ALMOST MINIMIZERS FOR THE THIN OBSTACLE PROBLEMS

by

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To my family

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TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

In this dissertation, we consider almost minimizers for the thin obstacle problems in different settings: Laplacian, fractional Laplacian and equation with variable coefficients.

In Chapter 1, we consider Anzellotti-type almost minimizers for the thin obstacle (or Signorini) problem with zero thin obstacle and establish their $C^{1,\beta}$ regularity on the either side of the thin manifold, the optimal growth away from the free boundary, the $C^{1,\gamma}$ regularity of the regular part of the free boundary, as well as a structural theorem for the singular set. The analysis of the free boundary is based on a successful adaptation of energy methods such as a one-parameter family of Weiss-type monotonicity formulas, Almgren-type frequency formula, and the epiperimetric and logarithmic epiperimetric inequalities for the solutions of the thin obstacle problem. This chapter is based on a joint work with Arshak Petrosyan [1].

In Chapter 2, we study almost minimizers for the thin obstacle problem with variable Hölder continuous coefficients and zero thin obstacle and establish their $C^{1,\beta}$ regularity on the either side of the thin space. Under an additional assumption of quasisymmetry, we establish the optimal growth of almost minimizers as well as the regularity of the regular set and a structural theorem on the singular set. The proofs are based on the generalization of Weiss- and Almgren-type monotonicity formulas for almost minimizers established earlier in the case of constant coefficients (Chapter 1). This chapter is based on recent joint work with Arshak Petrosyan and Mariana Smit Vega Garcia [2].

In Chapter 3, we introduce a notion of almost minimizers for certain variational problems governed by the fractional Laplacian, with the help of the Caffarelli-Silvestre extension. In particular, we study almost fractional harmonic functions and almost minimizers for the fractional obstacle problem with zero obstacle. We show that for a certain range of parameters, almost minimizers are almost Lipschitz or $C^{1,\beta}$-regular. This is based on a work in collaboration with Arshak Petrosyan [3].

1. ALMOST MINIMIZERS FOR THE THIN OBSTACLE PROBLEM

1.1 Introduction and main results

1.1.1 The thin obstacle (or Signorini) problem

Let $D \subset \mathbb{R}^n$ be an open set and $\mathcal{M} \subset \mathbb{R}^n$ a smooth $(n-1)$-dimensional manifold (the *thin space*) and consider the problem of minimizing the Dirichlet energy

$$J_D(u) := \int_D |\nabla u(x)|^2 dx \tag{1.1.1}$$

among all functions $u \in W^{1,2}(D)$ satisfying

$$u = g \text{ on } \partial D, \quad u \geq \psi \text{ on } \mathcal{M} \cap D,$$

where $\psi : \mathcal{M} \to \mathbb{R}$ is the so-called *thin obstacle* and $g : \partial D \to \mathbb{R}$ is the prescribed boundary data with $g \geq \psi$ on $\mathcal{M} \cap \partial D$. This problem is known as *the thin obstacle problem.* In other words, it is a constrained minimization problem for the energy functional J_D on a closed convex set

$$\mathfrak{K}_{\psi,g}(D, \mathcal{M}) := \{u \in W^{1,2}(D) : u = g \text{ on } \partial D,\, u \geq \psi \text{ on } \mathcal{M} \cap D\}.$$

This problem can be viewed as a scalar version of the Signorini problem with unilateral constraint from elastostatics [4] and is often referred to as *the Signorini problem.* It goes back to the origins of variational inequalities and is considered as one of the prototypical examples of such problems, see [5]. An equivalent formulation is given in the form

$$\begin{aligned} \Delta u = 0 &\quad \text{on } D \setminus \mathcal{M}, \\ u = g &\quad \text{on } \partial D, \\ u \geq \psi, \quad \partial_{\nu^+} u + \partial_{\nu^-} u \geq 0, \quad (\partial_{\nu^+} u + \partial_{\nu^-} u)(u - \psi) = 0 &\quad \text{on } \mathcal{M} \cap D, \end{aligned}$$

where the conditions on $\mathcal{M}\cap D$ are known as the *Signorini complementarity* (or *ambiguous*) *conditions*. Here, $\partial_{\nu^\pm}$ are the exterior normal derivatives from the either side of $\mathcal{M}$. In particular, at points on $\mathcal{M}\cap D$ we must have one of the two boundary conditions satisfied: either $u=\psi$ or $\partial_{\nu^+}u+\partial_{\nu^-}u=0$. The set

$$\Gamma(u):=\partial_{\mathcal{M}}\{x\in\mathcal{M}\cap D: u(x)=\psi(x)\}, \tag{1.1.2}$$

which separates the regions where different boundary conditions are satisfied, is known as the *free boundary* and plays a central role in the analysis of the problem.

Because of the presence of the thin obstacle, it is not hard to realize that the solutions u of the Signorini problem are at most Lipschitz across $\mathcal{M}$, even if both $\mathcal{M}$ and ψ are smooth, as we may have $\partial_{\nu^+}u+\partial_{\nu^-}u>0$ at some points on $\mathcal{M}$[1]. However, it has been known since the works [6]–[8] that the solutions of the thin obstacle problem are $C^{1,\beta}$ on $\mathcal{M}$ and consequently on the either side of $\mathcal{M}$, up to $\mathcal{M}$. In recent years, there has been a renewed interest in this problem, following the breakthrough result of Athanasopoulos and Caffarelli [9] on the optimal $C^{1,1/2}$ regularity of the minimizers (on the either side of $\mathcal{M}$) as well as its relation to the obstacle-type problems for the fractional Laplacian through the Caffarelli-Silvestre extension [10]. There has also been a significant effort in understanding the structure and the regularity of the free boundary. The results have been obtained in many settings, such as for the equations with variable coefficients, time-dependent versions, problems for fractional Laplacian and other nonlocal equations, both regarding the regularity of minimizers, as well as the properties of the free boundary; see e.g., [11]–[30] and many others.

1.1.2 Almost minimizers

In [31], Anzellotti introduced the notion of almost minimizers for energy functionals. Given $r_0>0$, we say that $\omega:(0,r_0)\to[0,\infty)$ is a modulus of continuity or a *gauge function*, if $\omega(r)$ is monotone nondecreasing in r and $\omega(0+)=0$.

[1]↑This can be seen on the explicit example $u(x)=\mathrm{Re}(x_1+i|x_n|)^{3/2}$, which is a solution of the obstacle problem with $\psi=0$ on $\mathcal{M}=\{x_n=0\}$.

Definition 1.1.1 (Almost minimizers)**.** *Given $r_0 > 0$ and a gauge function $\omega(r)$ on $(0, r_0)$, we say that $u \in W^{1,2}_{\mathrm{loc}}(D)$ is an* almost minimizer *(or ω-*minimizer*) for the functional J_D, if, for any ball $B_r(x_0) \Subset D$ with $0 < r < r_0$, we have*

$$J_{B_r(x_0)}(u) \leq (1 + \omega(r))J_{B_r(x_0)}(v) \quad \text{for any } v \in u + W^{1,2}_0(B_r(x_0)). \tag{1.1.3}$$

The idea is that the Dirichlet energy of u on the ball $B_r(x_0)$ is not necessarily minimal among all competitors $v \in u + W^{1,2}_0(B_r(x_0))$ but *almost minimal* in the sense that it cannot decrease more than by a factor of $1 + \omega(r)$. In the specific case of the energy functional J_D in (1.1.1), i.e., the Dirichlet energy, we refer to the almost minimizers of J_D as *almost harmonic functions* in D.

Results on almost minimizers for more general energy functionals can be found in [32]–[35]. Similar notions were considered earlier in the context of the geometric measure theory [36], [37], see also [38]. Almost minimizers are also related to quasiminimizers, introduced in [39], [40], see also [41]. For energy functionals exhibiting free boundaries, almost minimizers have been considered only recently in [42]–[46].

Almost minimizers can be viewed as perturbations of minimizers of various nature, but their study is motivated also by the observation that the minimizers with certain constrains, such as the ones with fixed volume or solutions of the obstacle problem, are realized as almost minimizers of unconstrained problems, see e.g. [31]. Yet another motivation is that the study of almost minimizers reveals a unique perspective on the problem and leads to the development of methods relying on less technical assumptions, thus allowing further generalization.

In this chapter we extend the notion of almost minimizers to the thin obstacle problem. Essentially, in (1.1.3), we restrict the function u and its competitors v to stay above the thin obstacle ψ on $\mathcal{M}$.

Definition 1.1.2 (Almost minimizer for the thin obstacle (or Signorini) problem)**.** *Given $r_0 > 0$ and a gauge function $\omega(r)$ on $(0, r_0)$, we say that $u \in W^{1,2}_{\mathrm{loc}}(D)$ is an* almost minimizer for the thin obstacle (*or* Signorini) problem*, if* $u \geq \psi$ *on* $\mathcal{M} \cap D$ *and, for any ball* $B_r(x_0) \Subset D$ *with* $0 < r < r_0$*, we have*

$$J_{B_r(x_0)}(u) \leq (1 + \omega(r)) J_{B_r(x_0)}(v), \quad \textit{for any } v \in \mathfrak{K}_{\psi,u}(B_r(x_0), \mathcal{M}). \tag{1.1.4}$$

Note that in the case when $\mathcal{M} \cap B_r(x_0) = \emptyset$, the condition (1.1.4) is the same as (1.1.3) and thus almost minimizers of the Signorini problem are almost harmonic in $D \setminus \mathcal{M}$. As in the case of the solutions of the Signorini problem, we are interested in the regularity properties of almost minimizers as well as the structure and the regularity of the free boundary $\Gamma(u) \subset \mathcal{M}$ as defined in (1.1.2).

Some examples of almost minimizers are given in Appendix 1.A. We would also like to mention here that a related notion of almost minimizers for the fractional obstacle problem has been considered by the authors in [3].

1.1.3 Main results

Because of the technical nature of the problem, in this chapter we restrict ourselves only to the case when $\omega(r) = r^\alpha$ for some $\alpha > 0$, $\mathcal{M}$ is flat, specifically $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$, and the thin obstacle $\psi = 0$. As we are mainly interested in local properties of almost minimizers and their free boundaries, we assume that D is the unit ball B_1, $u \in W^{1,2}(B_1)$, and the constant $r_0 = 1$ in Definition 1.1.2. We also assume that u is even in x_n-variable:

$$u(x', x_n) = u(x', -x_n), \quad \text{for any } x = (x', x_n) \in B_1.$$

Our first main result is then as follows.

Theorem A ($C^{1,\beta}$-regularity of almost minimizers)**.** *Let* u *be an almost minimizer for the Signorini problem in* B_1*, under the assumptions above. Then,* $u \in C^{1,\beta}_{\mathrm{loc}}(B_1^\pm \cup B_1')$ *for* $\beta = \beta(\alpha, n)$ *and*

$$\|u\|_{C^{1,\beta}(K)} \leq C \|u\|_{W^{1,2}(B_1)},$$

for any $K \Subset B_1^\pm \cup B_1'$ *and* $C = C(n, \alpha, K)$*.*

The proof is obtained by using Morrey and Campanato space estimates, following the original idea of Anzellotti [31]. However, in our case the proof is much more elaborate and, in a sense, based on the idea that the solutions of the Signorini problem are 2-valued harmonic functions, as we have to work with both even and odd extensions of u and ∇u from B_1^+ to B_1.

While the optimal regularity for the minimizer (or solutions) of the Signorini problem is $C^{1,1/2}$, we do not expect such regularity for almost minimizers. However, we are able to establish the optimal growth for almost minimizers, which then allows to study the local properties of the free boundary

$$\Gamma(u) = \partial\{u(\cdot,0)=0\} \cap B_1'.$$

Theorem B (Optimal growth near free boundary)**.** *Let u be as in Theorem A. Then,*

$$\int_{\partial B_r(x_0)} u^2 \le C(n,\alpha)\|u\|^2_{W^{1,2}(B_1)} r^{n+2},$$

for $x_0 \in B'_{1/2} \cap \Gamma(u)$, $0<r<r_0(n,\alpha)$.

One of the ingredients in the proof is an Almgren-type monotonicity formula, which we describe below. For an almost minimizer u, *Almgren's frequency* [47] is defined by

$$N(r,u,x_0) := \frac{r\int_{B_r(x_0)} |\nabla u|^2}{\int_{\partial B_r(x_0)} u^2}, \quad x_0 \in \Gamma(u).$$

It is one of the most important monotone quantities in the analysis of the free boundary for the Signorini problem, see e.g. Chapter 9 in [48]. We show that for almost minimizers a small modification of N is monotone.

Theorem C (Monotonicity of the truncated frequency)**.** *Let u be as in Theorem A. Then for any $\kappa_0 \ge 2$, there is $b = b(n,\alpha,\kappa_0)$ such that*

$$r \mapsto \widehat{N}_{\kappa_0}(r,u,x_0) := \min\left\{\frac{1}{1-br^\alpha} N(r,u,x_0), \kappa_0\right\}$$

is monotone for $x_0 \in B'_{1/2} \cap \Gamma(u)$*, and* $0 < r < r_0(n, \alpha, \kappa_0)$*. Moreover, we have that either*

$$\widehat{N}_{\kappa_0}(0+, u, x_0) = 3/2 \quad \textit{or} \quad \widehat{N}_{\kappa_0}(0+, u, x_0) \geq 2.$$

We give an indirect proof of this fact, based on an one-parametric family of Weiss-type energy functionals $\{W_\kappa\}_{0<\kappa<\kappa_0}$, see Theorem 1.5.1, that go back to the work [14] for the solutions of the Signorini problem and Weiss [49] for the classical obstacle problem. The fact that $\widehat{N} \geq 3/2$ at free boundary points is crucial for the proof of the optimal growth (Theorem B), however, the proof of Theorem B requires also an application of so-called *epiperimetric inequality* for Weiss's energy functional $W_{3/2}$ (see [20]), to remove a remaining logarithmic term.

Our next result concerns the subset of the free boundary

$$\mathcal{R}(u) := \{x_0 \in \Gamma(u) : \widehat{N}(0+, u, x_0) = 3/2\},$$

where Almgren's frequency is minimal, known as the *regular set* of u.

Theorem D (Regularity of the regular set)**.** *Let* u *be as in Theorem A. Then* $\mathcal{R}(u)$ *is a relatively open subset of the free boundary* $\Gamma(u)$ *and is a* $(n-2)$*-dimensional manifold of class* $C^{1,\gamma}$*.*

Our proof of this theorem is based on the use of the epiperimetric inequality and is similar to the one for the solutions of the Signorini problem in [20].

Finally, we state our main result for the so-called *singular set*. A free boundary point $x_0 \in \Gamma(u)$ is called *singular* if the *coincidence set* $\Lambda(u) := \{u(\cdot, 0) = 0\}$ has H^{n-1}-density zero at x_0, i.e.,

$$\lim_{r\to 0+} \frac{H^{n-1}(\Lambda(u) \cap B'_r(x_0))}{H^{n-1}(B'_r)} = 0.$$

If $\widehat{N}_{\kappa_0}(0+, u, x_0) = \kappa < \kappa_0$, then x_0 is singular if and only if $\kappa = 2m$, $m \in \mathbb{N}$ (see Proposition 1.10.1). For such κ, we then define

$$\Sigma_\kappa(u) := \{x_0 \in \Gamma(u) : \widehat{N}_{\kappa_0}(0+, u, x_0) = \kappa\}.$$

Theorem E (Structure of the singular set)**.** *Let u be as in Theorem A. Then, for any $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, $\Sigma_\kappa(u)$ is contained in a countable union of $(n-2)$-dimensional manifolds of class $C^{1,\log}$.*

A more refined version of this theorem is given in Theorem 1.10.10. The proof is based on the *logarithmic epiperimetric inequality* of Colombo-Spolaor-Velichkov [27] for Weiss's energy functional W_κ, with $\kappa = 2m$, $m \in \mathbb{N}$. We also point out that this inequality is instrumental in the proof of the optimal growth at singular points, which is rather immediate for the solutions of the Signorini problem, but far more complicated for the almost minimizers (see Lemmas 1.10.3–1.10.6).

Proofs of Theorems A–E

While we don't give formal proofs of Theorems A–E, in the main body of the chapter, they follow from the combination of results there. More specifically,

- Theorem A follows by combining Theorems 1.3.1 and 1.4.1.
- The statement of Theorem B is contained in that of Lemma 1.7.4.
- Theorem C follows by combining Theorem 1.5.4 and Corollary 1.9.2.
- The statement of Theorem D is contained in that of Theorem 1.9.5.
- The statement of Theorem E is contained in that of Theorem 1.10.10.

1.1.4 Notation

Throughout the thesis we use the following notation. $\mathbb{R}^n$ stands for the n-dimensional Euclidean space. We denote the points of $\mathbb{R}^n$ by $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. We routinely identify $x' \in \mathbb{R}^{n-1}$ with $(x', 0) \in \mathbb{R}^{n-1} \times \{0\}$. $\mathbb{R}^n_\pm$ stand for open halfspaces $\{x \in \mathbb{R}^n : \pm x_n > 0\}$.

For $x \in \mathbb{R}^n$, $r > 0$, we use the following notations for balls of radius r, centered at x.

$$\begin{aligned}
B_r(x) &= \{y \in \mathbb{R}^n : |x - y| < r\}, && \text{ball in } \mathbb{R}^n,\\
B_r^\pm(x') &= B_r(x', 0) \cap \{\pm x_n > 0\}, && \text{half-ball in } \mathbb{R}^n,\\
B'_r(x') &= B_r(x', 0) \cap \{x_n = 0\}, && \text{ball in } \mathbb{R}^{n-1}\text{, or thin ball.}
\end{aligned}$$

We typically drop the center from the notation if it is the origin. Thus, $B_r = B_r(0)$, $B'_r = B'_r(0)$, etc.

Next, for a direction $\mathrm{e} \in \mathbb{R}^n$, we denote

$$\partial_{\mathrm{e}} u = \nabla u \cdot \mathrm{e},$$

the directional derivative of u in the direction e. For the standard coordinate directions e_i, $i = 1, \ldots, n$, we simply write

$$u_{x_i} = \partial_{x_i} u = \partial_{\mathrm{e}_i} u.$$

Moreover, by $\partial^{\pm}_{x_n} u(x', 0)$ we mean the limit of $\partial_{x_n} u$ from within $B^{\pm}_r$, specifically,

$$\partial^{+}_{x_n} u(x', 0) = \lim_{\substack{y \to (x',0)\\ y \in B^+_r}} \partial_{x_n} u(y) = -\partial_{\nu^+} u(x', 0),$$

$$\partial^{-}_{x_n} u(x', 0) = \lim_{\substack{y \to (x',0)\\ y \in B^-_r}} \partial_{x_n} u(y) = \partial_{\nu^-} u(x', 0),$$

where $\nu^{\pm} = \mp \mathrm{e}_n$ are unit outward normal vectors for $B^{\pm}_r$ on B'_r.

In integrals, we often drop the variable and the measure of integration if it is with respect to the Lebesgue measure or the surface measure. Thus,

$$\int_{B_r} u = \int_{B_r} u(x)dx, \quad \int_{\partial B_r} u = \int_{\partial B_r} u(x) dS_x,$$

where S_x stands for the surface measure.

If E is a set of positive and finite Lebesgue measure, we indicate by $\langle u \rangle_E$ the integral mean value of a function u over E. That is,

$$\langle u \rangle_E := \fint_E u = \frac{1}{|E|} \int_E u.$$

In particular, we indicate by $\langle u \rangle_{x,r}$ the integral mean value of a function u over $B_r(x)$. That is,

$$\langle u \rangle_{x,r} := \fint_{B_r(x)} u = \frac{1}{\omega_n r^n} \int_{B_r(x)} u,$$

where $\omega_n = |B_1|$ is the volume of unit ball in $\mathbb{R}^n$. Similarly to the other notations, we drop the origin if it is 0 and write $\langle u\rangle_r$ for $\langle u\rangle_{0,r}$.

1.2 Almost harmonic functions

In this section we recall some results of Anzellotti [31] on almost harmonic functions, i.e., almost minimizers of the Dirichlet integral $J_D(v) = \int_D |\nabla v|^2$. We also state here some of the relevant auxiliary results that we will need also in the treatment of almost minimizers for the Signorini problem.

Theorem 1.2.1. *Let u be an almost harmonic function in an open set D with a gauge function ω. Then*

(i) *u is locally almost Lipschitz, i.e., $u \in C^{0,\sigma}_{\mathrm{loc}}(D)$ for all $\sigma \in (0,1)$.*

(ii) *If $\omega(r) \le Cr^\alpha$ for some $\alpha \in (0,2)$, then $u \in C^{1,\alpha/2}_{\mathrm{loc}}(D)$.*

While we refer to [31] for the full proof of this theorem, we would like to outline the key steps in Anzellotti's argument. The idea to prove $C^{0,\sigma}$ and $C^{1,\alpha/2}$ regularity of u is through the Morrey and Campanato space estimates, namely, by establishing that

$$\int_{B_\rho(x_0)} |\nabla u|^2 \le C\rho^{n-2+2\sigma} \tag{1.2.1}$$

$$\int_{B_\rho(x_0)} |\nabla u - \langle \nabla u\rangle_{x_0,\rho}|^2 \le C\rho^{n+\alpha} \tag{1.2.2}$$

for $x_0 \in K \Subset D$, and $0 < \rho < \rho_0$, with C and ρ_0 depending on n, r_0, $d = \operatorname{dist}(K, \partial D)$, the gauge function ω, and $\|u\|_{W^{1,2}(D)}$.

To obtain the estimates above, one starts by choosing a special competitor v in (1.1.3). Namely, we take $v = h$ which solves the Dirichlet problem

$$\Delta h = 0 \quad \text{in } B_r(x_0), \qquad h = u \quad \text{on } \partial B_r(x_0).$$

Equivalently, h is the minimizer of the Dirichlet energy $\int_{B_r(x_0)} |\nabla v|^2$ among all functions in $u + W^{1,2}_0(B_r(x_0))$. We call this h the *harmonic replacement* of u in $B_r(x_0)$. We then have the following concentric ball estimates for h.

Proposition 1.2.1. *Let h be harmonic in $B_r(x_0)$ and $0 < \rho < r$. Then*

$$\int_{B_\rho(x_0)} |\nabla h|^2 \le \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |\nabla h|^2 \tag{1.2.3}$$

$$\int_{B_\rho(x_0)} |\nabla h - \langle \nabla h\rangle_{x_0,\rho}|^2 \le \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |\nabla h - \langle \nabla h\rangle_{x_0,r}|^2. \tag{1.2.4}$$

Proof. The estimates above follow from the monotonicity in ρ of the quantities

$$\frac{1}{\rho^n}\int_{B_\rho(x_0)} |\nabla h|^2, \quad \frac{1}{\rho^{n+2}}\int_{B_\rho(x_0)} |\nabla h - \langle \nabla h\rangle_{x_0,\rho}|^2.$$

Noticing that $\langle \nabla h\rangle_{x_0,\rho} = \nabla h(x_0)$, an easy proof is obtained by decomposing h into the sum of the series of homogeneous harmonic polynomials. $\square$

We next use the almost minimizing property of u to deduce perturbed versions of the estimates above.

Proposition 1.2.2. *Let u be an almost harmonic function in D. Then for any ball $B_r(x_0) \Subset D$ with $r < r_0$ and $0 < \rho < r$ we have*

$$\int_{B_\rho(x_0)} |\nabla u|^2 \le 2\left[\left(\frac{\rho}{r}\right)^n + \omega(r)\right] \int_{B_r(x_0)} |\nabla u|^2 \tag{1.2.5}$$

$$\begin{aligned}\int_{B_\rho(x_0)} |\nabla u - \langle \nabla u\rangle_{x_0,\rho}|^2 \le 9\left(\frac{\rho}{r}\right)^{n+2} &\int_{B_r(x_0)} |\nabla u - \langle \nabla u\rangle_{x_0,r}|^2 \\ &+ 24\,\omega(r)\int_{B_r(x_0)} |\nabla u|^2.\end{aligned} \tag{1.2.6}$$

Proof. If h is a harmonic replacement of u in $B_r(x_0)$, we first note that

$$\begin{aligned}\int_{B_r(x_0)} |\nabla(u-h)|^2 &= \int_{B_r(x_0)} |\nabla u|^2 - |\nabla h|^2 - 2\int_{B_r(x_0)} \nabla h \nabla(u-h) \\ &= \int_{B_r(x_0)} |\nabla u|^2 - |\nabla h|^2 \le \omega(r)\int_{B_r(x_0)} |\nabla h|^2 \le \omega(r)\int_{B_r(x_0)} |\nabla u|^2.\end{aligned}$$

Then, combined with (1.2.3), we estimate

$$\int_{B_\rho(x_0)} |\nabla u|^2 \le 2\int_{B_\rho(x_0)} |\nabla h|^2 + 2\int_{B_\rho(x_0)} |\nabla(u-h)|^2$$

$$\leq 2\left[\left(\frac{\rho}{r}\right)^n + \omega(r)\right]\int_{B_r(x_0)} |\nabla u|^2,$$

which gives (1.2.5). To obtain (1.2.6), we argue very similarly by using additionally that by Jensen's inequality

$$\int_{B_\rho(x_0)} |\langle \nabla u\rangle_{x_0,\rho} - \langle \nabla h\rangle_{x_0,\rho}|^2 \leq \int_{B_\rho(x_0)} |\nabla u - \nabla h|^2.$$

For more details we refer to the proof of Theorem 1.4.1, Case 1.1. $\square$

From here, one deduces the estimates (1.2.1)–(1.2.2) with the help of the following useful lemma. The proof can be found e.g. in [50].

