STATE OBSERVATION AND UNKNOWN INPUT AND OUTPUT DISTURBANCE ESTIMATION IN NETWORKED CONTROL SYSTEMS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

Badriah Alenezi

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

December 2020

Purdue University

West Lafayette, Indiana

THE PURDUE UNIVERSITY GRADUATE SCHOOL
STATEMENT OF DISSERTATION APPROVAL

Prof. Stanislaw H. Żak, Chair

School of Electrical and Computer Engineering

Prof. Inseok Hwang

School of Aeronautics and Astronautics

Prof. Jinghai Hu

School of Electrical and Computer Engineering

Prof. Oleg Wasynczuk

School of Electrical and Computer Engineering

Approved by:

Dimitrios Peroulis,

Head of the School of Electrical and Computer Engineering

To my parents Hana and Abdulhakeem, who support me always.

To my husband Mahmoud, who always is on my side.

To my daughters, Zahrah and Rania, who bring the joy to my life.

ACKNOWLEDGMENTS

”He has not thanked God, who has not thank people” Prophet Mohammed (PBUH).

The words are not enough to express my deep appreciation and gratitude to my advisor, Professor Stanislaw Żak for his intelligent guidance, kind treatment, constant encouragement and sense of humor. I am lucky to have a great advisor like Professor Żak. He always pulls the best out of me, guide me to find better ideas, does not under estimate ideas I have, and always has a great positive attitude which encourages me through the hard points of research. Despite all the research work we have completed together, he does not hesitate to ask me about my family at the end of our meetings, which makes me feel more comfortable as a Ph.D. student-mother.

I would like to extend my sincere gratitude to Professor Mohammed Zeribi, my bachelor and master’s thesis advisor in Kuwait University, for his belief in me and for supporting me on pursuing my Ph.D. degree at Purdue university. Also, I would like to thank my committee members for their guidance. I wish to acknowledge Professor Jianghai Hu for his very inspiring classes, for his insightful comments on my first research paper, and for his support at the 2019 American Control Conference. I am also grateful to Professor Oleg Wasynczuk for his kindness and his very interesting Electric Hybrid Vehicle class. I would also like to acknowledge Professor Inseok Hwang for his two very helpful classes in my research. I am especially grateful to Professor Stefen Hui for his discussions and insightful comments that contributed to this research. I am also thankful to all my school teachers, my university professors who prepared me to conduct research.

I am grateful to Mukai Zhang for his collaboration on our research papers. It was a wonderful experience working with him on our many successful projects.

I would like to thank my family and my parents, Hana and Abdulhakeem, who always support me in every step in my life. They always have been on my side and help me to achieve my goals in every possible way. I am thankful to my husband, Mahmoud, for encouraging me to pursue my dreams and for his sacrifices to help me to complete my Ph.D. degree. Thanks to my daughters, Zahrah and Rania, for lightening my life with joy and happiness. Thanks to God for all of his blessings.

Last of all, thanks to my freinds, Fatema, Noura, and Fajer for believing in me and encouraging me with their warm words. Thanks to all my friends in Indiana for being a great family during my time here in the USA.

— *Badriah Alenezi*

TABLE OF CONTENTS

Page

Page

Page

LIST OF TABLES

LIST OF FIGURES

Figure Page

ABSTRACT

Alenezi, Badriah Ph.D., Purdue University, December 2020. State Observation and Unknown Input and Output Disturbance Estimation in Networked Control Systems. Major Professor: Stanislaw H. Żak.

In modern control systems, actuators and sensors are connected over communication networks. Such systems, referred to as networked control systems, can be subjected to various disturbances. These disturbances can have form of malicious attacks on the communication channels between the plant sensors and the controller and between the controller and the actuators. To protect the networked control system against such attacks, detectors of incoming attacks are needed. Attacks on the plant actuators are modeled as unknown bounded inputs, while attacks on the plant sensors are represented as output disturbances.

Continuous-time (CT) and discrete-time (DT) unknown input observer (UIO) architectures are developed to estimate the state, unknown inputs, and output disturbances in networked control systems. Adaptive CT schemes for unknown input and state estimation are proposed. Novel DT state and unknown input observers are proposed for a class of nonlinear networked control systems whose nonlinearities can be characterized by incremental multiplier matrices. Then, DT unknown input and output disturbance estimators are developed for the detection of attacks on the plant input and output channels. Delayed unknown input and output disturbance estimators are proposed for DT networked control systems for which the matrix rank condition for the existence of UIOs is not satisfied. An observer-based decentralized control design method is proposed for networked control systems where the communication network is modeled as a pure time-delay.

The results obtained can be applied to the observer-based decentralized control of networked control systems in the presence of time-delays and disturbances resulting from the presence of the communication networks.

1. INTRODUCTION

1.1 Motivation

In modern control systems, actuators and sensors are connected over communication networks. Such systems, referred to as networked control systems, can be subjected to various disturbances. These disturbances can have the form of adversarial attacks. To guarantee safe operation of the entire system, it is desirable to detect such attacks. The attacks could occur on the communication channels between the plant sensors and the controller and between the controller and actuators. We model the attacks on the plant actuators as bounded unknown inputs, while we model the attacks on the plant sensors as output disturbances.

In our activity, we will be using state observers. A state observer or state estimator or just observer is a dynamical system that generates an estimate of the plant state using only the input and output measurements of the plant. The state observer was first proposed by D. G. Luenberger for linear systems in the early sixties [1–3]. A block diagram of the state observer is shown in Figure 1.1.

As we mention above, we model adversarial attacks on the networked control system as unknown inputs and output disturbances. We propose to use unknown input observers (UIOs) to detect and monitor such attacks. An unknown input observer (UIO) is an observer that estimates the plant state in the presence of unknown input. The design of UIOs can be categorized into two classes: (i) UIOs for the state estimation in the presence of unknown inputs, and (ii) UIOs for the state and unknown input estimation. A block diagram of an unknown input observer is given in Figure 1.2.

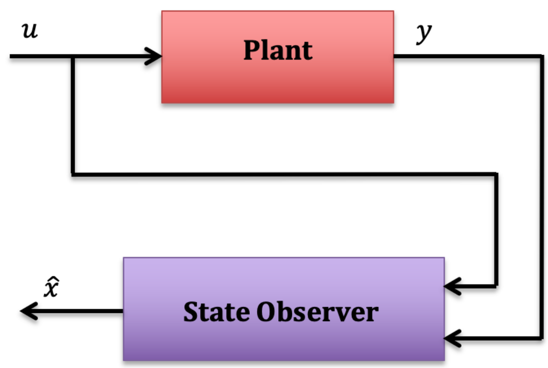

Fig. 1.1.: A block diagram of a state observer, where u is the known input, y is the output signal, $\hat{x}$ is the state estimate of the plant.

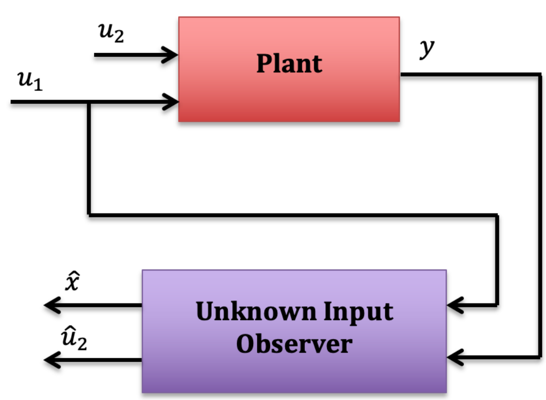

Fig. 1.2.: A block diagram of an unknown input observer, where u_1 is the known input, u_2 is the unknown input, y is the plant output, $\hat{x}$ is the state estimate of the plant, and $\hat{u}_2$ is the estimate of the unknown input.

The problem of designing observers for linear systems with unknown inputs was already studied by Basile and Marro [4] in 1969. Since then, different UIO architectures were developed, see [5] for an overview of early UIO developments, [6–8] for early UIO designs, and [9] for a comparative study of some UIO architectures.

The unknown input observer has been used in many applications, such as, fault detection [10–14], stress estimation in humans [15], secure state estimation in cyber physical systems (CPS) [16–18], fault detection in hydraulic valves [19], detection of sensor faults in a wind turbine [20], and recovery of hidden messages in the transmitted signals [21].

We use observer-based decentralized controller to control large scale networked control systems. Examples of large scale decentralized systems are power networks [22], urban traffic networks [23], and robotic systems [24].

1.2 Problem Statement

We propose continuous-time (CT) and discrete-time (DT) unknown input observers for the state, unknown input, and output disturbance estimation in networked control systems. We also propose observer-based decentralized controller to control large scale networked control systems.

1.3 Literature Overview

In this section, we review the literature related to the research reported in this thesis. We divide the review into parts corresponding to the types of the observers analyzed in the thesis.

1.3.1 Adaptive unknown input observers

Different unknown input observer architectures were proposed in the literature. Sliding mode and higher-order sliding mode observers for unknown input reconstruction were used for fault detection in [10, 11, 13] and for stress estimation in humans in [15]. A Lyapunov-type conditions were developed for the existence of an estimator that can estimate the state and the unknown input to any degree of accuracy in [25]. These conditions are also sufficient for the existence of a sliding mode unknown input observer that asymptotically estimates the plant state and the unknown input. In [26], high-gain approximate differentiator based sliding mode observer architecture were proposed for linear systems with unknown inputs that do not satisfy the so-called observer matching condition. The estimation error was proved to be uniformly ultimately bounded. A reduced-order observer for linear systems with unknown inputs was presented in [6], where the state and the unknown input were estimated. Another reduced-order unknown input observer was presented in [27]. A full-order state observer for linear systems with unknown input was proposed in [7]. In [28], a distributed decoupled observer was presented using an equivalent "free of unknown input" system to simplify the design procedure.

The design of observers for a class of nonlinear systems in the presence of bounded disturbance inputs were reported in [29]. Linear matrix inequalities were given for the design of state and unknown input observer that guarantees the state estimation error to satisfy a prescribed degree of accuracy using the $\mathscr{L}_\infty$-stability concept.

Adaptive unknown input observers were proposed recently in [30–32]. The adaptive unknown input observer proposed in [30] uses multiple observers. It is a modified form of the standard UIO where a bank of parallel observers are constructed to generate residual signals, which are used to detect and isolate actuator faults. To apply this scheme, n independent measurements should be available for the n-th order plant which limits the applicability of this approach. In [31], unknown input observer was used to estimate the torque in a vehicle engine. In [32], the plant state was estimated using a robust adaptive UIO for secure communication.

In our design, bounded adaptive unknown input estimators are proposed to estimate the plant state and the unknown input.

1.3.2 State observers and unknown input estimators for nonlinear systems characterized by incremental multiplier matrices

Unknown input observers (UIOs) were proposed for different classes of dynamical systems. Continuous-time (CT) UIOs were reported in [33, 34]. In [8], state and unknown input estimation was considered for a class of uncertain linear systems and a state dependent, time varying unknown input. In [35–37], discrete-time (DT) UIOs were constructed for linear DT systems. A DT state observer for a class of nonlinear systems using dissipativity theory was proposed in [38]. In [9], a CT UIO was designed using a projection operator approach. In our design, we adopt this projection method to propose a novel DT UIO architecture.

The UIO design for a class of DT nonlinear systems with locally Lipschitz nonlinearities was considered in [39] and this was generalized in [40] to a class of DT nonlinear systems characterized by incremental multiplier matricesIn [41], sliding mode observers were proposed for the state and unknown input estimation of CT nonlinear systems characterized by incremental multiplier matrices. We consider a class of DT nonlinear systems characterized by incremental multiplier matrices. We propose an observer architecture to estimate the state and the unknown input for such class of plants.

1.3.3 State, unknown input, and output disturbances estimation in DT linear network systems

A state observer for DT systems in the presence of disturbances on the sensors and actuators of the plant was proposed in [42]. In [43], a DT UIO was proposed to estimate the plant state in the presence of unknown inputs. In [44], designs of estimators and controllers were proposed for linear systems with the plant actuators or sensors under malicious attacks. A secure state estimator was proposed in [45] when the communication channel between a sensor and a remote estimator is corrupted by jamming attacks. In [46], a distributed attacks detectors and distributed state estimators were proposed for networked control systems under malicious attacks. A secure state estimator of distributed power systems under cyber-physical attacks and communication failure was presented in [47]. A so-called fixed-time observer for DT singular systems corrupted by unknown inputs was considered in [48], where the observer matching condition and the strong observability condition were satisfied.

We propose an observer architectures to simultaneously estimate the state, unknown input, and output disturbance for linear network systems corrupted by actuator and sensor attacks.

1.3.4 Delayed estimation of unknown input and output disturbances

One of the conditions for the existence of an UIO is the matrix rank condition, $\text{rank}(CB_2) = \text{rank}(B_2)$, where B_2 is the input matrix corresponding to the unknown input and the matrix C is the output measurement matrix. However, in many cases, the plant model does not satisfy this condition. When the matrix rank condition is failed, the vector recovery method reported in [49] was recently used. In [50, 51], the vector recovery method is applied to detect malicious packet drop attacks in a networked control system. The main idea of the method is to transform the state and unknown input estimation problem into a 0-norm minimization problem with equality constraints. The resulting problem is then solved using the 1-norm approximation of the 0-norm minimization problem. However, this approach has its limitations in that the unknown input has to be sparse. That is, the disturbance vector must have more zero entries than non-zero entries. The sparsity requirement is needed in order to obtain accurate approximation of 0-norm minimization by the 1-norm minimization [52]. In [53], DT UIO was constructed using the vector recovery method for DT linear network systems corrupted by sparse unknown input and output errors.

In [54], Hautus observed that if the plant is strongly detectable but the matrix rank condition is not satisfied, a delayed observer can be constructed such that the state is estimated with a delay. Following this idea, a sequence of extra output measurements is collected in order to relax the matrix rank condition in [55, 56]. In [57], the actuators and sensors of the plant are assumed to be corrupted by the same disturbance signals. For the same type of plant models, a DT UIO was constructed in [58], where the observation error was guaranteed to be bounded with a prescribed performance level.

We propose unknown input and output disturbance estimators for DT linear network systems when the matrix rank condition for the existence of an UIO is not satisfied.

1.3.5 Decentralized networked control systems

The design of decentralized controllers were reported in [59–63]. In [59, 64], local feedback control laws that depend only on partial system outputs were used to stabilize a linear time-invariant multivariable system. This gives a good starting point for the investigation of the stabilization of decentralized control systems. In [61], decentralized dynamic pole placement was used along feedback stabilization to the design of decentralized controllers. In [62], the decentralized controller design was formulated as a linear quadratic optimal regulator problem. In [63], an observer-based controller was implemented using decentralized functional observers at each local station. This results in a global state feedback controller using only local input-output information available at each station.

Data transmission error in communication networks is inevitable [65, 66]. The unknown errors occur when the control signals and measurements are transferred between the plant and the decentralized control system (DCS) through a communication network. Different communication network models were proposed in the literature. For example, in [67] a model of communication network was proposed that combines Gaussian random noise with sparse malicious packet drops for a remotely controlled cyber physical system. Another model of the networked system was reported in [68] which considers the packet dropout and transmission delays.

In [69–72], communication networks were modeled as pure time-delays. This type of communication network model simplifies the stability analysis of the decentralized networked control system (DNCS). It was shown in [73] and [74] that large time-delays degrade the performance of NCS or even destabilize the system. When the stability is not affected by the duration of the time-delay, then the control system stability is independent of the time-delay duration, see for example, [75]. Usually, however, the stability of the DNCS is dependent on the communication time-delays [76].

In our thesis, we propose a new approach to the design of a dynamical decentralized controller that implements the static centralized controller selected by the designer. We use results reported by Schoen [77] to obtain an upper bound on the communication network time-delay that will guarantee the stability of the decentralized networked control system (DNCS).

1.4 Thesis Organization

The rest of the document is organized as follows. In Chapter 2, an adaptive continuous-time CT scheme for unknown input and state estimation for a class of uncertain systems is presented. We consider two types of unknown inputs: constant and bounded not necessarily constant unknown input. Linear matrix inequality (LMI) conditions for both cases are derived. The state and bounded unknown input estimation with guaranteed performance is proved using $\mathscr{L}_\infty$-stability approach. In Chapter 2.6, a novel DT unknown input observer for a class of nonlinear systems characterized by incremental multiplier matrices is presented. Then, linear matrix inequality condition is derived for the proposed UIO. Also, an unknown input estimator is proposed that reconstructs the unknown input with one sampling period time-delay.

In Chapter 3.7, a state observer and unknown input and output disturbance estimators are proposed for discrete-time (DT) linear network systems. The state, unknown input, and output disturbance estimation errors are guaranteed to be l_∞-stable with prescribed performance level. The design of the state observer and disturbance estimators are given in terms of LMIs. In Chapter 4.11, delayed unknown input and output disturbance estimators are proposed. The delayed observers are required when the existence condition of unknown input observer (UIO) is not satisfied. Thus, a backward sequence of the output is used to relax the existence condition. The unknown input and output disturbance estimation errors are l_∞-stable with performance level.

In Chapter 5.7, an observer-based decentralized controller is proposed for the decentralized networked control system (DNCS), where the control loop is closed by a communication network. The communication network is modeled as pure time-delay. An observer-based decentralized controller is designed using LMIs. Finally, in Chapter 7, we present open problems that we will work on to extend our research in the future.

1.5 Publications

- B. Alenezi, J. Hu, S.H. Żak, *Adaptive Unknown Input and State Observers*. In 2019 American Control Conference (ACC), Philadelphia, PA, USA, 2019 pp. 2434–2439.
- B. Alenezi, M. Zhang, S. Hui, S.H. Żak, *State Observers and Unknown Input Estimators for Discrete-Time Nonlinear Systems Characterized by Incremental Multiplier Matrices*. In 59^{th} Conference on Decision and Control (CDC), Jeju Island, Republic of Korea, 2020.
- B. Alenezi, M. Zhang, S. Hui, S.H. Żak, *Simultaneous Estimation of the State, Unknown Input, and Output Disturbance in Discrete-Time Linear Systems*. Submitted to IEEE Transactions on Automatic Control.
- B. Alenezi, M. Zhang, S. Hui, S.H. Żak, *Delayed Estimation of Unknown Input and Output Disturbances in Discrete-Time Linear Systems*. Submitted to the 2021 American Control Conference (ACC), New Orleans, Louisiana, USA.

- B. Alenezi, M. Zhang, S.H. Żak, *Observer-Based Controller Synthesis for Decentralized Networked Systems*. In 2020 IEEE Conference on Control Technology and Applications (CCTA), Montreal, QC, Canada, 2020, pp. 732-737.
- M. Zhang, B. Alenezi, S. Hui, S.H. Żak, *State Estimation of Networked Control Systems Corrupted by Unknown Input and Output Sparse Errors*. In 2020 American Control Conference (ACC), Denver, CO, USA, 2020, pp. 4393–4398.
- M. Zhang, B. Alenezi, S. Hui, S.H. Żak, *Unknown Input Observers for Discretized Systems with Application to Networked Systems Corrupted by Sparse Malicious Packet Drops*. In IEEE Control Systems Letters (L-CSS), vol. 5, no. 4, pp. 1261-1266.

2. ADAPTIVE UNKNOWN INPUT AND STATE OBSERVERS

2.1 Introduction

This chapter presents a novel adaptive continuous-time (CT) observer architectures for simultaneous unknown input and state estimation for a class of uncertain systems. The adaptive unknown input estimator is bounded. The unknown input in the system plant is considered to be constant or bounded not necessarily constant unknown inputs. Using a Lyapunov approach, conditions are derived that ensure the state and unknown input estimation errors converge to zero for a constant unknown input. Next, combining a Lyapunov approach and linear matrix inequalities (LMIs), conditions are given that guarantee a prescribed performance level for state and unknown input estimation for a bounded not necessarily constant unknown input. We use $\mathscr{L}_\infty$-stability approach presented in [29], where a linear-in-state-error estimator is used.

2.2 Problem Statement

We consider a class of dynamical systems modeled by

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t) \tag{2.1a}$$

$$y(t) = Cx(t), \tag{2.1b}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u_1(t) \in \mathbb{R}^{m_1}$ is the control input, $u_2(t) \in \mathbb{R}^{m_2}$ is the unknown input, and $y(t) \in \mathbb{R}^p$ is the measured output. The system matrices are $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, and $C \in \mathbb{R}^{p\times n}$.

Our objective is to design adaptive state and unknown input observers for the dynamical system with constant and bounded not necessarily constant unknown inputs using available input-output information.

2.3 Observer Design

In this section, we propose an observer for adaptive state estimation in the presence of constant unknown input. The proposed method also allows for reconstruction of the constant unknown input.

2.3.1 Proposed observer architecture

The proposed observer for system model (2.1) is given by

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}) + B_1 u_1 + B_2 \hat{u}_2 \tag{2.2a}$$

$$\hat{y} = C\hat{x}, \tag{2.2b}$$

where $\hat{x}(t)$ is the state $x(t)$ estimate, and $\hat{u}_2$ is an adaptive estimator of u_2. The observer gain matrices L and F are obtained from the following conditions

$$(A - LC)^\top P + P(A - LC) \prec 0, \tag{2.3a}$$

$$B_2^\top P = FC, \tag{2.3b}$$

$$P = P^\top \succ 0, \tag{2.3c}$$

where the matrix F will be defined later. For system theoretical interpretation of conditions (2.3a), and (2.3b), we refer to [78, 79]. The adaptive estimator of the unknown input has the form

$$\dot{\hat{u}}_2 = \Gamma\sigma, \tag{2.4}$$

where

$$\Gamma = \text{diag}\{\Gamma_1, \Gamma_2, \cdots, \Gamma_{m_2}\}, \Gamma_i > 0, \text{ for } i = 1, 2, \cdots, m_2,$$
$$\sigma = \begin{bmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_{m_2} \end{bmatrix}^\top = F(y - \hat{y}).$$

Remark 1 *Conditions* (2.3a) *and* (2.3b) *have been proven by Corless and Tu [8] and Edwards and Spurgeon [25] to be equivalent to the following two conditions:*

Condition 1 $\text{rank}(CB_2) = \text{rank}(B_2)$.

Condition 2 *For every complex number λ with nonnegative real part,*

$$\text{rank}\begin{bmatrix} A - \lambda I & B_2 \\ C & O \end{bmatrix} = n + \text{rank}(B_2).$$

The existence of observers for continuous-time systems where the system has two types of inputs and outputs (measured and unmeasured) has been investigated by Hautus in [54], in which the concepts of strong and strong* detectability have been introduced. Hautus showed that the strong observability implies the strong detectability and that the existence of the state observer is equivalent to the strong* detectability. Hautus gave the conditions for the existence of a strong observer to estimate unknown input using only measured output. The existence conditions for our proposed adaptive unknown input and state observers are the same as Hautus' conditions for strong* detectability. Conditions 1 and 2 are necessary and sufficient for the existence of the strong observer of Hautus [54].

2.3.2 Error dynamics

Let the state estimation error be $e = x - \hat{x}$. Then, the observation error dynamics have the form

$$\dot{e} = (A - LC)e + B_2(u_2 - \hat{u}_2). \tag{2.5}$$

We now present Lyapunov-Like Lemma from [80, Subsection 4.12] that we use through our proof of the results in this section.

Lemma 1 *("Lyapunov-Like Lemma") Given a real-valued function $V(t,e)$ such that*

1. *$V(t,e)$ is bounded below,*
2. *$\dot{V}(t,e)$ is negative semidefinite, and*
3. *$\dot{V}(t,e)$ is uniformly continuos in time,*

then

$$\dot{V}(t,e) \to 0 \ \ \textit{as } t \to \infty$$

We give conditions for an adaptive state and unknown input reconstruction in the presence of constant unknown input in the form of the following theorem.

Theorem 1 *Suppose u_2 in the plant model given by* (2.1) *is constant and B_2 is a full column rank matrix. If there exist a symmetric matrix $P \succ 0$ and matrices L and F such that the conditions in* (2.3) *are satisfied, then the state observation error e converges to zero.*

Proof Consider the ideal error system dynamics

$$\dot{e} = (A - LC)e. \tag{2.6}$$

By (2.3a), $V = e^\top Pe$ is a Lyapunov function of (2.6). Let

$$Q = -((A - LC)^\top P + P(A - LC)).$$

Note that by (2.3a), $Q = Q^\top \succ 0$. Then, $\dot{V} = -e^\top Qe < 0$. We proceed by evaluating the derivative of V on the trajectories of (2.5) to obtain

$$\dot{V} = -e^\top Qe + 2e^\top PB_2(u_2 - \hat{u}_2).$$

Let the augmented Lyapunov function candidate be

$$V_a = V + (u_2 - \hat{u}_2)^\top \Gamma^{-1}(u_2 - \hat{u}_2) > 0$$

in the augmented space $(e, u_2 - \hat{u}_2)$. Evaluating the time derivative of V_a on the trajectories of (2.5) gives

$$\dot{V}_a = \dot{V} + \frac{d}{dt}((u_2 - \hat{u}_2)^\top \Gamma^{-1}(u_2 - \hat{u}_2)). \tag{2.7}$$

Let

$$\begin{aligned}
\Delta u_2 &= [\Delta u_{2_1} \;\; \Delta u_{2_2} \;\; \cdots \;\; \Delta u_{2_{m_2}}]^\top = u_2 - \hat{u}_2, \\
\sigma &= [\sigma_1 \;\; \sigma_2 \;\; \cdots \;\; \sigma_{m_2}]^\top = B_2^\top Pe.
\end{aligned}$$

Then,

$$\dot{V} = -e^\top Qe + 2\sigma^\top \Delta u_2 = -e^\top Qe + 2\sum_{i=1}^{m_2} \sigma_i \Delta u_{2_i}. \tag{2.8}$$

Taking into account the assumption that u_2 is constant, the second part of (2.7) becomes

$$\frac{d}{dt}(\Delta u_2^\top \Gamma^{-1} \Delta u_2) = -2\Delta u_2^\top \Gamma^{-1} \dot{\hat{u}}_2 = -2\sum_{i=1}^{m_2} \frac{1}{\Gamma_i} \Delta u_{2_i} \dot{\hat{u}}_{2_i}. \tag{2.9}$$

Combining (2.8) and (2.9) gives

$$\dot{V}_a = -e^\top Qe + 2\sum_{i=1}^{m_2} \Delta u_{2_i}(\sigma_i - \frac{1}{\Gamma_i}\dot{\hat{u}}_{2_i}). \tag{2.10}$$

Note that if $\dot{\hat{u}}_{2_i} = \Gamma_i \sigma_i$, then

$$\dot{V}_a = -e^\top Qe \le 0 \tag{2.11}$$

in the $(e, \Delta u_2)^\top$ space, which implies that e and Δu_2 are bounded. We now use the Lyapunov-like lemma, see, for example, [80, 81]. For this, we need to show that $\dot{V}_a(e(t), \Delta u_2(t))$ is uniformly continuous in time. Taking the second time derivative of V_a gives $\ddot{V}_a = -2e^\top Q\dot{e}$, which is bounded, since e and $\dot{e}$ are bounded. Therefore, $\dot{V}_a$ is uniformly continuous, and by the Lyapunov-like lemma,

$$\lim_{t\to\infty} \dot{V}_a \to 0. \tag{2.12}$$

From (2.11) and (2.12), we have to have $\lim_{t\to\infty} e(t) \to 0$, which completes the proof. ■

2.3.3 Practical implementation of the adaptation law

To ensure the boundedness of the estimates, we employ the following unknown input estimator

$$\begin{aligned}\frac{d\hat{u}_{2_i}}{dt} &= \begin{cases} 0 & \text{if } \ \hat{u}_{2_i} \ge \overline{\hat{u}}_{2_i} \ \text{ and } \ \sigma_i > 0 \\ 0 & \text{if } \ \hat{u}_{2_i} \le \underline{\hat{u}}_{2_i} \ \text{ and } \ \sigma_i < 0 \\ \Gamma_i \sigma_i & \text{otherwise} \end{cases} \\ &\triangleq \mathrm{Proj}_{\hat{u}_{2_i}}(\Gamma_i \sigma_i),\end{aligned} \tag{2.13}$$

where $\overline{\hat{u}}_2 = [\overline{\hat{u}}_{2_1} \ \ \overline{\hat{u}}_{2_2} \ \ \cdots \ \ \overline{\hat{u}}_{2_{m_2}}]$ and $\underline{\hat{u}}_2 = [\underline{\hat{u}}_{2_1} \ \ \underline{\hat{u}}_{2_2} \ \ \cdots \ \ \underline{\hat{u}}_{2_{m_2}}]$ are the upper and lower bounds of the unknown input u_2.

We now show that we also have $\lim_{t\to\infty} e(t) \to 0$ for the above u_2 estimator. Substituting (2.13) into (2.10) yields

$$\dot{V}_a = -e^\top Qe + 2\sum_{i=1}^{m_2} \Delta u_{2_i}(\sigma_i - \frac{1}{\Gamma_i}\text{Proj}_{\hat{u}_{2_i}}(\Gamma_i \sigma_i)).$$

It is easy to verify that $\Delta u_{2_i}(\sigma_i - \frac{1}{\Gamma_i}\text{Proj}_{\hat{u}_{2_i}}(\Gamma_i \sigma_i)) \leq 0$. Therefore, $\dot{V}_a \leq -e^\top Qe$. By the Lyapunov-like lemma, $\lim_{t\to\infty} e(t) \to 0$.

2.3.4 Estimating the unknown input

Applying Theorem 1 with the adaptation law (2.13), we now show that $\hat{u}_2 \to u_2$ as $t \to \infty$. To proceed, we need to show that $\dot{e}$ is uniformly continuous. Note that $\dot{e}$ is uniformly continuous if $\ddot{e}$ is bounded. Taking the second derivative of e, we obtain $\ddot{e} = (A-LC)\dot{e} - B_2\dot{\hat{u}}_2$. Since $\dot{\hat{u}}_{2_i} = \text{Proj}_{\hat{u}_{2_i}}(\Gamma_i\sigma_i)$ and $\dot{e}$ are bounded, $\ddot{e}$ is bounded and hence, $\dot{e}$ is uniformly continuous. By the Lyapunov-like lemma, $\lim_{t\to\infty} \dot{e}(t) \to 0$. In the steady state, $e = 0$ and $\dot{e} = 0$. But $\dot{e} = (A-LC)e + B_2(u_2 - \hat{u}_2)$, so $B_2(u_2 - \hat{u}_2) = 0$. For B_2 of full column rank, $\hat{u}_2 = u_2$ in the steady state.

2.4 State and Unknown Input Estimation with Guaranteed Performance

In this section, we extend our adaptive state and unknown input estimation to the case when u_2 is a bounded unknown input not necessarily constant.

2.4.1 Guaranteed performance

Assumption 1 *The unknown input u_2 is bounded with bounded derivative.*

Letting $\zeta = [e,\ \ \Delta u_2]^\top$ and then combining (2.4) and (2.5), we obtain

$$\dot{\zeta} = \tilde{A}\ \zeta + \tilde{B}\ \dot{u}_2, \tag{2.14}$$

where

$$\tilde{A} = \begin{bmatrix} A - LC & B_2 \\ -\Gamma B_2^\top P & O \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} O \\ I_{m_2} \end{bmatrix}, \tag{2.15}$$

and where O denotes a zero matrix. To proceed, we define $\mathscr{L}_\infty$-stability with performance level (p.l.) γ for the system (2.14).

