \gamma_2)$ . Note that for a fixed  $z \in \mathbb{C} \setminus \{\pm 1, z_*, -\overline{z_*}\}$ , the set  $\pi^{-1}(z)$  contains exactly two elements, one on each sheet, and for  $z \in \mathbb{C} \setminus (\gamma_1 \cup \gamma_2)$  we denote  $z^{(k)} := \pi^{-1}(z) \cap \mathfrak{R}^{(k)}$ .

Denote by  $\alpha$  a cycle on  $\mathfrak{R}$  that passes through  $\pi^{-1}(-\overline{z_*})$  and  $\pi^{-1}(z_*)$  and whose natural projection is an arc  $\hat{\gamma}$  that smoothly meets  $\gamma_1, \gamma_2$  at  $z_*, -\overline{z_*}$  and belongs to the region delimited by infinite trajectories in Figure 7.1. We assume that  $\pi(\alpha) \cap (\gamma_1 \cup \gamma_2) = \{z_*, -\overline{z_*}\}$  and orient  $\alpha$  towards  $-\overline{z_*}$  within  $\mathfrak{R}^{(0)}$ . Similarly, we define  $\beta = \pi^{-1}(\gamma_1)$ . We orient  $\beta$  so that  $\alpha, \beta$  form the right pair at  $\pi^{-1}(-\overline{z_*})$ . Figure 7.2 below is a schematic representation of  $\mathfrak{R}$ .

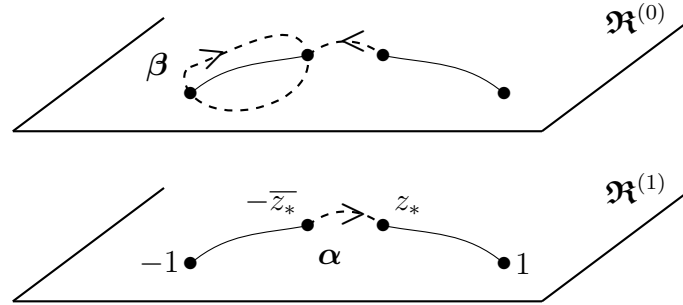


Fig. 7.2. Schematic plot of the Riemann surface  $\mathfrak{R}$  and the cycles  $\alpha$  and  $\beta$ .

Since this is a surface of genus 1, the linear space of holomorphic differentials is of dimension 1, and is generated by

$$\mathcal{H} \equiv \mathcal{H}_\lambda(z) = \left( \oint_{\alpha} \frac{dz}{w(z)} \right)^{-1} \frac{dz}{w(z)} \quad (7.3.9)$$

where  $w(z^{(k)}) = (-1)^k [(z^2 - 1)(z - z_*)(z + \overline{z_*})]^{1/2}(z)$  and the branch of the square root is such that  $w(z^{(k)}) = (-1)^k z^2 + \mathcal{O}(z)$  as  $z \rightarrow \infty$ .  $\mathcal{H}$  is normalized so that  $\oint_{\alpha} \mathcal{H} = 1$ , and under this normalization, Riemann showed (see [57, Theorem 2.1], for example) that

$$\text{Im}(\mathbf{B}) > 0, \quad \text{where} \quad \mathbf{B} := \oint_{\beta} \mathcal{H}(z) \quad (7.3.10)$$

Given this normalized differential, we can define the **Abel Map**  $\mathcal{A}(z)$  as

$$\mathcal{A}(z) := \int_1^z \mathcal{H}(s) \quad (7.3.11)$$

where the path of integration is chosen to lie in  $\mathfrak{R}_{\alpha, \beta} := \mathfrak{R} \setminus \{\alpha, \beta\}$ . This function is holomorphic on  $\mathfrak{R}_{\alpha, \beta}$  that satisfies

$$(\mathcal{A}_+ - \mathcal{A}_-)(z) = \begin{cases} 1, & z \in \beta \setminus \pi^{-1}(-\overline{z_*}), \\ -\mathbf{B}, & z \in \alpha \setminus \pi^{-1}(-\overline{z_*}) \end{cases} \quad (7.3.12)$$

### Szegő Function

We define a Szegő function entirely analogously to what has been done in Section 5.3. Let

$$\tilde{S}_h(z^{(k)}) := \exp \left\{ \frac{1}{4\pi i} \oint_{\pi^{-1}(\gamma_1 \cup \gamma_2)} \log(h) \Omega_{z^{(k)}, z^{(1-k)}} \right\} \quad \text{for } k = 0, 1 \quad (7.3.13)$$

where  $\Omega_{z^{(k)}, z^{(1-k)}}$  is the meromorphic differential on  $\mathfrak{R}$  with simple poles at  $z^{(k)}, z^{(1-k)}$  with residues 1,  $-1$ , respectively, and zero period on  $\alpha$ . By identical reasoning to that employed in Proposition 5.3.1, we have the following

**Proposition 7.3.2.** *Let  $\tilde{S}_h$  be as above and  $h(z) = h^*(z)(1-z)^\alpha(1+z)^\beta$  where  $h^*(z)$  is holomorphic, non-vanishing in a neighborhood of  $\gamma_1 \cup \gamma_2 \cup \hat{\gamma}$  and  $h(z)$  is holomorphic in a neighborhood of each point of  $(\gamma_1 \cup \gamma_2) \setminus \{\pm 1\}$ . Furthermore, define*

$$c_h = c_h(\lambda) := \frac{1}{2\pi i} \oint_{\pi^{-1}(\gamma_1 \cup \gamma_2)} \log(h) \mathcal{H}. \quad (7.3.14)$$

Then  $\tilde{S}_h$  is holomorphic and non-vanishing on  $\mathfrak{R} \setminus (\alpha \cup \pi^{-1}(\gamma_1 \cup \gamma_2))$  and satisfies the relation  $\tilde{S}_h(z^{(k)}) \cdot \tilde{S}_h(z^{(1-k)}) \equiv 1$ . Furthermore,  $\tilde{S}_h$  possesses continuous traces on  $(\alpha \cup \pi^{-1}(\gamma_1 \cup \gamma_2)) \setminus \{\pi^{-1}(\pm 1), \pi^{-1}(z_*), \pi^{-1}(-\overline{z_*})\}$  that satisfy

$$\tilde{S}_{h,+}(s) = \tilde{S}_{h,-}(s) \begin{cases} e^{2\pi i c_h}, & s \in \alpha \setminus \{\pi^{-1}(z_*), \pi^{-1}(-\overline{z_*})\}, \\ 1/h(s), & s \in \pi^{-1}(\gamma_1 \cup \gamma_2) \setminus \{\pi^{-1}(\pm 1)\}. \end{cases} \quad (7.3.15)$$

Furthermore, we have

$$\tilde{S}_h(z^{(0)}) \sim |z - e|^{-\alpha_e/2}, \quad e \in \{\pm 1, z_*, -\overline{z_*}\}, \quad (7.3.16)$$

where  $\alpha_e = 0$  for  $e = z_*, -\overline{z_*}$ ,  $\alpha_e = \alpha$  when  $e = 1$  and  $\alpha_e = \beta$  when  $e = -1$ .

## Theta Functions

Just as in Chapter 5, we denote by  $\theta(z)$  the function defined by the sum

$$\theta(u) = \sum_{k \in \mathbb{Z}} \exp \{ \pi i B k^2 + 2\pi i u k \}.$$

For convenience, we remind the reader of its properties here. This function is holomorphic in  $\mathbb{C}$  and satisfies the quasi-periodicity relations

$$\theta(u + j + Bm) = \exp \{ -\pi i B m^2 - 2\pi i u m \} \theta(u), \quad j, m \in \mathbb{Z}. \quad (7.3.17)$$

It is also known that  $\theta(u)$  vanishes only at the points of the lattice  $[\frac{B+1}{2}]$ , where we remind the reader of the notation  $[s] = \{s + l + Bm : l, m \in \mathbb{Z}\}$ , see 5.4. Furthermore, let  $\tilde{\mathcal{A}}$  denote the continuation of  $\mathcal{A}$  onto  $\alpha, \beta$  by  $\mathcal{A}_+$  and define  $z_{n,k}$  by the equation

$$\tilde{\mathcal{A}}(z_{n,k}) = \tilde{\mathcal{A}}(p^{(k)}) + c_h + n \left( \frac{1}{2} + B\tau \right) + j_{n,k} + m_{n,k}B, \quad j_{n,k}, m_{n,k} \in \mathbb{Z} \quad (7.3.18)$$

where  $p = i\text{Im}(z_*)/(1 - \text{Re}(z_*))$  and

$$\tau = \tau(\lambda) := -\frac{1}{\pi i} \int_{\hat{\gamma}} Q_{\lambda}^{1/2}(s) ds. \quad (7.3.19)$$

Since  $\mathfrak{R}$  is of genus one,  $\mathcal{A}$  is bijective and equation (7.3.18) defines  $z_{n,k}$  uniquely. In fact, the following holds

**Proposition 7.3.3.** *Let  $\tau$  be given by (7.3.19),  $z_{n,k} = z_{n,k}(\lambda)$  as in (7.3.18), and  $p$  as above. Then for any subsequence  $\mathbb{N}_*$  the point  $\infty^{(0)}$  is a topological limit point of  $\{z_{n,1}\}_{n \in \mathbb{N}_*}$  if and only if  $\infty^{(1)}$  is a topological limit point of  $\{z_{n,0}\}_{n \in \mathbb{N}_*}$ .*

The proof of this proposition is deferred to Appendix B. Next, we define

$$\Theta_{n,k}(z) = \exp \left\{ -2\pi i (m_{n,k} + \tau n) \mathcal{A}(z) \right\} \frac{\theta \left( \mathcal{A}(z) - \tilde{\mathcal{A}}(z_{n,k}) - \frac{B+1}{2} \right)}{\theta \left( \mathcal{A}(z) - \tilde{\mathcal{A}}(p^{(k)}) - \frac{B+1}{2} \right)} \quad (7.3.20)$$

and  $F^{(i)}(z)$ ,  $i \in \{0, 1\}$ , stands for the pull-back under  $\pi(z)$  of a function  $F(z)$  from  $\mathfrak{R}^{(i)}$  into  $\overline{\mathbb{C}} \setminus (\gamma_1 \cup \gamma_2)$ .

The functions  $\Theta_{n,k}(z)$  are meromorphic on  $\mathfrak{R}_{\alpha,\beta}$  with exactly one pole, which is simple and located at  $p^{(k)}$ , and exactly one zero, which is also simple and located at  $z_{n,k}$  (observe that the functions  $\Theta_{n,k}(z)$  can be analytically continued as multiplicatively multivalued functions on the whole surface  $\mathfrak{R}$ ; thus, we can talk about simplicity of a pole or zero regardless whether it belongs to the cycles of a homology basis or not). Moreover, according to (7.3.12), (7.3.18), and periodicity properties of  $\theta$ , they possess continuous traces on  $\alpha, \beta$  away from  $\pi^{-1}(-\overline{z_*})$  that satisfy

$$\Theta_{n,k+}(s) = \Theta_{n,k-}(s) \begin{cases} \exp \left\{ -\pi i (n + 2c_h) \right\}, & s \in \alpha \setminus \{\pi^{-1}(-\overline{z_*})\}, \\ \exp \left\{ -2\pi i \tau n \right\}, & s \in \beta \setminus \{\pi^{-1}(-\overline{z_*})\}. \end{cases} \quad (7.3.21)$$

### Subsequences $\mathbb{N}(\lambda, \epsilon)$

It will be important for our analysis (see section 8.3.1) that  $\Theta_{n,1}(z)$ , defined in (7.3.20), does not vanish near  $\infty^{(0)}$ . Hence, we will consider subsequences  $\mathbb{N}(\epsilon) = \mathbb{N}(\lambda, \epsilon)$  defined by

$$\mathbb{N}(\lambda, \epsilon) \equiv \mathbb{N}(\epsilon) := \left\{ n \in \mathbb{N} : z_{n,1} \notin \mathfrak{R}^{(0)} \cap \pi^{-1}(\{|z| \geq 1/\epsilon\}) \right\}.$$

Then there exists a constant  $c(\lambda, \epsilon) > 0$  such that  $|\Theta_{n,1}^{(1)}(\infty)| \geq c(\lambda, \epsilon)$  for  $n \in \mathbb{N}(\lambda, \epsilon)$ . Indeed, by its very definition,  $\mathcal{A}(z_{n,k})$  is a bounded quantity for all  $n$ , and hence using (7.3.18) and the fact that  $\text{Im}(B) > 0$ , we conclude that  $n\tau + m_{n,k}$  is a bounded



quantity. Consider a sequence of constants  $c(n; \lambda, \epsilon)$  so that  $|\Theta_{n,1}^{(1)}(\infty)| \geq c(n, \lambda, \epsilon)$ . Since  $\Theta_{n,1}^{(1)}(\infty) \neq 0$  whenever  $z_{n,1} \neq \infty^{(0)}$ , and by definition of  $\mathbb{N}(\epsilon)$ ,  $z_{n,1} \in \mathfrak{R} \setminus \left( \mathfrak{R}^{(0)} \cap \pi^{-1}(\{|z| > 2/\epsilon\}) \right)$ , which is compact, we conclude the existence of  $c(\lambda, \epsilon)$  with  $c(n, \lambda, \epsilon) > c(n, \lambda) > 0$ .

Note that  $\mathbb{N}(\lambda, \epsilon)$  contains either  $n$  or  $n - 1$  for all  $n \geq N_\epsilon$  for some natural number  $N_\epsilon$ . To prove this, suppose to the contrary that for any  $\epsilon > 0$ , there exists  $n_\epsilon$  such that  $n_\epsilon, n_\epsilon - 1 \notin \mathbb{N}(\lambda, \epsilon)$ . By the very definition of  $\mathbb{N}(\lambda, \epsilon)$ , it then holds that  $z_{n_\epsilon-1,1}, z_{n_\epsilon,1} \rightarrow \infty^{(0)}$  as  $\epsilon \rightarrow 0$ . This implies  $1/2 + B\tau = m + nB$  for some  $m, n \in \mathbb{Z}$ , which is false. We are ready to state the asymptotic formula for  $P_n^\lambda(z)$ .

**Theorem 7.3.4 (Supercritical Case ( $\lambda > \lambda_{cr}$ )).** *Let  $\lambda > \lambda_{cr}$ ,  $V_\lambda(z) = -i\lambda z$ ,  $h(z)$  as in Proposition 7.3.2, and  $\phi_1(z) = \int_1^z Q_\lambda^{1/2}(s)ds$ . Then, there exists a constant  $\ell_\lambda^*$  (defined in (8.3.2)) so that*

$$P_n^\lambda(z) = e^{n(V_\lambda(z) - \ell_\lambda^* + \phi_1(z))} \left( \left( A\Theta_{n,1}^{(0)} \tilde{S}_h^{(0)} \right) (z) + \mathcal{O}\left(\frac{1}{n}\right) \right) \quad \text{for } n \rightarrow \infty, n \in \mathbb{N}(\lambda, \epsilon) \quad (7.3.22)$$

locally uniformly for  $z \in \mathbb{C} \setminus \gamma_\lambda$ .

In the next chapter, we prove Theorems 7.2.3, 7.2.4, and 7.3.4.

## 8. RIEMANN-HILBERT ANALYSIS: VARYING ORTHOGONALITY WITH LINEAR POTENTIAL

A version of this chapter will appear in [46].

Just as discussed in Chapter 4, supposing

$$\deg P_n^\lambda = n, \quad \mathcal{C}(P_n^\lambda w_n(z)) \sim z^{-n-1} \quad \text{as } z \rightarrow \infty$$

where we use  $w_n(z) := h(z)e^{in\lambda z}$ ,  $h(z)$  as in (7.2.1), and

$$\mathcal{C}(f)(z) = \frac{1}{2\pi i} \int_{\gamma_\lambda} \frac{f(s) \, ds}{s - z},$$

then the matrix

$$\mathbf{Y}(z) = \begin{pmatrix} P_n^\lambda(z) & \mathcal{C}(P_n^\lambda w_n)(z) \\ k_{n-1}P_{n-1}^\lambda(z) & k_{n-1}\mathcal{C}(P_{n-1}^\lambda w_n)(z) \end{pmatrix} \quad (8.0.1)$$

solves **RHP-Y**. To proceed with the analysis, we follow the standard sequence of transformations that appears in [48] and was outlined in Chapter 4.

### 8.1 Subcritical Case; $0 \leq \lambda < \lambda_{cr}$

#### 8.1.1 Global Analysis

##### First Transformation

In this setting, and unlike the analysis carried out in Chapter 6, we start with the construction of the  $g$ -function. Let

$$g(z) := \int \log(z - s) d\mu_\lambda(s), \quad z \in \mathbb{C} \setminus ((-\infty, -1) \cup \gamma_\lambda) \quad (8.1.1)$$

where the branch of  $\log(\cdot - s)$  is holomorphic outside the curve connecting  $-\infty$  and  $s$  along  $(-\infty, 1] \cup \gamma_\lambda$  and  $d\mu_\lambda$  is given by Theorem 7.2.1. Observe that by its very definition,

$$\partial_z g(z) = \int \frac{d\mu_\lambda(s)}{z - s} \quad (8.1.2)$$

and via (7.2.3), we deduce that

$$g(z) = \frac{V_\lambda(z) - \ell}{2} + \int_1^z Q_\lambda^{1/2}(s) \, ds, \quad (8.1.3)$$

where  $Q_\lambda^{1/2}(z) = i\lambda/2 + \mathcal{O}(z^{-1})$  as  $z \rightarrow \infty$ , integral is taken along a smooth curve in  $\mathbb{C} \setminus \gamma_\lambda$ , and  $\ell$  is chosen so that  $g(z) = \log z + \mathcal{O}(z^{-1})$ . In fact, since  $Q_\lambda$  is fairly explicit, we can calculate  $\ell = 2 \log 2$ . Using the Plemelj-Sokhotski formulas yields

$$(g_+ - g_-)(s) = \begin{cases} \pm \phi_\pm(s) & \text{for } s \in \gamma_\lambda, \\ 2\pi i & \text{for } s \in (-\infty, -1), \end{cases} \quad (8.1.4)$$

where we denote  $\phi(z) := 2 \int_1^z Q_\lambda^{1/2}(s) \, ds$ . One can compute this integral explicitly to arrive at (7.2.5) in Theorem 7.2.1. Furthermore,

$$(g_+ + g_-)(s) = V_\lambda(s) - \ell \quad \text{for } s \in \gamma_\lambda. \quad (8.1.5)$$

We are now ready to make the first transformation: let

$$\mathbf{T}(z) := 2^{n\sigma_3} \mathbf{Y}(z) e^{-n[g(z) + \log 2]\sigma_3}. \quad (8.1.6)$$

Then,  $\mathbf{T}(z)$  solves the following RH problem (RHP- $\mathbf{T}$ ):

(a)  $\mathbf{T}(z)$  is analytic in  $\mathbb{C} \setminus \gamma_\lambda$  and  $\lim_{z \rightarrow \infty} \mathbf{T} = \mathbf{I}$ .

(b)  $\mathbf{T}$  has continuous traces as  $z \rightarrow \gamma_\lambda \setminus \{\pm 1\}$  and

$$\mathbf{T}_+(s) = \mathbf{T}_-(s) \begin{pmatrix} e^{-n\phi_+(s)} & h(s) \\ 0 & e^{n\phi_+(s)} \end{pmatrix} \quad \text{for } s \in \gamma_\lambda \setminus \{\pm 1\}.$$

(c)  $\mathbf{T}$  behaves the same way as  $\mathbf{Y}$  as  $z \rightarrow \pm 1$ .

Indeed, RHP- $\mathbf{T}$ (a), (c) follow from analyticity properties of  $g$  while RHP- $\mathbf{T}$ (b) follows by explicit calculation and using (8.1.5).

## Opening the Lenses

Let  $\gamma_{\pm}$  be arcs within the domain of holomorphy of  $h(z)$  as shown in Figure 8.1 and define

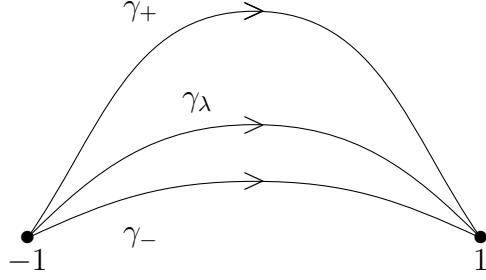


Fig. 8.1. RHP for Kissing Polynomials: Curves  $\gamma_{\pm}$  and  $\gamma_{\lambda}$

$$\mathbf{X}(z) = \begin{cases} \mathbf{T}(z) & z \text{ outside the lens} \\ \mathbf{T}(z) \begin{pmatrix} 1 & 0 \\ -e^{-n\phi(z)/h(z)} & 1 \end{pmatrix} & z \text{ on the upper lens} \\ \mathbf{T}(z) \begin{pmatrix} 1 & 0 \\ e^{-n\phi(z)/h(z)} & 1 \end{pmatrix} & z \text{ on the lower lens} \end{cases}. \quad (8.1.7)$$

Then  $\mathbf{X}(z)$  solves (RHP- $\mathbf{X}$ ):

- (a)  $\mathbf{X}(z)$  is analytic in  $\mathbb{C} \setminus (\gamma_{\lambda} \cup \gamma_{\pm})$  and  $\lim_{z \rightarrow \infty} \mathbf{X}(z) = \mathbf{I}$
- (b)  $\mathbf{X}$  has continuous traces on  $(\gamma_{\lambda} \cup \gamma_{\pm}) \setminus \{\pm 1\}$  that satisfy

$$\mathbf{X}_+(s) = \mathbf{X}_-(s) \begin{cases} \begin{pmatrix} 0 & h(s) \\ -1/h(s) & 0 \end{pmatrix} & \text{for } s \in \gamma_{\lambda}, \\ \begin{pmatrix} 1 & 0 \\ e^{-n\phi(s)/h(s)} & 1 \end{pmatrix} & \text{for } s \in \gamma_{\pm}. \end{cases}$$

(c) As  $z \rightarrow 1$  from outside the lenses,

$$\mathbf{X}(z) = \begin{cases} \mathcal{O} \begin{pmatrix} 1 & |z-1|^\alpha \\ 1 & |z-1|^\alpha \end{pmatrix} & \text{for } -1 < \alpha < 0 \\ \mathcal{O} \begin{pmatrix} 1 & \log|z-1| \\ 1 & \log|z-1| \end{pmatrix} & \text{for } \alpha = 0 \\ \mathcal{O} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{for } \alpha > 0 \end{cases}$$

while if  $z \rightarrow 1$  from inside the lenses,

$$\mathbf{X}(z) = \begin{cases} \mathcal{O} \begin{pmatrix} 1 & |z-1|^\alpha \\ 1 & |z-1|^\alpha \end{pmatrix} & \text{for } -1 < \alpha < 0 \\ \mathcal{O} \begin{pmatrix} \log|z-1| & \log|z-1| \\ \log|z-1| & \log|z-1| \end{pmatrix} & \text{for } \alpha = 0 \\ \mathcal{O} \begin{pmatrix} |z-1|^\alpha & 1 \\ |z-1|^\alpha & 1 \end{pmatrix} & \text{for } \alpha > 0 \end{cases}$$

with similar behavior for  $z \rightarrow -1$ .

Note that  $\operatorname{Re}(2\phi(z)) = \ell - \operatorname{Re}(V_\lambda(z)) - 2U^\mu(z)$  is a non-constant subharmonic function that vanishes on  $\gamma_\lambda$  (the last claim follows from the variational condition (7.1.1)). Since critical trajectories are exactly the set where  $\operatorname{Re}(2\phi(z)) = 0$ , we conclude that the sign of  $\operatorname{Re}(2\phi(z))$  must be fixed (locally) on either side of  $\gamma_\lambda$ . Due to the S-property (7.1.2), this sign must be the same on either side of  $\gamma_\lambda$ . Hence, we deduce from the maximum principle for subharmonic functions that  $\operatorname{Re}(\phi(z)) > 0$  in some neighborhood of  $\gamma_\lambda$ . Therefore, the jumps of  $\mathbf{X}$  on  $\gamma_\pm$  are exponentially small.

## Global Parametrix

Since jumps on the lenses  $\gamma_\pm$  are exponentially small, we temporarily ignore them and focus on solving the resulting RHP:

(a)  $\mathbf{N}(z)$  is analytic in  $\mathbb{C} \setminus \gamma_\lambda$  and  $\lim_{z \rightarrow \infty} \mathbf{N}(z) = \mathbf{I}$ ,

(b)  $\mathbf{N}$  has continuous traces as  $z \rightarrow \gamma_\lambda \setminus \{\pm 1\}$  and satisfies

$$\mathbf{N}_+(s) = \mathbf{N}_-(s) \begin{pmatrix} 0 & h(s) \\ -1/h(s) & 0 \end{pmatrix} \quad \text{for } s \in \gamma_\lambda \setminus \{\pm 1\}$$

We can solve this RH problem with the help of the Szegő function (cf. Section (5.3)):

$$S(z) := \exp \left\{ \frac{w(z)}{2\pi i} \int_{\gamma_\lambda} \frac{\log[(w_+h)(x)]}{z-x} \frac{dx}{w_+(x)} \right\}, \quad (8.1.8)$$

where we use the notation  $w(z) := (z^2 - 1)^{1/2}$  for the branch holomorphic in  $\mathbb{C} \setminus \gamma_\lambda$  with  $w(z) = z + \mathcal{O}(1)$  as  $z \rightarrow \infty$ . Observe that  $S$  is analytic and non-vanishing in  $\mathbb{C} \setminus \gamma_\lambda$  and satisfies

$$S_+(t)S_-(t) = (w_+h)(t) \quad \text{for } t \in \gamma_\lambda \setminus \{\pm 1\}, \quad (8.1.9)$$

where the above follows by application of the Plemelj-Sokhotski formula. Then, one can check that  $\mathbf{N}(z)$  can be written down explicitly as

$$\mathbf{N}(z) := (S(\infty))^{\sigma_3} \begin{pmatrix} 1 & 1/w(z) \\ 1/2\varphi(z) & \varphi(z)/2w(z) \end{pmatrix} S^{-\sigma_3}(z), \quad (8.1.10)$$

where  $\varphi$  is as in (7.2.5). Indeed, **RHP-N**(a) follows from analyticity properties of  $S, \varphi, w$  and the identity

$$\lim_{z \rightarrow \infty} \frac{\varphi(z)}{w(z)} = 2$$

while **RHP-N**(b) follows from (8.1.9) and

$$\varphi_+(t)\varphi_-(t) = 1 \quad \text{for } t \in \gamma_\lambda \setminus \{\pm 1\}. \quad (8.1.11)$$

Although we will construct separate local parametrices near  $z = \pm 1$ , we will need to note the behavior of  $\mathbf{N}(z)$  as  $z \rightarrow \pm 1$ , but this can be easily deduced from [28, equations (8.8) and (8.35)] (cf. proof of Proposition 5.3.1 in Appendix B) and turns out to be

$$\mathbf{N}(z) = \mathcal{O} \begin{pmatrix} |z-1|^{-(2\alpha+1)/4} & |z-1|^{(2\alpha-1)/4} \\ |z-1|^{-(2\alpha+1)/4} & |z-1|^{(2\alpha-1)/4} \end{pmatrix} \quad \text{as } z \rightarrow 1 \quad (8.1.12)$$

and the same formula ( $\alpha$  replaced by  $\beta$  and 1 by  $-1$ ) holds for  $z \rightarrow -1$ .

### 8.1.2 Local Analysis

Next, we solve the local RHPs at the end points  $z = \pm 1$ . The local parametrices that are involved are common in the literature, and appear in the already mentioned [40], for example.

#### Local Parametrix around $z = 1$

Let  $U_1$  be a disk centered at  $z = 1$  small enough so that  $h^*(z)$  (see (7.2.2)) is holomorphic in  $\overline{U}_1$ . We seek a matrix  $\mathbf{P}_\alpha(z)$  to solve the following RH problem (RHP- $\mathbf{P}_\alpha$ ):

(a, b, c)  $\mathbf{P}_\alpha(z)$  satisfies RHP- $\mathbf{X}$ (a, b, c) within  $U_1$ ,

(d) It holds uniformly for  $z \in \partial U_1$  that  $\mathbf{N}^{-1}(z)\mathbf{P}_\alpha(z) = \mathbf{I} + \mathcal{O}\left(\frac{1}{n}\right)$ .

While this is different from the local problem that appeared at the end points in Chapter 8, the simple transformation  $\tilde{\mathbf{P}}(z) = \mathbf{P}e^{-n\phi(z)\sigma_3/2}$  reduces it to the same problem, and hence we will refer the reader to Section 6.3.1 for the model problem and use the same notation,  $\Psi_\alpha(z)$  for its solution.

#### Conformal Map

Since  $w(z)$  has a square root singularity at  $z = 1$  and satisfies  $w_+(s) = -w_-(s)$ ,  $s \in \gamma_\lambda$ , the function

$$\zeta_1(z) := \left( \frac{1}{4} \int_1^z \frac{(2 + i\lambda s)ds}{w(s)} \right)^2 = \frac{1}{16} \phi^2(z), \quad z \in U_1, \quad (8.1.13)$$

is holomorphic in  $U_\delta$  with a simple zero at 1. Thus, the radius of  $U_\delta$  can be made small enough so that  $\zeta_1(z)$  is conformal on  $\overline{U}_\delta$ . Since  $\gamma_\lambda$  is exactly the curve where  $\text{Re}(\phi) = 0$ , it follows that  $\zeta_1$  maps  $\gamma_\lambda$  into the negative reals. Since we had some freedom in our choice of  $\gamma_\pm$ , we now define them as the pre-images of  $I_\pm$ , respectively, under  $\zeta_1$ . It is clear in the case  $\lambda = 0$  that  $\gamma_\pm$  are in the correct half-planes, and by

continuity of  $\zeta_1$  w.r.t  $\lambda$ , and since for  $\lambda > 0$ ,  $\gamma_\lambda$  is the unique arc emanating from  $z = 1$  with  $\operatorname{Re}(\phi(z)) = 0 \ \forall z \in \gamma_\lambda$ , we see that  $\gamma_\pm$  are in the correct half-plane for all  $\lambda \geq 0$ . In what follows, we consider the branch  $\zeta_1^{1/2}(z) = \frac{1}{4}\phi(z)$ .

### Matrix $\mathbf{P}_\alpha$

Recall that

$$h(z) = h^*(z)(1-z)^\alpha(1+z)^\beta, \quad z \in U_1,$$

where  $(1-z)^\alpha$ ,  $(1+z)^\beta$  be functions holomorphic in  $U_1 \setminus [1, \infty)$ ,  $U_1 \setminus (-\infty, -1]$ , respectively, and  $h(z)$  is holomorphic and non-vanishing in  $\overline{U_1}$ . Define

$$r_1(z) := \sqrt{h^*(z)(1+z)^\beta} \cdot (z-1)^{\alpha/2}, \quad z \in U_1 \setminus \gamma_\lambda,$$

where  $(z-1)^{\alpha/2}$  has branch cut along  $\gamma_\lambda$ . It holds that

$$(z-1)^\alpha = e^{\pm\pi i\alpha}(1-z)^\alpha, \quad z \in U_1^\pm,$$

where  $U_1^\pm$  are the domains within  $U_1$  to the left, respectively right, of  $\gamma_\lambda \cup [1, \infty)$ .

Then

$$\begin{cases} r_{1,+}(s)r_{1,-}(s) = h(s), & s \in \gamma_\lambda \cap U_1, \\ r_1^2(z) = h(z)e^{\pm\pi i\alpha}, & z \in U_1^\pm. \end{cases} \quad (8.1.14)$$

The above relations and **RHP- $\Psi_\alpha$**  imply that

$$\mathbf{P}_\alpha(z) := \mathbf{E}_\alpha(z)\Psi_\alpha(n^2\zeta_1(z))r_1^{-\sigma_3}(z)e^{-\frac{n}{2}\phi(z)\sigma_3} \quad (8.1.15)$$

satisfied **RHP- $\mathbf{P}_\alpha$** (a, b, c) for any  $\mathbf{E}_\alpha(z)$  holomorphic in  $U_1$ .

### Matrix $\mathbf{E}_\alpha$

We will use the freedom of choosing  $\mathbf{E}_\alpha$  to satisfy **RHP- $\mathbf{P}_\alpha$** (d). To this end, let

$$\mathbf{E}_\alpha(z) := \mathbf{N}(z)r_1^{\sigma_3}(z)\mathbf{S}^{-1}(n^2\zeta_1(z)). \quad (8.1.16)$$

where  $\mathbf{S}$  is defined in **RHP- $\Psi_\alpha$** (d). It follows from **RHP- $\mathbf{N}$** , (8.1.14), and (6.3.1) that  $\mathbf{E}_\alpha$  is holomorphic in  $U_1 \setminus \{1\}$ . The fact that  $\mathbf{S}^{-1}(n^2\zeta_1) \sim |z-1|^{-\sigma_3/4}$ ,  $r_1^{\sigma_3}(z) \sim$



$|z - 1|^{\alpha\sigma_3}$ , coupled with (8.1.12), imply that  $z = 1$  is a removable singularity of  $\mathbf{E}_\alpha$ , which establishes the holomorphy of  $\mathbf{E}_\alpha$  in  $U_1$ .

### Local Parametrix around $z = -1$

A similar construction can be carried out in a neighborhood  $U_{-1}$  of  $z = -1$  defined in a fashion similar to  $U_1$  to arrive at the local parametrix

$$\tilde{\mathbf{P}}_\beta(z) := \mathbf{E}_\beta(z)\sigma_3\mathbf{\Psi}_\beta(n^2\zeta_{-1}(z))\sigma_3r_{-1}^{-\sigma_3}e^{-\frac{n}{2}\tilde{\phi}(z)\sigma_3} \quad (8.1.17)$$

where  $\tilde{\phi}(z) = \phi(z) - 2\pi i$ ,  $\zeta_{-1}(z) = \tilde{\phi}^2(z)/16$

$$r_{-1}(z) := \sqrt{h^*(z)(1-z)^\alpha(z+1)^{\beta/2}}, \quad z \in U_{-1} \setminus \gamma_\lambda, \quad (8.1.18)$$

where the branch  $(z+1)^{\beta/2}$  is taken with cut along  $\gamma_\lambda$ . The correct choice of  $\mathbf{E}_\beta(z)$  turns out to be

$$\mathbf{E}_\beta(z) := \mathbf{N}(z)r_{-1}^{\sigma_3}(z)\mathbf{S}^{-1}(n^2\zeta_{-1}(z)). \quad (8.1.19)$$

### 8.1.3 Final Riemann-Hilbert Problem

We now define

$$\mathbf{R}(z) := \mathbf{X}(z) \begin{cases} \mathbf{N}^{-1}(z), & z \in \mathbb{C} \setminus (\overline{U_1} \cup \overline{U_{-1}} \cup \gamma_\lambda \cup \gamma_\pm), \\ \mathbf{P}_\alpha^{-1}(z), & z \in U_1 \setminus (\gamma_\lambda \cup \gamma_\pm), \\ \tilde{\mathbf{P}}_\beta^{-1}(z), & z \in U_{-1} \setminus (\gamma_\lambda \cup \gamma_\pm). \end{cases} \quad (8.1.20)$$

where we orient  $\partial U_{\pm 1}$  clockwise. It follows that  $\mathbf{R}(z)$  is analytic in  $U_1, U_{-1}$  and  $\overline{C} \setminus (\overline{U_1} \cup \overline{U_{-1}} \cup \gamma_\pm)$  and that, for  $s \in \gamma_\pm \cup \partial U_1 \cup \partial U_{-1}$

$$\mathbf{R}_+(s) = \mathbf{R}_-(s) \begin{cases} \mathbf{I} + \mathcal{O}(e^{-cn}) & \text{for } z \in \gamma_\pm \setminus U_{\pm 1} \\ \mathbf{I} + \mathcal{O}\left(\frac{1}{n}\right) & \text{for } z \in (\partial U_1 \cup \partial U_{-1}) \end{cases}. \quad (8.1.21)$$

Indeed, for  $s \in \gamma_\pm \cap (\mathbb{C} \setminus (\overline{U_1} \cup \overline{U_{-1}}))$ ,

$$(\mathbf{R}_-^{-1}\mathbf{R}_+)(s) = \mathbf{N}(s) \begin{pmatrix} 1 & 0 \\ e^{-n\phi(s)/h(s)} & 1 \end{pmatrix} \mathbf{N}^{-1}(s) = \mathbf{I} + \mathbf{N}(s) \begin{pmatrix} 0 & 0 \\ e^{-n\phi(s)/h(s)} & 0 \end{pmatrix} \mathbf{N}^{-1}(s).$$

and since  $\mathbf{N}$  is independent of  $n$  and  $\operatorname{Re}(\phi) > 0$  on  $\gamma_{\pm}$ , see discussion in Section 8.1.1 right after **RHP-X**, it follows that

$$\|\mathbf{R}_-^{-1}\mathbf{R}_+ - \mathbf{I}\|_{L^\infty(\gamma_{\pm} \cap (\mathbb{C} \setminus (\overline{U_1} \cup \overline{U_{-1}}))} = \mathcal{O}(e^{-cn}).$$

As for the second equality, for  $s \in \partial U_1$  ( $s \in \partial U_{-1}$  can be handled similarly)

$$(\mathbf{R}_-^{-1}\mathbf{R}_+)(s) = \mathbf{P}_\alpha(s)\mathbf{N}^{-1}(s) = \mathbf{N}(s) \left( \mathbf{I} + \mathcal{O}\left(\frac{1}{n}\right) \right) \mathbf{N}^{-1}(s)$$

where the last equality is due to **RHP- $\mathbf{P}_\alpha$** (d). Therefore,

$$\|\mathbf{R}_-^{-1}\mathbf{R}_+ - \mathbf{I}\|_{L^\infty(\partial U_1 \cup \partial U_{-1})} = \mathcal{O}\left(\frac{1}{n}\right)$$

Since all curves involved are fixed with  $n$  and of finite length, all estimates hold in  $L^2$ -norm as well. It now follows from [26, Corollary 7.108] that

$$\mathbf{R}(z) = \mathbf{I} + \mathcal{O}\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow \infty, \quad (8.1.22)$$

uniformly for  $z \in \mathbb{C} \setminus (\gamma_{\pm} \cup \partial U_1 \cup \partial U_{-1})$ . The asymptotic formula of  $P_n^\lambda(z)$  outside the lenses and away from endpoints follow from the observation

$$P_n^\lambda(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{Y}(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{ng(z)} \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{R}\mathbf{N}(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (8.1.23)$$

$$= e^{ng(z)} ([\mathbf{R}(z)]_{11}[\mathbf{N}(z)]_{11} + [\mathbf{R}(z)]_{12}[\mathbf{N}(z)]_{21}) \quad (8.1.24)$$

$$= e^{ng(z)} \left( [\mathbf{N}(z)]_{11} + \mathcal{O}\left(\frac{1}{n}\right) \right), \quad (8.1.25)$$

where the last equality follows from (8.1.22). Finally, observing that by (8.1.3),

$$e^{ng(z)} = \left( \frac{\varphi(z)}{2} \right)^n \exp \left( -\frac{in\lambda}{\varphi(z)} \right),$$

from which (7.2.8) follows.