Lemma 1.2.2. *Let $r_0 > 0$ be a positive number and let $\varphi : (0, r_0) \to (0, \infty)$ be a nondecreasing function. Let a, β, and γ be such that $a > 0$, $\gamma > \beta > 0$. There exist two positive numbers $\varepsilon = \varepsilon(a, \gamma, \beta)$, $c = c(a, \gamma, \beta)$ such that, if*

$$\varphi(\rho) \leq a\left[\left(\frac{\rho}{r}\right)^\gamma + \varepsilon\right]\varphi(r) + b\, r^\beta$$

for all ρ, r with $0 < \rho \leq r < r_0$, where $b \geq 0$, then one also has, still for $0 < \rho < r < r_0$,

$$\varphi(\rho) \leq c\left[\left(\frac{\rho}{r}\right)^\beta \varphi(r) + b\rho^\beta\right].$$

We can now give a formal proof of Theorem 1.2.1.

Proof of Theorem 1.2.1. (i) Taking r_0 small enough so that $\omega(r_0) < \varepsilon$, a direct application of Lemma 1.2.2 to (1.2.5) produces the estimate (1.2.1), which in turn implies that $u \in C^{0,\sigma}_{\text{loc}}(D)$, by the Morrey space embedding theorem.

(ii) Using that $\omega(r) \leq Cr^\alpha$, combined with the estimate (1.2.1), we first obtain

$$\int_{B_\rho(x_0)} |\nabla u - \langle \nabla u\rangle_{x_0,\rho}|^2 \leq 9\left(\frac{\rho}{r}\right)^{n+2}\int_{B_r(x_0)} |\nabla u - \langle \nabla u\rangle_{x_0,r}|^2 + Cr^{n-2+2\sigma+\alpha}.$$

If σ is so that $\alpha' = -2 + 2\sigma + \alpha > 0$, Lemma 1.2.2 implies that

$$\int_{B_\rho(x_0)} |\nabla u - \langle \nabla u \rangle_{x_0,\rho}|^2 \leq C\rho^{n+\alpha'}.$$

By the Campanato space embedding, we therefore obtain that $\nabla u \in C^{0,\alpha'/2}_{\mathrm{loc}}(D)$. However, it is easy to bootstrap the regularity up to $C^{0,\alpha/2}_{\mathrm{loc}}$ by noticing that we now know that ∇u is locally bounded in D and thus $\int_{B_r(x_0)} |\nabla u|^2 \leq Cr^n$. Plugging that in the last term of (1.2.6), we obtain that

$$\int_{B_\rho(x_0)} |\nabla u - \langle \nabla u \rangle_{x_0,\rho}|^2 \leq 9 \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |\nabla u - \langle \nabla u \rangle_{x_0,r}|^2 + Cr^{n+\alpha}$$

and repeating the arguments above conclude that $u \in C^{1,\alpha/2}_{\mathrm{loc}}$. $\square$

1.3 Almost Lipschitz regularity of almost minimizers

In this section we prove the first regularity results for the almost minimizers for the Signorini problem, see Definition 1.1.2. Recall that we assume $D = B_1$, $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$, $\psi = 0$, $r_0 = 1$, and $\omega(r) = r^\alpha$ for some $\alpha > 0$. Furthermore we assume that u is even symmetric in x_n-variable.

Theorem 1.3.1. *Let u be an almost minimizer for the Signorini problem in B_1. Then $u \in C^{0,\sigma}(B_1)$ for all $0 < \sigma < 1$. Moreover, for any $K \Subset B_1$,*

$$\|u\|_{C^{0,\sigma}(K)} \leq C\|u\|_{W^{1,2}(B_1)} \tag{1.3.1}$$

with $C = C(n, \alpha, \sigma, K)$.

The idea of the proof is to follow that of Anzellotti [31] that we outlined in Section 1.2 and to prove an estimate similar to (1.2.5). The proof of the latter estimate followed by a perturbation argument from a similar estimate for the harmonic replacement of u. However, in the case of the Signorini problem, the harmonic replacements are not necessarily admissible competitors. Instead, for $B_r(x_0) \Subset B_1$, we consider the *Signorini replacements* h of u in $B_r(x_0)$, which solve the Signorini problem in $B_r(x_0)$ with the thin obstacle 0 on $\mathcal{M}$

and boundary values $h = u$ on $\partial B_r(x_0)$. Equivalently, Signorini replacements are the minimizers of $J_{B_r(x_0)}$ on the constraint set $\mathfrak{K}_{0,u}(B_r(x_0), \mathcal{M})$ and they also satisfy the variational inequality[2]

$$\int_{B_r(x_0)} \nabla h \cdot \nabla(v - h) \geq 0 \quad \text{for any } v \in \mathfrak{K}_{0,u}(B_r(x_0), \mathcal{M}). \tag{1.3.2}$$

We then have the following concentric ball estimates for Signorini replacements similar to the one for harmonic replacements, at least when the center of the balls is on $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$.

Proposition 1.3.1. *Let $x_0 \in \mathcal{M}$ and let h be a solution of the Signorini problem in $B_r(x_0)$ with zero obstacle on $\mathcal{M}$, even in x_n-variable. Then,*

$$\int_{B_\rho(x_0)} |\nabla h|^2 \leq \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |\nabla h|^2, \quad 0 < \rho < r. \tag{1.3.3}$$

Proof. We claim that $|\nabla h|^2$ is subharmonic in $B_r(x_0)$. This follows from the fact that $h^\pm_{x_i}$, $\mathrm{i} = 1, \ldots, n-1$, are subharmonic in $B_r(x_0)$, see [48], and similarly that the even extensions $\widetilde{h}^\pm_{x_n}$ of $h^\pm_{x_n}$ in x_n-variable from $B^+_R(x_0)$ to all of $B_R(x_0)$ are also subharmonic. These are all consequences of the fact that a continuous nonnegative function, subharmonic in its positivity set is subharmonic, see Ex. 2.6 in [48].

The subharmonicity of $|\nabla h|^2$ in $B_r(x_0)$ then implies, by the sub-mean value property, that the function

$$\rho \mapsto \frac{1}{\rho^n} \int_{B_\rho(x_0)} |\nabla h|^2$$

is monotone nondecreasing. This readily implies (1.3.3). □

We next have the perturbed version of Proposition 1.3.1.

Proposition 1.3.2. *Let u be an almost minimizer for the Signorini problem in B_1, and $B_r(x_0) \subset B_1$. Then, there is $C_1 = C_1(n) > 1$ such that*

$$\int_{B_\rho(x_0)} |\nabla u|^2 \leq C_1 \left[\left(\frac{\rho}{r}\right)^n + r^\alpha\right] \int_{B_r(x_0)} |\nabla u|^2, \quad 0 < \rho < r. \tag{1.3.4}$$

[2] ↑which follows from the inequality $\int_{B_r(x_0)} |\nabla h|^2 \leq \int_{B_r(x_0)} |\nabla((1-\varepsilon)h + \varepsilon v)|^2$, $\varepsilon \in (0,1)$ by a first variation argument.

Proof. By using the continuity argument, we may assume that $B_r(x_0) \Subset B_1$. We first prove the estimate when x_0 is in the thin space, i.e., $x_0 \in B_1'$ and then extend it to arbitrary $x_0 \in B_1$.

Case 1. Suppose $x_0 \in B_1'$ and let h be the Signorini replacement of u in $B_r(x_0)$. Recall that h satisfies (1.3.2). Then, plugging $v = u$, we obtain

$$\int_{B_r(x_0)} \nabla h \cdot \nabla u - |\nabla h|^2 \geq 0. \tag{1.3.5}$$

Using this, we can estimate

$$\begin{aligned} \int_{B_r(x_0)} |\nabla(u-h)|^2 &= \int_{B_r(x_0)} \Big(|\nabla u|^2 + |\nabla h|^2 - 2\nabla u \cdot \nabla h \Big) \\ &\leq \int_{B_r(x_0)} |\nabla u|^2 - \int_{B_r(x_0)} |\nabla h|^2 \\ &\leq \Big(1 + r^\alpha\Big) \int_{B_r(x_0)} |\nabla h|^2 - \int_{B_r(x_0)} |\nabla h|^2 \\ &= r^\alpha \int_{B_r(x_0)} |\nabla h|^2 \leq r^\alpha \int_{B_r(x_0)} |\nabla u|^2, \end{aligned} \tag{1.3.6}$$

where in the very last step we have used that h minimizes the Dirichlet integral among all functions in $\mathfrak{K}_{0,u}(B_r(x_0), \mathcal{M})$.

Next, we use the same perturbation argument as in the proof of (1.2.5). By using (1.3.3) and (1.3.6), we estimate

$$\begin{aligned} \int_{B_\rho(x_0)} |\nabla u|^2 &\leq 2 \int_{B_\rho(x_0)} |\nabla h|^2 + 2 \int_{B_\rho(x_0)} |\nabla(u-h)|^2 \\ &\leq 2 \Big(\frac{\rho}{r}\Big)^n \int_{B_r(x_0)} |\nabla h|^2 + 2r^\alpha \int_{B_r(x_0)} |\nabla u|^2 \\ &\leq 2 \Big[\Big(\frac{\rho}{r}\Big)^n + r^\alpha \Big] \int_{B_r(x_0)} |\nabla u|^2. \end{aligned}$$

Thus, (1.3.4) follows in this case.

Case 2. Consider now the case $x_0 \in B_1^+$. If $\rho \geq r/4$, then we simply have

$$\int_{B_\rho(x_0)} |\nabla u|^2 \leq 4^n \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |\nabla u|^2.$$

Thus, we may assume $\rho < r/4$. Then, let $d := \mathrm{dist}(x_0, B_1') > 0$ and choose $x_1 \in \partial B_d(x_0) \cap B_1'$.

Case 2.1. If $\rho \geq d$, then we use $B_\rho(x_0) \subset B_{2\rho}(x_1) \subset B_{r/2}(x_1) \subset B_r(x_0)$ and the result of Case 1 to write

$$\begin{aligned}\int_{B_\rho(x_0)} |\nabla u|^2 \leq \int_{B_{2\rho}(x_1)} |\nabla u|^2 &\leq C\left[\left(\frac{2\rho}{r/2}\right)^n + (r/2)^\alpha\right]\int_{B_{r/2}(x_1)} |\nabla u|^2 \\ &\leq C\left[\left(\frac{\rho}{r}\right)^n + r^\alpha\right]\int_{B_r(x_0)} |\nabla u|^2.\end{aligned}$$

Case 2.2. Suppose now $d > \rho$. If $d > r$, then $B_r(x_0) \Subset B_1^+$. Since u is almost harmonic in B_1^+, we can apply Proposition 1.2.2 to obtain

$$\int_{B_\rho(x_0)} |\nabla u|^2 \leq 2\left[\left(\frac{\rho}{r}\right)^n + r^\alpha\right]\int_{B_r(x_0)} |\nabla u|^2.$$

Thus, we may assume $d \leq r$. Then we note that $B_d(x_0) \subset B_1^+$ and by a limiting argument from the previous estimate, we obtain

$$\int_{B_\rho(x_0)} |\nabla u|^2 \leq 2\left[\left(\frac{\rho}{d}\right)^n + r^\alpha\right]\int_{B_d(x_0)} |\nabla u|^2.$$

Case 2.2.1. If $r/4 \leq d$, then

$$\int_{B_d(x_0)} |\nabla u|^2 \leq 4^n\left(\frac{d}{r}\right)^n\int_{B_r(x_0)} |\nabla u|^2,$$

which immediately implies (1.3.4).

Case 2.2.2. It remains to consider the case $\rho < d < r/4$. Using Case 1 again, we have

$$\begin{aligned}\int_{B_d(x_0)} |\nabla u|^2 \leq \int_{B_{2d}(x_1)} |\nabla u|^2 &\leq C\left[\left(\frac{2d}{r/2}\right)^n + (r/2)^\alpha\right]\int_{B_{r/2}(x_1)} |\nabla u|^2 \\ &\leq C\left[\left(\frac{d}{r}\right)^n + r^\alpha\right]\int_{B_r(x_0)} |\nabla u|^2,\end{aligned}$$

which also implies (1.3.4). This concludes the proof of the proposition. □

We can now give the proof of the almost Lipschitz regularity of almost minimizers.

Proof of Theorem 1.3.1. Let $K \Subset B_1$ and $x_0 \in K$. Take $\delta = \delta(n, \alpha, \sigma, K) > 0$ such that $\delta < \operatorname{dist}(K, \partial B_1)$ and $\delta^\alpha \le \varepsilon(C_1, n, n + 2\sigma - 2)$, where $\varepsilon = \varepsilon(C_1, n, n + 2\sigma - 2)$ is as in Lemma 1.2.2. Then for all $0 < \rho < r < \delta$, by (1.3.4),

$$\int_{B_\rho(x_0)} |\nabla u|^2 \le C_1 \left[\left(\frac{\rho}{r} \right)^n + \varepsilon \right] \int_{B_r(x_0)} |\nabla u|^2.$$

By applying Lemma 1.2.2, we obtain

$$\int_{B_\rho(x_0)} |\nabla u|^2 \le C(n, \sigma) \left(\frac{\rho}{r} \right)^{n+2\sigma-2} \int_{B_r(x_0)} |\nabla u|^2.$$

Taking $r \nearrow \delta$, we can therefore conclude

$$\int_{B_\rho(x_0)} |\nabla u|^2 \le C(n, \alpha, \sigma, K) \|\nabla u\|^2_{L^2(B_1)} \rho^{n+2\sigma-2}. \tag{1.3.7}$$

From here, we use the Morrey space embedding to obtain $u \in C^{0,\sigma}(K)$ with the norm estimate

$$\|u\|_{C^{0,\sigma}(K)} \le C(n, \alpha, \sigma, K) \|u\|_{W^{1,2}(B_1)},$$

as required. $\square$

1.4 $C^{1,\beta}$ regularity of almost minimizers

In this section we establish the $C^{1,\beta}$ regularity of almost minimizers for some $\beta > 0$. The idea is again to use Signorini replacements and an appropriate version of the concentric ball estimate (1.2.4) for solutions of the Signorini problem.

As we saw in the proof of the almost Lipschitz regularity of almost minimizers, it is enough to obtain such estimates when balls are centered at x_0 on the thin space $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$. It turns out that to prove a proper version of (1.2.4), we have to work with both even and odd extensions in x_n-variable of Signorini replacements h from $B_r^+(x_0)$ to $B_r(x_0)$. The reason is that even extensions are harmonic across the positivity set $\{h(\cdot, 0) > 0\}$, while the odd extensions are harmonic across the interior of the coincidence set $\{h(\cdot, 0) = 0\}$.

Proposition 1.4.1. *Let h be a solution of the Signorini problem in $B_r(x_0)$ with $x_0 \in \mathcal{M}$, even in x_n-variable. Define*

$$\widehat{\nabla h} := \begin{cases} \nabla h(x', x_n), & x_n \geq 0 \\ \nabla h(x', -x_n), & x_n < 0, \end{cases}$$

the even extension of ∇h from $B_r^+(x_0)$ to $B_r(x_0)$. Then for $0 < \alpha < 1$, there is $C = C(n)$ such that for all $0 < \rho \leq (3/4)r$,

$$\int_{B_\rho(x_0)} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{x_0,\rho}|^2 \leq C \, \frac{\rho^{n+1}}{r^{n+3}} \int_{B_r(x_0)} h^2.$$

Proof. This is an immediate corollary of the estimate

$$\|\nabla h\|_{C^{0,1/2}\left(B^{\pm}_{(3/4)r}(x_0) \cup B_{(3/4)r}(x_0)\right)} \leq C(n) r^{-\frac{n+3}{2}} \|h\|_{L^2(B_r^+(x_0))}, \tag{1.4.1}$$

see e.g. Theorem 9.13 in [48]. Indeed, for $0 < \rho \leq (3/4)r$, we have

$$\begin{aligned} \int_{B_\rho(x_0)} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{x_0,\rho}|^2 &= 2 \int_{B_\rho^+(x_0)} |\nabla h - \langle \nabla h \rangle_{B_r^+(x_0)}|^2 \\ &\leq C(n) \rho^{n+1} \|\nabla h\|^2_{C^{0,1/2}(B^+_{(3/4)r}(x_0))} \\ &\leq C(n) \frac{\rho^{n+1}}{r^{n+3}} \int_{B_r(x_0)} h^2. \end{aligned}$$

□

We now prove the $C^{1,\beta}$ regularity of almost minimizers.

Theorem 1.4.1. *Let u be an almost minimizer of the Signorini problem in B_1. Define*

$$\widehat{\nabla u}(x', x_n) := \begin{cases} \nabla u(x', x_n), & x_n \geq 0 \\ \nabla u(x', -x_n), & x_n < 0. \end{cases}$$

Then

$$\widehat{\nabla u} \in C^{0,\beta}(B_1) \quad \textit{with } \beta = \frac{\alpha}{4(2n + \alpha)}.$$

Moreover, for any $K \Subset B_1$ there holds

$$\|\widehat{\nabla u}\|_{C^{0,\beta}(K)} \leq C(n,\alpha,K)\|u\|_{W^{1,2}(B_1)}. \tag{1.4.2}$$

Proof. Without loss of generality, we may assume that K is a ball centered at 0. Fix a small $r_0 = r_0(n,\alpha,K) > 0$ to be chosen later. Particularly, we will ask $R_0 := r_0^{\frac{2n}{2n+\alpha}} \leq (1/2)\operatorname{dist}(K,\partial B_1)$, which will imply that

$$\widetilde{K} := \{x \in B_1 : \operatorname{dist}(x,K) \leq R_0\} \Subset B_1.$$

Our goal now is to show that for $x_0 \in K$, $0 < \rho < r < r_0$,

$$\begin{aligned}\int_{B_\rho(x_0)} |\widehat{\nabla u} - \langle\widehat{\nabla u}\rangle_{x_0,\rho}|^2 \leq C(n,\alpha)\left(\frac{\rho}{r}\right)^{n+\alpha}\int_{B_r(x_0)} |\widehat{\nabla u} - \langle\widehat{\nabla u}\rangle_{x_0,r}|^2 \\ + C(n,\alpha,K)\|u\|^2_{W^{1,2}(B_1)}r^{n+2\beta}, \quad (1.4.3)\end{aligned}$$

which readily gives the estimate (1.4.2) by applying Lemma 1.2.2 and using the Campanato space embedding.

We first prove (1.4.3) for $x_0 \in K \cap B_1'$, by taking the advantage of the symmetry of $\widehat{\nabla u}$, and then we argue as in the proof of Proposition 1.3.2 to extend it to all $x_0 \in K$.

Case 1. Suppose $x_0 \in K \cap B_1'$. For notational simplicity, we assume $x_0 = 0$ (by shifting the center of the domain $D = B_1$ to $-x_0$) and let $0 < r < r_0$ be given. Let us also denote

$$\alpha' := 1 - \frac{\alpha}{8n} \in (0,1), \quad R := r^{\frac{2n}{2n+\alpha}}.$$

We then split our proof into two cases:

$$\sup_{\partial B_R} |u| \leq C_3 R^{\alpha'} \quad \text{and} \quad \sup_{\partial B_R} |u| > C_3 R^{\alpha'},$$

with $C_3 = 2[u]_{0,\alpha',\widetilde{K}} = 2\sup_{\substack{x,y\in\widetilde{K}\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha'}}$.

Case 1.1. Assume first that $\sup_{\partial B_R} |u| \le C_3 R^{\alpha'}$. Let h be the Signorini replacement of u on B_R. Then, for any $0 < \rho < r$, we have

$$\int_{B_\rho} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_\rho|^2 \le 3 \int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_\rho|^2 + 3 \int_{B_\rho} |\widehat{\nabla u} - \widehat{\nabla h}|^2 + 3 \int_{B_\rho} |\langle \widehat{\nabla u} \rangle_\rho - \langle \widehat{\nabla h} \rangle_\rho|^2.$$

Besides, by Jensen's inequality, we have

$$\int_{B_\rho} |\langle \widehat{\nabla u} \rangle_\rho - \langle \widehat{\nabla h} \rangle_\rho|^2 \le \int_{B_\rho} |\widehat{\nabla u} - \widehat{\nabla h}|^2.$$

Hence, combining the estimates above, we obtain

$$\int_{B_\rho} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_\rho|^2 \le 3 \int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_\rho|^2 + 6 \int_{B_\rho} |\widehat{\nabla u} - \widehat{\nabla h}|^2. \tag{1.4.4}$$

Similarly

$$\int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_r|^2 \le 3 \int_{B_r} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_r|^2 + 6 \int_{B_r} |\widehat{\nabla u} - \widehat{\nabla h}|^2. \tag{1.4.5}$$

Next, note that if $r_0 \le (3/4)^{\frac{2n+\alpha}{\alpha}}$, then $r \le (3/4)R$, and thus by Proposition 1.4.1,

$$\int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_\rho|^2 \le C(n,\alpha) \frac{\rho^{n+1}}{R^{n+3}} \int_{B_R} h^2. \tag{1.4.6}$$

Then, using (1.4.4), (1.4.5), and (1.4.6), we obtain

$$\begin{aligned}\int_{B_\rho} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_\rho|^2 &\le 3 \int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_\rho|^2 + 6 \int_{B_\rho} |\widehat{\nabla u} - \widehat{\nabla h}|^2 \\ &\le C(n,\alpha) \frac{\rho^{n+1}}{R^{n+3}} \int_{B_R} h^2 + 6 \int_{B_\rho} |\widehat{\nabla u} - \widehat{\nabla h}|^2.\end{aligned} \tag{1.4.7}$$

Now take $\delta = \delta(n,\alpha,K) > 0$ such that $\delta < \operatorname{dist}(K, \partial B_1)$ and $\delta^\alpha \le \varepsilon = \varepsilon(C_1, n, n + 2\alpha' - 2)$, where C_1 is as in Theorem 1.3.1 and ε is as in Lemma 1.2.2. If $r_0 \le \delta^{\frac{2n+\alpha}{2n}}$, then $R < \delta$, and therefore by (1.3.7),

$$\int_{B_R} |\widehat{\nabla u}|^2 \le C(n,\alpha,K) \|\nabla u\|^2_{L^2(B_1)} R^{n+2\alpha'-2}.$$

Thus, using the above inequality, combined with (1.3.5), we obtain

$$
\begin{aligned}
\int_{B_R} |\widehat{\nabla u} - \widehat{\nabla h}|^2 &\leq \int_{B_R} |\widehat{\nabla u}|^2 - \int_{B_R} |\widehat{\nabla h}|^2 \\
&\leq R^\alpha \int_{B_R} |\widehat{\nabla h}|^2 \leq R^\alpha \int_{B_R} |\widehat{\nabla u}|^2 \\
&\leq C(n,\alpha,K)\|\nabla u\|^2_{L^2(B_1)} R^{n+\alpha+2\alpha'-2} \\
&= C(n,\alpha,K)\|\nabla u\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2n+\alpha}(n-\frac{1}{2})}.
\end{aligned}
\tag{1.4.8}
$$

We next use that h^2 is subharmonic in B_R. (This can be seen for instance by a direct computation $\Delta(h^2) = 2\left(|\nabla h|^2 + h\Delta h\right) = 2|\nabla h|^2 \geq 0$, or by using the fact that $h^\pm$ are subharmonic.) Then,

$$
\langle h^2 \rangle_R \leq \sup_{B_R} h^2 = \sup_{\partial B_R} h^2 = \sup_{\partial B_R} u^2 \leq C_3^2 R^{2\alpha'}. \tag{1.4.9}
$$

Also note that by (1.3.1), $C_3 \leq C(n,\alpha,K)\|u\|_{W^{1,2}(B_1)}$. Hence,

$$
\frac{r^{n+1}}{R^{n+3}} \int_{B_R} h^2 = C(n)\,\frac{r^{n+1}}{R^3} \langle h^2 \rangle_R \leq C(n,\alpha,K)\|u\|^2_{W^{1,2}(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}. \tag{1.4.10}
$$

Now (1.4.7), (1.4.8), (1.4.10) give

$$
\begin{aligned}
\int_{B_\rho} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_\rho|^2 &\leq C(n,\alpha,K)\|u\|^2_{W^{1,2}(B_1)} \left(r^{n+\frac{\alpha}{2(2n+\alpha)}} + r^{n+\frac{\alpha}{2n+\alpha}(n-\frac{1}{2})} \right) \\
&\leq C(n,\alpha,K)\|u\|^2_{W^{1,2}(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}.
\end{aligned}
\tag{1.4.11}
$$

Case 1.2. Now suppose $\sup_{\partial B_R} |u| > C_3 R^{\alpha'}$. By the choice of $C_3 = 2[u]_{0,\alpha,\widetilde{K}}$, we have either $u \geq (C_3/2)R^{\alpha'}$ in all of B_R or $u \leq -(C_3/2)R^{\alpha'}$ in all of B_R. However, from the inequality $u(0) \geq 0$, the only possibility is

$$
u \geq \frac{C_3}{2} R^{\alpha'} \quad \text{in } B_R.
$$

Let h again be the Signorini replacement of u in B_R. Then from positivity of $h = u > 0$ on ∂B_R and superharmonicity of h in B_R, it follows that $h > 0$ in B_R and is therefore harmonic there. Thus,

$$\int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 \leq \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\nabla h - \langle \nabla h \rangle_r|^2, \quad 0 < \rho < r.$$