Definition 1 *The system*

$$\dot{\zeta} = \tilde{A}\ \zeta + \tilde{B}\ \dot{u}_2 \tag{2.16a}$$

$$z = H\zeta = [O \ \ I_{m2}]\ \zeta, \tag{2.16b}$$

where z is the output and $H \in \mathbb{R}^{m_2 \times (n+m_2)}$, is globally uniformly $\mathscr{L}_\infty$-stable with performance level γ if the following conditions are satisfied:

1. *$\tilde{A}$ has eigenvalues in the open left half plane.*
2. *For every initial condition $\zeta(t_0) = \zeta_0$, where $t_0 \geq 0$, and every bounded unknown input derivative $\dot{u}_2(\cdot)$, there exists a bound $\beta(\zeta_0, ||\dot{u}_2(\cdot)||_\infty)$ such that*

$$||\zeta(t)|| \leq \beta(\zeta_0, ||\dot{u}_2(\cdot)||_\infty), \quad \forall t \geq t_0. \tag{2.17}$$

3. *For zero initial condition, $\zeta(t_0) = 0$, and every bounded unknown input derivative $\dot{u}_2(\cdot)$, we have*

$$||z(t)|| \leq \gamma ||\dot{u}_2(\cdot)||_\infty, \quad \forall t \geq t_0. \tag{2.18}$$

4. *For every initial condition, $\zeta(t_0) = \zeta_0$, and every bounded unknown input derivative $\dot{u}_2(\cdot)$, we have*

$$\limsup_{t\to\infty} ||z(t)|| \leq \gamma ||\dot{u}_2(\cdot)||_\infty. \tag{2.19}$$

For more details on the $\mathscr{L}_\infty$-stability with level of performance, we refer to [82]. For zero initial error, γ is defined as the upper bound on the $\mathscr{L}_\infty$ gain.

We now present a lemma from [29] that we use in our proof of the main result of this Chapter.

Lemma 2 *Consider a system with bounded input w and performance output z described by*

$$\dot{e} = F(t,e,w) \tag{2.20a}$$

$$z = G(t,e), \tag{2.20b}$$

where $e(t) \in \mathbb{R}^n$, $w \in \mathbb{R}^{n_w}$, and $z(t) \in \mathbb{R}^{n_z}$. Suppose there exists a differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ and scalars $\alpha, \beta_1, \beta_2 > 0$ and $\mu_1, \mu_2 \geq 0$ such that

$$\beta_1 \|e\|^2 \leq V(e) \leq \beta_2 \|e\|^2, \tag{2.21}$$

and

$$DV(e)F(t,e,w) \leq -2\alpha(V(e) - \mu_1 ||w||^2), \tag{2.22a}$$

$$||G(t,e)||^2 \leq \mu_2 V(e), \tag{2.22b}$$

for all $t \geq 0$, where DV denotes the derivative of V. Then system (2.20) is globally uniformly $\mathscr{L}_\infty$-stable with performance level $\gamma = \sqrt{\mu_1 \mu_2}$.

The proof of Lemma 2 is given in [29].

2.4.2 Stability of $\tilde{A}$

The stability of $\tilde{A}$ in (2.15) is critical in the state and unknown input estimation. We investigated if the stability of $\tilde{A}$ is implied by the stability of $(A - LC)$. In the following, we provide a couple of examples to illustrate our discussion.

Example 1 Consider the *RLC* circuit shown in Figure 2.1. Let x_1 be the current through the inductor and x_2 be the capacitor voltage. The *RLC* circuit is modeled by the following equations,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_0+R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{CR_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u_2, \; y = \begin{bmatrix} R_0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $L = 1$ H, $R_1 = R_2 = 1\Omega$, $C = 1$ F, and $R_0 = 0.1\Omega$.

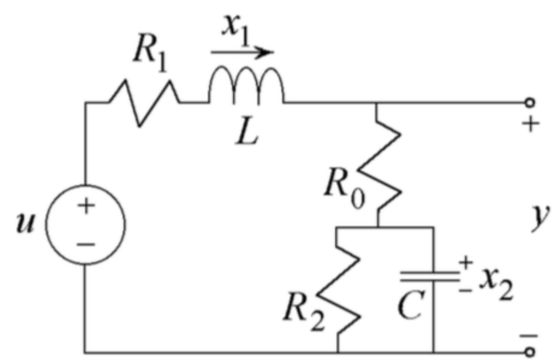

Fig. 2.1.: *RLC* circuit of Example 1.

Solving (2.3), we obtain

$$P = \begin{bmatrix} 1.815 & 18.15 \\ 18.15 & 405.2 \end{bmatrix}, \quad L = \begin{bmatrix} 239.52 \\ -10.65 \end{bmatrix}.$$

For $\Gamma = 20$, the matrix $\tilde{A}$ has the form

$$\tilde{A} = \begin{bmatrix} -25.05 & -240.52 & 1 \\ 2.06 & 9.65 & 0 \\ -36.3 & -362.98 & 0 \end{bmatrix}.$$

The eigenvalues of $\tilde{A}$ are located at $-6.96 \pm -j14.91$, and -1.48. This $\tilde{A}$ is Hurwitz.

Next, we give an example where the matrix $(A - LC)$ is Hurwitz while $\tilde{A}$ is not Hurwitz.

Example 2 Consider the induction motor model in [41], where

$$A = \begin{bmatrix} -2379.2 & 0 & 0 & 0 & 0 \\ 0 & -2.3 & 0 & 0.21 & 0 \\ 0 & 0 & -2.3 & 0 & 0.21 \\ 0 & 267.5 & 0 & -43.83 & 0 \\ 0 & 0 & 267.54 & 0 & -43.83 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 68245 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 232.75 & 0 \\ 0 & 0 & -232.75 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solving (2.3), we obtain

$$P = \begin{bmatrix} 13.13 & 0 & 0 & 0 & 0 \\ 0 & 6597 & -137.57 & 56.69 & -1.18 \\ 0 & -137.58 & 6597 & -1.18 & 56.69 \\ 0 & 56.69 & -1.18 & 59.25 & -0.059 \\ 0 & -1.18 & 56.69 & -0.059 & 59.25 \end{bmatrix},$$

$$L = \begin{bmatrix} -1489.5 & 0 & 0 \\ 0 & 0.912 & 0.0467 \\ 0 & 0.0467 & 0.9119 \\ 0 & 151.97 & 0.125 \\ 0 & 0.125 & 151.97 \end{bmatrix}.$$

Therefore, the matrix $(A-LC)$ is Hurwitz. For $\Gamma = 20I_3$, the matrix $\tilde{A}$ has its eigenvalues located at $-444.876 \pm j1.1059\times 10^6,\ -98.988 \pm j7981.95,\ -99.113 \pm j7975.34,\ 0$, and 0. Thus, in this example the matrix $\tilde{A}$ is not Hurwitz.

In conclusion, the stability of $(A-LC)$ does not imply the stability of $\tilde{A}$. The stability of $\tilde{A}$ as a function of its parameters requires further investigation.

2.4.3 Sufficient conditions for the state and unknown input estimation

Using results of [29], we present now sufficient conditions for the design of the state and unknown input observer when u_2 is a bounded unknown input. We also provide the performance level of the proposed observer.

Theorem 2 *Suppose* $\dot{u}_2$ *is bounded,* $\tilde{A}$ *is asymptotically stable in the plant model given by* (2.14)*, and there exist a symmetric matrix* $P \succ 0$*, matrices* L *and* F *such that conditions* (2.3) *are satisfied. If there exist* $\alpha > 0,\ \mu \geq 0$*, a symmetric matrix* $\tilde{P} \succ 0$ *such that the matrix inequalities*

$$\phi \preceq 0 \tag{2.23a}$$

$$\begin{bmatrix} \tilde{P} & * \\ H & \mu I \end{bmatrix} \succeq 0 \tag{2.23b}$$

are satisfied where

$$\phi = \begin{bmatrix} \phi_{11} & \tilde{P}\tilde{B} \\ * & -2\alpha I \end{bmatrix} \tag{2.24}$$

and

$$\phi_{11} = \tilde{P}\tilde{A} + \tilde{A}^\top \tilde{P} + 2\alpha\tilde{P}, \tag{2.25}$$

then observer (2.2) *yields $\mathscr{L}_\infty$-stable state and unknown input error dynamics with performance level $\gamma = \sqrt{\mu}$ for the performance output $z = H\zeta$.*

Proof We evaluate the Lyapunov derivative of $\tilde{V}(\zeta) = \zeta^\top \tilde{P}\zeta$ on the trajectories of (2.14) to obtain

$$\dot{\tilde{V}}(\zeta) = D\tilde{V}(\zeta)\dot{\zeta} = 2\zeta^\top \tilde{P}(\tilde{A}\zeta + \tilde{B}\dot{u}_2).$$

Let $q = [\zeta^\top \;\; \dot{u}_2^\top]^\top$. Performing manipulations gives

$$\begin{aligned} q^\top \phi q &= \begin{bmatrix} \zeta^\top & \dot{u}_2^\top \end{bmatrix} \begin{bmatrix} \phi_{11} & \tilde{P}\tilde{B} \\ * & -2\alpha I \end{bmatrix} \begin{bmatrix} \zeta \\ \dot{u}_2 \end{bmatrix} \\ &= 2\zeta^\top \tilde{P}\tilde{A}\zeta + 2\zeta^\top \tilde{P}\tilde{B}\dot{u}_2 + 2\alpha\zeta^\top \tilde{P}\zeta - 2\alpha\dot{u}_2^\top \dot{u}_2 \\ &= D\tilde{V}(\zeta)\dot{\zeta} - 2\alpha||\dot{u}_2||^2 + 2\alpha\tilde{V}(\zeta). \end{aligned}$$

Since $\phi \preceq 0$, then

$$D\tilde{V}(\zeta)\dot{\zeta} - 2\alpha||\dot{u}_2||^2 + 2\alpha\tilde{V}(\zeta) = q^\top \phi q \leq 0. \tag{2.26}$$

Rearranging (2.26) gives

$$D\tilde{V}(\zeta)\dot{\zeta} \leq -2\alpha(\tilde{V}(\zeta) - ||\dot{u}_2||^2). \tag{2.27}$$

Therefore, condition (2.22a) in Lemma 2 holds with $\mu_1 = 1$.

Next, taking the Schur complement of (2.23b), we obtain

$$\tilde{P} - H^\top \mu^{-1} H = \tilde{P} - \mu^{-1} H^\top H \succeq 0. \tag{2.28}$$

Pre-multiplying (2.28) by $\zeta^\top$ and post-multiplying it by ζ gives

$$\zeta^\top \tilde{P}\zeta - \mu^{-1}\zeta^\top H^\top H\zeta \geq 0. \tag{2.29}$$

Rearranging the above gives

$$||H\zeta||^2 \leq \mu\tilde{V}(\zeta). \tag{2.30}$$

So condition (2.22b) in Lemma 2 holds for $\mu_2 = \mu$. From (2.27) and (2.30), we conclude that the assumptions of Lemma 2 are satisfied. Therefore the state and unknown input error dynamics are $\mathscr{L}_\infty$-stable with performance level $\gamma = \sqrt{\mu}$. ■

We summarize our discussion with Algorithm 1 for the design of the adaptive observer.

Algorithm 1: Adaptive unknown input observer design

1. For the dynamical system (2.1), solve conditions (2.3) for (P, L, F) by letting $Y = PL$ and solving the following LMIs for (P, Y, F) using CVX,

$$\begin{aligned} A^\top P + PA - C^\top Y^\top - YC &\prec 0, \\ B_2^\top P = FC, \;\; P = P^\top &\succ 0. \end{aligned}$$

2. Choose the estimator gain Γ and set $\dot{\hat{u}}_2 = \Gamma F(y - \hat{y})$.
3. Construct state and unknown input error dynamics system (2.14) and check that $\tilde{A}$ is Hurwitz.
4. Let $H = [O \;\; I_{m2}]$, choose the design parameter α and solve LMIs (2.23) for $\tilde{P}$ and μ.

Note that Theorem 2 uses the same first set of conditions (2.3) as Theorem 1 for the uncertain system (2.1) with extra conditions (2.23) for the linear system in (2.14). Theorem 1 technique allows us to prove in addition that the unknown input has bounded unknown input estimate dynamics given by (2.13). While in Theorem 2, we use a linear unknown input estimator in state error e given by (2.4).

Now we consider the case when u_2 is constant. If we apply Theorem 2, since $\dot{u}_2$ is zero, then the plant model (2.14) reduces to the plant $\dot{\zeta} = \tilde{A}\,\zeta$, where $\tilde{A}$ is asymptotically stable and conditions (2.17) and (2.19) reduce to

$$||\zeta(t)|| \leq \beta(\zeta_0),$$

and

$$\limsup_{t\to\infty} ||z(t)|| \leq 0.$$

Therefore, the estimation error of the state and unknown input in Theorem 2 will converge to zero in the constant unknown input case. Thus, Theorem 2 is a generalization of Theorem 1 for the case when u_2 is any bounded unknown input not necessarily constant.

2.5 Example

In this section, we present an example to illustrate the effectiveness of the proposed observer to estimate the state and unknown input for a nonlinear system with bounded unknown input and bounded unknown input derivative. LMIs of Theorem 2 have been solved using CVX [83, 84].

Example 3 We consider the flight path rate demand missile (two-loop) autopilot system from [85]. The state variables of the system are:

x_1: flight path rate demand,

x_2: pitch rate,

x_3: elevator deflection,

x_4: rate of change of elevator deflection.

The output of the system are state variables x_1 and x_2. The state space model of the two-loop autopilot system has the form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_a} & \frac{a+\sigma^2 w_b^2}{T_a} & \frac{-k_b \sigma^2 w_b^2}{T_a} & -k_b \sigma^2 {w_b}^2 \\ -\frac{1+w_b^2 T_a^2}{T_a(1+\sigma^2 w_b^2)} & \frac{1}{T_a} & \frac{(T_a^2-\sigma^2)k_b w_b^2}{T_a(1+\sigma^2 w_b^2)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -w_a^2 & -2\zeta_a w_a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_q w_a^2 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2,$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

The numerical values of the model parameters are shown in Table 2.1. The unknown input u_2 is taken to be $u_2 = \sin(10t)$.

Table 2.1.: Parameter values for the two-loop autopilot example.

Parameter	Value
T_a	0.36 s
σ^2	0.00029 s^2
w_b	11.77 rad/s
ζ_a	0.6
k_b	-9.91 s^{-1}
w_a	180 rad/s
k_q	-0.07

We apply Theorem 2 for the autopilot model. Note that the conditions of the existence of UIO are satisfied, where $\text{rank}(B2) = \text{rank}(CB_2) = 1$. We use the CVX software to compute (L, F, P) that satisfy the conditions in (2.3). We obtain

$$L = \begin{bmatrix} -16.088 & -66.420 \\ 69.632 & 222.372 \\ -28.867 & -98.993 \\ 3692.687 & 9301.315 \end{bmatrix}, \quad F = \begin{bmatrix} 8899.1 & 2651.55 \end{bmatrix}.$$

We set $\Gamma = 20$, $\bar{\hat{u}}_2 = 10$, and $\underline{\hat{u}}_2 = -10$. The error estimation dynamics (2.14) take the form

$$\begin{aligned} \dot{\zeta} &= \tilde{A}\,\zeta + \tilde{B}\,\dot{u}_2 \\ &= \begin{bmatrix} 13.3 & 69.3 & 1.1 & 0.4 & 1 \\ -120.24 & -219.6 & -474.1 & 0 & 0 \\ 28.9 & 99 & 0 & 1 & 1 \\ -3692.7 & -9301.3 & -32400 & -216 & 0 \\ -177981.5 & -53031.1 & 0 & 0 & 0 \end{bmatrix} \zeta + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{u}_2, \end{aligned}$$

where $\tilde{A}$ is asymptotically stable with the eigenvalues located at $-101.535 \pm j474.156, -32.823 \pm j133.833$, and -153.569.

We solve the LMIs given by (2.23), and obtain the observation error performance level $\gamma = 0.124$. In our simulation, we use the initial condition of the system to be $x(0) = [0.5, -10, 5, -3]^\top$ and the initial conditions on the adaptive observer dynamics are zero. The design parameter $\alpha = 1$. We can see from Figures 2.2 and 2.3 that the adaptive observer estimates the system states well. The unknown input is reconstructed accurately as can be seen in Figure 2.4 and 2.5.

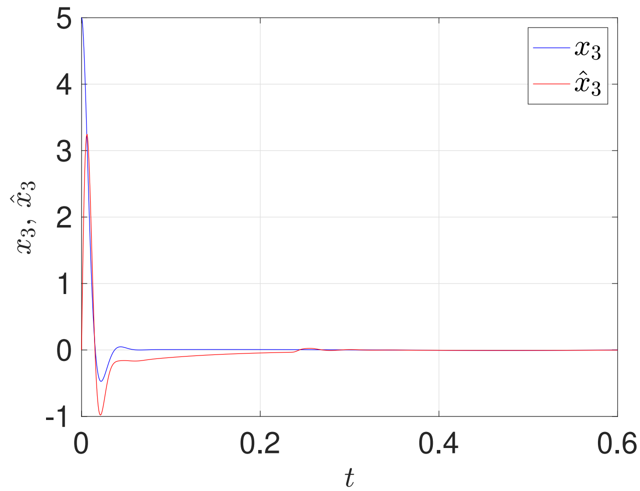

Fig. 2.2.: Plot of the state x_3 and its estimate in Example 3.

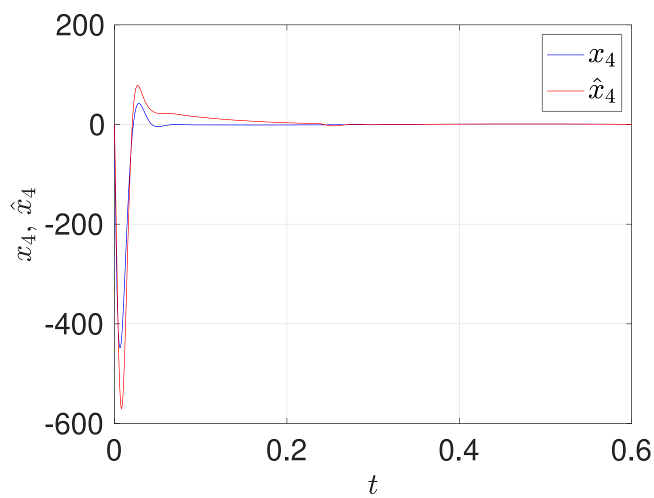

Fig. 2.3.: Plot of the state x_4 and its estimate in Example 3.

2.6 Conclusions

We propose adaptive CT state and unknown input observers for uncertain systems when the unknown input is constant or bounded not necessarily constant. We give conditions for existence of the proposed observers in terms of LMIs. The state estimation and unknown input estimation errors are shown to have a degree of accuracy with a guaranteed level of performance for the class of bounded unknown inputs. An open problem is to investigate

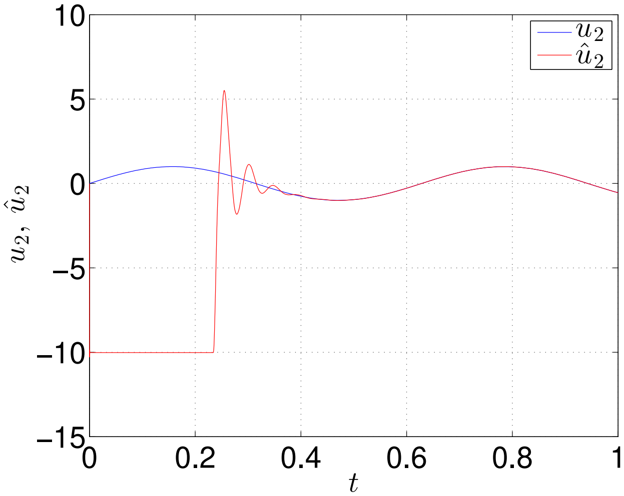

Fig. 2.4.: Plot of unknown input estimation in Example 3.

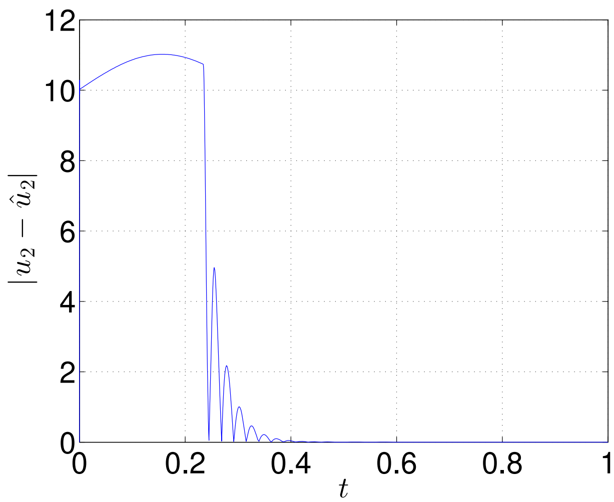

Fig. 2.5.: Plot of the absolute value of the unknown input estimation error in Example 3.

the conditions under which the matrix $\tilde{A}$ in (2.15) is Hurwitz. At present, the design parameters α, and μ in Theorem 2 are selected by trial and error. More systematic procedure to select the parameters is desired.

3. STATE OBSERVERS AND UNKNOWN INPUT ESTIMATORS FOR DISCRETE-TIME NONLINEAR SYSTEMS CHARACTERIZED BY INCREMENTAL MULTIPLIER MATRICES

3.1 Introduction

In this chapter, we propose a novel discrete-time state and unknown input observers for a class of continuous-time (CT) nonlinear systems whose nonlinearity can be characterized by incremental multiplier matrices. We represent these nonlinear models as linear models by treating the nonlinearity as a nonlinear input with a known structure. The input to the original nonlinear system is allowed to have an unknown component. Then, the linear model is discretized using the exact discretization method. We present a novel discrete-time state observer for such systems. The condition for the existence of the observer is presented in terms of a linear matrix inequality (LMI). A novel unknown input estimator is then used to estimate the system unknown input with one sampling period time-delay.

3.2 Background Results and Problem Statement

We consider a CT nonlinear model of the form

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 u_1(t) + B_2 u_2(t) + B_\phi \Phi(x(t)) \\
y(t) &= Cx(t),
\end{aligned}
$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, $B_\phi \in \mathbb{R}^{n\times m_3}$, and $C \in \mathbb{R}^{p\times n}$. We assume $m_2 \leq p \leq n$. The control input is $u_1(t) \in \mathbb{R}^{m_1}$, while $u_2(t) \in \mathbb{R}^{m_2}$ is an unknown input. The vector-valued function $\Phi(x(t)) \in \mathbb{R}^{m_3}$ models the plant's nonlinearities.

We discretize the CT plant model. Using a DT model allows one to construct an unknown input estimator. We can use the Euler discretization method. If all inputs are approximately constant over the sampling time T_s, then we could use the exact discretization method. In either case, the discretized model has the form

$$\left.\begin{aligned} x[k+1] &= A_d x[k] + B_{1d} u_1[k] + B_{2d} u_2[k] + B_{\phi d} \Phi(x[k]) \\ y[k] &= C x[k]. \end{aligned}\right\} \tag{3.1}$$

We assume that the matrix C is full row rank, that is, $\operatorname{rank}(C) = p$, and that the pair (A_d, B_{1d}) is controllable and the pair (A_d, C) is detectable—see [54] for a discussion of the role of detectability in the existence of both CT and DT UIOs. For more information on the modeling and properties of discrete-time systems, see, for example [86, Subsection 1.1.2 and Chapter 2] or [87].

We further assume that the matrix B_{2d} has full column rank and $\operatorname{rank}(CB_{2d}) = \operatorname{rank}(B_{2d}) = m_2$.

For notational convenience, we introduce the following. Let M be a $\mathbb{R}^{(n+m_3)\times(n+m_3)}$ matrix. We define the quadratic form, for $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^{m_3}$,

$$Q_M(a,b) = \begin{bmatrix} a^\top & b^\top \end{bmatrix} M \begin{bmatrix} a \\ b \end{bmatrix} \tag{3.2}$$

Definition 2 *(Compare with [40]) A symmetric $(n+m_3)\times(n+m_3)$ matrix M is an incremental multiplier matrix (δMM) for Φ if for every $x, \delta \in \mathbb{R}^n$,*

$$Q_M(\delta, \Phi(x+\delta) - \Phi(x)) \geq 0.$$

The usefulness of this concept will be demonstrated in Theorem 3. Note that the existence of a δMM for Φ provides useful information on Φ only when the δMM is not positive definite.

Remark 2 *Lipschitz continuity is a special case of δMM. Let*

$$M = \begin{bmatrix} \beta^2 I & 0 \\ 0 & -I \end{bmatrix}.$$

Then, it is easy to see that

$$Q_M(\delta, \Phi(x+\delta) - \Phi(x)) = \beta^2 \|\delta\|^2 - \|\Phi(x+\delta) - \Phi(x)\|^2$$

and thus $Q_M(\delta, \Phi(x+\delta) - \Phi(x)) \geq 0$ is equivalent to $\|\Phi(x+\delta) - \Phi(x)\| \leq \beta \|\delta\|$.

We say that a nonlinear system is δMM if its nonlinearity has a δMM matrix. We assume in the remainder of the chapter that the systems we consider are δMM. It is interesting to note that the concept of δMM nonlinearity is related to the dissipativity concept proposed in [38, 88].

Our objective is to construct state and unknown input estimators/observers for DT nonlinear systems whose nonlinearities have incremental multiplier matrices using DT observers.

3.3 State Observer Design

In this section, we propose a novel DT observer architecture for DT systems modeled by (3.1). We begin by representing $x[k]$ as

$$x[k] = x[k] - Hy[k] + Hy[k] = (I - HC)x[k] + Hy[k],$$

where $H \in \mathbb{R}^{n \times p}$. The selection of H is discussed below. Let

$$z[k] = (I - HC)x[k],$$

then we have

$$x[k] = z[k] + Hy[k].$$

Recall that our objective is to estimate $x[k]$. We use the above relation to obtain the state estimate, that is,

$$\hat{x}[k] = z[k] + Hy[k],$$

where we obtain $z[k]$ from the relation,

$$z[k+1] = (I - HC)x[k+1].$$

Substituting into the above the dynamics of the DT system gives

$$z[k+1] = (I - HC)\big(A_d z[k] + A_d Hy[k] + B_{1d}u_1[k] + B_{2d}u_2[k] + B_{\phi d}\Phi(x[k])\big).$$

In the above equation for $z[k+1]$, we replace $\Phi(x[k])$ with $\Phi(\hat{x}[k])$ in our construction of the state observer. To proceed, let $e[k] = x[k] - \hat{x}[k]$ be the state estimation error. Then after simple manipulations, we obtain

$$e[k+1] = (I - HC)A_d e[k] + (I - HC)B_{2d}u_2[k] + (I - HC)B_{\phi d}\big(\Phi(x[k]) - \Phi(\hat{x}[k])\big).$$

If we select H so that $(I - HC)B_{2d} = 0$, then we obtain

$$e[k+1] = (I - HC)A_d e[k] + (I - HC)B_{\phi d}\big(\Phi(x[k]) - \Phi(\hat{x}[k])\big).$$

We can see from the above that we do not have any control over the estimation error convergence dynamics, which is determined by the matrix $(I - HC)A_d$. To improve the estimation error convergence dynamics, we add the term $L(y[k] - \hat{y}[k])$, where $L \in \mathbb{R}^{n\times p}$ and

$$\hat{y}[k] = C\hat{x}[k] = C(z[k] + Hy[k]). \tag{3.3}$$

With the introduction of L, we obtain our projection-based closed-loop observer:

$$\begin{aligned} z[k+1] =& (I-HC)(A_d z[k] + A_d H y[k] + B_{1d} u_1[k] + B_{\phi d}\Phi(\hat{x}[k])) \\ & + L(y[k] - Cz[k] - CHy[k]) \end{aligned} \tag{3.4a}$$

$$\hat{x}[k] = z[k] + Hy[k], \tag{3.4b}$$

The state estimation error dynamics are,

$$e[k+1] = \big((I-HC)A_d - LC\big)e[k] + (I-HC)B_{2d}u_2[k] + (I-HC)B_{\phi d}(\Phi(x[k]) - \Phi(\hat{x}[k])).$$

Recall that H was chosen so that $(I-HC)B_{2d} = 0$. Let

$$\tilde{A} = (I-HC)A_d \text{ and } \tilde{B} = (I-HC)B_{\phi d},$$

then the estimation error dynamics can be represented as,

$$\begin{aligned} e[k+1] =& (\tilde{A} - LC)e[k] + \tilde{B}\big(\Phi(x[k]) - \Phi(\hat{x}[k])\big) \\ =& (\tilde{A} - LC)e[k] + \tilde{B}\Delta\Phi[k], \end{aligned} \tag{3.5}$$

where $\Delta\Phi[k] = \Phi(x[k]) - \Phi(\hat{x}[k])$.

We next present a necessary and sufficient condition for the solvability of the matrix equation $(I-HC)B_{2d} = 0$.

Lemma 3 *The equation $(I-HC)B_{2d} = 0$ is solvable if and only if rank(CB_{2d}) = rank(B_{2d}), where*

$$H = B_{2d}\big((CB_{2d})^{\dagger} + H_0(I_p - (CB_{2d})(CB_{2d})^{\dagger})\big). \tag{3.6}$$

The superscript † denotes the Moore-Penrose pseudo-inverse operation and $H_0 \in \mathbb{R}^{m_2 \times p}$ is a design parameter matrix.

Proof See [9]. ∎

Our objective is to select the observer gain L so that $e[k] \to 0$ as $k \to \infty$ for a class of nonlinearities characterized by the incremental quadratic constraints given by Definition 2. We have the following theorem.

Theorem 3 *Suppose $(I - HC)B_{2d} = 0$ and that $M \in \mathbb{R}^{(n+m_2)\times(n+m_2)}$ is an δMM for Φ. If there exist matrices $P = P^\top \succ 0$, L, and a scalar $\kappa \geq 0$, such that*

$$\Gamma + \kappa M \prec 0, \tag{3.7}$$

where

$$\Gamma = \begin{bmatrix} (\tilde{A} - LC)^\top P(\tilde{A} - LC) - P & (\tilde{A} - LC)^\top P\tilde{B} \\ \tilde{B}^\top P(\tilde{A} - LC) & \tilde{B}^\top P\tilde{B} \end{bmatrix},$$

then $e[k] \to 0$ as $k \to \infty$.