## 8.2 Critical Case; $\lambda = \lambda_{cr}$

In the case  $\lambda = \lambda_{cr}$  the zero-attracting curve seizes to be smooth, and we must modify the lenses we consider to the figure shown below.

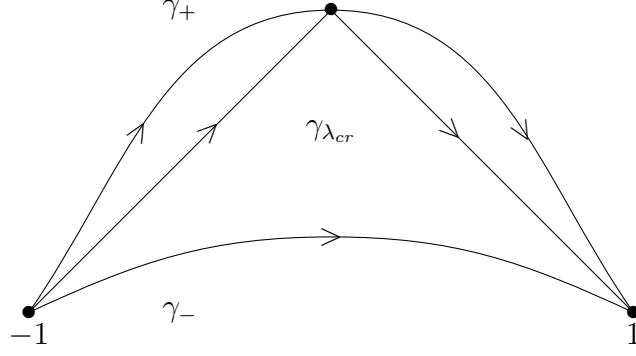


Fig. 8.2. RHP for Kissing Polynomials: lenses in the critical case

We will define matrices  $\mathbf{T}$ ,  $\mathbf{X}$ , and  $\mathbf{N}$  in the same way as way done in the sub-critical case. However, we will need to perform some local analysis at the midpoint of  $\gamma_{\lambda_{cr}}$ , which lies at  $z_* = 2i/\lambda_{cr}$ .

### 8.2.1 Local Parametrix around $z_* = 2i/\lambda_{cr}$

Let  $U_c$  be a disk centered at  $z_* = 2i/\lambda_{cr}$  small enough so that  $h(z)$  (see (7.2.2)) is holomorphic in  $\overline{U}_c$ . We seek a matrix  $\mathbf{P}_c(z)$  to solve the following RH problem (RHP- $\mathbf{P}_c$ ):

- (a)  $\mathbf{P}_c(z)$  satisfies is holomorphic in  $U_c \setminus (\gamma_{\lambda_{cr}} \cup \gamma_+)$ ,
- (b)  $\mathbf{P}_c(z)$  has continuous traces on  $\gamma_{\lambda_{cr}} \cup \gamma_+$  that satisfy

$$\mathbf{P}_{c,+}(s) = \mathbf{P}_{c,-}(s) \begin{cases} \begin{pmatrix} 1 & 0 \\ -e^{-n\phi(s)}/h(s) & 1 \end{pmatrix}, & s \in \gamma_+ \cap U_c \\ \begin{pmatrix} 0 & h(s) \\ -1/h(s) & 0 \end{pmatrix}, & s \in \gamma_{\lambda_{cr}} \cap U_c \end{cases} \quad (8.2.1)$$

- (c)  $\mathbf{P}_c(z)$  is bounded as  $z \rightarrow 2i/\lambda_{cr}$ . Furthermore, it holds uniformly for  $z \in \partial U_c$  that  $\mathbf{N}^{-1}(z)\mathbf{P}_c(z) = \mathbf{I} + \mathcal{O}(n^{-1/2})$ .

### Model Problem

We seek a matrix  $\mathbf{C}(\zeta)$  that solves the following RHP:

- (a)  $\mathbf{C}$  is holomorphic in  $\mathbb{C} \setminus \mathbb{R}$
- (b)  $\mathbf{C}$  has continuous traces on  $\mathbb{R}$  that satisfy

$$\mathbf{C}_+(s) = \mathbf{C}_-(s) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad (8.2.2)$$

- (c)  $\mathbf{C}(\zeta)$  is bounded as  $\zeta \rightarrow 0$  and

$$\mathbf{C}(\zeta) \sim \left( \mathbf{I} + \sum_{k=0}^{\infty} \begin{pmatrix} 0 & b_k \\ 0 & 0 \end{pmatrix} \zeta^{-(2k+1)} \right) e^{-\zeta^2 \sigma_3} \quad (8.2.3)$$

for some  $b_k \neq 0$ .

This problem appears in [50] and is solved by the matrix

$$\mathbf{C}(\zeta) = \begin{pmatrix} e^{-\zeta^2} & b(\zeta) \\ 0 & e^{\zeta^2} \end{pmatrix}, \quad (8.2.4)$$

where

$$b(\zeta) := \frac{1}{2} e^{\zeta^2} \cdot \begin{cases} \operatorname{erfc}(-i\sqrt{2}\zeta), & \operatorname{Im}(\zeta) > 0, \\ \operatorname{erfc}(i\sqrt{2}\zeta), & \operatorname{Im}(\zeta) < 0. \end{cases} \quad (8.2.5)$$

With this definition and using [44, Equation (7.12.1)], we see that  $b_k = \frac{i}{\sqrt{2\pi}} \frac{\Gamma(k+1/2)}{2^{k+1}\Gamma(1/2)}$ .

### Conformal Map

Let

$$\phi_c(z) = \begin{cases} \phi(z), & z \in U_{c,+}, \\ -\phi(z), & z \in U_{c,-}, \end{cases} \quad (8.2.6)$$

where  $U_{c,+}$  (resp.,  $U_{c,-}$ ) is the component of  $U_c$  to the left (resp., right) of  $\gamma_{\lambda_{cr}}$ . Then,  $\phi_c$  is holomorphic in  $U_c$  and since  $z_* = 2i/\lambda_{cr}$  is a simple zero of  $Q_{\lambda_{cr}}^{1/2}$ , we have that

$$|\phi_c(z) - \phi_c(z_*)| \sim |z - z_*|^2 \quad \text{as } z \rightarrow z_*.$$

Furthermore, by Theorem 7.2.1, we have that

$$\phi_{\pm}(s) = \pm 2\pi i \mu_{\lambda_{cr}}([s, 1]) \quad \text{for } s \in \gamma_{\lambda_{cr}} \quad (8.2.7)$$

and we can see that  $\phi_c(z)$  is purely imaginary and positive on  $\gamma_{\lambda_{cr}}(-1, z_*)$  and negative purely imaginary on  $\gamma_{\lambda_{cr}}(z_*, 1)$  where  $\gamma_{\lambda_{cr}}(z_1, z_2)$ ,  $z_1, z_2 \in \gamma_{\lambda_{cr}}$  is the segment of  $\gamma_{\lambda_{cr}}$  that proceeds from  $z_1$  to  $z_2$ . With this in mind, we can define a branch of  $(\phi_c(z) - \phi_c(z_*))^{1/2}$  that is holomorphic and, WLOG (up to restricting  $U_c$  to a smaller neighborhood) conformal in  $U_c$  and maps  $\gamma_{\lambda_{cr}}(-1, z_*) \cap U_c \rightarrow \{z \mid \arg(z) = \pi/4\}$ ,  $\gamma_{\lambda_{cr}}(z_*, 1) \cap U_c \rightarrow \{z \mid \arg(z) = 3\pi/4\}$ . Using this branch, the map

$$\zeta_c(z) := -(\phi_c(z) - \phi_c(z_*))^{1/2} \quad (8.2.8)$$

is conformal, maps  $\gamma_{\lambda_{cr}}(-1, z_*) \cap U_c$  into  $\{z \mid \arg(z) = 5\pi/4\}$  and  $\gamma_+$  into  $\mathbb{R}$ .

### Matrix $\mathbf{P}_c$

Since  $h(z)$  is holomorphic and nonvanishing in  $U_c$ , we can define a holomorphic branch of  $r(z) := \sqrt{h(z)}$ . Furthermore, let

$$\mathbf{J}(z) := \begin{cases} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & z \in U_{c,+}, \\ \mathbf{I}, & z \in U_{c,-}. \end{cases} \quad (8.2.9)$$

Then,

$$\mathbf{P}_c(z) = \mathbf{E}_c(z) \mathbf{C} \left( \sqrt{n/2} \cdot \zeta_c(z) \right) \mathbf{J}^{-1}(z) r^{-\sigma_3}(z) e^{-n\phi(z)\sigma_3/2} \quad (8.2.10)$$

satisfies **RHP- $\mathbf{P}_c$** (a, b) for any  $\mathbf{E}_c(z)$  holomorphic in  $U_c$ . Furthermore, by the very definition of  $\mathbf{C}, \mathbf{J}, r$ , it follows that  $\mathbf{P}_c$  is bounded as  $z \rightarrow z_*$ . **RHP- $\mathbf{P}_c$** (d) follows by letting

$$\mathbf{E}_c(z) := \mathbf{N}(z) r^{\sigma_3}(z) \mathbf{J}(z) \quad (8.2.11)$$

and noting the holomorphy of all matrices in  $U_c$ , expansion (8.2.3), that  $\phi_c(z_*) \in i\mathbb{R}$ , and the relation

$$e^{-n\phi(z)\sigma_3/2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e^{n\phi(z)\sigma_3/2}$$

yield the desired result.

### 8.2.2 Final Riemann-Hilbert Problem

The final RHP can now be constructed in a manner completely analogous to (8.1.20) and yields the asymptotic formula for  $P_n^{\lambda_{cr}}(z)$  in Theorem 7.2.4. Note that the worse error term is due to a worse matching between the local solution at  $z = z_*$  and the global solution  $\mathbf{N}$ .

### 8.3 Supercritical Case; $\lambda > \lambda_{cr}$

We begin our analysis by deforming  $[-1, 1]$  to a curve  $\gamma_\lambda$  that goes along  $\gamma_1$ , starting at  $-1$  smoothly proceeds in the sector shown in Figure 7.1 from  $-\overline{z_*}$  to  $z_*$  along  $\hat{\gamma}$  and goes along  $\gamma_2$  to  $1$ . The initial RHP is as in RHP-Y with  $\rho(s; n) := h(z)e^{in\lambda z}$  and again we require  $h(z)$  to be as in (7.2.2). See Section 7.3.2 for the definition of the aforementioned curves.

#### 8.3.1 Global Analysis

Because the zero-attracting curve has two connected components, the global analysis will drastically change, and more complicated functions will appear. Nonetheless, we still follow the general layout of the steepest descent method.

#### First Transformation

In the same spirit as the one-cut case, let

$$g(z) := \int \log(z - s) d\mu_\lambda(s), \quad z \in \mathbb{C} \setminus ((-\infty, -1) \cup \gamma_\lambda), \quad (8.3.1)$$

where  $\log(\cdot - s)$  is holomorphic outside the curve connecting  $-\infty$  and  $s$  along  $(-\infty, 1] \cup \gamma_\lambda$ . Then it follows from (7.2.3) that there is  $\ell^* \in \mathbb{C}$  so that

$$g(z) = \frac{V_\lambda(z) - \ell^*}{2} + \phi_1(z) \quad \text{and} \quad \phi_e(z) := 2 \int_e^z Q_\lambda^{1/2}(s) ds, \quad e \in \{\pm 1, z_*, -\overline{z_*}\}, \quad (8.3.2)$$

where the domain of holomorphy for  $\phi_e$  is  $\mathbb{C} \setminus ((-\infty, -1) \cup \gamma_\lambda)$  for  $e = 1$ ,  $\mathbb{C} \setminus (\gamma_\lambda \cup [1, \infty))$  for  $e = -1$ , and  $\mathbb{C} \setminus ((-\infty, -1) \cup \gamma_\lambda(-1, -\overline{z_*}) \cup \gamma_\lambda(z_*, 1) \cup [1, \infty))$  for  $e \in \{z_*, -\overline{z_*}\}$ . From Figure 7.1, we immediately deduce that  $\tau \in \mathbb{R}$  (see (7.3.19)) and

$$\phi_{1,\pm}(s) = \begin{cases} \pm 2\pi i \mu_\lambda([s, 1]), & s \in \gamma_2, \\ \pm 2\pi i \mu_\lambda([s, 1]) + 2\pi i \tau, & s \in \gamma_1. \end{cases} \quad (8.3.3)$$

Furthermore, using the fact that  $\mu_\lambda$  is a probability measure and definition (7.3.19) yields

$$\phi_1(z) = \begin{cases} \phi_{z_*}(z) \pm \pi i \\ \phi_{-\overline{z_*}}(z) \pm \pi i + 2\pi i \tau \\ \phi_{-1}(z) \pm 2\pi i + 2\pi i \tau \end{cases}, \quad z \in \mathbb{C} \setminus ((-\infty, -1) \cup \gamma_\lambda \cup (1, \infty)), \quad (8.3.4)$$

and  $+$  (resp.  $-$ ) is chosen when  $z$  belongs to the left (resp. right) of  $(-\infty, -1) \cup \gamma_\lambda \cup (1, \infty)$ , oriented from  $-\infty$  to  $\infty$ , and we use the fact that

$$\frac{1}{2} = -\frac{1}{\pi i} \int_{\gamma_1} Q_{\lambda,+}^{1/2}(s) ds. \quad (8.3.5)$$

The later follows from a residue calculation and the reflection symmetry of  $\gamma_1, \gamma_2$ , see [56, Proposition 3.5]. With this, (8.3.3), and (8.3.2) in mind, we can write

$$(g_+ - g_-)(s) = \begin{cases} 0, & s \in (1, \infty), \\ \pm \phi_{1,\pm}(s), & s \in \gamma_2, \\ \pi i, & s \in \hat{\gamma}, \\ \pm(\phi_{1,\pm} - 2\pi i \tau), & s \in \gamma_1, \\ 2\pi i, & s \in (-\infty, -1). \end{cases} \quad (8.3.6)$$

Furthermore,

$$(g_+ + g_- - V + \ell^*)(s) = \begin{cases} \phi_1(s), & s \in (1, \infty), \\ 0, & s \in \gamma_2, \\ \phi_{z_*}(s), & s \in \hat{\gamma}, \\ 2\pi i\tau, & s \in \gamma_1, \\ \phi_{-1}(s) + 2\pi i\tau, & s \in (-\infty, -1). \end{cases} \quad (8.3.7)$$

We are now ready to start with our first transformation (cf. Section 8.1.1)

$$\mathbf{T}(z) := e^{n\ell^*\sigma_3} \mathbf{Y}(z) e^{-n(g(z)+\ell^*/2)\sigma_3}. \quad (8.3.8)$$

Then,  $\mathbf{T}$  solves

(a)  $\mathbf{T}(z)$  is holomorphic in  $\mathbb{C} \setminus \gamma_\lambda$  and  $\lim_{z \rightarrow \infty} \mathbf{T} = \mathbf{I}$ ,

(b)  $\mathbf{T}(z)$  has continuous traces on  $\gamma_\lambda \setminus \{\pm 1, z_*, -\overline{z_*}\}$  that satisfy

$$\mathbf{T}_+(s) = \mathbf{T}_-(s) \begin{cases} \begin{pmatrix} e^{-n(\phi_{1,+}-2\pi i\tau)} & h(s)e^{2n\pi i\tau} \\ 0 & e^{-n(\phi_{1,-}-2\pi i\tau)} \end{pmatrix}, & s \in \gamma_1, \\ \begin{pmatrix} e^{n\pi i} & h(s)e^{n\phi_{z_*}(s)} \\ 0 & e^{-n\pi i} \end{pmatrix}, & s \in \hat{\gamma} \\ \begin{pmatrix} e^{-n\phi_{1,+}} & h(s) \\ 0 & e^{-n\phi_{1,-}} \end{pmatrix}, & s \in \gamma_2, \end{cases}$$

(c)  $\mathbf{T}(z)$  behaves the same as  $\mathbf{Y}$  as  $z \rightarrow \pm 1$ .

## Opening the Lenses

Motivated by the factorization

$$\begin{pmatrix} e^{-n(\phi_{1,+}(s)-C)} & h(z)e^{nC} \\ 0 & e^{-n(\phi_{1,-}(s)-C)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-n\phi_{1,-}(s)}/h(s) & 1 \end{pmatrix} \times \\ \begin{pmatrix} 0 & h(z)e^{nC} \\ -e^{-nC}/h(z) & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-n\phi_{1,+}(s)}/h(s) & 1 \end{pmatrix},$$



where  $C = 2\pi i\tau, 0$  on  $\gamma_1, \gamma_2$ , respectively, we make the following definitions. Let  $\gamma_{i,\pm}$  be arcs within the neighborhood of holomorphy of  $h(z)$  as shown in Figure 8.3 below.  $\gamma_{1,\pm}$  proceed from  $z = -1$  to  $z = -\bar{z}_*$  while  $\gamma_{2,\pm}$  proceed from  $z = z_*$  to  $z = 1$ .

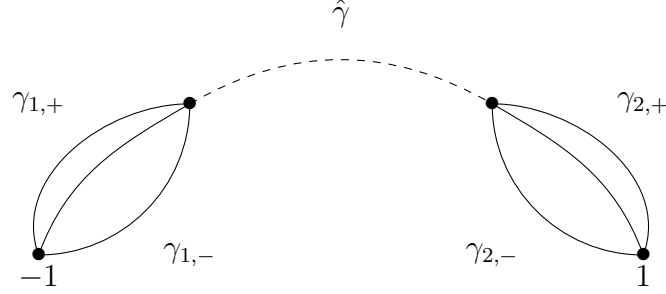


Fig. 8.3. Opening the lenses: supercritical regime for kissing polynomials

Denote by  $\Omega_{i,\pm}$  the open sets delimited by  $\gamma_{i,\pm}$  and  $\gamma_i$ . Set

$$\mathbf{X}(z) := \mathbf{T}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ \mp e^{-n\phi_1(z)/h(z)} & 1 \end{pmatrix}, & z \in \Omega_{i,\pm}, \\ \mathbf{I}, & \text{otherwise.} \end{cases} \quad (8.3.9)$$

Then  $\mathbf{X}$  solves

- (a)  $\mathbf{X}(z)$  is analytic in  $\mathbb{C} \setminus (\gamma_\lambda \cup \gamma_{i,\pm})$  and  $\lim_{z \rightarrow \infty} \mathbf{X}(z) = \mathbf{I}$ ,
- (b)  $\mathbf{X}(z)$  has continuous traces on  $\gamma_\lambda \setminus \{\pm 1, -\bar{z}_*, z_*\}$  that satisfy **RHP- $\mathbf{T}$** (b) on  $\hat{\gamma}$ , as well as

$$\mathbf{X}_+(s) = \mathbf{X}_-(s) \begin{cases} e^{2n\pi i\tau\sigma_3} \begin{pmatrix} 0 & h(s) \\ -1/h(s) & 0 \end{pmatrix}, & s \in \gamma_1, \\ \begin{pmatrix} 0 & h(s) \\ -1/h(s) & 0 \end{pmatrix}, & s \in \gamma_2, \\ \begin{pmatrix} 1 & 0 \\ e^{-n\phi_1(s)/h(s)} & 1 \end{pmatrix}, & s \in \gamma_{i,\pm}, \quad i = 1, 2. \end{cases}$$

(c) As  $z \rightarrow 1$  from outside the lenses,

$$\mathbf{X}(z) = \begin{cases} \mathcal{O} \begin{pmatrix} 1 & |z-1|^\alpha \\ 1 & |z-1|^\alpha \end{pmatrix} & \text{for } -1 < \alpha < 0, \\ \mathcal{O} \begin{pmatrix} 1 & \log |z-1| \\ 1 & \log |z-1| \end{pmatrix} & \text{for } \alpha = 0, \\ \mathcal{O} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \text{for } \alpha > 0. \end{cases}$$

Furthermore, if  $z \rightarrow 1$  from inside the lenses,

$$\mathbf{X}(z) = \begin{cases} \mathcal{O} \begin{pmatrix} 1 & |z-1|^\alpha \\ 1 & |z-1|^\alpha \end{pmatrix} & \text{for } -1 < \alpha < 0, \\ \mathcal{O} \begin{pmatrix} \log |z-1| & \log |z-1| \\ \log |z-1| & \log |z-1| \end{pmatrix} & \text{for } \alpha = 0, \\ \mathcal{O} \begin{pmatrix} |z-1|^\alpha & 1 \\ |z-1|^\alpha & 1 \end{pmatrix} & \text{for } \alpha > 0. \end{cases}$$

with similar behavior for  $z \rightarrow -1$  where  $\beta$  replaces  $\alpha$ .

### Global Parametrix

To discuss boundedness properties of  $\Theta_{n,k}(z)$  and for the asymptotic analysis in the following section it will be convenient to define

$$M_{n,0}(z) = \Theta_{n,0}(z) \begin{cases} B(z), & z \in \mathfrak{R}^{(0)}, \\ A(z), & z \in \mathfrak{R}^{(1)}, \end{cases} \quad \text{and} \quad M_{n,1}(z) = \Theta_{n,1}(z) \begin{cases} A(z), & z \in \mathfrak{R}^{(0)}, \\ -B(z), & z \in \mathfrak{R}^{(1)}. \end{cases} \quad (8.3.10)$$

These functions are holomorphic on  $\mathfrak{R} \setminus \{\alpha \cup \beta \cup \pi^{-1}(\gamma_\lambda)\}$  since the pole of  $\Theta_{n,k}(z)$  is canceled by the zero of  $\beta(z)$ . Each function  $M_{n,k}(z)$  has exactly two zeros, namely,  $z_{n,k}$  and  $\infty^{(k)}$ . It follows from (B.3.3) and (7.3.21) that

$$\begin{cases} M_{n,k\pm}^{(0)}(s) = \mp M_{n,k\mp}^{(1)}(s), & s \in \gamma_2, \\ M_{n,k\pm}^{(0)}(s) = \mp e^{-2\pi i \tau n} M_{n,k\mp}^{(1)}(s), & s \in \gamma_1, \\ M_{n,k\pm}^{(i)}(s) = e^{(-1)^i 2\pi i (n\omega + c_h)} M_{n,k\mp}^{(i)}(s), & s \in \hat{\gamma}. \end{cases} \quad (8.3.11)$$

By arguing in the same way as we did in the one cut case (see Section 8.1.1, see also (8.3.3), (7.3.19)), we see that the jumps on  $\gamma_{i,\pm}$  and off diagonal entry in the jump on  $\hat{\gamma}$  are exponentially small. Hence, the Riemann-Hilbert problem for the global parametrix is obtained from **RHP-X** by removing those quantities. Thus, we are seeking the solution of **RHP-N**:

- (a)  $\mathbf{N}(z)$  is analytic in  $\mathbb{C} \setminus (\gamma_\lambda)$  and  $\lim_{z \rightarrow \infty} \mathbf{N}(z) = \mathbf{I}$ ;
- (b)  $\mathbf{N}(z)$  has continuous traces on  $\gamma_\lambda \setminus \{\pm 1, -\overline{z_*}, z_*\}$  that satisfy

$$\mathbf{N}_+(s) = \mathbf{N}_-(s) \begin{cases} e^{2n\pi i \tau \sigma_3} \begin{pmatrix} 0 & h(s) \\ -1/h(s) & 0 \end{pmatrix}, & s \in \gamma_1, \\ \begin{pmatrix} 0 & h(s) \\ -1/h(s) & 0 \end{pmatrix}, & s \in \gamma_2, \\ e^{n\pi i \sigma_3}, & s \in \hat{\gamma}. \end{cases}$$

We shall solve this problem only for  $n \in \mathbb{N}(\epsilon) = \mathbb{N}(\lambda, \epsilon)$  from Section 7.3.2.

Let the functions  $M_{n,k}(z)$  be given by (8.3.10) and  $S_h$  be defined by (7.2.7). With the notation introduced right after (7.3.20), a solution of **RHP-N** is given by

$$\mathbf{N}(z) = \mathbf{M}^{-1}(\infty) \mathbf{M}(z), \quad \mathbf{M}(z) := \begin{pmatrix} M_{n,1}^{(0)}(z) & M_{n,1}^{(1)}(z) \\ M_{n,0}^{(0)}(z) & M_{n,0}^{(1)}(z) \end{pmatrix} \tilde{S}_h^{\sigma_3}(z^{(0)}). \quad (8.3.12)$$

Indeed, **RHP-N**(a) follows from holomorphy of  $\tilde{S}_h(z)$  and  $M_{n,k}(z)$  discussed in Proposition 7.3.2 and right after (8.3.10), respectively. Fulfillment of **RHP-N**(b) can be

checked by using (7.3.15) and (8.3.11). It will be important for our analysis that  $\mathbf{N}$  be invertible, which it is. Indeed, observe that  $\det(\mathbf{N}(z)) \equiv 1$ . Indeed, as the jump matrices in  $\text{RHP-}\mathbf{N}(\text{b})$  have determinants 1,  $\det(\mathbf{N}(z))$  is holomorphic through  $\gamma_1$ ,  $\hat{\gamma}$ , and  $\gamma_2$ . It also has at most square root singularities at  $\{\pm 1, z_*, -\bar{z}_*\}$  as explained right after (8.3.11). Thus, it is holomorphic throughout  $\overline{\mathbb{C}}$  and therefore is a constant. The normalization at infinity implies that this constant is 1.

The behavior of  $\mathbf{N}$  near the end points of  $\gamma_1, \gamma_2$  will be important for the analysis, and so we note it here. It follows from (7.3.6), (7.3.7) that

$$|M_{n,k}(\mathbf{z})| \sim |z - e|^{-1/4} \quad \text{as } \mathbf{z} \rightarrow \mathbf{e} \in \{\pi^{-1}(1), \pi^{-1}(-1), \pi^{-1}(z_*), \pi^{-1}(-\bar{z}_*)\}. \quad (8.3.13)$$

Combining this and (7.3.16) yields

$$\mathbf{N} \sim |z - e|^{-1/4} \cdot |z - e|^{\alpha_e \sigma_3/2}, \quad e \in \{\pm 1, z_*, -\bar{z}_*\} \quad (8.3.14)$$

where where  $\alpha_e = 0$  for  $e = z_*, -\bar{z}_*$ ,  $\alpha_e = \alpha$  when  $e = 1$  and  $\alpha_e = \beta$  when  $e = -1$ . In fact, for  $n \in \mathbb{N}(\lambda, \epsilon)$  and for  $\mathbf{z} \in \mathfrak{A} \setminus \pi^{-1}(\cup_e U_{e,\delta})$  where  $e \in \{\pm 1, z_*, -\bar{z}_*\}$  and  $U_e$  is a neighborhood of  $e$  of radius  $\delta > 0$ , we can argue in the same way as was done in Section 7.3.2 to arrive at constants  $c(\epsilon)$ ,  $C(\delta) > 0$  that satisfy

$$0 < c(\epsilon) < |M_{n,k}(\mathbf{z})| < C(\delta). \quad (8.3.15)$$

### 8.3.2 Local Analysis

In this section, we solve local RHP near points  $z = \pm 1$  and  $z = z_*, -\bar{z}_*$ . These local parametrices are standard in the literature, and involve Bessel/Hankel functions (near  $z = \pm 1$ ) and Airy functions (near  $z = z_*, -\bar{z}_*$ ). In fact, the parametrices near  $z = \pm 1$  have already appeared multiple times in this work, see Sections 6.3.1, 8.1.2 for example.

### Local Parametrices at $z = -1, 1$

The local analysis near  $z = \pm 1$  is very similar to what had been done in the one-cut case. Let  $U_e$ ,  $e \in \{\pm 1\}$  be an open disk centered at  $e$  with fixed radius  $\Delta$  small enough so that it is in the domain of holomorphy of  $h^*(z)$  (see the line below (7.2.1)). We seek a matrix  $\mathbf{P}_e$ , that solves the following RHP- $\mathbf{P}_e$ :

- (a)  $\mathbf{P}_e$  satisfies the same analyticity properties as  $\mathbf{X}$  within  $U_e$ ,
- (b)  $\mathbf{P}_e$  satisfies the same jump relations as  $\mathbf{X}$  within  $U_e$ ,
- (c)  $\mathbf{P}_e(z) = \mathbf{N}(z) (\mathbf{I} + \mathcal{O}(n^{-1}))$  uniformly on  $\partial U_e$  as  $n \rightarrow \infty$ .

### Conformal Map

Let  $\phi_e$  be as defined in (8.3.2), and define

$$\zeta_e(z) := \left( \frac{1}{4} \phi_e(z) \right)^2, \quad e \in \{\pm 1\} \quad (8.3.16)$$

Then, since  $\phi_e \sim |z - e|^{1/2}$ , it follows that  $\phi_e^2$  is conformal in a neighborhood of  $e$  (WLOG, we suppose  $U_e$  is small enough for this). Furthermore,  $\phi_e^2$  maps  $\gamma_1, \gamma_2$  into  $(-\infty, 0)$  (see Figure 7.1), and we choose  $\gamma_{i,\pm}$  to be preimages of  $I_\pm := \{z : \arg(\zeta) = \pm 2\pi/3\}$ .

### Matrix $\mathbf{P}_e$

For this problem, we will reuse the model RHP that appeared in Section 6.3.1, and denote  $\Psi_{-1}(\zeta) := \sigma_3 \Psi_\alpha(\zeta) \sigma_3$  and  $\Psi_1(\zeta) := \Psi_\beta(\zeta)$ . Furthermore, let

$$\mathbf{J}_e = \begin{cases} \mathbf{I}, & e = 1, \\ e^{-n\pi i \tau \sigma_3}, & e = -1. \end{cases} \quad (8.3.17)$$

Then it follows from RHP- $\Psi_\alpha$ , definition of  $r_e$  in Section 6.3.1 (defined analogously to  $r_{a_1}$  with  $\rho$  replaced by  $h$ ), (8.3.2), (8.3.2), (8.3.17), and (8.3.4) that

$$\mathbf{P}_e(z) = \mathbf{E}_e(z) \Psi_e(n^2 \zeta_e(z)) r_e^{-\sigma_3}(z) e^{-n\phi_e(z)\sigma_3/2} \mathbf{J}_e \quad (8.3.18)$$

satisfies  $\text{RHP-}\mathbf{P}_e(\text{a, b})$ . The choice of  $\mathbf{E}_e$  to ensure  $\text{RHP-}\mathbf{P}_e(\text{c})$  holds is made below.

### Matrix $\mathbf{E}_e$

Finally, to satisfy the matching condition  $\text{RHP-}\mathbf{P}_e(\text{c})$ , we simply need to choose

$$\mathbf{E}_e(z) := \mathbf{N}(z) \mathbf{J}_e^{-1} r_e^{\sigma_3}(z) \mathbf{S}_e^{-1}(n^2 \zeta_e(z)), \quad (8.3.19)$$

where  $\mathbf{S}_e = \sigma_3 \mathbf{S} \sigma_3$  (see section 6.3.1 for the definition of  $\mathbf{S}$ ) for  $e = -1$  and  $\mathbf{S}_e = \mathbf{S}$  for  $e = 1$ . Holomorphy in  $U_e \setminus \{e\}$  follows from  $\text{RHP-}\mathbf{N}(\text{b})$ , definition of  $\mathbf{S}$ , while the behavior of  $\mathbf{N}$  near  $e \in \{\pm 1\}$ , see (8.3.14), the behavior of  $r_e$  near  $e$ , and the fact that  $\zeta_e(z)$  possesses a simple zero at  $e$  yield holomorphy in  $U_e$ .

### Local Parametrices at $z = z_*, -\overline{z}_*$

Let  $U_e$ ,  $e \in \{z_*, -\overline{z}_*\}$  be an open disk centered at  $e$  small enough to be within the domain of holomorphy of  $h^*$ . We seek a matrix  $\mathbf{P}_e$  that solves the following  $\text{RHP-}\mathbf{P}_e$ :

- (a)  $\mathbf{P}_e$  satisfies the same analyticity properties as  $\mathbf{X}$  within  $U_e$ ,
- (b)  $\mathbf{P}_e$  satisfies the same jump relations as  $\mathbf{X}$  within  $U_e$ ,
- (c)  $\mathbf{P}_e(z) = \mathbf{N}(z) (\mathbf{I} + \mathcal{O}(n^{-1}))$  uniformly on  $\partial U_e$  as  $n \rightarrow \infty$ .

### Model Problem

In this setting, we will yet another well-known model problem. Let  $I_{\pm}$  be as in the previous section and consider the following  $\text{RHP} - \mathbf{A}$

- (a)  $\mathbf{A}$  is analytic in  $\mathbb{C} \setminus ((-\infty, \infty) \cup I_- \cup I_+)$

(b)  $\mathbf{A}$  possess continuous traces on  $(-\infty, \infty) \cup I_{\pm}$  and satisfies

$$\mathbf{A}_+(s) = \mathbf{A}_-(s) \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in (-\infty, 0), \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & s \in I_{\pm}, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & s \in (0, \infty), \end{cases}$$

(c) It holds uniformly in  $\mathbb{C} \setminus ((-\infty, \infty) \cup I_- \cup I_+)$  that

$$\mathbf{A}(\zeta)e^{\frac{2}{3}\zeta^{3/2}\sigma_3} \sim \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \sum_{k=0}^{\infty} \begin{pmatrix} s_k & 0 \\ 0 & t_k \end{pmatrix} \begin{pmatrix} (-1)^k & i \\ (-1)^k i & 1 \end{pmatrix} \left(\frac{2}{3}\zeta^{3/2}\right)^{-k},$$

$$\text{where } s_0 = t_0 = 1 \text{ and } s_k = \frac{\Gamma(3k+1/2)}{54^k k! \Gamma(k+1/2)}, \quad t_k = -\frac{6k+1}{6k-1} s_k, \quad k \geq 1.$$

This problem is solved by the Airy matrix [33, 58]. We will write  $\mathbf{A}_e := \mathbf{A}$  for  $e = -\overline{z}_*$  and  $\mathbf{A}_e := \sigma_3 \mathbf{A} \sigma_3$  for  $e = z_*$ . Furthermore, let

$$\mathbf{J}_e(z) := \begin{cases} e^{\pm \pi i n \sigma_3 / 2}, & e = z_*, \\ e^{\pi i (\pm 1 - 2\tau) n \sigma_3 / 2}, & e = -\overline{z}_*, \end{cases} \quad (8.3.20)$$

where we use  $+$  (resp.,  $-$ ) for  $z$  to the left (resp. right) of  $\gamma_\lambda$ .

## Conformal Map

Let  $\phi_e$  be as defined in (8.3.2), and define

$$\zeta_e(z) := \left(-\frac{3}{4}\phi_e(z)\right)^{2/3}, \quad e \in \{z_*, -\overline{z}_*\} \quad (8.3.21)$$

Then, since  $\phi_e \sim |z - e|^{3/2}$ , it follows that a branch of  $\phi_e^{2/3}$  can be chosen so that  $\zeta_e$  is conformal in a neighborhood of  $e$  (WLOG, we suppose  $U_e$  is small enough for this). Furthermore, we fix the branch so that  $\phi_e^{2/3}$  maps  $\gamma_1 \cap \gamma_2$  into  $(-\infty, 0)$  (see Figure 7.1), and we choose  $\gamma_{i,\pm}$  to be preimages of  $I_{\pm} := \{z : \arg(\zeta) = \pm 2\pi/3\}$ . In fact, we had some freedom in choosing  $\hat{\gamma}$ , and we now fix it to (locally) go along the orthogonal trajectory of  $-Q(z)(dz)^2$ , so that it is mapped by  $\phi_e$  into  $(0, \infty)$ .