We now want to obtain a version of this estimate for $\widehat{\nabla h}$. We start by observing that ∇h and $\widehat{\nabla h}$ differ only in the n-th component. The n-th component of ∇h, h_{x_n}, is odd in x_n. On the other hand, the n-th component of $\widehat{\nabla h}$, is even in x_n and is given by

$$\widehat{h_{x_n}}(x) = \begin{cases} h_{x_n}(x', x_n), & x_n \geq 0 \\ h_{x_n}(x', -x_n), & x_n < 0 \end{cases}.$$

Then we have

$$\int_{B_\rho} |\widehat{h_{x_n}} - \langle \widehat{h_{x_n}} \rangle_\rho|^2 = \int_{B_\rho} \widehat{h_{x_n}}^2 - \frac{1}{|B_\rho|}\left(\int_{B_\rho} \widehat{h_{x_n}}\right)^2 = \int_{B_\rho} |h_{x_n} - \langle h_{x_n} \rangle_\rho|^2 - \frac{1}{|B_\rho|}\left(\int_{B_\rho} \widehat{h_{x_n}}\right)^2,$$

where we have used that $\langle h_{x_n} \rangle_\rho = 0$. Hence, we arrive at

$$\int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_\rho|^2 = \int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 - \frac{1}{|B_\rho|}\left(\int_{B_\rho} \widehat{h_{x_n}}\right)^2. \tag{1.4.12}$$

Similarly, we have

$$\int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_r|^2 = \int_{B_r} |\nabla h - \langle \nabla h \rangle_r|^2 - \frac{1}{|B_r|}\left(\int_{B_r} \widehat{h_{x_n}}\right)^2. \tag{1.4.13}$$

Now, using (1.4.12) and (1.4.13), we have for all $0 < \rho < r$

$$\begin{aligned} \int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_\rho|^2 &\leq \int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 \leq \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\nabla h - \langle \nabla h \rangle_r|^2 \\ &\leq \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_r|^2 + \frac{1}{|B_r|}\left(\int_{B_r} \widehat{h_{x_n}}\right)^2. \end{aligned} \tag{1.4.14}$$

Next, note that if $r_0 \le (1/2)^{\frac{2n+\alpha}{\alpha}}$, then $r \le R/2$. Then, for $\gamma := 1 - \frac{3\alpha}{8n}$,

$$\begin{aligned}\sup_{B_{R/2}} |D^2 h| &\le \frac{C(n)}{R} \sup_{B_{(3/4)R}} |\nabla h| \le \frac{C(n)}{R^{1+\frac{n}{2}}} \left(\int_{B_R} |\nabla h|^2 \right)^{1/2} \\ &\le \frac{C(n)}{R^{1+\frac{n}{2}}} \left(\int_{B_R} |\nabla u|^2 \right)^{1/2} \le C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)} R^{\gamma-2},\end{aligned}$$

where the last inequality follows from (1.3.7). Thus, for $x = (x', x_n) \in B_r$, we have

$$\begin{aligned}|h_{x_n}| &\le |x_n| \sup_{B_{R/2}} |D^2 h| \le C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)} r R^{\gamma-2} \\ &\le C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)} r^{1+\frac{2n}{2n+\alpha}(\gamma-2)},\end{aligned}$$

and hence

$$\begin{aligned}\frac{1}{|B_r|} \left(\int_{B_r} \widehat{h_{x_n}} \right)^2 &\le C(n, \alpha, K) \|\nabla u\|^2_{L^2(B_1)} r^{n+2+\frac{4n}{2n+\alpha}(\gamma-2)} \\ &= C(n, \alpha, K) \|\nabla u\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}.\end{aligned} \tag{1.4.15}$$

Combining (1.4.14) and (1.4.15), we obtain

$$\int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_\rho|^2 \le \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_r|^2 + C(n, \alpha, K) \|\nabla u\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}. \tag{1.4.16}$$

Finally, (1.4.4), (1.4.5), (1.4.8), and (1.4.16) give

$$\begin{aligned}\int_{B_\rho} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_\rho|^2 &\le 3 \int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_\rho|^2 + 6 \int_{B_\rho} |\widehat{\nabla u} - \widehat{\nabla h}|^2 \\ &\le 3 \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_r|^2 \\ &\qquad + C(n, \alpha, K) \|\nabla u\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}} + 6 \int_{B_\rho} |\widehat{\nabla u} - \widehat{\nabla h}|^2 \\ &\le 9 \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_r|^2 \\ &\qquad + C(n, \alpha, K) \|\nabla u\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}} + 24 \int_{B_r} |\widehat{\nabla u} - \widehat{\nabla h}|^2\end{aligned}$$

$$
\begin{aligned}
&\le 9\Big(\frac{\rho}{r}\Big)^{n+2}\int_{B_r}|\widehat{\nabla u}-\langle\widehat{\nabla u}\rangle_r|^2 + C(n,\alpha,K)\|\nabla u\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}\\
&\qquad + C(n,\alpha,K)\|\nabla u\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2n+\alpha}(n-\frac{1}{2})}\\
&\le 9\Big(\frac{\rho}{r}\Big)^{n+2}\int_{B_r}|\widehat{\nabla u}-\langle\widehat{\nabla u}\rangle_r|^2 + C(n,\alpha,K)\|\nabla u\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}.
\end{aligned}
$$

From this and (1.4.11) we obtain (1.4.3) for $x_0 \in K \cap B_1$.

Case 2. To extend (1.4.3) to any $x_0 \in K$, we now assume $x_0 \in K \cap B_1^+$. We use an argument similar to the one in Case 2 in the proof of Proposition 1.3.2.

Now, if $\rho \ge r/4$, then

$$
\int_{B_\rho(x_0)}|\widehat{\nabla u}-\langle\widehat{\nabla u}\rangle_{x_0,\rho}|^2 \le \int_{B_\rho(x_0)}|\widehat{\nabla u}-\langle\widehat{\nabla u}\rangle_{x_0,r}|^2 \le 4^{n+\alpha}\left(\frac{\rho}{r}\right)^{n+\alpha}\int_{B_r(x_0)}|\widehat{\nabla u}-\langle\widehat{\nabla u}\rangle_{x_0,r}|^2,
$$

and thus we may assume $\rho < r/4$. Let $d := \operatorname{dist}(x_0, B_1') > 0$ and choose $x_1 \in \partial B_d(x_0) \cap B_1'$. Note that from the assumption that K is a ball centered at 0, we have $x_1 \in K \cap B_1'$.

Case 2.1. If $\rho \ge d$, then from $B_\rho(x_0) \subset B_{2\rho}(x_1) \subset B_{r/2}(x_1) \subset B_r(x_0)$, we have

$$
\begin{aligned}
\int_{B_\rho(x_0)}|\widehat{\nabla u}-\langle\widehat{\nabla u}\rangle_{x_0,\rho}|^2 &\le \int_{B_{2\rho}(x_1)}|\widehat{\nabla u}-\langle\widehat{\nabla u}\rangle_{x_1,2\rho}|^2\\
&\le C(n,\alpha)\left(\frac{\rho}{r}\right)^{n+\alpha}\int_{B_{r/2}(x_1)}|\widehat{\nabla u}-\langle\widehat{\nabla u}\rangle_{x_1,r/2}|^2\\
&\qquad\qquad + C(n,\alpha,K)\|u\|^2_{W^{1,2}(B_1)} r^{n+2\beta}\\
&\le C(n,\alpha)\left(\frac{\rho}{r}\right)^{n+\alpha}\int_{B_r(x_0)}|\widehat{\nabla u}-\langle\widehat{\nabla u}\rangle_{x_0,r}|^2\\
&\qquad\qquad + C(n,\alpha,K)\|u\|^2_{W^{1,2}(B_1)} r^{n+2\beta},
\end{aligned}
$$

which gives (1.4.3) in this case.

Case 2.2. Now we suppose $d > \rho$. If also $d > r$, then $B_r(x_0) \subset B_1^+$ and since u is almost harmonic in B_1^+, we can apply Proposition 1.2.2, together with the growth estimate (1.3.7) in the proof of Theorem 1.3.1, to conclude

$$
\int_{B_\rho(x_0)}|\widehat{\nabla u}-\langle\widehat{\nabla u}\rangle_{x_0,\rho}|^2 \le C(n,\alpha)\left(\frac{\rho}{r}\right)^{n+\alpha}\int_{B_r(x_0)}|\widehat{\nabla u}-\langle\widehat{\nabla u}\rangle_{x_0,r}|^2
$$

$$+ C(n,\alpha,K)\|u\|^2_{W^{1,2}(B_1)} r^{n+2\beta}.$$

Thus, we may assume $d \le r$. Then, $B_d(x_0) \subset B_1^+$, and hence, again by the combination of Proposition 1.2.2 and the growth estimate (1.3.7), we have

$$\int_{B_\rho(x_0)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_0,\rho}|^2 \le C(n,\alpha) \left(\frac{\rho}{d}\right)^{n+\alpha} \int_{B_d(x_0)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_0,d}|^2$$
$$+ C(n,\alpha,K)\|u\|^2_{W^{1,2}(B_1)} d^{n+2\beta}.$$

We need to consider further subcases.

Case 2.2.1. If $r/4 \le d$, then (since also $d \le r$)

$$\int_{B_d(x_0)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_0,d}|^2 \le 4^{n+\alpha} \left(\frac{d}{r}\right)^{n+\alpha} \int_{B_r(x_0)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_0,r}|^2$$

and combined with the previous inequality, we obtain (1.4.3) in this subcase.

Case 2.2.2. If $d < r/4$, then we also have

$$\begin{aligned}
\int_{B_d(x_0)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_0,d}|^2 &\le \int_{B_{2d}(x_1)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_1,2d}|^2 \\
&\le C(n,\alpha) \left(\frac{d}{r}\right)^{n+\alpha} \int_{B_{r/2}(x_1)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_1,r/2}|^2 \\
&\qquad + C(n,\alpha,K)\|u\|^2_{W^{1,2}(B_1)} r^{n+2\beta} \\
&\le C(n,\alpha) \left(\frac{d}{r}\right)^{n+\alpha} \int_{B_r(x_0)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_0,r}|^2 \\
&\qquad + C(n,\alpha,K)\|u\|^2_{W^{1,2}(B_1)} r^{n+2\beta}.
\end{aligned}$$

Hence, the estimate (1.4.3) has been established in all possible cases.

To complete the proof of the theorem, we now apply Lemma 1.2.2 to the estimate (1.4.3) to obtain

$$\int_{B_\rho(x_0)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_0,\rho}|^2 \le C(n,\alpha) \Bigg[\left(\frac{\rho}{r}\right)^{n+2\beta} \int_{B_r(x_0)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_0,r}|^2$$

$$+ C(n,\alpha,K)\|u\|^2_{W^{1,2}(B_1)}\rho^{n+2\beta}\Big].$$

Taking $r \nearrow r_0 = r_0(n,\alpha,K)$, we have

$$\int_{B_\rho(x_0)} |\widehat{\nabla u} - \langle\widehat{\nabla u}\rangle_{x_0,\rho}|^2 \leq C(n,\alpha,K)\|u\|^2_{W^{1,2}(B_1)}\rho^{n+2\beta}.$$

Then by the Campanato space embedding we conclude that

$$\widehat{\nabla u} \in C^{0,\beta}(K)$$

with

$$\|\widehat{\nabla u}\|_{C^{0,\beta}(K)} \leq C(n,\alpha,K)\|u\|_{W^{1,2}(B_1)}. \qquad \square$$

Having the $C^{1,\beta}$ regularity of almost minimizers, we can now talk about pointwise values of

$$\partial^+_{x_n} u(x',0) = \lim_{\substack{y\to(x',0)\\ y\in B^+_r}} \partial_{x_n} u(y)$$

for $x' \in B'_1$. The following complementarity condition is of crucial importance in the study of the free boundary.

Lemma 1.4.2 (Complementarity condition)**.** *Let u be an almost minimizer for the Signorini problem in B_1. Then u satisfies the following* complementarity condition

$$u\,\partial^+_{x_n} u = 0 \quad \textit{on} \quad B'_1.$$

Moreover, if $x_0 \in \Gamma(u)$ then

$$u(x_0) = 0 \quad \textit{and} \quad |\widehat{\nabla u}(x_0)| = 0.$$

Proof. Since $u \geq 0$ on B'_1, the complementarity condition will follow once we show that $\partial^+_{x_n} u$ vanishes where $u > 0$ on B'_1. To this end, let $u(x',0) > 0$ for some $x' \in B'_1$. By the continuity of u in B_1, (see Theorem 1.3.1), we have $u > 0$ in some open neighborhood $U \subset B_1$ of $(x',0)$.

If $B_r(y) \Subset U$ (not necessarily centered on B'_1) and v is a harmonic replacement of u in $B_r(y)$, then by the minimum principle $v > 0$ in $\overline{B_r(y)}$, and particularly $v > 0$ on set $B_r(y) \cap B'_1$. Then $v \in \mathfrak{K}_{0,u}(B_r(y), \mathcal{M})$ and therefore we must have

$$\int_{B_r(y)} |\nabla u|^2 \le (1 + r^\alpha) \int_{B_r(y)} |\nabla v|^2.$$

This means that u is an almost harmonic function in U. Hence $u \in C^{1,\alpha/2}(U)$ by Theorem 1.2.1. From the even symmetry of u in x_n, it is then immediate that $\partial^+_{x_n} u(x', 0) = \partial_{x_n} u(x', 0) = 0$.

The second part of the lemma now follows by the $C^{1,\beta}$ regularity and the complementarity condition. $\square$

1.5 Weiss- and Almgren-type monotonicity formulas

In the rest of this chapter we study the free boundary of almost minimizers. In this section we introduce important technical tools, so-called *Weiss-* and *Almgren-type monotonicity formulas*, which play a significant role in our analysis.

We start with Weiss-type monotonicity formulas. They go back to the works of Weiss [49], [51] in the case of the classical obstacle problem and Alt-Caffarelli minimum problem, respectively, and to [14] for the solutions of the thin obstacle problems. In the context of almost minimizers, this type of monotonicity formulas has been used in a recent paper [43].

Theorem 1.5.1 (Weiss-type monitonicity formula)**.** *Let u be an almost minimizer for the Signorini problem in B_1. For $x_0 \in B'_{1/2}$ and $0 < \kappa < \kappa_0$ with a fixed $\kappa_0 \ge 2$ set*

$$W_\kappa(t, u, x_0) := \frac{e^{at^\alpha}}{t^{n+2\kappa-2}} \left[\int_{B_t(x_0)} |\nabla u|^2 - \kappa \frac{1 - bt^\alpha}{t} \int_{\partial B_t(x_0)} u^2 \right],$$

with

$$a = a_\kappa = \frac{n + 2\kappa - 2}{\alpha}, \quad b = \frac{n + 2\kappa_0}{\alpha}.$$

Then, for $0 < t < t_0 = t_0(n, \alpha, \kappa_0)$,

$$\frac{d}{dt} W_\kappa(t, u, x_0) \geq \frac{\mathrm{e}^{at^\alpha}}{t^{n+2\kappa-2}} \int_{\partial B_t(x_0)} \left(u_\nu - \frac{\kappa(1 - bt^\alpha)}{t} u \right)^2.$$

In particular, $W_\kappa(t, u, x_0)$ *is nondecreasing in* t *for* $0 < t < t_0$.

Remark 1.5.2. It is important to observe that while $a = a_\kappa$ depends on κ, the constant b depends only on α, n and κ_0. We also note that in our version of Weiss's monotonicity formula, perturbations (from the case of the thin obstacle problem) appear in the form of multiplicative factors, rather than additive errors as in [43]. Because of the multiplicative nature of the perturbations, we can then use the one-parametric family of monotonicity formulas $\{W_\kappa\}_{0<\kappa<\kappa_0}$ to derive an Almgren-type monotonicity formula, see Theorem 1.5.4.

Remark 1.5.3. To avoid bulky notations, we will write $W_\kappa(t, u)$ for $W_\kappa(t, u, x_0)$ when $x_0 = 0$ or even simply $W_\kappa(t)$, when both u and x_0 are clear from the context.

Proof. The proof uses an argument similar to the one in Theorem 1.2 in [51]. Essentially, it follows from a comparison (1.1.4) with special competitors, described below. Without loss of generality, assume $x_0 = 0$. Then for $t \in (0, 1/2)$, define w by

$$w(x) := \left(\frac{|x|}{t} \right)^\kappa u \left(t \frac{x}{|x|} \right), \quad \text{for } x \in B_t.$$

Note that w is κ-homogeneous in B_t, i.e., $w(\lambda x) = \lambda^\kappa w(\lambda x)$ for $\lambda > 0$, $x, \lambda x \in B_t$, and coincides with u on ∂B_t. Also note that $w \geq 0$ on B_t' and is therefore a valid competitor for u in (1.1.4). We refer to this w as the *κ-homogeneous replacement* of u in B_t.

Now, in B_t, we have

$$\nabla w(x) = \left(\frac{|x|}{t} \right)^{\kappa-1} \left[\frac{\kappa}{t} u \left(t \frac{x}{|x|} \right) \frac{x}{|x|} + \nabla u \left(t \frac{x}{|x|} \right) - \nabla u \left(t \frac{x}{|x|} \right) \cdot \frac{x}{|x|} \frac{x}{|x|} \right],$$

which gives

$$\int_{B_t} |\nabla w|^2 dx = \int_0^t \int_{\partial B_r} |\nabla w(x)|^2 dS_x dr$$

$$
\begin{aligned}
&= \int_0^t \int_{\partial B_r} \Big(\frac{r}{t}\Big)^{2\kappa-2} \Big| \frac{\kappa}{t} u\Big(t\frac{x}{r}\Big)\nu - \Big(\nabla u\Big(t\frac{x}{r}\Big)\cdot\nu\Big)\nu + \nabla u\Big(t\frac{x}{r}\Big)\Big|^2 dS_x dr \\
&= \int_0^t \int_{\partial B_t} \Big(\frac{r}{t}\Big)^{n+2\kappa-3} \Big| \frac{\kappa}{t} u\nu - \Big(\nabla u\cdot\nu\Big)\nu + \nabla u\Big|^2 dS_x dr \\
&= \frac{t}{n+2\kappa-2} \int_{\partial B_t} \Big|\nabla u - \Big(\nabla u\cdot\nu\Big)\nu + \frac{\kappa}{t} u\nu\Big|^2 dS_x \\
&= \frac{t}{n+2\kappa-2} \int_{\partial B_t} \Big(|\nabla u|^2 - \Big(\nabla u\cdot\nu\Big)^2 + \Big(\frac{\kappa}{t}\Big)^2 u^2\Big) dS_x.
\end{aligned}
$$

The latter equality can be rewritten as

$$
\int_{\partial B_t} u^2 dS_x = \Big(\frac{t}{\kappa}\Big)^2 \Big[\frac{n+2\kappa-2}{t} \int_{B_t} |\nabla w|^2 dx + \int_{\partial B_t} \Big(u_\nu^2 - |\nabla u|^2\Big)\, dS_x\Big]. \tag{1.5.1}
$$

Since w is a competitor for u, we have

$$
\int_{B_t} |\nabla w|^2 dx \geq \frac{1}{1+t^\alpha} \int_{B_t} |\nabla u|^2 dx \geq (1-t^\alpha) \int_{B_t} |\nabla u|^2 dx \tag{1.5.2}
$$

and combining (1.5.1) and (1.5.2) yields

$$
\int_{\partial B_t} u^2 dS_x \geq \Big(\frac{t}{\kappa}\Big)^2 \Big[(n+2\kappa-2)\frac{1-t^\alpha}{t} \int_{B_t} |\nabla u|^2 dx + \int_{\partial B_t} \Big(u_\nu^2 - |\nabla u|^2\Big)\, dS_x\Big]. \tag{1.5.3}
$$

Multiplying this by $\kappa^2 \mathrm{e}^{at^\alpha} t^{-n-2\kappa}$ and rearranging terms, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big(\mathrm{e}^{at^\alpha} t^{-n-2\kappa+2}\Big) \int_{B_t} |\nabla u|^2 dx \\
&\quad = -(n+2\kappa-2)\mathrm{e}^{at^\alpha} t^{-n-2\kappa}\Big(t - t^{\alpha+1}\Big) \int_{B_t} |\nabla u|^2 dx \\
&\quad \geq \mathrm{e}^{at^\alpha} t^{-n-2\kappa+2} \int_{\partial B_t} \Big(u_\nu^2 - |\nabla u|^2\Big)\, dS_x - \kappa^2 \mathrm{e}^{at^\alpha} t^{-n-2\kappa} \int_{\partial B_t} u^2 dS_x.
\end{aligned} \tag{1.5.4}
$$

Define now an auxiliary function

$$
\psi(t) = \frac{\kappa \mathrm{e}^{at^\alpha}(1-bt^\alpha)}{t^{n+2\kappa-1}}.
$$

Then we write

$$W_\kappa(t,u,0) = \mathrm{e}^{at^\alpha} t^{-n-2\kappa+2} \int_{B_t} |\nabla u|^2 dx - \psi(t) \int_{\partial B_t} u^2 dS_x$$

and, using (1.5.4), obtain

$$\begin{aligned}
\frac{d}{dt} W_\kappa(t,u,0) &= \frac{d}{dt}\left(\mathrm{e}^{at^\alpha} t^{-n-2\kappa+2}\right) \int_{B_t} |\nabla u|^2 dx + \mathrm{e}^{at^\alpha} t^{-n-2\kappa+2} \int_{\partial B_t} |\nabla u|^2 dS_x \\
&\quad - \psi'(t) \int_{\partial B_t} u^2 dS_x - 2\psi(t) \int_{\partial B_t} u u_\nu dS_x - (n-1)\frac{\psi(t)}{t} \int_{\partial B_t} u^2 dS_x \\
&\geq \mathrm{e}^{at^\alpha} t^{-n-2\kappa+2} \int_{\partial B_t} \left(u_\nu^2 - |\nabla u|^2\right) dS_x - \kappa^2 \mathrm{e}^{at^\alpha} t^{-n-2\kappa} \int_{\partial B_t} u^2 dS_x \\
&\quad + \mathrm{e}^{at^\alpha} t^{-n-2\kappa+2} \int_{\partial B_t} |\nabla u|^2 dS_x - \psi'(t) \int_{\partial B_t} u^2 dS_x \\
&\quad - 2\psi(t) \int_{\partial B_t} u u_\nu dS_x - (n-1)\frac{\psi(t)}{t} \int_{\partial B_t} u^2 dS_x \\
&= \mathrm{e}^{at^\alpha} t^{-n-2\kappa+2} \int_{\partial B_t} u_\nu^2 dS_x - 2\psi(t) \int_{\partial B_t} u u_\nu dS_x \\
&\quad - \left(\kappa^2 \mathrm{e}^{at^\alpha} t^{-n-2\kappa} + \psi'(t) + (n-1)\frac{\psi(t)}{t}\right) \int_{\partial B_t} u^2 dS_x.
\end{aligned}$$

Now observe that $\psi(t)$ satisfies the inequality

$$-\frac{\mathrm{e}^{at^\alpha}}{t^{n+2\kappa-2}} \left(\kappa^2 \mathrm{e}^{at^\alpha} t^{-n-2\kappa} + \psi'(t) + (n-1)\frac{\psi(t)}{t}\right) - \psi^2(t) \geq 0$$

for $0 < t < t_0(n,\alpha,\kappa_0)$ and $0 < \kappa < \kappa_0$. Indeed, a direct computation shows that the above inequality is equivalent to

$$2\alpha^2(1+\kappa_0-\kappa) - (n+2\kappa_0)[(n+2\kappa_0)\kappa - \alpha(n+2\kappa-2)]t^\alpha \geq 0,$$

which holds for $0 < \kappa < \kappa_0$ and small $t > 0$ such that

$$2\alpha^2 - 4(n+2\kappa_0)^2 \kappa_0 t^\alpha \geq 0.$$

Hence, recalling also the formula for $\psi(t)$, we can conclude that

$$\begin{aligned}\frac{d}{dt}W_\kappa(t,u,0) &\geq \frac{\mathrm{e}^{at^\alpha}}{t^{n+2\kappa-2}}\left[\int_{\partial B_t} u_\nu^2 dS_x - 2\frac{\kappa(1-bt^\alpha)}{t}\int_{\partial B_t} uu_\nu dS_x + \left(\frac{\kappa(1-bt^\alpha)}{t}\right)^2\int_{\partial B_t} u^2 dS_x\right]\\ &= \frac{\mathrm{e}^{at^\alpha}}{t^{n+2\kappa-2}}\int_{\partial B_t}\left(u_\nu - \frac{\kappa(1-bt^\alpha)}{t}u\right)^2,\end{aligned}$$

for $0 < t < t_0(n, \alpha, \kappa_0)$. □

Next, for an almost minimizer u in B_1 and $x_0 \in B'_{1/2}$, consider the quantity

$$N(t,u,x_0) := \frac{t\int_{B_t(x_0)}|\nabla u|^2}{\int_{\partial B_t(x_0)} u^2}, \quad 0 < t < 1/2$$

which is known as *Almgren's frequency* and goes back to Almgren's *Big Regularity Paper* [47]. This kind of quantities have also been used in unique continuation for a class of elliptic operators [52], [53] and have been instrumental in thin obstacle-type problems, starting with the works [12]–[14].