Proof Let $V[k] = e[k]^\top Pe[k]$ be a Lyapunov function candidate for the estimation error dynamics given by (3.5). We evaluate the difference $\Delta V[k] = V[k+1] - V[k]$ on the trajectories of (3.5) to obtain

$$\begin{aligned}\Delta V[k] =& e[k]^\top \left((\tilde{A} - LC)^\top P(\tilde{A} - LC) - P\right) e[k] + 2e[k]^\top (\tilde{A} - LC)^\top P\tilde{B}\Delta\Phi[k] \\ &+ \Delta\Phi[k]^\top \tilde{B}^\top P\tilde{B}\Delta\Phi[k].\end{aligned}$$

Let $\Delta q = [e[k]^\top \ \Delta\Phi[k]^\top]^\top$. Premultiplying and postmultiplying the matrix inequality (3.7) by $\Delta q^\top$ and Δq, respectively, and taking into account the above equality, we obtain

$$\Delta V[k] + \kappa \Delta q[k]^\top M \Delta q[k] < 0.$$

Since M is an δMM for $\Phi(x[k])$, we have $\Delta V[k] < 0$. ∎

3.3.1 LMI synthesis

We present a method to solve matrix inequality (3.7) using an LMI. Since $P = P^\top \succ 0$, it is nonsingular and we can write $P = PP^{-1}P$. Therefore, solving the discrete-time Lyapunov inequality,

$$(\tilde{A} - LC)^\top P(\tilde{A} - LC) - P \prec 0$$

for $P = P^\top \succ 0$, is equivalent to solving the matrix inequality

$$(\tilde{A} - LC)^\top PP^{-1}P(\tilde{A} - LC) - P \prec 0.$$

Let $Y = PL$. Then using the Schur complement, we represent the above matrix inequality as an LMI,

$$\begin{bmatrix} -P & \tilde{A}^\top P - C^\top Y^\top \\ P\tilde{A} - YC & -P \end{bmatrix} \prec 0. \tag{3.8}$$

Partition the incremental multiplier matrix as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^\top & M_{22} \end{bmatrix}.$$

Let

$$\begin{aligned}
\Omega_{11} &= \begin{bmatrix} -P + \kappa M_{11} & \tilde{A}^\top P - C^\top Y^\top \\ P\tilde{A} - YC & -P \end{bmatrix}, \\
\Omega_{12} &= \begin{bmatrix} (\tilde{A}^\top P - C^\top Y^\top)\tilde{B} + \kappa M_{12} \\ O \end{bmatrix}, \\
\Omega_{22} &= \tilde{B}^\top P\tilde{B} + \kappa M_{22}
\end{aligned}$$

where O is a matrix of zeros of compatible dimensions. Then, we can represent matrix inequality (3.7) as an LMI,

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \star & \Omega_{22} \end{bmatrix} \prec 0. \tag{3.9}$$

We solve (3.9) for $P = P^\top \succ 0$, $Y \in \mathbb{R}^{n \times p}$, and $\kappa \geq 0$.

We summarize our discussion of the DT state observer design in Algorithm 2.

Algorithm 2: State observer design

Check if $\mathrm{rank}(CB_{2d}) = \mathrm{rank}(B_{2d}) = m_2$ for plant model (3.1).
Compute H using (3.6).
Solve (3.9) for $P = P^\top \succ 0$, $L = P^{-1}Y$, and $\kappa \geq 0$.
Construct state observer given by (3.4).

3.4 Unknown Input Estimation

In this section, we propose a DT estimator of the unknown input $u_2(t)$ of the CT plant.

Remark 3 *If rank$(CB_{2d}) =$ rank$(B_{2d}) = m_2$, then there exists a matrix $(CB_{2d})^\dagger \in \mathbb{R}^{m_2 \times p}$ such that $(CB_{2d})^\dagger CB_2 = I_{m_2}$, where $(CB_{2d})^\dagger = \left((CB_{2d})^\top CB_{2d}\right)^{-1}(CB_{2d})^\top$.*

In the remainder of the Chapter, we let $T = (CB_{2d})^\dagger$ for notational convenience. Premultiplying both sides of (3.1) by the matrix TC, we obtain

$$TCx[k+1] = TCA_d x[k] + TCB_{1d}u_1[k] + TCB_{2d}u_2[k] + TCB_{\phi d}\Phi(x[k]). \tag{3.10}$$

From Remark 3, $TCB_{2d} = I_{m_2}$. We can thus rewrite (3.10) as

$$u_2[k] = Ty[k+1] - TCA_d x[k] - TCB_{1d}u_1[k] - TCB_{\phi d}\Phi(x[k]).$$

Using the above equation, we propose the following unknown input estimator:

$$\hat{u}_2[k] = Ty[k+1] - TCA_d\hat{x}[k] - TCB_{1d}u_1[k] - TCB_{\phi d}\Phi(\hat{x}[k]). \tag{3.11}$$

Since the unknown input estimate in (3.11) depends on $y[k+1]$, we can estimate the unknown input with one sampling period time-delay as,

$$\hat{u}_2[k-1] = Ty[k] - TCA_d\hat{x}[k-1] - TCB_{1d}u_1[k-1] - TCB_{\phi d}\Phi(\hat{x}[k-1]). \tag{3.12}$$

Let $e_{u2}[k] = u_2[k] - \hat{u}_2[k]$ be the unknown input estimation error. Then, we have

$$e_{u2}[k] = TCA_d(x[k] - \hat{x}[k]) + TCB_{\phi d}(\Phi(x[k]) - \Phi(\hat{x}[k])).$$

In our subsequent analysis, we assume that $\tilde{B} = (I - HC)B_{\phi d}$ has full column rank. By Theorem 3, $x[k] - \hat{x}[k] \to 0$ as $k \to \infty$. It follows from (3.5) that $\tilde{B}(\Phi(x[k]) - \Phi(\hat{x}[k])) \to 0$. Since by assumption $\tilde{B} = (I - HC)B_{\phi d}$ has full column rank, we can conclude that $\Phi(x[k]) - \Phi(\hat{x}[k]) \to 0$. Therefore $e_{u2}[k] \to 0$ as $k \to \infty$ and $u_2[k] - \hat{u}_2[k] \to 0$. Hence we can asymptotically estimate the unknown input with one sampling period time-delay. We summarize our discussion in Algorithm 3.

Algorithm 3: DT unknown input estimator design

Check if $\operatorname{rank}(CB_{2d}) = \operatorname{rank}(B_{2d})$ for DT plant (3.1).
Check if $\tilde{B}$ has full column rank.
Compute $T = (CB_{2d})^{\dagger}$.
Set $\hat{x}[-1] = 0$, and $u_1[-1] = 0$.
Collect $y[k]$. Compute $\hat{x}[k]$ and $u_1[k]$, where $k \geq 0$, using Algorithm 2.
Construct the unknown input estimator given by (3.12).

3.5 An Example of a State and Unknown Input Estimation for a Nonlinear System

A block diagram of a continues-time (CT) nonlinear plant controlled by a discrete-time (DT) observer-based controller is depicted in Figure 3.1. The output measurement $y(t)$ of the CT nonlinear plant is sampled and the sampled signal is transmitted to the state observer and the unknown input estimator. The control signal is $u_1[k] = -K_d\hat{x}[k]$, where K_d is a constant state feedback gain obtained, for example, using the discrete linear-quadratic regulator (LQR) method. Then, the unknown input estimator uses the estimated DT state and the control input with one sampling period time-delay to reconstruct the unknown input. The DT control signal $u_1[k]$ is passed through the zero-order-hold (ZOH) element and then sent to the CT nonlinear plant.

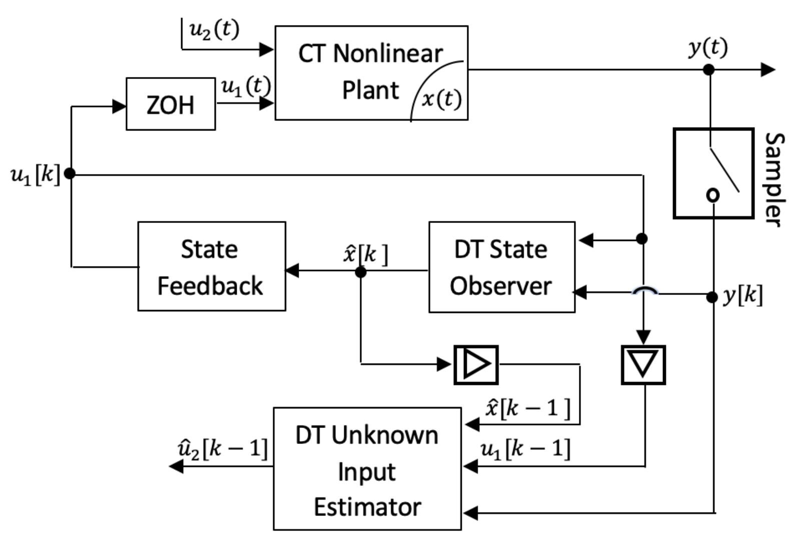

Fig. 3.1.: Closed-loop system with the combined controller-observer compensator and an unknown input estimator.

We consider a single-link manipulator with revolute joints from [39]. The nonlinear system model is described by

$$\left.\begin{aligned} \dot{\theta}_m &= \omega_m + \tfrac{mgb}{J_l} u_2 \\ \dot{\omega}_m &= \tfrac{k}{J_m}(\theta_l - \theta_m) - \tfrac{B_f}{J_m}\omega_m + \tfrac{K_\tau}{J_m} u_1 \\ \dot{\theta}_l &= \omega_l \\ \dot{\omega}_l &= \tfrac{k}{J_l}(\theta_l - \theta_m) - \tfrac{mgb}{J_l}\sin(\theta_l), \end{aligned}\right\} \tag{3.13}$$

where θ_m, ω_m are the angular rotation and angular velocity of the motor, θ_l, ω_l are the angular rotation and angular velocity of the link, u_1 is the input motor torque, and u_2 is the unknown input affecting the angular rotation of the motor. The parameters of the model are shown in Table 3.1. The state vector of the plant model is $x = [\theta_m \ \omega_m \ \theta_l \ \omega_l]^\top$. We

Table 3.1.: Parameter values for the single-link flexible robot.

Parameter	Description	Value
J_m	Inertia of dc motor	$0.0037\ \mathrm{kg\,m^2}$
J_l	Inertia of the controlled link	$0.0093\ \mathrm{kg\,m^2}$
m	Link mass	0.21 kg
b	Center of mass	0.15 m
k	Elastic constant	$0.18\ \mathrm{Nm/rad}$
B_f	Viscous friction coefficient	$0.0083\ \mathrm{Nm/V}$
K_τ	Amplifier gain	$0.08\ \mathrm{Nm/V}$
g	Acceleration due to gravity	$9.81\ \mathrm{m/s^2}$

represent the the nonlinear plant model in the form,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_m}{J_m} & -\frac{B_f}{J_m} & \frac{k_m}{J_m} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_m}{J_l} & 0 & -\frac{k_m}{J_l} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} \frac{mgb}{J_l} \\ 0 \\ 0 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{mgb}{J_l} \end{bmatrix} \sin(x_3(t)),$$

and the output is

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x.$$

The above system is "slow." Therefore, for sufficiently small sampling period T_s, we can use the exact discretization method to discretize the above model because the inputs to the system are approximately constant over sufficiently small sampling period. We use MATLAB's function, c2d, with $T_s = 0.02$ sec, to obtain the DT model (3.1) with

$$A_d = 10^{-2} \times \begin{bmatrix} 99.0423 & 1.9495 & 0.9564 & 0.0064 \\ -94.9625 & 94.6692 & 94.7146 & 0.9564 \\ -0.3862 & -0.0025 & 99.6125 & 1.9974 \\ -38.5357 & -0.3805 & -38.7837 & 99.6125 \end{bmatrix},$$

$$B_{1d} = 10^{-2} \times \begin{bmatrix} 0.4253 \\ 42.1505 \\ -0.0003 \\ -0.0551 \end{bmatrix}, \; B_{2d} = 10^{-2} \times \begin{bmatrix} 66.2418 \\ -31.8204 \\ -0.0856 \\ -12.8333 \end{bmatrix},$$

$$B_{\phi d} = 10^{-2} \times \begin{bmatrix} -0.0011 \\ -0.2128 \\ -0.6641 \\ -66.3690 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

where the nonlinearity of the model is $\Phi(x[k]) = \sin(x_3[k])$. Note that C is full row rank and the rank condition of our DT plant model is satisfied, that is, $\operatorname{rank}(CB_{2d}) = \operatorname{rank}(B_{2d}) = 1$. By solving the discrete-time algebraic Riccati equation, we obtain the feedback controller

$$u_1[k] = -\begin{bmatrix} 2.2263 & 0.4538 & 2.6579 & -0.5152 \end{bmatrix} \hat{x}[k].$$

To proceed with our observer design given by (3.4), we solve for H so that $(I-HC)B_{2d}=0$ to obtain

$$H=\begin{bmatrix}0.8125 & -0.3903\\ -0.3903 & 0.1875\\ -0.0011 & 0.0005\\ -0.1574 & 0.0756\end{bmatrix}.$$

Since $|\sin(x_3[k])-\sin(\hat{x}_3[k])|\leq|x_3[k]-\hat{x}_3[k]|$ by the Mean Value Theorem, $\Phi(x[k])=\sin(x_3[k])$ is Lipschitz. Therefore, we can use the following incremental multiplier matrix,

$$M=\begin{bmatrix}I & 0\\ 0 & -I\end{bmatrix}.$$

Solving (3.9) for P, L, and κ using the `cvx` toolbox, we obtain

$$P=\begin{bmatrix}94.33051 & 18.0754 & -36.0293 & 0.1537\\ 18.0754 & 96.8613 & -61.7687 & 0.2902\\ -36.0293 & -61.7687 & 54.7671 & -0.3795\\ 0.1537 & 0.2902 & -0.3795 & 0.0050\end{bmatrix},$$

$$L=\begin{bmatrix}0.4317 & 1.3569\\ 1.0290 & 2.7607\\ 2.2369 & 3.5281\\ 63.3664 & 85.5940\end{bmatrix},\text{ and }\kappa=8.1364\times10^{-4}.$$

In our simulation, we let $x[0]=[3\ 2\ 3\ -2]^{\top}$, zero initial condition for the state observer, $\hat{x}[-1]=[0\ 0\ 0\ 0]^{\top}$, and $u[-1]=0$. A plot of the unknown input u_2 is shown in the top subfigure of Figure 3.4. We simulate the closed-loop system shown in Figure 3.1 with the state observer given by (3.4), and the unknown input estimator given by (3.12). To show the results of the state estimates more clearly, we use 150 samples. Figure refstates contains the plots of the true state components, and their estimates. The control input is shown in Figure 3.3, where the top plot shows the DT control signal, while the bottom plot shows the input to the actuator after zero-order hold (ZOH) operation on the DT control signal. The unknown input estimate is shown in the bottom subfigure of Figure 3.4. We note that the state and the unknown input estimators perform as expected.

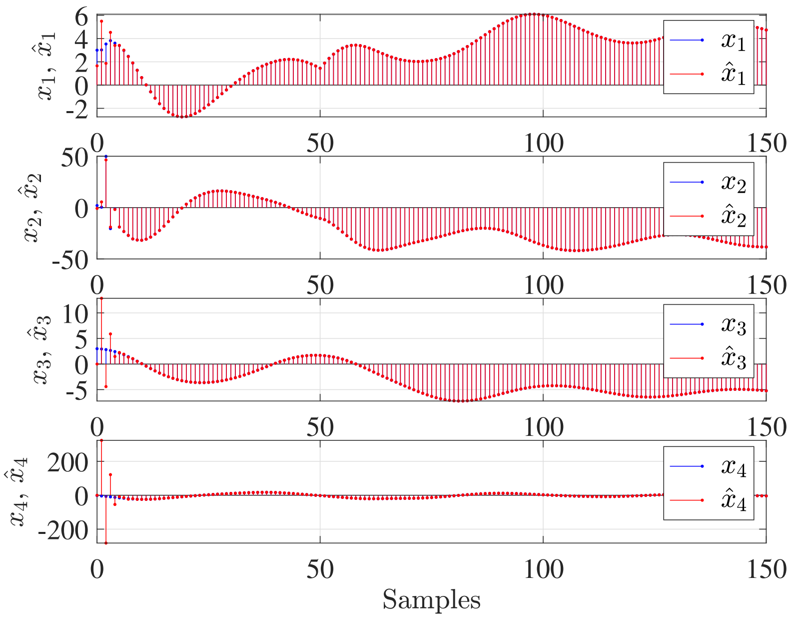

Fig. 3.2.: State estimates of the nonlinear plant model.

3.6 An Application of the Proposed Unknown Input Estimator to Reconstruct Malicious Packet Drops During the Control Signal Transmission

The proposed unknown input estimator in this Chapter can also be used to reconstruct malicious packet drops in the communication between the controller and the actuators. We illustrate this on a linear DT plant model,

$$\left.\begin{array}{rcl} x[k+1] & = & A_d x[k] + B_d u^a[k] \\ y[k] & = & Cx[k], \end{array}\right\} \tag{3.14}$$

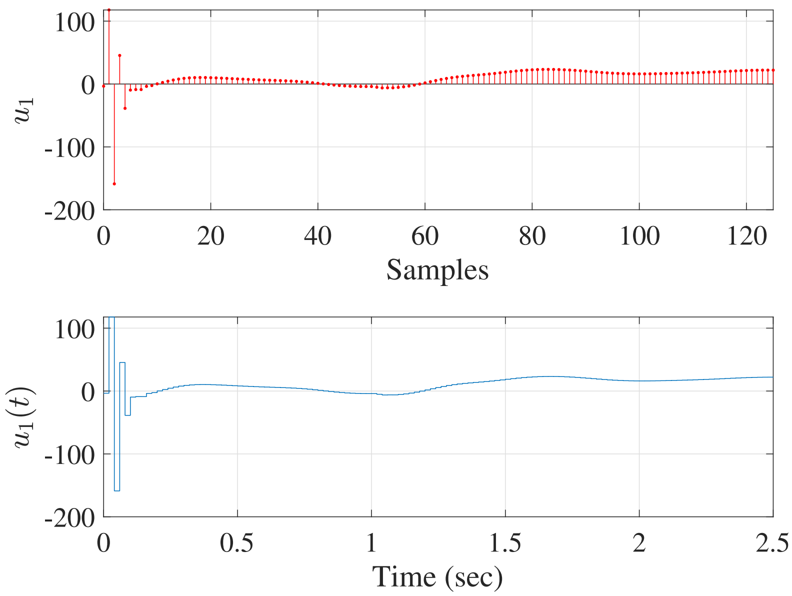

Fig. 3.3.: The control signal and the input to the actuator.

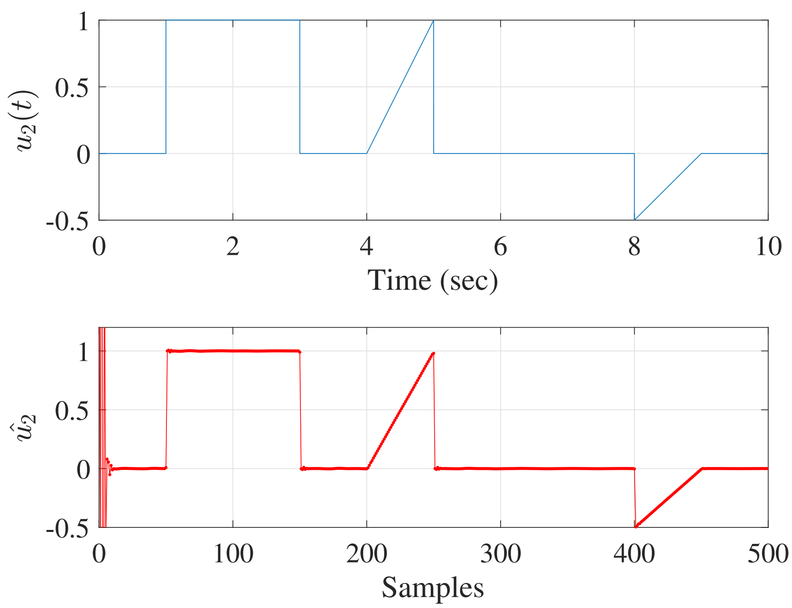

Fig. 3.4.: A plot of the unknown input signal and its estimate.

where $A_d \in \mathbb{R}^{n\times n}$, $B_d \in \mathbb{R}^{n\times m}$ has full column rank, and $C \in \mathbb{R}^{p\times n}$. $u^a[k]$ is the input received by actuators, $y[k] \in \mathbb{R}^p$ is the output measured by sensors.

A block diagram of the network control system under consideration is shown in Figure 3.5. The plant is to be remotely controlled through a communication network: the sensor measurements, $y[k]$, are sent to the controller through a reliable part of network so that the sensor measurements are always received correctly, while the control signals, $u^c[k]$, are sent to the plant through a part of the network that causes an additive error $e_a[k]$. As

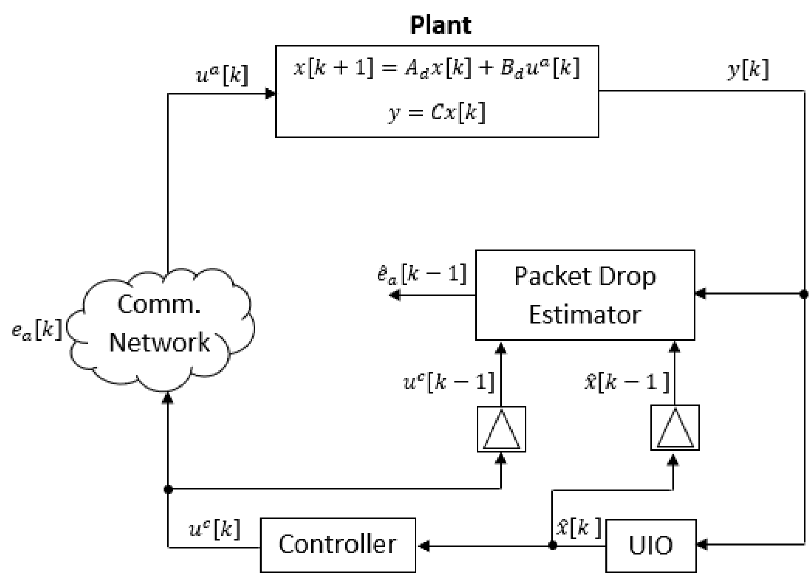

Fig. 3.5.: A network control system experiencing malicious packet drops between the controller and the actuators.

in [50], we note that the communication error $e_a[k]$ can be used to model the presence of malicious attackers that causes packet drops during the control signal transmission. We model these packet drops using the matrix $\Lambda(k) = \text{diag}\{\lambda_1(k), \lambda_2(k), \ldots, \lambda_m(k)\}$, where $\lambda_i(k), i = 1, \ldots, m$, are Boolean variables, with 1 for packet received and 0 for packet dropped. Therefore, the signal received by the actuators is $u^a[k] = \Lambda(k)u^c[k]$. Substituting the above and $u^a[k] = u^c[k] + e_a[k]$ into (3.14), we obtain the closed-loop network control system dynamics,

$$\left.\begin{aligned} x[k+1] &= A_d x[k] + B_d(u^c[k] + e_a[k]) \\ y[k] &= Cx[k], \end{aligned}\right\} \tag{3.15}$$

where $e_a[k] = \overline{\Lambda}(k)u^c[k]$, and $\overline{\Lambda}(k) = \Lambda(k) - I_m$.

We assume the output measurements $y[k]$ and the control signal $u^c[k]$ are known at every time instant k. Using Algorithm 3, we obtain estimates of the malicious packet drops of the form,

$$\hat{e}_a[k-1] = Ty[k] - TCA_d\hat{x}[k-1] - TCB_d u^c[k-1].$$

We now illustrate the proposed unknown input estimator reconstructing the malicious packet drops between the controller and actuators with one sampling period time-delay.

Example 4 We consider a DT state-space model of a coupled mass-spring-damper system, where

$$A_d = \begin{bmatrix} 0.9907 & 0.0047 & 0.0903 & 0.0002 \\ 0.0047 & 0.9907 & 0.0002 & 0.0903 \\ -0.1805 & 0.0900 & 0.8100 & 0.0044 \\ 0.0900 & -0.1805 & 0.0044 & 0.8100 \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0 \\ 0.0047 \\ 0.0002 \\ 0.0903 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

See [89, p. 148] for modeling equations of such a system.

In our simulation, we assumed $u^c[-1] = 0$, $\hat{x}[-1] = [0\ 0\ 0\ 0]^\top$, and 30% input transmission packet drops. We can see in Figure 3.6 that the unknown input estimator reconstructs well the unknown input with one sampling period time-delay.

3.7 Conclusions

We proposed a novel unknown input observer (UIO) architecture for a class of continuous-time nonlinear systems. The proposed UIO design is in the DT domain. The design of the UIO was formulated in terms of an LMI. A novel unknown input estimator was also proposed. We showed how this unknown input estimator can be used to reconstruct malicious

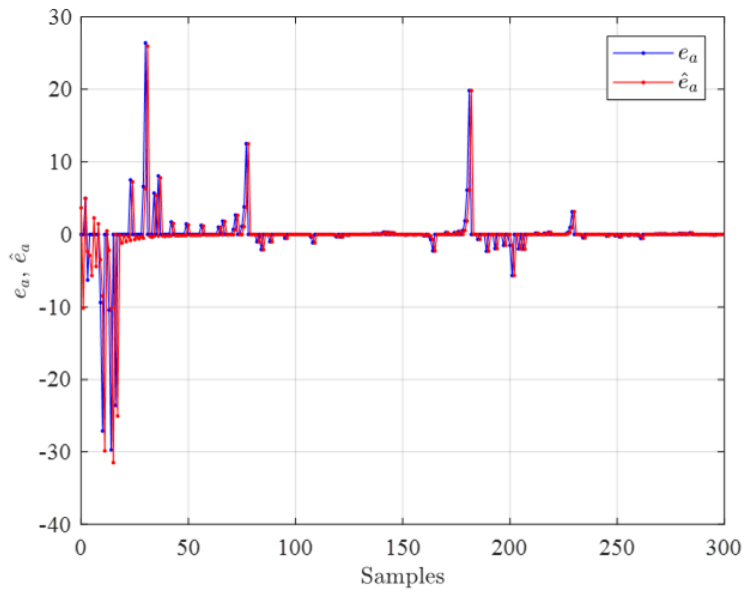

Fig. 3.6.: Malicious packet drops reconstruction with 30% input transmission packet drops.

packet drops in a network control system experiencing malicious packet drops during the control signal transmission. The proposed unknown input estimator can also be used to detect faulty actuators.

4. SIMULTANEOUS ESTIMATION OF THE STATE, UNKNOWN INPUT, AND OUTPUT DISTURBANCE IN DISCRETE-TIME LINEAR SYSTEMS

4.1 Introduction

In this Chapter, a state observer and unknown input and output disturbance estimators are proposed for discrete-time (DT) linear systems corrupted by bounded unknown inputs and output disturbances. Sufficient conditions for the existence of the state observer and disturbance estimators are given. Relationships with the strong observer of Hautus are investigated. The state, unknown input, and output disturbance estimation errors are guaranteed to be l_∞-stable with prescribed performance level. The design of the state observer and disturbance estimators are given in terms of linear matrix inequalities (LMIs). The proposed estimators can be applied to detect adversarial attacks on the communication channels between the controller and actuators and between the plant sensors and the controller. The unknown input can represent the attacks between the controller and actuator, while the output disturbances can represent the attacks between the sensor and the controller.

4.2 Notations

In our analysis, we use the following notation. For a vector $v \in \mathbb{R}^n$, we use standard notation for the Euclidean norm of a vector: $\|v\| = \sqrt{v^\top v}$. For a sequence of vectors $\{v[k]\}_{k=0}^{\infty}$, we denote $\|v[k]\|_\infty \triangleq \sup_{k\geq 0} \|v[k]\|$. We say that a sequence $\{v[k]\} \in l_\infty$ if $\|v[k]\|_\infty \leq \infty$.

4.3 Problem Statement

We consider a class of DT dynamical systems modeled by

$$\left.\begin{aligned} x[k+1] &= Ax[k] + B_1u[k] + B_2w[k] \\ y[k] &= Cx[k] + Dv[k], \end{aligned}\right\} \tag{4.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times r}$. The control input is $u[k] \in \mathbb{R}^{m_1}$. The unknown input and output disturbance to the system are modeled by $w[k] \in \mathbb{R}^{m_2}$ and $v[k] \in \mathbb{R}^{r}$, respectively. See, for example [86, Subsection 1.1.2 and Chapter 2] or [87] for a discussion on modeling of DT systems.

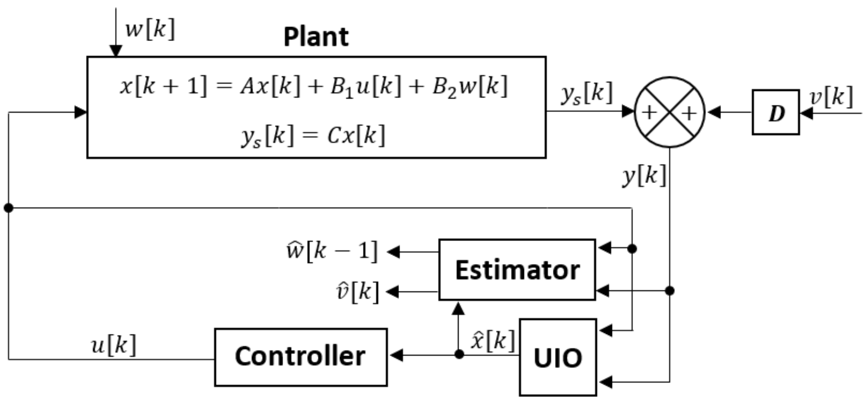

Fig. 4.1.: A combined UIO-controller compensator and an estimator of unknown input and output disturbance for system modeled by (4.1).

Our objective is to construct an observer to estimate the system state in the presence of unknown input $w[k]$ and output disturbance $v[k]$. In addition, we also wish to estimate the unknown input and output disturbance.

We make the following assumptions:

Assumption 2 *The pair (A,C) is detectable.*

Assumption 3 *Matrices B_2 and D have full column rank.*

Assumption 4 *The unknown input $w[k]$ and output disturbance $v[k]$ are uniformly bounded as functions of k.*

4.4 Proposed UIO Architecture

In this section, we propose an observer architecture to estimate the state of system given by (4.1). The estimated state can then be used to synthesize a combined UIO-controller compensator as shown in Figure 4.1.

We begin by representing $x[k]$ as

$$x[k] = x[k] - MCx[k] + MCx[k] = (I - MC)x[k] + M(y[k] - Dv[k]), \tag{4.2}$$

where $M \in \mathbb{R}^{n \times p}$ is to be determined. We select M such that

$$MD = O_{n \times r}, \tag{4.3}$$

where $O_{n \times r}$ is an n-by-r matrix of zeros. We discuss a method for finding M in the following section.