### Matrix $P_e$

Let  $r(z) = \sqrt{h(z)}$  be a branch holomorphic in  $U_e$ . It can be readily verified by using (8.3.4) that

$$P_e(z) := E_e(z) A_e(n^{2/3} \zeta_e(z)) e^{-n\phi_e \sigma_3/2} r^{-\sigma_3}(z) J_e(z), \quad (8.3.22)$$

satisfies **RHP- $P_e$** (a, b). The choice of  $E_e$  to ensure **RHP- $P_e$** (c) holds is made below.

### Matrix $E_e$

Let

$$E_e(z) = N(z) J_e^{-1} r^{\sigma_3}(z) S_e^{-1}(n^{2/3} \zeta_e(z)) \quad (8.3.23)$$

where we let  $S_e = \sigma_3 S \sigma_3$  when  $e = z_*$  and  $S_e = S$  when  $z = -\bar{z}_*$  and define  $(-\phi_e)^{1/6}$  to be positive on  $\hat{\gamma}$ . Then, holomorphy of  $E_e$  in  $U_e \setminus \{e\}$  follows from (8.3.20), definition of  $S_e$ ,  $r$ , and **RHP- $N$** (b). The behavior of  $N$  near  $z = z_*, -\bar{z}_*$ , described in (8.3.14), and the fact that  $\zeta_e$  possesses a simple zero at  $\zeta_e$  yields holomorphy in  $U_e$ .

### 8.3.3 Final Riemann-Hilbert Problem

Let  $\Sigma := ([(\gamma_\lambda \setminus (\gamma_1 \cup \gamma_2)) \cup \gamma_{i,\pm}] \cap D) \cup (\cup_e \partial U_e)$ , where  $i \in \{1, 2\}$  and  $D := \mathbb{C} \setminus \cup_e \bar{U}_e$ , and define

$$R(z) := X(z) \begin{cases} N^{-1}(z), & z \in \mathbb{C} \setminus (\cup_e U_e \cup \gamma_\lambda \cup \gamma_{i,\pm}), \\ P_e^{-1}(z), & z \in U_e \setminus (\gamma_\lambda \cup \gamma_{i,\pm}). \end{cases} \quad (8.3.24)$$

where  $e \in \{\pm 1, z_*, -\bar{z}_*\}$  and  $i = 1$  when  $e \in \{-1, -\bar{z}_*\}$ ,  $i = 2$  when  $e \in \{1, z_*\}$ . Then, it follows that  $R$  solves the following **RHP- $R$** :

(a)  $R(z)$  is holomorphic in  $\mathbb{C} \setminus \Sigma$  and  $\lim_{\mathbb{C} \setminus \Sigma \ni z \rightarrow \infty} R(z) = I$ ;



(b)  $\mathbf{R}(z)$  has continuous traces on  $\Sigma^\circ$  that satisfy

$$\mathbf{R}_+(s) = \mathbf{R}_-(s) \begin{cases} \mathbf{P}_e(s) \mathbf{N}^{-1}(s), & s \in \partial U_e, \\ \mathbf{N}(s) \begin{pmatrix} 1 & 0 \\ e^{-n\phi_{b_2}(s)}/h(s) & 1 \end{pmatrix} \mathbf{N}^{-1}(s), & s \in \gamma_{i,\pm} \cap D, \\ \mathbf{N}_-(s) \begin{pmatrix} e^{\pi i n} & h(s) e^{n\phi_{z_*}(s)} \\ 0 & e^{-\pi i n} \end{pmatrix} \mathbf{N}_+^{-1}(s), & s \in \hat{\gamma} \cap D \end{cases}$$

where  $\partial U_e$  is oriented clockwise.

It follows that  $\mathbf{R}(z)$  is analytic in  $\mathbb{C} \setminus (\gamma_{i,\pm} \cup (\cup_e \partial U_e))$  and that, for  $s \in \gamma_{i,\pm} \cup (\cup_e \partial U_e)$  and  $n \in \mathbb{N}(\epsilon)$  we have

$$\mathbf{R}_+(s) = \mathbf{R}_-(s) \begin{cases} \mathbf{I} + \mathcal{O}_\epsilon(e^{-cn}) & \text{for } z \in (\gamma_{i,\pm}) \setminus U_e, \\ \mathbf{I} + \mathcal{O}_\epsilon(n^{-1}) & \text{for } z \in \cup_e \partial U_e. \end{cases} \quad (8.3.25)$$

The first equality follows from the fact that  $\operatorname{Re}(\phi_1) > 0$  on  $\Gamma_\pm$ , which follows from noting that the formula  $\operatorname{Re}(2\phi_1(z)) = \operatorname{Re}(V_\lambda(z)) - \ell - U^\mu(z)$  implies  $\operatorname{Re}(\phi_1)$  is a non-constant subharmonic function that vanishes on  $\gamma_\lambda$  (the last claim follows from the variational condition (7.1.1)). Since critical trajectories are exactly the set where  $\operatorname{Re}(2\phi_1(z)) = 0$ , we conclude that the sign of  $\operatorname{Re}(2\phi(z))$  must be fixed (locally) on either side of  $\gamma_1, \gamma_2$ . Furthermore, due to the S-property (7.1.2), the sign of  $\operatorname{Re}(2\phi_1(z))$  must be the same (locally) on either side of  $\gamma_1, \gamma_2$ . Hence, we deduce from the maximum principle for subharmonic functions that  $\operatorname{Re}(\phi_1) > 0$  in some neighborhood of  $\gamma_1 \cup \gamma_2$ . Furthermore, as argued in Section 8.3.1, see (8.3.15),  $\mathbf{N}$  is bounded as  $\mathbb{N}(\epsilon) \ni n \rightarrow \infty$ . Therefore, the jumps of  $\mathbf{X}$  on  $\gamma_{i,\pm}$  are exponentially small. The second equality holds again by boundedness of  $\mathbf{N}$  with  $n \in \mathbb{N}(\epsilon)$  and construction of  $\mathbf{P}_e$ , see RHP- $\mathbf{P}_e(c)$ . Since all contours are fixed with  $n$  and are of finite length, we deduce that

$$\|\mathbf{R}_-^{-1} \mathbf{R}_+ - \mathbf{I}\|_{L^\infty(\Sigma) \cap L^2(\Sigma)} = \mathcal{O}_\epsilon(n^{-1}),$$

Finally, from [26, Corollary 7.108] we conclude that

$$\mathbf{R}(z) = \mathbf{I} + \mathcal{O}_\epsilon\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow \infty, \quad (8.3.26)$$

uniformly for  $z \in \mathbb{C} \setminus (\gamma_{i,\pm} \cup (\cup_e \partial U_e))$ . The asymptotic formula of  $P_n^\lambda(z)$  outside the lenses and away from endpoints follows by undoing the above transformations as was done in [48].

## 8.4 Concluding Remarks and Future Work

As we will see in Chapter 9 (see Section 9.3), much of the work of attaining strong asymptotics of orthogonal polynomials via Riemann-Hilbert analysis relies on identifying the zero-attracting curve associated to the family of polynomials being investigated. In the setting of the kissing polynomials we studied here, this work was done in [48] and [56]. However, introducing an algebraic singularity

$$\rho_n(x) = |x|^\gamma (1-x)^\alpha (1+x)^\beta e^{i\omega x}, \quad \omega = \lambda n, \quad \lambda \geq 0,$$

changes the geometry of the attracting curve in a non-trivial way, since now the curve must pass through  $z = 0$ . This was not the case for kissing polynomials considered above since the weight of orthogonality was required to be analytic in a region specifically designed to allow us to deform  $[-1, 1]$  to  $\gamma_\lambda$ . Once the geometry of this new zero-attracting curve is established, the problem becomes amenable to Riemann-Hilbert analysis.

## 9. VARYING ORTHOGONALITY IN POLYNOMIAL EXTERNAL FIELDS

A version of this chapter will appear in [59].

In the previous chapter, we discussed varying orthogonality in a polynomial external field of degree 1. In this present chapter, we move up in degree and investigate some of the complications that arise. Consider polynomials satisfying the orthogonality relation

$$\int_{\Gamma} P_n(z; N) z^k e^{-NV(z)} dz = 0 \quad \text{for } k = 0, 1, \dots, n-1 \quad (9.0.1)$$

where  $V(z)$  is a polynomial,  $N$  is an integer parameter, and  $\Gamma$  is a curve tending to infinity in both senses in such a way that (9.0.1) converges. Such polynomials appear in the study of random matrices [50, 51, 60]. While the RHP 4.1.1 is still valid, its analysis now relies on deforming  $\Gamma$  so as to contain the zero-attracting curve(s) of  $P_n$ . The zeros of polynomials satisfying (9.0.1) asymptotically distribute as the weighted equilibrium measure on an associated S-contour corresponding to the weight function  $V$ . We consider the class of curves

**Definition.** We say a curve  $\Gamma \in \mathcal{T}$  if  $\Gamma$  is an unbounded smooth contour such that for any parametrization  $z(s)$ ,  $s \in \mathbb{R}$ , of  $\Gamma$  there exists  $\epsilon \in (0, \pi/6)$  and  $s_0 > 0$  for which

$$\begin{cases} |\arg(z(s)) - \pi/3| \leq \pi/6 - \epsilon, & s \geq s_0, \\ |\arg(z(s)) - \pi| \leq \pi/6 - \epsilon, & s \leq -s_0, \end{cases} \quad (9.0.2)$$

where  $\arg(z(s)) \in [0, 2\pi)$ . The above conditions ensure that integral (9.0.1) is finite and due to analyticity of the integrand does not depend on a particular  $\Gamma$  satisfying (9.0.2).

Of course one can still seek a solution of the energy-minimization introduced in section 7.1. The equilibrium measure  $\mu = \mu_\Gamma$  is characterized by the Euler–Lagrange variational conditions:

$$2U^\mu(z) + \operatorname{Re}V(z) \begin{cases} = \ell, & z \in J_\Gamma, \\ \geq \ell, & z \in \Gamma \setminus J_\Gamma, \end{cases} \quad (9.0.3)$$

where  $\ell = \ell_\Gamma$  is a constant, the Lagrange multiplier, and

$$U^\mu(z) = - \int \log |z - s| d\mu(s)$$

is the logarithmic potential of  $\mu$  as before. Any  $\Gamma \in \mathcal{T}$  can be used to define  $P_n(z; N)$  in (9.0.1), nevertheless, in our tour of the theory of non-Hermitian orthogonal polynomials, starting with the works of Stahl [15–17] and Gonchar and Rakhmanov [55] that one we saw that one should use the contour whose equilibrium measure has support with the S-property (7.1.2) in the external field  $\operatorname{Re}V$ . We shall say that a curve  $\Gamma \in \mathcal{T}$  is an S-curve in the field  $\operatorname{Re}V$ , if  $J_\Gamma$  has the S-property in this field.

Much like we saw in Chapter 7, it is also understood that geometrically  $J_\Gamma$  is comprised of critical trajectories of quadratic differentials. Recall that if  $Q$  is a meromorphic function, a *trajectory* (resp. *orthogonal trajectory*) of a quadratic differential  $-Q(z)dz^2$  is a maximal regular arc on which

$$-Q(z(s))(z'(s))^2 > 0 \quad (\text{resp.} \quad -Q(z(s))(z'(s))^2 < 0)$$

for any local uniformizing parameter. A trajectory is called *critical* if it is incident with a *finite critical point* (zero or a simple pole of  $-Q(z)dz^2$ ) and it is called *short* if it is incident only with finite critical points. We designate the expression *critical (orthogonal) graph of  $-Q(z)dz^2$*  for the totality of the critical (orthogonal) trajectories  $-Q(z)dz^2$ .

Henceforth, we will specialize consideration to a cubic potential of the form

$$V(z; t) = -\frac{1}{3}z^3 + tz, \quad t \in \mathbb{C} \quad (9.0.4)$$

In this setting, [61, Theorem 2.3], reproduced below, asserts the existence of such S-contours and characterizes them:

**Theorem 9.0.1.** *Let  $V(z; t)$  be given by (9.0.4).*

1. *There exists a contour  $\Gamma_t \in \mathcal{T}$  such that*

$$\mathcal{I}_V(\Gamma_t) = \sup_{\Gamma \in \mathcal{T}} I_V(\Gamma). \quad (9.0.5)$$

2. *The equilibrium measure  $\mu_t := \mu_{\Gamma_t}$  is the same for every  $\Gamma_t$  satisfying (9.0.5). The support  $J_t$  of  $\mu_t$  has the S-property in the external field  $\text{Re}V(z; t)$ .*

3. *The function*

$$Q(z; t) = \left( \frac{V'(z; t)}{2} - \int \frac{d\mu_t(s)}{z - s} \right)^2, \quad z \in \mathbb{C} \setminus J_t, \quad (9.0.6)$$

*is a polynomial of degree 4.*

4. *The support  $J_t$  consists of some short critical trajectories of the quadratic differential  $-Q(z; t)dz^2$  and the equation*

$$d\mu_t(z) = -\frac{1}{\pi i} Q_+^{1/2}(z; t)dz, \quad z \in J_t, \quad (9.0.7)$$

*holds on each such critical trajectory, where  $Q^{1/2}(z; t) = \frac{1}{2}z^2 + \mathcal{O}(z)$  as  $z \rightarrow \infty$  (in what follows,  $Q^{1/2}(z; t)$  will always stand for such a branch).*

*Remark 9.0.2.* Although the equilibrium measure  $\mu_\Gamma$  is unique, the S-contour  $\Gamma_t$  is not. Indeed, we can slightly perturb  $\Gamma_t$  outside of the support of  $\mu_\Gamma$  while preserving the equilibrium measure and the min-max property (9.0.5).

Much information on the structure of the critical graphs of a quadratic differential can be found in the excellent monographs [62–64]. Since  $\deg Q = 4$ ,  $J_t$  consists of one or two arcs, corresponding (respectively) to the cases where  $Q(z; t)$  has two simple zeros and one double zero, and the case where it has four simple zeros. Away from  $J_t$ , one has freedom in choosing  $\Gamma_t$ . In particular, let

$$\mathcal{U}(z; t) := \text{Re} \left( \int_e^z Q^{1/2}(s; t)ds \right) = \ell_t - \text{Re}(V(z; t)) - U^{\mu_t}(z), \quad (9.0.8)$$

where  $e \in J_t$  is any and the second equality follows from (9.0.6) (since the constant  $\ell_t := \ell_{\Gamma_t}$  in (9.0.3) is the same for both connected components of  $J_t$  and is purely imaginary on  $J_t$ , the choice of  $e$  is indeed not important). Clearly,  $\mathcal{U}(z; t)$  is a subharmonic function (harmonic away from  $J_t$ ) which is equal to zero  $J_t$  by (9.0.3). The trajectories of  $-Q(z; t)dz^2$  emanating out of the endpoints  $J_t$  belong to the set  $\{z : \mathcal{U}(z; t) = 0\}$  and it follows from the variational condition (9.0.3) that  $\Gamma_t \setminus J_t \subset \{z : \mathcal{U}(z; t) < 0\}$ . However, within the region  $\{z : \mathcal{U}(z; t) < 0\}$  the set  $\Gamma_t \setminus J_t$  can be varied freely. The geometry of the set  $\{z : \mathcal{U}(z; t) < 0\}$  is described further below in Theorems 9.1.1 and 9.1.2.

### 9.1 Geometry of $\Gamma_t$

The structure of  $\Gamma_t$  and its dependence on  $t$  has been heuristically described in [65, 66] and rigorously in [50], but only in the one-cut region. Let us quickly recall the important notions from [50].

Denote by  $\mathcal{C}$  the critical graph of an auxiliary quadratic differential

$$-(1 + 1/s)^3 ds^2, \quad (9.1.1)$$

see Figure 9.1(a). It was shown in [50, Section 5] that  $\mathcal{C}$  consists of 5 critical trajectories emanating from  $-1$  at the angles  $2\pi k/5$ ,  $k \in \{0, 1, 2, 3, 4\}$ , one of them being  $(-1, 0)$ , other two forming a loop crossing the real line approximately at 0.635, and the last two approaching infinity along the imaginary axis without changing the half-plane (upper or lower). Given  $\mathcal{C}$ , define

$$\Delta := \{x : 2x^3 \in \mathcal{C}\}.$$

Further, put  $\Omega_{\text{one-cut}}$  to be the shaded region on Figure 9.1(b) and set

$$\partial\Omega_{\text{one-cut}} = \Delta_{\text{birth}}^b \cup \{-2^{-1/3}\} \cup \Delta_{\text{split}} \cup \{e^{\pi i/3} 2^{-1/3}\} \cup \Delta_{\text{birth}}^a,$$

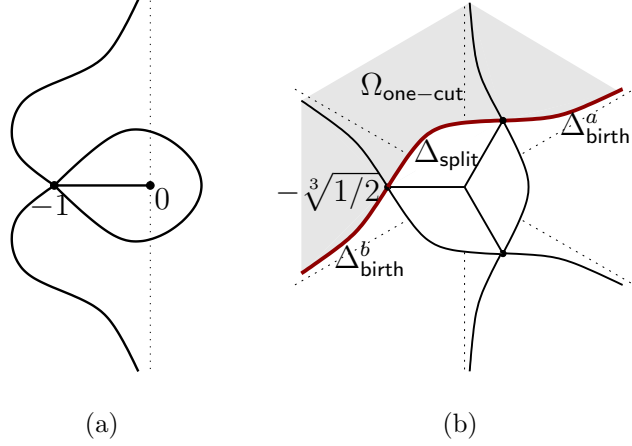


Fig. 9.1. Schematic representation of (a) the critical graph  $\mathcal{C}$ ; (b) the set  $\Delta$  (solid lines) and the domain  $\Omega_{\text{one-cut}}$  (shaded region).

where  $\Delta_{\text{split}}$  connects  $-2^{-1/3}$  and  $e^{\pi i/3}2^{-1/3}$ ,  $\Delta_{\text{birth}}^b$  extends to infinity in the direction of the angle  $7\pi/6$  while  $\Delta_{\text{birth}}^a$  extends to infinity in the direction of the angle  $\pi/6$ . Let

$$t(x) := (x^3 - 1)/x$$

and set

$$\left\{ \begin{array}{l} t_{\text{cr}} := 3 \cdot 2^{-2/3} = t(-2^{-1/3}), \\ O_{\text{one-cut}} := t(\Omega_{\text{one-cut}}), \\ C_{\text{split}} := t(\Delta_{\text{split}}), \quad C_{\text{birth}}^b := t(\Delta_{\text{birth}}^b), \quad C_{\text{birth}}^a := t(\Delta_{\text{birth}}^a), \\ S := (t_{\text{cr}}, \infty), \quad e^{2\pi i/3}S := \{z : e^{-2\pi i/3}z \in S\}, \end{array} \right. \quad (9.1.2)$$

see Figure 9.2. The function  $t(x)$  is holomorphic in  $\Omega_{\text{one-cut}}$  with non-vanishing derivative there. It maps  $\Omega_{\text{one-cut}}$  onto  $O_{\text{one-cut}}$  in a one-to-one fashion. Hence, the inverse map  $x(t)$  exists and is holomorphic.

Below, we adapt the following convention:  $\Gamma(z_1, z_2)$  (resp.  $\Gamma[z_1, z_2]$ ) stands for the trajectory or orthogonal trajectory (resp. the closure of) of the differential  $-Q(z; t)dz^2$  connecting  $z_1$  and  $z_2$ , oriented from  $z_1$  to  $z_2$ , and  $\Gamma(z, e^{i\theta}\infty)$  (resp.

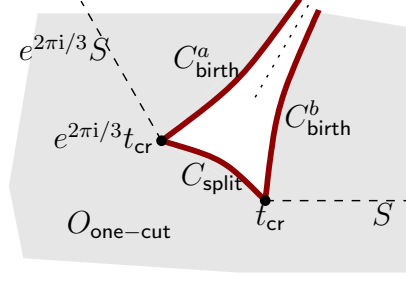


Fig. 9.2. Domain  $O_{\text{one-cut}}$  (shaded region);  $\partial O_{\text{one-cut}}$  consisting of the open bounded arc  $C_{\text{split}}$ , two open semi-unbounded arcs  $C_{\text{birth}}^a$  and  $C_{\text{birth}}^b$ , and two points  $t_{\text{cr}}$  and  $e^{2\pi i/3} t_{\text{cr}}$ ; the semi-unbounded open horizontal rays  $S$  and  $e^{2\pi i/3} S$  (dashed lines).

$\Gamma(e^{i\theta}\infty, z)$ ) stands for the orthogonal trajectory ending at  $z$ , approaching infinity at the angle  $\theta$ , and oriented away from  $z$  (resp. oriented towards  $z$ ).<sup>1</sup>

The following theorem has been proven in [50, Theorem 3.2] and it describes the geometry of  $\Gamma_t$  when  $t \in \overline{O_{\text{one-cut}}}$ .

**Theorem 9.1.1.** *Let  $\mu_t$  and  $Q(z; t)$  be as in Theorem 9.0.1,  $J_t = \text{supp}(\mu_t)$ . When  $t \in \overline{O_{\text{one-cut}}}$ , the polynomial  $Q(z; t)$  is of the form*

$$Q(z; t) = \frac{1}{4}(z - a(t))(z - b(t))(z - c(t))^2. \quad (9.1.3)$$

with  $a(t)$ ,  $b(t)$ , and  $c(t)$  given by

$$\begin{cases} a(t) &:= x(t) - i\sqrt{2}/\sqrt{x}(t), \\ b(t) &:= x(t) + i\sqrt{2}/\sqrt{x}(t), \\ c(t) &:= -x(t), \end{cases} \quad (9.1.4)$$

where  $\sqrt{x}(t)$  is the branch holomorphic in  $O_{\text{one-cut}}$  satisfying  $\sqrt{x}(0) = e^{\pi i/3}$ . The set  $J_t$  consists of a single arc and

(I) if  $t \in O_{\text{one-cut}}$ , then  $J_t = \Gamma[a, b]$  and an  $S$ -curve  $\Gamma_t \in \mathcal{T}$  can be chosen as

<sup>1</sup>This notation is unambiguous as the corresponding trajectories are unique for polynomial differentials as follows from Teichmüller's lemma.



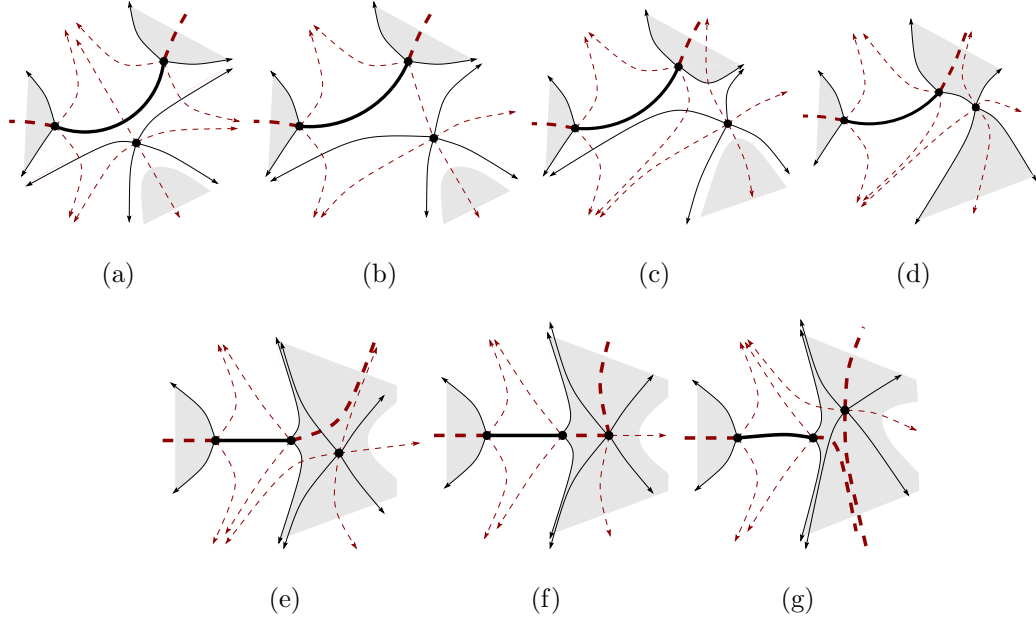


Fig. 9.3. Schematic representation of the critical (solid) and critical orthogonal (dashed) graphs of  $-Q(z; t)dz^2$  when  $t \in O_{\text{one-cut}}$ . The bold curves represent the preferred S-curve  $\Gamma_t$ . Shaded region is the set  $\{U(z; t) < 0\}$ .

(a)  $\Gamma(e^{\pi i}\infty, a) \cup J_t \cup \Gamma(b, e^{\pi i/3}\infty)$  when  $t$  belongs to the connected component bounded by  $S \cup C_{\text{split}} \cup e^{2\pi i/3}S$ , see Figure 9.3(a-e);

(b)  $\Gamma(e^{\pi i}\infty, a) \cup J_t \cup \Gamma(b, c) \cup \Gamma(c, e^{\pi i/3}\infty)$  when  $t \in S$ , see Figure 9.3(f);

(c)  $\Gamma(e^{\pi i}\infty, c) \cup \Gamma(c, a) \cup J_t \cup \Gamma(b, e^{\pi i/3}\infty)$  when  $t \in e^{2\pi i/3}S$ ;

(d)  $\Gamma(e^{\pi i}\infty, a) \cup J_t \cup \Gamma(b, e^{-\pi i/3}\infty) \cup \Gamma(e^{-\pi i/3}\infty, c) \cup \Gamma(c, e^{\pi i/3}\infty)$  when  $t$  belongs to the connected component bounded by  $S \cup C_{\text{birth}}^b$ , see Figure 9.3(g);

(e)  $\Gamma(e^{\pi i}\infty, c) \cup \Gamma(c, e^{-\pi i/3}) \cup \Gamma(e^{-\pi i/3}\infty, a) \cup J_t \cup \Gamma(b, e^{\pi i/3}\infty)$  when  $t$  belongs to the connected component bounded by  $e^{2\pi i/3}S \cup C_{\text{birth}}^a$ .

(II) if  $t = t_{\text{cr}}$  (resp.  $t = e^{2\pi i/3}t_{\text{cr}}$ ), then  $J_t = \Gamma[a, b]$ ,  $c$  coincides with  $b$  (resp.  $a$ ), and an S-curve  $\Gamma_t \in \mathcal{T}$  can be chosen as in Case I(a), see Figure 9.4(a).

(III) if  $t \in C_{\text{split}}$ , then  $J_t = \Gamma[a, c] \cup \Gamma[c, b]$  and an S-curve  $\Gamma_t \in \mathcal{T}$  can be chosen as in Case I(a), see Figure 9.4(b).

(IV) if  $t \in C_{\text{birth}}^b$  (resp.  $t \in C_{\text{birth}}^a$ ), then  $J_t = \Gamma[a, b]$  and an  $S$ -curve  $\Gamma_t \in \mathcal{T}$  can be chosen as in Case I(d) (resp. Case I(e)), see Figure 9.4(c).

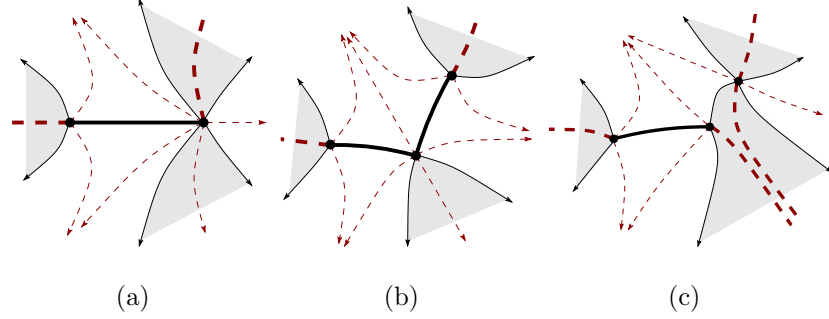


Fig. 9.4. This is a continuation of Figure 9.3 for the case  $t \in \partial O_{\text{one-cut}}$ .

Now, let  $O_{\text{two-cut}} := \mathbb{C} \setminus \overline{O_{\text{one-cut}}}$ . Then the following theorem holds.

**Theorem 9.1.2.** *Let  $\mu_t$  and  $Q(z; t)$  be as in Theorem 9.0.1,  $J_t = \text{supp}(\mu_t)$ . When  $t \in O_{\text{two-cut}}$ , the polynomial  $Q(z; t)$  is of the form*

$$Q(z; t) = \frac{1}{4}(z - a_1(t))(z - b_1(t))(z - a_2(t))(z - b_2(t)) \quad (9.1.5)$$

with  $a_1(t)$ ,  $b_1(t)$ ,  $a_2(t)$ , and  $b_2(t)$  all distinct. The real and imaginary parts of  $a_i(t)$ ,  $b_i(t)$  are real analytic functions of  $\text{Re}(t)$  and  $\text{Im}(t)$  when  $t \in O_{\text{two-cut}}$  while the functions  $a_i(t)$ ,  $b_i(t)$  themselves are not analytic functions of  $t$ . Moreover, it holds that

$$a_1(t), b_1(t) \rightarrow a(t^*), \quad b_1(t), a_2(t) \rightarrow c(t^*), \quad \text{and} \quad a_2(t), b_2(t) \rightarrow b(t^*) \quad (9.1.6)$$

as  $t \rightarrow t^*$  with  $t^* \in C_{\text{birth}}^a \cup \{e^{2\pi i/3} t_{\text{cr}}\}$ ,  $t^* \in C_{\text{split}}$ , and  $t^* \in C_{\text{birth}}^b \cup \{t_{\text{cr}}\}$ , respectively.

The  $S$ -curve  $\Gamma_t$  can be chosen as

$$\Gamma(e^{\pi i} \infty, a_1(t)) \cup J_{t,1} \cup \Gamma(b_1(t), e^{-\pi i/3} \infty) \cup \Gamma(e^{-\pi i/3} \infty, a_2(t)) \cup J_{t,2} \cup \Gamma(b_2(t), e^{\pi i/3} \infty),$$

where  $J_t = J_{t,1} \cup J_{t,2}$  and  $J_{t,i} = \Gamma[a_i(t)b_i(t)]$ ,  $i \in \{1, 2\}$ , see Figure 9.5 (this also explains how we choose the labeling of the zeros of  $Q(z; t)$  in the considered case).

*Remark 9.1.3.* Theorem 9.1.2 is the justification for the notation  $O_{two-cut}$ : combined with Theorem 9.0.1 we see that  $\mu_t$  is supported on two analytic arcs, which appear as a special choice of branch cut for  $Q^{1/2}$ .

We prove Theorem 9.1.2 in Section 9.3.

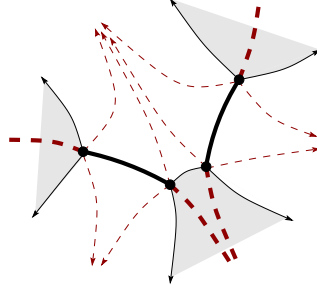


Fig. 9.5. The schematic representation of the critical and critical orthogonal graphs of  $-Q(z;t)dz^2$  when  $t \in O_{two-cut}$ . The bold curves represent the preferred S-curve  $\Gamma_t$ . Shaded region is the set  $\{\mathcal{U}(z;t) < 0\}$ .

## 9.2 Main Results

In this section we assume that  $t \in O_{two-cut}$  and  $Q(z;t)$ ,  $\Gamma_t$ , and  $J_t$  are as in Theorem 9.1.2. When it comes to the definition of the contour  $\Gamma_t$ , it will be more practical for us to change the choice of  $\Gamma_t$  from the one made in Theorem 9.1.2 by dropping the unbounded trajectories  $\Gamma(b_1(t), \infty e^{-\pi i/3})$  and  $\Gamma(e^{-\pi i \pi/3} \infty, a_2(t))$  and replacing them with a smooth Jordan arc, say  $I_t$ , connecting  $b_1(t)$  and  $a_2(t)$  such that  $I_t^\circ := I_t \setminus E_t$  lies entirely in the set  $\{\mathcal{U}(z;t) < 0\}$  in such a way that there exists  $s_1(t) \in \Gamma(b_1(t), e^{-\pi i/3} \infty)$ ,  $s_2(t) \in \Gamma(e^{-\pi i/3} \infty, a_2(t))$  for which  $\Gamma(b_1(t), s_1(t)) \cup \Gamma(s_2(t), a_2(t)) \subset I_t^\circ$ . In what follows we shall write  $J_t := J_{t,1} \cup J_{t,2}$ ,  $J_t^\circ := J_{t,1}^\circ \cup J_{t,2}^\circ$ , and  $E_t := J_t \setminus J_t^\circ = \{a_1(t), b_1(t), a_2(t), b_2(t)\}$ , where  $J_{t,i}^\circ := \Gamma(a_i(t), b_i(t))$ .

### 9.2.1 Asymptotics of $P_n(z; t, N)$

To describe the asymptotics of the orthogonal polynomials themselves, we need to construct the Szegő function of  $e^{V(z;t)}$ .

**Proposition 9.2.1.** *Let the constant  $\varsigma(t)$  be given by*

$$\varsigma(t) := \frac{2t}{3} \left( \int_{I_t} \frac{ds}{Q^{1/2}(s;t)} \right)^{-1}, \quad (9.2.1)$$

where, as usual, we use the branch  $Q^{1/2}(z;t) = \frac{1}{2}z^2 + \mathcal{O}(z)$  as  $z \rightarrow \infty$ . Then the function

$$\mathcal{D}(z;t) := \exp \left\{ \frac{1}{2}V(z;t) + \frac{1}{3} \left( z + \int_{I_t} \frac{3\varsigma(t)}{s-z} \frac{ds}{Q^{1/2}(s;t)} \right) Q^{1/2}(z;t) \right\} \quad (9.2.2)$$

is holomorphic and non-vanishing in  $\overline{\mathbb{C}} \setminus (J_t \cup I_t)$  with continuous traces on  $J_t^\circ \cup I_t^\circ$  that satisfy

$$\begin{cases} \mathcal{D}_+(s;t)\mathcal{D}_-(s;t) = e^{V(s;t)}, & s \in J_t^\circ, \\ \mathcal{D}_+(s;t) = \mathcal{D}_-(s;t)e^{2\pi i\varsigma(t)}, & s \in I_t^\circ. \end{cases} \quad (9.2.3)$$

We shall also denote by  $D(z;t) := \mathcal{D}(z;t)/\mathcal{D}(\infty;t)$  the normalized Szegő function.

We prove Proposition 9.2.1 in Appendix B.

To describe the geometric growth of orthogonal polynomials, let us define

$$\mathcal{Q}(z;t) := \int_{b_2(t)}^z Q^{1/2}(s;t)ds, \quad z \in \mathbb{C} \setminus \Gamma_t(e^{\pi i}\infty, b_2(t)]. \quad (9.2.4)$$

Observe that  $\mathcal{U}(z;t) = \operatorname{Re}(\mathcal{Q}(z;t))$  as defined in (9.0.8). This function has the following properties.

**Proposition 9.2.2.** *Let the constants  $\tau(t), \omega(t)$  be given by*

$$\tau(t) := -\frac{1}{\pi i} \int_{I_t} Q^{1/2}(s;t)ds \quad \text{and} \quad \omega(t) := -\frac{1}{\pi i} \int_{J_{t,1}} Q_+^{1/2}(s;t)ds. \quad (9.2.5)$$

These constants are necessarily real (in fact,  $\omega(t) = \mu_t(J_{t,1})$ , see (9.0.7)). The function  $e^{\mathcal{Q}(z;t)}$  is holomorphic in  $\mathbb{C} \setminus (J_t \cup I_t)$  and there exists a constant  $\ell_*(t)$  such that

$$\exp \left\{ \frac{V(z;t) - \ell_*(t)}{2} + \mathcal{Q}(z;t) \right\} = z + \mathcal{O}(1) \quad \text{as } z \rightarrow \infty. \quad (9.2.6)$$

Moreover,  $\mathcal{Q}(z; t)$  possesses continuous traces on  $J_t^\circ \cup I_t^\circ$  that are purely imaginary on  $J_t$  and satisfy

$$\begin{cases} e^{\mathcal{Q}_+(s; t) + \mathcal{Q}_-(s; t)} = 1, & s \in J_{t,2}^\circ, \\ e^{\mathcal{Q}_+(s; t) + \mathcal{Q}_-(s; t)} = e^{2\pi i \tau(t)}, & s \in J_{t,1}^\circ, \\ e^{\mathcal{Q}_+(s; t)} = e^{\mathcal{Q}_-(s; t) - 2\pi i \omega(t)}, & s \in I_t^\circ. \end{cases} \quad (9.2.7)$$

We prove Proposition 9.2.2 in Section 10.1. Observe that it follows from Theorem 9.1.2 that  $|e^{\mathcal{Q}(z; t)}|$  is less than 1 when  $\mathcal{U}(z; t) < 0$  (the shaded areas of Figure 9.5), is equal to 1 on critical trajectories (black curves), and otherwise is greater than 1.