Before proceeding, we observe that Almgren's frequency is well defined when x_0 is a free boundary point, since $\int_{\partial B_t(x_0)} u^2 > 0$. Indeed, otherwise $u = 0$ on $\partial B_t(x_0)$ and we can use $h \equiv 0$ in $B_t(x_0)$ as a competitor, to obtain that $\int_{B_t(x_0)}|\nabla u|^2 \leq (1+t^\alpha)0 = 0$, implying $u \equiv 0$ in $B_t(x_0)$, contradicting the assumption that x_0 is a free boundary point. Next, we also consider a modification of N:

$$\widetilde{N}(t,u,x_0) := \frac{1}{1-bt^\alpha}N(t,u,x_0),$$

where b is as in Theorem 1.5.1, as well as

$$\widehat{N}_{\kappa_0}(t,u,x_0) := \min\{\widetilde{N}(t), \kappa_0\}, \quad 0 < t < t_0,$$

which we call the *truncated frequency*.

For the frequencies N, $\widetilde{N}$, and $\widehat{N}_{\kappa_0}$, we will follow the same notational conventions as outlined in Remark 1.5.3 for Weiss's functionals W_κ.

With the Weiss type monotonicity formula at hand, we easily obtain the following monotonicity of $\widehat{N}_{\kappa_0}$.

Theorem 1.5.4 (Almgren-type monotonicity formula)**.** *Let u, κ_0, and t_0 be as in Theorem 1.5.1, and x_0 a free boundary point. Then $\widehat{N}_{\kappa_0}(t,u,x_0)$ is nondecreasing in $0<t<t_0$.*

Proof. We assume $x_0=0$. It is quite important to observe that t_0 depends only on n, α, and κ_0. Then, if $\widetilde{N}(t)<\kappa$ for some $t\in(0,t_0)$ and $\kappa\in(0,\kappa_0)$, then

$$\begin{aligned}W_\kappa(t)&=\frac{e^{at^\alpha}}{t^{n+2\kappa-1}}\left(\int_{\partial B_t}u^2\right)(N(t)-\kappa(1-bt^\alpha))\\&=\frac{e^{at^\alpha}}{t^{n+2\kappa-1}}\left(\int_{\partial B_t}u^2\right)(1-bt^\alpha)\left(\widetilde{N}(t)-\kappa\right)<0.\end{aligned}$$

By Theorem 1.5.1 we also have $W_\kappa(s)\le W_\kappa(t)<0$ for all $s\in(0,t)$, and thus $\widetilde{N}(s)<\kappa$. This completes the proof. $\square$

Remark 1.5.5. The proof above is rather indirect and establishes the monotonicity of $\widehat{N}_{\kappa_0}$ from that of Weiss-type formulas in one-parametric family $\{W_\kappa\}_{0<\kappa<\kappa_0}$. This kind of relation has been first observed in [14].

1.6 Almgren rescalings and blowups

In this section we prove a lower bound on Almgren's frequency for almost minimizers at free boundary points. The idea is to consider appropriate rescalings and blowups of almost minimizers to obtain solutions of the Signorini problem, for which a bound $N(0+)\ge 3/2$ is known.

Now, let u be an almost minimizer for the Signorini problem in B_1, and $x_0\in B'_{1/2}$ a free boundary point. For $0<r<1/2$ consider the *Almgren rescaling*[3] of u at x_0

$$u^A_{x_0,r}(x):=\frac{u(rx+x_0)}{\left(\frac{1}{r^{n-1}}\int_{\partial B_r(x_0)}u^2\right)^{\frac12}},\quad x\in B_{1/(2r)}.$$

[3]↑We use the superscript A to distinguish this rescaling from the other rescalings, namely, homogeneous and almost homogeneous rescalings that we consider later.

When $x_0 = 0$, we also write u_r^A instead of $u_{0,r}^A$. The Almgren rescalings have the following normalization and scaling properties

$$\begin{aligned}
&\|u_{x_0,r}^A\|_{L^2(\partial B_1)} = 1\\
&N(\rho, u_{x_0,r}^A) = N(\rho r, u, x_0), \quad \rho < 1/(2r).
\end{aligned}$$

We will call the limits of $u_{x_0,r}^A$ over any sequence $r = r_{\mathrm{j}} \to 0+$ *Almgren blowups* of u at x_0 and denote by $u_{x_0,0}^A$.

Proposition 1.6.1 (Existence of Almgren blowups). *Let $x_0 \in B'_{1/2} \cap \Gamma(u)$ be such that $\widehat{N}_{\kappa_0}(0+, u, x_0) = \kappa < \kappa_0$. Then every sequence of Almgren rescalings $u_{x_0,r_{\mathrm{j}}}^A$, with $r_{\mathrm{j}} \to 0+$ contains a subsequence, still denoted r_{j}, such that for a function $u_{x_0,0}^A \in W^{1,2}(B_1) \cap C^1_{\mathrm{loc}}(B_1^{\pm} \cup B'_1)$*

$$\begin{aligned}
u_{x_0,r_{\mathrm{j}}}^A &\to u_{x_0,0}^A \quad \text{in } W^{1,2}(B_1),\\
u_{x_0,r_{\mathrm{j}}}^A &\to u_{x_0,0}^A \quad \text{in } L^2(\partial B_1),\\
u_{x_0,r_{\mathrm{j}}}^A &\to u_{x_0,0}^A \quad \text{in } C^1_{\mathrm{loc}}(B_1^{\pm} \cup B'_1).
\end{aligned}$$

Moreover, $u_{x_0,0}^A$ is a nonzero solution of the Signorini problem in B_1, even in x_n, and homogeneous of degree κ in B_1, i.e.,

$$u_{x_0,0}^A(\lambda x) = \lambda^\kappa u_{x_0,0}^A(x),$$

for $\lambda > 0$, provided $x, \lambda x \in B_1$.

Proof. Without loss of generality, we assume $x_0 = 0$. From the fact that $\widehat{N}(0+, u) = \kappa < \kappa_0$, it follows also that $N(0+, u) = \widehat{N}(0+, u) = \kappa$. In particular, $N(r_{\mathrm{j}}, u) < \kappa_0$ for large j. Then, for such j

$$\int_{B_1} |\nabla u_{r_{\mathrm{j}}}^A|^2 = N(1, u_{r_{\mathrm{j}}}^A) = N(r_{\mathrm{j}}, u) \leq \kappa_0$$

and combined with the normalization $\fint_{\partial B_1}(u^A_{r_j})^2 = 1$, we see that the sequence $u^A_{r_j}$ is bounded in $W^{1,2}(B_1)$. Hence, there is a function $u^A_0 \in W^{1,2}(B_1)$ such that, over a subsequence,

$$\begin{aligned} u^A_{r_j} &\to u^A_0 \quad \text{weakly in } W^{1,2}(B_1),\\ u^A_{r_j} &\to u^A_0 \quad \text{strongly in } L^2(\partial B_1). \end{aligned}$$

In particular, $\fint_{\partial B_1}(u^A_0)^2 = 1$, implying that $u^A_0 \not\equiv 0$ in B_1.

Next, we observe that since u is an almost minimizer in B_1 with gauge function $\omega(t) = t^\alpha$, u^A_r is also an almost minimizer in $B_{1/(2r)}$ with gauge function $\omega_r(t) = (rt)^\alpha$. This is rather easy to see, since $u^A_r(x)$ up to a positive constant factor is $u(rx)$ and the multiplication (or the division) by a positive number preserves the almost minimizing property. Since $\omega_r(t) \le \omega(t)$, Theorem 1.4.1 is applicable to rescalings $u^A_{r_j}$, from where we can deduce that over yet another subsequence,

$$u^A_{r_j} \to u^A_0 \quad \text{in } C^1_{\text{loc}}(B^\pm_1 \cup B'_1). \tag{1.6.1}$$

Now, we claim that since the gauge functions $\omega_r(t) = (rt)^\alpha \to 0$ as $r \to 0$, the blowup u^A_0 is a solution of the Signorini problem in B_1. Indeed, for a fixed r_j, let h_{r_j} be the Signorini replacement of $u^A_{r_j}$ in B_1. Then, by repeating the argument as in the proof of Proposition 1.3.2

$$\int_{B_1} |\nabla(u^A_{r_j} - h_{r_j})|^2 \le r_j^\alpha \int_{B_1} |\nabla u^A_{r_j}|^2.$$

This implies that $h_{r_j} \to u^A_0$ weakly in $W^{1,2}(B_1)$. On the other hand, by the boundedness of the sequence h_{r_j} in $W^{1,2}(B_1)$, we have also boundedness in $C^{1,1/2}_{\text{loc}}(B^\pm_1 \cup B'_1)$ and hence, over a subsequence, $h_{r_j} \to u^A_0$ in $C^1_{\text{loc}}(B^\pm_1 \cup B'_1)$. By this convergence we then conclude that u^A_0 satisfies

$$\begin{aligned} \Delta u^A_0 = 0 \quad &\text{in } B_1 \setminus B'_1\\ u^A_0 \ge 0, \quad -\partial^+_{x_n} u^A_0 \ge 0, \quad u^A_0 \partial^+_{x_n} u^A_0 = 0 \quad &\text{on } B'_1, \end{aligned}$$

and hence u^A_0 itself solves the Signorini problem in B_1.

Using the C^1_{loc} convergence again, we have that for any $0 < \rho < 1$

$$N(\rho, u_0^A) = \lim_{r_{\mathrm{j}} \to 0} N(\rho, u_{r_{\mathrm{j}}}^A) = \lim_{r_{\mathrm{j}} \to 0} N(\rho r_{\mathrm{j}}, u) = N(0+, u) = \kappa.$$

Thus, the Almgren frequency of u_0^A is constant κ, which is possible only if u_0^A is a κ-homogeneous solution of the Signorini problem in B_1, see Theorem 9.4 in [48]. □

In what follows, it will be sufficient for us to fix $\kappa_0 \geq 2$ (say $\kappa_0 = 2$), in the definition of $\widehat{N}_{\kappa_0}$ and we will simply write

$$\widehat{N} = \widehat{N}_{\kappa_0}.$$

Lemma 1.6.1 (Minimal frequency)**.** *Let u be an almost minimizer for the Signorini problem in B_1. If $x_0 \in B'_{1/2} \cap \Gamma(u)$, then*

$$\widehat{N}(0+, u, x_0) = \lim_{r \to 0+} \widehat{N}(r, u, x_0) \geq \frac{3}{2}.$$

Consequently, we also have

$$\widehat{N}(t, u, x_0) \geq 3/2 \quad \text{for } 0 < t < t_0.$$

Proof. As before, let $x_0 = 0$. Assume to the contrary that $\widehat{N}(0+, u) = \kappa < 3/2$. Since $\kappa < \kappa_0$ we can apply Proposition 1.6.1 to obtain that over a sequence $r_{\mathrm{j}} \to 0+$, $u_{r_{\mathrm{j}}}^A \to u_0^A$ in $C^1_{\mathrm{loc}}(B_1^{\pm} \cup B'_1)$, where u_0^A is a nonzero κ-homogeneous solution of the Signorini problem in B_1, even in x_n. Moreover, since $0 \in \Gamma(u)$, by Lemma 1.4.2 we have that $u(0) = |\widehat{\nabla u}(0)| = 0$, implying that $u_{r_{\mathrm{j}}}^A(0) = |\widehat{\nabla u_{r_{\mathrm{j}}}^A}(0)| = 0$ and, by passing to the limit, $u_0^A(0) = |\widehat{\nabla u_0^A}(0)| = 0$. Now, to arrive at a contradiction, we argue as in the proof of Proposition 9.9 in [48] to reduce the problem to dimension $n = 2$, where we can classify all possible homogeneous solutions of the Signorini problem, even in x_n. The only nonzero homogeneous solutions with $\kappa < 3/2$ in dimension $n = 2$ are possible for $\kappa = 1$ and have the form $u_0^A(x) = -cx_n$ for some $c > 0$, but they fails to satisfy the condition $|\widehat{\nabla u_0^A}(0)| = 0$. Thus, we arrived at contradiction, implying that $\widehat{N}(0+, u) \geq 3/2$. Finally, applying Theorem 1.5.4, we obtain $\widehat{N}(t, u) \geq \widehat{N}(0+, u) \geq 3/2$, for $0 < t < t_0$. □

Corollary 1.6.2. *Let u be an almost minimizer for the Signorini problem in B_1 and x_0 a free boundary points. Then*

$$W_{3/2}(t, u, x_0) \geq 0 \quad \textit{for } 0 < t < t_0.$$

Proof. We simply observe that $\widetilde{N}(t) \geq \widehat{N}(t) \geq 3/2$ for $0 < t < t_0$ and hence

$$W_{3/2}(t, u, x_0) = \frac{\mathrm{e}^{at^\alpha}}{t^{n+2\kappa-1}} \left(\int_{\partial B_t} u^2\right)(1 - bt^\alpha)\left(\widetilde{N}(t) - \frac{3}{2}\right) \geq 0. \qquad \square$$

1.7 Growth estimates

An important step in the study of the free boundary in the Signorini problem (and in many other free boundary problems) is the proof of the optimal regularity of solutions, which in this case is $C^{1,1/2}$ on each side of the thin space. This allows to make proper blowup arguments to establish the regularity of the so-called regular part of the free boundary. However, in the case of almost minimizers, we only know $C^{1,\beta}$ regularity for some small $\beta > 0$ and do not expect to have anything better. Yet, in this section, we establish the *optimal growth* of the almost minimizers at free boundary points with the help of the Weiss-type monotonicity formula and the epiperimetric inequality.

Finally, we want to point out that the results in this section are rather immediate in the case of minimizers, as they follow easily from the differentiation formulas for the quantities involved in the Almgren's frequency formula. This is completely unavailable for almost minimizers.

We start by defining a new type of rescalings. Fix $\kappa \geq 3/2$. For a free boundary point x_0 in $B'_{1/2}$ and $r > 0$, we define the *κ-homogeneous rescaling* by

$$u_{x_0,r}(x) := u^{(\kappa)}_{x_0,r}(x) = \frac{u(rx + x_0)}{r^\kappa}, \quad x \in B_{1/(2r)}.$$

To take advantage of the Weiss-type monotonicity formula, we need a slight modification of this rescaling. With the help of an auxiliary function

$$\phi(r) = \phi_\kappa(r) := \mathrm{e}^{-(\kappa b/\alpha) r^\alpha} r^\kappa, \quad r > 0,$$

which is a solution of the differential equation

$$\phi'(r) = \kappa\,\phi(r)\frac{1 - br^\alpha}{r}, \quad r > 0$$

we define the κ*-almost homogeneous rescalings* by

$$u^\phi_{x_0,r}(x) := \frac{u(rx + x_0)}{\phi(r)}, \quad x \in B_{1/(2r)}.$$

Lemma 1.7.1 (Weak growth estimate)**.** *Let u be an almost minimizer of the Signorini problem in B_1 and $x_0 \in B'_{1/2} \cap \Gamma(u)$ be such that $\widehat{N}(0+, u, x_0) \geq \kappa$ for $\kappa \leq \kappa_0$. Then*

$$\begin{aligned}
\int_{\partial B_t(x_0)} u^2 &\leq C(n, \alpha, \kappa_0)\|u\|^2_{W^{1,2}(B_1)}\left(\log\frac{1}{t}\right) t^{n+2\kappa-1},\\
\int_{B_t(x_0)} |\nabla u|^2 &\leq C(n, \alpha, \kappa_0)\|u\|^2_{W^{1,2}(B_1)}\left(\log\frac{1}{t}\right) t^{n+2\kappa-2},
\end{aligned}$$

for $0 < t < t_0 = t_0(n, \alpha, \kappa_0)$.

Proof. Without loss of generality, assume $x_0 = 0$. We first note that the condition $\widehat{N}(0+, u) \geq \kappa$ implies that $\widehat{N}(t, u) \geq \kappa$ for $0 < t < t_0 = t_0(n, \alpha, \kappa_0)$. Then also $\widetilde{N}(t, u) \geq \kappa$ for such t and consequently,

$$W_\kappa(t, u) = \frac{\mathrm{e}^{at^\alpha}}{t^{n+2\kappa-1}}\left(\int_{\partial B_t} u^2\right)(1 - bt^\alpha)\left(\widetilde{N}(t, u) - \kappa\right) \geq 0.$$

Next, for $\phi = \phi_\kappa$, we have that

$$\begin{aligned}
\frac{d}{dr}u^\phi_r(x) &= \frac{\nabla u(rx)\cdot x}{\phi(r)} - \frac{u(rx)[\phi'(r)/\phi(r)]}{\phi(r)}\\
&= \frac{1}{\phi(r)}\left(\nabla u(rx)\cdot x - \frac{\kappa(1 - br^\alpha)}{r}u(rx)\right).
\end{aligned}$$

Now let

$$m(r) = \left(\int_{\partial B_1} (u_r^\phi(\xi))^2 dS_\xi\right)^{1/2}, \quad r > 0.$$

Then,

$$m'(r) = \left(\int_{\partial B_1} u_r^\phi(\xi)\frac{d}{dr}u_r^\phi(\xi)dS_\xi\right)\left(\int_{\partial B_1} (u_r^\phi(\xi))^2 dS_\xi\right)^{-1/2}$$

and consequently, by Cauchy-Schwarz,

$$|m'(r)| \le \left(\int_{\partial B_1}\left[\frac{d}{dr}u_r^\phi(\xi)\right]^2 dS_\xi\right)^{1/2}.$$

Hence,

$$\begin{aligned}
|m'(r)| &\le \frac{1}{\phi(r)}\left(\int_{\partial B_1}\left(\nabla u(r\xi)\cdot\xi - \frac{\kappa(1-br^\alpha)}{r}u(r\xi)\right)^2 dS_\xi\right)^{1/2} \\
&= \frac{1}{\phi(r)}\left(\frac{1}{r^{n-1}}\int_{\partial B_r}\left(\partial_\nu u(x) - \frac{\kappa(1-br^\alpha)}{r}u(x)\right)^2 dS_x\right)^{1/2} \\
&\le \frac{1}{\phi(r)}\left(\frac{1}{r^{n-1}}\frac{r^{n+1}}{\mathrm{e}^{ar^\alpha}}\frac{d}{dr}W_\kappa(r)\right)^{1/2} = \frac{\mathrm{e}^{cr^\alpha}}{r^{1/2}}\left(\frac{d}{dr}W_\kappa(r)\right)^{1/2}, \quad c = \kappa\frac{b}{\alpha} - \frac{a}{2},
\end{aligned}$$

for $0 < r < t_0 = t_0(n, \alpha, \kappa_0)$. Thus, we have shown

$$|m'(r)| \le \frac{\mathrm{e}^{cr^\alpha}}{r^{1/2}}\left(\frac{d}{dr}W_\kappa(r)\right)^{1/2}, \quad 0 < r < t_0.$$

Integrating in r over the interval $(s, t) \subset (0, t_0)$, we obtain

$$\begin{aligned}
|m(t) - m(s)| &\le \int_s^t \frac{\mathrm{e}^{cr^\alpha}}{r^{1/2}}\left(\frac{d}{dr}W_\kappa(r)\right)^{1/2} dr \le \left(\int_s^t \frac{\mathrm{e}^{2cr^\alpha}}{r}dr\right)^{1/2}\left(\int_s^t \frac{d}{dr}W_\kappa(r)\right)^{1/2} \\
&\le C_0\left(\log\frac{t}{s}\right)^{1/2}[W_\kappa(t) - W_\kappa(s)]^{1/2}.
\end{aligned}$$

In particular (recalling that $W_\kappa(s) \ge 0$), we obtain

$$m(t) \le m(t_0) + C_0\left(\log\frac{t_0}{t}\right)^{1/2}[W_\kappa(t_0)]^{1/2}.$$

Varying t_0 by an absolute factor, we can guarantee that

$$m(t_0) \leq C(n, \alpha, \kappa_0)\|u\|_{L^2(B_1)}, \quad W_\kappa(t_0) \leq C(n, \alpha, \kappa_0)\|u\|^2_{W^{1,2}(B_1)}.$$

Hence, we can conclude

$$\int_{\partial B_t} u^2 \leq C(n, \alpha, \kappa_0)\|u\|^2_{W^{1,2}(B_1)} \left(\log \frac{1}{t}\right) t^{n+2\kappa-1},$$

for $0 < t < t_0 = t_0(n, \alpha, \kappa_0)$. This implies the first bound. The second bound follows immediately from the first one by using that $W_\kappa(t, u) \leq W_\kappa(t_0, u)$:

$$\begin{aligned}
\frac{1}{t^{n+2\kappa-2}} \int_{B_t} |\nabla u|^2 &\leq \frac{\kappa(1 - bt^\alpha)}{t^{n+2\kappa-1}} \int_{\partial B_t} u^2 + \mathrm{e}^{-at^\alpha} W_\kappa(t_0, u) \\
&\leq C(n, \alpha, \kappa_0)\|u\|^2_{W^{1,2}(B_1)} \left(\log \frac{1}{t}\right) + \frac{\mathrm{e}^{at_0^\alpha}}{t_0^{n+2\kappa-2}} \int_{B_{t_0}} |\nabla u|^2 \\
&\leq C(n, \alpha, \kappa_0)\|u\|^2_{W^{1,2}(B_1)} \left(\log \frac{1}{t}\right).
\end{aligned}$$

□

The logarithmic term in Lemma 1.7.1 does not allow to conclude that the sequence of κ-homogeneous or almost homogeneous rescaling is uniformly bounded say in $W^{1,2}(B_1)$. In the rest of this section we show that in the case of the minimal frequency $\kappa = 3/2$ we can do that with the help of the so-called epiperimetric inequality for the Signorini problem for the Weiss energy

$$W^0_{3/2}(w) := \int_{B_1} |\nabla w|^2 - \frac{3}{2} \int_{\partial B_1} w^2.$$

To state this result, we let

$$\mathcal{A} := \{w \in W^{1,2}(B_1) : w \geq 0 \text{ on } B_1',\ w(x', x_n) = w(x', -x_n)\} \tag{1.7.1}$$

Theorem 1.7.2 (Epiperimetric inequality). *There exists $\eta \in (0,1)$ such that if $w \in \mathcal{A}$ is homogeneous of degree $3/2$ in B_1, then there exists $v \in \mathcal{A}$ with $v = w$ on ∂B_1 such that*

$$W^0_{3/2}(v) \leq (1-\eta) W^0_{3/2}(w).$$

This kind of inequalities go back to the work of Weiss [49], in the case of the classical obstacle problem. For the Signorini problem, a version of this theorem was proved in [20] and [54]. In fact, the theorem above is the version in [55]. The inequality in [20] and [54] requires w to be close to the blowup profile, but this can be easily removed by a scaling argument (see [55]). We also refer to [27], for a more direct proof of this inequality with an explicit constant $\eta = 1/(2n+3)$.

Now, with the help of the epiperimetric inequality, we can prove a decay estimate for the Weiss-type energy functional $W_{3/2}$. For the rest of the section, we will assume

$$\kappa_0 = 2,$$

which will make some of the constants independent of κ_0, but the results hold also for any other value of $\kappa_0 \geq 2$, with possible added dependence of constants on κ_0.

Lemma 1.7.3. *Let $x_0 \in B'_{1/2}$ be a free boundary point. Then, there exist $\delta = \delta(n, \alpha) > 0$ such that*

$$0 \leq W_{3/2}(t, u, x_0) \leq C t^{\delta}, \quad 0 < t < t_0 = t_0(n, \alpha),$$

with $C = C(n, \alpha) \|u\|^2_{W^{1,2}(B_1)}$.

Proof. As before, without loss of generality we assume that $x_0 = 0$.