To proceed, let $z[k] = (I - MC)x[k]$. Taking into account (4.3), we represent (4.2) as

$$x[k] = z[k] + My[k]. \tag{4.4}$$

We will now show that an estimate of the state $x[k]$ can be obtained from

$$\hat{x}[k] = z[k] + My[k]. \tag{4.5}$$

The signal $z[k]$ is obtained from the equation, $z[k+1] = (I - MC)x[k+1]$. Substituting (4.1) into the above gives

$$z[k+1] = (I - MC)(Ax[k] + B_1u[k] + B_2w[k]) \tag{4.6}$$

We select M such that (4.3) is satisfied and

$$(I - MC)B_2 = O_{n\times m_2}. \tag{4.7}$$

Next, substituting (4.4) and (4.7) into (4.6), we obtain

$$z[k+1] = (I - MC)(Az[k] + AMy[k] + B_1u[k]).$$

To proceed, let $e[k] = x[k] - \hat{x}[k]$ be the state estimation error. Performing some manipulations gives

$$e[k+1] = (I - MC)Ae[k].$$

We can see from the above that we do not have any control over the estimation error convergence dynamics, which is determined by the matrix $(I - MC)A$. To improve the estimation error convergence dynamics, we add the innovation term $L(y[k] - \hat{y}[k])$, where $L \in \mathbb{R}^{n\times p}$ and

$$\hat{y}[k] = C\hat{x}[k] = C(z[k] + My[k]).$$

Then,

$$z[k+1] = (I - MC)(A\hat{x}[k] + B_1u[k]) + L(y[k] - \hat{y}[k]). \tag{4.8}$$

Combining (4.5) and (4.8), we obtain the proposed UIO architecture,

$$\left.\begin{array}{rcl} z[k+1] & = & (I - MC)(Az[k] + AMy[k] + B_1u[k]) + L(y[k] - Cz[k] - CMy[k]) \\ \hat{x}[k] & = & z[k] + My[k]. \end{array}\right\} \tag{4.9}$$

The error dynamics for the UIO given by (4.9) are

$$\begin{aligned} e[k+1] =& x[k+1]-\hat{x}[k+1] \\ =& Ax[k]+B_1u[k]+B_2w[k]-z[k+1]-My[k+1] \\ =& Ax[k]+B_1u[k]+B_2w[k]-(I-MC)(A\hat{x}[k]+B_1u[k])-L(y[k]-Cz[k]-CMy[k]) \\ & -M(Cx[k+1]+Dv[k+1]). \end{aligned}$$

Substituting the state dynamics of (4.1), equation (4.3), and (4.5) into the above, we obtain

$$\begin{aligned} e[k+1] =& Ax[k]+B_1u[k]+B_2w[k]-(I-MC)(A\hat{x}[k]+B_1u[k])-L(y[k]-C\hat{x}[k]) \\ & -MC(Ax[k]+B_1u[k]+B_2w[k]). \end{aligned}$$

Performing some manipulations and taking into account (4.7) gives

$$e[k+1] = ((I-MC)A-LC)\,e[k]-LDv[k].$$

Let $\tilde{A} = (I-MC)A$, then

$$e[k+1] = (\tilde{A}-LC)e[k]-LDv[k]. \tag{4.10}$$

A necessary condition for the stability of the above error dynamics is that $(\tilde{A}-LC)$ is Schur stable. To solve for such an L, the pair $(\tilde{A},C)$ is required to be detectable.

Remark 4 *We note that if an L exists such that $(\tilde{A}-LC)$ is Schur stable and $LD = O_{n\times r}$, then the error dynamics in* (4.10) *are asymptotically stable. However, for general systems, it may not be feasible to satisfy the condition $LD = O_{n\times r}$. In our further analysis, we do not impose the constraint $LD = O_{n\times r}$. We present conditions for the stability of the error dynamics* (4.10) *and conditions for the existence of the UIO gain matrix L in Section 4.8.*

In the following section, we give the existence condition for a matrix M to satisfy (4.3) and (4.7).

4.5 Solving for M

In this section, we present a sufficient condition for the existence of a matrix M with the required properties. This sufficient condition would be one of the conditions for the existence of our proposed UIO. To proceed, we first prove the following lemma.

Lemma 4 *A necessary and sufficient condition for $(I-MC)B_2 = O$ to have a solution M is that $rank(CB_2) = rank(B_2)$.*

Proof Suppose there exists a matrix M such that $(I-MC)B_2 = O$. Equivalently,

$$MCB_2 = B_2.$$

Hence,

$$\text{rank}(MCB_2) \leq \text{rank}(CB_2) \leq \text{rank}(B_2).$$

Therefore, we must have $\text{rank}(CB_2) = \text{rank}(B_2)$. Conversely, if $\text{rank}(CB_2) = \text{rank}(B_2)$, then the row spaces of B_2 and CB_2 must be the same and so the rows of B_2 are in the row space of CB_2. It follows that there is a matrix M such that $B_2 = MCB_2$. ■

We now give a condition for the existence of M that solves matrix equations (4.3) and (4.7) simultaneously.

Theorem 4 *If*

$$rank[CB_2 \;\; D] = rank(B_2) + rank(D), \tag{4.11}$$

then there exists a solution M to

$$(I - MC)B_2 = O_{n\times m_2}, \tag{4.12a}$$

$$MD = O_{n\times r}. \tag{4.12b}$$

Proof We represent (4.12a) and (4.12b) as

$$M\begin{bmatrix} CB_2 & D \end{bmatrix} = \begin{bmatrix} B_2 & O_{n\times r} \end{bmatrix}. \tag{4.13}$$

By (4.11), the matrix $[CB_2 \;\; D]$ has full column rank and therefore it is left invertible. For example, $[CB_2 \;\; D]^\dagger$ is a left inverse of $[CB_2 \;\; D]$. Therefore,

$$M = [B_2 \;\; O_{n\times r}][CB_2 \;\; D]^\dagger \tag{4.14}$$

is a solution to (4.12a) and (4.12b), which concludes the proof. ■

Note that a class of solutions to (4.12a) and (4.12b) has the form

$$M = [B_2 \;\; O_{n\times r}]\big([CB_2 \;\; D]^\dagger + H_0(I - [CB_2 \;\; D][CB_2 \;\; D]^\dagger)\big), \tag{4.15}$$

where $H_0 \in \mathbb{R}^{n\times p}$ is a design parameter matrix.

Remark 5 *For the matrix $[CB_2 \;\; D]$ to have full column rank, it is necessary that $r \le p - m_2$. Equivalently, the number of outputs should be greater than or equal the number of unknown inputs and output disturbances.*

4.6 The UIO Synthesis Conditions

In our further discussion, we use the following lemma.

Lemma 5 *If the pair $(\tilde{A}, MC)$ is detectable, then the pair $(\tilde{A}, C)$ is detectable. Furthermore, if M has full column rank, then the converse is also true. (Note that in our application, we need $MD = O$ and thus unless D is the zero matrix, M can never have full column rank.)*

Proof We prove the lemma by contraposition. Assume that the pair $(\tilde{A}, C)$ is non-detectable. Then, there exists an eigenvalue $|z_1| \geq 1$ such that

$$\text{rank} \begin{bmatrix} z_1 I - \tilde{A} \\ C \end{bmatrix} < n.$$

Therefore, there exist a vector $v_1 \in \mathbb{C}^n$ such that

$$\begin{bmatrix} z_1 I - \tilde{A} \\ C \end{bmatrix} v_1 = 0,$$

and hence, $Cv_1 = 0$. Premultiplying the above equation by M gives $MCv_1 = 0$. We conclude from the above that

$$\begin{bmatrix} z_1 I - \tilde{A} \\ MC \end{bmatrix} v_1 = 0. \tag{4.16}$$

Thus, z_1 also an unobservable eigenvalue of the pair $(\tilde{A}, MC)$, that is, the pair $(\tilde{A}, MC)$ is non-detectable.

Note that,

$$\begin{bmatrix} z_1 I - \tilde{A} \\ MC \end{bmatrix} = \begin{bmatrix} I & O \\ O & M \end{bmatrix} \begin{bmatrix} z_1 I - \tilde{A} \\ C \end{bmatrix}$$

and if M has full column rank, then its pseudoinverse $M^\dagger$ exists and

$$\begin{bmatrix} z_1 I - \tilde{A} \\ C \end{bmatrix} = \begin{bmatrix} I & O \\ O & M^\dagger \end{bmatrix} \begin{bmatrix} z_1 I - \tilde{A} \\ MC \end{bmatrix}.$$

Hence, if M has full column rank, then the pair $(\tilde{A}, MC)$ is detectable $\iff$ the pair $(\tilde{A}, C)$ is detectable. ■

We now present a theorem that gives sufficiency conditions for the existence of the proposed UIO.

Theorem 5 1. *If* (4.11) *holds, then there exists M that satisfies* (4.12) *and thus the UIO given by* (4.9) *can be constructed with this M.*

2. *Suppose in addition, for this M, the pair $(\tilde{A}, C) = \big((I - MC)A, C\big)$ is detectable. Then there is a matrix L and a constant γ such that $(\tilde{A} - LC)$ is Schur stable and the observer error given by equation* (4.10) *satisfies*

$$\limsup_{k\to\infty} \|e[k]\| \leq \gamma \limsup_{k\to\infty} \|v[k]\|. \tag{4.17}$$

Proof By Theorem 4, condition (4.11) is sufficient for the existence of M that solves (4.12).

The detectability of the pair $(\tilde{A}, C)$ is necessary and sufficient for the existence of L such that $(\tilde{A} - LC)$ is Schur stable. By Assumption 4, $v[k]$ is uniformly bounded as a function of k. Combining this with $(\tilde{A} - LC)$ being Schur stable yields (4.17) which complete the proof. ■

The following lemma gives conditions for the detectability of the pair $(\tilde{A}, MC)$ of the system given by

$$\left.\begin{aligned} x[k+1] &= Ax[k] + B_1 u[k] + B_2 w[k] \\ My[k] &= MCx[k] + MDv[k]. \end{aligned}\right\} \tag{4.18}$$

We will use this lemma in our further analysis. To proceed, note that by construction, $MD = O$.

Lemma 6 *If*

- $rank(CB_2) = rank(B_2) = m_2$,

- *$rank(I_n - MC) = n - m_2$,*

then the following conditions are equivalent

1. *$(\tilde{A}, MC)$ is detectable,*
2. *$rank \begin{bmatrix} z(I_n - MC) - \tilde{A} \\ MC \end{bmatrix} = n$ for all $|z| \geq 1$,*
3. *$rank \begin{bmatrix} zI_n - A & -B_2 \\ MC & O_{n \times m_2} \end{bmatrix} = n + m_2$ for all $|z| \geq 1$.*

Proof First, we show that conditions 1 and 2 are equivalent. The pair $(\tilde{A}, MC)$ being detectable is equivalent to

$$\text{rank} \begin{bmatrix} zI_n - \tilde{A} \\ MC \end{bmatrix} = n \text{ for all } |z| \geq 1,$$

which is equivalent to

$$\text{rank}\left(\begin{bmatrix} I_n & -zI_n \\ O & I_n \end{bmatrix} \begin{bmatrix} zI_n - \tilde{A} \\ MC \end{bmatrix} \right) = \text{rank} \begin{bmatrix} z(I_n - MC) - \tilde{A} \\ MC \end{bmatrix} = n, \text{ for all } |z| \geq 1,$$

which proves that conditions 1 and 2 are equivalent.

Next, we will show that conditions 2 and 3 are equivalent. Since B_2 has full column rank, it is left invertible, for example, we can take ${B_2}^\dagger$ as a left inverse of B_2. Then, we have ${B_2}^\dagger B_2 = I_{m_2}$. Therefore,

$$\ker\,({B_2}^\dagger) \cap \ker\,(I_n - MC) = \{0\}$$

and hence

$$\text{rank} \begin{bmatrix} I_n - MC \\ {B_2}^\dagger \end{bmatrix} = n.$$

Let

$$S = \begin{bmatrix} I_n - MC & O_{n\times p} \\ B_2{}^{\dagger} & O_{m_2\times p} \\ O_{p\times n} & I_p \end{bmatrix}, \; T = \begin{bmatrix} I_n & O_{n\times m_2} \\ B_2{}^{\dagger}(zI_n - A) & I_{m_2} \end{bmatrix},$$

where $S \in \mathbb{R}^{(n+p+m_2)\times(n+p)}$, $T \in \mathbb{R}^{(n+m_2)\times(n+m_2)}$, and $\text{rank}(S) = n+p$. We then have

$$\begin{aligned} \text{rank} \begin{bmatrix} zI_n - A & -B_2 \\ MC & O \end{bmatrix} &= \text{rank}\left(S \begin{bmatrix} zI_n - A & -B_2 \\ MC & O \end{bmatrix} T \right) \\ &= \text{rank} \begin{bmatrix} z(I_n - MC) - \tilde{A} & O \\ O & -I_{m_2} \\ MC & O \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} z(I_n - MC) - \tilde{A} \\ MC \end{bmatrix} + m_2 \\ &= n + m_2. \end{aligned}$$

This proves that conditions 2 and 3 are equivalent. ■

The following theorem gives us insight into the role of system zeros on the existence of the proposed UIO.

Theorem 6 *If*

- *the matrix rank condition* (4.11) *is satisfied,*
- *the matrix* $\begin{bmatrix} -B_2 \\ D \end{bmatrix}$ *is defined and has full column rank,*
- *rank* $\begin{bmatrix} I - MC & O \\ O & M \end{bmatrix} = n,$
- *rank* $\begin{bmatrix} z(I_n - MC) - \tilde{A} \\ MC \end{bmatrix} = n$ *for all* $|z| \geq 1,$

then

$$rank \begin{bmatrix} zI_n - A & -B_2 \\ C & D \end{bmatrix} = n + m_2 \text{ for all } |z| \geq 1. \tag{4.19}$$

Proof By Theorem 4, if matrix rank condition (4.11) is satisfied then there exists a solution M that satisfies

$$\begin{bmatrix} I - MC & O \\ O & M \end{bmatrix} \begin{bmatrix} -B_2 \\ D \end{bmatrix} = O.$$

Let

$$\tilde{M} = \begin{bmatrix} I - MC & O \\ O & M \end{bmatrix}.$$

There exists $M_1 \in \mathbb{R}^{(p-m_2)\times(n+p)}$ such that $M_1 \begin{bmatrix} -B_2 \\ D \end{bmatrix} = O$ and rank $\begin{bmatrix} \tilde{M} \\ M_1 \end{bmatrix} = n + p - m_2$. Then, since $\begin{bmatrix} -B_2 \\ D \end{bmatrix}$ has full column rank, its pseudoinverse is also its left inverse, that is,

$$\begin{bmatrix} -B_2 \\ D \end{bmatrix}^{\dagger} \begin{bmatrix} -B_2 \\ D \end{bmatrix} = I_{m_2}.$$

Therefore,

$$\ker \begin{bmatrix} \tilde{M} \\ M_1 \end{bmatrix} \cap \ker \begin{bmatrix} -B_2 \\ D \end{bmatrix}^{\dagger} = \{0\},$$

and

$$\operatorname{rank}\begin{bmatrix}\tilde{M}\\ M_1\\ \begin{bmatrix}-B_2\\ D\end{bmatrix}^{\dagger}\end{bmatrix}=n+p.$$

Let

$$S=\begin{bmatrix}\tilde{M}\\ M_1\\ \begin{bmatrix}-B_2\\ D\end{bmatrix}^{\dagger}\end{bmatrix},$$

and

$$T=\begin{bmatrix}I_n & O\\ -\begin{bmatrix}-B_2\\ D\end{bmatrix}^{\dagger}\begin{bmatrix}zI_n-A\\ C\end{bmatrix} & I_{m_2}\end{bmatrix}.$$

Then,

$$\operatorname{rank}S\begin{bmatrix}zI_n-A & -B_2\\ C & D\end{bmatrix}T=\operatorname{rank}\begin{bmatrix}\tilde{M}\begin{bmatrix}zI_n-A\\ C\end{bmatrix} & O\\ M_1\begin{bmatrix}zI_n-A\\ C\end{bmatrix} & O\\ O & I_{m_2}\end{bmatrix}=\operatorname{rank}\begin{bmatrix}\tilde{M}\begin{bmatrix}zI_n-A\\ C\end{bmatrix}\\ M_1\begin{bmatrix}zI_n-A\\ C\end{bmatrix}\end{bmatrix}+m_2. \tag{4.20}$$

Note that

$$\tilde{M}\begin{bmatrix} zI_n - A \\ C \end{bmatrix} = \begin{bmatrix} z(I - MC) - \tilde{A} \\ MC \end{bmatrix}.$$

Hence, (4.19) holds if

$$\text{rank}\begin{bmatrix} z(I - MC) - \tilde{A} \\ MC \end{bmatrix} = n, \text{ for all } |z| \geq 1.$$

■

Remark 6 *By Lemma 5 and Theorem 5, the above condition also implies that the pair* $(\tilde{A}, C)$ *is detectable.*

4.7 Relations With the Strong Observer of Hautus

In this section, we discuss relationships between our UIO existence conditions and the strong observer existence conditions of Hautus [54]. Our conditions are applicable to a general class of linear systems when the unknown input and output disturbance are different as given by (4.1). Hautus, on the other hand, gives the strong observer existence conditions for a class of linear systems with the same unknown input and output disturbance. The Hautus' necessary and sufficient conditions for the existence of his strong observer are:

$$\text{rank}\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix} = \text{rank}(D) + \text{rank}\begin{bmatrix} B_2 \\ D \end{bmatrix}, \tag{4.21}$$

and system zeros of the system defined by quadruple (A, B_2, C, D) are in the open unit disk.

Our plant model is more general than that of Hautus. Therefore, we cannot apply his conditions to our plant model. However, we can represent our plant model in the form of the Hautus plant model as follows,

$$\left.\begin{aligned} x[k+1] &= Ax[k] + B_1u[k] + \begin{bmatrix} B_2 & O \end{bmatrix} \begin{bmatrix} w[k] \\ v[k] \end{bmatrix} \\ y[k] &= Cx[k] + \begin{bmatrix} O & D \end{bmatrix} \begin{bmatrix} w[k] \\ v[k] \end{bmatrix}, \end{aligned}\right\} \tag{4.22}$$

where O is used to represent matrices of zeros with compatible dimensions. Note that the above plant model has the same unknown input and output disturbance. Thus, we can apply the conditions of Hautus to this plant model for the existence of the strong observer of Hautus. We first concern ourselves with the matrix rank condition of Hautus given by (4.21). Applying (4.21) to (4.22) gives

$$\operatorname{rank} \begin{bmatrix} CB_2 & O & O & D \\ O & D & O & O \end{bmatrix} = \operatorname{rank} \begin{bmatrix} O & D \end{bmatrix} + \operatorname{rank} \begin{bmatrix} B_2 & O \\ O & D \end{bmatrix}. \tag{4.23}$$

Rearranging the left hand side of the above gives

$$\operatorname{rank} \begin{bmatrix} CB_2 & O & O & D \\ O & D & O & O \end{bmatrix} = \operatorname{rank} \begin{bmatrix} CB_2 & D & O \\ O & O & D \end{bmatrix} = \operatorname{rank} \begin{bmatrix} CB_2 & D \end{bmatrix} + \operatorname{rank}(D). \tag{4.24}$$

Rearrange the right hand side of the rank condition (4.23) as

$$\operatorname{rank} \begin{bmatrix} O & D \end{bmatrix} + \operatorname{rank} \begin{bmatrix} B_2 & O \\ O & D \end{bmatrix} = 2\operatorname{rank}(D) + \operatorname{rank}(B_2) \tag{4.25}$$

Comparing (4.24) and (4.25). We see that the matrix rank condition (4.23) is the same as the matrix rank condition (4.11).

Next, the system zeros condition for the system modeled by (4.22) is

$$\operatorname{rank}\begin{bmatrix} zI-A & -B_2 & O \\ C & O & D \end{bmatrix} = n+m_2+r \text{ for all } |z| \geq 1. \tag{4.26}$$

If the rank condition (4.19) is not satisfied, and B_2 and D have the same number of columns, then there are β_1, β_2, not both zero, such that

$$\begin{bmatrix} zI-A & -B_2 \\ C & D \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = 0.$$

Then, it is easy to see that

$$\begin{bmatrix} zI-A & -B_2 & O \\ C & O & D \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_2 \end{bmatrix} = 0.$$

That is, the system zeros condition (4.26) implies the system zeros condition (4.19).

The following example shows that there are general systems given by (4.1) such that our proposed UIO can be constructed, however, the strong observer of Hautus cannot be constructed for the equivalent augmented system given by (4.22).

Example 5 We consider a plant model (4.1) with

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -0.3 \end{bmatrix},\ B_2 = \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ D = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

To design our proposed UIO for the above system, we first check the UIO existence conditions given in Theorem 5. The matrix rank condition (4.11) is satisfied, where

$$\operatorname{rank}\begin{bmatrix} CB_2 & D \end{bmatrix} = \operatorname{rank}\begin{bmatrix} -2 & 2 \\ -3 & 2 \end{bmatrix} = \operatorname{rank}(B_2) + \operatorname{rank}(D) = 2.$$

We solve (4.14) to obtain,

$$M = \begin{bmatrix} -2 & 2 \\ -3 & 3 \\ -4 & 4 \end{bmatrix}.$$

Next, we construct the matrix $\tilde{A} = (I_3 - MC)A$ to obtain

$$\tilde{A} = \begin{bmatrix} -3 & 4 & 0 \\ -3 & 4 & 0 \\ -4 & 8 & -0.3 \end{bmatrix}.$$

which has the eigenvalues $\{-0.3,\, 1,\, 0\}$. We check the detectability of the pair $(\tilde{A}, C)$, by checking that the unstable eigenvalue is observable, that is,

$$\operatorname{rank}\begin{bmatrix} I_3 - \tilde{A} \\ C \end{bmatrix} = 3.$$

Therefore, the pair $(\tilde{A}, C)$ is detectable. Hence, there exists a matrix L such that $(\tilde{A} - LC)$ is Schur stable. Since $v[k]$ is uniformly bounded, then there exists a constant γ such that (4.17) is satisfied. Therefore, both existence conditions of our proposed observer are satisfied.

If one wishes to construct the strong observer of Hautus for this example, it is required that the unknown input and output disturbance are the same. It is easy to check that, in this case, the strong observer of Hautus can be constructed.

Now suppose that the unknown input and output disturbance are not the same. To construct the strong observer, we represent the system of this example in the Hautus format given by (4.22). We next check the Hautus' existence conditions. It is easy to check that the matrix rank condition of Hautus is satisfied. Then, we check the system zeros condition. The system zero condition is not satisfied for $z_1 = 1$. Indeed,

$$\det \begin{bmatrix} I_3 - A & -B & O \\ C & O & D \end{bmatrix} = 0.$$

In conclusion, one can construct our proposed observer for this example but not the strong observer of Hautus.

In our further discussion, we will use the following notation for the various matrix rank conditions:

1. $\mathscr{S} \Longleftrightarrow \operatorname{rank}\begin{bmatrix} CB_2 \end{bmatrix} = \operatorname{rank} B_2$.
2. $\mathscr{G} \Longleftrightarrow \operatorname{rank}\begin{bmatrix} CB_2 & D \end{bmatrix} = \operatorname{rank} B_2 + \operatorname{rank} D$.
3. $\mathscr{M} \Longleftrightarrow \operatorname{rank}(CB_2 + D) = \operatorname{rank}\begin{bmatrix} B_2 \\ D \end{bmatrix}$, when $CB_2 + D$ is defined.
4. $\mathscr{H} \Longleftrightarrow$ Hautus' matrix rank condition:

$$\operatorname{rank}\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_2 \\ D \end{bmatrix} + \operatorname{rank} D.$$

For any matrix M, we let

$$\begin{aligned} \mathfrak{c}(M) &= \text{Number of columns of } M, \\ \ker M &= \{v : Mv = 0\}. \end{aligned}$$

We will make frequent use of the well known equality

$$\mathfrak{c}(M) = \text{rank}M + \dim\ker M. \tag{4.27}$$

First note that $\mathscr{G}$ implies that $\text{rank}(CB_2) = \text{rank}B_2$ and thus $CB_2v = 0$ if and only if $B_2v = 0$, or equivalently, $\ker(CB_2) = \ker B_2$.

If u,v are column vectors, we let $u \oplus v = \begin{bmatrix} u \\ v \end{bmatrix}$ and if U,V are vector spaces of column vectors, we let

$$U \oplus V = \{u \oplus v | u \in U, v \in V\}.$$

It is easy to see that $\dim(U \oplus V) = \dim U + \dim V$. (Note that if $\{u_1,\ldots,u_p\}$ and $\{v_1,\ldots,v_q\}$ are bases of U,V, respectively, then $\{u_1 \oplus 0,\ldots,u_p \oplus 0, 0 \oplus v_1,\ldots,0 \oplus v_q\}$ is a basis for $U \oplus V$.)

Next, we give the following lemmas that we will use in the proof of our main theorem in this section.

Lemma 7 *If $\mathscr{G}$, then* $\ker(\begin{bmatrix} CB_2 & D \end{bmatrix}) = \ker B_2 \oplus \ker D$.

Proof We have from $\mathscr{G}$ and equality (4.27) that

$$\begin{aligned}
\dim\ker(\begin{bmatrix} CB_2 & D \end{bmatrix}) &= \mathfrak{c}(\begin{bmatrix} CB_2 & D \end{bmatrix}) - \text{rank}(\begin{bmatrix} CB_2 & D \end{bmatrix}) \\
&= \mathfrak{c}(CB_2) + \mathfrak{c}(D) - \text{rank}(B_2) - \text{rank}D \\
&= \dim\ker B_2 + \dim\ker D \\
&= \dim(\ker B_2 \oplus \ker D).
\end{aligned}$$

Since it is immediate that $\ker B_2 \oplus \ker D \subset \ker(\begin{bmatrix} CB_2 & D \end{bmatrix})$, we must have $\ker(\begin{bmatrix} CB_2 & D \end{bmatrix}) = \ker B_2 \oplus \ker D$. ∎

Lemma 8 *If $\mathscr{G}$, then* $\ker(CB_2 + D) = \ker B_2 \cap \ker D$.

Proof Clearly we have

$$\ker B_2 \cap \ker D \subset \ker(CB_2 + D).$$

Suppose $(CB_2 + D)v = 0$. Then

$$\begin{bmatrix} CB_2 & D \end{bmatrix} \begin{bmatrix} v \\ v \end{bmatrix} = 0.$$

It follows from Lemma 7 that

$$\begin{bmatrix} v \\ v \end{bmatrix} \in \ker B_2 \oplus \ker D.$$

Therefore $B_2 v = 0$ and $Dv = 0$ and the lemma follows. ■

Lemma 9 *If $\mathscr{G}$, then $\mathscr{M}$.*

Proof Observe that

$$\ker \begin{bmatrix} B_2 \\ D \end{bmatrix} = \ker B_2 \cap \ker D.$$

Thus by Lemma 8, we have

$$\ker(CB_2 + D) = \ker \begin{bmatrix} B_2 \\ D \end{bmatrix},$$

which is equivalent to the claim of the lemma since the two matrices have the same number of columns. ■

The following theorem states that the matrix rank condition for the existence of our proposed UIO is also sufficient for the matrix rank condition of the strong observer of Hautus.

Theorem 7 *The matrix rank condition* (4.11) *implies the matrix rank condition* (4.21) *of Hautus.*

Proof Assume $\mathscr{G}$. Then the above lemmas hold (and all will be used.) Hautus matrix rank condition is

$$\operatorname{rank}\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_2 \\ D \end{bmatrix} + \operatorname{rank} D.$$

By Lemma 9, this is equivalent to

$$\operatorname{rank}\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix} = \operatorname{rank}(CB_2 + D) + \operatorname{rank} D,$$

which is equivalent to

$$\operatorname{rank}\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix} = \operatorname{rank}\begin{bmatrix} CB_2 + D & O \\ O & D \end{bmatrix}.$$

We prove the above by showing that the two matrices have the same kernel. Suppose

$$\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Then,

$$\begin{bmatrix} CB_2 & D \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0, \text{ and } Du = 0.$$

By Lemma 7,

$$CB_2u = 0, \ Dv = 0, \text{ and } Du = 0,$$

and it follows that

$$\begin{bmatrix} CB_2 + D & O \\ O & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Conversely, suppose

$$\begin{bmatrix} CB_2 + D & O \\ O & D \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Then,

$$(CB_2 + D)u = 0, \text{ and } Dv = 0.$$

By Lemma 8,

$$CB_2u = 0, \; Du = 0, \text{ and } Dv = 0.$$

Thus

$$\begin{bmatrix} CB_2 & D \\ D & O \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Therefore the two matrices in question have the same kernel and therefore the same rank (since they clearly have the same number of columns.) ■

In the next section, we analyze the stability of the error dynamics given by (4.10).

4.8 Stability of the Error Dynamics

In this section, we analyze the error dynamics stability and give the conditions for finding the UIO gain matrix L in terms of linear matrix inequalities. To proceed, we define l_∞-stability with performance level (p.l.) γ.

Definition 3 *The system*

$$e[k+1] = f(k, e[k], v[k]) \tag{4.28}$$

is globally uniformly l_∞-stable with performance level γ if the following conditions are satisfied:

1. *The undisturbed system, (that is, $v[k] = 0$ for all $k \geq 0$) is globally uniformly exponentially stable with respect to the origin.*
2. *For zero initial condition, $e[k_0] = 0$, and every bounded unknown input $v[k]$, we have*

$$\|e[k]\| \leq \gamma \|v[k]\|_\infty, \quad \forall k \geq k_0.$$

3. *For every initial condition, $e[k_0] = e_0$, and every bounded unknown input $v[\cdot]$, we have*

$$\limsup_{k\to\infty} \|e[k]\| \leq \gamma \|v[k]\|_\infty.$$

For more details on the l_∞-stability for discrete-time systems with level of performance, we refer to [40]. We now present a lemma from [40] that we use in our proof of the theorem that gives the design condition of the proposed UIO in LMI format.

Lemma 10 *Suppose that for the error dynamics given by* (4.28) *there exists a function $V : \mathbb{R}^n \to \mathbb{R}$ and scalars $\delta \in (0,1)$, $\beta_1, \beta_2 > 0$ and $\mu_1, \mu_2 \geq 0$ such that*

$$\beta_1 \|e[k]\|^2 \leq V(e[k]) \leq \beta_2 \|e[k]\|^2, \tag{4.29}$$

and

$$\Delta V[k] \leq -\delta(V(e[k]) - \mu_1 \|v[k]\|^2), \tag{4.30a}$$

$$\|e[k]\|^2 \leq \mu_2 V(e[k]), \tag{4.30b}$$

for all $k \geq 0$*, where* $\Delta V[k] = V(e[k+1]) - V(e[k])$*. Then system* (4.28) *is globally uniformly* l_∞*-stable with performance level* $\gamma = \sqrt{\mu_1 \mu_2}$ *with respect to the output disturbance* $v[k]$*.*

Our objective is to select the UIO gain matrix L so that the state error dynamics (4.10) is l_∞-stable with performance level γ.

To proceed, consider the error dynamics equation given by (4.10). Let $E = \tilde{A} - LC$, and $N = -LD$. Then, we have the following theorem.