Another auxiliary function we need is given by (cf. (7.3.7))

$$A(z; t) := \frac{1}{2} \left( \left( \frac{z - b_2(t)}{z - a_2(t)} \frac{z - b_1(t)}{z - a_1(t)} \right)^{1/4} + \left( \frac{z - b_2(t)}{z - a_2(t)} \frac{z - b_1(t)}{z - a_1(t)} \right)^{-1/4} \right) \quad (9.2.8)$$

for  $z \in \overline{\mathbb{C}} \setminus J_t$ , where the branches are chosen so that the summands are holomorphic in  $\overline{\mathbb{C}} \setminus J_t$  and have value 1 at infinity. As explained in Section 10.3.2, this function can be analytically continued through each side of  $J_t^\circ$  and is non-vanishing in the domain of the definition.

Finally, given a sequence  $\{N_n\}_{n \in \mathbb{N}}$ , we define further below in (7.3.20) functions  $\Theta_n(z; t)$ , which are certain ratios of Riemann theta functions on the Riemann surface of  $Q^{1/2}(z; t)$ . To shorten the presentation of the main results, we only discuss main properties of the functions  $\Theta_n(z; t)$  and defer to Section 10.3 for the detailed construction and description of further properties (cf. Section 7.3.2).

**Proposition 9.2.3.** *Functions  $\Theta_n(z; t)$  are holomorphic in  $\overline{\mathbb{C}} \setminus J_t \cup I_t$  with at most one zero there. These functions have continuous traces on  $J_t^\circ \cup I_t^\circ$  that satisfy*

$$\Theta_{n+}(s; t) = \Theta_{n-}(s; t) e^{2\pi i (n\omega(t) + (n - N_n)\varsigma(t))}.$$

Assume that there exists a constant  $N_*$  such that  $|n - N_n| \leq N_*$  for all  $n \in \mathbb{N}$ . Then for any  $\delta > 0$  there exists a constant  $C(t, \delta, N_*)$  such that

$$|A(z; t) \Theta_n(z; t)| \leq C(t, \delta, N_*), \quad z \in \overline{\mathbb{C}} \setminus \bigcup_{e \in E_t} \{|z - e| < \delta\},$$

that is, including the traces on  $J_t \cup I_t$ . Given  $\epsilon > 0$ , let  $\mathbb{N}(t, \epsilon)$  be a subsequence of indices  $n$  such that  $\Theta_n(z; t)$  is non-vanishing in  $\{|z| \geq 1/\epsilon\}$ . Then there exists a constant  $c(t, \epsilon) > 0$  such that

$$|\Theta_n(\infty; t)| \geq c(t, \epsilon), \quad n \in \mathbb{N}(t, \epsilon).$$

As in the case of Szegő functions, we denote the renormalized functions by  $\vartheta_n(z; t) = \Theta(z; t)/\Theta(\infty; t)$ . Observe that the functions  $\vartheta_n(z; t)D^{N_n-n}(z; t)e^{n\mathcal{Q}(z; t)}$  are holomorphic in  $\mathbb{C} \setminus J_t$ .

Proposition 9.2.3 has substance only if the sets  $\mathbb{N}(t, \epsilon)$  have infinite cardinality. To describe when this happens, let us define

$$\mathbf{B} := - \left( \int_{J_{t,1}} \frac{ds}{Q_+^{1/2}(s; t)} \right) / \left( \int_{I_t} \frac{ds}{Q^{1/2}(s; t)} \right). \quad (9.2.9)$$

It follows from the general theory of Riemann surfaces, see Section 10.3.1, that  $\text{Im}(\mathbf{B}) > 0$ . In particular, any  $s \in \mathbb{C}$  can be uniquely written as  $x + \mathbf{B}y$  for some  $x, y \in \mathbb{R}$ .

**Proposition 9.2.4.** *Given  $\{N_n\}_{n \in \mathbb{N}}$  such that  $|n - N_n| \leq N_*$  for some  $N_* \geq 0$ , the subsequence  $\mathbb{N}(t, \epsilon)$  is infinite for all  $\epsilon > 0$  small enough unless there exist integers  $d > 0, k, i_1, i_2, m_1, m_2$  such that*

$$\varsigma(t) = (i_1 + \mathbf{B}i_2)/d, \quad \omega(t)d = (k-1)i_1 + m_1d, \quad \text{and} \quad \tau(t)d = (k-1)i_2 + m_2d, \quad (9.2.10)$$

where at least one of the fractions  $i_1/d, i_2/d$  is irreducible, and the sequence  $\{N_n\}$  is such that every  $nk - N_n$  is either divisible by  $d$  or  $d/2$  when the latter is an integer.

Write  $\varsigma(t) = x(t) + \mathbf{B}y(t)$ ,  $x(t), y(t) \in \mathbb{R}$ . If one of the triples  $\omega(t), x(t), 1$  or  $\tau(t), y(t), 1$  is rationally independent, then at least one of the integers  $n, n+1$  belongs to  $\mathbb{N}(t, \epsilon)$  for all  $0 < \epsilon \leq \epsilon(N_*)$ . Furthermore, if there exists an infinite subsequence  $\{n_l\}$  such that  $N_{n_l+1} - N_{n_l} \in \{0, 1\}$ , then at least one of the integers  $n_l, n_l+1$  belongs to  $\mathbb{N}(t, \epsilon)$  for all  $0 < \epsilon \leq \epsilon_*$ .

We prove Proposition 9.2.4 in Section 10.3.3. With all the functions defined above, the following theorem holds.

**Theorem 9.2.5.** *Let  $t \in O_{\text{two-cut}}$  and  $\{N_n\}_{n=1}^\infty$  be a sequence such that  $|n - N_n| \leq N_*$  for some  $N_*$  fixed. Let  $P_n(z; t, N)$  be the minimal degree polynomial satisfying (9.0.1) and (9.0.4) and*

$$\psi_n(z; t) := P_n(z; t, N_n) e^{-n(V(z; t) - \ell_*)/2}.$$

*Given  $\epsilon > 0$ , let  $\mathbb{N}(t, \epsilon)$  be as in Proposition 9.2.3. Then for all  $n \in \mathbb{N}(t, \epsilon)$  large enough it holds that*

$$\psi_n(z; t) = \left( (A\vartheta_n D^{N_n-n}) (z; t) + \mathcal{O}_\epsilon(n^{-1}) \right) e^{n\mathcal{Q}(z; t)} \quad (9.2.11)$$

*locally uniformly in  $\mathbb{C} \setminus J_t$ ; moreover,*

$$\psi_n(s; t) = \left( (A\vartheta_n D^{N_n-n})_+ (s; t) e^{n\mathcal{Q}_+(s; t)} + (A\vartheta_n D^{N_n-n})_- (s; t) e^{n\mathcal{Q}_-(s; t)} + \mathcal{O}_\epsilon(n^{-1}) \right) \quad (9.2.12)$$

*locally uniformly on  $J_t^\circ$ .*

Recall that each function  $\vartheta_n(z)$  might have a single zero in  $\mathbb{C} \setminus J_t$ . If these zeros accumulate to some point  $z_*$  along some subsequence of  $\mathbb{N}(t, \epsilon)$ , then the polynomials  $P_n(z; t, N_n)$  will have a single zero approaching  $z_*$  along this subsequence by (9.2.11) and Rouché's theorem. With this exception, it also follows from (9.2.11) that  $P_n(z; t, N_n)$  are eventually zero free on compact subsets  $\mathbb{C} \setminus J_t$ .

The proof of Theorem 9.2.5 is carried out in Chapter 10.

### 9.3 S-curves

In this section we prove Theorem 9.1.2. We do it in several steps. In Section 9.3.1 we gather results about quadratic differentials that will be important to us throughout the proof. In Section 9.3.2 we show the validity of formula (9.1.5); that is, we prove that we are indeed in the two-cut case when  $t \in O_{\text{two-cut}}$ . In Section 9.3.3 we show that the critical and critical orthogonal graphs of

$$\varpi_t(z) := -Q(z, t) dz^2$$

do look like as depicted on Figure 9.5. In Section 9.3.4 we describe the dependence of the zeros of  $Q(z; t)$  on  $t$  by showing that the variational condition (9.0.3) and the S-property (9.0.6) yield that the zeros satisfy a certain system of real equations with non-zero Jacobian, see (9.3.16) and (9.3.17), and that this system, in fact, is uniquely solved by them. Finally, in Section 9.3.5 we establish the limits in (9.1.6).

### 9.3.1 On Quadratic Differentials

To start, let us also recall the following important result, known as Teichmüller's lemma, see [64, Theorem 14.1]. Let  $P$  be a geodesic polygon of a quadratic differential, that is, a Jordan curve in  $\overline{\mathbb{C}}$  that consists of a finite number of trajectories and orthogonal trajectories of this differential. Then it holds that

$$\sum_{z \in P} \left( 1 - \theta(z) \frac{2 + \text{ord}(z)}{2\pi} \right) = 2 + \sum_{z \in \text{int}(P)} \text{ord}(z), \quad (9.3.1)$$

where  $\text{ord}(z)$  is the order of  $z$  with respect to the considered differential and  $\theta(z) \in [0, 2\pi]$ ,  $z \in P$ , is the interior angle of  $P$  at  $z$ . Both sums in (9.3.1) are finite since only critical points of the differential have a non-zero contribution.

Let us briefly recall the main properties of the differential  $\varpi_t(z)$ . The only critical points of  $\varpi_t(z)$  are the zeros of  $Q(z; t)$  and the point at infinity. Regular points have order 0, the order of a zero of  $Q(z; t)$  is equal to its multiplicity, and infinity is a critical point of order  $-8$ . Through each regular point passes exactly one trajectory and one orthogonal trajectory of  $\varpi_t(z)$ , which are orthogonal to each other at the point. Two distinct (orthogonal) trajectories meet only at critical points [64, Theorem 5.5]. As  $Q(z; t)$  is a polynomial, no finite union of (orthogonal) trajectories can form a closed Jordan curve while a trajectory and an orthogonal trajectory can intersect at most once [63, Lemma 8.3]. Furthermore, (orthogonal) trajectories of  $\varpi_t(z)$  cannot be recurrent (dense in two-dimensional regions) [62, Theorem 3.6]. From each critical point of order  $m > 0$  there emanate  $m + 2$  critical trajectories whose consecutive tangent lines at the critical point form an angle  $2\pi/(m + 2)$ . Furthermore, since infinity is a pole of order 8, the critical trajectories can approach infinity only in six



distinguished directions, namely, asymptotically to the lines  $L_{-\pi/6}$ ,  $L_{\pi/6}$ , and  $L_{\pi/2}$ , where  $L_\theta = \{z : z = re^{i\theta}, r \in (-\infty, \infty)\}$ . In fact, there exists a neighborhood of infinity such that any trajectory entering it necessarily tends to infinity [64, Theorem 7.4]. This discussion also applies to orthogonal trajectories. In particular, they can approach infinity only asymptotically to the lines  $L_0$ ,  $L_{\pi/3}$ , and  $L_{2\pi/3}$ .

Denote by  $\mathcal{G}$  the critical graph of  $\varpi_t(z)$ , that is, the totality of all the critical trajectories of  $\varpi_t(z)$ . Then, see [62, Theorem 3.5], the complement of  $\mathcal{G}$  can be written as a disjoint union of either half-plane or strip domains. Recall that a *half-plane (or end) domain* is swept by trajectories unbounded in both directions that approach infinity along consecutive critical directions. Its boundary is connected and consists of a union of two unbounded critical trajectories and a finite number (possibly zero) of short trajectories of  $\varpi_t(z)$ . The map  $z \mapsto \int^z \sqrt{-\varpi_t}$  maps end domains conformally onto half planes  $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > c\}$  for some  $c \in \mathbb{R}$  that depends on the domain, and extends continuously to the boundary. Similarly, a *strip domain* is again swept by trajectories unbounded in both directions, but its boundary consists of two disjoint  $\varpi_t(z)$ -paths, each of which is comprised of two unbounded critical trajectories and a finite number (possibly zero) of short trajectories. The map  $z \mapsto \int^z \sqrt{-\varpi_t}$  maps strip domains conformally onto vertical strips  $\{w \in \mathbb{C} \mid c_1 < \operatorname{Re}(w) < c_2\}$  for some  $c_1, c_2 \in \mathbb{R}$  depending on the domain, and extends continuously to their boundaries. The number  $c_2 - c_1$  is known as the *width of a strip domain* and can be calculated in terms of  $\varpi_t(z)$  as

$$\left| \operatorname{Re} \int_p^q \sqrt{-\varpi} \right| \quad (9.3.2)$$

where  $p, q$  belong to different components of the boundary of the domain.

### 9.3.2 Two-Cut Region

We now prove expression (9.1.5). Assume to the contrary that we are in one-cut case. That is, there exists a choice of  $a$ ,  $b$ , and  $c$  such that the polynomial  $Q(z; t)$

from (9.0.6) has the form (9.1.3). It follows from (9.0.6) in conjunction with (9.0.4) that

$$Q(z; t) = \left( \frac{-z^2 + t}{2} - \frac{1}{z} + \mathcal{O}(z^{-2}) \right)^2 = \frac{(z^2 - t)^2}{4} + z + C \quad (9.3.3)$$

for some constant  $C$ . Then, by equating the coefficients in (9.1.3) and (9.3.3), we obtain a system of equations

$$\begin{cases} a + b + 2c = 0, \\ ab + c^2 + 2(a + b)c = -2t, \\ 2abc + (a + b)c^2 = -4. \end{cases} \quad (9.3.4)$$

Setting  $x := (a + b)/2$  and eliminating the product  $ab$  from the second and third equations yields

$$x^3 - tx - 1 = 0, \quad (9.3.5)$$

which is exactly the equation appearing before (9.1.2). Given any solution of (9.3.5), say  $x(t)$ , then  $a(t)$ ,  $b(t)$ , and  $c(t)$  are necessarily expressed via (9.1.4). Theorem 9.0.1 and the variational condition (9.0.3) imply that there must exist a contour  $\Gamma_t \in \mathcal{T}$  (this class of contours was defined right after (9.0.2)) such that

$$\mathcal{U}(z; t) \leq 0 \quad \text{for all } z \in \Gamma_t, \quad (9.3.6)$$

see (9.0.8). In what follows, we shall show that no such contour exists in  $\mathcal{T}$  for any of the three possible choices of  $x(t)$  solving (9.3.5) when  $t \in O_{\text{two-cut}}$  and  $Q(z; t)$  is given by (9.1.3) and (9.1.4).

In accordance with the above strategy, observe that the solutions of (9.3.5) can be written as

$$x_k(t) = u_k(t) + \frac{t}{3u_k(t)}, \quad u_k(t) := \left( \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{t^3}{27}} \right)^{1/3} e^{2k\pi i/3}, \quad (9.3.7)$$

$k \in \{0, 1, 2\}$ , with all branches being principal. It can be readily verified that  $x_1(t)$  is analytic in  $\mathbb{C} \setminus (\overline{e^{2\pi i/3}S}, \overline{S})$  (here,  $\bar{\cdot}$  means topological closure), see (9.1.2) for the definition of the ray  $S$ , and

$$\begin{cases} x_0(t) &= e^{4\pi i/3} x_1(te^{-2\pi i/3}), \\ x_1(t) &= e^{4\pi i/3} \overline{x_1(\bar{t}e^{2\pi i/3})}, \\ x_2(t) &= \overline{x_1(\bar{t})}. \end{cases} \quad (9.3.8)$$

Furthermore, noting that the function  $x(t)$ , defined after (9.1.2), maps  $\Omega_{\text{one-cut}}$  onto  $O_{\text{one-cut}}$ , it can be easily checked that  $x(t)$  is evaluated as shown on Figure 9.6, where the dashed lines are the chosen branch cuts of  $x_1(t)$ . In particular,  $x(t)$  can be analytically continued across  $C_{\text{split}}$ ,  $C_{\text{birth}}^a$ , and  $C_{\text{birth}}^b$ , see (9.1.2) and Figure 9.6. In what follows, we consider what happens in the case of each of these continuations.

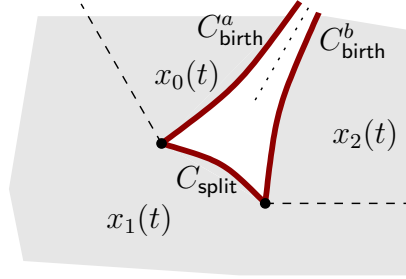


Fig. 9.6. Determination of  $x(t)$

Continue  $x(t)$  into  $O_{\text{two-cut}}$  by either  $x_2(t)$  or  $x_0(t)$ , that is, analytically across either  $C_{\text{birth}}^b$  or  $C_{\text{birth}}^a$ , see Figure 9.6. The first and the last symmetries in (9.3.8) then yield that  $\varpi_t(z)$  is either equal to

$$\overline{\varpi_{\bar{t}}(\bar{z})} \quad \text{or} \quad \varpi_{te^{-2\pi i/3}}(ze^{-4\pi i/3}).$$

Since the set  $O_{\text{two-cut}}$  is symmetric with respect to the line  $L_{\pi/3}$ , its rotation by  $-2\pi/3$  is equal to its reflection across the real axis. Thus, the critical graph  $\varpi_t(z)$  when  $t \in O_{\text{two-cut}}$  and  $x(t)$  is continued by either  $x_2(t)$  or  $x_0(t)$  is equal to the reflection across the real axis or the rotation by  $4\pi/3$  of the critical graph of  $\varpi_{t_*}(z)$  for some  $t_*$

such that  $\bar{t}_* \in O_{\text{two-cut}}$ . These graphs were studied in [50, Theorems 3.2 and 3.4] and determined to have the structure as depicted on Figure 9.3(a–c) (or the reflection of these three panels across the line  $L_{2\pi/3}$ ). A direct examination shows that none of these critical graphs form a curve in  $\mathcal{T}$  for which (9.3.6) holds (such a curve must belong to the closure of the gray regions on Figure 9.3).

Suppose now that we continue  $x(t)$  by  $x_1(t)$ , that is, analytically across  $C_{\text{split}}$ . For such a choice of  $x(t)$ , the critical graph of  $\varpi_t(z)$  was studied in [67] when  $t \in L_{\pi/3}$  (in which case the critical graph is symmetric with respect to  $L_{2\pi/3}$ ). In particular, it was shown that  $x(t) \in L_{2\pi/3}$ , no union of critical trajectories join  $a$  and  $b$ , and no critical trajectory of  $\varpi_t(z)$  crosses the line  $L_{2\pi/3}$  when  $t \in L_{\pi/3} \cap O_{\text{two-cut}}$ , see [67, Lemma 3.2]. Since critical trajectories cannot intersect, can approach infinity only asymptotically to the lines  $L_{\pi/6}$ ,  $L_{\pi/2}$ , and  $L_{-\pi/6}$ , and must obey Teichmüller’s lemma (9.3.1), the critical graph of  $\varpi_t(z)$  must be as on Figure 9.7.

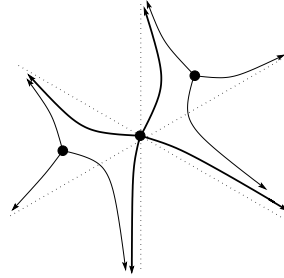


Fig. 9.7. The critical graph of  $\varpi_t(z)$  when  $x(t)$  is analytically continued across  $C_{\text{split}}$ .

Clearly, this critical graph does not yield a curve in  $\mathcal{T}$  for which (9.3.6) holds. Thus, to complete the proof, we need to argue that the structure of the critical trajectories of  $\varpi_t(z)$  remains the same for all  $t \in O_{\text{two-cut}}$ . Observe that it is enough to show that all the trajectories out of  $b$  approach infinity.

Recall that the trajectories emanating out of  $b$  are part of the level set  $\{\mathcal{U}(z; t) = 0\}$ , see (9.0.8). When  $t \in L_{\pi/3} \cap O_{\text{two-cut}}$ , these trajectories approach infinity at the angles  $-\pi/6$ ,  $\pi/6$ , and  $\pi/2$ , see Figure 9.7. Since the values of  $\mathcal{U}(z; t)$  analytically

depend on  $t$ , the same must be true in a neighborhood of each such  $t$ . This will remain so until one of the trajectories hits a critical point different from the one at infinity. As can be seen from Figure 9.7, this critical point must necessarily be  $c$ . That is, as long as  $\mathcal{U}(-x; t) \neq 0$ , the trajectories out of  $b$  will asymptotically behave as on Figure 9.7. When  $x(t)$  is continued into  $O_{\text{two-cut}}$  by  $x_1(t)$ , its values lie within the

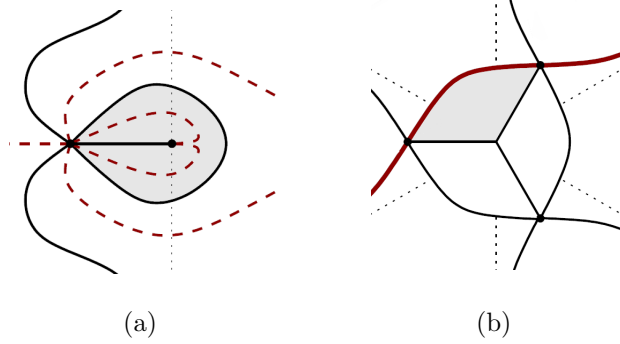


Fig. 9.8. Shaded regions represent the domains within which  $2x_1^3(t)$  (panel a) and  $x_1(t)$  (panel b) change when  $t \in O_{\text{two-cut}}$ .

gray region on Figure 9.8(b), see also Figure 9.1(b). Respectively, the values  $2x^3(t)$  lie within the gray region on Figure 9.8(a), see also Figure 9.1(a). It was verified in [50, Section 5.3] that

$$\mathcal{U}(-x; t) = \operatorname{Re} \left( \frac{2}{3} \int_{-1}^{2x^3} \left( 1 + \frac{1}{s} \right)^{3/2} ds \right),$$

where the path of integration lies within the shaded domain on Figure 9.8(a). Hence,  $\mathcal{U}(-x; t) = 0$  if and only if  $2x^3$  belongs to a trajectory of  $-(1 + 1/s)^3 ds^2$  emanating from  $-1$ . These trajectories are drawn on Figures 9.1 and 9.8(a) (black lines). Thus,  $\mathcal{U}(-x; t) \neq 0$  in the considered case as claimed.

### 9.3.3 Critical Graph of $\varpi_t(z)$

Let, as usual,  $Q(z; t)$  be the polynomial guaranteed by Theorem 9.0.1. According to what precedes, it has the form (9.1.5) when  $t \in O_{\text{two-cut}}$ . Recall the properties

of the differential  $\varpi_t(z) = -Q(z; t)dz^2$  described at the beginning of Section 9.3.1. In particular, it has four critical points of order 1, which, for a moment, we label as  $z_1(t), z_2(t), z_3(t), z_4(t)$  (these are the zeros of  $Q(z; t)$ ), a critical point of order  $-8$  at infinity, and no other critical points. It follows from Theorem 9.0.1(4) and (9.0.3) with (9.0.8) that

$$\operatorname{Re} \left( \int_{\Gamma_t[z_i(t), z_j(t)]} Q_+^{1/2}(z; t) dz \right) = 0, \quad (9.3.9)$$

where  $\Gamma_t[z_i(t), z_j(t)]$  is the subarc of  $\Gamma_t$  with endpoints  $z_i(t), z_j(t)$  and  $Q_+^{1/2}(z; t)$  is the trace of  $Q^{1/2}(z; t)$  on the positive side of  $\Gamma_t$ . Equations (9.3.9) imply existence of three short critical trajectories of  $\varpi_t(z)$ . Indeed, if all three critical trajectories out of a zero  $z_i(t)$  approach infinity, then  $z_i(t)$  must belong to a boundary of at least one strip domain. Let  $z_j(t)$  be a different zero of  $Q(z; t)$  belonging to the other component of the boundary of this strip domain. Then it follows from (9.3.2) and (9.3.9) that the width of this strip domain is 0, which is impossible. Thus, each zero of  $Q(z; t)$  must be coincident with at least one short trajectory. Therefore, either there is a zero, say  $z_4(t)$ , connected by short trajectories to the remaining three zeros or there are at least two short trajectories connecting two pairs of zeros. In the latter case, label these zeros by  $a_1(t), b_1(t)$  and  $a_2(t), b_2(t)$ . If the other two trajectories out of both  $a_1(t)$  and  $b_1(t)$  approach infinity, one these zeros again must belong to the boundary of a strip domain with either  $a_2(t)$  or  $b_2(t)$  belonging to the other component of the boundary. As before, (9.3.9) yields that the width of this strip domain is 0, which, again, is impossible. Thus, in this case there also exists a third short critical trajectory. Then we choose a labeling of the zeros so that  $b_1(t)$  and  $a_2(t)$  are connected by this trajectory.

Since short critical trajectories cannot form closed curves, there cannot be any more of them. That is, the remaining critical trajectories are unbounded. Consider the two unbounded critical trajectories out of  $z_1(t)$  in the case where short ones form a threefold. Since critical trajectories cannot intersect and the remaining zeros are connected to  $z_1(t)$  by short critical trajectories, they delimit a half-plane domain and,

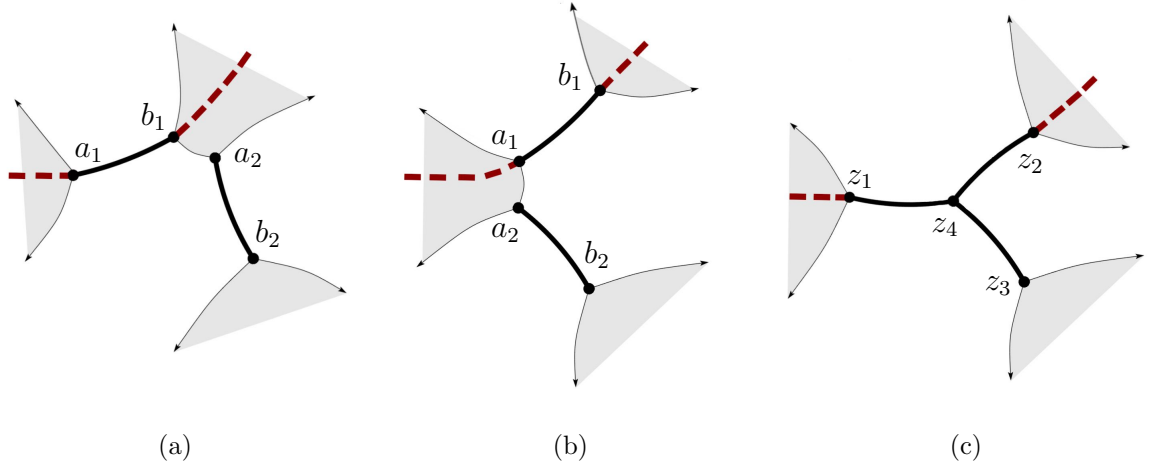


Fig. 9.9. Geometries of the critical graph of  $\varpi_t(z)$ . Shaded regions represent the open set  $\{\mathcal{U}(z; t) < 0\}$ , the white regions represent the open  $\{\mathcal{U}(z; t) > 0\}$ , and the red, dashed arcs form  $\Gamma_t \setminus J_t$ .

in particular, must approach infinity along consecutive critical directions (those are given by the angles  $(2k + 1)\pi/6$ ,  $k \in \{0, \dots, 5\}$ , see Section 9.3.1). Clearly, the same is true for the unbounded critical trajectories out of  $z_2(t)$  and  $z_3(t)$  as well as for the unbounded critical trajectories out of  $a_1(t)$ ,  $b_2(t)$ , and the union of the unbounded critical trajectory out of  $b_1(t)$ , the short critical trajectory connecting  $b_1(t)$  to  $a_2(t)$ , and the unbounded critical trajectory out of  $a_2(t)$  in the case where short critical trajectories form a Jordan arc.

Now, let  $\mathcal{U}(z; t)$  be given by (9.0.8). Clearly,  $\mathcal{U}(z; t)$  is a subharmonic function which is equal to zero on  $J_t$ , see (9.0.3). Since  $\mathcal{U}(z; t)$  must have the same sign on both sides of each subarc of  $J_t$  by the S-property (7.1.2), it follows from the maximum principle for subharmonic functions that it is positive there. Further, since trajectories of  $\varpi_t(z)$  cannot form a closed Jordan curve, all the connected components of the open set  $\{\mathcal{U}(z; t) < 0\}$  must necessarily extend to infinity. Since  $\operatorname{Re}(V(z; t))$  is

the dominant term of  $\mathcal{U}(z; t)$  around infinity, see (9.0.8), for any  $\delta > 0$  there exists  $R > 0$  sufficiently large such that

$$\begin{cases} (S_{\pi/3, \delta} \cup S_{\pi, \delta} \cup S_{-\pi/3, \delta}) \cap \{|z| > R\} & \subset \{\mathcal{U}(z; t) < 0\}, \\ (S_{0, \delta} \cup S_{2\pi/3, \delta} \cup S_{-2\pi/3, \delta}) \cap \{|z| > R\} & \subset \{\mathcal{U}(z; t) > 0\}, \end{cases}$$

where  $S_{\theta, \delta} := \{|\arg(z) - \theta| < \pi/6 - \delta\}$ . Altogether, the critical graph of  $\varpi_t(z)$  must look like either on Figure 9.5 or on Figure 9.9.

It remains to show that  $\varpi_t(z)$  cannot have the critical graph as on any of the panels of Figure 9.9. To this end, recall that the contour  $\Gamma_t$  must contain  $J_t$  and two unbounded arcs extending to infinity in the directions  $\pi/3$  and  $\pi$  (blue unbounded arcs on Figure 9.9). Let  $\Gamma_*$  be obtained from  $\Gamma_t$  by dropping the short trajectory that is a part of  $J_t$  and whose removal keeps  $\Gamma_*$  connected (this can be done for any of the panels on Figure 9.9). Observe that  $\Gamma_*$  also belongs to  $\mathcal{T}$ . Let  $\mu_*$  be the weighted equilibrium distribution on  $\Gamma_*$  as defined in Definition 7.1. Since  $\Gamma_* \subset \Gamma_t$ , it holds that  $\mu_* \in \mathcal{M}(\Gamma_t)$ . Moreover, since  $\mu_* \neq \mu_t$ ,  $I_V(\mu_*) > I_V(\mu_t)$ , see Definition (7.1). However, the last inequality clearly contradicts (9.0.5).

We have shown that the critical graph of  $\varpi_t(z)$  must look like on Figure 9.5. As the critical orthogonal and critical trajectories cannot intersect, the structure of the critical orthogonal graph is uniquely determined by structure of the critical graph. Now, we can completely fix the labeling of the zeros of  $Q(z; t)$  by given the label  $a_1(t)$  to one that is incident with the orthogonal critical trajectory extending to infinity asymptotically to the ray  $\arg(z) = \pi$ .

#### 9.3.4 Dependence on $t$

We start with some general considerations. Let  $f(z)$  and  $g(z)$  be analytic functions of  $z = x + iy$ . Consider a determinant of the form

$$D = \begin{vmatrix} \partial_x \operatorname{Re}(f) & \partial_y \operatorname{Re}(f) & * \\ \partial_x \operatorname{Im}(f) & \partial_y \operatorname{Im}(f) & * \\ \partial_x \operatorname{Re}(g) & \partial_y \operatorname{Re}(g) & * \end{vmatrix},$$



where the entries of the third column are not important for the forthcoming computation. Due to Cauchy-Riemann relations it holds that  $f' = \partial_x \operatorname{Re}(f) + i\partial_x \operatorname{Im}(f) = \partial_y \operatorname{Im}(f) - i\partial_y \operatorname{Re}(f)$ . Therefore,

$$D = \begin{vmatrix} \operatorname{Re}(f') & -\operatorname{Im}(f') & * \\ \operatorname{Im}(f') & \operatorname{Re}(f') & * \\ \operatorname{Re}(g') & -\operatorname{Im}(g') & * \end{vmatrix} = \begin{vmatrix} f' & if' & * \\ \operatorname{Im}(f') & \operatorname{Re}(f') & * \\ \operatorname{Re}(g') & -\operatorname{Im}(g') & * \end{vmatrix} = \frac{i}{2} \begin{vmatrix} f' & if' & * \\ \bar{f}' & -i\bar{f}' & * \\ \operatorname{Re}(g') & -\operatorname{Im}(g') & * \end{vmatrix}$$

by adding the second row times  $i$  to the first one and then multiplying the second row by  $-2i$  and adding the first row to it. It further holds that

$$D = \frac{i}{2} \begin{vmatrix} 2f' & if' & * \\ 0 & -i\bar{f}' & * \\ g' & -\operatorname{Im}(g') & * \end{vmatrix} = \frac{i}{2} \begin{vmatrix} 2f' & 0 & * \\ 0 & -i\bar{f}' & * \\ g' & -i\bar{g}'/2 & * \end{vmatrix} = \frac{1}{2} \begin{vmatrix} f' & 0 & * \\ 0 & \bar{f}' & * \\ g' & \bar{g}' & * \end{vmatrix},$$

where we added the second column times  $-i$  to the first one, then added the first column times  $-i/2$  to the second one, and then factored 2 from the first column,  $-i$  from the second one, and  $1/2$  from the third row.