The proof will follow from a differential inequality that we derive by using our earlier computations and the epiperimetric inequality. Recalling the proof of the Weiss-type monotonicity formula (Theorem 1.5.1), for small $t > 0$, we have

$$\begin{aligned}\frac{d}{dt} W_{3/2}(t,u) = \frac{\mathrm{e}^{at^{\alpha}}}{t^{n+1}} \int_{\partial B_t} |\nabla u|^2 - \frac{(n+1)(1-t^{\alpha})\mathrm{e}^{at^{\alpha}}}{t^{n+2}} \int_{B_t} |\nabla u|^2 \\ - \psi'(t) \int_{\partial B_t} u^2 - (n-1)\frac{\psi(t)}{t} \int_{\partial B_t} u^2 - 2\psi(t) \int_{\partial B_t} u \partial_{\nu} u\end{aligned}$$

$$
\begin{aligned}
&= -\frac{(n+1)(1-t^\alpha)}{t} W_{3/2}(t,u) + \frac{\mathrm{e}^{at^\alpha}}{t^{n+1}} \int_{\partial B_t} |\nabla u|^2 \\
&\quad - \left([(n+1)(1-t^\alpha) + (n-1)] \frac{\psi(t)}{t} + \psi'(t) \right) \int_{\partial B_t} u^2 - 2\psi(t) \int_{\partial B_t} u \partial_\nu u \\
&\geq -\frac{(n+1)(1-t^\alpha)}{t} W_{3/2}(t,u) \\
&\quad + \frac{\mathrm{e}^{at^\alpha}(1-bt^\alpha)}{t^{n+1}} \int_{\partial B_t} \Bigg(|\nabla u|^2 - \frac{3}{t} u \partial_\nu u \\
&\qquad\qquad - \frac{3}{2t} \left[\frac{(n+1)(1-t^\alpha) + (n-1)}{t} + \frac{\psi'(t)}{\psi(t)} \right] u^2 \Bigg).
\end{aligned}
$$

To proceed, note that

$$
\frac{(n+1)(1-t^\alpha) + (n-1)}{t} + \frac{\psi'(t)}{\psi(t)} = \frac{(n-2) + O(t^\alpha)}{t}.
$$

Now, for the homogeneous rescalings

$$
u_t(x) = \frac{u(tx)}{t^{3/2}},
$$

we can write

$$
\begin{aligned}
&\int_{\partial B_t} |\nabla u|^2 - \frac{3}{t} u \partial_\nu u - \frac{3}{2} \frac{(n-2) + O(t^\alpha)}{t^2} u^2 \\
&\quad = t^n \int_{\partial B_1} |\nabla u_t|^2 - 3 u_t \partial_\nu u_t - \frac{3}{2} [(n-2) + O(t^\alpha)] u_t^2 \\
&\quad = t^n \int_{\partial B_1} \left(\partial_\nu u_t - \frac{3}{2} u_t \right)^2 + (\partial_\tau u_t)^2 - \frac{3}{2} \left[\left(n - \frac{1}{2} \right) + O(t^\alpha) \right] u_t^2,
\end{aligned}
$$

where $\partial_\tau u_t$ is the tangential component of ∇u_t on the unit sphere. We can summarize for now that

$$
\begin{aligned}
\frac{d}{dt} W_{3/2}(t,u) \geq &-\frac{(n+1)(1-t^\alpha)}{t} W_{3/2}(t,u) \\
&+ \frac{\mathrm{e}^{at^\alpha}(1-bt^\alpha)}{t} \int_{\partial B_1} \left[\left(\partial_\nu u_t - \frac{3}{2} u_t \right)^2 + (\partial_\tau u_t)^2 - \frac{3}{2} \left(n - \frac{1}{2} \right) u_t^2 \right] \\
&+ O(t^{\alpha-1}) \int_{\partial B_1} u_t^2.
\end{aligned}
$$

On the other hand, if w_t is a 3/2-homogeneous replacement of u_t in B_1, i.e.,

$$w_t(x) = |x|^{3/2} u_t(x/|x|)$$

then

$$\int_{\partial B_1} (\partial_\tau u_t)^2 - \frac{3}{2}\left(n - \frac{1}{2}\right) u_t^2 = \int_{\partial B_1} (\partial_\tau w_t)^2 - \frac{3}{2}\left(n - \frac{1}{2}\right) w_t^2 = (n+1) W^0_{3/2}(w_t),$$

where

$$W^0_{3/2}(w_t) = \int_{B_1} |\nabla w_t|^2 - \frac{3}{2}\int_{\partial B_1} w_t^2.$$

The last equality follows by repeating the arguments in the beginning of the proof of Theorem 1.5.1 with $\kappa = 3/2$. Let v_t be the solution of the Signorini problem in B_1 with $v_t = u_t = w_t$ on ∂B_1. Then by the epiperimetric inequality

$$W^0_{3/2}(v_t) \le (1 - \eta) W^0_{3/2}(w_t).$$

On the other hand, since u is an almost minimizer, we have

$$\int_{B_1} |\nabla u_t|^2 \le (1 + t^\alpha) \int_{B_1} |\nabla v_t|^2$$

and since also $u_t = v_t$ on ∂B_1, we have

$$\begin{aligned} W_{3/2}(t, u) &= \frac{\mathrm{e}^{at^\alpha}}{t^{n+1}} \left[\int_{B_t} |\nabla u|^2 - \frac{(3/2)(1 - bt^\alpha)}{t} \int_{\partial B_t} u^2\right] \\ &\le (1 + O(t^\alpha)) W^0_{3/2}(v_t) + O(t^\alpha) \int_{\partial B_1} u_t^2 \\ &\le \left(1 - \frac{\eta}{2}\right) W^0_{3/2}(w_t) + O(t^\alpha) \int_{\partial B_1} u_t^2, \quad \text{for } 0 < t < t_0 = t_0(n, \alpha). \end{aligned}$$

We can therefore write

$$\begin{aligned} \frac{d}{dt} W_{3/2}(t, u) \ge &-\frac{(n+1)(1 - t^\alpha)}{t} W_{3/2}(t, u) \\ &+ \frac{(n+1)\mathrm{e}^{at^\alpha}(1 - bt^\alpha)}{t} W^0_{3/2}(w_t) + O(t^{\alpha - 1}) \int_{\partial B_1} u_t^2 \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{n+1}{t}\left(-1+\frac{1}{1-\eta/2}+O(t^\alpha)\right) W_{3/2}(t,u)+\frac{O(t^\alpha)}{t^{n+3}}\int_{\partial B_t} u^2 \\
&\geq \frac{\eta}{4t} W_{3/2}(t,u) - Ct^{\alpha/2-1},
\end{aligned}
$$

for small t, where we have also used the growth estimate in Lemma 1.7.1. Taking now δ such that

$$
0<\delta<\min\left\{\frac{\eta}{4},\frac{\alpha}{2}\right\},
$$

we have

$$
\begin{aligned}
\frac{d}{dt}\left[W_{3/2}(t,u)t^{-\delta}+\frac{C}{\alpha/2-\delta}t^{\alpha/2-\delta}\right] &= t^{-\delta}\left(\frac{d}{dt}W_{3/2}(t,u)-\frac{\delta}{t}W_{3/2}(t,u)\right)+Ct^{\alpha/2-\delta-1} \\
&\geq t^{-\delta-1}\left[\frac{\eta}{4}-\delta\right]W_{3/2}(t,u)-Ct^{\alpha/2-\delta-1}+Ct^{\alpha/2-\delta-1} \\
&\geq 0,
\end{aligned}
$$

for small t, where we have used again that $W_{3/2}(t,u)\geq 0$. Thus, we can conclude that

$$
0\leq W_{3/2}(t,u)\leq Ct^\delta, \quad 0<t<t_0=t_0(n,\alpha),
$$

with $C=C(n,\alpha)\|u\|^2_{W^{1,2}(B_1)}$. □

Using the estimate on $W_{3/2}(t,u)$ in Lemma 1.7.3, we can improve on Lemma 1.7.1 in the case $\kappa = 3/2$.

Lemma 1.7.4 (Optimal growth estimate). *Let $x_0\in B'_{1/2}$ be a free boundary point. Then, for $0<t<t_0=t_0(n,\alpha)$,*

$$
\begin{aligned}
\int_{\partial B_t(x_0)} u^2 &\leq C(n,\alpha)\|u\|^2_{W^{1,2}(B_1)}t^{n+2}, \\
\int_{B_t(x_0)} |\nabla u|^2 &\leq C(n,\alpha)\|u\|^2_{W^{1,2}(B_1)}t^{n+1}.
\end{aligned}
$$

Proof. We proceed as in the proof of Lemma 1.7.1 up to the estimate

$$
|m(t)-m(s)|\leq C_0\left(\log\frac{t}{s}\right)^{1/2}[W_{3/2}(t,u)-W_{3/2}(s,u)]^{1/2}.
$$

From there, using Lemma 1.7.3, we now have an improved bound

$$|m(t) - m(s)| \leq C \left(\log \frac{t}{s}\right)^{1/2} t^{\delta/2}, \quad s < t < t_0,$$

with $C = C(n, \alpha)\|u\|_{W^{1,2}(B_1)}$. Then, by a dyadic argument, we can conclude that

$$|m(t) - m(s)| \leq C t^{\delta/2}.$$

Indeed, let $k = 0, 1, 2, \ldots$ be such that $t/2^{k+1} \leq s < t/2^k$. Then,

$$\begin{aligned}
|m(t) - m(s)| &\leq \sum_{\mathrm{j}=1}^{k} |m(t/2^{\mathrm{j}-1}) - m(t/2^{\mathrm{j}})| + |m(t/2^k) - m(s)| \\
&\leq C(\log 2)^{1/2} \sum_{\mathrm{j}=1}^{k+1} (t/2^{\mathrm{j}-1})^{\delta/2} \leq C(\log 2)^{1/2} \frac{t^{\delta/2}}{1 - 2^{-\delta/2}} = C t^{\delta/2}.
\end{aligned}$$

In particular, we have

$$m(t) \leq m(t_0) + C t_0^{\delta/2} \leq C(n, \alpha)\|u\|_{W^{1,2}(B_1)}, \quad t < t_0.$$

This implies the first bound. The second bound follows immediately from the first one by using that $W_{3/2}(t, u) \leq W_{3/2}(t_0, u)$:

$$\begin{aligned}
\frac{1}{t^{n+1}} \int_{B_t} |\nabla u(x)|^2 dx &\leq \frac{(3/2)(1 - b t^{\alpha})}{t^{n+2}} \int_{\partial B_t} u(x)^2 dS_x + \mathrm{e}^{-a t^{\alpha}} W_{3/2}(t_0, u) \\
&\leq C(n, \alpha)\|u\|^2_{W^{1,2}(B_1)} + \frac{\mathrm{e}^{a t_0^{\alpha}}}{t_0^{n+1}} \int_{B_{t_0}} |\nabla u(x)|^2 dx \\
&\leq C(n, \alpha)\|u\|^2_{W^{1,2}(B_1)}.
\end{aligned}$$

□

1.8 3/2-Homogeneous blowups

For a free boundary point $x_0 \in B'_{1/2}$, we consider again the 3/2-almost homogeneous rescalings

$$u^{\phi}_{x_0,t}(x) = \frac{u(tx+x_0)}{\phi(t)}, \quad x \in B_{1/(2t)},$$

with $\phi = \phi_{3/2}$. We now observe that the optimal growth estimates in Lemma 1.7.4 implies the boundedness of this family of rescalings in $W^{1,2}(B_R)$ for any $R > 1$. Indeed, the rescalings above will be defined in B_R if $t < 1/(2R)$, and by Lemma 1.7.4, we will have

$$\begin{aligned}
\int_{B_R} |\nabla u^{\phi}_{x_0,t}|^2 &= \frac{\mathrm{e}^{\frac{3b}{\alpha}t^{\alpha}}}{t^{n+1}} \int_{B_{Rt}(x_0)} |\nabla u|^2 \le C(n,\alpha)\|u\|^2_{W^{1,2}(B_1)} R^{n+1}, \\
\int_{\partial B_R} (u^{\phi}_{x_0,t})^2 &= \frac{\mathrm{e}^{\frac{3b}{\alpha}t^{\alpha}}}{t^{n+2}} \int_{\partial B_{Rt}(x_0)} u^2 \le C(n,\alpha)\|u\|^2_{W^{1,2}(B_1)} R^{n+2},
\end{aligned}$$

for $0 < t < t_0/R$. Arguing as in the proof of Proposition 1.6.1, we have for a sequence $t = t_{\mathrm{j}} \to 0+$

$$u^{\phi}_{x_0,t_{\mathrm{j}}} \to u^{\phi}_{x_0,0} \quad \text{in} \quad C^1_{\mathrm{loc}}(B^{\pm}_R \cup B'_R).$$

By letting $R \to \infty$ and using Cantor's diagonal argument, we therefore have that over a subsequence $t = t_{\mathrm{j}} \to 0+$

$$u^{\phi}_{x_0,t_{\mathrm{j}}} \to u^{\phi}_{x_0,0} \quad \text{in} \quad C^1_{\mathrm{loc}}(\mathbb{R}^n_{\pm} \cup \mathbb{R}^{n-1}).$$

We call such $u^{\phi}_{x_0,0}$ a 3/2-*homogeneous blowup* of u at x_0. The name is explained by the fact that

$$\lim_{t\to 0} \frac{\phi(t)}{t^{3/2}} = 1,$$

which implies that if we consider the 3/2-homogeneous rescalings

$$u^{(3/2)}_{x_0,t}(x) = \frac{u(tx+x_0)}{t^{3/2}},$$

then we will have

$$u^{\phi}_{x_0,0} = \lim_{t_{\mathrm{j}}\to 0} u^{\phi}_{x_0,t_{\mathrm{j}}} = \lim_{t_{\mathrm{j}}\to 0} u^{(3/2)}_{x_0,t_{\mathrm{j}}} =: u^{(3/2)}_{x_0,0}$$

and thus $u^{\phi}_{x_0,0} = u^{(3/2)}_{x_0,0}$.

Remark 1.8.1. Because of the logarithmic term in the weak growth estimates in Lemma 1.7.1, at the moment we are unable to consider κ-homogeneous blowups as above for frequencies other than $\kappa = 3/2$. However, once the logarithmic term is removed, the same construction as for $\kappa = 3/2$ applies. In particular, we note that in Lemma 1.10.6 we prove the optimal growth estimates for frequencies $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, enabling us to consider the κ-homogeneous blowups for these values of κ.

We show next that the 3/2-homogeneous blowups are unique at free boundary points. This is achieved by the control on the "rotation" of the rescalings $u^{\phi}_{x_0,r}(x)$.

Lemma 1.8.2 (Rotation estimate)**.** *Let u be an almost minimizer for the Signorini problem in B_1, $x_0 \in B'_{1/2}$ a free boundary point, and δ as in Lemma 1.7.3. Then for $\kappa = 3/2$ and $\phi = \phi_{3/2}$*

$$\int_{\partial B_1} |u^{\phi}_{x_0,t} - u^{\phi}_{x_0,s}| \le C t^{\delta/2}, \quad s < t < t_0 = t_0(n,\alpha),$$

for $C = C(n,\alpha)\|u\|_{W^{1,2}(B_1)}$.

Proof. The proof uses computations similar to the proof of Lemma 1.7.1 combined with the growth estimated for $W_{3/2}(t,u)$ in Lemma 1.7.3. We assume $x_0 = 0$, and have

$$\begin{aligned}
\int_{\partial B_1} |u^{\phi}_t - u^{\phi}_s| &\le \int_{\partial B_1} \int_s^t \left| \frac{d}{dr} u^{\phi}_r \right| dr = \int_s^t \int_{\partial B_1} \left| \frac{d}{dr} u^{\phi}_r \right| dr \\
&\le C_n \int_s^t \left(\int_{\partial B_1} \left| \frac{d}{dr} u^{\phi}_r \right|^2 \right)^{1/2} \\
&\le C_n \left(\int_s^t \frac{1}{r} dr \right)^{1/2} \left(\int_s^t r \int_{\partial B_1} \left| \frac{d}{dr} u^{\phi}_r \right|^2 \right)^{1/2} \\
&\le C_n e^{ct^{\alpha}} \left(\log \frac{t}{s} \right)^{1/2} \left(\int_s^t \frac{d}{dr} W_{3/2}(r,u) dr \right)^{1/2}, \qquad c = \frac{3b}{2\alpha} - \frac{a}{2},
\end{aligned}$$

where we have re-used the computation made in the proof of Lemma 1.7.1. Thus, we obtain

$$\int_{\partial B_1} |u^{\phi}_t - u^{\phi}_s| \le C(n,\alpha) \left(\log \frac{t}{s} \right)^{1/2} (W_{3/2}(t,u) - W_{3/2}(s,u))^{1/2} \le C \left(\log \frac{t}{s} \right)^{1/2} t^{\delta/2}.$$

Then, using a dyadic argument as Lemma 1.7.4, we can conclude that

$$\int_{\partial B_1} |u^\phi_t - u^\phi_s| \le C t^{\delta/2}, \quad s < t < t_0,$$

as required. Indeed, let $k = 0, 1, 2, \ldots$ be such that $t/2^{k+1} \le s < t/2^k$. Then

$$\begin{aligned}\int_{\partial B_1} |u^\phi_t - u^\phi_s| &\le \sum_{\mathrm{j}=1}^{k} \int_{\partial B_1} \left|u^\phi_{t/2^{\mathrm{j}-1}} - u^\phi_{t/2^{\mathrm{j}}}\right| + \int_{\partial B_1} \left|u^\phi_{t/2^k} - u^\phi_s\right| \\ &\le C(\log 2)^{1/2} \sum_{\mathrm{j}=1}^{k+1} (t/2^{\mathrm{j}-1})^{\delta/2} \le C(\log 2)^{1/2} \frac{t^{\delta/2}}{1 - 2^{-\delta/2}}.\end{aligned}$$

This completes the proof. □

The uniqueness of 3/2-homogeneous blowup now follows.

Lemma 1.8.3. *Let $u^\phi_{x_0,0}$ be a blowup at a free boundary point $x_0 \in B'_{1/2}$. Then for $\kappa = 3/2$*

$$\int_{\partial B_1} |u^\phi_{x_0,t} - u^\phi_{x_0,0}| \le C t^{\delta/2}, \quad 0 < t < t_0,$$

where $C = C\left(n, \alpha, \|u\|_{W^{1,2}(B_1)}\right)$ and $\delta = \delta(n, \alpha) > 0$ are as in Lemma 1.8.2. In particular, the blowup $u^\phi_{x_0,0}$ is unique.

Proof. If $u_{x_0,0}$ is the limit of $u^\phi_{x_0,t_{\mathrm{j}}}$ for $t_{\mathrm{j}} \to 0$, then first part of the lemma follows immediately from Lemma 1.8.2, by taking $s = t_{\mathrm{j}} \to 0$ and passing to the limit.

To see the uniqueness of blowup, we observe that $u^\phi_{x_0,0}$ is a solution of the Signorini problem in B_1, by arguing as in the proof of Lemma 1.6.1 for Almgren blowups. Now, if $\tilde{u}^\phi_{x_0,0}$ is another blowup, then from the first part of the lemma we will have

$$\int_{\partial B_1} |\tilde{u}^\phi_{x_0,0} - u^\phi_{x_0,0}|^2 = 0,$$

implying that both $\tilde{u}^\phi_{x_0,0}$ and $u^\phi_{x_0,0}$ are solutions of the Signorini problem in B_1 with the same boundary values on ∂B_1. By the uniqueness of such solutions, we have $\tilde{u}^\phi_{x_0,0} = u^\phi_{x_0,0}$ in B_1. The equality propagates to all of $\mathbb{R}^n$ by the unique continuation of harmonic functions in $\mathbb{R}^n_\pm$. □

We next show that not only the blowups are unique, but also depend continuously on a free boundary point.

Lemma 1.8.4 (Continuous dependence of blowups)**.** *There exists $\rho = \rho(n,\alpha) > 0$ such that if $x_0, y_0 \in B_\rho$ are free boundary points, then*

$$\int_{\partial B_1} |u^\phi_{x_0,0} - u^\phi_{y_0,0}| \leq C|x_0 - y_0|^\gamma,$$

with $C = C\left(n, \alpha, \|u\|_{W^{1,2}(B_1)}\right)$ and $\gamma = \gamma(n,\alpha) > 0$.

Proof. Let $d = |x_0 - y_0|$ and $d^\mu \leq r \leq 2d^\mu$ with $\mu \in (0,1]$ to be determined. By Lemma 1.8.3 we have

$$\begin{aligned}\int_{\partial B_1} |u^\phi_{x_0,0} - u^\phi_{y_0,0}| &\leq 2Cr^{\delta/2} + \int_{\partial B_1} |u^\phi_{x_0,r} - u^\phi_{y_0,r}| \\ &\leq Cd^{\mu\delta/2} + \frac{C}{d^{\mu(n+1/2)}} \int_{\partial B_r} |u(x_0+z) - u(y_0+z)| dS_z\end{aligned}$$

and taking the average over $d^\mu \leq r \leq 2d^\mu$, we have

$$\int_{\partial B_1} |u^\phi_{x_0,0} - u^\phi_{y_0,0}| \leq Cd^{\mu\delta/2} + \frac{C}{d^{\mu(n+3/2)}} \int_{B_{2d^\mu}\setminus B_{d^\mu}} |u(x_0+z) - u(y_0+z)| dz.$$

On the other hand, by using Lemma 1.7.4,

$$\begin{aligned}\int_{B_{2d^\mu}\setminus B_{d^\mu}} |u(x_0+z) - u(y_0+z)| dz &\leq \int_{B_{2d^\mu}\setminus B_{d^\mu}} \left| \int_0^1 \frac{d}{ds} u(z + x_0(1-s) + y_0 s) ds \right| dz \\ &\leq |x_0 - y_0| \int_0^1 \int_{B_{2d^\mu}} |\nabla u(z + x_0(1-s) + y_0 s)| dz ds \\ &\leq d \int_0^1 \left(\int_{B_{2d^\mu}(x_0(1-s)+y_0 s)} |\nabla u| \right) ds \\ &\leq d \int_{B_{2d^\mu + d}(x_0)} |\nabla u| \leq d \int_{B_{3d^\mu}(x_0)} |\nabla u| \\ &\leq Cd^{1+\mu n/2} \left(\int_{B_{3d^\mu}(x_0)} |\nabla u|^2 \right)^{1/2} \leq Cd^{1+\mu n/2} d^{\mu(n+1)/2} \\ &\leq Cd^{1+\mu(n+1/2)},\end{aligned}$$

provided $3d^\mu < t_0$, which will hold if $d < \rho(n,\alpha)$.

Combining the estimates, we infer that

$$\int_{\partial B_1} |u^{\phi}_{x_0,0} - u^{\phi}_{y_0,0}| \le C d^{\mu\delta/2} + C d^{1-\mu}.$$

Now choosing μ so that $\mu\delta/2 = 1-\mu$, that is $\mu = 1/(1+\delta/2)$, we obtain

$$\int_{\partial B_1} |u^{\phi}_{x_0,0} - u^{\phi}_{y_0,0}| \le C|x_0 - y_0|^{\gamma}, \quad x_0, y_0 \in B'_\rho$$

with

$$\gamma = \frac{\delta}{\delta+2}. \qquad \square$$

1.9 Regularity of the regular set

In this section we establish one of the main result of this chapter, the $C^{1,\gamma}$ regularity of the regular set. In fact, the most technical part of the proof has already been done in the previous section, where we proved the uniqueness of the 3/2-homogeneous blowups, as well as their Hölder continuous dependence on the free boundary points.

We start by defining the regular set.

Definition 1.9.1 (Regular points)**.** *For an almost minimizer u for the Signorini problem in B_1, we say that a free boundary point x_0 is* regular *if*

$$\widehat{N}(0+, u, x_0) = 3/2.$$

Note that since $3/2 < 2 \le \kappa_0$, we will have that $\widehat{N}(r) < \kappa_0$ for small $r > 0$, implying that $\widetilde{N}(r) = \widehat{N}(r)$ for such r and consequently that

$$N(0+) = \widetilde{N}(0+) = \widehat{N}(0+) = 3/2.$$

In particular, the condition above does not depend on the choice of $\kappa_0 \ge 2$.

We denote the set of all regular points of u by $\mathcal{R}(u)$ and call it the regular set*.*

An important ingredient in the analysis of the regular set is the following nondegeneracy lemma.

Lemma 1.9.1 (Nondegeneracy at regular points)**.** *Let $x_0 \in B'_{1/2} \cap \mathcal{R}(u)$ for an almost minimizer u for the Signorini problem in B_1. Then, for $\kappa = 3/2$,*

$$\liminf_{t\to 0} \int_{\partial B_1} (u^{\phi}_{x_0,t})^2 = \liminf_{t\to 0} \frac{1}{t^{n+2}} \int_{\partial B_t(x_0)} u^2 > 0.$$

Proof. As before, assume $x_0 = 0$. In terms of the quantities defined in the proofs of Lemmas 1.7.1 and 1.7.4, we want to prove that

$$\liminf_{t\to 0} m(t) > 0.$$

Assume, towards a contradiction, that $m(t_{\mathrm{j}}) \to 0$ for some sequence $t_{\mathrm{j}} \to 0$. Recall that by the proof of Lemma 1.7.4, we have

$$|m(t) - m(s)| \leq Ct^{\delta/2}, \quad 0 < s < t < t_0.$$

Now, setting $s = t_{\mathrm{j}} \to 0$, we conclude that

$$|m(t)| \leq Ct^{\delta/2}, \quad 0 < t < t_0.$$

Equivalently, we can rewrite this as

$$\int_{\partial B_t} u^2 \leq Ct^{n+2+\delta}.$$

Next, take $\tilde{\kappa} = 3/2 + \delta/4$ and consider Weiss's monotonicity formula

$$W_{\tilde{\kappa}}(t, u) = \frac{\mathrm{e}^{a_{\tilde{\kappa}} t^{\alpha}}}{t^{n+2\tilde{\kappa}-2}} \left[\int_{B_t} |\nabla u|^2 - \tilde{\kappa} \frac{1 - bt^{\alpha}}{t} \int_{\partial B_t} u^2 \right].$$

Now observe that

$$\frac{1}{t^{n+2\tilde{\kappa}-1}} \int_{\partial B_t} u^2 \leq Ct^{\delta/2} \to 0,$$

which readily implies that

$$W_{\tilde{\kappa}}(0+,u) \geq 0.$$

In particular, by monotonicity, $W_{\tilde{\kappa}}(t,u) \geq 0$, for small $t > 0$, which also implies that $\widetilde{N}(t,u) \geq \tilde{\kappa}$. But then $N(0+,u) = \widetilde{N}(0+,u) \geq \tilde{\kappa} = 3/2 + \delta/4$ contrary to the assumption in the lemma. This completes the proof. $\square$

The next result provides two important facts: a gap in possible values of Almgren's frequency $N(0+)$ as well as the classification of $3/2$-homogeneous blowups.

Proposition 1.9.1. *If $\widehat{N}(0+,u,x_0) = \kappa < 2$, then $\kappa = 3/2$ and*

$$u^{\phi}_{x_0,0}(x) = a_{x_0} \operatorname{Re}(x' \cdot \nu_{x_0} + i|x_n|)^{3/2}$$

for some $a_{x_0} > 0$, $\nu_{x_0} \in \partial B'_1$.