Theorem 8 *Suppose Assumption 4 is satisfied. If there exist matrices* $P = P^\top \succ 0$ *and* L*, and* $\alpha \in (0,1)$ *such that the matrix inequalities*

$$\begin{bmatrix} E^\top PE - (1-\alpha)P & * \\ N^\top PE & N^\top PN - \alpha I \end{bmatrix} \preceq 0 \tag{4.31a}$$

$$\begin{bmatrix} P & * \\ I & \mu I \end{bmatrix} \succeq 0, \tag{4.31b}$$

are satisfied, then the state observation error dynamics are l_∞*-stable with performance level* $\gamma = \sqrt{\mu}$*.*

Proof Let $V[k] = e[k]^\top Pe[k]$ be a Lyapunov function candidate for the estimation error dynamics given by (4.10). We evaluate the first forward difference $\Delta V[k] = V[k+1] - V[k]$ on the trajectories of (4.10) to obtain

$$\Delta V[k] = e[k]^\top (E^\top PE - P)e[k] + 2e[k]^\top E^\top PNv[k] + v[k]^\top N^\top PNv[k].$$

Let $\zeta = [e[k]^\top \;\; v[k]^\top]^\top$. Premultiplying and postmultiplying the matrix inequality (4.31a) by $\zeta^\top$ and ζ, respectively, and taking into account the above equality, we obtain

$$\Delta V[k] + \alpha(V[k] - \|v[k]\|^2) \preceq 0.$$

Therefore, condition (4.30a) in Lemma 10 holds with $\mu_1 = 1$. Next, taking the Schur complement of (4.31b), we obtain

$$P - \mu^{-1} I \succeq 0.$$

Premultiplying the above inequality by $e[k]^\top$ and postmultiplying it by $e[k]$ gives

$$e[k]^\top P e[k] - \mu^{-1} e[k]^\top e[k] \geq 0.$$

Rearranging the above yields

$$\|e[k]\|^2 \leq \mu V(e[k]).$$

By Lemma 10, the state error dynamics (4.10) are l_∞-stable with performance level $\gamma = \sqrt{\mu}$.
■

We present a method to solve matrix inequality (4.31a) in Theorem 8 using an LMI. Let $Z = PL$, then solving the matrix inequality (4.31a) is equivalent to solving the following LMI for P and Z,

$$\begin{bmatrix} -P & * \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \preceq 0, \tag{4.32}$$

where

$$\Omega_{21}^\top = \begin{bmatrix} P\tilde{A} - ZC & -ZD \end{bmatrix}$$

and

$$\Omega_{22} = \begin{bmatrix} -(1-\alpha)P & O_{n\times r} \\ O_{r\times n} & -\alpha I \end{bmatrix}.$$

Since $P = P^\top \succ 0$, taking Schur complement of (4.32) gives,

$$\Omega_{22} + \Omega_{21}P^{-1}\Omega_{21}^\top \preceq 0,$$

which, in turn, yields matrix inequality (4.31a).

4.9 Unknown Input and Output Disturbance Reconstruction

In this section, we propose estimators for the unknown input $w[k]$ and the output disturbance $v[k]$ of system model (4.1).

4.9.1 Unknown input reconstruction

By Assumption 3, the matrix B_2 has full column rank, therefore, $B_2^\dagger = (B_2^\top B_2)^{-1}(B_2)^\top$ exists. Premultiplying both sides of the state dynamics given by (4.1) by the matrix $B_2^\dagger$, we obtain

$$B_2^\dagger x[k+1] = B_2^\dagger Ax[k] + B_2^\dagger B_1 u[k] + B_2^\dagger B_2 w[k].$$

Since $B_2^\dagger B_2 = I_{m_2}$, we rewrite the above equation as

$$w[k] = B_2^\dagger x[k+1] - B_2^\dagger Ax[k] - B_2^\dagger B_1 u[k].$$

Using this equation, we obtain the following unknown input estimator:

$$\hat{w}[k] = B_2^\dagger \hat{x}[k+1] - B_2^\dagger A\hat{x}[k] - B_2^\dagger B_1 u[k].$$

The above unknown input estimator depends on $\hat{x}[k+1]$. Therefore, we delay the argument by one sampling period time-delay to obtain our proposed estimator of $w[k]$ of the form

$$\hat{w}[k-1] = B_2^{\dagger}\hat{x}[k] - B_2^{\dagger}A\hat{x}[k-1] - B_2^{\dagger}B_1u[k-1]. \tag{4.33}$$

To prove l_∞-stability of the unknown input estimates, we let $e_w[k] = w[k] - \hat{w}[k]$ be the unknown input estimation error. Then, we have

$$e_w[k] = B_2^{\dagger}e[k+1] - B_2^{\dagger}Ae[k].$$

By Theorem 8, we have

$$\limsup_{k\to\infty} \|e[k]\| \leq \gamma\|v[k]\|_\infty.$$

Hence the bound on the unknown input estimation steady-state error satisfies

$$\begin{aligned}\limsup_{k\to\infty} \|e_w[k]\| &\leq \|B_2^{\dagger}\|(\gamma\|v[k+1]\|_\infty + \|A\|\gamma\|v[k]\|_\infty) \\ &\leq \|B_2^{\dagger}\|(1+\|A\|)\sqrt{\mu}\|v[k]\|_\infty.\end{aligned}$$

Thus, the unknown input estimator estimates the unknown input with performance level $\gamma_w = \|B_2^{\dagger}\|(1+\|A\|)\sqrt{\mu}$.

4.9.2 Output disturbance reconstruction

By Assumption 3, the matrix D has full column rank, therefore it has left inverse. Premultiplying both sides of the output equation of model (4.1) by $D^{\dagger}$ gives

$$D^{\dagger}y[k] = D^{\dagger}Cx[k] + D^{\dagger}Dv[k].$$

Rearranging the above equation, we obtain

$$v[k] = D^{\dagger} y[k] - D^{\dagger} C x[k].$$

From the above, we obtain the output disturbance estimator,

$$\hat{v}[k] = D^{\dagger} y[k] - D^{\dagger} C \hat{x}[k]. \tag{4.34}$$

To prove the l_∞-stability of the output disturbance estimate, we let $e_v[k] = v[k] - \hat{v}[k]$ be the output disturbance estimation error. Then, we have

$$e_v[k] = -D^{\dagger} C e[k].$$

By Theorem 8,

$$\limsup_{k\to\infty} \|e[k]\| \leq \gamma \|v[k]\|_\infty.$$

We use this to obtain a bound on the output disturbance steady-state estimation error,

$$\limsup_{k\to\infty} \|e_v[k]\| \leq \|D^{\dagger}\| \|C\| \gamma \|v[k]\|_\infty = \|D^{\dagger}\| \|C\| \sqrt{\mu} \|v[k]\|_\infty.$$

Thus, the performance level of the output disturbance estimator is $\gamma_v = \|D^{\dagger}\| \|C\| \sqrt{\mu}$.

We summarize our discussion in the form the following observer design algorithm.

Algorithm 4: Unknown input estimator synthesis

Check if matrix rank condition (4.11) is satisfied for system model given by (4.1).
Solve (4.14) for M.
Solve LMIs (4.32) and (4.31b) for L and P using, for example, CVX toolbox.
Set $\hat{x}[-1]$ and $u[k-1]$ to zero.
Construct the state observer, unknown input estimator, and output disturbance estimator given by (4.9), (4.33), and (4.34), respectively.

4.10 Examples

In this section, we consider two examples to illustrate our obtained results.

Example 6 We consider a discrete-time dynamical system model, where

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}.$$

In this example, the control input is set to zero. The matrix rank condition (4.11) is satisfied, thus we can proceed with our proposed UIO design. Solving (4.14) for M yields

$$M = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ 0 & 0 \end{bmatrix}.$$

Next, we construct the matrix $\tilde{A} = (I_3 - MC)A$ to obtain

$$\tilde{A} = \begin{bmatrix} 0.5 & -0.5 & -0.25 \\ 0 & 0 & -0.25 \\ 0 & 0 & 0.5 \end{bmatrix}.$$

It is easy to see that the pair $(\tilde{A}, C)$ is detectable.

We set $\alpha = 0.95$ and solving LMIs (4.32) and (4.31b), we obtain $P = P^\top \succ 0$ and the observer gain matrix

$$L = \begin{bmatrix} -0.066086 & 0.783868 \\ -0.151477 & -0.027697 \\ 0.739215 & 1.017137 \end{bmatrix}.$$

The performance level $\gamma = 1.4901 \times 10^{-8}$.

In our simulations, we randomly selected the state initial condition,

$$x[0]^\top = \begin{bmatrix} 0.967710 & 0.086768 & 0.173477 \end{bmatrix}.$$

We selected the observer state initial condition and $\hat{x}[-1]$ to be zero. The unknown input and output disturbance are:

$$w[k] = 0.2\cos\sqrt{5k} \text{ and } v[k] = 0.25\cos\sqrt{k}.$$

We see in Figure 4.2 that the states are being estimated correctly with negligible estimation error. Figure 4.4 and Figure 4.5 show plots of the unknown input and the output disturbances and their estimates.

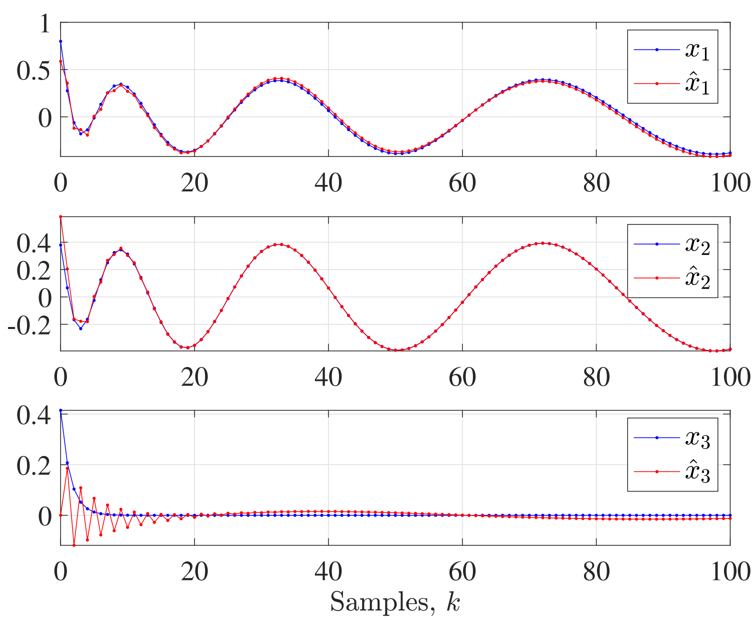

Fig. 4.2.: The state and its estimate.

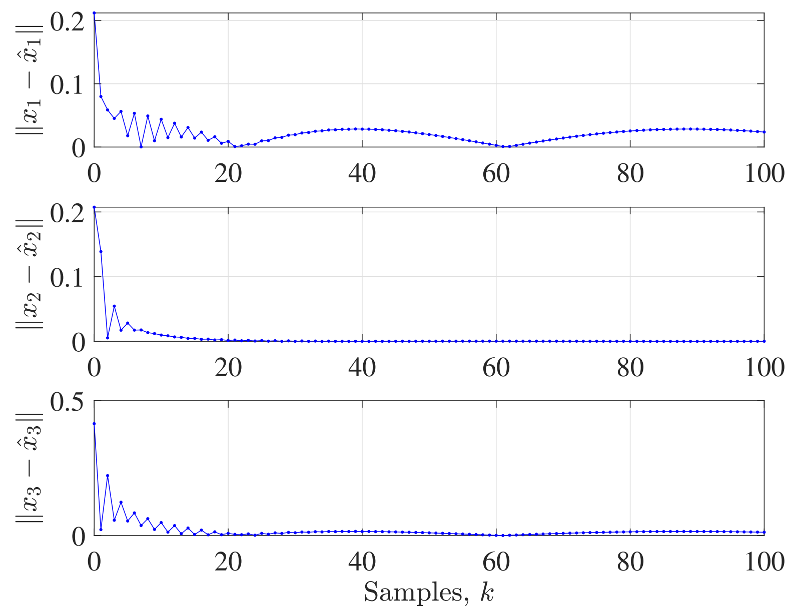

Fig. 4.3.: The state estimation error.

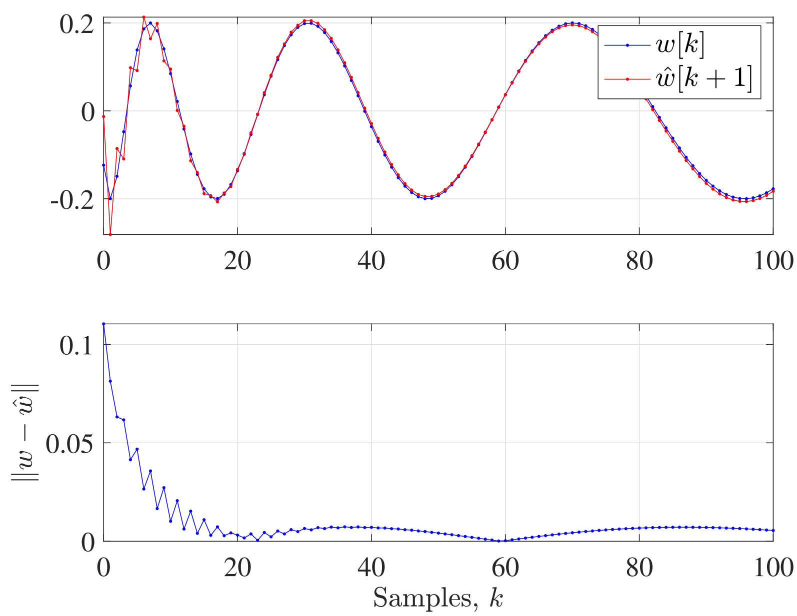

Fig. 4.4.: Top plot shows the unknown input and its estimate. Bottom plot shows the unknown input reconstruction error norm.

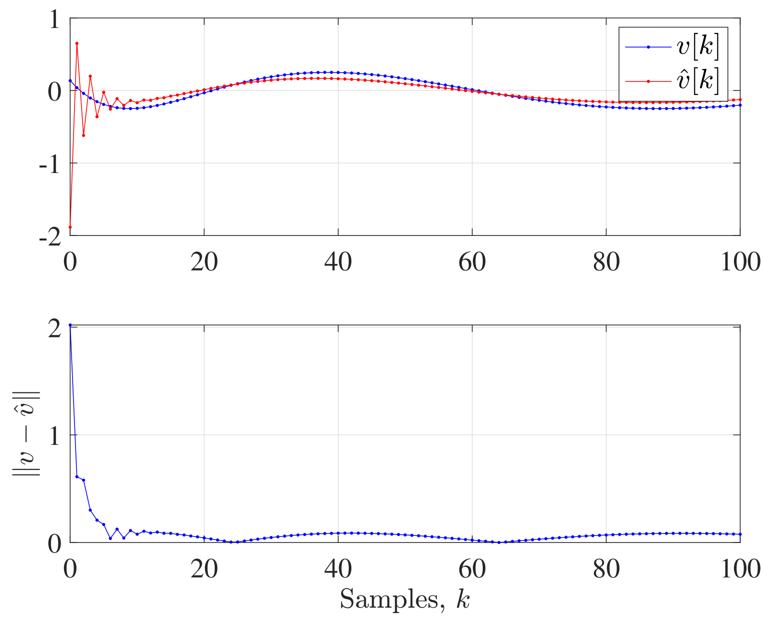

Fig. 4.5.: Top plot shows the sensor disturbance and its estimate. Bottom plot shows the output disturbance reconstruction error norm.

4.11 Conclusions

We proposed a novel unknown input observer (UIO) architecture for a class of discrete-time linear systems in the presence of unknown input and output disturbance. An UIO existence condition was given and proven to be a generalization of existence condition in the literature. The design of the UIO was formulated in terms of an LMI. Unknown input and output disturbance estimators were also proposed.

5. DELAYED ESTIMATION OF UNKNOWN INPUT AND OUTPUT DISTURBANCES IN DISCRETE-TIME LINEAR SYSTEMS

5.1 Introduction

Unknown input and output disturbance estimators are proposed for DT linear network systems corrupted by bounded unknown inputs and output disturbances. One of the necessary conditions for the existence of unknown input observers (UIOs) is a matrix rank condition. The proposed estimator architectures are for the plants for which the matrix rank condition for the existence of UIOs is not satisfied.

In our approach, we first analyze the case when the unknown input is bounded and there is no output disturbance. We then consider the case where the plant is subjected to bounded unknown input and output disturbances. We collect δ observations that are used to form a delayed system that satisfies the matrix rank condition, where δ is a design parameter. The design of the unknown input and output disturbance estimators are given in terms of linear matrix inequalities (LMIs). The unknown input and output disturbance estimation errors are guaranteed to be l_∞-stable with prescribed performance level. The proposed estimators can be applied to detect adversarial attacks on the communication channels between the controller and actuators and between the plant sensors and the controller. Adversarial attacks on the plant actuators can be modeled as unknown plant inputs, while attacks on the plant sensors are modeled as output disturbances.

5.2 Problem Statement

We consider a class of DT dynamical systems modeled by

$$\left.\begin{aligned} x[k+1] &= Ax[k]+B_1u[k]+B_2w[k] \\ y[k] &= Cx[k]+Dv[k], \end{aligned}\right\} \tag{5.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times r}$. The control input is $u[k] \in \mathbb{R}^{m_1}$. The unknown input and output disturbance to the system are modeled by $w[k] \in \mathbb{R}^{m_2}$ and $v[k] \in \mathbb{R}^{r}$, respectively. See, for example [86, Subsection 1.1.2 and Chapter 2] or [87] for a discussion on modeling of DT systems.

Our objective is to construct unknown input and output disturbance estimators for system modeled by (5.1).

We make the following assumptions:

Assumption 5 *The pair (A,C) is observable.*

Assumption 6 *Matrices B_2 and D have full column rank.*

Assumption 7 *The unknown input $w[k]$ and output disturbance $v[k]$ are uniformly bounded as a function of k.*

5.3 Delayed Unknown Input Observer For Plants With Uncorrupted Output Measurements

In this section, we consider the plant model given by (5.1) with unknown input $w[k]$ and uncorrupted output measurements. Thus, the plant model we analyze has the form,

$$\left.\begin{aligned} x[k+1] &= Ax[k]+B_1u[k]+B_2w[k] \\ y[k] &= Cx[k]. \end{aligned}\right\} \tag{5.2}$$

We assume that the matrix rank condition for the existence of an UIO is not satisfied, that is, $\text{rank}(CB_2) \neq \text{rank}(B_2)$.

5.3.1 Delayed system model

Since the matrix rank condition for system (5.2) is not satisfied, we use a sequence of output measurements to relax this condition. We collect $(\delta+1)$ measurements of y and represent them in the following format:

$$\begin{bmatrix} y[k] \\ y[k-1] \\ \vdots \\ y[k-\delta] \end{bmatrix} = \begin{bmatrix} CA^{\delta} \\ \vdots \\ CA \\ C \end{bmatrix} x[k-\delta] + \begin{bmatrix} CB_1 & \cdots & CA^{\delta-1}B_1 \\ \vdots & \ddots & \vdots \\ O_{p\times m_1} & \cdots & CB_1 \\ O_{p\times m_1} & \cdots & O_{p\times m_1} \end{bmatrix} \begin{bmatrix} u[k-1] \\ \vdots \\ u[k-\delta] \end{bmatrix}$$
$$+ \begin{bmatrix} CB_2 & \cdots & CA^{\delta-1}B_2 \\ \vdots & \ddots & \vdots \\ O_{p\times m_2} & \cdots & CB_2 \\ O_{p\times m_2} & \cdots & O_{p\times m_2} \end{bmatrix} \begin{bmatrix} w[k-1] \\ \vdots \\ w[k-\delta] \end{bmatrix}.$$

We represent the above equation in a compact form as

$$Y_{k-\delta} = \mathscr{O}^{\delta} x[k-\delta] + \mathscr{J}_1 U_{k-\delta} + \mathscr{J}_2 W_{k-\delta}, \tag{5.3}$$

where $\mathscr{O}^{\delta} \in \mathbb{R}^{(\delta+1)p\times n}$, $\mathscr{J}_1 \in \mathbb{R}^{(\delta+1)p\times \delta m_1}$, and $\mathscr{J}_2 \in \mathbb{R}^{(\delta+1)p\times \delta m_2}$.

To proceed, let $F_i = \begin{bmatrix} O_{n\times(\delta-1)m_i} & B_i \end{bmatrix} \in \mathbb{R}^{n\times \delta m_i}$ for $i=1,2$, and

$$t = k-\delta. \tag{5.4}$$

Then we rewrite $x[k+1] = Ax[k] + B_1 u[k] + B_2 w[k]$ as

$$x[t+1] = Ax[t] + F_1 U_t + F_2 W_t. \tag{5.5}$$

Let now $\tilde{Y}_t = Y_t - \mathscr{J}_1 U_t$. Combining (5.3) and (5.5), we obtain the delayed plant model of the form

$$\left.\begin{aligned} x[t+1] &= Ax[t] + F_1 U_t + F_2 W_t \\ \tilde{Y}_t &= \mathscr{O}^{\delta} x[t] + \mathscr{J}_2 W_t, \end{aligned}\right\} \tag{5.6}$$

where $x[t] \in \mathbb{R}^n$, $U_t \in \mathbb{R}^{\delta m_1}$, $W_t \in \mathbb{R}^{\delta m_2}$, and $\tilde{Y}_t \in \mathbb{R}^{(\delta+1)p}$ are the state, control input sequence, unknown input sequence, and output measurements, respectively. A block diagram of the unknown input and output disturbance estimators to be constructed is shown in Figure 5.1.

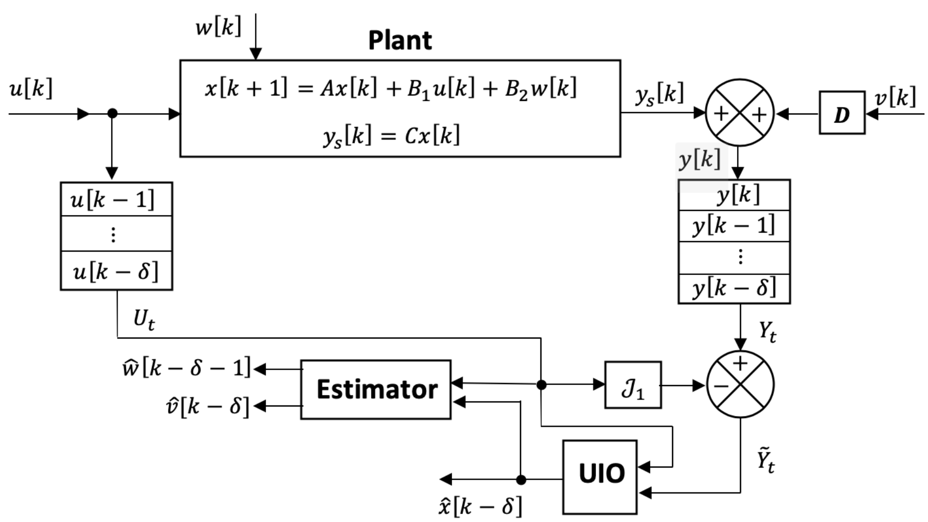

Fig. 5.1.: A block diagram of the delayed unknown input and output disturbance estimators for system model given by (5.1).

Remark 7 *$\tilde{Y}_t$ is a combination of the output and control input measurements over the intervals $[k, k-\delta]$ and $[k-1, k-\delta]$, respectively. We assume that the system starts its operation at $k=0$. We let $y[-1]$ to $y[-\delta]$ and $u[-1]$ to $u[-\delta]$ all equal to zero, then $\tilde{Y}_t$ is known at all k. Since $t = k-\delta$, the delayed system given by* (5.6) *is delayed by δ sampling periods compared with the DT plant modeled by* (5.2).

We select the number of past measurements δ so that the delayed plant (5.6) satisfies the matrix rank condition of Hautus [54] for the existence of the strong observer, that is,

$$\operatorname{rank}\begin{bmatrix}\mathscr{O}^{\delta}F_2 & \mathscr{J}_2\\ \mathscr{J}_2 & O_{(\delta+1)p\times\delta m_2}\end{bmatrix} = \operatorname{rank}\mathscr{J}_2 + \operatorname{rank}\begin{bmatrix}F_2\\ \mathscr{J}_2\end{bmatrix}. \tag{5.7}$$

The idea here is that we collect enough output measurements until the above matrix rank condition is satisfied. For a discussion of this matrix rank condition in the context of the delayed systems, we refer to [55–57].

5.3.2 Delayed UIO architecture

In this subsection, we propose an UIO architecture for the delayed plant model (5.6). We begin by representing $x[t]$ as

$$x[t] = x[t] - M\mathscr{O}^{\delta}x[t] + M\mathscr{O}^{\delta}x[t] = (I - M\mathscr{O}^{\delta})x[t] + M(\tilde{Y}_t - \mathscr{J}_2 W_t), \tag{5.8}$$

where $M \in \mathbb{R}^{n\times(\delta+1)p}$. We select M such that

$$M\mathscr{J}_2 = O, \tag{5.9}$$

where O is a matrix of zeros with appropriate dimensions. A necessary condition for (5.9) to have a solution is that $(\delta+1)p \geq \delta m_2$. Equivalently, the delayed system (5.6) should have at least as many outputs as the number of unknown inputs. Let

$$z[t] = (I - M\mathscr{O}^{\delta})x[t]. \tag{5.10}$$

Then we represent (5.8) as $x[t] = z[t] + M\tilde{Y}_t$. Recall that our objective is to estimate $x[t]$. We will show that the state estimate has the form,

$$\hat{x}[t] = z[t] + M\tilde{Y}_t, \tag{5.11}$$

where we obtain $z[t]$ from the dynamical system,

$$z[t+1] = (I - M\mathscr{O}^{\delta})x[t+1].$$

Substituting into above (5.5) and performing manipulations gives

$$z[t+1] = (I - M\mathscr{O}^{\delta})(Az[t] + AM\tilde{Y}_t + F_1U_t + F_2W_t). \tag{5.12}$$

Note that W_t is unknown to us, so we select M such that in addition to (5.9), we have

$$(I - M\mathscr{O}^{\delta})F_2 = O. \tag{5.13}$$

Then (5.12) takes the form,

$$z[t+1] = (I - M\mathscr{O}^{\delta})(Az[t] + AM\tilde{Y}_t + F_1U_t). \tag{5.14}$$

To proceed, let $e[t] = x[t] - \hat{x}[t]$ be the state estimation error. Performing simple manipulations gives

$$e[t+1] = x[t+1] - \hat{x}[t+1] = (I - M\mathscr{O}^{\delta})Ae[t].$$

We can see from the above that we do not have any control over the estimation error convergence dynamics, which is determined by the matrix $(I - M\mathscr{O}^{\delta})A$. To improve the estimation error convergence dynamics, we add the term $L(\tilde{Y}_t - \hat{Y}_t)$ to the right hand side of (5.14), where $L \in \mathbb{R}^{n\times\delta p}$ and

$$\hat{Y}_t = \mathscr{O}^{\delta}\hat{x}[t] = \mathscr{O}^{\delta}(z[t] + M\tilde{Y}_t),$$

to obtain

$$z[t+1] = (I - M\mathscr{O}^{\delta})(Az[t] + AM\tilde{Y}_t + F_1U_t) + L(\tilde{Y}_t - \mathscr{O}^{\delta}z[t] - \mathscr{O}^{\delta}M\tilde{Y}_t). \tag{5.15}$$

Combining (5.11) and (5.15), we obtain the proposed UIO architecture,

$$\left.\begin{array}{rcl} z[t+1] & = & (I-M\mathscr{O}^{\delta})(Az[t]+AM\tilde{Y}_t+F_1U_t)+L(\tilde{Y}_t-\mathscr{O}^{\delta}z[t]-\mathscr{O}^{\delta}M\tilde{Y}_t) \\ \hat{x}[t] & = & z[t]+M\tilde{Y}_t. \end{array}\right\} \tag{5.16}$$

The state estimation error dynamics for the above UIO are

$$e[t+1]=\left((I-M\mathscr{O}^{\delta})A-L\mathscr{O}^{\delta}\right)e[t]-L\mathscr{J}_2W_t. \tag{5.17}$$

For notational convenience, we let $\tilde{A}=(I-M\mathscr{O}^{\delta})A$, $E=\tilde{A}-L\mathscr{O}^{\delta}$ and $N=-L\mathscr{J}_2$. Then, the error dynamics become

$$e[t+1]=Ee[t]+NW_t. \tag{5.18}$$

Our objective is to find L such that the matrix $E=\tilde{A}-L\mathscr{O}^{\delta}$ is Schur stable. A sufficient condition for the existence of such an L is that the pair $(\tilde{A},\mathscr{O}^{\delta})$ is detectable. Note that the observability implies the detectability. We will show that Assumption 5 implies that the pair $(\tilde{A},\mathscr{O}^{\delta})$ is observable.

Lemma 11 *If the pair (A,C) is observable, then the pair $(\tilde{A},\mathscr{O}^{\delta})$ is observable.*

Proof If the pair (A,C) is observable then for some δ, rank$(\mathscr{O}^{\delta})=n$. Therefore,

$$\text{rank}\begin{bmatrix} z_1I-\tilde{A} \\ \mathscr{O}^{\delta} \end{bmatrix}=n, \text{ for any } z_1\in\mathbb{C}.$$

That is, the pair $(\tilde{A},\mathscr{O}^{\delta})$ is observable.

■

Remark 8 *A necessary and sufficient condition for $(I-M\mathscr{O}^{\delta})F_2=O$ to have a solution M is that $rank(\mathscr{O}^{\delta}F_2)=rank(F_2)$. This is because*

$$rank(\mathscr{O}^{\delta}F_2)\leq\min\{rank(\mathscr{O}^{\delta}),rank(F_2)\}\leq rank(F_2).$$

5.4 Stability of the Error Dynamics

In this section, we analyze the error dynamics stability and give the conditions for finding the UIO gain matrix L. To proceed, we define l_∞-stability with performance level γ.

Definition 4 *The system*

$$e[t+1] = f(t, e[t], W_t) \tag{5.19}$$

is globally uniformly l_∞-stable with performance level γ if the following conditions are satisfied:

1. *The undisturbed system, (that is, $W_t = 0$ for all $t \geq 0$) is globally uniformly exponentially stable with respect to the origin.*
2. *For zero initial condition, $e[t_0] = 0$, and every bounded unknown input W_t, we have*

$$\|e[t]\| \leq \gamma \|W_t\|_\infty, \quad \forall t \geq t_0.$$

3. *For every initial condition, $e[t_0] = e_0$, and every bounded unknown input $w(\cdot)$, we have*

$$\limsup_{t \to \infty} \|e[t]\| \leq \gamma \|W_t\|_\infty.$$

For more details on the l_∞-stability with level of performance, we refer to [40].

Theorem 9 *The error dynamics system given by* (5.18) *is globally uniformly l_∞-stable with performance level γ if E is Schur stable and either condition 2) or 3) of Definition 4 is satisfied.*

Proof It is easy to verify that for linear systems, conditions 2), and 3) of Definition 4 are equivalent. ■

We now present a lemma from [40] that we use in our proof of the stability of the error dynamics system.

Lemma 12 *Consider the error system in* (5.19). *Suppose there exists a function* $V : \mathbb{R}^n \to \mathbb{R}$ *and scalars* $\xi \in (0,1)$, $\beta_1, \beta_2 > 0$ *and* $\mu_1 \geq 0$ *such that*

$$\beta_1 \|e[t]\|^2 \leq V(e[t]) \leq \beta_2 \|e[t]\|^2, \tag{5.20}$$

and

$$\Delta V[t] \leq -\xi(V(e[t]) - \mu_1 \|W_t\|^2), \tag{5.21}$$

for all $t \geq 0$, *where* $\Delta V[t] = V(e[t+1]) - V(e[t])$. *Then system* (5.18) *is globally uniformly* l_∞*-stable with performance level* $\gamma = \sqrt{\mu_1/\beta_1}$ *with respect to disturbance sequence* W_t.

Our objective is to select the UIO gain matrix L so that the state error dynamics (5.18) are l_∞-stable with performance level γ. We now state and prove the following theorem.

Theorem 10 *Suppose Assumption 7 is satisfied. If there exist matrices* $P = P^\top \succ 0$, L, *and scalars* $\alpha \in (0,1)$ *and* $\mu > 0$ *such that*

$$\begin{bmatrix} E^\top PE - (1-\alpha)P & * \\ N^\top PE & N^\top PN - \alpha I \end{bmatrix} \preceq 0 \tag{5.22a}$$

$$\begin{bmatrix} P & * \\ I & \mu I \end{bmatrix} \succeq 0, \tag{5.22b}$$

then the state observation error dynamics are l_∞*-stable with performance level* $\gamma = \sqrt{\mu}$.