Now, let  $f_j(z_1, z_2, z_3, z_4)$ ,  $j \in \{1, 2, 3, 4, 5\}$ , be analytic functions in each variable  $z_i = x_i + iy_i$ . We would like to compute the Jacobian of the following system of real-valued functions of  $x_1, y_1, \dots, x_4, y_4$ :

$$\operatorname{Re}(f_1), \operatorname{Im}(f_1), \operatorname{Re}(f_2), \operatorname{Im}(f_2), \operatorname{Re}(f_3), \operatorname{Im}(f_3), \operatorname{Re}(f_4), \operatorname{Re}(f_5). \quad (9.3.10)$$

That is, we are interested in

$$Jac = \begin{vmatrix} \operatorname{Re}(f_{11}) & -\operatorname{Im}(f_{11}) & \operatorname{Re}(f_{12}) & -\operatorname{Im}(f_{12}) & \operatorname{Re}(f_{13}) & -\operatorname{Im}(f_{13}) & \operatorname{Re}(f_{14}) & -\operatorname{Im}(f_{14}) \\ \operatorname{Im}(f_{11}) & \operatorname{Re}(f_{11}) & \operatorname{Im}(f_{12}) & \operatorname{Re}(f_{12}) & \operatorname{Im}(f_{13}) & \operatorname{Re}(f_{13}) & \operatorname{Im}(f_{14}) & \operatorname{Re}(f_{14}) \\ \operatorname{Re}(f_{21}) & -\operatorname{Im}(f_{21}) & \operatorname{Re}(f_{22}) & -\operatorname{Im}(f_{22}) & \operatorname{Re}(f_{23}) & -\operatorname{Im}(f_{23}) & \operatorname{Re}(f_{24}) & -\operatorname{Im}(f_{24}) \\ \operatorname{Im}(f_{21}) & \operatorname{Re}(f_{21}) & \operatorname{Im}(f_{22}) & \operatorname{Re}(f_{22}) & \operatorname{Im}(f_{23}) & \operatorname{Re}(f_{23}) & \operatorname{Im}(f_{24}) & \operatorname{Re}(f_{24}) \\ \operatorname{Re}(f_{31}) & -\operatorname{Im}(f_{31}) & \operatorname{Re}(f_{32}) & -\operatorname{Im}(f_{32}) & \operatorname{Re}(f_{33}) & -\operatorname{Im}(f_{33}) & \operatorname{Re}(f_{34}) & -\operatorname{Im}(f_{34}) \\ \operatorname{Im}(f_{31}) & \operatorname{Re}(f_{31}) & \operatorname{Im}(f_{32}) & \operatorname{Re}(f_{32}) & \operatorname{Im}(f_{33}) & \operatorname{Re}(f_{33}) & \operatorname{Im}(f_{34}) & \operatorname{Re}(f_{34}) \\ \operatorname{Re}(f_{41}) & -\operatorname{Im}(f_{41}) & \operatorname{Re}(f_{42}) & -\operatorname{Im}(f_{42}) & \operatorname{Re}(f_{43}) & -\operatorname{Im}(f_{43}) & \operatorname{Re}(f_{44}) & -\operatorname{Im}(f_{44}) \\ \operatorname{Re}(f_{51}) & -\operatorname{Im}(f_{51}) & \operatorname{Re}(f_{52}) & -\operatorname{Im}(f_{52}) & \operatorname{Re}(f_{53}) & -\operatorname{Im}(f_{53}) & \operatorname{Re}(f_{54}) & -\operatorname{Im}(f_{54}) \end{vmatrix},$$

where  $f_{ji} := \partial_{z_i} f_j$ . By performing the same row and column operations as for the determinant  $D$  above, we get that

$$Jac = \frac{1}{2i} \begin{vmatrix} f_{11} & 0 & f_{12} & 0 & f_{13} & 0 & f_{14} & 0 \\ 0 & \bar{f}_{11} & 0 & \bar{f}_{12} & 0 & \bar{f}_{13} & 0 & \bar{f}_{14} \\ f_{21} & 0 & f_{22} & 0 & f_{23} & 0 & f_{24} & 0 \\ 0 & \bar{f}_{21} & 0 & \bar{f}_{22} & 0 & \bar{f}_{23} & 0 & \bar{f}_{24} \\ f_{31} & 0 & f_{32} & 0 & f_{33} & 0 & f_{34} & 0 \\ 0 & \bar{f}_{31} & 0 & \bar{f}_{32} & 0 & \bar{f}_{33} & 0 & \bar{f}_{34} \\ f_{41} & \bar{f}_{41} & f_{42} & \bar{f}_{42} & f_{43} & \bar{f}_{43} & f_{44} & \bar{f}_{44} \\ f_{51} & \bar{f}_{51} & f_{52} & \bar{f}_{52} & f_{53} & \bar{f}_{53} & f_{54} & \bar{f}_{54} \end{vmatrix}.$$

Assume further that

$$\begin{cases} f_1(z_1, z_2, z_3, z_4) = z_1 + z_2 + z_3 + z_4, \\ f_2(z_1, z_2, z_3, z_4) = z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4, \\ f_3(z_1, z_2, z_3, z_4) = z_2 z_3 z_4 + z_1 z_3 z_4 + z_1 z_2 z_4 + z_1 z_2 z_3. \end{cases} \quad (9.3.11)$$

Then, by using the above explicit expressions and subtracting the first (resp. second) column from the third, fifth, and seventh (resp. fourth, sixth, and eighth), we get that

$$Jac = (2i)^{-1} \times \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & z_1 - z_2 & 0 & z_1 - z_3 & 0 & z_1 - z_4 & 0 \\ * & * & 0 & \bar{z}_1 - \bar{z}_2 & 0 & \bar{z}_1 - \bar{z}_3 & 0 & \bar{z}_1 - \bar{z}_4 \\ * & * & \frac{z_1 - z_2}{(z_3 + z_4)^{-1}} & 0 & \frac{z_1 - z_3}{(z_2 + z_4)^{-1}} & 0 & \frac{z_1 - z_4}{(z_2 + z_3)^{-1}} & 0 \\ * & * & 0 & \frac{\bar{z}_1 - \bar{z}_2}{(\bar{z}_3 + \bar{z}_4)^{-1}} & 0 & \frac{\bar{z}_1 - \bar{z}_3}{(\bar{z}_2 + \bar{z}_4)^{-1}} & 0 & \frac{\bar{z}_1 - \bar{z}_4}{(\bar{z}_2 + \bar{z}_3)^{-1}} \\ * & * & \frac{z_1 - z_2}{g_{42}^{-1}} & \frac{\bar{z}_1 - \bar{z}_2}{\bar{g}_{42}^{-1}} & \frac{z_1 - z_3}{g_{43}^{-1}} & \frac{\bar{z}_1 - \bar{z}_3}{\bar{g}_{43}^{-1}} & \frac{z_1 - z_4}{g_{44}^{-1}} & \frac{\bar{z}_1 - \bar{z}_4}{\bar{g}_{44}^{-1}} \\ * & * & \frac{z_1 - z_2}{g_{52}^{-1}} & \frac{\bar{z}_1 - \bar{z}_2}{\bar{g}_{52}^{-1}} & \frac{z_1 - z_3}{g_{53}^{-1}} & \frac{\bar{z}_1 - \bar{z}_3}{\bar{g}_{53}^{-1}} & \frac{z_1 - z_4}{g_{54}^{-1}} & \frac{\bar{z}_1 - \bar{z}_4}{\bar{g}_{54}^{-1}} \end{vmatrix},$$

where  $g_{ji}(z_1, z_2, z_3, z_4) := (z_1 - z_i)^{-1}(f_{ji} - f_{j1})(z_1, z_2, z_3, z_4)$ ,  $j \in \{4, 5\}$  and  $i \in \{2, 3, 4\}$ .

Hence,

$$Jac = \frac{|z_1 - z_2|^2 |z_1 - z_3|^2 |z_1 - z_4|^2}{2i} \times \begin{vmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ z_3 + z_4 & 0 & z_2 + z_4 & 0 & z_2 + z_3 & 0 \\ 0 & \bar{z}_3 + \bar{z}_4 & 0 & \bar{z}_2 + \bar{z}_4 & 0 & \bar{z}_2 + \bar{z}_3 \\ g_{42} & \bar{g}_{42} & g_{43} & \bar{g}_{43} & g_{44} & \bar{g}_{44} \\ g_{52} & \bar{g}_{52} & g_{53} & \bar{g}_{53} & g_{54} & \bar{g}_{54} \end{vmatrix}.$$

Absolutely analogous computation now implies that

$$Jac = \frac{|z_1 - z_2|^2 |z_1 - z_3|^2 |z_1 - z_4|^2 |z_2 - z_3|^2 |z_2 - z_4|^2}{2i} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ h_{43} & \bar{h}_{43} & h_{44} & \bar{h}_{44} \\ h_{53} & \bar{h}_{53} & h_{54} & \bar{h}_{54} \end{vmatrix},$$

where  $h_{ji}(z_1, z_2, z_3, z_4) := (z_2 - z_i)^{-1}(g_{ji} - g_{j2})(z_1, z_2, z_3, z_4)$ ,  $j \in \{4, 5\}$  and  $i \in \{3, 4\}$ .

The above expression immediately yields that

$$Jac = \frac{\prod_{i < j} |z_i - z_j|^2}{2i} \begin{vmatrix} k_4 & \bar{k}_4 \\ k_5 & \bar{k}_5 \end{vmatrix} = \prod_{i < j} |z_i - z_j|^2 \text{Im}(k_4 \bar{k}_5), \quad (9.3.12)$$

where  $k_j(z_1, z_2, z_3, z_4) := (z_3 - z_4)^{-1}(g_{j4} - g_{j3})(z_1, z_2, z_3, z_4)$ ,  $j \in \{4, 5\}$ . Finally, let

$$w(z) := \sqrt{(z - z_1)(z - z_2)(z - z_3)(z - z_4)}$$

be a branch such that  $w(z) = z^2 + \mathcal{O}(z)$  as  $z \rightarrow \infty$  with branch cuts  $\gamma_{12}$  and  $\gamma_{34}$  that are bounded, disjoint, and smooth, and where  $\gamma_{ij}$  connects  $z_i$  to  $z_j$ . Further, select a smooth arc  $\gamma_{32}$  disjoint (except for the endpoints) from the previous two. Set

$$f_4(z_1, z_2, z_3, z_4) := 4 \int_{\gamma_{12}} w(z) dz \quad \text{and} \quad f_5(z_1, z_2, z_3, z_4) := 4 \int_{\gamma_{32}} w(z) dz, \quad (9.3.13)$$

where we integrate  $w(z)$  on the positive side of  $\gamma_{12}$ . Let  $O \subset \{z_i \neq z_j, i \neq j, i, j \in \{1, 2, 3, 4\}\}$  be a domain such that there exist arcs  $\gamma_{ij}(z_1, z_2, z_3, z_4)$  with the above

properties for each  $(z_1, z_2, z_3, z_4) \in O$ , which, in addition, possess parameterizations that depend continuously on each variable  $z_1, z_2, z_3, z_4$ . Then the functions  $f_j(z_1, z_2, z_3, z_4)$ ,  $j \in \{4, 5\}$ , are analytic in each variable  $z_i$  for  $(z_1, z_2, z_3, z_4) \in O$ . Furthermore, it can be readily computed that

$$g_{ji} = 2 \int \frac{w(z)dz}{(z - z_1)(z - z_i)}, \quad h_{ji} = -2 \int \frac{w(z)dz}{(z - z_1)(z - z_2)(z - z_i)}, \quad k_j = 2 \int \frac{dz}{w(z)},$$

where the integrals are taken over  $\gamma_{12}$  when  $j = 4$  and  $\gamma_{32}$  when  $j = 5$ . Trivially, (9.3.12) can be rewritten as

$$Jac = 4 \prod_{i < j} |z_i - z_j|^2 \text{Im} \left( \int_{\gamma_{12}} \frac{dz}{w(z)} \overline{\int_{\gamma_{32}} \frac{dz}{w(z)}} \right). \quad (9.3.14)$$

Now, consider the Riemann surface  $\mathfrak{R} := \{z := (z, w) : w^2 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)\}$ . Denote by  $\pi : \mathfrak{R} \rightarrow \mathbb{C}$  the natural projection  $\pi(z) = z$  and write  $w(z)$  for a rational function on  $\mathfrak{R}$  such that  $z = (z, w(z))$ . Let  $\beta := \pi^{-1}(\gamma_{12})$  and  $\alpha := \pi^{-1}(\gamma_{32})$ . Orient these cycles so that

$$2 \int_{\gamma_{12}} \frac{dz}{w(z)} = \oint_{\beta} \frac{dz}{w(z)} \quad \text{and} \quad 2 \int_{\gamma_{32}} \frac{dz}{w(z)} = \oint_{\alpha} \frac{dz}{w(z)}.$$

Observe that the cycles  $\alpha, \beta$  form the right pair at the point of their intersection and that  $\mathfrak{R} \setminus \{\alpha \cup \beta\}$  is simply connected. Hence, the cycles  $\alpha, \beta$  form a homology basis on  $\mathfrak{R}$ . Since the genus of  $\mathfrak{R}$  is 1, it has a unique (up to multiplication by a constant) holomorphic differential. It is quite easy to check that this differential is  $dz/w(z)$ . Hence, we get from (9.3.14) that

$$Jac = \prod_{i < j} |z_i - z_j|^2 \text{Im} \left( \oint_{\beta} \frac{dz}{w(z)} \overline{\oint_{\alpha} \frac{dz}{w(z)}} \right) > 0 \quad (9.3.15)$$

when  $(z_1, z_2, z_3, z_4) \in O$ , where the last inequality was shown by Rimeann.

Now, let  $Q(z; t) = \frac{1}{4}(z - a_1(t))(z - b_1(t))(z - a_2(t))(z - b_2(t))$  be the polynomial from Theorem 9.0.1. It can be easily deduced from (9.0.6) that

$$\begin{cases} f_1(a_1, b_1, a_2, b_2) &= 0, \\ f_2(a_1, b_1, a_2, b_2) &= -2t, \\ f_3(a_1, b_1, a_2, b_2) &= -4, \end{cases} \quad (9.3.16)$$

where the functions  $f_i(z_1, z_2, z_3, z_4)$ ,  $i \in \{1, 2, 3\}$ , are given by (9.3.11).

Fix  $t^* \in O_{\text{two-cut}}$  and let  $\delta^* > 0$  be small enough so that all four disks  $\{|z - z_i^*| \leq \delta^*\}$  are disjoint, where  $z_1^* = a_1(t^*)$ ,  $z_2^* = b_1(t^*)$ ,  $z_3^* = a_2(t^*)$ , and  $z_4^* = b_2(t^*)$ . Let  $O := \{(z_1, z_2, z_3, z_4) : |z_i - z_i^*| < \delta^*\}$ ,  $s_i^*$  be the point of intersection of  $\{|z - z_i^*| = \delta^*\}$  and  $J_{t^*}$ ,  $i \in \{1, 2, 3, 4\}$ , and  $u_i^*$  be the point of intersection of  $\{|z - z_i^*| = \delta^*\}$  and  $\Gamma(b_1(t^*), a_2(t^*))$ ,  $i \in \{2, 3\}$ , where, as usual,  $\Gamma(a, b)$  is the subarc of the trajectories of  $\varpi_t(z)$  connecting  $a$  and  $b$ . Then we can choose  $\gamma_{12} = [z_1, s_1^*] \cup \Gamma(s_1^*, s_2^*) \cup [s_2^*, z_2]$ ,  $\gamma_{32} = [z_3, u_3^*] \cup \Gamma(u_3^*, u_2^*) \cup [u_2^*, z_2]$ , and  $\gamma_{34} = [z_3, s_3^*] \cup \Gamma(s_3^*, s_4^*) \cup [s_4^*, z_4]$ , where  $[a, b]$  is the line segment connecting  $a$  and  $b$  in  $\mathbb{C}$ . Clearly, the arcs  $\gamma_{ij}$  continuously depend on  $(z_1, z_2, z_3, z_4) \in O$ . Now, the relations (9.3.9) can be rewritten as

$$\operatorname{Re}(f_4(a_1, b_1, a_2, b_2)) = 0 \quad \text{and} \quad \operatorname{Re}(f_5(a_1, b_1, a_2, b_2)) = 0 \quad (9.3.17)$$

with  $f_j(z_1, z_2, z_3, z_4)$ ,  $j \in \{4, 5\}$ , given by (9.3.13), where we set  $w(z) := Q(z; t)$ .

It follows from (9.3.15) and the implicit function theorem that there exists a neighborhood of  $t^*$  in which system (9.3.16) and (9.3.17) is uniquely solvable and the solution, say  $(a_1^*(t), b_1^*(t), a_2^*(t), b_2^*(t))$ , is such that the real and imaginary parts of  $a_i^*(t), b_i^*(t)$  are real analytic functions of  $\operatorname{Re}(t)$  and  $\operatorname{Im}(t)$  for  $t$  in this neighborhood. These solutions are unique only locally around the point  $(a_1(t^*), b_1(t^*), a_2(t^*), b_2(t^*))$  and we still need to argue that they do coincide with the zeros  $a_i(t), b_i(t)$  of  $Q(z; t)$  (of course, it holds that  $a_i^*(t^*) = a_i(t^*)$  and  $b_i^*(t^*) = b_i(t^*)$ ).

In what follows, we always assume that  $t$  belongs to a disk centered at  $t^*$  of small enough radius so that the functions  $a_i^*(t), b_i^*(t)$  are defined and continuous in this disk. Let

$$Q^*(z; t) := \frac{1}{4}(z - a_1^*(t))(z - b_1^*(t))(z - a_2^*(t))(z - b_2^*(t)) \quad \text{and} \quad \varpi_t^*(z) := -Q^*(z; t)dz^2.$$

Further, let  $\mathcal{U}^*(z; t)$  be defined as in (9.0.8) with  $Q(z; t)$  replaced by  $Q^*(z; t)$ . For the moment, choose the branch cut for  $Q^*(z; t)^{1/2}$  as in the paragraph between (9.3.16) and (9.3.17). Each function  $\mathcal{U}^*(z; t)$  is harmonic off the chosen branch cut and can be continued harmonically across it by  $-\mathcal{U}^*(z; t)$ . Moreover, it follows immediately from

their definition that the functions  $\mathcal{U}^*(z; t)$  are uniformly bounded above and below on any compact set for all considered values of the parameter  $t$ . Thus, they converge to  $\mathcal{U}(z; t^*)$  locally uniformly in  $\mathbb{C} \setminus \{a_1(t^*), b_1(t^*), a_2(t^*), b_2(t^*)\}$  as  $t \rightarrow t^*$ . Since the critical graph of  $\varpi_t^*(z)$  is the zero-level set of  $\mathcal{U}^*(z; t)$ , it converges to the critical graph of  $\varpi_{t^*}^*(z)$  in any disk  $\{|z| < R\}$ . Due to relations (9.3.17), the argument at the beginning of Section 9.3.3 also shows that the critical graph of  $\varpi_t^*(z)$  has three short critical trajectories, which, due to uniform convergence, necessarily connect  $a_1^*(t)$  to  $b_1^*(t)$ ,  $b_1^*(t)$  to  $a_2^*(t)$ , and  $a_2^*(t)$  to  $b_2^*(t)$  (the disk around  $t^*$  can be decreased if necessary). Thus, arguing as in Section 9.3.3 and using uniform convergence, we can show that Figure 9.5 also schematically represents the critical and critical orthogonal graphs of  $\varpi_t^*(z)$ . Moreover, let us now take the branch cut for  $Q^*(z; t)^{1/2}$ , say  $J_t^*$ , along the short critical trajectories of  $\varpi_t^*(z)$  connecting  $a_i^*(t)$  to  $b_i^*(t)$ . Then the shading on Figure 9.5 corresponds to regions where  $\mathcal{U}^*(z; t)$  is positive (white) and negative (gray).

Define  $\Gamma_t^*$  to be the union of the critical orthogonal trajectory  $\varpi_t^*(z)$  that connects infinity to  $a_1^*(t)$ , its short critical trajectories, and the critical orthogonal trajectory that connects  $b_2^*(t)$  to infinity. Orient it so that the positive direction proceeds from  $a_1^*(t)$  to  $b_2^*(t)$ . Let the measures  $\mu_t^*$  be given by (9.0.7) with  $Q(z; t)$  replaced by  $Q^*(z; t)$  and  $J_t$  replaced by  $J_t^*$ . Clearly, each  $\mu_t^*$  is a positive measure. Moreover, it has a unit mass by the Cauchy theorem and since  $Q^*(z; t)^{1/2} = (z^2 - t)/2 + 1/z + \mathcal{O}(1/z^2)$  due to (9.3.16), see also (9.3.3). Thus, it holds that

$$F^*(z; t) := Q^*(z; t)^{1/2} + \frac{V'(z; t)}{2} - \int \frac{d\mu_t^*(s)}{z - s} = \mathcal{O}(z^{-2})$$

as  $z \rightarrow \infty$  and  $F^*(z; t)$  is holomorphic in  $\overline{\mathbb{C}} \setminus J_t^*$ . It follows from the well known behavior of Cauchy integrals of smooth densities, see [28, Section I.8], that the traces

of  $F^*(z; t)$  on  $J_t^*$  are bounded. It further follows from the Sokhotski-Plemelj formulae, see [28, Section I.4], that

$$\begin{aligned} F_+^*(s; t) - F_-^*(s; t) \\ &= Q_+^*(s; t)^{1/2} - Q_-^*(s; t)^{1/2} - \left( \int \frac{Q_+^*(w; t)^{1/2} dw}{w - z} \frac{1}{\pi i} \right)_+ + \left( \int \frac{Q_+^*(w; t)^{1/2} dw}{w - z} \frac{1}{\pi i} \right)_- \\ &= Q_+^*(s; t)^{1/2} - Q_-^*(s; t)^{1/2} - 2Q_+^*(s; t)^{1/2} \equiv 0 \end{aligned}$$

for  $s \in J_t^*$ . Hence,  $F^*(z; t)$  is an entire function and therefore is identically zero. This observation, in particular, yields that

$$\mathcal{U}^*(z; t) := \operatorname{Re} \left( 2 \int_{b_2^*(t)}^z Q^*(s; t)^{1/2} ds \right) = \ell_t^* - \operatorname{Re}(V(z; t)) - 2U^{\mu_t^*}(z)$$

for some constant  $\ell_t^*$ , see also (9.0.8). Since  $\mathcal{U}^*(z; t)$  can be harmonically continued across  $J_t^*$  by  $-\mathcal{U}^*(z; t)$ , we get that  $\mu_t^*$  satisfies (7.1.2); that is  $J_t^*$  has the S-property in the field  $\operatorname{Re}(V(z; t))$ . Since  $\Gamma_t^* \in \mathcal{T}$ , it follows from the uniqueness part of Theorem 9.0.1(2) that  $\mu_t^* = \mu_t$ . In particular,  $a_i^*(t) = a_i(t)$  and  $b_i^*(t) = b_i(t)$ ,  $i \in \{1, 2\}$ .

Since any compact subset of  $O_{\text{two-cut}}$  can be covered by finitely many disks where the above considerations hold, the functions  $a_i(t), b_i(t)$  continuously depend on  $t \in O_{\text{two-cut}}$  and, moreover, their real and imaginary parts are real analytic functions of  $\operatorname{Re}(t)$  and  $\operatorname{Im}(t)$ .

### 9.3.5 Degeneration of the Support at the Boundary

Fix a  $t^* \in \partial O_{\text{two-cut}}$ . Then, it follows that all branch points and short trajectories remain in a compact subset of the  $z$ -plane as  $t \rightarrow t^*$  along any path. Indeed, it was shown in [53, Theorem 5.11] that for a path  $t(s) \in \overline{O_{\text{two-cut}}}$ ,  $s \in [0, 1]$  functions  $a_1(t), a_2(t), b_1(t), b_2(t)$  satisfying (9.3.9) are uniformly bounded for  $s \in [0, 1]$ . Suppose the points  $a_1(t^*), b_1(t^*), a_2(t^*), b_2(t^*)$  are distinct. Implicit function theorem and the calculation resulting in (9.3.15) implies that  $a_1(t), b_1(t), a_2(t), b_2(t)$  continuously extend to  $\partial O_{\text{two-cut}}$ . Assuming  $a_1(t^*), b_1(t^*), a_2(t^*), b_2(t^*)$  are distinct and combining this with the reasoning of Section 9.3.3 yields the following two possibilities

- (i) degenerates to one of the three critical graphs described in Figure 9.9, or
- (ii) maintains the structure described in Figure 9.5.

We start by arguing that option (i) is impossible.

**Lemma 9.3.1.** *Suppose that as  $t \rightarrow t^* \in \partial O_{\text{two-cut}}$  the points  $\{a_1(t), b_1(t), a_2(t), b_2(t)\}$  remain separated. Then, subject to all the assumptions of Chapter 9, there exists a neighborhood  $U$  of infinity such that any trajectory entering  $U_t$  within a sector defined by orthogonal critical directions must tend to infinity along this direction. In particular, short trajectories of  $\varpi_t$  must remain in a compact subset of the plane as  $t \rightarrow O_{\text{two-cut}}$ .*

*Proof.* Since  $a_1(t), a_2(t), b_1(t), b_2(t)$  remain in a compact set, there exists a neighborhood  $U = \{z \mid |z| > 1/\epsilon, \epsilon > 0\}$  such that for  $z \in U$ , we may define  $\zeta(z)$  by the equation

$$\int^z Q^{1/2}(s; t) ds = \zeta^3 + \log(\zeta) + c$$

where  $c$  is an arbitrary constant. Indeed, it follows from (10.1.2) below that if we write  $g(z) = f(z) + \log(z)$ ,  $f(z) = a_0 + \frac{a_1}{z} + \dots$  as  $z \rightarrow \infty$ . It follows from (8.1.4) that  $f(z)$  is holomorphic outside any compact set containing  $\Gamma_t[a_1, b_2]$ . Furthermore,

$$\int_{b_2(t)}^z Q^{1/2}(s; t) ds = \frac{\ell^* - V(z)}{2} + \log(z) + f(z).$$

For a fixed path  $t(s)$ ,  $s \in [0, 1]$ , there is a neighborhood  $U_t = \{z \mid |z| > R(t), R(t) > 0\}$  such that  $\frac{\ell^* - V(z)}{2} + f(z)$  is meromorphic, non-vanishing in  $U_t$ . Since we need only ensure the convergence and meromorphy of  $f(z)$ , it suffices to take  $R(t) = 2^{-1} \max_{i=1,2} \{|a_i(t)|, |b_i(t)|\}$ . Since  $R(t)$  is bounded above, we may choose an optimal radius  $R > 0$  and use it to define the neighborhood  $U = \{z \mid |z| > R\}$ . Hence, one may find a function  $\zeta'(z)$  meromorphic in  $U_t$  and with  $\zeta'(z) = z(b_0 + b_1 z^{-1} + \dots)$  such that  $\frac{\ell^* - V(z)}{2} + f(z) = (\zeta'(z))^3$ . Finally, the parameter  $\zeta(z)$  is defined by the equation

$$(\zeta'(z))^3 = (\zeta(z))^3 + \log(\zeta(z)/z)$$



for  $z \in U$ , which implies that  $\zeta(z)$  is meromorphic in  $U$  with  $\zeta(z) = z(c_0 + c_1 z^{-1} + \dots)$ . In this variable, the quadratic differential can be represented as

$$Q^{1/2}(z)dz = \left(3\zeta^2 + \frac{1}{\zeta}\right)d\zeta$$

The study of the trajectory structure of this differential was done in [64, Section 7.4] and in even more detail in [62, Theorem 3.3] and allows us to make the important conclusion: *a trajectory entering  $U$  intersects  $\partial U$  once and tends to infinity along a critical direction.*  $\square$

In fact, it holds that trajectories approaching infinity may not change their asymptotic direction as  $t \rightarrow t^*$ , since they separate domains where  $\mathcal{U}(z; t)$  changes signs and since the set  $\{\mathcal{U}(z; t) < 0\}$  (respectively,  $\{\mathcal{U}(z; t) > 0\}$ ) contain sectors of the form  $S_{\theta, \delta} \cap \{|z| > r\}$  (see Section 9.3.2) where  $r$  is independent of  $t$ . Finally, note that due to the specific topology of the critical graph of  $\varpi_t$  shown in Figure 9.5, short trajectories may not approach branch points since trajectories of polynomials differentials cannot have loops. Since tangent vectors to trajectories near branch points may not become parallel nor intersect, we conclude that option (i) is impossible.

Next, we show that option (ii) is impossible. Suppose to the contrary that (ii) were true. This yields a function  $Q^*(z; t)$  and an associated measure  $\mu^*$  defined as in (9.0.7) that satisfy the relation (9.0.6). However, as discussed in Section 9.3.4, these relations imply the S-property and characterize the equilibrium measure. Since (ii) produces a measure that is inconsistent with what was shown in [50], we conclude that option (ii) cannot hold.

From the previous discussion, we conclude that some branch points must coincide on  $\partial O_{\text{two-cut}}$ , where the corresponding critical graphs are shown in Figure 9.4. Since all branch points satisfy equations (9.3.16), it follows from the first two equations and the fact that  $0 \notin \overline{O_{\text{two-cut}}}$  that all four branch points cannot collapse to one point. Similar considerations of the first and third equation of (9.3.16) yield that we cannot simultaneously have  $a_1(t^*) = b_1(t^*)$  and  $a_2(t^*) = b_2(t^*)$ . Furthermore, the three branch points coincide only when  $t = 3 \cdot 2^{-2/3}$ ,  $t = 3 \cdot 2^{-2/3} e^{2\pi i/3}$  (the only

solutions accessible from  $O_{\text{two-cut}}$ ). To see that the degeneration is of the correct type (i.e. produces the correct critical graph from Figure 9.4), we allude to the uniqueness of the S-curves provided in Theorem 9.0.1 and the particular structure of the support and the differences in the function  $Q$ . If  $t \rightarrow t^* \in C_{\text{split}}$ , the support of  $\mu_t$  must be a union of two analytic arcs. If  $t^* \in C_{\text{birth}}^a$ , the support must be a union of two disjoint analytic arcs and  $Q(z; t)$  has a double root. Finally, if  $t^* \in C_{\text{split}} \cap C_{\text{birth}}^a$ , then  $Q(z; t)$  has a single root of order 3.

## 10. RIEMANN-HILBERT ANALYSIS: VARYING ORTHOGONALITY WITH CUBIC POTENTIAL

A version of this chapter will appear in [59].

We start our analysis with the construction of the  $g$ -function, whose properties were discussed in Proposition 9.2.2. Note that this construction is analogous to what has been carried out in the analysis of kissing polynomials in the supercritical regime, see Section 8.3

### 10.1 Proof of Proposition 9.2.2

Since the arc  $\Gamma[b_1, a_2]$  is homologous to a short critical trajectory of  $-Q(z)dz^2$  and  $\Gamma[a_1, b_1]$  is such a trajectory, see Figure 9.5, these constants  $\tau, \omega$  are indeed real. Let

$$g(z) := \int \log(z - s) d\mu(s), \quad z \in \mathbb{C} \setminus \Gamma(e^{\pi i} \infty, b_2], \quad (10.1.1)$$

where we take the principal branch of  $\log(\cdot - s)$  holomorphic outside of  $\Gamma(e^{\pi i} \infty, s]$  and  $\mu$  is the equilibrium measure defined in (9.0.7). It follows directly from definition (10.1.1) that

$$\partial_z g(z) = \int \frac{d\mu(s)}{z - s},$$

where, as usual,  $\partial_z := (\partial_x - i\partial_y)/2$ . Therefore, it can be deduced from (9.0.3) and (9.0.6) that

$$g(z) = \frac{V(z) - \ell_*}{2} + \int_{b_2}^z Q^{1/2}(s) ds = \frac{V(z) - \ell_*}{2} + \mathcal{Q}(z), \quad (10.1.2)$$

where, as usual, we take the branch  $Q^{1/2}(z) = \frac{1}{2}z^2 + \mathcal{O}(z)$ ,  $\ell_*$  is a constant such that the equality holds that  $b_2$  (notice that  $\operatorname{Re}(\ell_*) = \ell$ , see (9.0.3)), and  $\mathcal{Q}(z)$  is given by

(9.2.4). Property (9.2.6) clearly follows from (10.1.1) and (10.1.2). In the view of (10.1.2), let us define

$$\phi_e(z) := 2 \int_e^z Q^{1/2}(s) ds, \quad e \in \{a_1, b_1, a_2, b_2\}, \quad (10.1.3)$$

holomorphically in  $\mathbb{C} \setminus \Gamma(e^{\pi i} \infty, b_2]$  when  $e = b_2$ , in  $\mathbb{C} \setminus \Gamma[a_1, e^{\pi i/3} \infty)$  when  $e = a_1$ , and in  $\mathbb{C} \setminus (\Gamma(e^{\pi i} \infty, b_1) \cup \Gamma(a_2, e^{\pi i/3} \infty))$  when  $e \in \{b_1, a_2\}$ . Clearly,  $\phi_{b_2}(z) = 2\mathcal{Q}(z)$ . One can readily check that

$$\phi_{b_2}(z) = \begin{cases} \phi_{a_2}(z) \pm 2\pi i(1 - \omega), \\ \phi_{b_1}(z) \pm 2\pi i(1 - \omega) + 2\pi i\tau, \\ \phi_{a_1}(z) \pm 2\pi i + 2\pi i\tau, \end{cases} \quad z \in \mathbb{C} \setminus \Gamma, \quad (10.1.4)$$

where the plus sign is used if  $z$  lies to the left of  $\Gamma$  and the minus sign if  $z$  lies to the right of it, and

$$\phi_{b_2\pm}(s) = \begin{cases} \pm 2\pi i\mu(\Gamma[s, b_2]), & s \in \Gamma(a_2, b_2), \\ \pm 2\pi i\mu(\Gamma[s, b_2]) + 2\pi i\tau, & s \in \Gamma(a_1, b_1). \end{cases} \quad (10.1.5)$$

The jump relations in (9.2.7) now easily follow from (10.1.4) and (10.1.5).

For future use let us record that (10.1.2), (10.1.4), and (10.1.5) imply that

$$g_+(s) - g_-(s) = \begin{cases} 0, & s \in \Gamma(b_2, e^{\pi i/3} \infty), \\ \pm \phi_{b_2\pm}(s), & s \in \Gamma(a_2, b_2), \\ 2\pi i(1 - \omega), & s \in \Gamma(b_1, a_2), \\ \pm(\phi_{b_2\pm}(s) - 2\pi i\tau), & s \in \Gamma(a_1, b_1), \\ 2\pi i, & s \in \Gamma(e^{\pi i} \infty, a_1), \end{cases} \quad (10.1.6)$$

and that

$$g_+(s) + g_-(s) - V(s) + \ell_* = \begin{cases} \phi_{b_2}(s), & s \in \Gamma(b_2, e^{\pi i/3}\infty), \\ 0, & s \in \Gamma(a_2, b_2), \\ \phi_{a_2}(s), & s \in \Gamma(b_1, a_2), \\ 2\pi i\tau, & s \in \Gamma(a_1, b_1), \\ \phi_{a_1}(s) + 2\pi i\tau, & s \in \Gamma(e^{\pi i}\infty, a_1). \end{cases} \quad (10.1.7)$$

## 10.2 Local Analysis at $e \in \{a_1, b_1, a_2, b_2\}$

Given  $e \in \{a_1, b_1, a_2, b_2\}$ , let

$$U_e := \{z : |z - e| < \delta_e \rho(t)/3\}, \quad (10.2.1)$$

where  $\delta_e \in (0, 1]$  to be adjusted later and we shall specify the function  $\rho(t)$  at the end of this subsection. Set

$$J_e := U_e \cap J \quad \text{and} \quad I_e := U_e \cap (\Gamma \setminus J), \quad (10.2.2)$$

where the arcs  $J_e$  and  $I_e$  inherit their orientation from  $\Gamma$  and we assume that the value of  $\rho(t)$  is small enough so that these arcs are connected. We shall suppose that  $I_e$  is a subarc of the orthogonal critical trajectory of  $-Q(z)dz^2$  emanating from  $e$ . The latter fact and Theorem 9.1.2 yield that

$$\phi_e(s) < 0, \quad s \in I_e, \quad (10.2.3)$$

see Figure 9.5. In fact, the same reasoning shows that (10.2.3) holds not only on  $I_e$ , but on  $\Gamma(e^{\pi i}\infty, a_1)$  when  $e = a_1$ , on  $\Gamma(b_2, e^{\pi i/3}\infty)$  when  $e = b_2$ , and, for  $\text{Re}(\phi_e(z))$  on  $\Gamma(b_1, a_2)$  when  $e \in \{b_1, a_2\}$  (observe that these functions are also monotone on the respective arcs). Furthermore, each function  $\phi_e(z)$  is analytic  $U_e \setminus J_e$  and its traces on  $J_e$  satisfy

$$\phi_{e\pm}(s) = \pm 2\pi i \nu_e \mu(J_{s,e}) = 2\pi e^{\pm 3\pi i \nu_e/2} \mu(J_{s,e}), \quad (10.2.4)$$

where  $J_{s,e}$  is the subarc of  $J_e$  with endpoints  $e$  and  $s$ ,

$$\nu_e := \begin{cases} 1, & e \in \{b_1, b_2\}, \\ -1, & e \in \{a_1, a_2\}, \end{cases} \quad (10.2.5)$$

and the second equality follows from (9.0.7) and (10.1.3). Since  $|\phi_e(z)| \sim |z - e|^{3/2}$  as  $z \rightarrow e$ , it follows from (10.2.3) and (10.2.4) that we can define an analytic branch of  $(-\phi_e)^{2/3}(z)$  in  $U_e$  that is positive on  $I_e$  and satisfies  $(-\phi_e)^{2/3}(s) = -(2\pi\mu_\Gamma(J_{s,e}))^{2/3}$ ,  $s \in J_e$ . Since  $(-\phi_e)^{2/3}(z)$  has a simple zero at  $e$ , it is conformal in  $U_e$  for all radii small enough. Altogether,  $(-\phi_e)^{2/3}(z)$  maps  $e$  into the origin, is conformal in  $U_e$ , and satisfies

$$\begin{cases} (-\phi_e)^{2/3}(J_e) \subset (-\infty, 0), \\ (-\phi_e)^{2/3}(I_e) \subset (0, \infty). \end{cases} \quad (10.2.6)$$

Furthermore, if we define  $(-\phi_e)^{1/6}(z)$  to be holomorphic in  $U_e \setminus J_e$  and positive on  $I_e$ , then

$$(-\phi_e)_+^{1/6}(s) = \nu_e i (-\phi_e)_-^{1/6}(s), \quad s \in J_e. \quad (10.2.7)$$

To specify  $\rho(t)$ , let  $\rho_e(t)$  be the radius of the largest disk around  $e$  for which  $J_e, I_e$  are connected and in which  $(-\phi_e)^{2/3}(z)$  is conformal. Observe that the disk around  $e$  of radius  $\rho_e(t)$  cannot contain other endpoints of  $J$  besides  $e$ . We set  $\rho(t) := \min_e \{\rho_e(t)\}$ . Then the disks  $U_e$  in (10.2.1) are necessarily disjoint. Observe also that  $\rho(t)$  is non-zero for all  $t \in O_{\text{two-cut}}$  and continuously depends on  $t$  due to continuous dependence on  $t$  of  $\phi_e(z)$ , which in itself follows from Theorem 9.1.2 and (10.1.3).

### 10.3 Functions $A_n(z; t)$

In this section we prove Proposition 9.2.3 and discuss some related results. Below, we denote by  $\mathfrak{R}$  the Riemann surface defined in Section 10.3.1 with  $Q(z) = Q(z; t)$ . We further specify that  $\pi(\beta) = \Gamma[a_1, b_1]$ ,  $\pi(\alpha) = \Gamma[b_1, a_2]$ , and we consider the realization of  $\mathfrak{R}$  with respect to  $\Delta = \pi^{-1}(J)$ , where, as before,  $J = \Gamma[a_1, b_1] \cup \Gamma[a_2, b_2]$ .

### 10.3.1 Riemann Surface

To describe the dependence of  $a_1(t), b_1(t), a_2(t), b_2(t)$  on  $t$ , it will be convenient to work on a certain Riemann surface, rather than the plane. Below, we define this surface.