Proof. Without loss of generality, we may assume $x_0 = 0$. Let $r_j \to 0+$ be a sequence such that $u^{\phi}_{r_j} \to u^{\phi}_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n_{\pm} \cup \mathbb{R}^{n-1})$. Comparing $3/2$-almost homogeneous and Almgren rescalings, we have

$$u^{\phi}_r(x) = u^A_r(x)\mu(r), \quad \mu(r) := \frac{\left(\frac{1}{r^{n-1}}\int_{\partial B_r} u^2\right)^{1/2}}{\phi(r)}.$$

By the optimal growth estimate (Lemma 1.7.4) and the nondegeneracy at regular points (Lemma 1.9.1) we have

$$0 < \liminf_{r\to 0+} \mu(r) \leq \limsup_{r \to 0+} \mu(r) < \infty.$$

Thus, we may assume that, over a subsequence of r_j, $\mu(r_j) \to \mu_0 \in (0,\infty)$, and therefore

$$u^{\phi}_{r_j} \to \mu_0 u^A_0 \quad \text{in } C^1_{\mathrm{loc}}(B^{\pm}_1 \cup B'_1),$$

where u^A_0 is an Almgren blowup of u at $x_0 = 0$. Now, since $\kappa < \kappa_0$, we can apply Proposition 1.6.1 to obtain that u^A_0 is a nonzero κ-homogeneous solution of the Signorini problem in B_1, even in x_n-variable. Next, applying Lemma 1.6.1, we have $3/2 \le \kappa < 2$ and thus by Proposition 9.9 in [48], we must have $\kappa = 3/2$ and

$$u_0^A(x) = C_n \operatorname{Re}(x' \cdot \nu_0 + i|x_n|)^{3/2}$$

for some $C_n > 0$, $\nu_0 \in \partial B_1$. (The constant C_n comes from the normalization $\int_{\partial B_1} (u_0^A)^2 = 1$.) Thus,

$$u_0^\phi(x) = a_0 \operatorname{Re}(x' \cdot \nu_0 + i|x_n|)^{3/2} \quad \text{in } B_1$$

with $a_0 = C_n \mu_0$. By the unique continuation of harmonic functions in $\mathbb{R}^n_\pm$, we obtain that the above formula for u_0^ϕ propagates to all of $\mathbb{R}^n$. $\square$

Proposition 1.9.1 has an immediate corollary.

Corollary 1.9.2 (Almgren frequency gap)**.** *Let u be an almost minimizer for the Signorini problem in B_1 and x_0 a free boundary point. Then either*

$$\widehat{N}(0+, u) = 3/2 \quad \textit{or} \quad \widehat{N}(0+, u) \ge 2.$$

Yet another important fact is as follows.

Corollary 1.9.3. *The regular set $\mathcal{R}(u)$ is a relatively open subset of the free boundary.*

Proof. For a fixed $0 < t < t_0$, the mapping $x \mapsto \widehat{N}(t, u, x)$ is continuous on $\Gamma(u)$. Then, by the monotonicity of $\widehat{N}$, the mapping $x \mapsto \widehat{N}(0+, u, x_0)$ is upper semicontinuous on $\Gamma(u)$. Moreover, by Proposition 1.9.1,

$$\mathcal{R}(u) = \{x \in \Gamma(u) : \widehat{N}(0+, u, x) < 2\},$$

which implies that $\mathcal{R}(u)$ is relatively open in $\Gamma(u)$. $\square$

The combination of Proposition 1.9.1 and Lemma 1.8.4 implies the following lemma.

Lemma 1.9.4. *Let u be an almost minimizer for the Signorini problem in B_1, and $x_0 \in \mathcal{R}(u)$. Then there exists $\rho > 0$, depending on x_0 such that $B'_\rho(x_0) \cap \Gamma(u) \subset \mathcal{R}(u)$ and if*

$u^{\phi}_{\bar{x},0}(x) = a_{\bar{x}} \operatorname{Re}(x' \cdot \nu_{\bar{x}} + i|x_n|)^{3/2}$ *is the unique* $3/2$*-homogeneous blowup at* $\bar{x} \in B'_\rho(x_0) \cap \Gamma(u)$, *then*

$$|a_{\bar{x}} - a_{\bar{y}}| \leq C_0 |\bar{x} - \bar{y}|^{\gamma},$$
$$|\nu_{\bar{x}} - \nu_{\bar{y}}| \leq C_0 |\bar{x} - \bar{y}|^{\gamma},$$

for any $\bar{x}, \bar{y} \in B'_\rho(x_0) \cap \Gamma(u)$ *with a constant* C_0 *depending on* x_0.

Proof. The proof follows by repeating the argument in Lemma 7.5 in [20]. □

Now we are ready to prove the main result on the regularity of the regular set.

Theorem 1.9.5 ($C^{1,\gamma}$ regularity of the regular set)**.** *Let* u *be an almost minimizer for the Signorini problem in* B_1*. Then, if* $x_0 \in B'_{1/2} \cap \mathcal{R}(u)$, *there exists* $\rho > 0$, *depending on* x_0 *such that, after a possible rotation of coordinate axes in* $\mathbb{R}^{n-1}$, *one has* $B'_\rho(x_0) \cap \Gamma(u) \subset \mathcal{R}(u)$, *and*

$$B'_\rho(x_0) \cap \Gamma(u) = B'_\rho(x_0) \cap \{x_{n-1} = g(x_1, \ldots, x_{n-2})\},$$

for $g \in C^{1,\gamma}(\mathbb{R}^{n-2})$ *with an exponent* $\gamma = \gamma(n, \alpha) \in (0, 1)$.

Proof. The proof of the theorem is similar to that of Theorem 1.2 in [20]. However, we provide full details since there are technical differences.

Step 1. By relative openness of $\mathcal{R}(u)$ in $\Gamma(u)$, for small $\rho > 0$ we have $B'_{2\rho}(x_0) \cap \Gamma(u) \subset \mathcal{R}(u)$. We then claim that for any $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that for $\bar{x} \in B'_\rho(x_0) \cap \Gamma(u)$, $r < r_\varepsilon$, we have that for $\phi = \phi_{3/2}$

$$\|u^{\phi}_{\bar{x},r} - u^{\phi}_{\bar{x},0}\|_{C^1(\overline{B_1^{\pm}})} < \varepsilon.$$

Assuming the contrary, there is a sequence of points $\bar{x}_j \in B'_\rho(x_0) \cap \Gamma(u)$ and radii $r_j \to 0$ such that

$$\|u^{\phi}_{\bar{x}_j,r_j} - u^{\phi}_{\bar{x}_j,0}\|_{C^1(\overline{B_1^{\pm}})} \geq \varepsilon_0$$

for some $\varepsilon_0 > 0$. Taking a subsequence, if necessary, we may assume $\bar{x}_{\rm j} \to \bar{x}_0 \in \overline{B'_\rho(x_0)} \cap \Gamma(u)$. Using estimates (1.3.1), (1.4.2) and Lemma 1.7.4, we can see that $u^\phi_{\bar{x}_{\rm j},r_{\rm j}}$ are uniformly bounded in $C^{1,\beta}(B_2^\pm \cup B'_2)$. Thus, we may assume that for some w

$$u^\phi_{\bar{x}_{\rm j},r_{\rm j}} \to w \quad \text{in } C^1(\overline{B_1^\pm}).$$

By arguing as in the proof of Proposition 1.6.1, we see that the limit w is a solution of the Signorini problem in B_1. Further, by Lemma 1.8.3, we have

$$\|u^\phi_{\bar{x}_{\rm j},r_{\rm j}} - u^\phi_{\bar{x}_{\rm j},0}\|_{L^1(\partial B_1)} \to 0.$$

On the other hand, by Lemma 1.9.4, we have

$$u^\phi_{\bar{x}_{\rm j},0} \to u^\phi_{\bar{x}_0,0} \quad \text{in } C^1(\overline{B_1^\pm}),$$

and thus

$$w = u^\phi_{\bar{x}_0,0} \quad \text{on } \partial B_1.$$

Since both w and $u^\phi_{\bar{x}_0,0}$ are solutions of the Signorini problem, they must coincide also in B_1. Therefore

$$u^\phi_{\bar{x}_{\rm j},r_{\rm j}} \to u^\phi_{\bar{x}_0,0} \quad \text{in } C^1(\overline{B_1^\pm}),$$

implying also that

$$\|u^\phi_{\bar{x}_{\rm j},r_{\rm j}} - u^\phi_{\bar{x}_{\rm j},0}\|_{C^1(\overline{B_1^\pm})} \to 0,$$

which contradicts our assumption.

Step 2. As [20], for a given $\varepsilon > 0$ and a unit vector $\nu \in \mathbb{R}^{n-1}$ define the cone

$$\mathcal{C}_\varepsilon(\nu) = \{x' \in \mathbb{R}^{n-1} : x' \cdot \nu > \varepsilon |x'|\}.$$

By Lemma 1.9.4, we may assume $a_{\bar{x}} \geq \frac{a_{x_0}}{2}$ for $\bar{x} \in B'_\rho(x_0) \cap \Gamma(u)$ by taking ρ small. For such ρ we then claim that for any $\varepsilon > 0$ there is $r_\varepsilon > 0$ such that for any $\bar{x} \in B'_\rho(x_0) \cap \Gamma(u)$ we have

$$\bar{x} + \left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right) \subset \{u(\cdot, 0) > 0\}.$$

Indeed, denoting $\mathcal{K}_\varepsilon(\nu) = \mathcal{C}_\varepsilon \cap \partial B'_{1/2}$, we have for some universal $C_\varepsilon > 0$

$$\mathcal{K}_\varepsilon(\nu_{\bar{x}}) \Subset \{u^\phi_{\bar{x},0}(\cdot, 0) > 0\} \cap B'_1 \quad \text{and} \quad u^\phi_{\bar{x},0}(\cdot, 0) \geq a_{\bar{x}} C_\varepsilon \geq \frac{a_{x_0}}{2} C_\varepsilon \quad \text{on } \mathcal{K}_\varepsilon(\nu_{\bar{x}}).$$

Since $\frac{a_{x_0}}{2} C_\varepsilon$ is independent of $\bar{x}$, by Step 1 we can find $r_\varepsilon > 0$ such that for $r < 2r_\varepsilon$,

$$u^\phi_{\bar{x},r}(\cdot, 0) > 0 \quad \text{on } \mathcal{K}_\varepsilon(\nu_{\bar{x}}).$$

This implies that for $r < 2r_\varepsilon$,

$$u(\cdot, 0) > 0 \quad \text{on} \quad \bar{x} + r\mathcal{K}_\varepsilon(\nu_{\bar{x}}) = \bar{x} + \left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap \partial B'_{r/2}\right).$$

Taking the union over all $r < 2r_\varepsilon$, we obtain

$$u(\cdot, 0) > 0 \quad \text{on } \bar{x} + \left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right).$$

Step 3. We claim that for given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for any $\bar{x} \in B'_\rho(x_0) \cap \Gamma(u)$ we have $\bar{x} - \left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right) \subset \{u(\cdot, 0) = 0\}$.

Indeed, we first note that

$$-\partial^+_{x_n} u^\phi_{\bar{x},0} \geq a_{\bar{x}} C_\varepsilon > \left(\frac{a_{x_0}}{2}\right) C_\varepsilon \quad \text{on } -\mathcal{K}_\varepsilon(\nu_{\bar{x}})$$

for a universal constant $C_\varepsilon > 0$. From Step 1, there exists $r_\varepsilon > 0$ such that for $r < 2r_\varepsilon$,

$$-\partial^+_{x_n} u^\phi_{\bar{x},r}(\cdot, 0) > 0 \quad \text{on } -\mathcal{K}_\varepsilon(\nu_{\bar{x}}).$$

By arguing as in Step 2, we obtain

$$-\partial_{x_n}^{+} u(\cdot,0) > 0 \quad \text{on } \bar{x} - \left(\mathcal{C}(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right).$$

By the complementarity condition in Lemma 1.4.2, we therefore conclude that

$$\bar{x} - \left(\mathcal{C}(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right) \subset \{-\partial_{x_n}^{+} u(\cdot,0) > 0\} \subset \{u(\cdot,0) = 0\}.$$

Step 4. By rotation in $\mathbb{R}^{n-1}$ we may assume $\nu_{x_0} = \mathrm{e}_{n-1}$. For any $\varepsilon > 0$, by Lemma 1.9.4 again, we can take $\rho_\varepsilon = \rho(x_0, \varepsilon)$, possibly smaller than ρ in the previous steps, such that

$$\mathcal{C}_{2\varepsilon}(\mathrm{e}_{n-1}) \cap B'_{r_\varepsilon} \subset \mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon} \quad \text{for } \bar{x} \in B'_{\rho_\varepsilon}(x_0) \cap \Gamma(u).$$

By Step 2 and Step 3, for $\bar{x} \in B'_{\rho_\varepsilon}(x_0) \cap \Gamma(u)$,

$$\begin{aligned} \bar{x} + \left(\mathcal{C}_{2\varepsilon}(\mathrm{e}_{n-1}) \cap B'_{r_\varepsilon}\right) &\subset \{u(\cdot,0) > 0\}, \\ \bar{x} - \left(\mathcal{C}_{2\varepsilon}(\mathrm{e}_{n-1}) \cap B'_{r_\varepsilon}\right) &\subset \{u(\cdot,0) = 0\}. \end{aligned}$$

Now, fixing $\varepsilon = \varepsilon_0$, by the standard arguments, we conclude that there exists a Lipschitz function $g : \mathbb{R}^{n-2} \to \mathbb{R}$ with $|\nabla g| \leq C_n/\varepsilon_0$ such that

$$\begin{aligned} B'_{\rho_{\mathrm{e}_0}}(x_0) \cap \{u(\cdot,0) = 0\} &= B'_{\rho_{\mathrm{e}_0}}(x_0) \cap \{x_{n-1} \leq g(x'')\}, \\ B'_{\rho_{\mathrm{e}_0}}(x_0) \cap \{u(\cdot,0) > 0\} &= B'_{\rho_{\mathrm{e}_0}}(x_0) \cap \{x_{n-1} > g(x'')\}. \end{aligned}$$

Step 5. Taking $\varepsilon \to 0$ in Step 4, $\Gamma(u)$ is differentiable at x_0 with normal ν_{x_0}. Recentering at any $\bar{x} \in B'_{\rho_{\varepsilon_0}}(x_0) \cap \Gamma(u)$, we see that $\Gamma(u)$ has a normal $\nu_{\bar{x}}$ at $\bar{x}$. By Lemma 1.9.4, we conclude that g in Step 4 is $C^{1,\gamma}$. This completes the proof of the theorem. □

1.10 Singular points

In this section we study the set of so-called singular free boundary points. An important technical tool to accomplish this is the logarithmic epiperimetric inequality of [27]. We use it for two purposes: to establish the optimal growth at singular points as well as the rate of convergence of rescalings to blowups, ultimately implying a structural theorem for the singular set.

Definition 1.10.1 (Singular points)**.** *Let u be an almost minimizer for the Signorini problem in B_1. We say that a free boundary point x_0 is* singular *if the coincidence set $\Lambda(u) = \{u(\cdot, 0) = 0\} \subset B_1'$ has zero H^{n-1}-density at x_0, i.e.,*

$$\lim_{r\to 0+} \frac{H^{n-1}\left(\Lambda(u) \cap B_r'(x_0)\right)}{H^{n-1}(B_r'(x_0))} = 0.$$

By using Almgren's rescalings $u^A_{x_0,r}$, we can rewrite this condition as

$$\lim_{r\to 0+} H^{n-1}(\Lambda(u^A_{x_0,r}) \cap B_1') = 0.$$

We denote the set of all singular points by $\Sigma(u)$ and call it the singular set*.*

Throughout the section we will assume that

$$\kappa_0 > 2.$$

We can take κ_0 as large as we like, however, we have to remember that the constants in $\widehat{N} = \widehat{N}_{\kappa_0}$ and W_κ do depend on κ_0.

We then have the following characterization of singular points, similar to Proposition 9.22 in [48] for the solutions of the Signorini problem.

Proposition 1.10.1 (Characterization of singular points)**.** *Let u be an almost minimizer for the Signorini problem in B_1, and $x_0 \in B_{1/2}' \cap \Gamma(u)$ be such that $\widehat{N}(0+, u, x_0) = \kappa < \kappa_0$. Then the following statements are equivalent.*

(i) $x_0 \in \Sigma(u)$.

(ii) *any Almgren blowup of u at x_0 is a nonzero polynomial from the class*

$$\mathcal{Q}_\kappa = \{q : q \text{ is homogeneous polynomial of degree } \kappa \text{ such that} \\ \Delta q = 0,\ q(x',0) \geq 0,\ q(x',x_n) = q(x',-x_n)\}.$$

(iii) *$\kappa = 2m$ for some $m \in \mathbb{N}$.*

Note that for $\kappa < \kappa_0$, the condition $\widehat{N}(0+) = \kappa$ is equivalent to $N(0+) = \kappa$.

Proof. Without loss of generality we may assume $x_0 = 0$. By Proposition 1.6.1, any Almgren blowup u_0^A of u at 0 is a nonzero global solution of the Signorini problem, homogeneous of degree κ. Moreover u_0^A is a C^1_{loc} limit of Almgren rescalings $u_{r_j}^A$ in $B_1^\pm \cup B_1'$. Because of that, most parts of the proof of this proposition are just the repetitions of Proposition 9.22 in [48]. Thus, by following Proposition 9.22 in [48], we can easily see the implications (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (i). Moreover, in the proof of the remaining implication (i) $\Rightarrow$ (ii), the only nontrivial part is that any blowup u_0^A is harmonic in B_1. But this comes from the complementarity condition in Lemma 1.4.2. Indeed, assuming (i), we claim that

$$\partial_{x_n}^+ u_0^A = 0 \quad \text{in} \quad B_1'.$$

Otherwise,

$$H^{n-1}\left(\{-\partial_{x_n}^+ u_0^A(\cdot,0) > 0\} \cap B_1'\right) \geq \delta$$

for some $\delta > 0$. Then using the continuity from the below we also have that for some $\rho > 0$,

$$H^{n-1}\left(\{-\partial_{x_n}^+ u_0^A(\cdot,0) > \rho\} \cap B_{1-\rho}'\right) \geq \delta/2.$$

Using C^1_{loc} convergence $u_{r_j}^A \to u_0^A$ in $B_1^\pm \cup B_1'$ and applying the complementarity condition in Lemma 1.4.2 to rescalings $u_{r_j}^A$, we obtain that for small r_j,

$$H^{n-1}\left(\Lambda(u_{r_j}^A) \cap B_1'\right) \geq H^{n-1}\left(\{-\partial_{x_n}^+ u_{r_j}^A(\cdot,0) > 0\} \cap B_1'\right) \geq \delta/4,$$

which contradicts (i). Now recalling that u_0^A is a solution of the Signorini problem, even in x_n-variable, it satisfies

$$\Delta u_0^A = 2(\partial_{x_n}^+ u_0^A) H^{n-1}|_{\Lambda(u_0^A)} = 0 \quad \text{in} \quad B_1.$$

By homogeneity, we obtain that u_0^A is harmonic in all of $\mathbb{R}^n$, and we complete the proof as in [48]. $\square$

In order to study the singular set, in view of Proposition 1.10.1, we need to refine the growth estimate in Lemma 1.7.1 by removing the logarithmic term in the case when $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$. In the case $\kappa = 3/2$ we were able to do so by proving a decay estimate for $W_{3/2}$ with the help of the epiperimetric inequality. In the case $\kappa = 2m$ we will use the so-called *logarithmic epiperimetric inequality* for the Weiss energy

$$W_\kappa^0(w) = \int_{B_1} |\nabla w|^2 - \kappa \int_{\partial B_1} w^2, \quad \kappa = 2m,\ m \in \mathbb{N}$$

that first appeared in [27]. To state this result, we recall the notation

$$\mathcal{A} = \{w \in W^{1,2}(B_1) : w \geq 0 \text{ on } B_1',\ w(x', x_n) = w(x', -x_n)\}.$$

Theorem 1.10.1 (Logarithmic epiperimetric inequality). *Let $\kappa = 2m$, $m \in \mathbb{N}$ and $w \in \mathcal{A}$ be homogeneous of degree κ in B_1 such that $w \in W^{1,2}(\partial B_1)$ and*

$$\int_{\partial B_1} w^2 \leq 1, \quad |W_\kappa^0(w)| \leq 1.$$

There is constant $\varepsilon = \varepsilon(n, \kappa) > 0$ and a function $v \in \mathcal{A}$ with $v = w$ on ∂B_1 such that

$$W_\kappa^0(v) \leq W_\kappa^0(w)(1 - \varepsilon |W_\kappa^0(w)|^\gamma), \quad \textit{where } \gamma = \frac{n-2}{n}.$$

To simplify the notations, in the results below all constants will depend on n, α, κ, κ_0, as well as $\|u\|_{W^{1,2}(B_1)}$, unless stated otherwise, in addition to other quantities. Thus, when we write $C = C(\sigma)$, we mean $C = C(n, \alpha, \kappa, \kappa_0, \|u\|_{W^{1,2}(B_1)}, \sigma)$.

The next lemma allows to apply the logarithmic epiperimetric inequality, without the constraints.

Lemma 1.10.2. *Let u be an almost minimizer for the Signorini problem in B_1 such that $0 \in \Gamma(u)$ and $\widehat{N}(0+, u) = \kappa < \kappa_0$, $\kappa = 2m$, $m \in \mathbb{N}$. For $0 < r < 1$, let*

$$u_r(x) = u_r^{(\kappa)}(x) = \frac{u(rx)}{r^\kappa}, \quad w_r(x) = |x|^\kappa u_r\left(\frac{x}{|x|}\right).$$

Suppose that for a given $0 \le \sigma \le 1$, there is $C = C(\sigma)$ such that

$$\int_{\partial B_r} u^2 \le C\left(\log\frac{1}{r}\right)^\sigma r^{n+2\kappa-1}.$$

Then there is a constant $\varepsilon = \varepsilon(\sigma) > 0$ and $h \in \mathcal{A}$ with $h = w_r$ on ∂B_1 such that

(i) *If $|W_\kappa^0(w_r)| \ge \int_{\partial B_1} w_r^2$, then*

$$W_\kappa^0(h) \le (1-\varepsilon)\, W_\kappa^0(w_r)$$

(ii) *If $|W_\kappa^0(w_r)| \le 2\int_{\partial B_1} w_r^2$, then*

$$W_\kappa^0(h) \le W_\kappa^0(w_r)\left(1 - \varepsilon\left(\log\frac{1}{r}\right)^{-\sigma\gamma}|W_\kappa^0(w_r)|^\gamma\right), \quad \textit{where } \gamma = \frac{n-2}{n}.$$

Proof. Let $A = \int_{\partial B_1} w_r^2 + |W_\kappa^0(w_r)|$. Then by Theorem 1.10.1 applied to $w_r/A^{1/2}$, there is $h \in \mathcal{A}$ such that $h = w_r$ on ∂B_1 and

$$W_\kappa^0(h) \le W_\kappa(w_r)^0\left(1 - \varepsilon A^{-\gamma}|W_\kappa^0(w_r)|^\gamma\right).$$

If $|W_\kappa^0(w_r)| \ge \int_{\partial B_1} w_r^2$, then $A \le 2|W_\kappa^0(w_r)|$, implying

$$W_\kappa^0(h) \le W_\kappa^0(w_r)\left(1 - \varepsilon 2^{-\gamma}\right).$$

If $|W^0_\kappa(w_r)| \leq 2\int_{\partial B_1} w_r^2$, then

$$A \leq 3\int_{\partial B_1} w_r^2 = \frac{3}{r^{n+2\kappa-1}}\int_{\partial B_r} u^2 \leq C(\sigma)\left(\log\frac{1}{r}\right)^\sigma.$$

This completes the proof. □

Now we show that the logarithmic epiperimetric inequality, combined with a growth estimate for u, implies a growth estimate on $W_\kappa(t,u)$. This is the first part of a bootstrapping argument that gradually decreases the power of $\log(1/t)$ in the bound for u.

Lemma 1.10.3. *Let u be an almost minimizer for the Signorini problem in B_1 such that $0 \in \Gamma(u)$ and $\widehat{N}(0+,u) = \kappa < \kappa_0$, $\kappa = 2m$, $m \in \mathbb{N}$. Suppose that for some $0 \leq \sigma \leq 1$*

$$\int_{\partial B_r} u^2 \leq C(\sigma)\left(\log\frac{1}{r}\right)^\sigma r^{n+2\kappa-1}, \quad 0 < r < r_0(\sigma).$$

Then,

$$0 \leq W_\kappa(t,u) \leq C(\sigma)\left(\log\frac{1}{t}\right)^{-\frac{1-\sigma\gamma}{\gamma}}, \quad 0 < t < t_0(\sigma).$$

Proof. We first observe that $W_\kappa(t,u) \geq 0$ for $0 < t < t_0$, which follows easily from the condition $\widehat{N}(0+,u) = \kappa < \kappa_0$, see the beginning of the proof of Lemma 1.7.1.

Next, recall that in the proof of Lemma 1.7.3, we have used epiperimetric inequality to show that $0 \leq W_{3/2}(t,u) \leq Ct^\delta$. This followed by obtaining a differential inequality for $W_{3/2}$. Thus, if for $0 < t < t_0$, if alternative (i) holds in Lemma 1.10.2, i.e., $W^0_\kappa(h) \leq (1-\varepsilon)W^0_\kappa(w_t)$, by arguing in the same way, we can show that

$$\frac{d}{dt}W_\kappa(t,u) \geq \frac{\varepsilon/4}{t}W_\kappa(t,u) - Ct^{\alpha/2-1}, \tag{1.10.1}$$

for $C = C(\sigma)$.