Proof Let $V[t] = e[t]^\top Pe[t]$ be a Lyapunov function candidate for the estimation error dynamics given by (5.18). We evaluate the first forward difference $\Delta V[t] = V[t+1] - V[t]$ on the trajectories of (5.18) to obtain

$$\Delta V[t] = e[t]^\top (E^\top PE - P)e[t] + 2e[t]^\top E^\top PNW_t + W_t^\top N^\top PNW_t. \tag{5.23}$$

Let $\zeta = [e[t]^\top \ \ W_t^\top]^\top$. Premultiplying and postmultiplying the matrix inequality (5.22a) by $\zeta^\top$ and ζ, respectively, and taking into account (5.23), we obtain

$$\Delta V[t] + \alpha(V[t] - \|W_t\|^2) \preceq 0.$$

Therefore, condition (5.21) in Lemma 12 holds with $\mu_1 = 1$. Next, taking the Schur complement of (5.22b), we obtain

$$P - \mu^{-1} I \succeq 0.$$

Premultiplying the above inequality by $e[t]^\top$ and postmultiplying it by $e[t]$ gives

$$e[t]^\top P e[t] - \mu^{-1} e[t]^\top e[t] \geq 0.$$

Rearranging the above gives

$$||e[t]||^2 \leq \mu V(e[t]).$$

Hence by (5.20), we can take $\mu = 1/\beta_1$. By Lemma 12, the state error dynamics are l_∞-stable with performance level $\gamma = \sqrt{\mu}$. ■

We now present a method to solve matrix inequality (5.22a) in Theorem 10 using an equivalent LMI. Let $Z = PL$, then solving the matrix inequality (5.22a) is equivalent to solving the following LMI for P and Z,

$$\begin{bmatrix} -P & * \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \preceq 0, \tag{5.24}$$

where

$$\Omega_{21}^\top = \begin{bmatrix} P\tilde{A} - Z\mathscr{O}^\delta & -Z\mathscr{J}_2 \end{bmatrix},$$

and

$$\Omega_{22} = \begin{bmatrix} -(1-\alpha)P & O_{n\times\delta m_2} \\ O_{\delta m_2 \times n} & -\alpha I \end{bmatrix}.$$

Since $P = P^\top \succ 0$, taking Schur complement of (5.24) gives

$$\Omega_{22} + \Omega_{21} P^{-1} \Omega_{21}^\top \preceq 0,$$

which, in turn, yields matrix inequality (5.22a).

5.5 Unknown Input Reconstruction

In this section, we propose a DT estimator of the bounded unknown input w. By Assumption 6, the matrix B_2 has full column rank, then $B_2^\dagger$ is a left inverse of B_2. Premultiplying both sides of the state dynamics given by (5.6) by the matrix $B^\dagger$, we obtain

$$B_2^\dagger x[t+1] = B_2^\dagger A x[t] + B_2^\dagger F_1 U_t + B^\dagger F_2 W_t. \tag{5.25}$$

Since $F_2 W_t = B_2 w[t]$ and $B_2^\dagger B_2 = I_{m_2}$, we rewrite (5.25) as

$$w[t] = B_2^\dagger x[t+1] - B_2^\dagger A x[t] - B_2^\dagger F_1 U_t.$$

Using this equation, we obtain the following unknown input estimator,

$$\hat{w}[t] = B_2^\dagger \hat{x}[t+1] - B_2^\dagger A \hat{x}[t] - B_2^\dagger F_1 U_t.$$

Since the above unknown input estimator depends on $\hat{x}[t+1]$, and $\hat{x}$ is estimated with δ sampling periods time-delay. Recall that $t = k - \delta$. We therefore can estimate the unknown input with $(\delta+1)$ sampling periods time-delay. That is, the proposed estimator has the form

$$\hat{w}[k-\delta-1] = B_2^\dagger \hat{x}[k-\delta] - B_2^\dagger A \hat{x}[k-\delta-1] - B_2^\dagger F_1 U_{t-1}, \tag{5.26}$$

To prove the l_∞-stability of the unknown input estimates, we let $e_w[t-1] = w[t-1] - \hat{w}[t-1]$ be the unknown input estimation error. Then,

$$e_w[t-1] = B_2^\dagger e[t] - B_2^\dagger A e[t-1].$$

By Theorem 10, $\limsup_{t\to\infty} \|e[t]\| \leq \gamma \|W_t\|_\infty$. Hence the bound on the unknown input estimation steady-state error satisfies

$$\|e_w[t-1]\| \leq \|B_2^\dagger\|(\gamma\|W_t\|_\infty + \|A\|\gamma\|W_{t-1}\|_\infty) = \|B_2^\dagger\|(1+\|A\|)\sqrt{\mu}\|W_t\|_\infty. \tag{5.27}$$

Hence the performance level of the unknown input estimator is $\gamma_w = \|B^\dagger\|(1+\|A\|)\sqrt{\mu}$.

We summarize our discussion in the form the algorithm.

Algorithm 5: Unknown input estimator synthesis

1 If $\text{rank}(CB_2) < \text{rank}(B_2)$, determine δ such that condition given by (5.7) is satisfied.
2 Collect $(\delta+1)$ output measurements and construct model (5.6).
3 Solve for M such that (5.9) and (5.13) are satisfied.
4 Solve LMs (5.24), and (5.22b) for L and P.
5 Set all past values of state $x[k-\delta]$, control input $u[k-\delta]$, unknown input $w[k-\delta]$, and state estimates $\hat{x}[k-\delta]$ to zero.
6 Construct the state and the unknown input estimator given by (5.16) and (5.26).

Example 7 We consider a plant from [58, 90], where

$$A = \begin{bmatrix} 0.8 & 0 & 0 \\ 2.1 & -1.3 & -0.6481 \\ 0 & 0.6481 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 0 & 1.543 \end{bmatrix}.$$

The above linear system does not satisfy the matrix rank condition for the existence of an UIO. Indeed, $\text{rank}(CB_2) < \text{rank}(B_2)$. In this example, the control $u[k]$ is zero. We proceed with Algorithm 5. We collect δ-output measurements to form the system given by (5.6), where $\delta = 6$. The matrix rank condition (5.7) is satisfied for the delayed system (5.6). We solve (5.9) and (5.13) to obtain

$$M = \begin{bmatrix} O_{1\times 5} & 0.999981700334884 & 0 \\ O_{1\times 5} & 0.999981700334884 & 0 \\ O_{1\times 5} & 0 & 0 \end{bmatrix}$$

Next, using a line search method, we find $\alpha = 0.82472919$. Then, using this α, we solve LMIs (5.24) and (5.22b) using CVX toolbox. We obtain $P = P^\top \succ 0$, the observer gain matrix L, and performance level $\gamma = 1.4901 \times 10^{-8}$ and the unknown input performance level $\gamma_w = 3.8545 \times 10^{-8}$. Comparing with [58], our performance level is much smaller than the performance level obtained in [58], which was 0.083.

In our simulations, we use the initial state as in [58, 90], that is, $x[0] = [5\ \ -15\ \ 10]^\top$ and we set zero initial condition for the observer state. We set all past δ values of the plant state $x[k]$, the estimated state $\hat{x}[k]$, as well as the unknown input $w[k]$ to zero. We generate the unknown input $w[k]$ as in [58, 90]. Figure 5.2 shows the reconstructed unknown input and the norm of the unknown input reconstruction error. Our proposed UIO was able to reconstruct the unknown input with negligible reconstruction error.

5.6 Unknown Input Observer for Linear Systems with Unknown Input and Output Disturbances

We consider the plant model given by (5.1) with both unknown input and output disturbance. Our objective now is to reconstruct both the unknown input and output disturbance when the matrix rank condition for the existence of an UIO is not satisfied.

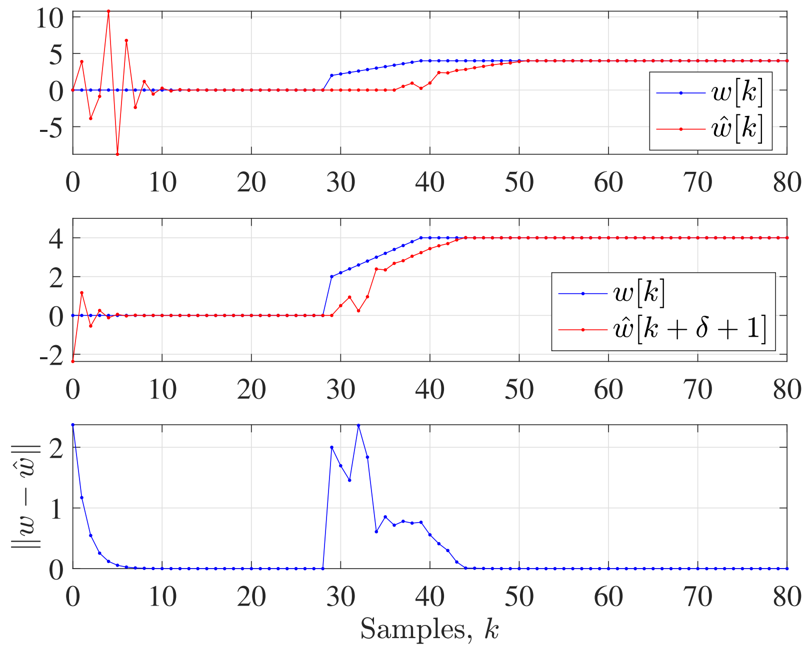

Fig. 5.2.: Top and middle plots shows the unknown input and its estimate. Bottom plot shows the unknown input reconstruction error norm. In the middle plot, $\hat{w}$ has been shifted by $(\delta+1)$ sampling periods to compare with the true values of w.

5.6.1 Delayed system model

We proceed by collecting $(\delta+1)$ observations to obtain

$$Y_{k-\delta} = \begin{bmatrix} y[k] \\ y[k-1] \\ \vdots \\ y[k-\delta] \end{bmatrix} \triangleq \mathscr{O}^{\delta} x[k-\delta] + \mathscr{J}_1 U_{k-\delta} + \tilde{\mathscr{J}}_2 \tilde{\mathscr{U}}_{k-\delta},$$

where $Y_{k-\delta} \in \mathbb{R}^{(\delta+1)p}$, $\tilde{\mathscr{J}}_2 \in \mathbb{R}^{(\delta+1)p\times((\delta+1)r+\delta m_2)}$, and $\tilde{\mathscr{U}}_{k-\delta} \in \mathbb{R}^{(\delta+1)r+\delta m_2}$. The matrices $\mathscr{O}^{\delta}$, $\mathscr{J}_1$, and the vector $U_{k-\delta}$ are defined in Section 5.2. The remaining matrices and vectors are

$$\tilde{D} = \begin{bmatrix} D & O_{p\times r} & \dots & O_{p\times r} \\ O_{p\times r} & D & \dots & O_{p\times r} \\ \vdots & \vdots & \ddots & \vdots \\ O_{p\times r} & O_{p\times r} & \dots & D \end{bmatrix}, \; V_{k-\delta} = \begin{bmatrix} v[k] \\ \vdots \\ v[k-\delta] \end{bmatrix}.$$

Let

$$\tilde{\mathscr{J}}_2 = [\tilde{D} \vdots \mathscr{J}_2]$$

$$\tilde{\mathscr{U}}_{k-\delta} = [V_{k-\delta} \vdots W_{k-\delta}]$$

$$\tilde{Y}_{k-\delta} = Y_{k-\delta} - \mathscr{J}_1 U_{k-\delta}$$

and

$$\tilde{F}_2 = \begin{bmatrix} O_{n\times(\delta-1)m_2+(\delta+1)r} & \cdots & B_2 \end{bmatrix}.$$

Recall that $t = k-\delta$. We represent the delayed system as

$$\left.\begin{aligned} x[t+1] &= Ax[t] + F_1 U_t + \tilde{F}_2 \tilde{\mathscr{U}}_t, \\ \tilde{Y}_t &= \mathscr{O}^{\delta} x[t] + \tilde{\mathscr{J}}_2 \tilde{\mathscr{U}}_t. \end{aligned}\right\} \tag{5.28}$$

Since the delayed model (5.28) has the same structure as the delayed model given by (5.6), we can use the observer structure given by (5.16). The UIO for (5.28) has the form

$$\left.\begin{aligned} z[t+1] &= (I-\tilde{M}\mathscr{O}^{\delta})(Az[t] + A\tilde{M}\tilde{Y}_t + F_1 U_t) + \tilde{L}(\tilde{Y}_t - \mathscr{O}^{\delta} z[t] - \mathscr{O}^{\delta}\tilde{M}\tilde{Y}_t) \\ \hat{x}[t] &= z[t] + \tilde{M}\tilde{Y}_t. \end{aligned}\right\} \tag{5.29}$$

We solve for a matrix $\tilde{M} \in \mathbb{R}^{n\times(\delta+1)p}$ such that $\tilde{M}\tilde{\mathscr{J}}_2 = O$ and $(I-\tilde{M}\mathscr{O}^{\delta})\tilde{F}_2 = O$. The state error dynamics become

$$e[t+1] = \tilde{E}e[t] + \tilde{N}\tilde{\mathscr{U}}_t, \tag{5.30}$$

where $\tilde{E} = (I - \tilde{M}\mathscr{O}^{\delta})A - \tilde{L}\mathscr{O}^{\delta}$, and $\tilde{N} = -\tilde{L}\tilde{\mathscr{J}}_2$. Since this state error dynamics system has the same structure as (5.18), we use Theorem 10 to calculate the observer gain matrix $\tilde{L}$. We then use the unknown input estimator given by (5.26) to reconstruct $w[k]$ with $(\delta+1)$ sampling periods time-delay.

In the next subsection, we propose an estimator for the output disturbance v.

5.6.2 Output disturbance reconstruction

Since by Assumption 6, the matrix D has full column rank, therefore we can take $D^{\dagger}$ as its left inverse. Premultiplying both sides of the output equation of model (5.1) by $D^{\dagger}$ gives

$$D^{\dagger}y[k] = D^{\dagger}Cx[k] + v[k].$$

Rearranging the above equation, we obtain

$$v[k] = D^{\dagger}y[k] - D^{\dagger}Cx[k].$$

From the above, we obtain the output disturbance estimator,

$$\hat{v}[k] = D^{\dagger}y[k] - D^{\dagger}C\hat{x}[k].$$

Since we have only available $\hat{x}[k-\delta]$, the output disturbance estimator is

$$\hat{v}[k-\delta] = D^{\dagger}y[k-\delta] - D^{\dagger}C\hat{x}[k-\delta]. \tag{5.31}$$

To prove the l_{∞}-stability of the output disturbance estimate, we let $e_v[k-\delta] = v[k-\delta] - \hat{v}[k-\delta]$ be the output disturbance estimation error. Then, we have

$$e_v[k-\delta] = -D^{\dagger}Ce[k-\delta].$$

By Theorem 10,

$$\limsup_{k\to\infty} \|e[k-\delta]\| \leq \gamma \|\tilde{\mathscr{U}}_{k-\delta}\|_\infty.$$

The bound on the output disturbance steady-state estimation error is

$$\|e_v[k-\delta]\| \leq \|D^\dagger\| \|C\| \gamma \|\tilde{\mathscr{U}}_{k-\delta}\|_\infty = \|D^\dagger\| \|C\| \sqrt{\mu} \|\tilde{\mathscr{U}}_{k-\delta}\|_\infty.$$

The performance level of the output disturbance estimator is $\gamma_v = \|D^\dagger\| \|C\| \sqrt{\mu}$.

Example 8 We consider a system, where

$$A = \begin{bmatrix} 1 & -0.5 & -0.5 & -0.5 \\ 0 & 0.5 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -0.5 & 1.5 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.4 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}.$$

We have $\operatorname{rank}(CB_2) < \operatorname{rank}(B_2)$. The matrix rank condition (5.7) is satisfied for the delayed system (5.28) for $\delta = 1$. We solve for $\tilde{M}$ such that $\tilde{M}\tilde{\mathscr{J}}_2 = O$, and $(I - \tilde{M}\mathscr{O}^\delta)\tilde{F}_2 = O$, where

$$\tilde{M} = \begin{bmatrix} 0.000000001336419 & -0.000000000334105 & O_{1,4} \\ 0.000000000428138 & -0.000000000107035 & O_{1,4} \\ -0.000000000060461 & 0.000000000015115 & O_{1,4} \\ -0.111738036588382 & -0.472065490852906 & O_{1,4} \end{bmatrix}.$$

Now select $\alpha = 0.85$ and solve LMIs (5.24) and (5.22b) to obtain $P = P^\top \succ 0$, the observer gain matrix $\tilde{L}$, and the performance level indicator $\gamma = 1.4901 \times 10^{-8}$. In our simulations, we select the state initial condition randomly, $x[0] = \begin{bmatrix} 0.7537 & 0.3804 & 0.5678 & 0.0758 \end{bmatrix}^\top$, and set the observer state initial condition to zero. We set all past δ values of the state $x[k]$, the estimated states $\hat{x}[k]$, the unknown input $w[k]$, and the output disturbance $v[k]$ to zero. The unknown input and the output disturbance have the form

$$w[k] = 0.2\cos(\sqrt{5k}) \text{ and } v[k] = 0.25\cos(\sqrt{k}).$$

Figures 5.3 and 5.4 show plots of the unknown input and the output disturbance and their estimates.

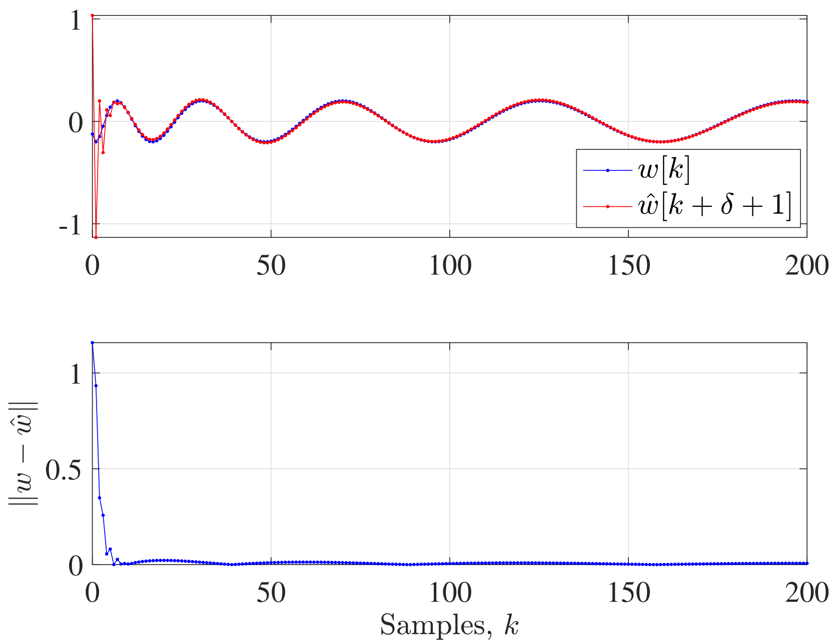

Fig. 5.3.: Top plot shows the unknown input and its estimate with time delay $(\delta+1)$. Bottom plot shows the unknown input reconstruction error norm.

5.7 Conclusions

We proposed a novel unknown input and output disturbance estimators for DT linear systems when the existences condition for an UIO is not satisfied. We collect output measurements and formulate an delayed system for which the existence condition of an UIO is satisfied. The conditions for the existence of the estimators are presented as LMIs. The observation error of the unknown input and output measurements are guarantee with a prescribe performance level.

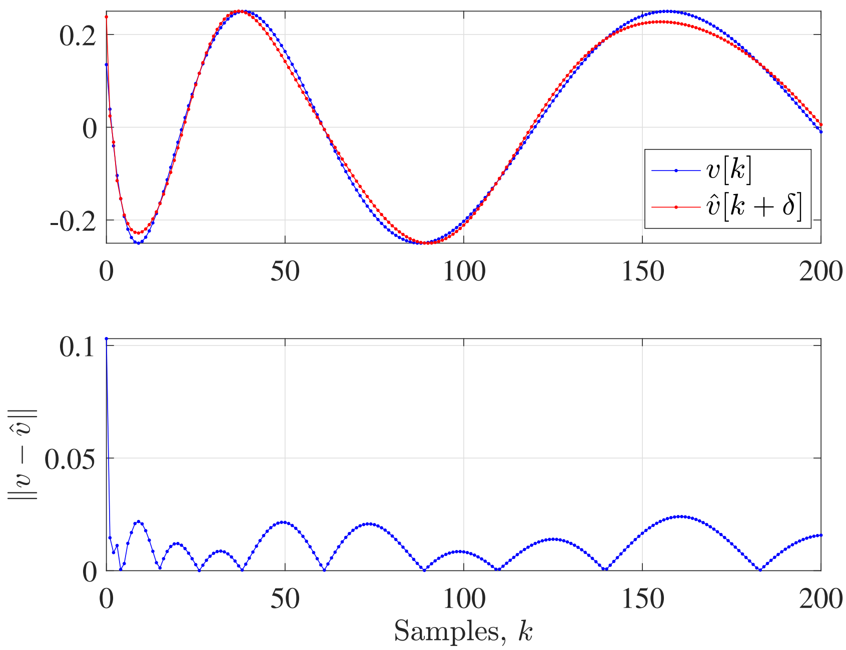

Fig. 5.4.: Top plot shows the sensor disturbance and its estimate with time delay δ. Bottom plot shows the output disturbance reconstruction error norm.

6. OBSERVER-BASED CONTROLLER SYNTHESIS FOR DECENTRALIZED NETWORKED SYSTEMS

6.1 Introduction

We use observer-based decentralized controller to control large scale networked control systems. A decentralized networked control system (DNCS) is a system with a feedback decentralized control loop closed by a communication network. A block diagram of the decentralized observer-based controller is shown in Figure 6.1.

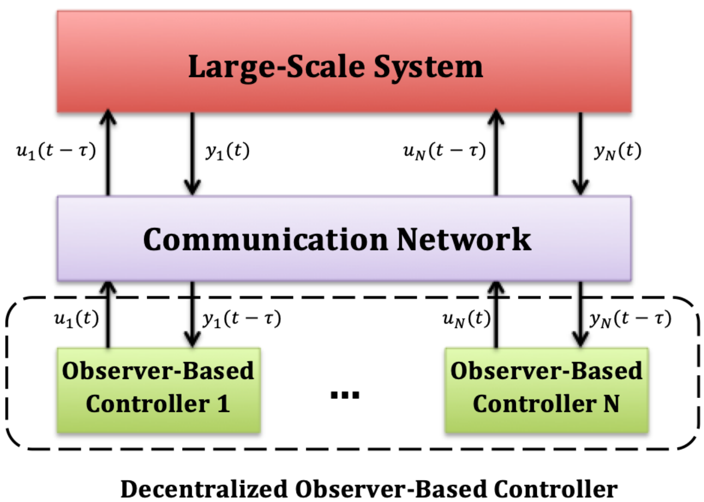

Fig. 6.1.: A block diagram of a decentralized networked control system, where y_i are the outputs of the system, u_i are the local control inputs, and τ is the communication network time-delay.

The efficient control of such systems uses only local information available to each local controller. This, in turn, reduces significantly the cost of communication needed to implement local controllers as compared with communication needed to implement a centralized controller.

In this chapter, a new approach to the DNCS design is proposed, where the control loop is closed by a communication network. The communication network is modeled as a pure time-delay. We adopt the design of the observer-based controller in [63] which uses decentralized functional observers at each local station. An observer-based decentralized controller is designed using linear matrix inequalities (LMIs). The stability of the DNCS is analyzed using results reported by Schoen [77] to obtain a sufficient upper bound on the communication network time-delay that guarantees the stability of the DNCS. The obtained results are applied to the design of a decentralized system consisting of two remotely controlled mobile robots.

6.2 Problem Statement

We consider a plant that is a large scale system with N local control stations modeled by

$$\dot{x}_p(t) = A_p x_p(t) + \sum_{i=1}^{N} B_i u_i(t) \tag{6.1a}$$

$$y_i(t) = C_i x_p(t), \qquad i = 1, 2, \cdots, N, \tag{6.1b}$$

where $x_p(t) \in \mathbb{R}^n$ is the state vector of the plant, $u_i(t) \in \mathbb{R}^{m_i}$, and $y_i(t) \in \mathbb{R}^{r_i}$ are the control input and the measured output of each local station, respectively. The system matrices are $B_i \in \mathbb{R}^{n \times m_i}$ and $C_i \in \mathbb{R}^{r_i \times n}$. Let

$$\begin{aligned}
B_p &= [B_1 \;\; B_2 \;\; \cdots \;\; B_N] \\
C_p &= [C_1^\top \;\; C_2^\top \;\; \cdots \;\; C_N^\top]^\top \\
u(t) &= [u_1^\top(t) \;\; u_2^\top(t) \;\; \cdots \;\; u_N^\top(t)]^\top \\
y(t) &= [y_1^\top(t) \;\; y_2^\top(t) \;\; \cdots \;\; y_N^\top(t)]^\top .
\end{aligned}$$

Then, the plant model can be described in a centralized format as

$$\dot{x}_p(t) = A_p x_p(t) + B_p u(t) \tag{6.2a}$$

$$y(t) = C_p x_p(t). \tag{6.2b}$$

The plant given by (6.2) will be controlled by an observer-based decentralized controller through a communication network modeled as a pure time-delay. We first present a method for the design of the decentralized controller of the above decentralized control system (DCS). Then, we give an upper bound on the network time-delay that guarantees the stability of the closed-loop decentralized networked control system (DNCS).

6.3 Controller Design and Stability Analysis of the Decentralized Control System

In this section, we first review the observer-based decentralized control design proposed by Ha and Trinh in [63] on which our design method is based. Then, we provide LMIs for the design of the decentralized controller. Finally, we formulate a sufficiency condition for the stability of the closed-loop decentralized system.

As in [63], we make the following assumptions that are required for the implementation of the proposed decentralized controller.

Assumption 8 *The plant model given by* (6.2) *is controllable and observable.*

Assumption 9 *There are no unstable decentralized fixed modes associated with* (A_p, B_i, C_i)*,* $i = 1, 2, \ldots, N$.

Assumption 10 *Information available at the i-th control station contains only the output and input of that station.*

Assumption 11 *A global, centralized, state-feedback controller of the form* $u(t) = -F x_p(t)$*, where* $F \in R^{m \times n}$*, can be constructed so that the closed-loop system possesses desired properties.*

We can design an optimal state-feedback $u(t) = -Fx_p(t)$ using, for example, the linear quadratic regulator approach.

6.3.1 Observer-based decentralized controller design

As in [63], we assume that the decentralized local inputs in (6.1) have the form,

$$u_i(t) = -F_i x_p(t) = -K_i z_i(t) - W_i y_i(t), \quad i = 1,2,...,N, \tag{6.3}$$

where $F_i \in \mathbb{R}^{m_i \times n}$ is a sub-matrix of $F \in \mathbb{R}^{m \times n}$, $K_i \in \mathbb{R}^{m_i \times p_i}$ and $W_i \in \mathbb{R}^{m_i \times r_i}$ are constant design matrices to be computed. The vector $z_i(t) = L_i x_p(t) \in \mathbb{R}^{p_i}$ is a state vector of the dynamical system

$$\dot{z}_i(t) = E_i z_i(t) + L_i B_i u_i(t) + G_i y_i(t), \;\; i = 1,2,\ldots,N, \tag{6.4}$$

where $E_i \in \mathbb{R}^{p_i \times p_i}$ is a real constant design matrix chosen to be asymptotically stable according to the required dynamics of the observer. The matrices $L_i \in \mathbb{R}^{p_i \times n}$ and $G_i \in \mathbb{R}^{p_i \times r_i}$ are real constant matrices to be determined.

Note that each i-th decentralized controller in (6.3) depends only on the local input and output of the i-th station.

As in [63], we define the error vector,

$$e_i(t) = z_i(t) - L_i x_p(t),$$

for $i = 1,2,\ldots,N$. Combining (6.3) and (6.4), we obtain the following error dynamics,

$$\dot{e}_i(t) = E_i e_i(t) + (G_i C_i + E_i L_i - L_i A_p) x_p(t) - L_i B_{r_i} u_{r_i}, \tag{6.5}$$

for $i = 1, 2, \ldots, N$. Note that $B_{r_i} \in \mathbb{R}^{n \times (m-m_i)}$ is the sub-matrix of the matrix B_p after removing from B_p the sub-matrix B_i. The vector u_{r_i} is composed of $N-1$ input vectors of the remaining $N-1$ control stations.

Following [63], we select in (6.5) the matrix E_i to be Hurwitz and the design matrices G_i, L_i, K_i, and W_i, $i = 1, 2, \ldots, N$, are chosen to satisfy the following conditions:

$$L_i B_{r_i} = 0, \tag{6.6a}$$

$$K_i L_i + W_i C_i = F_i, \tag{6.6b}$$

$$G_i C_i + E_i L_i - L_i A_p = 0. \tag{6.6c}$$

Remark 9 *The above equations are the synthesis equations of a dynamic decentralized controller approximating the static centralized controller selected by the designer. The controller design using the above equations directly are highly nontrivial. In the following subsection, we present a novel method for solving the above equations using LMIs.*

6.3.2 LMIs for the design of the observer-based decentralized controller

In this subsection, we present a novel systematic approach for the design of the observer-based decentralized controller using convex optimization approach.

First, to satisfy constraint (6.6a), $L_i^\top$ is chosen to be in the null space of $B_{r_i}^\top$.

Solving for G_i, K_i, and W_i in (6.6b) and (6.6c) is represented as solving two matrix inequalities,

$$(K_i L_i + W_i C_i - F_i)^\top (K_i L_i + W_i C_i - F_i) \prec {\varepsilon_{i_1}}^2 I_{m_i}, \tag{6.7a}$$

$$(G_i C_i + E_i L_i - L_i A_p)^\top (G_i C_i + E_i L_i - L_i A_p) \prec {\varepsilon_{i_2}}^2 I_{p_i}, \tag{6.7b}$$

where $\varepsilon_{i_1}, \varepsilon_{i_2} > 0$, $i = 1, 2, \dots, N$ are design parameters.

Using the Schur complements, we formulate the design problem as an optimization problem of the form,

$$\left.\begin{aligned}
&\min \textstyle\sum_{i=1}^{N} (\varepsilon_{i_1} + \varepsilon_{i_2}) \\
&\text{subject to} \\
&\begin{bmatrix} \varepsilon_{i_1} I_{m_i} & (K_i L_i + W_i C_i - F_i) \\ (K_i L_i + W_i C_i - F_i)^\top & \varepsilon_{i_1} I_{m_i} \end{bmatrix} \succ 0, \\
&\begin{bmatrix} \varepsilon_{i_2} I_{p_i} & (G_i C_i + E_i L_i - L_i A_p) \\ (G_i C_i + E_i L_i - L_i A_p)^\top & \varepsilon_{i_2} I_{p_i} \end{bmatrix} \succ 0, \\
&\varepsilon_{i_1},\ \varepsilon_{i_2} > 0, \quad i = 1, 2, \dots, N.
\end{aligned}\right\} \tag{6.8}$$

Solving (6.8), we obtain G_i, K_i, W_i, ε_{i_1}, and ε_{i_2}.