Let  $a_1, b_1, a_2, b_2$  be distinct points in  $\mathbb{C}$  and  $Q(z) = \frac{1}{4}(z-a_1)(z-b_1)(z-a_2)(z-b_2)$ . Consider

$$\mathfrak{R} := \{z := (z, w) : w^2 = Q(z)\}. \quad (10.3.1)$$

We denote by  $\pi : \mathfrak{R} \rightarrow \overline{\mathbb{C}}$  the natural projection  $\pi(z) = z$  and by  $\cdot^*$  a holomorphic involution on  $\mathfrak{R}$  acting according to the rule  $z^* = (z, -w)$ . In general, we use notation  $\mathbf{z}, \mathbf{s}, \mathbf{a}$  for point on  $\mathfrak{R}$  with natural projections  $z, s, a$ .

The function  $w(z)$  defined by  $w^2(z) = Q(z)$  is a meromorphic function on  $\mathfrak{R}$  with simple zeros at the ramification points  $\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2$ , double poles at the points on top of infinity, and is otherwise non-vanishing and finite. Fix a branch cut  $J$  for  $Q^{1/2}(z)$ . Then  $\mathfrak{R}$  can be written as  $D^{(0)} \cup \Delta \cup D^{(1)}$ , where  $\Delta := \pi^{-1}(J)$  and the domains  $D^{(k)}$  project onto  $\overline{\mathbb{C}} \setminus J$  with labels chosen so that  $2w(\mathbf{z}) = (-1)^k z^2 + \mathcal{O}(z)$  as  $\mathbf{z}$  approaches the point on top of infinity within  $D^{(k)}$ . For  $z \in \overline{\mathbb{C}} \setminus J$  we let  $z^{(k)}$  stand for a point in  $D^{(k)}$  with a natural projection  $z$ .

Denote by  $\alpha$  a cycle on  $\mathfrak{R}$  that passes through  $\mathbf{b}_1$  and  $\mathbf{a}_2$  and whose natural projection is an arc connecting  $b_1$  and  $a_2$ . We assume that  $\pi(\alpha) \cap J = \{b_1, a_2\}$  and orient  $\alpha$  towards  $\mathbf{b}_1$  within  $D^{(0)}$ . Similarly, we define  $\beta$  to be a cycle on  $\mathfrak{R}$  that passes through  $\mathbf{a}_1$  and  $\mathbf{b}_1$  and whose natural projection is an arc connecting  $a_1$  and  $b_1$ . We orient  $\beta$  so that  $\alpha, \beta$  form the right pair at  $\mathbf{b}_1$ .

The surface  $\mathfrak{R}$  has genus one. Thus, there exists a unique holomorphic differential on  $\mathfrak{R}$  normalized to have a unit period on  $\alpha$ , say  $\mathcal{H}$ . In fact, it can be explicitly expressed as

$$\mathcal{H}(z) = \left( \oint_{\alpha} w^{-1}(z) dz \right)^{-1} w^{-1}(z) dz. \quad (10.3.2)$$

We denote by  $B$  the other period of  $\mathcal{H}$  and recall (as shown by Riemann) that

$$\text{Im}(B) > 0, \quad B := \oint_{\beta} \mathcal{H}. \quad (10.3.3)$$

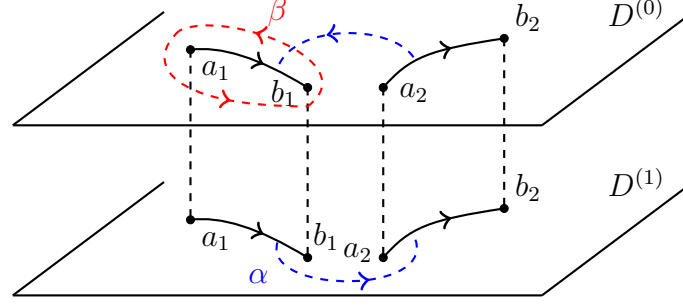


Fig. 10.1. Schematic plot of the Riemann surface  $\mathfrak{R}$  and the cycles  $\alpha$  and  $\beta$ .

Given the normalized holomorphic differential, we can define Abel's map as

$$\mathfrak{a}(z) := \int_{b_2}^z \mathcal{H}, \quad (10.3.4)$$

where we restrict  $z$  as well as the path of integration to the simply connected region  $\mathfrak{R}_{\alpha,\beta} := \mathfrak{R} \setminus \{\alpha, \beta\}$ . It is a holomorphic function in  $\mathfrak{R}_{\alpha,\beta}$  with continuous traces on  $\alpha, \beta$  away from the point of their intersection that satisfy

$$\mathfrak{a}_+(s) - \mathfrak{a}_-(s) = \begin{cases} -B, & s \in \alpha \setminus \{b_1\}, \\ 1, & s \in \beta \setminus \{b_1\}, \end{cases} \quad (10.3.5)$$

by the normalization of  $\mathcal{H}$  and the definition of  $B$ . Moreover, observe that  $\mathfrak{a}(z)$  continuously extends to  $\partial\mathfrak{R}_{\alpha,\beta}$ , the topological boundary of  $\mathfrak{R}_{\alpha,\beta}$ . Similarly to (9.2.5), let us set

$$\tau = \frac{1}{2\pi i} \oint_{\alpha} w(s) ds \quad \text{and} \quad \omega = -\frac{1}{2\pi i} \oint_{\beta} w(s) ds. \quad (10.3.6)$$

It readily follows from (10.3.5) and (10.3.6) that

$$\oint_{\partial\mathfrak{R}_{\alpha,\beta}} (w\mathfrak{a})(s) ds = \oint_{\beta} w(s)(\mathfrak{a}_+ - \mathfrak{a}_-)(s) ds - \oint_{\alpha} w(s)(\mathfrak{a}_+ - \mathfrak{a}_-)(s) ds = -2\pi i(\omega + B\tau),$$

where  $\partial\mathfrak{R}_{\alpha,\beta}$  is oriented counter-clockwise, that is,  $\mathfrak{R}_{\alpha,\beta}$  remains on the left when  $\partial\mathfrak{R}_{\alpha,\beta}$  is traversed in the positive direction. On the other hand, the function  $(w\mathfrak{a})(z)$  is meromorphic in  $\mathfrak{R}_{\alpha,\beta}$  with only two singularities, both polar, at  $\infty^{(0)}$  and  $\infty^{(1)}$ .



Moreover, since  $w(\mathbf{z}^*) = -w(\mathbf{z})$  and  $\mathfrak{a}(\mathbf{z}^*) = -\mathfrak{a}(\mathbf{z})$ , the residues at those poles coincide. Therefore, it holds that

$$\omega + \mathbf{B}\tau = -\frac{1}{2\pi i} \oint_{\partial \mathfrak{R}_{\alpha, \beta}} (w\mathfrak{a})(s) ds = -2\text{res}_{z=\infty(0)}(w\mathfrak{a})(\mathbf{z}). \quad (10.3.7)$$

As we show in the next section, the above residue has a rather explicit expression when  $Q(z)$ , which, obviously, defines  $w(\mathbf{z})$ , satisfies (9.0.6).

### 10.3.2 Jacobi Inversion Problem

Let  $\mathbf{B}$  be given by (10.3.3). Denote by  $\text{Jac}(\mathfrak{R}) := \mathbb{C}/\{\mathbb{Z} + \mathbf{B}\mathbb{Z}\}$  the Jacobi variety of  $\mathfrak{R}$ . We shall represent elements of  $\text{Jac}(\mathfrak{R})$  as equivalence classes  $[s] = \{s + j + m\mathbf{B} : j, m \in \mathbb{Z}\}$ , where  $s \in \mathbb{C}$ . Since  $\mathfrak{R}$  has genus one, Abel's map

$$\mathbf{z} \in \mathfrak{R} \mapsto \left[ \int_{b_2}^{\mathbf{z}} \mathcal{H} \right] \in \text{Jac}(\mathfrak{R}) \quad (10.3.8)$$

is a holomorphic bijection. Thus, given  $s \in \mathbb{C}$  there exists a unique  $\mathbf{z}_{[s]} \in \mathfrak{R}$  such that  $\left[ \int_{b_2}^{\mathbf{z}_{[s]}} \mathcal{H} \right] = [s]$ .

**Proposition 10.3.1.** *Let  $\tau, \omega$  be given by (9.2.5) and  $\varsigma$  by (9.2.1). Further, let  $\{N_n\}$  be a sequence as in Theorem 9.2.5. Denote by  $\mathbf{z}_{n,k} = \mathbf{z}_{n,k}(t)$  the unique solution  $\mathbf{z}_{[s_{n,k}(t)]}$  of the Jacobi inversion problem with*

$$s_{n,k}(t) := \int_{b_2}^{p^{(k)}} \mathcal{H} + (n - N_n)\varsigma + (\omega + \mathbf{B}\tau)n, \quad p = p(t) := \frac{b_1b_2 - a_1a_2}{(b_2 - a_2) + (b_1 - a_1)}, \quad (10.3.9)$$

$k \in \{0, 1\}$ . Then for any subsequence  $\mathbb{N}_*$  the point  $\infty^{(0)}$  is a topological limit point of  $\{\mathbf{z}_{n,1}\}_{n \in \mathbb{N}_*}$  if and only if  $\infty^{(1)}$  is a topological limit point of  $\{\mathbf{z}_{n,0}\}_{n \in \mathbb{N}_*}$ .

To prove the first claim, define

$$\gamma(z) := \left( \frac{z - b_2}{z - a_2} \frac{z - b_1}{z - a_1} \right)^{1/4}, \quad z \in \overline{\mathbb{C}} \setminus J, \quad (10.3.10)$$

where  $\gamma(z)$  is holomorphic off  $J$  and the branch is chosen so that  $\gamma(\infty) = 1$ . Further, set

$$A(z) = \frac{\gamma(z) + \gamma^{-1}(z)}{2} \quad \text{and} \quad B(z) := \frac{\gamma(z) - \gamma^{-1}(z)}{-2i}. \quad (10.3.11)$$

Observe that the function  $A(z)$  was already defined in (9.2.8). The proof of Proposition 10.3.1 is exactly the same as the proof Proposition 7.3.3 with the correct formulas for  $\gamma, A, B$ , see Appendix B. The behavior of the points  $\mathbf{z}_{n,k}$  with respect to  $n$  can be extremely chaotic. Assuming that  $n - N_n$  is constant, it is known that if the numbers  $\omega$  and  $\tau$  are rational, then there exist only finitely many distinct points  $\mathbf{z}_{n,k}$ ; when  $\omega$  and  $\tau$  are irrational, all the points  $\mathbf{z}_{n,k}$  are distinct, lie on a Jordan curve if 1,  $\omega$  and  $\tau$  are rationally dependent, and are dense on the whole surface  $\mathfrak{R}$  otherwise [68].

### 10.3.3 Subsequences $\mathbb{N}(t, \epsilon)$

As we shall show further below, the functions  $\Theta_n(z; t)$  from Proposition 9.2.3 vanish at  $\mathbf{z}_{n,1}$  when it belongs to  $D^{(0)}$  and do not vanish at all when  $\mathbf{z}_{n,1}$  does not belong to  $D^{(0)}$ . Hence, the subsequences  $\mathbb{N}(\epsilon) = \mathbb{N}(t, \epsilon)$  from Proposition 9.2.3 can be equivalently defined as

$$\mathbb{N}(\epsilon) := \{n \in \mathbb{N} : \mathbf{z}_{n,1} \notin D^{(0)} \cap \pi^{-1}(\{|z| \geq 1/\epsilon\})\}.$$

Set  $\mathbb{K} := \{k \in \mathbb{Z} : r_k = 0\}$ , where  $r_k := \min_{j,m \in \mathbb{Z}} |(1-k)\varsigma + \omega + \mathbf{B}\tau + j + \mathbf{B}m|$ . Let  $k = N_{n+1} - N_n$ . Then it follows from (10.3.9) that

$$[\mathbf{a}(\mathbf{z}_{n+1,1}) - \mathbf{a}(\mathbf{z}_{n,1})] = [(1-k)\varsigma + \omega + \mathbf{B}\tau]. \quad (10.3.12)$$

If  $r_k = 0$ , then  $[(1-k)\varsigma + \omega + \mathbf{B}\tau] = [0]$  and  $\mathbf{z}_{n+1,1} = \mathbf{z}_{n,1}$  due to the unique solvability of the Jacobi inversion problem. Thus,  $[(1-k)\varsigma + \omega + \mathbf{B}\tau] = [0]$ , means that both triples  $\omega, x, 1$  and  $\tau, y, 1$  are rationally dependent,  $\varsigma = x + \mathbf{B}y$ . If  $r_k > 0$ , choose  $\epsilon_k$  to be the largest positive number such that  $U_{\epsilon_k} \subset \mathfrak{R}_{\alpha,\beta}$  and  $|\mathbf{a}(\mathbf{z}) - \mathbf{a}(\infty^{(0)})| < r_k/3$ , where  $U_\epsilon := D^{(0)} \cap \pi^{-1}(\{|z| > 1/\epsilon\})$ . If neither  $n$  nor  $n+1$  were to belong to  $\mathbb{N}(\epsilon_k)$ , then we would have that  $\mathbf{z}_{n,1}, \mathbf{z}_{n+1,1} \in U_{\epsilon_k}$ . This, in conjunction with (10.3.12), would imply that  $0 < r_k \leq |\mathbf{a}(\mathbf{z}_{n+1,1}) - \mathbf{a}(\mathbf{z}_{n,1})| \leq (2/3)r_k$ , which, of course, is impossible. Altogether, we proved the following.

**Lemma 10.3.2.** *If  $\mathbb{K} = \emptyset$  and  $|n - N_n| \leq N_*$ , then at least one of the integers  $n, n + 1$  belongs to  $\mathbb{N}(\epsilon)$  for all  $\epsilon \leq \min_{|k| \leq 2N_*+1} \epsilon_k$ . If one of the triples  $\omega, x, 1$  or  $\tau, y, 1$  is rationally independent, then  $\mathbb{K} = \emptyset$ .*

If there exists an infinite subsequence  $\{n_l\}_{l \in \mathbb{N}}$  such that  $N_{n_l+1} - N_{n_l} \in \{0, 1\}$  for all  $l \in \mathbb{N}$ , then at least one of the integers  $n_l, n_l + 1$  belongs to  $\mathbb{N}(\epsilon)$  for all  $\epsilon \leq \min\{\epsilon_0, \epsilon_1\}$  if we show that  $r_0, r_1 > 0$ . This is true since  $[\omega + \mathbf{B}\tau] \neq [0]$  and  $[\varsigma + \omega + \mathbf{B}\tau] \neq [0]$ , where the first conclusion holds since  $\omega = \mu_t(J_{t,1}) \in (0, 1)$  by the very definition in (9.2.5) and the second one holds by (10.3.7) and the unique solvability of the Jacobi inversion problem.

Assume that  $k', k'' \in \mathbb{K}$ ,  $k' \neq k''$ . Then it follows from (10.3.12) that  $[\omega + \mathbf{B}\tau] = [(k' - 1)\varsigma]$  and  $[(k'' - k')\varsigma] = [0]$ . The latter relation implies the first representation in (9.2.10) while the former gives the other two. It is easy to see in this case that  $\mathbb{K} = k' + d\mathbb{Z}$ . That is, if  $\mathbb{K}$  has at least two elements, then it is an arithmetic progression,  $\omega, \tau$  are rational numbers,  $\varsigma$  has rational coordinates in the basis  $1, \mathbf{B}$ , and the second and third relations of (9.2.10) must be satisfied. Thus, we can claim the following.

**Lemma 10.3.3.** *If  $\mathbb{K} = \{k'\}$  and  $|n - N_n| \leq N_*$ , then there exists an infinite subsequence  $\{n_l\}$  such that  $N_{n_l+1} - N_{n_l} \neq k'$  (recall that  $k' \neq 0$ ). Hence, at least one of the integers  $n_l, n_l + 1$  belongs to  $\mathbb{N}(\epsilon)$  for all  $\epsilon \leq \min_{|k| \leq 2N_*+1, k \neq k'} \epsilon_k$ . If not all numbers  $\omega, \tau, x, y$  are rational or they all rational but the second and third relations of (9.2.10) do not hold, then either  $\mathbb{K} = \emptyset$  or  $\mathbb{K} = \{k'\}$ .*

Assume now that all three relations of (9.2.10) take place. That is,  $[\omega + \mathbf{B}\tau] = [(k - 1)\varsigma]$  and  $[d\varsigma] = [0]$  for some integers  $k, d$ . It follows from (B.3.5) that

$$\int_{b_2}^{p^{(1)}} \mathcal{H} = \frac{1}{2} \int_{p^{(0)}}^{p^{(1)}} \mathcal{H} = \frac{1}{2} \left( \int_{\infty^{(1)}}^{\infty^{(0)}} \mathcal{H} + j + \mathbf{B}m \right) = \int_{b_2}^{\infty^{(0)}} \mathcal{H} + \frac{j + \mathbf{B}m}{2}$$

for some  $j, m \in \mathbb{Z}$ , where we use involution-symmetric paths of integration. Notice that  $j, m$  cannot be simultaneously even as this would contradict unique solvability of the Jacobi inversion problem. Hence,

$$\left[ \int_{\infty^{(0)}}^{p^{(1)}} \mathcal{H} \right] = \left[ \frac{\kappa_1 + \mathbf{B}\kappa_2}{2} \right]$$

for some  $\kappa_1, \kappa_2 \in \{0, 1\}$ ,  $\kappa_1 + \kappa_2 > 0$ . Therefore, adding  $\int_{\infty^{(0)}}^{b_2} \mathcal{H}$  to both sides of (10.3.9) gives us

$$\left[ \int_{\infty^{(0)}}^{z_{n,1}} \mathcal{H} \right] = \left[ \frac{\kappa_1 + \mathbf{B}\kappa_2}{2} + (n - N_n)\varsigma + (\omega + \mathbf{B}\tau)n \right] = \left[ \frac{\kappa_1 + \mathbf{B}\kappa_2}{2} + (nk - N_n)\varsigma \right]. \quad (10.3.13)$$

Since  $\varsigma$  has rational coordinates in the basis  $1, \mathbf{B}$  with denominator  $d$ , the right-hand side of (10.3.13) has at most  $d$  distinct values that depend only on  $\varrho \in \{0, \dots, d-1\}$ , the remainder of the division of  $nk - N_n$  by  $d$ . Let  $\mathbf{z}_\varrho$ ,  $\varrho \in \{0, \dots, d-1\}$ , be such that

$$\left[ \int_{\infty^{(0)}}^{z_\varrho} \mathcal{H} \right] = \left[ \frac{\kappa_1 + \mathbf{B}\kappa_2}{2} + \varrho\varsigma \right].$$

Clearly,  $\{\mathbf{z}_{n,1}\}_{n \in \mathbb{N}} \subseteq \{\mathbf{z}_\varrho\}_{\varrho=0}^{d-1}$ . Thus, it only remains to investigate when  $\mathbf{z}_\varrho = \infty^{(0)}$ , or equivalently, when  $[(\kappa_1 + \mathbf{B}\kappa_2)/2 + \varrho\varsigma] = [0]$ . Trivially, it must hold that  $\varrho = d(2l_j - \kappa_j)/(2i_j)$ ,  $j \in \{1, 2\}$ , for some  $l_1, l_2 \in \mathbb{Z}$ . Since one of the pairs  $(i_j, d)$  is co-prime and  $\varrho \in \{0, \dots, d-1\}$ , this is possible only if  $\varrho = 0$  or  $\varrho = d/2$  (in the second case, of course,  $d$  must be even).

**Lemma 10.3.4.** *If all three relations of (9.2.10) take place, then Jacobi inversion problem (10.3.9) for  $\mathbf{z}_{n,1}$  has only finitely many distinct solutions and  $\infty^{(0)}$  is one of them if and only if  $nk - N_n$  is divisible by either  $d$  or  $d/2$  (in this case  $d$  must be even).*

### 10.3.4 Theta Functions

In this section we will prove Proposition 9.2.3. Recall that Abel's map (10.3.8) is essentially carried out by the function  $\mathbf{a}(\mathbf{z})$  defined in (10.3.4). We shall consider the

extension  $\tilde{\mathbf{a}}(\mathbf{z})$  of  $\mathbf{a}(\mathbf{z})$  to the whole surface  $\mathfrak{R}$  defined by setting  $\tilde{\mathbf{a}}(\mathbf{s}) := \mathbf{a}_+(\mathbf{s})$  for  $\mathbf{s} \in \boldsymbol{\alpha}$  and  $\mathbf{s} \in \boldsymbol{\beta} \setminus \{\mathbf{b}_1\}$ . Given such an extension and (10.3.9), there exist unique integers  $j_{n,k}, m_{n,k}$  such that

$$\tilde{\mathbf{a}}(\mathbf{z}_{n,k}) = \tilde{\mathbf{a}}(p^{(k)}) + (n - N_n)\varsigma + (\omega + \mathbf{B}\tau)n + j_{n,k} + \mathbf{B}m_{n,k}, \quad k \in \{0, 1\}. \quad (10.3.14)$$

Denote by  $\theta(u)$  the theta function of one variable associated with  $\mathbf{B}$ . That is,

$$\theta(u) = \sum_{k \in \mathbb{Z}} \exp \{ \pi i \mathbf{B} k^2 + 2\pi i u k \}, \quad u \in \mathbb{C}.$$

The function  $\theta(u)$  is holomorphic in  $\mathbb{C}$  and enjoys the following periodicity properties:

$$\theta(u + j + \mathbf{B}m) = \exp \{ -\pi i \mathbf{B} m^2 - 2\pi i u m \} \theta(u), \quad j, m \in \mathbb{Z}. \quad (10.3.15)$$

It is also known that  $\theta(u)$  vanishes only at the points of the lattice  $\left[ \frac{\mathbf{B}+1}{2} \right]$ .

Now, we define  $\Theta_n(\mathbf{z}; t)$  from Proposition 9.2.3 by  $\Theta_n(\mathbf{z}; t) := \Theta_{n,1}^{(0)}(\mathbf{z})$ , where

$$\Theta_{n,k}(\mathbf{z}) = \exp \{ -2\pi i (m_{n,k} + \tau n) \mathbf{a}(\mathbf{z}) \} \frac{\theta(\mathbf{a}(\mathbf{z}) - \tilde{\mathbf{a}}(\mathbf{z}_{n,k}) - \frac{\mathbf{B}+1}{2})}{\theta(\mathbf{a}(\mathbf{z}) - \tilde{\mathbf{a}}(p^{(k)}) - \frac{\mathbf{B}+1}{2})} \quad (10.3.16)$$

and  $F^{(i)}(\mathbf{z})$ ,  $i \in \{0, 1\}$ , stands for the pull-back under  $\pi(\mathbf{z})$  of a function  $F(\mathbf{z})$  from  $D^{(i)}$  into  $\overline{\mathbb{C}} \setminus J$ .

The functions  $\Theta_{n,k}(\mathbf{z})$  are meromorphic on  $\mathfrak{R}_{\boldsymbol{\alpha}, \boldsymbol{\beta}}$  with exactly one pole, which is simple and located at  $p^{(k)}$ , and exactly one zero, which is also simple and located at  $\mathbf{z}_{n,k}$  (observe that the functions  $\Theta_{n,k}(\mathbf{z})$  can be analytically continued as multiplicatively multivalued functions on the whole surface  $\mathfrak{R}$ ; thus, we can talk about simplicity of a pole or zero regardless whether it belongs to the cycles of a homology basis or not). Moreover, according to (10.3.5), (10.3.14), and (10.3.15), they possess continuous traces on  $\boldsymbol{\alpha}, \boldsymbol{\beta}$  away from  $\mathbf{b}_1$  that satisfy

$$\Theta_{n,k+}(\mathbf{s}) = \Theta_{n,k-}(\mathbf{s}) \begin{cases} \exp \{ -2\pi i (\omega n + (n - N_n)\varsigma) \}, & \mathbf{s} \in \boldsymbol{\alpha} \setminus \{\mathbf{b}_1\}, \\ \exp \{ -2\pi i \tau n \}, & \mathbf{s} \in \boldsymbol{\beta} \setminus \{\mathbf{b}_1\}. \end{cases} \quad (10.3.17)$$

To discuss boundedness properties of  $\Theta_{n,k}(\mathbf{z})$  and for the asymptotic analysis in the following section it will be convenient to define

$$M_{n,0}(\mathbf{z}) = \Theta_{n,0}(\mathbf{z}) \begin{cases} B(z), & \mathbf{z} \in D^{(0)}, \\ A(z), & \mathbf{z} \in D^{(1)}, \end{cases} \quad \text{and} \quad M_{n,1}(\mathbf{z}) = \Theta_{n,1}(\mathbf{z}) \begin{cases} A(z), & \mathbf{z} \in D^{(0)}, \\ -B(z), & \mathbf{z} \in D^{(1)}. \end{cases} \quad (10.3.18)$$

These functions are holomorphic on  $\mathfrak{R} \setminus \{\alpha \cup \beta \cup \Delta\}$  since the pole of  $\Theta_{n,k}(\mathbf{z})$  is canceled by the zero of  $\beta(z)$ . Each function  $M_{n,k}(\mathbf{z})$  has exactly two zeros, namely,  $\mathbf{z}_{n,k}$  and  $\infty^{(k)}$ . It follows from (B.3.3) and (10.3.17) that

$$\begin{cases} M_{n,k\pm}^{(0)}(s) = \mp(-1)^k M_{n,k\mp}^{(1)}(s), & s \in \Gamma(a_2, b_2), \\ M_{n,k\pm}^{(0)}(s) = \mp(-1)^k e^{-2\pi i \tau n} M_{n,k\mp}^{(1)}(s), & s \in \Gamma(a_1, b_1), \\ M_{n,k\pm}^{(i)}(s) = e^{(-1)^i 2\pi i (n\omega + (n-N_n)\varsigma)} M_{n,k\mp}^{(i)}(s), & s \in \Gamma(b_1, a_2). \end{cases} \quad (10.3.19)$$

It further follows from (B.3.1) and (B.3.2) that  $|M_{n,k}(\mathbf{z})| \sim |z - e|^{-1/4}$  as  $\mathbf{z} \rightarrow \mathbf{e} \in \mathbf{E} = \{\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2\}$  unless  $\mathbf{z}_{n,k}$  coincides with  $\mathbf{e}$  in which case the exponent becomes  $1/4$ . Assume now that there exists  $N_* \geq 0$  such that  $|n - N_n| \leq N_*$  for all  $n \in \mathbb{N}$ . Then for each  $\delta > 0$  there exists  $C(\delta, N_*)$  independent of  $n$  such that

$$|M_{n,k}(\mathbf{z})| \leq C(\delta, N_*), \quad \mathbf{z} \in O_\delta := \mathfrak{R} \setminus \cup_{\mathbf{e} \in \mathbf{E}} \pi^{-1}\{|z - e| < \delta\}. \quad (10.3.20)$$

Indeed, let  $O_\delta^{(i)} := \pi^{-1}(O_\delta) \cap (D^{(i)} \setminus \alpha)$ ,  $i \in \{0, 1\}$ . Observe that  $\{\log |M_{n,k}(\mathbf{z})|\}$  is a family of subharmonic functions in  $O_\delta^{(i)}$  (the jump of  $M_{n,k}(\mathbf{z})$  is unimodular on  $\beta$ ). By the maximum principle for subharmonic functions,  $\log |M_{n,k}(\mathbf{z})|$  reaches its maximum on  $\partial O_\delta^{(i)}$ , where the maximum is clearly finite. Since the sequence  $\{n - N_n\}$  is bounded by assumption and the range of  $\tilde{\mathbf{a}}(\mathbf{z})$  is bounded by construction, so is the sequences  $\{m_{n,k} + \tau n\}$  and  $\{j_{n,k} + \omega n\}$ , see (10.3.14) and recall that  $j_{n,k} + \omega n$  are real and  $\text{Im}(\mathbf{B}) > 0$ . Thus, any limit point of  $\{\log |M_{n,k}(\mathbf{z})|\}$  is obtained by taking simultaneous limit points of  $\{n - N_n\}$ ,  $\{m_{n,k} + \tau n\}$ , and  $\{j_{n,k} + \omega n\}$ , computing the corresponding solution  $\mathbf{z}_k$  of the Jacobi inversion problem (10.3.14) and plugging all of these quantities into the right-hand side of (10.3.16). Hence, all these limit functions

are also bounded above on the closure of  $O_\delta^{(i)}$ , which proves (10.3.20). Finally, it holds that

$$|M_{n,k}(\infty^{(1-k)})| \geq c_\epsilon, \quad n \in \mathbb{N}(\epsilon), \quad (10.3.21)$$

for some constant  $c_\epsilon > 0$  and all  $\epsilon > 0$  small enough by a similar compactness argument combined with the definition of  $\mathbb{N}(\epsilon)$  in Proposition 9.2.3, and the observation that  $M_{n,k}(\mathbf{z})$  is non-zero at  $\infty^{(1-k)}$  when  $\infty^{(1-k)} \neq z_{n,k}$ .

## 10.4 Asymptotic Analysis

With  $\Gamma_t$  as it was defined at the beginning of Section 9.2, we denote by  $\Gamma[z_1, z_2]$ , where  $z_1, z_2 \in \Gamma_t$ , the arc of  $\Gamma_t$  connecting  $z_1, z_2$ . This is a slight abuse of notation since  $I_t$  is not entirely contained in a union of critical trajectories, but will be convenient for the discussion ahead.

### 10.4.1 Initial Riemann-Hilbert Problem

As agreed before, we omit the dependence on  $t$ . We remind the reader of the initial Riemann-Hilbert problems for orthogonal polynomials, RHP- $\mathbf{Y}$  (see Chapter 4):

- (a)  $\mathbf{Y}(z)$  is analytic in  $\mathbb{C} \setminus \Gamma$  and  $\lim_{\mathbb{C} \setminus \Gamma \ni z \rightarrow \infty} \mathbf{Y}(z) z^{-n\sigma_3} = \mathbf{I}$ ;
- (b)  $\mathbf{Y}(z)$  has continuous traces on  $\Gamma \setminus \{a_1, b_1, a_2, b_2\}$  that satisfy

$$\mathbf{Y}_+(s) = \mathbf{Y}_-(s) \begin{pmatrix} 1 & e^{-N_n V(s)} \\ 0 & 1 \end{pmatrix},$$

where, as before,  $V(z)$  is given by (9.0.4).

The connection of **RHP- $\mathbf{Y}$**  to orthogonal polynomials was first demonstrated by Fokas, Its, and Kitaev in [35, 36] and lies in the following. If the solution of **RHP- $\mathbf{Y}$**  exists, then it is necessarily of the form

$$\mathbf{Y}(z) = \begin{pmatrix} P_n(z) & (\mathcal{C}P_n e^{-N_n V})(z) \\ -\frac{2\pi i}{h_{n-1}} P_{n-1}(z) & -\frac{2\pi i}{h_{n-1}} (\mathcal{C}P_{n-1} e^{-N_{n-1} V})(z) \end{pmatrix}, \quad (10.4.1)$$

where  $P_n(z) = P_n(z; t, N_n)$  are the polynomial satisfying orthogonality relations (9.0.1),  $h_n = h_n(t, N_n)$  are the constants that appear in the three-term recurrence relation (cf. Section 2.2.1)

$$zP_n(z; t, N) = P_{n+1}(z; t, N) + \beta_n(t, N)P_n(z; t, N) + \gamma_n^2(t, N)P_{n-1}(z; t, N), \quad (10.4.2)$$

granted all the polynomials in (10.4.2) have prescribed degrees, where

$$\begin{cases} \gamma_n^2(t, N) &= h_n(t, N)/h_{n-1}(t, N), \\ h_n(t, N) &= \int_{\Gamma} P_n^2(z; t, N) e^{-NV(z; t)} dz. \end{cases} \quad (10.4.3)$$

and  $\mathcal{C}f(z)$  is the Cauchy transform of a function  $f$  given on  $\Gamma$ , i.e.,

$$(\mathcal{C}f)(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{f(s)}{s - z} ds.$$

Observe that if  $P_n(z; t, N) = P_{n+1}(z; t, N)$  with both polynomials having degree  $n$ , then  $h_n(t, N) = 0$  and  $h_{n+1}(t, N) = \infty$ . More generally, it holds that  $h_n(t, N)$  is a meromorphic function of  $t$  and so is  $\gamma_n^2(t, N)$ .

Below, we show the solvability of **RHP- $\mathbf{Y}$**  for all  $n \in \mathbb{N}(t, \epsilon)$  large enough following the framework of the steepest descent analysis introduced by Dieft and Zhou [42]. The latter lies in a series of transformations which reduce **RHP- $\mathbf{Y}$**  to a problem with jumps asymptotically close to identity.

#### 10.4.2 Renormalized Riemann-Hilbert Problem

Suppose that  $\mathbf{Y}(z)$  is a solution of **RHP- $\mathbf{Y}$** . Put

$$\mathbf{T}(z) := e^{n\ell_*\sigma_3/2} \mathbf{Y}(z) e^{-n(g(z) + \ell_*/2)\sigma_3}, \quad (10.4.4)$$



where the function  $g(z)$  is defined by (10.1.1) and  $\ell_*$  appeared in (10.1.2). Then

$$\mathbf{T}_+(s) = \mathbf{T}_-(s) \begin{pmatrix} e^{-n(g_+(s)-g_-(s))} & e^{n(g_+(s)+g_-(s)-V(s)+\ell_*)+(n-N_n)V(s)} \\ 0 & e^{-n(g_-(s)-g_+(s))} \end{pmatrix},$$

$s \in \Gamma$ , and therefore we deduce from (9.2.6), (10.1.6), and (10.1.7) that  $\mathbf{T}(z)$  solves RHP- $\mathbf{T}$ :

(a)  $\mathbf{T}(z)$  is analytic in  $\mathbb{C} \setminus \Gamma$  and  $\lim_{\mathbb{C} \setminus \Gamma \ni z \rightarrow \infty} \mathbf{T}(z) = \mathbf{I}$ ;

(b)  $\mathbf{T}(z)$  has continuous traces on  $\Gamma \setminus \{a_1, b_1, a_2, b_2\}$  that satisfy

$$\mathbf{T}_+(s) = \mathbf{T}_-(s) \begin{cases} \begin{pmatrix} 1 & e^{n(2\pi i \tau + \phi_{a_1}(s)) + (n-N_n)V(s)} \\ 0 & 1 \end{pmatrix}, & s \in \Gamma(e^{i\pi} \infty, a_1), \\ \begin{pmatrix} 1 & e^{n\phi_{b_2}(s) + (n-N_n)V(s)} \\ 0 & 1 \end{pmatrix}, & s \in \Gamma(b_2, e^{\pi i/3} \infty), \\ \begin{pmatrix} e^{2\pi i \omega n} & e^{n\phi_{a_2}(s) + (n-N_n)V(s)} \\ 0 & e^{-2\pi i \omega n} \end{pmatrix}, & s \in \Gamma(b_1, a_2), \end{cases}$$

and

$$\mathbf{T}_+(s) = \mathbf{T}_-(s) \begin{cases} \begin{pmatrix} e^{-n\phi_{b_2+}(s)} & e^{(n-N_n)V(s)} \\ 0 & e^{-n\phi_{b_2-}(s)} \end{pmatrix}, & s \in \Gamma(a_2, b_2), \\ \begin{pmatrix} e^{-n(\phi_{b_2+}(s)-2\pi i \tau)} & e^{2\pi i \tau n + (n-N_n)V(s)} \\ 0 & e^{-n(\phi_{b_2-}(s)-2\pi i \tau)} \end{pmatrix}, & s \in \Gamma(a_1, b_1). \end{cases}$$

Clearly, if RHP- $\mathbf{T}$  is solvable and  $\mathbf{T}(z)$  is the solution, then by inverting (10.4.4) one obtains a matrix  $\mathbf{Y}(z)$  that solves RHP- $\mathbf{Y}$ .

### 10.4.3 Lens Opening

As usual in the steepest descent analysis of matrix Riemann-Hilbert problems for orthogonal polynomials, the next step is based on the identity

$$\begin{pmatrix} e^{-n(\phi_{b_2+}(s)-C)} & e^{nC+(n-N_n)V(s)} \\ 0 & e^{-n(\phi_{b_2-}(s)-C)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{-n\phi_{b_2-}(s)-(n-N_n)V(s)} & 1 \end{pmatrix} \times \\ \begin{pmatrix} 0 & e^{nC+(n-N_n)V(s)} \\ -e^{-nC-(n-N_n)V(s)} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{-n\phi_{b_2+}(s)-(n-N_n)V(s)} & 1 \end{pmatrix}$$

that follows from (10.1.5), where  $C = 2\pi i\tau$  when  $s \in \Gamma(a_1, b_1)$  and  $C = 0$  when  $s \in \Gamma(a_2, b_2)$ . To carry it out, we shall introduce two additional system of arcs.