Suppose now the alternative (ii) holds in Lemma 1.10.2 for some $0 < t < t_0$. Then, following the computations in Lemma 1.7.3, we have

$$\frac{d}{dt}W_\kappa(t,u) \geq -\frac{(n+2\kappa-2)(1-t^\alpha)}{t}W_\kappa(t,u)$$

$$+\frac{e^{at^\alpha}(1-bt^\alpha)}{t}\int_{\partial B_1}(\partial_\nu u_t-\kappa u_t)^2+(\partial_\tau u_t)^2-\kappa(n+\kappa-2)u_t^2$$
$$+(2\kappa_0+n)t^{\alpha-1}\int_{\partial B_1}u_t^2.$$

For w_t as in the statement of Lemma 1.10.2, by following the computations in the proof of Theorem 1.5.1, we have the identity

$$\int_{\partial B_1}(\partial_\tau u_t)^2-\kappa(n+\kappa-2)u_t^2=(n+2\kappa-2)W_\kappa^0(w_t).$$

This gives

$$\begin{aligned}\frac{d}{dt}W_\kappa(t,u)\geq &-\frac{(n+2\kappa-2)(1-t^\alpha)}{t}W_\kappa(t,u)\\ &+\frac{e^{at^\alpha}(1-bt^\alpha)}{t}(n+2\kappa-2)W_\kappa^0(w_t)+(2\kappa_0+n)t^{\alpha-1}\int_{\partial B_1}u_t^2.\end{aligned}\tag{1.10.2}$$

Let now v_t be the solution of the Signorini problem in B_1 with $v_t=u_t=w_t$ on ∂B_1. Then

$$\begin{aligned}(1+t^\alpha)W_\kappa^0(w_t)\geq(1+t^\alpha)W_\kappa^0(v_t)&\geq\int_{B_1}|\nabla u_t|^2-\kappa(1+t^\alpha)\int_{\partial B_1}u_t^2\\ &=W_\kappa^0(u_t)-\kappa t^\alpha\int_{\partial B_1}u_t^2=e^{-at^\alpha}W_\kappa(t,u)-\kappa(b+1)t^\alpha\int_{\partial B_1}u_t^2.\end{aligned}\tag{1.10.3}$$

Now, if

$$e^{-at^\alpha}W_\kappa(t,u)-\kappa(b+1)t^\alpha\int_{\partial B_1}u_t^2\leq 0,$$

then by Lemma 1.7.1 we have

$$W_\kappa(t,u)\leq e^{at^\alpha}\kappa(b+1)t^\alpha\int_{\partial B_1}u_t^2\leq Ct^\alpha\left(\log\frac{1}{t}\right)\leq Ct^{\alpha/2}.\tag{1.10.4}$$

We then proceed under the assumption

$$e^{-at^\alpha}W_\kappa(t,u)-\kappa(b+1)t^\alpha\int_{\partial B_1}u_t^2>0,$$

which also implies

$$W_\kappa^0(w_t)>0.$$

Now, applying Lemma 1.10.2, we have

$$
\begin{aligned}
W^0_\kappa(w_t) &\geq W^0_\kappa(v_t) + \varepsilon \left(\log \frac{1}{t}\right)^{-\sigma\gamma} W^0_\kappa(w_t)^{\gamma+1} \\
&\geq \frac{1}{1+t^\alpha} \left[\mathrm{e}^{-at^\alpha} W_\kappa(t,u) - \kappa(b+1)t^\alpha \int_{\partial B_1} u_t^2\right] \\
&\quad + \varepsilon \left(\log \frac{1}{t}\right)^{-\sigma\gamma} \left(\frac{1}{1+t^\alpha}\right)^{\gamma+1} \times \\
&\qquad \times \left[\mathrm{e}^{-at^\alpha} W_\kappa(t,u) - \kappa(b+1)t^\alpha \int_{\partial B_1} u_t^2\right]^{\gamma+1} \\
&\geq (1-t^\alpha) \left[\mathrm{e}^{-at^\alpha} W_\kappa(t,u) - \kappa(b+1)t^\alpha \int_{\partial B_1} u_t^2\right] \\
&\quad + \varepsilon \left(\log \frac{1}{t}\right)^{-\sigma\gamma} (1-t^\alpha)^{\gamma+1} \times \\
&\qquad \times \left[\frac{\left(\mathrm{e}^{-at^\alpha} W_\kappa(t,u)\right)^{\gamma+1}}{2^\gamma} - \left(\kappa(b+1)t^\alpha \int_{\partial B_1} u_t^2\right)^{\gamma+1}\right] \\
&= (1-t^\alpha)\mathrm{e}^{-at^\alpha} W_\kappa(t,u) \\
&\quad + \frac{\varepsilon}{2^\gamma} \left(\log \frac{1}{t}\right)^{-\sigma\gamma} (1-t^\alpha)^{\gamma+1} \mathrm{e}^{-a(\gamma+1)t^\alpha} W_\kappa(t,u)^{\gamma+1} \\
&\quad - (1-t^\alpha)\kappa(b+1)t^\alpha \int_{\partial B_1} u_t^2 \\
&\quad - \varepsilon \left(\log \frac{1}{t}\right)^{-\sigma\gamma} (1-t^\alpha)^{\gamma+1} \kappa^{\gamma+1} (b+1)^{\gamma+1} t^{\alpha(\gamma+1)} \left(\int_{\partial B_1} u_t^2\right)^{\gamma+1},
\end{aligned}
\tag{1.10.5}
$$

where we used (1.10.3) in the second inequality and the convexity of $x \mapsto x^{\gamma+1}$ on $\mathbb{R}_+$ in the third inequality. Now (1.10.2) and (1.10.5), together with Lemma 1.7.1, yield

$$
\frac{d}{dt} W_\kappa(t,u) \geq -C_1 t^{\alpha-1} W_\kappa(t,u) + C_2 t^{-1} \left(\log \frac{1}{t}\right)^{-\sigma\gamma} W_\kappa(t,u)^{\gamma+1} - C_3 t^{\alpha/2-1}, \tag{1.10.6}
$$

where $C_i = C_i(\sigma)$. Summarizing, we have that at every $0 < t < t_0(\sigma)$, either (1.10.1), (1.10.6), or the bound (1.10.4) holds. Further note that by the growth estimate in Lemma 1.7.1, the bound (1.10.1) implies (1.10.6) for sufficiently small t and thus we may assume that (1.10.6) holds for all $0 < t < t_0$ for which $W_\kappa(t,u) > Ct^{\alpha/2}$.

To proceed, let $0 < t < t_0$ be such that $W_\kappa(t,u) \geq t^{\alpha/8}$. Then the bound (1.10.6) holds and we can derive that for $C = \frac{\gamma C_2}{2(1-\sigma\gamma)}$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(-W_\kappa(t,u)^{-\gamma}\mathrm{e}^{-t^{\alpha/4}} + C\left(\log\frac{1}{t}\right)^{1-\sigma\gamma}\right) \\
&\quad = W_\kappa(t,u)^{-\gamma-1}\mathrm{e}^{-t^{\alpha/4}}\left(\gamma\frac{d}{dt}W_\kappa(t,u) + \frac{\alpha}{4}W_\kappa(t,u)t^{\alpha/4-1}\right) - C(1-\sigma\gamma)t^{-1}\left(\log\frac{1}{t}\right)^{-\sigma\gamma} \\
&\quad \geq W_\kappa(t,u)^{-\gamma}\mathrm{e}^{-t^{\alpha/4}}t^{\alpha/4-1}\left(\frac{\alpha}{4} - \gamma C_1 t^{3\alpha/4} - \frac{\gamma C_3 t^{\alpha/4}}{W_\kappa(t,u)}\right) \\
&\qquad + \left(\log\frac{1}{t}\right)^{-\sigma\gamma} t^{-1}\left(\mathrm{e}^{-t^{\alpha/4}}\gamma C_2 - C(1-\sigma\gamma)\right) \\
&\quad \geq 0,
\end{aligned}$$

$0 < t < t_0 = t_0(\sigma)$. Since also the function $-t^{-\gamma(\alpha/8)}\mathrm{e}^{-t^{\alpha/4}} + C\left(\log\frac{1}{t}\right)^{1-\sigma\gamma}$ is nondecreasing for small t, denoting

$$\widehat{W}_\kappa(t,u) = \max\{W_\kappa(t,u), t^{\alpha/8}\},$$

we obtain that the function

$$-\widehat{W}_\kappa(t,u)^{-\gamma}\mathrm{e}^{-t^{\alpha/4}} + C\left(\log\frac{1}{t}\right)^{1-\sigma\gamma}$$

is nondecreasing on $(0, t_0)$. Hence,

$$\begin{aligned}
-\widehat{W}_\kappa(t,u)^{-\gamma}\mathrm{e}^{-t^{\alpha/4}} + C\left(\log\frac{1}{t}\right)^{1-\sigma\gamma} &\leq -\widehat{W}_\kappa(t_0,u)^{-\gamma}\mathrm{e}^{-t_0^{\alpha/4}} + C\left(\log\frac{1}{t_0}\right)^{1-\sigma\gamma} \\
&\leq C\left(\log\frac{1}{t_0}\right)^{1-\sigma\gamma}.
\end{aligned}$$

If $0 < t < t_0^2$, then $\left(\log\frac{1}{t_0}\right)^{1-\sigma\gamma} < \left(\frac{1}{2}\right)^{1-\sigma\gamma}\left(\log\frac{1}{t}\right)^{1-\sigma\gamma}$, implying that

$$-\widehat{W}_\kappa(t,u)^{-\gamma}\mathrm{e}^{-t^{\alpha/4}} \leq C\left((1/2)^{1-\sigma\gamma} - 1\right)\left(\log\frac{1}{t}\right)^{1-\sigma\gamma}$$

and hence

$$W_\kappa(t,u) \leq \widehat{W}_\kappa(t,u) \leq C\left(1 - (1/2)^{1-\sigma\gamma}\right)^{-\frac{1}{\gamma}}\left(\log\frac{1}{t}\right)^{-\frac{1-\sigma\gamma}{\gamma}}. \qquad \square$$

Lemma 1.10.4. *If u is as in Lemma 1.10.3 with $\frac{2}{n-2} < \sigma \leq 1$, then there exist positive $C = C(\sigma)$, $t_0 = t_0(\sigma)$ such that*

$$\int_{\partial B_t} u^2 \leq C \left(\log \frac{1}{t}\right)^{\sigma - \frac{2}{n-2}} t^{n+2\kappa-1}, \quad 0 < t < t_0.$$

Proof. Going back to the proof and notations of Lemma 1.7.1, we have that for $0 < s < t < t_0$

$$|m(t) - m(s)| \leq C \left(\log \frac{t}{s}\right)^{1/2} \left(W_\kappa(t) - W_\kappa(s)\right)^{1/2}.$$

Let now $0 \leq \mathrm{j} \leq i$ be such that $2^{-2^{i+1}} < t \leq 2^{-2^i}$, $2^{-2^{\mathrm{j}+1}} < t_0 \leq 2^{-2^{\mathrm{j}}}$. Then

$$\begin{aligned}
|m(t_0) - m(t)| &\leq |m(t_0) - m(2^{-2^{\mathrm{j}+1}})| + |m(2^{-2^i}) - m(t)| + \sum_{k=\mathrm{j}+1}^{i-1} |m(2^{-2^k}) - m(2^{-2^{k+1}})| \\
&\leq \sum_{k=0}^{i} C \left[\log\left(2^{-2^k}\right) - \log\left(2^{-2^{k+1}}\right)\right]^{1/2} \left[W_\kappa\left(2^{-2^k}\right) - W_\kappa\left(2^{-2^{k+1}}\right)\right]^{1/2} \\
&\leq C \sum_{k=0}^{i} 2^{k/2} W_\kappa \left(2^{-2^k}\right)^{1/2} \leq C \sum_{k=0}^{i} 2^{(1-\frac{1-\sigma\gamma}{\gamma})k/2} \\
&\leq C 2^{(\sigma - \frac{2}{n-2})i/2} \leq C \left(\log \frac{1}{t}\right)^{\frac{1}{2}(\sigma - \frac{2}{n-2})}.
\end{aligned}$$

Note that in the fifth inequality we have used that $1 - \frac{1-\sigma\gamma}{\gamma} = \sigma - \frac{2}{n-2} > 0$. Thus

$$m(t) \leq m(t_0) + C \left(\log \frac{1}{t}\right)^{\frac{1}{2}(\sigma - \frac{2}{n-2})} \leq C \left(\log \frac{1}{t}\right)^{\frac{1}{2}(\sigma - \frac{2}{n-2})}.$$

This implies the desired result. $\square$

Lemma 1.10.3 and Lemma 1.10.4 imply the following.

Corollary 1.10.5 (Bootstraping). *Let u be an almost minimizer for the Signorini problem in B_1 such that $0 \in \Gamma(u)$ and $\widehat{N}(0+, u) = \kappa < \kappa_0$, $\kappa = 2m$, $m \in \mathbb{N}$. Suppose that for $\frac{2}{n-2} < \leq 1$*

$$\int_{\partial B_t} u^2 \leq C(\sigma) \left(\log \frac{1}{t}\right)^{\sigma} t^{n+2\kappa-1}, \quad 0 < t < t_0(\sigma).$$

Then

$$\int_{\partial B_t} u^2 \le C'(\sigma)\left(\log\frac{1}{t}\right)^{\sigma-\frac{2}{n-2}} t^{n+2\kappa-1}, \quad 0<t<t_0'(\sigma).$$

Lemma 1.10.6 (Optimal growth estimate at sigular points)**.** *Let u be an almost minimizer for the Signorini problem in B_1 such that $0\in\Gamma(u)$ and $\widehat{N}(0+,u)=\kappa<\kappa_0$, $\kappa=2m$, $m\in\mathbb{N}$. Then, for $0<t<t_0$,*

$$\int_{\partial B_t} u^2 \le Ct^{n+2\kappa-1},$$
$$\int_{B_t} |\nabla u|^2 \le Ct^{n+2\kappa-2}.$$

Proof. Starting with $\sigma=1$ in Lemma 1.7.1 and repeatedly applying Corollary 1.10.5, we find $0<\le\min\{\frac{2}{n-2},1\}$ such that

$$\int_{\partial B_t} u^2 \le C\left(\log\frac{1}{t}\right)^{\sigma} t^{n+2\kappa-1}, \quad 0<t<t_0.$$

In fact, we can make σ to be strictly less than $\frac{2}{n-2}$ by noticing that in Lemma 1.10.4 we can replace $\frac{2}{n-2}$ by any smaller positive number. Then by Lemma 1.10.3

$$0\le W_\kappa(t,u)\le C\left(\log\frac{1}{t}\right)^{-\frac{1-\sigma\gamma}{\gamma}}.$$

Recall also that for $0<s<t<t_0$

$$|m(t)-m(s)|\le C\left(\log\frac{t}{s}\right)^{1/2}\left(W_\kappa(t)-W_\kappa(s)\right)^{1/2}.$$

We then again consider the exponentially dyadic decomposition as in the proof of Lemma 1.10.4. Let $0\le \mathrm{j}\le i$ be such that $2^{-2^{i+1}}\le s/t_0<2^{-2^i}$ and $2^{-2^{\mathrm{j}+1}}\le t/t_0<2^{-2^{\mathrm{j}}}$. Then,

$$\begin{aligned}|m(t)-m(s)| &\le C\sum_{k=\mathrm{j}}^{i} 2^{k/2}W_\kappa(2^{-2^k}t_0)^{1/2}\le C\sum_{k=\mathrm{j}}^{\infty}2^{\left(1-\frac{1-\sigma\gamma}{\gamma}\right)k/2}\\ &\le C2^{\left(\sigma-\frac{2}{n-2}\right)\mathrm{j}/2}\le C\left(\log\frac{1}{t}\right)^{\left(\sigma-\frac{2}{n-2}\right)/2}.\end{aligned} \tag{1.10.7}$$

Particularly,

$$m(t) \leq m(t_0) + C\left(\log\frac{1}{t_0}\right)^{\left(\sigma - \frac{2}{n-2}\right)/2}.$$

This gives the first bound. The second bound is obtained from the first one by arguing as at the end of Lemma 1.7.1. □

Remark 1.10.7. The growth estimates in Lemma 1.10.6 enable us to consider *κ-homogeneous blowups*

$$u^{\phi}_{t_j} \to u^{\phi}_0 \quad \text{in} \quad C^1_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1}).$$

for $t = t_j \to 0+$, similar to $3/2$-homogeneous blowups, defined at the beginning of Section 1.7, see Remark 1.8.1.

Proposition 1.10.2. *Let u be an almost minimizer for the Signorini problem in B_1 such that $0 \in \Gamma(u)$ and $\widehat{N}(0+, u) = \kappa < \kappa_0$, $\kappa = 2m$, $m \in \mathbb{N}$. Then there exist $C > 0$ and $t_0 > 0$ such that*

$$\int_{\partial B_1} |u^\phi_t - u^\phi_s| \leq C\left(\log\frac{1}{t}\right)^{-\frac{1-\gamma}{2\gamma}}, \quad 0 < t < t_0.$$

In particular the blowup u^ϕ_0 is unique.

Proof. Using Lemma 1.10.6, we apply Lemma 1.10.3 with =0 to obtain

$$0 \leq W_\kappa(t, u) \leq C\left(\log\frac{1}{t}\right)^{-\frac{1}{\gamma}}.$$

Recall now the estimate

$$\int_{\partial B_1} |u^\phi_t - u^\phi_s| \leq C\left(\log\frac{t}{s}\right)^{1/2} (W_\kappa(t) - W_\kappa(s))^{1/2},$$

for $0 < s < t < t_0$, that we proved in Lemma 1.8.2 in the case $\kappa = 3/2$ – the proof actually works for any $0 < \kappa < \kappa_0$. Then, applying the exponentially dyadic argument as in the proof of Lemma 1.10.6, we obtain

$$\int_{\partial B_1} |u^\phi_t - u^\phi_s| \leq C\left(\log\frac{1}{t}\right)^{-\frac{1-\gamma}{2\gamma}}.$$

□

Lemma 1.10.8 (Nondegeneracy)**.** *Let* 0 *be a free boundary point of* u *such that* $\widehat{N}(0+,u)=\kappa$*,* $\kappa=2m$*,* $m\in\mathbb{N}$*. Then*

$$\liminf_{t\to 0}\int_{\partial B_1}(u_t^{\phi})^2=\liminf_{t\to 0}\frac{1}{t^{n+2\kappa-1}}\int_{\partial B_t}u^2>0.$$

Proof. We use the approach of Lemma 7.2 in [27]. Assume to the contrary that for some $r_{\mathrm{j}}\searrow 0+$

$$\lim_{\mathrm{j}\to\infty}\frac{1}{r_{\mathrm{j}}^{n+2\kappa-1}}\int_{\partial B_{r_{\mathrm{j}}}}u^2=0.$$

Consider then the corresponding Almgren rescalings $u^A_{r_{\mathrm{j}}}(x)$. By Proposition 1.6.1, over a subsequence, $u^A_{r_{\mathrm{j}}}\to q$ for some blowup q. By a characterization of singular points in Proposition 1.10.1, q is κ-homogeneous and is normalized by $\|q\|_{L^2(\partial B_1)}=1$. Next, for each Almgren rescaling $u^A_{r_{\mathrm{j}}}$ consider its κ-almost homogeneous rescalings

$$[u^A_{r_{\mathrm{j}}}]^{\phi}_t:=\frac{u^A_{r_{\mathrm{j}}}(tx)}{\phi(t)}.$$

Since $u^A_{r_{\mathrm{j}}}$ is an almost minimizer in $B_{1/r_{\mathrm{j}}}$ with gauge function $\omega(t)=(r_{\mathrm{j}}t)^{\alpha}$, we have

$$N(0+,u^A_{r_{\mathrm{j}}})=\lim_{s\to 0+}N(s,u^A_{r_{\mathrm{j}}})=\lim_{s\to 0+}N(r_{\mathrm{j}}s,u)=N(0+,u)=\kappa.$$

Thus, by Proposition 1.10.2, over subsequences, $[u^A_{r_{\mathrm{j}}}]^{\phi}_t$ converges to a unique blowup $q_{r_{\mathrm{j}}}$ and

$$\int_{\partial B_1}\left|[u^A_{r_{\mathrm{j}}}]^{\phi}_t-q_{r_{\mathrm{j}}}\right|\le C\left(\log\frac{1}{t}\right)^{-\frac{1-\gamma}{2\gamma}},\quad 0<t<t_0.$$

Notice that since $\|u^A_{r_{\mathrm{j}}}\|_{W^{1,2}(B_1)}$ is uniformly bounded, the constant C is independent of r_{j}, t. Now we fix r_{j}, and consider a sequence $\{\rho_i\}_{i=1}^{\infty}=\{r_i/r_{\mathrm{j}}\}_{i=1}^{\infty}$. Note that up to subsequence, $[u^A_{r_{\mathrm{j}}}]^{\phi}_{\rho_i}\to q_{r_{\mathrm{j}}}$ as $\rho_i\to 0$, by the uniqueness. Then

$$\begin{aligned}\int_{\partial B_1}q_{r_{\mathrm{j}}}^2&=\lim_{\rho_i\to 0}\frac{1}{\rho_i^{n+2\kappa-1}}\int_{\partial B_{\rho_i}}(u^A_{r_{\mathrm{j}}})^2\frac{r_{\mathrm{j}}^{n+2\kappa-1}}{\int_{\partial B_{r_{\mathrm{j}}}}u^2}\lim_{i\to\infty}\frac{1}{(r_{\mathrm{j}}\rho_i)^{n+2\kappa-1}}\int_{\partial B_{r_{\mathrm{j}}\rho_i}}u^2\\&=\frac{r_{\mathrm{j}}^{n+2\kappa-1}}{\int_{\partial B_{r_{\mathrm{j}}}}u^2}\lim_{i\to\infty}\frac{1}{r_i^{n+2\kappa-1}}\int_{\partial B_{r_i}}u^2=0\end{aligned}$$

by the contradiction assumption. Thus, $q_{r_{\mathrm{j}}} = 0$ on ∂B_1, and hence

$$\int_{\partial B_1} \left| [u^A_{r_{\mathrm{j}}}]^\phi_t \right| \le C \left(\log \frac{1}{t} \right)^{-\frac{1-\gamma}{2\gamma}}.$$

Now for any $\rho > 0$ and r_{j},

$$\begin{aligned}
1 &= \frac{1}{\rho^{n+2\kappa-1}} \int_{\partial B_\rho} q^2 \\
&\le \frac{\|q\|_{L^\infty(\partial B_\rho)}}{\rho^\kappa} \frac{1}{\rho^{n+\kappa-1}} \int_{\partial B_\rho} |q| \\
&\le \|q\|_{L^\infty(\partial B_1)} \left[\frac{1}{\rho^{n+\kappa-1}} \int_{\partial B_\rho} |q - u^A_{r_{\mathrm{j}}}| + \frac{1}{\rho^{n+\kappa-1}} \int_{\partial B_\rho} |u^A_{r_{\mathrm{j}}}| \right] \\
&\le \|q\|_{L^\infty(\partial B_1)} \left[\frac{1}{\rho^{n+\kappa-1}} C_n \rho^{\frac{n-1}{2}} \left(\int_{\partial B_\rho} |q - u^A_{r_{\mathrm{j}}}|^2 \right)^{1/2} + \mathrm{e}^{-\left(\frac{\kappa b}{\alpha}\right)\rho^\alpha} \int_{\partial B_1} \left| [u^A_{r_{\mathrm{j}}}]^\phi_\rho \right| \right] \\
&\le C \|q\|_{L^\infty(\partial B_1)} \left[\left(\frac{1}{\rho^{n+2\kappa-1}} \int_{\partial B_\rho} |q - u^A_{r_{\mathrm{j}}}|^2 \right)^{1/2} + \left(\log \frac{1}{\rho} \right)^{-\frac{1-\gamma}{2\gamma}} \right].
\end{aligned}$$

Note that $u^A_{r_{\mathrm{j}}} \to q$ in $C^1_{\mathrm{loc}}(B_1^\pm \cup B_1')$. We choose first $\rho > 0$ small and then $r_{\mathrm{j}} = r_{\mathrm{j}}(\rho) > 0$ small to reach a contradiction. □

The nondegeneracy implies the following important fact, which enables the use of the Whitney Extension Theorem in the proof of the structural theorem on the singular set (Theorem 1.10.10 below).