6.3.3 Stability analysis of the decentralized closed-loop system

Solving the problem presented in the previous subsection yields approximate solutions to the design matrices G_i, K_i, and W_i. This necessitates the need for the stability analysis of the closed-loop decentralized system. To proceed, we define error matrices,

$$\Delta M_i = G_i C_i + E_i L_i - L_i A_p, \tag{6.9a}$$

$$\Delta F_i = F_i - K_i L_i - W_i C_i. \tag{6.9b}$$

Then, the local error dynamics can be written as

$$\dot{e}_i(t) = E_i e_i(t) + \Delta M_i x_p(t), \quad i = 1, 2, \ldots, N. \tag{6.10}$$

The global error dynamics can be represented as

$$\dot{e}(t) = Ee(t) + \Delta M x_p(t), \tag{6.11}$$

where $E = \mathrm{diag}(E_i)$, for $i = 1, 2, \ldots, N$, and $\Delta M = [\Delta M_1 \ \Delta M_2 \ \cdots \ \Delta M_N]^\top$. Now, the local control inputs have the form

$$u_i(t) = -F_i x_p(t) + \Delta F_i x_p(t), \quad i = 1, 2, \ldots, N. \tag{6.12}$$

Therefore, the global input vector of the system becomes

$$u(t) = -F x_p(t) + \Delta F x_p(t), \tag{6.13}$$

where $F = [F_1 \ F_2 \ \cdots \ F_N]^\top$, and $\Delta F = [\Delta F_1 \ \Delta F_2 \ \cdots \ \Delta F_N]^\top$.

Let $w(t) = [x_p(t)^\top \ \ e(t)^\top]^\top$, then the decentralized closed-loop system has the form as in [63],

$$\dot{w}(t) = Jw(t) + \Delta J w(t) \tag{6.14}$$

where

$$J = \begin{bmatrix} A_p - B_p F & O \\ O & E \end{bmatrix}, \text{ and } \Delta J = \begin{bmatrix} B_p \Delta F & O \\ \Delta M & O \end{bmatrix}, \tag{6.15}$$

where O is a matrix of zeros with compatible dimensions. In [63], a sufficiency condition for the closed-loop system to be asymptotically stable is given using a classical Lyapunov-type argument. Note that J is asymptotically stable since $A_p - B_p F$ and E are designed to be asymptotically stable. Solving Lyapunov equation $J^\top P + PJ = -Q$, for some $Q = Q^\top \prec 0$, and then evaluating the Lyapunov derivative of $V = w^\top P w$ on the trajectories of (6.14), we obtain that for any ΔJ that satisfies the condition

$$\|\Delta J\| < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \tag{6.16}$$

the closed-loop system is asymptotically stable. Note that this is only a sufficiency condition for the closed-loop system to be asymptotically stable.

Rather than using condition (6.16), we represent (6.14) as

$$\dot{w}(t) = [J + \Delta J]\, w(t). \tag{6.17}$$

Then we have, a necessary and sufficiency condition for the closed-loop system to be asymptotically stable is that the matrix $[J + \Delta J]$ is Hurwitz.

We summarize the above considerations in the form of the decentralized control design algorithm.

Algorithm 6: Decentralized control design

1 Design a global optimal state feedback controller $u = -Fx_p$ for system (6.2)
2 Partition system (6.2) and F according to (6.1)
3 Select Hurwitz matrices E_i
4 Compute $L_i = (\mathrm{null}(\mathrm{B}_{\mathrm{r_i}}^\top))^\top$
5 Solve LMIs given by (6.8) for G_i, K_i, and W_i
6 Check if the matrix $[J + \Delta J]$ is Hurwitz
7 If the matrix $[J + \Delta J]$ is not Hurwitz, go to Step 3, else STOP.

6.4 Decentralized Networked Control System Stability Analysis

In the previous section, we presented a method for observer-based design of decentralized local controllers. In this section, we analyze the effects of the presence of the communication network between the plant and the local controllers. The architecture we analyze is depicted in Figure 6.2. We model the communication network as a pure time delay. We give an upper bound for the time delay duration for which the DNCS is guaranteed to be asymptotically stable.

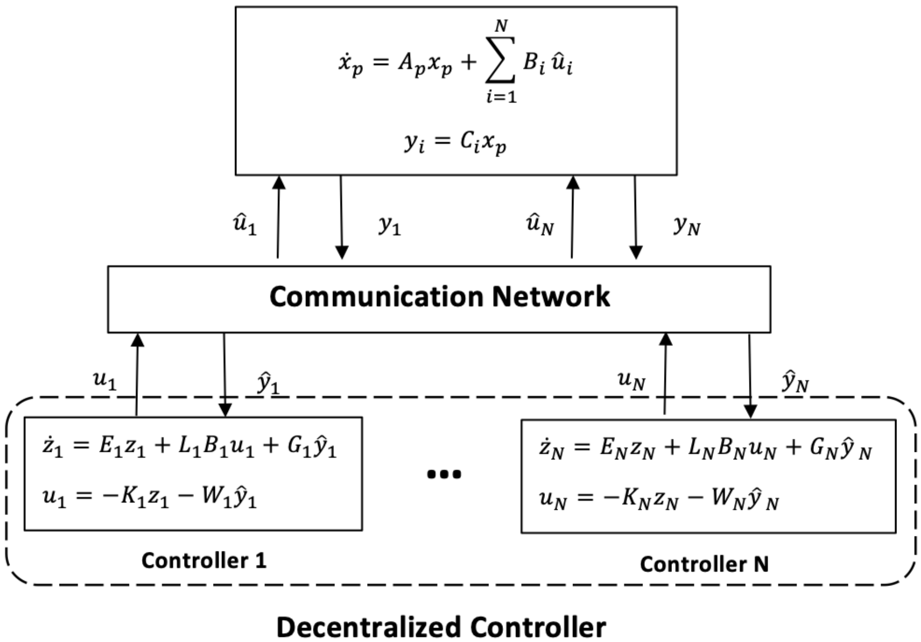

Fig. 6.2.: Decentralized networked control system.

The centralized state space plant model has the form,

$$\left.\begin{array}{rcl} \dot{x}_p & = & A_p x_p + B_p \hat{u} \\ y & = & C_p x_p, \end{array}\right\} \tag{6.18}$$

where $\hat{u} = [\hat{u}_1^\top \ \cdots \ \hat{u}_N^\top]^\top$ and $\hat{u}(t) = u(t-\tau)$. The centralized controller, on the other hand, has the form,

$$\left.\begin{array}{rcl} \dot{z} & = & A_c z + B_c \hat{y} \\ u & = & C_c z + D_c \hat{y}, \end{array}\right\} \tag{6.19}$$

where $\hat{y} = [\hat{y}_1^\top \ \cdots \ \hat{y}_N^\top]^\top$, $\hat{y}(t) = y(t-\tau)$, and $z = [z_1^\top \ \cdots \ z_N^\top]^\top$. The matrices A_c, B_c, C_c, and D_c of the above controller are obtained by substituting (6.3) into (6.4). We get

$$\left.\begin{array}{rcl} \dot{z} &=& (E - LB_pK)z + (G - LB_pW)\hat{y} \\ u &=& -Kz - W\hat{y}, \end{array}\right\} \tag{6.20}$$

where

$$K = \mathrm{diag}(K_1, K_2, \cdots, K_N)$$
$$L = [L_1^\top \ L_2^\top \ \cdots \ L_N^\top]^\top$$
$$G = \mathrm{diag}(G_1, G_2, \cdots, G_N)$$
$$W = \mathrm{diag}(W_1, W_2, \cdots, W_N).$$

Comparing (6.19) and (6.20) gives

$$A_c = E - LB_pK$$
$$B_c = G - LB_pW$$
$$C_c = -K$$
$$D_c = -W.$$

We are now ready to write down the dynamics equation of the closed-loop system. Combining (6.18) and (6.19) gives

$$\dot{x}_p(t) \quad = \quad A_p x_p(t) + B_p C_c z(t-\tau) + B_p D_c C_p x_p(t - 2\tau).$$

The centralized controller dynamics (6.19) can be written as

$$\dot{z}(t) \quad = \quad A_c z(t) + B_c C_p x_p(t - \tau).$$

Combining the above yields

$$\begin{aligned}\begin{bmatrix}\dot{x}_p(t)\\ \dot{z}(t)\end{bmatrix} &= \begin{bmatrix}A_p & O\\ O & A_c\end{bmatrix}\begin{bmatrix}x_p(t)\\ z(t)\end{bmatrix}+\begin{bmatrix}O & B_pC_c\\ B_cC_p & O\end{bmatrix}\begin{bmatrix}x_p(t-\tau)\\ z(t-\tau)\end{bmatrix}\\ &+\begin{bmatrix}B_pD_cC_p & O\\ O & O\end{bmatrix}\begin{bmatrix}x_p(t-2\tau)\\ z(t-2\tau)\end{bmatrix}.\end{aligned} \tag{6.21}$$

The closed-loop model of the DNCS is a linear time-delay dynamical system with two commensurate time delays. To analyze the stability of such a system, we use the results of Schoen [77]. To proceed, we transform the two time-delay system into single-delay system in the next section.

6.4.1 Transformation from multiple-delay system to a single-delay system

We consider a multiple time-delay system modeled as

$$\dot{x}(t) = A_0x(t)+\sum_{i=1}^{k}A_ix(t-i\tau), \tag{6.22}$$

where the delays are commensurate, that is, the subsequent delays are integer multiples of the fixed constant delay τ, where $\tau\in[0,\infty)$. We follow the method of Schoen [77] to obtain the single-delay transformed system corresponding to (6.22),

$$\dot{\bar{x}}(t) = \bar{A}_0\bar{x}(t)+\bar{A}_1\bar{x}(t-k\tau), \tag{6.23}$$

where

$$\bar{A}_0 = \begin{bmatrix} A_0 & A_1 & \cdot & \cdot & A_{k-1} & & & \\ & A_0 & A_1 & \cdot & \cdot & A_{k-1} & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot & \cdot & A_{k-1} \\ & & & & \cdot & \cdot & \cdot & \cdot \\ & O & & & & \cdot & \cdot & \cdot \\ & & & & & & \cdot & A_1 \\ & & & & & & & A_0 \end{bmatrix},$$

and

$$\bar{A}_1 = \begin{bmatrix} A_k & & & & & & & \\ A_{k-1} & A_k & & & & & & \\ \cdot & A_{k-1} & \cdot & & & & O & \\ \cdot & \cdot & \cdot & \cdot & & & & \\ A_1 & \cdot & \cdot & \cdot & \cdot & & & \\ & A_1 & \cdot & \cdot & \cdot & \cdot & & \\ & & A_1 & \cdot & \cdot & \cdot & \cdot & \\ & & & A_1 & \cdot & \cdot & A_{k-1} & A_k \end{bmatrix}.$$

Comparing the delay system (6.22) with the DNCS (6.21), we obtain the corresponding matrices A_0, A_1, and A_2 of the single-delay equivalent DNCS model,

$$A_0 = \begin{bmatrix} A_p & O \\ O & A_c \end{bmatrix}, \; A_1 = \begin{bmatrix} O & B_pC_c \\ B_cC_p & O \end{bmatrix}, \; A_2 = \begin{bmatrix} B_pD_cC_p & O \\ O & O \end{bmatrix}.$$

Therefore, the single-delay equivalent system model of the two-delay DNCS model is described by (6.23), where

$$\bar{A}_0 = \begin{bmatrix} A_0 & A_1 \\ O & A_0 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} A_2 & O \\ A_1 & A_2 \end{bmatrix}. \tag{6.24}$$

In the following subsection, we use an approach proposed by Schoen in [77] to derive an upper bound on the time-delay τ for which we are guaranteed the stability of the closed-loop time-delay system modeled by (6.21).

6.4.2 Stability analysis of the DNCS

We present an upper bound on τ that guarantees the stability of the DNCS modeled by (6.21). In our analysis, we use the following notation. The 2-norm of a matrix X is $||X||_2 = \sqrt{\lambda_{\max}(X^\top X)}$. The function $\mu(\cdot)_2$ of a real square matrix X is defined as $\mu(X)_2 = \frac{1}{2}\lambda_{\max}(X^\top + X)$.

We use the following lemmas in our analysis.

Lemma 13 *[91] Suppose $A_0 + A_1$ is asymptotically stable. Then, system* (6.22) *with $k = 1$ is asymptotically stable, if there exists a symmetric positive-definite matrix P such that*

$$\tau < \frac{-\mu(P(A_0 + A_1))_2}{||A_1(A_0 + A_1)||_2}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)^3}}. \tag{6.25}$$

Lemma 14 *[77] If the transformed system* (6.23) *is asymptotically stable, then the original system* (6.22) *is asymptotically stable.*

Remark 10 *By the Lyapunov theorem, if $(A_0 + A_1)$ is asymptotically stable, then, for a given $Q = Q^\top \succ 0$, a matrix $P = P^\top \succ 0$ is obtained from the Lyapunov equation, $(A_0 + A_1)^\top P + P(A_0 + A_1) = -Q$, see [80] for more details.*

Combining the above lemmas, we have the following

Proposition 1 *System* (6.21) *is asymptotically stable if* $\bar{A}_0 + \bar{A}_1$ *in* (6.24) *is asymptotically stable and if there exists a symmetric positive-definite matrix P such that*

$$\tau < \tau_{ub} \triangleq \frac{-\mu(P(\bar{A}_0 + \bar{A}_1))_2}{2||\bar{A}_1(\bar{A}_0 + \bar{A}_1)||_2} \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)^3}}. \tag{6.26}$$

We summarize the above discussion in the form of an algorithm for calculating an allowable time-delay upper bound τ_{ub}.

Algorithm 7: Calculating upper bound for allowable time-delay τ_{ub}

Transform the multiple-delay system (6.21) into a single time-delay system (6.23) with $k = 2$

Check if $(\bar{A}_0 + \bar{A}_1)$ is asymptotically stable

If $(\bar{A}_0 + \bar{A}_1)$ is asymptotically stable, find P using the method described in Remark 10

Compute τ_{ub} using (6.26).

6.5 Numerical Example

In this section, we apply the DNCS design method presented in the previous sections to a system consisting of two interconnected mobile robots that are remotely controlled through a communication network. We use the formation model proposed in [92]. The vectors (x_1, y_1, θ_1) and (x_2, y_2, θ_2) are the position and orientation of robot 1 and robot 2, respectively. The dynamics of each robot are

$$\begin{aligned} \dot{x}_i &= v_i \cos\theta_i \\ \dot{y}_i &= v_i \sin\theta_i \\ \dot{\theta}_i &= \omega_i, \end{aligned}$$

where $i = 1,2$ and v_i and ω_i are the velocity and angular velocity of the i-th robot, respectively.

The following constraints are imposed:

$$x_1 = x_2,\ \frac{y_1 + y_2}{2} = 0,\ \theta_1 = \theta_2 = 0.$$

Let $x_p = [\theta_1\ \theta_2\ h_1\ h_2]^\top$, where

$$\begin{aligned} h_1 &= x_1 - x_2 + \alpha(\theta_1 - \theta_2) \\ h_2 &= y_1 + y_2 + \beta(\theta_1 + \theta_2) \end{aligned}$$

are the so-called system (platoon) level functions, and α and β are constants that represent the level of perturbation in distance measurements due to the robot orientation [92]. We present the nonlinear formation trajectory of the two robots as:

$$\left.\begin{array}{lcl} \dot{\theta}_1 &=& \omega_1 \\ \dot{\theta}_2 &=& \omega_2 \\ \dot{h}_1 &=& v_1 cos\theta_1 - v_2 cos\theta_2 + \alpha(\omega_1 - \omega_2) \\ \dot{h}_2 &=& v_1 sin\theta_1 + v_2 sin\theta_2 + \beta(\omega_1 + \omega_2). \end{array}\right\} \tag{6.27}$$

The linearized formation trajectory of system (7.4) about $v_i = 2$ and $\theta_i = 0$ given in [92] can be then obtained as

$$\dot{x}_p = A_p x_p + B_1 \hat{u}_1 + B_2 \hat{u}_2, \tag{6.28}$$

where

$$A_p = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & \alpha \\ 0 & \beta \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ -1 & -\alpha \\ 0 & \beta \end{bmatrix}.$$

The control inputs are $\hat{u}_1 = [v_1 - 2\ \omega_1]^\top$ and $\hat{u}_2 = [v_2 - 2\ \omega_2]^\top$. The decentralized information are collected as $y_{out} = [y_{out1}^\top\ y_{out2}^\top]^\top = C_p x_p$, where $y_{out1} = [\theta_1\ h_1\ h_2]^\top$ and

$y_{out2} = [\theta_2\ h_1\ h_2]^\top$. We construct the plant dynamical model of the DNCS in (6.18), where $\hat{u} = [\hat{u}_1^\top\ \hat{u}_2^\top]^\top$, and $B_p = [B_1 \quad B_2]$.

Solving the Algebraic Riccati Equation gives a global controller $u = -Fx_p$ for the centralized model, where

$$F = \begin{bmatrix} -0.184 & 0.184 & 2.2209 & 0 \\ 3.3198 & 0.179 & 0.2602 & 2.2361 \\ 0.184 & -0.1840 & -2.2209 & 0 \\ 0.179 & 3.3198 & -0.2602 & 2.2361 \end{bmatrix}.$$

Let $E_i = -4I_2$. Solving (6.6a) and the LMIs in (6.8), for $N = 2$, we obtain the observer-based decentralized controller in (6.20). The controller matrices for Robot 1 are:

$$K_1 = \begin{bmatrix} -0.092 & -0.4114 \\ 1.6599 & -0.4002 \end{bmatrix}, W_1 = \begin{bmatrix} -0.092 & 2.2209 & 0.368 \\ 1.6599 & 0.2602 & 2.5941 \end{bmatrix},$$
$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -0.4472 & 0 & 0.8944 \end{bmatrix}, G_1 = \begin{bmatrix} 4 & 0 & 0 \\ 1.7889 & 0 & 3.5777 \end{bmatrix}.$$

The controller matrices for Robot 2 are:

$$K_2 = \begin{bmatrix} -0.4438 & -0.2381 \\ 2.1761 & -1.5355 \end{bmatrix}, W_2 = \begin{bmatrix} 0.1065 & -2.2209 & 0.368 \\ 0.6867 & -0.2602 & 2.594 \end{bmatrix},$$
$$L_2 = \begin{bmatrix} -0.2 & 0.8944 & 0 & 0.4 \\ -0.4 & -0.4472 & 0 & 0.8 \end{bmatrix}, G_2 = \begin{bmatrix} 4.3777 & 0 & 1.6 \\ -0.1889 & 0 & 3.2 \end{bmatrix}.$$

From (6.16), we obtain the upper bound on the $||\Delta J||$ as 1.5551. In our system, we have $||\Delta J|| = 3.4746 \times 10^{-6}$. Therefore, the sufficiency condition for the stability of the observer-based decentralized controller is satisfied.

We simulate the DNCS nonlinear plant model (6.27) using MATLAB's dde23 package. The initial state vector $x_p(0) = [1\ 2\ \ -21.2\ 11.5]^\top$. The initial state vector for the controller is zero. We obtain the matrix $(\bar{A}_0 + \bar{A}_1)$ which is Hurwitz with all its eigenvalues in the open left-hand complex plane: $-2.8674 \pm j0.8498$, -3.0503, -4.6364, -4. By Proposition 1, the sufficiency condition for the stability of the DNCS is satisfied for $\tau_{ub} = 0.0039$.

In Figure 6.3, we show plots of the plant state trajectories for $\tau = 0.003$ sec, where $\tau < \tau_{ub}$. As can be seen in Figure 6.4, the robots follow the desired paths. The upper bound on the time-delay given by Proposition 1 is sufficient for the stability of the closed-loop system but not necessary. Therefore, it is possible for the DNCS to be stable when τ is greater than τ_{ub}. To illustrate this, we simulate the system for $\tau = 0.01$ sec in Figure 6.5, where $\tau > \tau_{ub}$, and the DNCS still behaves in a stable fashion. However, if τ is much greater than τ_{ub}, then the DNCS becomes unstable as can be seen in Figure 6.6, where $\tau = 0.1$ sec. The above analysis motivates further research to obtain tighter upper bound on the allowable time-delay.

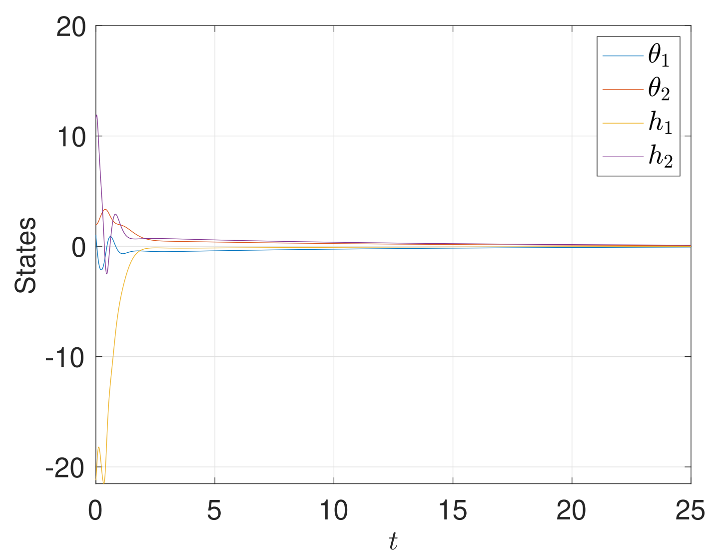

Fig. 6.3.: Plot of the states for $\tau = 0.003$.

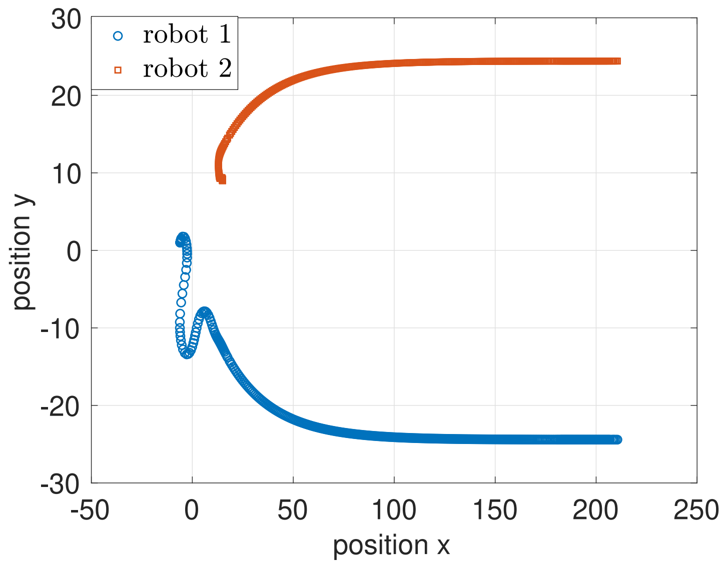

Fig. 6.4.: Plot of the path of robot 1 and robot 2 for $\tau = 0.003$.

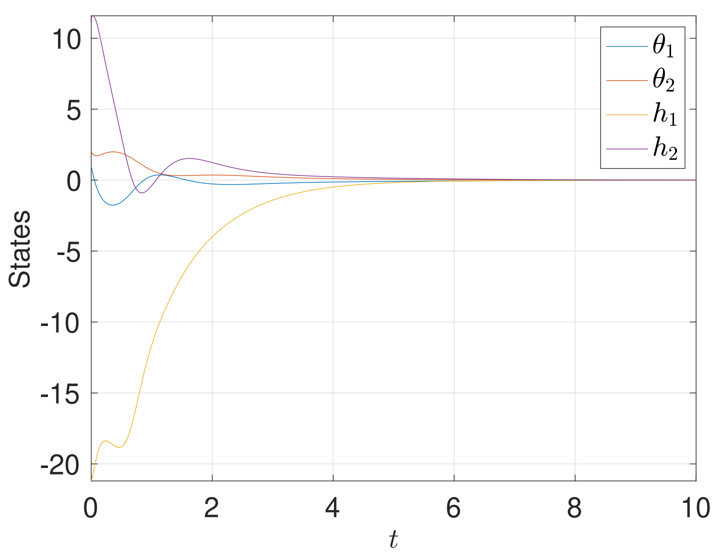

Fig. 6.5.: Plot of the states for $\tau = 0.01$.

6.6 Conclusions

We proposed a novel method for designing a decentralized controller using LMIs. We implemented the obtained controller design for the decentralized networked control system

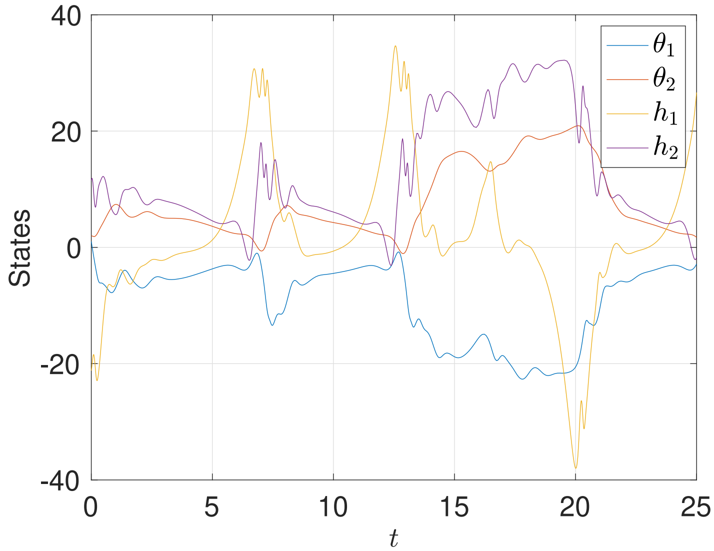

Fig. 6.6.: Plot of the states for $\tau = 0.1$.

(DNCS). The communication network was modeled as a pure time-delay. The stability of the DNCS has been analyzed and an upper bound on the allowable time-delay of the communication network was obtained. Open problems are to obtain a tighter upper bound on the allowable time-delay for the communication network and to consider different communication time-delays during the the local control input and output signal transmission.

7. SUMMARY AND OPEN PROBLEMS

7.1 Summary

In this thesis, we propose unknown input observers (UIOs) for the state, unknown input, and output disturbance estimation of networked control systems. We first consider the case when the networked system is linear with bounded unknown input. We then extend our UIO designs to a class of nonlinear systems whose nonlinearities can be characterized by incremental multiplier matrices. Then, we consider a linear networked system with unknown inputs and output disturbances. We construct delayed unknown input and output disturbance estimators for linear networked system for which the matrix rank condition for the existence of an UIO is not satisfied. Next, we design a decentralized observer-based controller for decentralized networked control system and provide a sufficient condition on the duration of the network time delay guaranteeing the stability of the decentralized networked control system. We illustrate the effectiveness of our proposed unknown input observer designs and the decentralized observer-based control design with simulations.

7.2 Open Problems

In this section, we describe two problems of designing unknown input observer-based controller for different types of networked control systems. In our opinion, these networked control systems have a potential for significant practical applications. We hypothesize that the design methods presented in this thesis can be used to design controllers for these systems. In the following subsections, we outline the proposed approach. In addition, the effects of time-delays during the output and input transmission and disturbances on the networked system performance need to be analyzed.

7.2.1 A multi-agent networked system under decentralized information structure

We consider a large-scale system with N local control stations modeled by

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} (B_{i1}u_{i1}(t) + B_{i2}u_{i2}(t)) \tag{7.1a}$$

$$y_i(t) = C_i x(t), \qquad i = 1, 2, \ldots, N, \tag{7.1b}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the plant, $u_{i1}(t) \in \mathbb{R}^{m_{i1}}$, $u_{i2}(t) \in \mathbb{R}^{m_{i2}}$, and $y_i(t) \in \mathbb{R}^{r_i}$ are the control input, unknown input, and the measured output of each local station, respectively. The system matrices are $B_{i1} \in \mathbb{R}^{n\times m_{i1}}$, $B_{i2} \in \mathbb{R}^{n\times m_{i2}}$, and $C_i \in \mathbb{R}^{r_i\times n}$, where $\sum_{i=1}^{N} m_{i1} = m_1$, $\sum_{i=1}^{N} m_{i2} = m_2$, and $\sum_{i=1}^{N} r_i = r$. Let

$$\begin{aligned}
B_1 &= [B_{11} \;\; B_{21} \;\; \cdots \;\; B_{N1}] \\
B_2 &= [B_{12} \;\; B_{22} \;\; \cdots \;\; B_{N2}] \\
C &= [C_1^\top \;\; C_2^\top \;\; \cdots \;\; C_N^\top]^\top \\
u_1(t) &= [u_{11}(t)^\top \;\; u_{21}(t)^\top \;\; \cdots \;\; u_{N1}(t)^\top]^\top \\
u_2(t) &= [u_{12}(t)^\top \;\; u_{22}(t)^\top \;\; \cdots \;\; u_{N2}(t)^\top]^\top \\
y(t) &= [y_1(t)^\top \;\; y_2(t)^\top \;\; \cdots \;\; y_N(t)^\top]^\top.
\end{aligned}$$

Then, the plant model can be described in a centralized format as

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t) \tag{7.2a}$$

$$y(t) = Cx(t). \tag{7.2b}$$

A block diagram of the multi-agent networked control system is shown in Figure 7.1. The communication networks in Figure 7.2 are modeled as pure time-delays τ_i. Therefore,

$$\hat{u}_{i1}(t) = u_{i1}(t - \tau_i)$$
$$\hat{y}_i(t) = y_i(t - \tau_i).$$

Our first objective is to design a distributed unknown input observer (UIO) to estimate the

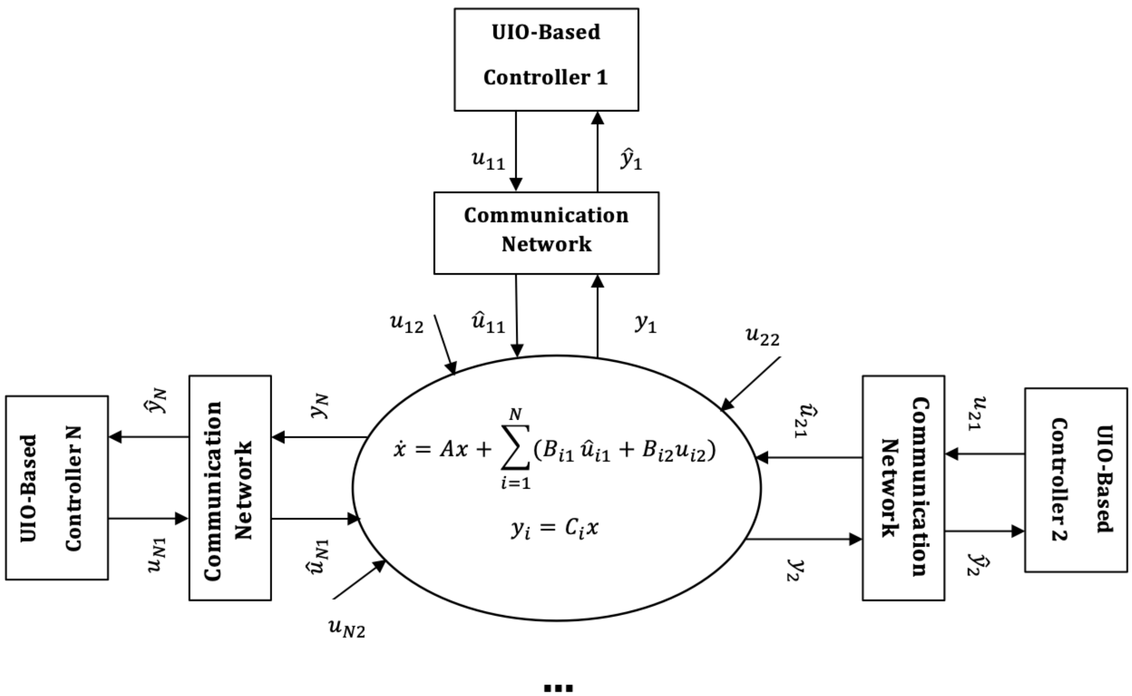

Fig. 7.1.: Decentralized networked UIO-based control for multi-agent system.

state and unknown input of the multi-agent networked system modeled by (7.1). Then, a distributed UIO-based controller need to be designed using the method developed in Chapter 5.7. The plant given by (7.2) will be controlled by the distributed UIO-based controller and the effects of the disturbances on the performance of this distributed networked control system will be analyzed.