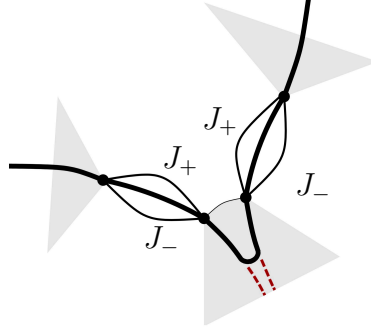


Fig. 10.2. The thick curves represent  $\Gamma$  and thinner black curves represent  $J_{\pm}$ . The shaded part represents regions where  $\text{Re}(\phi_e(z)) < 0$ . The dashed lines represent critical orthogonal trajectories.

Denote by  $J_{\pm}$  smooth homotopic deformations of  $J_{\Gamma}$  within the region  $\text{Re}(\phi_{b_2}(z)) > 0$  such that  $J_+$  lies to the left and  $J_-$  to the right of  $J_{\Gamma}$ , see Figure 10.2. We shall fix the way these arcs emanate from  $e \in \{a_1, b_1, a_2, b_2\}$ . Namely, let  $U_e$  be given by (10.2.1) and  $(-\phi_e)^{2/3}(z)$  be as in (10.2.6). Then we require that

$$\arg((-\phi_e)^{2/3}(z)) = \pm\nu_e(2\pi/3), \quad z \in U_e \cap J_{\pm}, \quad (10.4.5)$$

where  $\nu_e$  is defined by (10.2.5). This requirement always can be fulfilled due to conformality  $(-\phi_e)^{2/3}(z)$  in  $U_e$  and the choice of the branch in (10.2.6).

Denote by  $O_{\pm}$  the open sets delimited by  $J_{\pm}$  and  $J_{\Gamma}$ . Set

$$\mathbf{S}(z) := \mathbf{T}(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ \mp e^{-n\phi_{b_2}(z) - (n-N_n)V(z)} & 1 \end{pmatrix}, & z \in O_{\pm}, \\ \mathbf{I}, & \text{otherwise.} \end{cases} \quad (10.4.6)$$

Then, if  $\mathbf{T}(z)$  solves **RHP-T**,  $\mathbf{S}(z)$  solves **RHP-S**:

- (a)  $\mathbf{S}(z)$  is analytic in  $\mathbb{C} \setminus (\Gamma \cup J_+ \cup J_-)$  and  $\lim_{\mathbb{C} \setminus \Gamma \ni z \rightarrow \infty} \mathbf{S}(z) = \mathbf{I}$ ;
- (b)  $\mathbf{S}(z)$  has continuous traces on  $\Gamma \setminus \{a_1, b_1, a_2, b_2\}$  that satisfy **RHP-T**(b) on  $\Gamma(e^{\pi i} \infty, a_1)$ ,  $\Gamma(b_1, a_2)$ , and  $\Gamma(b_2, e^{\pi i/3} \infty)$ , as well as

$$\mathbf{S}_+(s) = \mathbf{S}_-(s) \begin{cases} \begin{pmatrix} 0 & e^{(n-N_n)V(s)} \\ -e^{-(n-N_n)V(s)} & 0 \end{pmatrix}, & s \in \Gamma(a_2, b_2), \\ \begin{pmatrix} 0 & e^{2\pi i \tau n + (n-N_n)V(s)} \\ -e^{-2\pi i \tau n - (n-N_n)V(s)} & 0 \end{pmatrix}, & s \in \Gamma(a_1, b_1), \\ \begin{pmatrix} 1 & 0 \\ e^{-n\phi_{b_2}(s) - (n-N_n)V(s)} & 1 \end{pmatrix}, & s \in J_{\pm}. \end{cases}$$

As before, since transformation (10.4.6) is invertible, a solution of **RHP-S** yields a solution of **RHP-T**.

#### 10.4.4 Global Parametrix

The Riemann-Hilbert problem for the global parametrix is obtained from **RHP-S** by removing the quantities that are asymptotically zero from the jump matrices in **RHP-S**(b). The latter can be easily identified with the help of (10.1.4) and by recalling that the constant  $\tau$  is real. Thus, we are seeking the solution of **RHP-N**:

- (a)  $\mathbf{N}(z)$  is analytic in  $\overline{\mathbb{C}} \setminus \Gamma[a_1, b_2]$  and  $\mathbf{N}(\infty) = \mathbf{I}$ ;

(b)  $\mathbf{N}(z)$  has continuous traces on  $\Gamma(a_1, b_2) \setminus \{b_1, a_2\}$  that satisfy

$$\mathbf{N}_+(s) = \mathbf{N}_-(s) \begin{cases} \begin{pmatrix} 0 & e^{(n-N_n)V(s)} \\ -e^{-(n-N_n)V(s)} & 0 \end{pmatrix}, & s \in \Gamma(a_2, b_2), \\ \begin{pmatrix} 0 & e^{2\pi i \tau n + (n-N_n)V(s)} \\ -e^{-2\pi i \tau n - (n-N_n)V(s)} & 0 \end{pmatrix}, & s \in \Gamma(a_1, b_1), \\ \begin{pmatrix} e^{2\pi i \omega n} & 0 \\ 0 & e^{-2\pi i \omega n} \end{pmatrix}, & s \in \Gamma(b_1, a_2). \end{cases}$$

We shall solve this problem only for  $n \in \mathbb{N}(\epsilon) = \mathbb{N}(t, \epsilon)$  from Proposition 9.2.3.

Let the functions  $M_{n,k}(\mathbf{z})$  be given by (10.3.18) and  $\mathcal{D}(z) = \mathcal{D}(z; t)$  be defined by (9.2.2). With the notation introduced right after (10.3.16), a solution of RHP- $\mathbf{N}$  is given by

$$\mathbf{N}(z) = \mathbf{M}^{-1}(\infty) \mathbf{M}(z), \quad \mathbf{M}(z) := \begin{pmatrix} M_{n,1}^{(0)}(z) & M_{n,1}^{(1)}(z) \\ M_{n,0}^{(0)}(z) & M_{n,0}^{(1)}(z) \end{pmatrix} \mathcal{D}^{(N_n-n)\sigma_3}(z). \quad (10.4.7)$$

Indeed, RHP- $\mathbf{N}$ (a) follows from holomorphy of  $\mathcal{D}(z)$  and  $M_{n,k}(\mathbf{z})$  discussed in Proposition 9.2.1 and right after (10.3.18). Fulfillment of RHP- $\mathbf{N}$ (b) can be checked by using (9.2.3) and (10.3.19). Observe also that  $\det(\mathbf{N}(z)) \equiv 1$ . Indeed, as the jump matrices in RHP- $\mathbf{N}$ (b) have unit determinants,  $\det(\mathbf{N}(z))$  is holomorphic through  $\Gamma(a_1, b_1)$ ,  $\Gamma(b_1, a_2)$ , and  $\Gamma(a_2, b_2)$ . It also has at most square root singularities at  $\{a_1, b_1, a_2, b_2\}$  as explained right after (10.3.19). Thus, it is holomorphic throughout  $\overline{\mathbb{C}}$  and therefore is a constant. The normalization at infinity implies that this constant is 1.

#### 10.4.5 Local Parametrices

The jumps discarded in RHP- $\mathbf{N}$  are not uniformly close to the identity around the points  $e \in \{a_1, b_1, a_2, b_2\}$ . The goal of this section is to solve RHP- $\mathbf{S}$  in the disks  $U_e$ , see (10.2.1), with a certain matching condition on the boundary of the disks. More precisely, we are looking for a matrix functions  $\mathbf{P}_e(z)$  that solves RHP- $\mathbf{P}_{a_1}$ :

- (a)  $\mathbf{P}_e(z)$  has the same analyticity properties as  $\mathbf{S}(z)$  restricted to  $U_e$ , see **RHP- $\mathbf{S}$ (a)**;
- (b)  $\mathbf{P}_e(z)$  satisfies the same jump relations as  $\mathbf{S}(z)$  restricted to  $U_e$ , see **RHP- $\mathbf{S}$ (b)**;
- (c)  $\mathbf{P}_e(z) = \mathbf{N}(z)(\mathbf{I} + \mathcal{O}(n^{-1}))$  holds uniformly on  $\partial U_e$  as  $n \rightarrow \infty$ .

Again, we shall solve **RHP- $\mathbf{P}_{a_1}$**  only for  $n \in \mathbb{N}(\epsilon)$ .

Let  $U_e$ ,  $J_e$ , and  $I_e$ ,  $e \in \{a_1, b_1, a_2, b_2\}$ , be as in (10.2.1) and (10.2.2). Further, let  $\mathbf{A}(\zeta)$  be the Airy matrix [33, 58]. That is, it is analytic in  $\mathbb{C} \setminus ((-\infty, \infty) \cup L_- \cup L_+)$ ,  $L_\pm := \{\zeta : \arg(\zeta) = \pm 2\pi/3\}$ , and satisfies

$$\mathbf{A}_+(s) = \mathbf{A}_-(s) \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & s \in (-\infty, 0), \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & s \in L_\pm, \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & s \in (0, \infty), \end{cases}$$

where the real line is oriented from  $-\infty$  to  $\infty$  and the rays  $L_\pm$  are oriented towards the origin. It is known that  $\mathbf{A}(\zeta)$  has the following asymptotic expansion at infinity:

$$\mathbf{A}(\zeta)e^{\frac{2}{3}\zeta^{3/2}\sigma_3} \sim \frac{\zeta^{-\sigma_3/4}}{\sqrt{2}} \sum_{k=0}^{\infty} \begin{pmatrix} s_k & 0 \\ 0 & t_k \end{pmatrix} \begin{pmatrix} (-1)^k & i \\ (-1)^k i & 1 \end{pmatrix} \left(\frac{2}{3}\zeta^{3/2}\right)^{-k}, \quad (10.4.8)$$

where the expansion holds uniformly in  $\mathbb{C} \setminus ((-\infty, \infty) \cup L_- \cup L_+)$ , and

$$s_0 = t_0 = 1, \quad s_k = \frac{\Gamma(3k + 1/2)}{54^k k! \Gamma(k + 1/2)}, \quad t_k = -\frac{6k + 1}{6k - 1} s_k, \quad k \geq 1.$$

Let us write  $\mathbf{A}_e := \mathbf{A}$  if  $e \in \{b_1, b_2\}$  and  $\mathbf{A}_e := \sigma_3 \mathbf{A} \sigma_3$  if  $e \in \{a_1, a_2\}$ . It can be easily checked that  $\sigma_3 \mathbf{A} \sigma_3$  has the same jumps as  $\mathbf{A}$  only with the reversed orientation of the rays. Moreover, one needs to replace  $i$  by  $-i$  in (10.4.8) when describing the

behavior of  $\sigma_3 \mathbf{A} \sigma_3$  at infinity. Let  $\zeta_e(z) := [-n(3/4)\phi_e(z)]^{2/3}$ , which is conformal in  $U_e$ , see (10.2.6). Further, put

$$\mathbf{J}_e(z) := e^{(N_n - n)V(z)\sigma_3/2} \begin{cases} \mathbf{I}, & e = b_2, \\ e^{\pi i(\pm\omega)n\sigma_3}, & e = a_2, \\ e^{\pi i(\pm\omega - \tau)n\sigma_3}, & e = b_1, \\ e^{-\pi i\tau n\sigma_3}, & e = a_1, \end{cases}$$

where we use  $\omega$  if  $z$  lies to the left of  $\Gamma$  and use  $-\omega$  if  $z$  lies to the right of  $\Gamma$ . Then it can be readily verified by using (10.1.4) that

$$\mathbf{P}_e(z) := \mathbf{E}_e(z) \mathbf{A}_e(\zeta_e(z)) e^{(2/3)\zeta_e^{3/2}(z)\sigma_3} \mathbf{J}_e(z), \quad (10.4.9)$$

satisfies **RHP- $\mathbf{P}_{a_1}$** (a,b) for any matrix  $\mathbf{E}_e(z)$  holomorphic in  $U_e$ . It follows immediately from (10.4.8) and the definition of  $\mathbf{J}_e$  that **RHP- $\mathbf{P}_{a_1}$** (c) will be satisfied if

$$\mathbf{E}_e(z) := (\mathbf{N} \mathbf{J}_e^{-1})(z) \begin{pmatrix} 1 & -\nu_e i \\ -\nu_e i & 1 \end{pmatrix} \frac{\zeta_e^{\sigma_3/4}(z)}{\sqrt{2}}, \quad (10.4.10)$$

provided this matrix function is holomorphic in  $U_e$ , where  $\nu_e$  was defined in (10.2.5). By using **RHP- $\mathbf{N}$** (b) and (10.2.7) one can readily check that  $\mathbf{E}_e(z)$  is holomorphic in  $U_e \setminus \{e\}$ . Since  $\zeta_e(z)$  has a simple zero at  $e$ , it also follows from (8.3.12) and the claim after (8.3.11) that  $\mathbf{E}_e(z)$  can have at most square root singularity at  $e$  and therefore is in fact holomorphic in the entire disk  $U_e$  as needed.

In fact, it follows from (10.4.8)–(10.4.10) that

$$\mathbf{P}_e(z) \sim \mathbf{N}(z) \left( \mathbf{I} + \frac{1}{n} \sum_{k=0}^{\infty} \frac{\mathbf{P}_{e,k}(z)}{n^k} \right), \quad (10.4.11)$$

where the expansion inside the parentheses holds uniformly on  $\partial U_e$  and locally uniformly for  $t \in O_{\text{two-cut}}$ , and for  $k \geq 1$

$$\mathbf{P}_{e,k-1}(z) = \mathbf{J}_e^{-1}(z) \begin{pmatrix} 1 & -\nu_e i \\ -\nu_e i & 1 \end{pmatrix} \begin{pmatrix} s_k & 0 \\ 0 & t_k \end{pmatrix} \begin{pmatrix} (-1)^k & \nu_e i \\ \nu_e (-1)^k i & 1 \end{pmatrix} \mathbf{J}_e(z) \left( -\frac{\phi_e(z)}{2} \right)^{-k}. \quad (10.4.12)$$

### 10.4.6 RH Problem with Small Jumps

Set  $\Sigma := ([(\Gamma \setminus J_\Gamma) \cup J_+ \cup J_-] \cap D) \cup (\cup_e \partial U_e)$ ,  $D := \mathbb{C} \setminus \cup_e \overline{U_e}$ . We shall show that for all  $n \in \mathbb{N}(\epsilon)$  large enough there exists a matrix function  $\mathbf{R}(z)$  that solves the following Riemann-Hilbert problem (RHP- $\mathbf{R}$ ):

(a)  $\mathbf{R}(z)$  is holomorphic in  $\mathbb{C} \setminus \Sigma$  and  $\lim_{\mathbb{C} \setminus \Gamma \ni z \rightarrow \infty} \mathbf{R}(z) = \mathbf{I}$ ;

(b)  $\mathbf{R}(z)$  has continuous traces on  $\Sigma^\circ$  that satisfy

$$(\mathbf{R}_-^{-1} \mathbf{R}_+)(s) = \begin{cases} \mathbf{P}_e(s) \mathbf{N}^{-1}(s), & s \in \partial U_e, \\ \mathbf{N}(s) \begin{pmatrix} 1 & 0 \\ e^{-n\phi_{b_2}(s) - (n-N_n)V(s)} & 1 \end{pmatrix} \mathbf{N}^{-1}(s), & s \in J_\pm \cap D, \end{cases}$$

where  $\partial U_e$  is oriented clockwise, and

$$(\mathbf{R}_-^{-1} \mathbf{R}_+)(s) = \begin{cases} \mathbf{N}(s) \begin{pmatrix} 1 & e^{n(2\pi i \tau + \phi_{a_1}(s)) + (n-N_n)V(s)} \\ 0 & 1 \end{pmatrix} \mathbf{N}^{-1}(s), & s \in \Gamma(e^{i\pi}\infty, a_1) \cap D, \\ \mathbf{N}_-(s) \begin{pmatrix} e^{2\pi i \omega n} & e^{n\phi_{a_2}(s) + (n-N_n)V(s)} \\ 0 & e^{-2\pi i \omega n} \end{pmatrix} \mathbf{N}_+^{-1}(s), & s \in \Gamma(b_1, a_2) \cap D, \\ \mathbf{N}(s) \begin{pmatrix} 1 & e^{n\phi_{b_2}(s) + (n-N_n)V(s)} \\ 0 & 1 \end{pmatrix} \mathbf{N}^{-1}(s), & s \in \Gamma(b_2, e^{\pi i/3}\infty) \cap D. \end{cases}$$

Observe that RHP- $\mathbf{R}$  is a well posed problem as  $\det(\mathbf{N}(z)) \equiv 1$ , as explained after (8.3.12), and therefore the matrix is invertible. Recall also that the entries of  $\mathbf{N}(z)$  and  $\mathbf{N}^{-1}(z)$  are uniformly bounded on  $\Sigma$  for  $n \in \mathbb{N}(\epsilon)$  according to (10.3.20) and (10.3.21).

To prove solvability of RHP- $\mathbf{R}$ , let us show that the jump matrices in RHP- $\mathbf{R}$ (b) are close to the identity. To this end, set

$$\Delta(s) := (\mathbf{R}_-^{-1} \mathbf{R}_+)(s) - \mathbf{I}, \quad s \in \Sigma. \quad (10.4.13)$$

Since the entries of  $\mathbf{N}(z)$  are uniformly bounded on each  $\partial U_e$  with respect to  $n \in \mathbb{N}(\epsilon)$ , it holds by **RHP- $\mathbf{P}_{a_1}$** (c) and (10.4.11) that

$$\Delta(s) \sim \frac{1}{n} \sum_{k=0}^{\infty} \frac{(\mathbf{N} \mathbf{P}_{e,k} \mathbf{N}^{-1})(s)}{n^k}, \quad (10.4.14)$$

where the expansion is valid uniformly on  $\partial U_e$ . Thus, it holds that

$$\|\Delta\|_{L^\infty(\cup_e \partial U_e)} = \mathcal{O}_\epsilon(n^{-1}). \quad (10.4.15)$$

Moreover, it follows from (10.2.3) and the sentence right after that there exists a constant  $C_D < 1$ , depending on the radii of the disks  $U_e$ , such that  $|e^{\phi_{b_2}(s)}| < C_D$  for  $s \in \Gamma(b_2, e^{\pi i/3} \infty) \cap D$ ,  $|e^{\phi_{a_1}(s)}| < C_D$  for  $s \in \Gamma(e^{\pi i} \infty, a_1) \cap D$ , and  $|e^{\phi_{a_2}(s)}| < C_D$  for  $s \in \Gamma(b_1, a_2) \cap D$ . Therefore,

$$\Delta(s) = \mathbf{N}_-(s) \begin{pmatrix} 0 & e^{n(\phi_e(s)+C)+(n-N_n)V(s)} \\ 0 & 0 \end{pmatrix} \mathbf{N}_+^{-1}(s) = \mathcal{O}(C_D^n) \quad (10.4.16)$$

on  $(\Sigma \cap D) \setminus (J_+ \cup J_-)$  since  $C$  is either zero or purely imaginary, the entries of  $\mathbf{N}(z)$  are bounded and so is the sequences  $n - N_n$ , where the subscripts  $\pm$  are needed only on  $\Gamma(b_1, a_2) \cap D$  and we used the fact

$$\mathbf{N}_-(s) e^{2\pi i \omega n \sigma_3} \mathbf{N}_+^{-1}(s) = \mathbf{I}, \quad s \in \Gamma(a_1, b_2),$$

see **RHP- $\mathbf{N}$** (b). Similarly, we get that

$$\Delta(s) = \mathbf{N}(s) \begin{pmatrix} 0 & 0 \\ e^{-n\phi_{b_2}(s)-(n-N_n)V(s)} & 0 \end{pmatrix} \mathbf{N}^{-1}(s) = \mathcal{O}(C_D^n) \quad (10.4.17)$$

on  $J_\pm \cap D$  for a possibly adjusted constant  $C_D$ , where we used the fact that  $\operatorname{Re}(\phi_{b_2}(s)) > 0$  for  $s \in J_\pm \setminus E$ , see Figure 6.1.

Equations (10.4.15), (10.4.16), and (10.4.17) show that  $\Delta(s)$  is uniformly close to zero. Since the entries of  $\mathbf{N}(z)$  are holomorphic at infinity and  $e^{n\phi_e(s)}$  is geometrically small as  $\Gamma \ni s \rightarrow \infty$ ,  $\Delta(s)$  is close to zero in  $L^2$ -norm as well. Then it follows from the same analysis as in [26, Corollary 7.108] that  $\mathbf{R}(z)$  exists for all  $n \in \mathbb{N}(\epsilon)$  and it holds uniformly in  $\mathbb{C}$  that

$$\mathbf{R}(z) = \mathbf{I} + \mathcal{O}_\epsilon(n^{-1}). \quad (10.4.18)$$



### 10.4.7 Solution of the Initial RHP

Given  $\mathbf{R}(z)$ ,  $\mathbf{N}(z)$ , and  $\mathbf{P}_e(z)$ , solutions of **RHP- $\mathbf{R}$** , **RHP- $\mathbf{N}$** , and **RHP- $\mathbf{P}_{a_1}$** , respectively, it is a trivial verification to check that **RHP- $\mathbf{S}$**  is solved by

$$\mathbf{S}(z) = \begin{cases} (\mathbf{RN})(z) & \text{in } D \setminus [(\Gamma \setminus J_\Gamma) \cup J_+ \cup J_-], \\ (\mathbf{RP}_e)(z) & \text{in } U_e, \quad e \in \{a_1, b_1, a_2, b_2\}. \end{cases} \quad (10.4.19)$$

Let  $K$  be a compact set in  $\mathbb{C} \setminus \Gamma$ . We can always arrange so that the set  $K$  lies entirely within the unbounded component of the contour  $\Sigma$ . Then it follows from (10.4.4), (10.4.6), and (10.4.19) that

$$\mathbf{Y}(z) = e^{-n\ell^*\sigma_3/2}(\mathbf{RN})(z)e^{n(g(z)+\ell^*/2)\sigma_3}, \quad z \in K. \quad (10.4.20)$$

Subsequently, by using (10.4.1) and (10.1.2), we see that

$$P_n(z) = [\mathbf{Y}(z)]_{11} = ([\mathbf{R}(z)]_{11}[\mathbf{N}(z)]_{11} + [\mathbf{R}(z)]_{12}[\mathbf{N}(z)]_{21})e^{n\mathcal{Q}(z)+\frac{n}{2}(V(z)-\ell_*)}.$$

Therefore, it follows from (8.3.12) and (10.4.18) that

$$\psi_n(z)e^{-n\mathcal{Q}(z)} = (1 + \mathcal{O}_\epsilon(n^{-1})) D^{N_n-n}(z) \frac{M_{n,1}^{(0)}(z)}{M_{n,1}^{(0)}(\infty)} + \mathcal{O}_\epsilon(n^{-1}) \frac{D^{N_n-n}(z)}{\mathcal{D}^{2(N_n-n)}(z)} \frac{M_{n,0}^{(0)}(z)}{M_{n,0}^{(1)}(\infty)}. \quad (10.4.21)$$

Asymptotic formula (9.2.11) now follows from (10.3.20) and (10.3.21), boundedness of  $\{N_n - m\}$ , and definitions of  $M_{n,1}(z)$  in (8.3.10), of  $\Theta_n(z)$  right before (7.3.20), and of  $\vartheta_n(z)$  right after Proposition 9.2.3.

Now, let  $K$  be any compact set in  $\mathbb{C} \setminus J$ . Write  $K = K_1 \cup K_2$ , where  $K_1, K_2$  are compact,  $K_1$  does not intersect  $\Gamma$  and  $K_2$  lies entirely within the region  $\{\operatorname{Re}(\phi_{b_2}(z)) < 0\}$ , see Figures 9.5 and 6.1. Again, the lens  $\Sigma$  can be adjusted so that  $K_1$  lies in the unbounded component of the complement of  $\Sigma$ . Hence, the estimate (9.2.11) on  $K_1$  follows as before. To obtain it on  $K_2$ , recall that we had a lot of freedom in choosing  $\Gamma$  away from  $J$ . That is,  $\Gamma$  can be deformed into  $\Gamma'$  that avoids  $K_2$  and belongs to  $\{\operatorname{Re}(\phi_{b_2}(z)) < 0\}$  away from  $J$ . Then **RHP- $\mathbf{Y}$** , formulated on  $\Gamma'$ , can be solved exactly as before since estimates (10.4.16) and (10.4.17) remain the same (with a possibly modified constant  $C_D$ ), and therefore (9.2.11) can be shown via (10.4.20)–(10.4.21).

Finally, take  $K \subset J^\circ$ . It again follows from (10.4.1), (10.4.4), (10.4.6), and (10.4.19) that

$$P_n(s) = [\mathbf{Y}(s)]_{11} = ([\mathbf{R}(s)]_{11} ([\mathbf{N}(s)]_{11+} + [\mathbf{N}(s)]_{12+} e^{-n\phi_{b_2+}(s) - (n-N_n)V(s)}) + \\ [\mathbf{R}(s)]_{12} ([\mathbf{N}(s)]_{21+} + [\mathbf{N}(s)]_{22+} e^{-n\phi_{b_2+}(s) - (n-N_n)V(s)})) e^{ng_+(s)}, \quad s \in K.$$

Now, (10.1.6) and RHP- $\mathbf{N}$ (b) yield that

$$[\mathbf{N}(s)]_{i2+} e^{-n\phi_{b_2+}(s) - (n-N_n)V(s) + ng_+(s)} = [\mathbf{N}(s)]_{i1-} e^{ng_-(s)}$$

for  $s \in J^\circ$  and  $i \in \{1, 2\}$ . Hence, we get from (10.1.2) and (10.4.18) that

$$\psi_n(s) = (1 + \mathcal{O}_\epsilon(n^{-1})) ([\mathbf{N}(s)]_{11+} e^{n\mathcal{Q}_+(s)} + [\mathbf{N}(s)]_{11-} e^{n\mathcal{Q}_-(s)}) + \\ \mathcal{O}_\epsilon(n^{-1}) ([\mathbf{N}(s)]_{21+} e^{n\mathcal{Q}_+(s)} + [\mathbf{N}(s)]_{21-} e^{n\mathcal{Q}_-(s)})$$

for  $s \in K$ . Since the traces  $Q_\pm(s)$ ,  $s \in J$ , are purely imaginary by (10.1.5), the above asymptotic formula yields (9.2.12) in the same way (10.4.21) yielded (9.2.11).

## 10.5 Concluding Remarks

Asymptotics of polynomials satisfying (9.0.1) were considered, in part, to attain a certain asymptotic expansion, the so-called *topological expansion*, of the partition function

$$Z_N(t) := \int_\Gamma \cdots \int_\Gamma \prod_{1 \leq j < k \leq N} (z_j - z_k)^2 \prod_{k=1}^N e^{-N(\frac{z^3}{3} + tz)} dz_1 dz_2 \cdots dz_N.$$

This was an extension of work done in [50] where parameters  $t$  associated with measures  $\mu_t$  supported on a single arc were considered. While we have chosen to focus on strong asymptotics of  $P_n(z; t, N)$ , much of the physically relevant quantities rely on attaining asymptotic formulas for recurrence coefficients  $\beta_n, \gamma_n^2$ , which of course can be done via the formulas

$$\begin{cases} \gamma_n^2(t, N) &= h_n(t, N)/h_{n-1}(t, N), \\ \beta_n(t, N) &= (P_n)_{n-1} - (P_{n+1})_n, \end{cases}$$

where we write  $P_n(z; t, N) = z^n + \sum_{k=0}^{n-1} (P_n)_k z^k$ . Such results will appear in [59].

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## APPENDICES

## A. THETA FUNCTION IDENTITIES

In this appendix we state a number of identities used in the analysis carried out in Chapter 5.

**Lemma A.0.1.** *Recall (5.4.2). It holds that*

$$\int_{\mathbf{a}_3}^{\mathbf{0}} \Omega = -\mathbf{K}_- \quad \text{and} \quad \int_{\mathbf{a}_3}^{\mathbf{0}^*} \Omega = \mathbf{K}_-, \quad (\text{A.0.1})$$

where the path of integration lies entirely in  $\mathfrak{R}_{\alpha, \beta}$ .

*Proof.* Exactly as in the case of (5.4.3), the symmetries of  $\Omega(\mathbf{z})$  imply that

$$-\int_{\mathbf{a}_3}^{\mathbf{0}} \Omega = \int_{\mathbf{a}_3}^{\mathbf{0}^*} \Omega = \frac{1}{2} \int_{\Delta_3} \Omega = \frac{1}{4} \int_{\Delta_3 - \Delta_1} \Omega.$$

The claim now follows from the fact that  $\Delta_3 - \Delta_1$  is homologous to  $\alpha - \beta$ .  $\square$

**Lemma A.0.2.** *It holds that*

$$\Phi(\mathbf{z}) = \exp \left\{ -\pi i \int_{\mathbf{a}_3}^{\mathbf{z}} \Omega \right\} \frac{\theta \left( \int_{\mathbf{a}_3}^{\mathbf{z}} \Omega - \mathbf{K}_+ \right)}{\theta \left( \int_{\mathbf{a}_3}^{\mathbf{z}} \Omega + \mathbf{K}_+ \right)}. \quad (\text{A.0.2})$$

*Proof.* It follows from (5.4.3) and (5.4.1) that the right hand side of (A.0.2) is a meromorphic functions with a simple pole at  $\infty^{(0)}$ , a simple zero at  $\infty^{(1)}$ , and otherwise non-vanishing and finite that satisfies (5.2.6). As only holomorphic functions on  $\mathfrak{R}$  are constants, the normalization at  $\mathbf{a}_3$  yields (A.0.2).  $\square$

**Lemma A.0.3.** *Let  $l_0, l_1, m_0, m_1$  be given by (5.4.5). Then it holds that*

$$\begin{cases} \Phi(\mathbf{z}_0) &= (-1)^{l_0+m_0} e^{-\pi i(c_\rho - \mathbf{K}_+)} \theta(c_\rho + 2\mathbf{K}_-) / \theta(c_\rho), \\ \Phi(\mathbf{z}_1) &= (-1)^{l_1+m_1} e^{-\pi i(c_\rho + \mathbf{K}_+)} \theta(c_\rho) / \theta(c_\rho + 2\mathbf{K}_+). \end{cases} \quad (\text{A.0.3})$$

In particular, when  $|\pi(\mathbf{z}_k)| < \infty$ , it holds that

$$\Phi(\mathbf{z}_0)\Phi(\mathbf{z}_1) = -(-1)^{l_0-l_1+m_0-m_1}. \quad (\text{A.0.4})$$

Moreover, we have that

$$\Phi(\mathbf{0}) = e^{\pi i \mathbf{K}_- \theta(1/2)/\theta(\mathbf{B}/2)}. \quad (\text{A.0.5})$$

*Proof.* Since  $-2\mathbf{K}_+ = 2\mathbf{K}_- - 1$ , we get from (A.0.2) that

$$\Phi(\mathbf{z}_0) = e^{\pi i (\mathbf{K}_+ - c_\rho - l_0 - m_0 \mathbf{B})} \frac{\theta(c_\rho + 2\mathbf{K}_- + m_0 \mathbf{B})}{\theta(c_\rho + m_0 \mathbf{B})}.$$

The first relation in (A.0.3) now follows from (5.4.1). Similarly, we have that

$$\Phi(\mathbf{z}_1) = e^{\pi i (-\mathbf{K}_+ - c_\rho - l_1 - m_1 \mathbf{B})} \frac{\theta(c_\rho + m_1 \mathbf{B})}{\theta(c_\rho + 2\mathbf{K}_+ + m_1 \mathbf{B})},$$

which yields the second relation in (A.0.3), again by (5.4.1). To get (A.0.4), observe that

$$\theta(c_\rho + 2\mathbf{K}_-) = \theta(c_\rho + 2\mathbf{K}_+ - \mathbf{B}) = -e^{2\pi i c_\rho} \theta(c_\rho + 2\mathbf{K}_+)$$

by (5.4.1). Finally, (A.0.5) follows from (A.0.2) and (A.0.1).  $\square$

**Lemma A.0.4.** *Let*

$$X_n := \lim_{z \rightarrow \infty} z^{-2} \Psi_n(z^{(0)}) \Psi_{n-1}(z^{(1)}). \quad (\text{A.0.6})$$

When  $|\pi(\mathbf{z}_k)| < \infty$ , it holds that

$$X_n = \frac{4}{a^2 + b^2} \frac{\theta^2(c_\rho)}{\theta^2(0)} \frac{(-1)^{i(n)}}{\Phi^{2i(n)}(\mathbf{z}_1)}. \quad (\text{A.0.7})$$

*Proof.* Since  $\Phi(\mathbf{z})\Phi(\mathbf{z}^*) \equiv 1$  and  $S_\rho(\mathbf{z})S_\rho(\mathbf{z}^*) \equiv 1$ , the desired limit is equal to

$$\frac{4}{a^2 + b^2} T_{i(n)}(\infty^{(0)}) \lim_{z \rightarrow \infty} \Phi(z^{(1)}) T_{i(n-1)}(z^{(1)}),$$

where we also used (5.2.9). Since  $-2\mathbf{K}_+ = 2\mathbf{K}_- - 1$ , it follows from (5.4.4) and (5.4.3) that

$$T_{i(n)}(\infty^{(0)}) = e^{\pi i i(n) \mathbf{K}_+} \frac{\theta(c_\rho + 2i(n) \mathbf{K}_-)}{\theta(0)}.$$

We further deduce from (5.4.4) and (A.0.2) that

$$(\Phi T_{i(n-1)})(\mathbf{z}) = \exp \left\{ -\pi i i(n) \int_{\mathbf{a}_3}^{\mathbf{z}} \Omega \right\} \frac{\theta(\int_{\mathbf{a}_3}^{\mathbf{z}} \Omega - c_\rho + (-1)^{i(n)} \mathbf{K}_+)}{\theta(\int_{\mathbf{a}_3}^{\mathbf{z}} \Omega + \mathbf{K}_+)}.$$

Therefore, it follows from (5.4.3) that

$$(\Phi T_{i(n-1)})(\infty^{(1)}) = e^{\pi i i(n) \mathbf{K}_+} \frac{\theta(c_\rho + 2i(n) \mathbf{K}_+)}{\theta(0)}.$$

Hence, we get from (A.0.3) that

$$X_n = \frac{4}{a^2 + b^2} \frac{\theta^2(c_\rho)}{\theta^2(0)} \left( (-1)^{l_0 - l_1 + m_0 - m_1} \frac{\Phi(z_0)}{\Phi(z_1)} \right)^{i(n)}.$$

The claim of the lemma now follows from (A.0.4).  $\square$

**Lemma A.0.5.** *It holds that*

$$\frac{d}{d\zeta} \left( e^{\pi i \zeta} \frac{\theta(\zeta + K_+)}{\theta(\zeta - K_+)} \right) = i\pi \theta^2(0) e^{\pi i \zeta} \frac{\theta(\zeta - K_-) \theta(\zeta + K_-)}{\theta^2(\zeta - K_+)}. \quad (\text{A.0.8})$$

*Proof.* See [44, Eq. (20.7.25)] (observe that  $\theta(\zeta) = \theta_3(\pi\zeta|\mathbf{B})$  in the notation of [44, Chapter 20]).  $\square$

**Lemma A.0.6.** *It holds that*

$$z = -\frac{\sqrt{a^2 + b^2}}{2} \frac{e^{-\pi i K_+} \theta^2(0)}{\theta(1/2) \theta(\mathbf{B}/2)} \frac{\theta(\int_{\mathbf{a}_3}^z \Omega - K_-) \theta(\int_{\mathbf{a}_3}^z \Omega + K_-)}{\theta(\int_{\mathbf{a}_3}^z \Omega - K_+) \theta(\int_{\mathbf{a}_3}^z \Omega + K_+)}. \quad (\text{A.0.9})$$

*Proof.* It follows from (5.4.1), (5.4.3), and (A.0.1) that

$$z = C \frac{\theta(\int_{\mathbf{a}_3}^z \Omega - K_-) \theta(\int_{\mathbf{a}_3}^z \Omega + K_-)}{\theta(\int_{\mathbf{a}_3}^z \Omega - K_+) \theta(\int_{\mathbf{a}_3}^z \Omega + K_+)}$$

for some normalizing constant  $C$ . It further follows from (5.2.9), (A.0.2), and (5.4.3) that

$$-\frac{\sqrt{a^2 + b^2}}{2} = \lim_{z \rightarrow \infty} z \Phi^{-1}(z^{(0)}) = C e^{\pi i K_+} \frac{\theta(1/2) \theta(\mathbf{B}/2)}{\theta^2(0)},$$

which yields the desired result.  $\square$

**Lemma A.0.7.** *It holds that*

$$e^{\pi i \mathbf{B}/2} \frac{\theta^2(1/2) \theta^2(\mathbf{B}/2)}{\theta^4(0)} = \frac{a^2 + b^2}{4ab}. \quad (\text{A.0.10})$$

*Proof.* To prove (A.0.10), evaluate (A.0.9) at  $\mathbf{a}_3$  to get

$$\frac{\theta(1/2) \theta(\mathbf{B}/2)}{\theta^2(0)} = \frac{\sqrt{a^2 + b^2}}{2a} e^{-\pi i K_+} \frac{\theta^2(K_-)}{\theta^2(K_+)}.$$

Since  $\Delta_3 - \Delta_1$  is homologous to  $\alpha - \beta$ , one can easily deduce from Figure 5.1 that it also holds that

$$\int_{\mathbf{a}_3}^{\mathbf{a}_2} \Omega = \left( \int_{\mathbf{a}_3}^{\mathbf{0}^*} + \int_{\mathbf{0}^*}^{\mathbf{a}_1} + \int_{\mathbf{a}_1}^{\mathbf{a}_2} \right) \Omega = \frac{1}{2} \int_{\Delta_3 - \Delta_1 + \beta} \Omega = \frac{1}{2},$$

where the initial path of integration (except for  $\mathbf{a}_2$ ) belongs to  $\mathfrak{R}_{\alpha,\beta}$ . Thus, evaluating (A.0.9) at  $\mathbf{a}_2$  gives us

$$\frac{\theta(1/2)\theta(\mathbf{B}/2)}{\theta^2(0)} = -\frac{\sqrt{a^2+b^2}}{2ib}e^{-\pi i\mathbf{K}_+}\frac{\theta^2(\mathbf{K}_+)}{\theta^2(\mathbf{K}_-)},$$

where we used (5.4.1). Multiplying two expressions for  $\theta(1/2)\theta(\mathbf{B}/2)/\theta^2(0)$  yields the desired result.  $\square$

**Lemma A.0.8.** *It holds that*

$$\oint_{\alpha} \frac{ds}{w(\mathbf{s})} = \frac{2\pi i}{\sqrt{a^2+b^2}}e^{\pi i\mathbf{K}_+}\theta(1/2)\theta(\mathbf{B}/2). \quad (\text{A.0.11})$$

*Proof.* We can deduce from (A.0.2), (A.0.8), and the evenness of the theta function that

$$\Phi'(z) = -i\pi\theta^2(0) \left( \oint_{\alpha} \frac{ds}{w(\mathbf{s})} \right)^{-1} \frac{\Phi(z)}{w(z)} \frac{\theta\left(\int_{\mathbf{a}_3}^z \Omega + \mathbf{K}_-\right)\theta\left(\int_{\mathbf{a}_3}^z \Omega - \mathbf{K}_-\right)}{\theta\left(\int_{\mathbf{a}_3}^z \Omega + \mathbf{K}_+\right)\theta\left(\int_{\mathbf{a}_3}^z \Omega - \mathbf{K}_+\right)}.$$

Since  $\Phi'(z) = z\Phi(z)/w(z)$  by (5.2.5), (A.0.11) follows from (A.0.9).  $\square$

**Lemma A.0.9.** *Let*

$$Y_n := (T'_{i(n)}T_{i(n-1)}/\Phi - T_{i(n)}(T_{i(n-1)}/\Phi)')(\mathbf{0}). \quad (\text{A.0.12})$$

When  $|\pi(\mathbf{z}_k)| = \infty$ , it holds that  $Y_n = 0$ , otherwise, we have that

$$Y_n = (-1)^{l_0+m_0+i(n)} \frac{2e^{\pi i c_\rho}}{\sqrt{a^2+b^2}} \frac{\Phi(\mathbf{z}_0)}{\Phi^2(\mathbf{0})} \frac{\theta^2(c_\rho)}{\theta^2(0)}, \quad (\text{A.0.13})$$

where the integers  $l_0, m_0$  were defined in (5.4.5).