Example 9 We consider the example presented in Chapter 5.7 of two interconnected mobile robots that are remotely controlled through a communication network. This system can be found in [92]. Given that (x_1, y_1, θ_1) and (x_2, y_2, θ_2) are the position and orientation of robot 1 and robot 2, respectively, then the dynamics of each robot can be modeled as:

$$\left.\begin{array}{rcl} \dot{x}_i & = & v_i \cos\theta_i \\ \dot{y}_i & = & v_i \sin\theta_i \\ \dot{\theta}_i & = & \omega_i, \end{array}\right\} \tag{7.3}$$

where $i = 1, 2$ and v_i and ω_i are the velocity and angular velocity of the i-th robot, respectively.

The two robots are to be remotely controlled in parallel formation with respect to the horizontal axis. To achieve this goal, the following constraints are imposed: $x_1 = x_2$, $\frac{y_1+y_2}{2} = 0$, and $\theta_1 = \theta_2 = 0$. Let $x = [\theta_1 \;\; \theta_2 \;\; h_1 \;\; h_2]^\top$, where $h_1 = x_1 - x_2 + \alpha(\theta_1 - \theta_2)$ and $h_2 = y_1 + y_2 + \beta(\theta_1 + \theta_2)$ contain the global information of the horizontal distance difference between the two robots and the average value of the vertical formation of the robots, respectively. The parameters α and β represent the level of perturbation in distance measurements due to the robot orientation [92]. To analyze the robustness of the proposed distributed UIO-based controller, we add bounded disturbances $u_{12}(t)$ and $u_{22}(t)$ to the system model. Then, the nonlinear formation model of the two robots with disturbances has the form:

$$\left.\begin{array}{rcl} \dot{\theta}_1 & = & \omega_1 \\ \dot{\theta}_2 & = & \omega_2 \\ \dot{h}_1 & = & v_1 cos\theta_1 - v_2 cos\theta_2 + \alpha(\omega_1 - \omega_2) + u_{12}(t) \\ \dot{h}_2 & = & v_1 sin\theta_1 + v_2 sin\theta_2 + \beta(\omega_1 + \omega_2) + u_{22}(t). \end{array}\right\} \tag{7.4}$$

The disturbances to the formation of the robots may have the form of skidding, slipping, friction force, and drift [93]. The linearized formation trajectory of system (7.4) about $v_i = 2$ and $\theta_i = 0$ can now be obtained as

$$\dot{x} = Ax + \sum_{i=1}^{N} (B_{i1}u_{i1}(t) + B_{i2}u_{i2}(t)), \tag{7.5}$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix}, B_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & \alpha \\ 0 & \beta \end{bmatrix}, B_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ -1 & -\alpha \\ 0 & \beta \end{bmatrix}, B_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, B_{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The control inputs are $\hat{u}_{11} = [v_1 - 2\ \omega_1]^\top$ and $\hat{u}_{21} = [v_2 - 2\ \omega_2]^\top$. The distributed information is collected using sensors on each robot as follows, $y_{out} = [y_{out1}^\top\ y_{out2}^\top]^\top$, where $y_{out1} = [\theta_1\ h_1\ h_2]^\top = C_1 x$ and $y_{out2} = [\theta_2\ h_1\ h_2]^\top = C_2 x$.

The open problem is to design a distributed UIO-based controller for the system consisting of two cooperating robots in the presence of disturbances. Then, use this design to control the two robots to achieve a parallel formation with respect to the horizontal axis through a communication networks modeled as pure time-delays.

7.2.2 A decentralized networked control system

We consider the linear time-invariant large-scale system of N interconnected subsystems with the $i-$th subsystem represented by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + \sum_{j=1, j\neq i}^{N} e_{ij} A_{ij} x_j(t) \tag{7.6a}$$

$$y_i(t) = C_i x_i(t), \qquad i = 1, 2, \ldots, N, \tag{7.6b}$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, and $y_i(t) \in \mathbb{R}^{r_i}$ are the state, input control, and the measured output of each local station, respectively, and $e_{ij}(t)$ are uncertain structural elements that determine the degree of coupling between the two subsystems i and j, such that $0 \leq e_{ij}(t) \leq 1$ [64].

The centralized format of the above model has the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{7.7a}$$

$$y(t) = Cx(t), \tag{7.7b}$$

where

$$\begin{aligned} x(t) &= [x_1(t)^\top \ \ x_2(t)^\top \ \cdots \ x_N(t)^\top]^\top \\ u(t) &= [u_1(t)^\top \ \ u_2(t)^\top \ \cdots \ u_N(t)^\top]^\top \\ y(t) &= [y_1(t)^\top \ \ y_2(t)^\top \ \cdots \ y_N(t)^\top]^\top. \end{aligned}$$

The open problem is to design a decentralized unknown input observer-based controller for a decentralized plant in (7.6). A block diagram of the decentralized networked control system is shown in Figure 7.2. The communication networks are modeled as pure time-delays τ_i, then,

$$\begin{aligned} \hat{u}_i(t) &= u_i(t - \tau_i) \\ \hat{y}_i(t) &= y_i(t - \tau_i). \end{aligned}$$

Example 10 We consider an example from the book by Šiljak [64] of two identical pendulums coupled by a spring and subject to two distinct inputs in Figure 7.3. The system

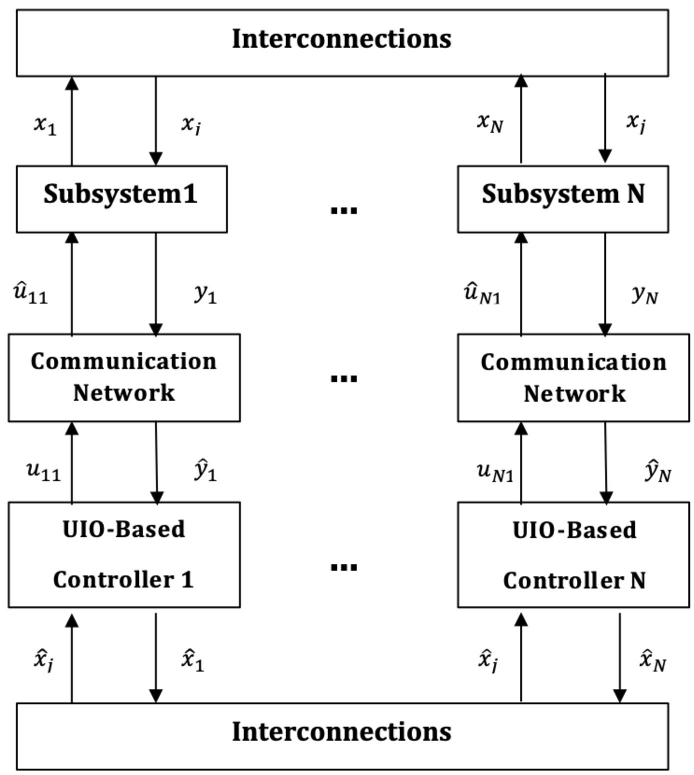

Fig. 7.2.: Decentralized networked UIO-based controller for an interconnected system.

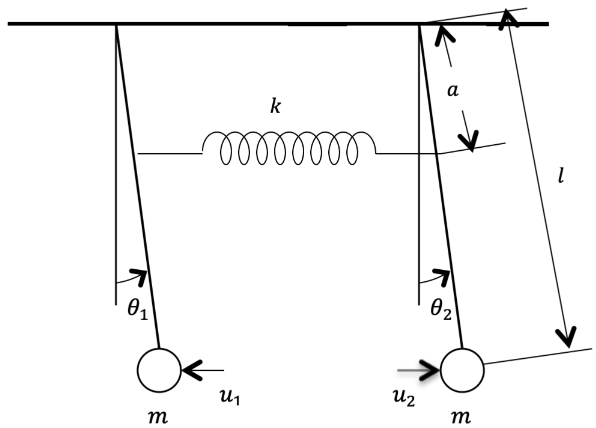

Fig. 7.3.: Interconnected pendulums.

equations are,

$$ml^2\ddot{\theta}_1 = -mgl\theta_1 - ka^2(\theta_1 - \theta_2) - u_1, \quad y_1 = \theta_1 \tag{7.8}$$

$$ml^2\ddot{\theta}_2 = -mgl\theta_2 - ka^2(\theta_2 - \theta_1) - u_2, \quad y_2 = \theta_2, \tag{7.9}$$

where the position of the spring $a(t)$ is considered uncertain. It can change along the length of the pendulum, that is, $0 \leq a(t) \leq l$.

Let $x = [\theta_1 \ \dot{\theta}_1 \ \theta_2 \ \dot{\theta}_2]^\top$, then the system model can be represented as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{g}{l} - \frac{ka^2}{ml^2} & 0 & \frac{ka^2}{ml^2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{ka^2}{ml^2} & 0 & -\frac{g}{l} - \frac{ka^2}{ml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ -\frac{1}{ml^2} & 0 \\ 0 & 0 \\ 0 & -\frac{1}{ml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x,$$

where $u = [u_1 \;\; u_2]^\top$. Let $x_1 = [\theta_1 \;\; \dot{\theta}_1]^\top$ and $x_2 = [\theta_2 \;\; \dot{\theta}_2]^\top$. The above system can be represented as two interconnected subsystems,

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ -\beta \end{bmatrix} u_1 + e_{11} \begin{bmatrix} 0 & 0 \\ -\gamma & 0 \end{bmatrix} x_1 + e_{12} \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} x_2$$
$$\dot{x}_2 = \begin{bmatrix} 0 & 1 \\ -\alpha & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ -\beta \end{bmatrix} u_2 + e_{21} \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix} x_1 + e_{22} \begin{bmatrix} 0 & 0 \\ -\gamma & 0 \end{bmatrix} x_2$$

where $\alpha = \frac{g}{l}$, $\beta = \frac{1}{ml^2}$, $\gamma = \frac{k}{m}$, and $e_{ij}(t) = \frac{a^2(t)}{l^2}$ for $i, j = 1, 2$.

The open problem is to design a decentralized UIO-based controller for a networked system consisting of two pendulums coupled by a spring with uncertain position of the spring. The effects of the time-delays due to presence of the communication networks and the system uncertainties on the DNCS performance need to be carefully analyzed.

REFERENCES

REFERENCES

[1] D. G. Luenberger, "Observing the state of a linear system," *IEEE Transactions on Military Electronics*, vol. 8, no. 2, pp. 74–80, 1964.

[2] D. Luenberger, "Observers for multivariable systems," *IEEE Transactions on Automatic Control*, vol. 11, no. 2, pp. 190–197, 1966.

[3] D. Luenberger, "An introduction to observers," *IEEE Transactions on Automatic Control*, vol. 16, no. 6, pp. 596–602, 1971.

[4] G. Basile and G. Marro, "On the observability of linear, time-invariant systems with unknown inputs," *Journal of Optimization Theory and Applications*, vol. 3, no. 6, pp. 410–415, 1969.

[5] R. Patton and J. Chen, *Robust Model-Based Fault Diagnosis for Dynamic Systems*. New York: Springer, 1999.

[6] M. Hou and P. C. Müller, "Design of observers for linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, pp. 632–635, June 1992.

[7] M. Darouach, M. Zasadzinski, and S. J. Xu, "Full-order observers for linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, no. 3, pp. 606–609, March 1994.

[8] M. Corless and J. Tu, "State and input estimation for a class of uncertain systems," *Automatica*, vol. 34, no. 6, pp. 757–764, 1998.

[9] S. Hui and S. H. Żak, "Observer design for systems with unknown inputs," *International Journal of Applied Mathematics and Computer Science*, vol. 15, no. 4, pp. 431–446, 2005.

[10] C. Edwards, S. K. Spurgeon, and R. J. Patton, "Sliding mode observers for fault detection and isolation," *Automatica*, vol. 36, no. 4, pp. 541–553, 2000.

[11] C. P. Tan and C. Edwards, "Sliding mode observers for detection and reconstruction of sensor faults," *Automatica*, vol. 38, no. 10, pp. 1815–1821, 2002.

[12] K. Zhou, Z. Ren, and W. Wang, "On the design of unknown input observers and fault detection filters," in *2006 6th World Congress on Intelligent Control and Automation*, vol. 2, June 2006, pp. 5638–5642.

[13] S. C. Johnson, A. Chakrabarty, J. Hu, S. H. Żak, and R. A. DeCarlo, "Dual-mode robust fault estimation for switched linear systems with state jumps," *Nonlinear Analysis: Hybrid Systems*, vol. 27, pp. 125–140, 2018.

[14] X. Liu, Z. Gao, and A. Zhang, "Robust fault tolerant control for discrete-time dynamic systems with applications to aero engineering systems," *IEEE Access*, vol. 6, pp. 18 832–18 847, 2018.

[15] S. Hui and S. H. Żak, "Stress estimation using unknown input observer," in *Proceedings of American Control Conference (ACC), Washington, DC*, June 17–19, 2013, pp. 259–264.

[16] M. S. Chong, M. Wakaiki, and J. P. Hespanha, "Observability of linear systems under adversarial attacks," in *2015 American Control Conference (ACC)*, July 2015, pp. 2439–2444.

[17] Y. H. Chang, Q. Hu, and C. J. Tomlin, "Secure estimation based Kalman filter for cyber-physical systems against adversarial attacks," *CoRR*, vol. abs/1512.03853, 2016.

[18] G. Fiore, A. Iovine, E. De Santis, and M. D. Di Benedetto, "Secure state estimation for dc microgrids control," in *2017 13th IEEE Conference on Automation Science and Engineering (CASE)*, Aug 2017, pp. 1610–1615.

[19] P. F. Odgaard, L. Skov, and R. Nielsen, "Unknown input observer based detection scheme for faults in hydraulic valves," in *Proceedings of the 1st Virtual Control Conference, Aalborg, Denmark*, Sept 21–23, 2010, pp. 81–86.

[20] P. F. Odgaard and J. Stoustrup, "Unknown input observer based detection of sensor faults in a wind turbine," in *2010 IEEE International Conference on Control Applications*, Sep. 2010, pp. 310–315.

[21] M. Chen and W. Min, "Unknown input observer based chaotic secure communication," *Physics Letters A*, vol. 372, no. 10, pp. 1595–1600, 2008.

[22] H. Iranmanesh and A. Afshar, "MPC-based control of a large-scale power system subject to consecutive pulse load variations," *IEEE Access*, vol. 5, pp. 26 318–26 327, 2017.

[23] T. Kato, Y. Kim, S. Okuma, and T. Narikiyo, "Large-scale traffic network control based on mixed integer non-linear system formulation," in *IECON 2006 - 32nd Annual Conference on IEEE Industrial Electronics*, Nov 2006, pp. 150–155.

[24] M. B. McMickell, B. Goodwine, and L. A. Montestruque, "Micabot: a robotic platform for large-scale distributed robotics," in *2003 IEEE International Conference on Robotics and Automation (Cat. No.03CH37422)*, vol. 2, Sep. 2003, pp. 1600–1605.

[25] C. Edwards and S. K. Spurgeon, *Sliding Mode Control*. Bristol, PA: Taylor and Francis Inc., 1998.

[26] K. Kalsi, J. Lian, S. Hui, and S. H. Żak, "Sliding-mode observers for systems with unknown inputs: A high-gain approach," *Automatica*, vol. 46, no. 2, pp. 347 – 353, 2010.

[27] Y. Guan and M. Saif, "A novel approach to the design of unknown input observer," *IEEE Transactions on Automatic Control*, no. 5, pp. 632–635, 1991.

[28] M. Hou and P. C. Müller, "Distributed decoupled observer design: A unified viewpoint," *IEEE Transactions on Automatic Control*, pp. 1338–1341, June 1994.

[29] A. Chakrabarty, M. J. Corless, G. T. Buzzard, S. H. Żak, and A. E. Rundell, "State and unknown input observers for nonlinear systems with bounded exogenous inputs,," *IEEE Transactions on Automatic Control*, no. 99, pp. 5497–5510, 2017.

[30] D. Wang and K. Lum, "Adaptive unknown input observer approach for aircraft actuator fault detection and isolation," *International Journal of Adaptive Control and Signal Processing*, vol. 21, no. 1, pp. 31–48, 2007.

[31] J. Na, A. S. Chen, G. Herrmann, R. Burke, and C. Brace, "Vehicle engine torque estimation via unknown input observer and adaptive parameter estimation," *IEEE Transactions on Vehicular Technology*, pp. 409–422, 2018.

[32] H. Dimassi and A. Loria, "Adaptive unknown-input observers-based synchronization of chaotic systems for telecommunication," *IEEE Transactions on Circuits and Systems*, vol. 58, no. 4, pp. 800–812, 2011.

[33] J. Anzurez-Marin, N. Pitalua-Diaz, O. Cuevas-Silva, and J. Villar-García, "Unknown inputs observers design for fault detection in a two-tank hydraulic system," in *2008 Electronics, Robotics and Automotive Mechanics Conference (CERMA '08)*, Sep. 2008, pp. 373–378.

[34] A. Termehchy, A. Afshar, and M. Javidsharifi, "A novel design of unknown input observer for fault diagnosis in the Tennessee-Eastman process system to solve non-minimum phase problem," in *2013 IEEE International Conference on Smart Instrumentation, Measurement and Applications (ICSIMA)*, Nov 2013, pp. 1–6.

[35] F. Xu, J. Tan, X. Wang, V. Puig, B. Liang, and B. Yuan, "A novel design of unknown input observers using set-theoretic methods for robust fault detection," in *2016 American Control Conference (ACC)*, July 2016, pp. 5957–5961.

[36] W. Gritli, H. Gharsallaoui, and M. Benrejeb, "Fault detection based on unknown input observers for switched discrete-time systems," in *2017 International Conference on Advanced Systems and Electric Technologies (IC ASET)*, Jan 2017, pp. 436–441.

[37] A. Ansari and D. S. Bernstein, "Deadbeat unknown-input state estimation and input reconstruction for linear discrete-time systems," *Automatica*, vol. 103, pp. 11–19, 2019.

[38] J. D. Avilés and J. A. Moreno, "Dissipative observers for discrete-time nonlinear systems," *Journal of the Franklin Institute*, vol. 355, no. 13, pp. 5759–5770, 2018.

[39] S. Sayyaddelshad and T. Gustafsson, "Observer design for a class of nonlinear systems subject to unknown inputs," in *2014 European Control Conference (ECC)*, June 2014, pp. 970–974.

[40] A. Chakrabarty, S. H. Żak, and S. Sundaram, "State and unknown input observers for discrete-time nonlinear systems," in *2016 IEEE 55th Conference on Decision and Control (CDC)*, Dec 2016, pp. 7111–7116.

[41] S. H. Żak, A. Chakrabarty, and G. T. Buzzard, "Robust state and unknown input estimation for nonlinear systems characterized by incremental multiplier matrices," in *Proceedings of the 2017 American Control Conference, Seattle, WA*, May 24–26 2017, pp. 3270–3275.

[42] M. E. Valcher, "State observers for discrete-time linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. 44, no. 2, pp. 397–401, Feb 1999.

[43] D. Ichalal and S. Mammar, "Asymptotic unknown input decoupling observer for discrete-time LTI systems," *IEEE Control Systems Letters*, vol. 4, no. 2, pp. 361–366, April 2020.

[44] H. Fawzi, P. Tabuada, and S. Diggavi, "Secure estimation and control for cyber-physical systems under adversarial attacks," *IEEE Transactions on Automatic Control*, vol. 59, no. 6, pp. 1454–1467, June 2014.

[45] Y. Li, L. Shi, P. Cheng, J. Chen, and D. E. Quevedo, "Jamming attacks on remote state estimation in cyber-physical systems: A game-theoretic approach," *IEEE Transactions on Automatic Control*, vol. 60, no. 10, pp. 2831–2836, Oct 2015.

[46] Y. Guan and X. Ge, "Distributed attack detection and secure estimation of networked cyber-physical systems against false data injection attacks and jamming attacks," *IEEE Transactions on Signal and Information Processing over Networks*, vol. 4, no. 1, pp. 48–59, 2017.

[47] Q. Hu, D. Fooladivanda, Y. H. Chang, and C. J. Tomlin, "Secure state estimation and control for cyber security of the nonlinear power systems," *IEEE Transactions on Control of Network Systems*, vol. 5, no. 3, pp. 1310–1321, Sep. 2018.

[48] J. Zhang, M. Chadli, and Y. Wang, "A fixed-time observer for discrete-time singular systems with unknown inputs," *Applied Mathematics and Computation*, vol. 363, 2019, article No. 124586.

[49] E. J. Candes and T. Tao, "Decoding by linear programming," *IEEE Transactions on Information Theory*, vol. 51, no. 12, pp. 4203–4215, 2005.

[50] G. Fiore, Y. H. Chang, Q. Hu, M. D. Di Benedetto, and C. J. Tomlin, "Secure state estimation for cyber physical systems with sparse malicious packet drop," in *2017 American Control Conference (ACC)*, Seattle, WA, May 2017, pp. 1898–1903.

[51] M. Zhang, S. Hui, M. R. Bell, and S. H. Żak, "Vector recovery for a linear system corrupted by unknown sparse error vectors with applications to secure state estimation," *IEEE Control Systems Letters*, vol. 3, no. 4, pp. 895–900, Oct 2019.

[52] D. L. Donoho and M. Elad, "For most large underdetermined systems of linear equations the minimal l_1-norm solution is also the sparsest solution," *SIAM Review*, vol. 56, no. 6, pp. 797–829, 2006.

[53] M. Zhang, B. Alenezi, S. Hui, and S. H. Żak, "State estimation of networked control systems corrupted by unknown input and output sparse errors," in *2020 American Control Conference (ACC)*, Denver, CO, July 2020, pp. 4393–4398.

[54] M. L. J. Hautus, "Strong detectability and observers," *Linear Algebra and Its Applications*, vol. 50, pp. 353–368, 1983.

[55] J. Jin, M.-J. Tahk, and C. Park, "Time-delayed state and unknown input observation," *International Journal of Control*, vol. 66, no. 5, pp. 733–746, 1997.

[56] J. Jin and M.-J. Tahk, "Time-delayed state estimator for linear systems with unknown inputs," *International Journal of Control, Automation, and Systems*, vol. 3, no. 1, pp. 117–121, 2005.

[57] S. Sundaram and C. N. Hadjicostis, "Delayed observers for linear systems with unknown inputs," *IEEE Transactions on Automatic Control*, vol. 52, no. 2, pp. 334–339, 2007.

[58] A. Chakrabarty, R. Ayoub, S. H. Żak, and S. Sundaram, "Delayed unknown input observers for discrete-time linear systems with guaranteed performance," *Systems & Control Letters*, vol. 103, pp. 9–15, 2017.

[59] S.-H. Wang and E. Davison, “On the stabilization of decentralized control systems,” *IEEE Transactions on Automatic Control*, vol. 18, no. 5, pp. 473–478, 1973.

[60] Z. Gong and M. Aldeen, “On the characterization of fixed modes in decentralized control,” *IEEE Transactions on Automatic Control*, vol. 37, no. 7, pp. 1046–1050, 1992.

[61] M. Ravi, J. Rosenthal, and X. Wang, “On decentralized dynamic pole placement and feedback stabilization,” *IEEE Transactions on Automatic Control*, vol. 40, no. 9, pp. 1603–1614, 1995.

[62] H. Trinh and M. Aldeen, “Decentralised feedback controllers for uncertain interconnected dynamic systems,” in *IEE Proceedings D-Control Theory and Applications*, vol. 140, no. 6. IET, 1993, pp. 429–434.

[63] Q. Ha and H. Trinh, “Observer-based control of multi-agent systems under decentralized information structure,” *International Journal of Systems Science*, vol. 35, no. 12, pp. 719–728, 2004.

[64] D. D. Šiljak, *Large-Scale Dynamic Systems*. Mineola, New York: Dover, 2007.

[65] Y. B. Zhao, X. M. Sun, J. Zhang, and P. Shi, “Networked control systems: The communication basics and control methodologies,” *Mathematical Problems in Engineering*, pp. 85–94, 2015.

[66] R. Bergman and M. Medard, “Fault isolation for communication networks for isolating the source of faults comprising attacks, failures, and other network propagating errors,” Aug 2002, US Patent 6,442,694.

[67] Y. H. Chang, Q. Hu, and C. J. Tomlin, “Secure estimation based kalman filter for cyber–physical systems against sensor attacks,” *Automatica*, vol. 95, pp. 399–412, 2018.

[68] D. Yue, Q. Han, and C. Peng, “State feedback controller design of networked control systems,” in *Proceedings of the 2004 IEEE International Conference on Control Applications, 2004.*, vol. 1, Sep. 2004, pp. 242–247.

[69] X. Jiang, Q.-L. Han, S. Liu, and A. Xue, “A new H_∞ stabilization criterion for networked control systems,” *IEEE Transactions on Automatic Control*, vol. 53, no. 4, pp. 1025–1032, 2008.

[70] G. C. Walsh, H. Ye, and L. G. Bushnell, “Stability analysis of networked control systems,” *IEEE Transactions on Control Systems Technology*, vol. 10, no. 3, pp. 438–446, 2002.

[71] M. S. Mahmoud and A. Ismail, “Role of delays in networked control systems,” in *10th IEEE International Conference on Electronics, Circuits and Systems, 2003. ICECS 2003. Proceedings of the 2003*, vol. 1, Dec 2003, pp. 40–43 Vol.1.

[72] J.-Q. Huang and F. L. Lewis, “Neural-network predictive control for nonlinear dynamic systems with time-delay,” *IEEE Transactions on Neural Networks*, vol. 14, no. 2, pp. 377–389, March 2003.

[73] A. F. Khalil and J. Wang, “Stability and time delay tolerance analysis approach for networked control systems,” *Mathematical Problems in Engineering*, vol. 2015, 2015.

[74] G. P. Liu, J. X. Mu, D. Rees, and S. C. Chai, “Design and stability analysis of networked control systems with random communication time delay using the modified mpc,” *International Journal of Control*, vol. 79, no. 4, pp. 288–297, 2006.

[75] M. S. Mahmoud, *Robust Control and Filtering for Time Delay Systems*. New York: Marcel Dekker, 2000.

[76] S. Brierley, J. Chiasson, E. Lee, and S. Żak, “On stability independent of delay for linear systems,” *IEEE Transactions on Automatic Control*, vol. 27, no. 1, pp. 252–254, February 1982.

[77] G. M. Schoen, “Stability and stabilization of time-delay systems,” Ph.D. dissertation, Swiss Federal Institute of Technology, Zurich, 1995.

[78] A. Steinberg and M. Corless, “Output feedback stabilization of uncertain dynamical systems,” *IEEE Transactions on Automatic Control*, vol. 30, no. 10, pp. 1025–1027, October 1985.

[79] B. Walcott and S. Żak, “State observation of nonlinear uncertain dynamical systems,” *IEEE Transactions on Automatic Control*, vol. 32, no. 2, pp. 166–170, February 1987.

[80] S. H. Żak, *Systems and Control*. New York: Oxford University Press, 2003.

[81] J. E. Slotine and W. Li, *Applied Nonlinear Control*. Englewood cliffs, New Jersey: Printice Hall, 1991.

[82] T. Pancake, M. J. Corless, and M. Brockman, “Analysis and control of polytopic uncertain/nonlinear systems in the presence of bounded disturbance inputs,” in *Proceedings of the 2000 American Control Conference, Chicago, IL*, June 28–30, 2000, pp. 159–163.

[83] C. R. Inc., “CVX: Matlab software for disciplined convex programming, version 2.0,” http://cvxr.com/cvx, Aug 2012.

[84] M. Grant and S. Boyd, “Graph implementations for non-smooth convex programs,” in *Recent Advances in Learning and Control*, ser. Lecture Notes in Control and Information Sciences, V. Blondel, S. Boyd, and H. Kimura, Eds. Springer-Verlag Limited, 2008, pp. 95–110.

[85] P. Samanta, S. Mitra, and G. Das, “Construction of a reduce order observer for linear time invariant system with unknown inputs,” in *Proceedings of the 2015 Third International Conference on Computer, Communication, Control and Information Technology*, Feb 7–8 2015.

[86] T. Kaczorek, K. M. Przyłuski, and S. H. Żak, *Wybrane Metody Analizy Liniowych Układów Dynamicznych (Selected Methods of Analysis of Linear Dynamical Systems)*. Warszawa (Warsaw): Państwowe Wydawnictwo Naukowe (Polish Scientific Publishers), 1984.

[87] C.-T. Chen, *Linear System Theory and Design*, 4th ed. New York: Oxford University Press, 2013.

[88] E. Rocha-Cózatl and J. A. Moreno, “Dissipative design of unknown input observers for systems with sector nonlinearities,” *International Journal of Robust and Nonlinear Control*, vol. 21, no. 14, pp. 1623–1644, 2011.

[89] K. J. Astrom and R. M. Murray, *Feedback Systems: An Introduction for Scientists and Engineers*. Princeton, NJ, USA: Princeton University Press, 2008.

[90] B. A. Charandabi and H. J. Marquez, "A novel approach to unknown input filter design for discrete-time linear systems," *Automatica*, vol. 50, no. 11, pp. 2835–2839, 2014.

[91] T. Su and C. Huang, "Robust stability of delay dependence for linear uncertain systems," *IEEE Transactions on Automatic Control*, vol. 37, no. 10, pp. 1656–1659, Oct 1992.

[92] A. Nguyen, Q. Ha, S. Huang, and H. Trinh, "Observer-based decentralized approach to robotic formation control," in *Australasian Conference on Robotics and Automation*. ARAA Australian Robotics & Automation Association, 2004.

[93] A. Liu, W. Zhang, L. Yu, H. Yan, and R. Zhang, "Formation control of multiple mobile robots incorporating an extended state observer and distributed model predictive approach," *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, pp. 1–11, 2018.

VITA

VITA

Badriah Alenezi received her B.S. and M.S. degrees in Electrical Engineering from Kuwait University, Kuwait in 2009 and 2013, respectively. In summer 2009, she was a teaching assistant for the Electric Circuit Lab in Kuwait University, Kuwait. From 2009 to 2013, she was an Engineer in the Construction Department of the Public Authority of Youth and Sport, Kuwait. She received her M.S. degree in Electrical and Computer Engineering from Purdue University, West Lafayette, IN, USA, in 2015. Her research interests include unknown input observers, networked control systems, decentralized controllers and fault detection.