*Proof.* Since  $\Phi'(z) = z\Phi(z)/w(z)$  by (5.2.5),  $\Phi'(\mathbf{0}) = 0$ . Therefore,

$$Y_n = (T_{i(n-1)}^2/\Phi)(\mathbf{0})(T_{i(n)}/T_{i(n-1)})'(\mathbf{0}).$$

Assume that  $|\pi(\mathbf{z}_k)| < \infty$ . Then it follows from (5.4.4), (A.0.8), and (A.0.11) that

$$\begin{aligned} \left( \frac{T_{i(n)}}{T_{i(n-1)}} \right)'(z) &= -(-1)^{i(n)} \frac{\sqrt{a^2+b^2}}{2w(z)} \frac{e^{-\pi i\mathbf{K}_+}\theta^2(0)}{\theta(1/2)\theta(\mathbf{B}/2)} \left( \frac{T_{i(n)}}{T_{i(n-1)}} \right)(z) \times \\ &\quad \times \frac{\theta\left(\int_{\mathbf{a}_3}^z \Omega - c_\rho + \mathbf{K}_-\right)\theta\left(\int_{\mathbf{a}_3}^z \Omega - c_\rho - \mathbf{K}_-\right)}{\theta\left(\int_{\mathbf{a}_3}^z \Omega - c_\rho + \mathbf{K}_+\right)\theta\left(\int_{\mathbf{a}_3}^z \Omega - c_\rho - \mathbf{K}_+\right)}. \end{aligned}$$

We further deduce from (5.4.4), (A.0.1), and (A.0.5) that

$$(T_{i(n-1)}T_{i(n)})(\mathbf{0}) = \frac{1}{\Phi(\mathbf{0})} \frac{\theta(c_\rho - \mathbf{B}/2)\theta(c_\rho + 1/2)}{\theta(1/2)\theta(\mathbf{B}/2)}.$$

Since  $w(\mathbf{0}) = iab$ , we therefore get from (A.0.1) that

$$Y_n = \frac{\sqrt{a^2 + b^2}}{2ab} \frac{i(-1)^{i(n)}}{\Phi^2(\mathbf{0})} \frac{e^{-\pi i \mathbf{K}_+} \theta^4(0)}{\theta^2(1/2)\theta^2(\mathbf{B}/2)} \frac{\theta(c_\rho)\theta(c_\rho + 2\mathbf{K}_-)}{\theta^2(0)}.$$

(A.0.13) now follows from (A.0.10) and the first formula in (A.0.3).

Let now  $\mathbf{z}_0 = \infty^{(1)}$ , in which case  $[c_\rho] = [0]$ . Since  $\Phi(\infty^{(1)}) = 0$ , we get that  $Y_n = 0$ . Finally, let  $\mathbf{z}_1 = \infty^{(1)}$ . Then we have that  $-c_\rho = -(-1)^k 2\mathbf{K}_+ + l_k + m_k \mathbf{B}$  and therefore

$$\begin{aligned} \frac{T_1(\mathbf{z})}{T_0(\mathbf{z})} &= \exp \left\{ \pi i \int_{a_3}^{\mathbf{z}} \Omega \right\} \frac{\theta \left( \int_{a_3}^{\mathbf{z}} \Omega + m_1 \mathbf{B} + 3\mathbf{K}_+ \right)}{\theta \left( \int_{a_3}^{\mathbf{z}} \Omega + (m_1 + 1) \mathbf{B} - 3\mathbf{K}_+ \right)} \\ &= \exp \left\{ \pi i \int_{a_3}^{\mathbf{z}} \Omega \right\} \frac{\theta \left( \int_{a_3}^{\mathbf{z}} \Omega + (m_1 + 1) \mathbf{B} - \mathbf{K}_+ \right)}{\theta \left( \int_{a_3}^{\mathbf{z}} \Omega + m_1 \mathbf{B} + \mathbf{K}_+ \right)} \\ &= e^{2\pi i (2m_1 + 1) \mathbf{K}_-} \Phi(\mathbf{z}) \end{aligned}$$

by (5.4.1) and (A.0.2). As  $\Phi'(\mathbf{0}) = 0$ , it also holds that  $Y_n = 0$ .  $\square$

**Lemma A.0.10.** *Let*

$$Z_n := (T'_{i(n)} T_{i(n-1)} / \Phi - T_{i(n)} (T_{i(n-1)} / \Phi)')(\mathbf{0}^*). \quad (\text{A.0.14})$$

When  $|\pi(\mathbf{z}_k)| = \infty$ , it holds that  $Z_n = 0$ , otherwise, we have that

$$Z_n = (-1)^{l_0 + m_0 + i(n)} \frac{2e^{-\pi i c_\rho}}{\sqrt{a^2 + b^2}} \frac{\Phi(\mathbf{z}_0)}{\Phi^2(\mathbf{0}^*)} \frac{\theta^2(c_\rho)}{\theta^2(0)}. \quad (\text{A.0.15})$$

*Proof.* The proof is the same as in the previous lemma.  $\square$

**Lemma A.0.11.** *Let  $\sigma_0, \sigma_1$  be as in (5.5.2). When  $|\pi(\mathbf{z}_k)| < \infty$ , it holds that*

$$Y_n X_n^{-1} = \sigma_{i(n)} e^{\pi i c_\rho} \frac{\sqrt{a^2 + b^2}}{2} \frac{\Phi(\mathbf{z}_{i(n)})}{\Phi^2(\mathbf{0})} \quad (\text{A.0.16})$$

and

$$Z_n X_n^{-1} = \sigma_{i(n)} e^{-\pi i c_\rho} \frac{\sqrt{a^2 + b^2}}{2} \frac{\Phi(\mathbf{z}_{i(n)})}{\Phi^2(\mathbf{0}^*)}, \quad (\text{A.0.17})$$

where  $X_n$ ,  $Y_n$ , and  $Z_n$  are given by (A.0.6), (A.0.12), and (A.0.14), respectively.

*Proof.* The claim follows immediately from (A.0.7), (A.0.13), (A.0.15), and (A.0.4).  $\square$

## B. PROOFS OF PROPOSITIONS

### B.1 Proof of Proposition 5.3.1

It follows from (5.3.4) that  $\Omega_{z,z^*} = -\Omega_{z^*,z}$  for all  $z \in \mathfrak{R}$  such that  $\pi(z) \in \mathbb{C}$  and therefore  $S_\rho(z)S_\rho(z^*) \equiv 1$  for such  $z$ . Clearly, this relation extends to the points on top of infinity by continuity. It is also immediate from (5.3.3) and (5.3.4) that

$$S_\rho(z^{(0)}) = \exp \left\{ - \sum_{i=1}^4 \frac{w(z)}{2\pi i} \int_{\Delta_i} \frac{\log(\rho_i w_+)(s)}{s-z} \frac{ds}{w_{|\Delta_i+}(s)} \right\} \times \\ \times \exp \{ 2\pi i (wH)(z) c_\rho \}, \quad (\text{B.1.1})$$

where, for emphasis, we write  $w_{|\Delta_i+}(s)$  for  $w_+(s)$  on  $s \in \Delta_i^\circ$  and

$$H(z) := \frac{1}{2\pi i} \int_{\pi(\alpha)} \frac{dt}{(t-z)w(t)}. \quad (\text{B.1.2})$$

Relations (5.3.6) now easily follow from (B.1.1), (B.1.2), and Plemelj-Sokhotski formulae [28, equations (4.9)]. As for the behavior near  $a_i$ , note that by [28, equation (8.8)], the function  $(wH)(z)$  is bounded as  $z \rightarrow a_i$ . Furthermore, [28, equations (8.8) and (8.35)] yield that

$$-\frac{w(z)}{2\pi i} \int_{\Delta_i} \frac{\log(\rho_i w_+)(s)}{s-z} \frac{ds}{w_{|\Delta_i+}(s)} = -\frac{1}{2} \log(z - a_i)^{\alpha_i+1/2} + \mathcal{O}(1).$$

Since the above integral is the only one with the singular contribution around  $a_i$ , the validity of the top line in (5.3.7) follows. As for the behavior near the origin, note that  $\lim_{\mathcal{Q}_j \ni z \rightarrow 0} w(z) = (-1)^{j-1} iab$ , where, as before,  $\mathcal{Q}_j$  stands for the  $j$ -th quadrant. Recall that each segment  $\Delta_i$  is oriented towards the origin, see Figure 5.1. Hence, it follows from [28, equation (8.2)] that

$$-\frac{w(z)}{2\pi i} \int_{\Delta_i} \frac{\log(\rho_i w_+)(s)}{s-z} \frac{ds}{w_{|\Delta_i+}(s)} = -\frac{w(z)}{2\pi i} \frac{\log(\rho_i w_+)(0)}{w_{|\Delta_i+}(0)} \log(z) + F_i(z) \\ = \frac{(-1)^{j+i}}{2\pi i} \log(\rho_i w_+)(0) \log(z) + F_i(z), \quad z \in \mathcal{Q}_j,$$

where  $F_i(z)$  is a bounded function around the origin tending to a definite limit as  $z \rightarrow 0$ . Thus, summing over  $i$  yields

$$-\frac{w(z)}{2\pi i} \int_{\Delta} \frac{\log(\rho_i w_+)(s)}{s-z} \frac{ds}{w_+(s)} = (-1)^j \nu \log(z) + \sum_{i=1}^4 F_i(z), \quad z \in \mathcal{Q}_j,$$

where  $\nu$  was defined in (5.3.1) and we used (5.3.2). Since  $(wH)(z)$  is holomorphic around the origin, the second line in (5.3.7) follows.

## B.2 Proof of Proposition 5.5.1

It readily follows from (5.4.5) and (5.4.3) that

$$[c_\rho] = [k(1+B)/2] \Leftrightarrow \mathbf{z}_1 = \infty^{(k)} \Leftrightarrow \mathbf{z}_0 = \infty^{(1-k)}$$

for  $k \in \{0, 1\}$  (in which case  $\Phi(\mathbf{z}_{i(n)}) = \Phi(\infty^{(1)}) = 0 = A_{\rho,n}$ ). On the other hand, because Abel's map is a bijection, we also get that  $|\pi(\mathbf{z}_1)| < \infty \Leftrightarrow |\pi(\mathbf{z}_0)| < \infty$ . This proves (5.5.4). Observe that

$$A_{\rho,n} = B_{\rho,i(n)} \Phi(\mathbf{o})^{2(n-1)} n^{\varsigma_\nu \nu - 1/2}, \quad (\text{B.2.1})$$

where  $B_{\rho,i(n)}$  depends only on the parity of  $n$  and  $|\Phi(\mathbf{o})| = 1$  by (5.2.10). Hence,  $A_{\rho,n} \rightarrow 0$  as  $n \rightarrow \infty$  when  $\text{Re}(\nu) \in (-1/2, 1/2)$ , which proves (5.5.5). In the remaining situation,

$$A_{\rho,n} = B_{\rho,i(n)} \exp \{ 2(n-1)i \arctan(a/b) + i \text{Im}(\nu) \log n \}$$

by (5.2.10). If  $|B_{i(n)}| \neq 1$ , then, in fact,  $\mathbb{N}_{\rho,\epsilon} = \mathbb{N}$ . Otherwise, we have that

$$A_{\rho,n+2}/A_{\rho,n} = \exp \{ 2i \arctan(a/b) + i \text{Im}(\nu) \log(1 + 2/n) \}.$$

As  $\arctan(a/b) \in (0, \pi/2)$  and  $\log(1 + 2/n) = o(1)$ , both constants  $A_{\rho,n+2}$  and  $A_{\rho,n}$  cannot be simultaneously close to 1.

## B.3 Proof of Proposition 7.3.3

To prove the first claim, define

$$\gamma(z) := \left( \frac{z-1}{z+1} \frac{z+\bar{z}_*}{z-\bar{z}_*} \right)^{1/4}, \quad z \in \overline{\mathbb{C}} \setminus (\gamma_1 \cup \gamma_2), \quad (\text{B.3.1})$$



where  $\gamma(z)$  is holomorphic off  $J$  and the branch is chosen so that  $\gamma(\infty) = 1$ . Further, set

$$A(z) := \frac{\gamma(z) + \gamma^{-1}(z)}{2} \quad \text{and} \quad B(z) := \frac{\gamma(z) - \gamma^{-1}(z)}{-2i}. \quad (\text{B.3.2})$$

The functions  $A(z)$  and  $B(z)$  are holomorphic in  $\overline{\mathbb{C}} \setminus (\gamma_1 \cup \gamma_2)$  and satisfy

$$A(\infty) = 1, \quad B(\infty) = 0, \quad \text{and}$$

$$A_{\pm}(s) = \pm B_{\mp}(s), \quad s \in (\gamma_1 \cup \gamma_2)^{\circ} := (\gamma_1 \cup \gamma_2) \setminus \{\pm 1, z_*, -\overline{z_*}\}. \quad (\text{B.3.3})$$

Notice that the equation  $(AB)(z) = 0$  can be rewritten as  $\gamma^4(z) = 1$  and has two solutions, namely,  $\infty$  and the point  $p$  from the line after (7.3.18). In fact, unless  $p \in (\gamma_1 \cup \gamma_2)^{\circ}$ , it is a zero of  $B(z)$ . Indeed, it is enough to show that  $\gamma(p) = 1$  in the latter case. Let  $L_i := \gamma^4(\gamma_i)$ ,  $i \in \{1, 2\}$ , which are unbounded arcs connecting the origin to the point at infinity. Let  $L \subset \overline{\mathbb{C}} \setminus (\gamma_1 \cup \gamma_2)$  be an arc connecting the point at infinity at  $p$ . Then  $\gamma^4(L)$  is a closed curve that contains 1 and does not intersect the arcs  $L_i$  and therefore does not wind around the origin. Thus, analytic continuation of the principal branch of the 1/4-root from 1 along  $\gamma^4(L)$  leads back to the value 1 at the point 1. However, this continuation is exactly the continuation of  $\gamma(z)$  from the point at infinity to  $p$  along  $L$ , which does imply that  $\gamma(p) = 1$  as claimed.

It follows from (B.3.3) that

$$\begin{cases} (B/A)(z), & z \in \mathfrak{R}^{(0)}, \\ -(A/B)(z), & z \in \mathfrak{R}^{(1)}, \end{cases} \quad (\text{B.3.4})$$

is a rational function on  $\mathfrak{R}$  with two simple zeros  $\infty^{(0)}$  and  $p^{(0)}$  and two simple poles  $\infty^{(1)}$  and  $p^{(1)}$  (if it happens that  $p \in (\gamma_1 \cup \gamma_2)^{\circ}$ , then we choose  $p^{(0)} \in \mathfrak{R}$  precisely in such a way that it is a zero of (B.3.4) and  $p^{(1)}$  so it is a pole of (B.3.4); it is, of course, still true that these points are distinct and  $\pi(p^{(k)}) = p$ ). Therefore, Abel's theorem yields that

$$\left[ \int_{p^{(0)}}^{\infty^{(1)}} \mathcal{H} \right] = \left[ \int_{p^{(1)}}^{\infty^{(0)}} \mathcal{H} \right] \quad (\text{B.3.5})$$

while the relations (7.3.18), in particular, imply that

$$\left[ \int_{p^{(0)}}^{z_{n,0}} \mathcal{H} \right] = \left[ \int_{p^{(1)}}^{z_{n,1}} \mathcal{H} \right]. \quad (\text{B.3.6})$$

Let  $z_k$  be a topological limit of a subsequence  $\{z_{n_i,k}\}$ . Holomorphy of the differential  $\mathcal{H}$  implies that

$$\int_{p^{(k)}}^{z_{n_i,k}} \mathcal{H} = \int_{p^{(k)}}^{z_k} \mathcal{H} + \int_{z_k}^{z_{n_i,k}} \mathcal{H} \rightarrow \int_{p^{(k)}}^{z_k} \mathcal{H}$$

as  $i \rightarrow \infty$ , where the integral from  $z_k$  to  $z_{n_i,k}$  is taken along the path that projects into a segment joining  $z_k$  and  $z_{n_i,k}$ . The desired claim now follows from (B.3.5), (B.3.6), and the unique solvability of the Jacobi inversion problem on  $\mathfrak{R}$ .

#### B.4 Proof of Proposition 9.2.1

It follows from (9.0.6) and the choice of the branch of  $Q^{1/2}(z)$  that

$$Q^{1/2}(z) = \frac{z^2 - t}{2} + \mathcal{O}\left(\frac{1}{z}\right)$$

as  $z \rightarrow \infty$ . It further follows from the choice of the constant  $\varsigma$  in (9.2.1) that

$$z + \int_{\Gamma(b_1, a_2)} \frac{3\varsigma}{s - z} \frac{ds}{Q^{1/2}(s)} = z - \frac{2t}{z} + \mathcal{O}\left(\frac{1}{z^2}\right)$$

as  $z \rightarrow \infty$ . Since the product of the above functions behaves like  $-3V(z)/2 + \mathcal{O}(1)$  as  $z \rightarrow \infty$ , the analyticity properties of  $D(z)$  follow. Moreover, since  $Q_+^{1/2}(s) = -Q_-^{1/2}(s)$  for  $s \in J$ , we get the first relation in (9.2.3). The second relation in (9.2.3) follows from Plemelj-Sokhotski formula

$$\left( \int_{\Gamma(b_1, a_2)} \frac{\varsigma}{s - z} \frac{ds}{Q^{1/2}(s)} \right)_+ - \left( \int_{\Gamma(b_1, a_2)} \frac{\varsigma}{s - z} \frac{ds}{Q^{1/2}(s)} \right)_- = \frac{2\pi i \varsigma}{Q^{1/2}(z)}$$

for  $z \in \Gamma(b_1, a_2)$ .

## C. EXAMPLES OF JACOBI-TYPE POLYNOMIALS ON THE CROSS

In this appendix, we illustrate Theorem 5.5.2 by three examples. In them, we shall not compute  $S_\rho(\mathbf{z})$  and  $c_\rho$  via their integral representations, (5.3.3) and (5.3.5), but rather construct a candidate  $\widehat{S}_\rho(\mathbf{z})$  with the desired jump over  $\Delta$  and the singular behavior as in (5.3.7). This construction will also determine a candidate constant  $\widehat{c}_\rho$ . It is simple to argue that

$$S_\rho(\mathbf{z}) = \widehat{S}_\rho(\mathbf{z}) \exp \left\{ 2\pi i m \int_{\mathbf{a}_3}^{\mathbf{z}} \Omega \right\}, \quad c_\rho = \widehat{c}_\rho - m\mathbf{B},$$

for some integer  $m$ . Using  $\widehat{c}_\rho$  in (5.4.4), we then construct  $\widehat{T}_{i(n)}(\mathbf{z})$  for which it holds that

$$T_{i(n)}(\mathbf{z}) = \widehat{T}_{i(n)}(\mathbf{z}) \exp \left\{ -2\pi i m \int_{\mathbf{a}_3}^{\mathbf{z}} \Omega - \pi i m^2 \mathbf{B} + 2\pi i (-1)^{i(n)} \mathbf{K}_+ \right\}$$

with the same integer  $m$ . This means that

$$(S_\rho T_{i(n)})(\mathbf{z}) / (S_\rho T_{i(n)})(\infty^{(0)}) = (\widehat{S}_\rho \widehat{T}_{i(n)})(\mathbf{z}) / (\widehat{S}_\rho \widehat{T}_{i(n)})(\infty^{(0)})$$

and therefore (5.5.8) and (5.6.2) remain valid with  $S_\rho(\mathbf{z})$ ,  $T_{i(n)}(\mathbf{z})$  replaced by  $\widehat{S}_\rho(\mathbf{z})$ ,  $\widehat{T}_{i(n)}(\mathbf{z})$ . Furthermore, the value of  $A_{\rho,n}$  in (5.5.2) will not change either as the limit in the definition of  $A'_{\rho,n}$  will be augmented by  $e^{\pi i m(1-\mathbf{B})}$ , see (A.0.1), that will be offset by the change in  $c_\rho$  and  $\sigma_k$  ( $\widehat{\sigma}_k = (-1)^m \sigma_k$ ). Thus, with a slight abuse of notation, we shall keep on writing  $S_\rho(\mathbf{z})$ ,  $T_{i(n)}(\mathbf{z})$  below.

### C.1 Chebyshev-type case

Let  $2\widehat{\rho}(z) = 1/w(z)$ , in which case it holds that

$$\rho(s) = 1/w_+(s), \quad s \in \Delta,$$

where  $\widehat{\rho}(z)$  and  $w(z)$  were defined in (5.0.2) and (5.1.1), respectively, and the implication follows from the Plemelj-Sokhotski formulae and Privalov's theorem. Using analytic continuations of  $w(z)$  one can easily see that  $\rho(s) \in \mathcal{W}_\infty$  and  $\nu = 0$ . Since  $(\rho w_+)(s) \equiv 1$ , we get that  $S_\rho(z) \equiv 1$  and necessarily  $c_\rho = 0$ . Thus,  $\mathbb{N}_{\rho, \epsilon} = 2\mathbb{N}$  and  $z_0 = \infty^{(1)}$  ( $z_1 = \infty^{(0)}$ ). Moreover, we get that  $T_0(z) \equiv 1$  and  $T_1(z) = 1/\Phi(z)$ , see (A.0.2). Hence, it follows from (5.2.8) and (5.5.8) that

$$Q_{2n}(z) = \frac{1 + o(1)}{2^n} \left( z^2 + \frac{b^2 - a^2}{2} + w(z) \right)^n,$$

where it holds that  $o(1)$  is geometrically small on closed subsets of  $\overline{\mathbb{C}} \setminus \Delta$  (see [23] for the error rate in this case). To show that the above result is in a way best possible, assume that  $a = b = 1$ . Recall that the  $n$ -th monic Chebyshev polynomial of the first kind is defined by

$$2^n T_n(z) = \left( z + \sqrt{z^2 - 1} \right)^n + \left( z - \sqrt{z^2 - 1} \right)^n$$

and is orthogonal to  $x^j$ ,  $j \in \{0, \dots, n-1\}$ , on  $(-1, 1)$  with respect to the weight  $1/\sqrt{1-x^2}$ . Hence,

$$\begin{aligned} i \int_{\Delta} s^k T_n(s^2) \rho(s) ds = \\ \left( \int_0^1 - \int_{-1}^0 \right) \frac{x^k T_n(x^2) dx}{\sqrt{1-x^4}} - i^{k+1} \left( \int_0^1 - \int_{-1}^0 \right) \frac{x^k T_n(-x^2) dx}{\sqrt{1-x^4}}. \end{aligned}$$

Clearly, the above expression is zero for all even  $k$ . Assume now that  $k = 2j + 1$ ,  $j \in \{0, \dots, n-1\}$ . Then we can continue the above chain of equalities by

$$\int_0^1 \frac{x^j T_n(x) dx}{\sqrt{1-x^2}} - (-1)^{j+1} \int_0^1 \frac{x^j T_n(-x) dx}{\sqrt{1-x^2}} = \int_{-1}^1 \frac{x^j T_n(x) dx}{\sqrt{1-x^2}} = 0,$$

where the last equality follows from the orthogonality properties of the Chebyshev polynomials. Thus, it holds that

$$Q_{2n+1}(z) = Q_{2n}(z) = T_n(z^2)$$

in this case, which justifies the exclusion of odd indices from  $\mathbb{N}_\rho = \mathbb{N}_{\rho, \epsilon}$  as for such indices polynomials can and do degenerate.

## C.2 Legendre-type case

Let  $\widehat{\rho}(z) = \frac{1}{2\pi i} (\log(z^2 - 1) - \log(z^2 + 1))$ , in which case it holds that

$$\rho(s) = (-1)^i, \quad s \in \Delta_i,$$

$i \in \{1, 2, 3, 4\}$ , where the justification for the implication is the same as before. As in the previous case, it holds that  $\nu = 0$ . Let  $\sqrt{w}(z)$  be the branch holomorphic in  $\mathbb{C} \setminus \Delta$  such that  $\sqrt{w}(z) = z + \mathcal{O}(1)$  as  $z \rightarrow \infty$ . Further, let

$$\Phi_*(z) := \sqrt{\frac{2}{a^2 + b^2}} \left( z^2 + \frac{b^2 - a^2}{2} + w(z) \right)^{1/2},$$

be the branch holomorphic in  $\mathbb{C} \setminus \Delta$  such that  $\Phi_*(z) = z + \mathcal{O}(1)$  as  $z \rightarrow \infty$ . It easily follows from (5.2.6), (5.2.8), and (5.2.9) that  $\Phi_*(z)$  is an analytic continuation of  $-\Phi(z^{(0)})$  across  $\pi(\boldsymbol{\alpha}) \cup \pi(\boldsymbol{\beta})$ . It is now straightforward to check that

$$S_\rho(z^{(0)}) = e^{-\pi i/4} \Phi_*(z) / \sqrt{w}(z)$$

and thus  $c_\rho = 0$ . Hence, as in the previous subsection,  $\mathbb{N}_{\rho, \epsilon} = 2\mathbb{N}$  and  $T_0(\mathbf{z}) \equiv 1$  while  $T_1(\mathbf{z}) = 1/\Phi(\mathbf{z})$ . Therefore, we again deduce from (5.2.8) and (5.5.8) that

$$Q_{2n}(z) = \frac{1 + \mathcal{O}(n^{-1/2})}{2^{n+1/2} \sqrt{w}(z)} \left( z^2 + \frac{b^2 - a^2}{2} + w(z) \right)^{n+1/2},$$

uniformly on closed subsets of  $\overline{\mathbb{C}} \setminus \Delta$ . Again, to show that the above result is best possible, assume that  $a = b = 1$ . Then we can check exactly as in the previous subsection that

$$Q_{2n+1}(z) = Q_{2n}(z) = L_n(z^2),$$

where  $L_n(x)$  is the  $n$ -th monic Legendre polynomial, that is, degree  $n$  polynomial orthogonal to  $x^j$ ,  $j \in \{0, \dots, n-1\}$ , on  $(-1, 1)$  with respect to a constant weight.

## C.3 Jacobi-1/4 case

Let  $\sqrt{2}\widehat{\rho}(z) = 1/\sqrt{w}(z)$ , in which case it holds that

$$\rho(s) = -i^{4-i} / |\sqrt{w}(s)|, \quad s \in \Delta_i, \quad i \in \{1, 2, 3, 4\},$$

where  $\sqrt{w}(z)$  is the branch defined in the previous subsection. Observe that

$$(\rho w_+)(s) = i^{i-1} |\sqrt{w}(s)|, \quad s \in \Delta_i,$$

and that  $\nu = 1/2$ . In particular, the constant  $A_\rho$  appearing in the definition of  $A_{\rho,n}$  in (5.5.2) is equal to  $A_\rho = \sqrt{2}e^{-\pi i/4}/\sqrt{ab}$ .

To construct a Szegő function of  $\rho(s)$ , let

$$\Theta^2(z) := \frac{\theta\left(\int_{a_3}^z \Omega + K_-\right) \theta\left(\int_{a_3}^z \Omega - K_+\right)}{\theta\left(\int_{a_3}^z \Omega - K_-\right) \theta\left(\int_{a_3}^z \Omega + K_+\right)}, \quad z \in \mathfrak{R}_{\alpha,\beta},$$

where the path of integration lies entirely in  $\mathfrak{R}_{\alpha,\beta}$ . It follows from (5.4.3) and (A.0.1) further below that  $\Theta^2(z)$  is a meromorphic function in  $\mathfrak{R}_{\alpha,\beta}$  with two simple poles, namely,  $\infty^{(0)}$ ,  $\mathbf{0}$ , and two simple zeros  $\infty^{(1)}$ ,  $\mathbf{0}^*$ . Moreover,  $\Theta^2(z)$  is continuous across  $\beta$  and satisfies  $\Theta_+^2(s) = \Theta_-^2(s)e^{-2\pi i B}$  on  $\alpha$  by (5.4.1) and  $\Theta^2(z)\Theta^2(z^*) \equiv 1$  by the symmetries of  $\theta(\zeta)$  and  $\Omega(z)$ . Since each individual fraction in the definition of  $\Theta^2(z)$  is injective, we can define a branch  $\Theta(z)$  such that

$$\Theta_+(s) = \Theta_-(s) \begin{cases} e^{-\pi i B}, & s \in \alpha, \\ -1, & s \in \Delta_3 \cup \pi^{-1}((-\infty, -a]), \end{cases}$$

and  $\Theta(z)\Theta(z^*) \equiv 1$ . Further, let  $w^{1/4}(z)$  be the branch holomorphic in  $\mathbb{C} \setminus (\Delta \cup (-\infty, a))$  that is positive for  $z > a$ . Now, one can verify that  $c_\rho = -B/2$  and

$$S_\rho(z^{(k)}) = \Theta(z^{(k)}) w^{\frac{2k-1}{4}}(z), \quad k \in \{0, 1\}.$$

Let us now compute  $A'_{\rho,n}$  appearing in (5.5.2). Since  $\sqrt{w}(z) \rightarrow e^{-3\pi i/4}\sqrt{ab}$  as  $\mathcal{Q}_3 \ni z \rightarrow 0$ , we get that

$$\begin{aligned} \lim_{z \rightarrow 0, \arg(z) = 5\pi/4} |z| S_\rho^2(z^{(0)}) &= \frac{e^{-\pi i/2}}{\sqrt{ab}} \lim_{\mathcal{Q}_3 \ni z \rightarrow 0} z \Theta^2(z^{(0)}) \\ &= e^{\pi i B/2} \frac{2\sqrt{ab}}{\sqrt{a^2 + b^2}} \Phi(\mathbf{0}), \end{aligned}$$

where the second equality follows from (A.0.1), (A.0.5), (A.0.9), and (A.0.10) further below. Therefore, it holds that  $A'_{\rho,n} = \Phi(\mathbf{0})$ . It is easy to see from (A.0.1) that

$\mathbf{z}_0 = \mathbf{0}$ ,  $l_0 = 0$ ,  $m_0 = 1$ , and  $\mathbf{z}_1 = \mathbf{0}^*$ ,  $l_1 = m_1 = 0$ . Therefore,  $\sigma_{i(n)} = -1$  and the condition defining  $\mathbb{N}_{\rho, \epsilon}$  in Proposition 5.5.1 specializes to

$$|1 + \exp \{2i(n - i(n)) \arctan(a/b)\}| > \epsilon$$

by (5.2.10) and since  $\Phi(\mathbf{z}_1)\Phi(\mathbf{z}_0) = 1$ , see (A.0.4) further below. As  $T_0(\mathbf{0}) = 0$  and respectively  $L_{n1} = 0$ , we then get that  $Q_n(z)$ ,  $n \in \mathbb{N}_{\rho, \epsilon}$ , is equal to

$$\gamma_n(S_\rho \Phi^n)(z^{(0)}) \begin{cases} (T_0(z^{(0)}) + \mathcal{O}_\epsilon(n^{-1})), & n \in 2\mathbb{N}, \\ (T_1(z^{(0)}) + z^{-1}L_{n2}(T_0/\Phi)(z^{(0)}) + \mathcal{O}_\epsilon(n^{-1})), & n \notin 2\mathbb{N}, \end{cases}$$

uniformly on closed subsets of  $\overline{\mathbb{C}} \setminus \Delta$ , where

$$L_{n2} = \frac{-1}{(T_0/T_1)'(\mathbf{0})} \frac{\Phi^{2n-1}(\mathbf{0})}{1 + \Phi^{2(n-1)}(\mathbf{0})}$$

for all odd  $n$ . When  $a = b$ , we further get that  $L_{n2} = -e^{\pi i/4}/[2(T_0/T_1)'(\mathbf{0})]$  for  $n \in \mathbb{N}_{\rho, \epsilon}$  and

$$\mathbb{N}_{\rho, \epsilon} = \{n = 4k, 4k + 1 : k \in \mathbb{N}\}.$$

Assume further that  $a = b = 1$  and let  $P_{n,1}(x)$  be the  $n$ -th degree monic polynomial orthogonal on  $[0, 1]$  to  $x^j$ ,  $j \in \{0, \dots, n-1\}$ , with respect to the weight function  $x^{-3/4}(1-x)^{-1/4}$ . Then

$$\int_{\Delta} s^k P_{n,1}(s^4) \rho(s) ds = (1 + i^k) \int_{-1}^1 y^k P_{n,1}(y^4) \frac{dy}{(1-y^4)^{1/4}},$$

which is equal to zero for all  $k$  odd by symmetry and for all  $k = 4j + 2$  due to the factor  $1 + i^k$ . When  $k = 4j$ ,  $j \in \{0, \dots, n-1\}$ , we can further continue the above equality by

$$4 \int_0^1 y^{4j} P_{n,1}(y^4) \frac{dy}{(1-y^4)^{1/4}} = \int_0^1 x^j P_{n,1}(x) \frac{dx}{x^{3/4}(1-x)^{1/4}} = 0,$$

where the last equality now holds by the very choice of  $P_{n,1}(z)$ . Hence, it holds that

$$Q_{4n}(z) = P_{n,1}(z^4) \quad \text{and} \quad Q_{4n+1}(z) = Q_{4n+2}(z) = Q_{4n+3}(z) = zP_{n,2}(z^4),$$

where the second set of relations can be shown similarly with  $P_{n,2}(x)$  being the  $n$ -th degree monic polynomial orthogonal on  $[0, 1]$  to  $x^j$ ,  $j \in \{0, \dots, n-1\}$ , with respect to

the weight function  $x^{1/4}(1-x)^{-1/4}$ . That is, the restriction to the sequence of indices  $\{n = 4k, 4k + 1 : k \in \mathbb{N}\}$  is not superfluous and the main term of the asymptotics of the polynomials does depend on the parity of  $n$ .