For $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, we denote

$$\Sigma_\kappa(u) := \{x_0 \in \Sigma(u) : N(0+, u, x_0) = \kappa\}.$$

Lemma 1.10.9. *The set $\Sigma_\kappa(u)$ is of topological type F_σ; i.e., it is a countable union of closed sets.*

Proof. For $\mathrm{j} \in \mathbb{N}$, $\mathrm{j} \ge 2$, let

$$E_{\mathrm{j}} := \left\{ x_0 \in \Sigma_\kappa(u) \cap \overline{B_{1-1/\mathrm{j}}} : \frac{1}{\mathrm{j}} \le \frac{1}{\rho^{n+2\kappa-1}} \int_{\partial B_\rho(x_0)} u^2 \le \mathrm{j} \text{ for } 0 < \rho < \frac{1}{2\mathrm{j}} \right\}.$$

Then by Lemma 1.10.6 and Lemma 1.10.8, $\Sigma_\kappa(u) = \bigcup_{\mathrm{j}=2}^{\infty} E_{\mathrm{j}}$. We now claim that E_{j} is closed for any $\mathrm{j} \geq 2$. Indeed, take a sequence $x_i \in E_{\mathrm{j}}$ such that $x_i \to x_0$ as $i \to \infty$. Then $x_0 \in \overline{B_{1-1/\mathrm{j}}}$ and for every $0 < \rho < 1/(2\mathrm{j})$, by the local uniform continuity of u,

$$\frac{1}{\rho^{n+2\kappa-1}} \int_{\partial B_\rho(x_0)} u^2 = \lim_{i\to\infty} \frac{1}{\rho^{n+2\kappa-1}} \int_{\partial B_\rho(x_i)} u^2 \in \left[\frac{1}{\mathrm{j}}, \mathrm{j}\right]. \tag{1.10.8}$$

Next, since $\Gamma(u)$ is relatively closed in B_1', we also know that $x_0 \in \Gamma(u)$. Moreover, since $N(0+, u, x_i) = \kappa$ and the function $x \mapsto \widehat{N}(0+, u, x)$ is upper semicontinuous, we have

$$\kappa = \limsup_{i\to\infty} \widehat{N}(0+, u, x_i) \leq \widehat{N}(0+, u, x_0).$$

If $\widehat{N}(0+, u, x_0) = \kappa' > \kappa$, then by Lemma 1.7.1,

$$\frac{1}{\rho^{n+2\kappa-1}} \int_{\partial B_\rho(x_0)} u^2 \leq C\rho^{2(\kappa'-\kappa)} \left(\log \frac{1}{\rho}\right) \to 0 \quad \text{as } \rho \to 0,$$

which contradicts (1.10.8). Therefore, $\widehat{N}(0+, u, x_0) = \kappa$ and consequently $x_0 \in E_{\mathrm{j}}$. Hence, E_{j} is closed, $\mathrm{j} = 2, 3, \ldots$, implying that $\Sigma_\kappa(u)$ is F_σ. □

To state the main result of this chapter concerning the singular points, we need to introduce the following notations. For $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$ and $x_0 \in \Sigma_\kappa(u)$, we define

$$d_{x_0}^{(\kappa)} := \dim\{\xi \in \mathbb{R}^{n-1} : \xi \cdot \nabla_x u_{x_0}^{\phi}(x, 0) \equiv 0 \text{ on } \mathbb{R}^{n-1}\},$$

which has the meaning of the dimension of $\Sigma_\kappa(u)$ at x_0, and where $u_{x_0}^{\phi}$ is the unique κ-homogeneous blowup at x_0. In fact, $d_{x_0}^{(\kappa)}$ is the dimension of the linear subspace $\Sigma_\kappa(u_{x_0}^{\phi}) \subset \mathbb{R}^{n-1}$. Since $u_{x_0}^{\phi}$ is a nonzero solution of the Signorini problem, it cannot vanish identically on $\mathbb{R}^{n-1}$ (see [14]) and therefore $d_{x_0}^{(\kappa)} < n-1$.

For $d = 0, 1, \ldots, n-2$, we denote

$$\Sigma_\kappa^d(u) := \{x_0 \in \Sigma_\kappa(u) : d_{x_0}^{(\kappa)} = d\}.$$

Theorem 1.10.10 (Structure of the singular set)**.** *Let u be an almost minimizer for the Signorini problem in B_1. Then for every $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, and $d = 0, 1, \ldots, n-2$, the set $\Sigma_\kappa^d(u)$ is contained in the union of countably many submanifolds of dimension d and class $C^{1,\log}$.*

Proof. Let $\kappa = 2m$, $m \in \mathbb{N}$. For $x \in \Sigma_\kappa(u) \cap B'_{1/2}$, let $q_x \in \mathcal{Q}_\kappa$ denote the unique κ-homogeneous blowup of u. By the optimal growth (Lemma 1.10.6) and the nondegeneracy (Lemma 1.10.8), we can write

$$q_x = \lambda_x q_x^A, \quad \lambda_x > 0, \quad \|q_x^A\|_{L^2(\partial B_1)} = 1,$$

where $q_x^A \in \mathcal{Q}_\kappa$ is the corresponding Almgren blowup. We want to show that the q_x, q_x^A, λ_x depend continuously on $x \in \Sigma_\kappa$, with a logarithmic modulus of continuity.

Let $x_1, x_2 \in \Sigma_\kappa(u) \cap B'_{1/2}$. Then for $t > 0$, to be chosen below, we can write

$$\|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} \leq \|q_{x_1} - u^\phi_{x_1,t}\|_{L^1(\partial B_1)} + \|u^\phi_{x_1,t} - u^\phi_{x_2,t}\|_{L^1(\partial B_1)} + \|u^\phi_{x_2,t} - q_{x_2}\|_{L^1(\partial B_1)}. \quad (1.10.9)$$

By Proposition 1.10.2, we have

$$\|q_x - u^\phi_{x,t}\|_{L^1(\partial B_1)} \leq C\left(\log\frac{1}{t}\right)^{-\frac{1}{n-2}} \qquad (1.10.10)$$

for $x \in \Sigma_\kappa(u) \cap B'_{1/2}$. This controls the first and third term on the right hand side of (1.10.9) To estimate the middle term, we observe that

$$\|u^\phi_{x_1,t} - u^\phi_{x_2,t}\|_{L^1(\partial B_1)} \leq \frac{\mathrm{e}^{\left(\frac{\kappa b}{\alpha}\right)t^\alpha}}{t^\kappa}\int_{\partial B_1}\int_0^1 |\nabla u(x_1 + tz + r(x_2 - x_1))|\,|x_1 - x_2|\,dr\,dS_z$$

for any $0 < t < 1/2$. Recalling that $\nabla u(x_1) = 0$ and $u \in C^{1,\beta}(B_1^\pm \cup B_1)$, we have

$$|\nabla u(x_1 + tz + r(x_2 - x_1))| \leq C|tz + r(x_2 - x_1)|^\beta \leq C(t + |x_1 - x_2|)^\beta \leq C|x_1 - x_2|^{\frac{\beta}{2(\kappa-\beta)}}$$

if we choose $t = |x_1 - x_2|^{\frac{1}{2(\kappa-\beta)}}$ and have $|x_1 - x_2| < (1/2)^{2(\kappa-\beta)}$. This gives

$$\|u^\phi_{x_1,t} - u^\phi_{x_2,t}\|_{L^1(\partial B_1)} \leq \frac{C}{t^\kappa}|x_1 - x_2|^{\frac{\beta}{2(\kappa-\beta)}}|x_1 - x_2| \leq C|x_1 - x_2|^{1/2}. \tag{1.10.11}$$

Combining (1.10.9), (1.10.11), and (1.10.10), we obtain

$$\|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} \leq C\left(\log\frac{1}{|x_1 - x_2|}\right)^{-\frac{1}{n-2}}. \tag{1.10.12}$$

Next, by Lemma 1.10.6, for any $x \in \Sigma_\kappa(u) \cap B'_{1/2}$ and small t

$$\int_{\partial B_1}(u^\phi_{x,t})^2 \leq C$$

with C independent of x, and passing to the limit as $t \to \infty$ obtain the bound

$$\lambda_x^2 = \int_{\partial B_1} q_x^2 \leq C$$

Moreover, since q_x is a κ-homogeneous harmonic polynomial, we also have

$$\|q_x\|_{L^\infty(B_1)} \leq C(n,\kappa)\|q_x\|_{L^2(\partial B_1)} \leq C. \tag{1.10.13}$$

Then, by combining (1.10.12) and (1.10.13), we have

$$\begin{aligned}
|\lambda_{x_1} - \lambda_{x_2}| &\leq |\lambda_{x_1}^2 - \lambda_{x_2}^2|^{1/2} \leq \left(\int_{\partial B_1}|q_{x_1}^2 - q_{x_2}^2|\right)^{1/2} \\
&\leq \|q_{x_1} + q_{x_2}\|^{1/2}_{L^\infty(B_1)}\|q_{x_1} - q_{x_1}\|^{1/2}_{L^1(\partial B_1)} \\
&\leq C\left(\log\frac{1}{|x_1 - x_2|}\right)^{-\frac{1}{2(n-2)}}.
\end{aligned} \tag{1.10.14}$$

Finally, we want to estimate $q^A_{x_1} - q^A_{x_2}$. By writing

$$\begin{aligned}
\|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} &= \int_{\partial B_1}|\lambda_{x_1}q^A_{x_1} - \lambda_{x_2}q^A_{x_2}| \\
&= \int_{\partial B_1}|\lambda_{x_1}(q^A_{x_1} - q^A_{x_2}) + (\lambda_{x_1} - \lambda_{x_2})q^A_{x_2}|
\end{aligned}$$

$$\geq \lambda_{x_1} \int_{\partial B_1} |q^A_{x_1} - q^A_{x_2}| - |\lambda_{x_1} - \lambda_{x_2}| \int_{\partial B_1} |q^A_{x_2}|,$$

we estimate

$$\begin{aligned} \lambda_{x_1} \int_{\partial B_1} |q^A_{x_1} - q^A_{x_2}| &\leq \|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} + |\lambda_{x_1} - \lambda_{x_2}| \int_{\partial B_1} |q^A_{x_2}| \\ &\leq \|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} + C(n)|\lambda_{x_1} - \lambda_{x_2}| \\ &\leq C\left(\log \frac{1}{|x_1 - x_2|}\right)^{-\frac{1}{2(n-2)}}, \end{aligned} \tag{1.10.15}$$

where we used $\|q^A_{x_2}\|_{L^2(\partial B_1)} = 1$ in the second inequality and (1.10.12) and the bound (1.10.14) in the third inequality. Next, using that q^A_x are κ-homogeneous harmonic polynomials, we have

$$\|q^A_{x_1} - q^A_{x_2}\|_{L^\infty(B_1)} \leq C\|q^A_{x_1} - q^A_{x_2}\|_{L^1(\partial B_1)},$$

which combined with (1.10.15) gives

$$\lambda_{x_1}\|q^A_{x_1} - q^A_{x_2}\|_{L^\infty(B_1)} \leq C\left(\log \frac{1}{|x_1 - x_2|}\right)^{-\frac{1}{2(n-2)}}. \tag{1.10.16}$$

Now we fix $x_0 \in \Sigma_\kappa(u) \cap B'_{1/4}$. Then by (1.10.14), there exists $\delta = \delta(x_0) \in \left(0, (1/2)^{2(\kappa-\beta)+1}\right)$ such that $\lambda_x \geq 1/2\lambda_{x_0}$ if $x \in \Sigma_\kappa(u) \cap B'_\delta(x_0)$. Then by (1.10.16), we conclude that

$$\|q^A_{x_1} - q^A_{x_2}\|_{L^\infty(B_1)} \leq C\left(\log \frac{1}{|x_1 - x_2|}\right)^{-\frac{1}{2(n-2)}}, \quad x_1, x_2 \in \Sigma_\kappa(u) \cap B'_\delta(x_0). \tag{1.10.17}$$

Notice that the constant C does not depend on x_1, x_2, but both C and δ do depend on x_0.

Once we have the estimates (1.10.14) and (1.10.17), as well as Lemma 1.10.9, we can apply the Whitney Extension Theorem of Fefferman's [56], to complete the proof, see e.g., the proof of Theorem 5 in [27]. $\square$

1.A Some examples of almost minimizers

Example 1.A.1. If u is a minimizer of the functional

$$\int_D a(x)|\nabla u|^2$$

over the set $\mathfrak{K}_{\psi,g}(D,\mathcal{M})$ with strictly positive $a \in C^{0,\alpha}(\overline{D})$, $0<\alpha\leq 1$, then u is an almost minimizer for the Signorini problem with a gauge function $\omega(r)=Cr^\alpha$.

Proof. This is rather immediate. □

Example 1.A.2. Let u be a solution of the Signorini problem for the Laplacian with drift with the velocity field $b\in L^p(B_1)$, $p>n$:

$$\begin{aligned} -\Delta u + b(x)\nabla u = 0 \quad &\text{in } B_1^\pm \\ -\partial_{x_n} u \geq 0, \quad u\geq 0, \quad u\partial_{x_n} u = 0 \quad &\text{on } B_1', \end{aligned}$$

even in x_n-variable. We understand this in the weak sense that u satisfies the variational inequality

$$\int_{B_1} \nabla u \nabla(w-u) + (b(x)\nabla u)(w-u) \geq 0,$$

for any competitor $w\in \mathfrak{K}_{0,u}(B_1,B_1')$, i.e. $w\in u+W_0^{1,2}(B_1)$ such that $w\geq 0$ on B_1' in the sense of traces. Then u is an almost minimizer for the Signorini problem with $\psi=0$ on $\mathcal{M}=\mathbb{R}^{n-1}\times\{0\}$ and a gauge function $\omega(r)=Cr^{1-n/p}$.

Proof. This example corresponds to Example 2.A.1 when $A=I$. Thus, for this proof, we refer to the proof of Example 2.A.1 □

2. ALMOST MINIMIZERS FOR THE THIN OBSTACLE PROBLEM WITH VARIABLE COEFFICIENTS

2.1 Introduction and Main Results

2.1.1 The thin obstacle (or Signorini) problem with variable coefficients

Let D be a domain in $\mathbb{R}^n$, $n \geq 2$, and Π a smooth hypersurface (the *thin space*), that splits D into two subdomains $D^\pm$: $D \setminus \Pi = D^+ \cup D^-$. Let $\psi : \Pi \to \mathbb{R}$ be a certain (smooth) function (the *thin obstacle*) and $g : \partial D \to \mathbb{R}$ (the *boundary values*). Let also $A(x) = (a_{\mathrm{ij}}(x))$ be an $n \times n$ symmetric uniformly elliptic matrix, α-Hölder continuous as a function of $x \in D$, for some $0 < \alpha < 1$, with ellipticity constants $0 < \lambda \leq 1 \leq \Lambda < \infty$:

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2, \quad x \in D,\ \xi \in \mathbb{R}^n.$$

Then consider the minimizer U of the energy functional

$$\mathcal{J}_{A,D}(V) = \int_D \langle A(x)\nabla V, \nabla V\rangle dx,$$

over a closed convex set $\mathfrak{K}_{\psi,g}(D, \Pi) \subset W^{1,2}(D)$ defined by

$$\mathfrak{K}_{\psi,g}(D, \Pi) := \{V \in W^{1,2}(D) : V = g \text{ on } \partial D,\ V \geq \psi \text{ on } \Pi \cap D\}.$$

Because of the unilateral constraint on the thin space Π, the problem is known as the *thin obstacle problem.* Away from Π, the minimizer solves a uniformly elliptic divergence form equation with variable coefficients

$$\operatorname{div}(A(x)\nabla U) = 0 \quad \text{in } D^+ \cup D^-.$$

On the thin space, the minimizers satisfy

$$\begin{gathered} U \geq \psi, \quad \langle A\nabla U, \nu^+\rangle + \langle A\nabla U, \nu^-\rangle \geq 0, \\ (U - \psi)(\langle A\nabla U, \nu^+\rangle + \langle A\nabla U, \nu^-\rangle) = 0 \quad \text{on } D \cap \Pi, \end{gathered}$$

in a certain weak sense, where $\nu^{\pm}$ are the exterior normals to $D^{\pm}$ on Π and $\langle A\nabla U, \nu^{\pm}\rangle$ are understood as the limits from inside $D^{\pm}$. These are known as the *Signorini complementarity conditions* and therefore the problem is often referred to as the *Signorini problem* with variable coefficients (or *A-Signorini problem*, for short). One of the main objects of the study is the *free boundary*

$$\Gamma(U) = \partial_{\Pi}\{x \in \Pi : U(x) = \psi(x)\} \cap D,$$

which separates the *coincidence set* $\{U = \psi\}$ from the *noncoincidence set* $\{U > \psi\}$ in $D \cap \Pi$. The set $\Gamma(U)$ is also called a *thin* free boundary as it lives in Π and is expected to be of codimension two with respect to the domain D.

These types of problems go back to the original Signorini problem in elastostatics [4], but also appear in many applications ranging from math biology (semipermeable membranes) to boundary heat control [5] or more recently in math finance, with connection to the obstacle problem for the fractional Laplacian, through the Caffarelli-Silvestre extension [10]. The presence of the free boundary makes the problem particularly challenging and while the $C^{1,\beta}$ regularity of the minimizers (on the either side of the thin space) was known already in [6]–[8], the study of the free boundary became possible only after the breakthrough work of [9] on the optimal $C^{1,1/2}$ regularity of the minimizers. Since then there has been a significant effort in the literature to understand the structure and regularity properties of the free boundary in many different settings including equations with variable coefficients, problems for the fractional Laplacian, as well as the time-dependent problems, see e.g. [11]–[30], [55], [57], [58], and many others.

2.1.2 Almost minimizers

The approach we take in this chapter is by considering almost minimizers of the functional $\mathcal{J}_{A,D}$ in the sense of Anzellotti [31]. For this we need a *gauge* function $\omega : (0, r_0) \to [0, \infty)$, $r_0 > 0$, which is a nondecreasing function with $\omega(0+) = 0$, as well as a family $\{E_r(x_0)\}_{0<r<r_0}$ of open sets for any $x_0 \in D$, comparable to balls centered at x_0 (in what comes next, we will take it to be a family of ellipsoids).

Definition 2.1.1 (Almost minimizers)**.** *We say U is an* almost minimizer for the A-Signorini problem *in D if $U \in W^{1,2}_{\mathrm{loc}}(D)$, $U \geq \psi$ on $D \cap \Pi$, and for any $E_r(x_0) \Subset D$ with $0 < r < r_0$, we have*

$$\int_{E_r(x_0)} \langle A\nabla U, \nabla U\rangle \leq (1+\omega(r)) \int_{E_r(x_0)} \langle A\nabla V, \nabla V\rangle, \tag{2.1.1}$$

for any competitor function $V \in \mathfrak{K}_{\psi,U}(E_r(x_0), \Pi)$, i.e., V satisfying

$$V = U \quad \text{on } \partial E_r(x_0), \quad V \geq \psi \quad \text{on } E_r(x_0) \cap \Pi.$$

In fact, observing that for $x, x_0 \in D$, and $\xi \in \mathbb{R}^n$, $\xi \neq 0$

$$(1 - C|x-x_0|^\alpha) \leq \frac{\langle A(x_0)\xi, \xi\rangle}{\langle A(x)\xi, \xi\rangle} \leq (1 + C|x-x_0|^\alpha),$$

with C depending on the ellipticity of A and $\|A\|_{C^{0,\alpha}(D)}$, we can rewrite (2.1.1) in the form *with frozen coefficients*

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \leq (1+\omega(r)) \int_{E_r(x_0)} \langle A(x_0)\nabla V, \nabla V\rangle, \tag{2.1.2}$$

by replacing the gauge $\omega(r)$ with $C(\omega(r) + r^\alpha)$ if necessary.

An example of an almost minimizer is given in Example 2.A.1. Generally, we view almost minimizers as perturbations of minimizers in a certain sense, but in the case of variable coefficients there are even some advantages of treating minimizers themselves as almost minimizers, particularly in the sense of frozen coefficients (2.1.2).

Almost minimizers for the Signorini problem have already been studied in Chapter 1 in the case $A(x) \equiv I$, where their $C^{1,\beta}$-regularity (on the either side of the thin space) has been established and a number of technical tools such as Weiss- and Almgren-type monotonicity formulas were proved. In combination with the epiperimetric and log-epiperimetric inequalities these tools allowed to establish the optimal growth and prove the $C^{1,\gamma}$-regularity of the regular set and a structural theorem on the singular set. The aim of this chapter is to extend these results to the variable coefficient case. It is noteworthy that the results that we obtain (see Theorems F–I below) for almost minimizers improve even on some of the results available for the minimizers. For example, we only need the coefficients $A(x)$ to be $C^{0,\alpha}$ with arbitrary $0 < \alpha < 1$ in order to study the free boundary, compared to $W^{1,p}$, $p > n$, in [26] or $C^{0,\alpha}$, $1/2 < \alpha < 1$, in [55] for the regular part of the free boundary and $C^{0,1}$ in [58] for the singular set.

2.1.3 Main results

Since we are interested in local regularity results, we will assume that $D = B_1$, the unit ball in $\mathbb{R}^n$, and that

$$\Pi = \mathbb{R}^{n-1} \times \{0\}$$

after a local diffeomorphism. In this chapter, we will consider only the case when the thin obstacle is identically zero: $\psi \equiv 0$.

Further, we will assume $r_0 = 1$ in Definition 2.1.1 and take $\{E_r(x_0)\}$ to be the family of ellipsoids associated with the positive symmetric matrix $A(x_0)$:

$$E_r(x_0) := A^{1/2}(x_0)(B_r) + x_0.$$

By the ellipticity of $A(x_0)$, we have

$$B_{\lambda^{1/2}r}(x_0) \subset E_r(x_0) \subset B_{\Lambda^{1/2}r}(x_0).$$

To simplify the tracking of the constants, we will assume that there is $M > 0$ such that

$$\|A\|_{C^{0,\alpha}(B_1)} \leq M, \quad \lambda^{-1}, \Lambda \leq M, \quad \omega(r) \leq M r^{\alpha}, \quad 0 < \alpha < 1. \tag{2.1.3}$$

Then we can go between almost minimizing properties (2.1.1) and (2.1.2) by changing M if necessary.

Then our first result is as follows.

Theorem F ($C^{1,\beta}$-regularity of almost minimizers)**.** *Let $U \in W^{1,2}(B_1)$ be an almost minimizer for the A-Signorini problem in B_1, under the assumptions above. Then, $U \in C^{1,\beta}_{\mathrm{loc}}(B_1^\pm \cup B_1)$ for $\beta = \beta(\alpha, n) \in (0,1)$ and*

$$\|U\|_{C^{1,\beta}(K)} \leq C\|U\|_{W^{1,2}(B_1)},$$

for any $K \Subset B_1^\pm \cup B_1'$ and $C = C(n, \alpha, M, K)$.

The proof is obtained by using Morrey and Campanato space estimates, following the original idea of Anzellotti [31] that was successfully used in the constant coefficient case of our problem in Chapter 1. We explicitly mention, however, that in the above theorem we do not require the even symmetry of the almost minimizer in the x_n-variable, so Theorem F extends the corresponding result in Chapter 1 also in that respect.

To state our results related to the free boundary, we need to assume the following quasisymmetry condition. For $x_0 \in B_1' = B_1 \cap \Pi$, let

$$P_{x_0} = I - 2\frac{A(x_0)\mathrm{e}_n \otimes \mathrm{e}_n}{a_{nn}(x_0)}$$

be a matrix corresponding to the reflection with respect to Π in the conormal direction $A(x_0)\mathrm{e}_n$ at x_0. Note that $P_{x_0}x = x$ for any $x \in \Pi$ and $P_{x_0}E_r(x_0) = E_r(x_0)$. Then, for a function U in B_1 define

$$U^*_{x_0}(x) := \frac{U(x) + U(P_{x_0}x)}{2}.$$

Note that $U^*_{x_0}$ may not be defined in all of B_1, but is defined in any ellipsoid $E_r(x_0)$ as long as it is contained in B_1. Note also that $U = U^*_{x_0}$ on Π.

Definition 2.1.2 (Quasisymmetry)**.** *We say that $U \in W^{1,2}(B_1)$ is A*-quasisymmetric *with respect to Π, if there is a constant Q such that*

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \leq Q \int_{E_r(x_0)} \langle A(x_0)\nabla U^*_{x_0}, \nabla U^*_{x_0}\rangle,$$

for any ellipsoid $E_r(x_0) \Subset B_1$ centered at any $x_0 \in B_1'$.

We will assume $Q \leq M$ throughout the chapter, in addition to (2.1.3).

Note that when $A(x) \equiv I$ and U is even in x_n, then it is automatically quasisymmetric in the sense of the above definition. The quasisymmetry condition will also hold for even minimizers if e_n is an eigenvector of $A(x_0)$ for any $x_0 \in B_1'$, i.e., when

$$a_{in}(x_0) = 0, \quad \text{for } i = 1, \dots, n-1, \ x_0 \in B_1'.$$

This condition is typically imposed in the existing literature and can be satisfied with an application of a local $C^{1,\alpha}$-diffeomorphism that preserves Π, see [15], [55], [59]. The reason for a quasisymmetry condition is that the growth rate of the symmetrization $U_{x_0}^*$ over the ellipsoids $E_r(x_0)$ captures that of $U = U_{x_0}^*$ on the thin space Π at $x_0 \in \Gamma(U)$, while in the nonsymmetric case there could be a mismatch in those rates caused by the odd component of U, vanishing on Π.

More specifically, the growth rate of U on Π at $x_0 \in \Gamma(U)$ is determined by the following quantity

$$N^A(r, U_{x_0}^*, x_0) := \frac{r \int_{E_r(x_0)} \langle A(x_0) \nabla U_{x_0}^*, \nabla U_{x_0}^* \rangle}{\int_{\partial E_r(x_0)} (U_{x_0}^*)^2 \mu_{x_0}(x - x_0)},$$

which is a version of *Almgren's frequency functional* [47] written in the geometric terms determined by $A(x_0)$, where $\mu_{x_0}(z) = \frac{|A^{-1/2}(x_0)z|}{|A^{-1}(x_0)z|}$ is the *conformal factor*. As in the constant coefficient case, this quantity is of paramount importance for the classification of free boundary points.

Theorem G (Monotonicity of the truncated frequency)**.** *Let U be as in Theorem F and assume additionally that U is A-quasisymmetric with respect to Π. Then for any $\kappa_0 \geq 2$, there is $b = b(n, \alpha, M, \kappa_0)$ such that the* truncated frequency

$$r \mapsto \widehat{N}_{\kappa_0}^A(r, U_{x_0}^*, x_0) := \min \left\{ \frac{1}{1 - br^\alpha} N^A(r, U_{x_0}^*, x_0), \kappa_0 \right\}$$

is monotone increasing for $x_0 \in B_{1/2}' \cap \Gamma(u)$, and $0 < r < r_0(n, \alpha, M, \kappa_0)$. Moreover, if we define

$$\kappa(x_0) := \widehat{N}_{\kappa_0}^A(0+, U_{x_0}^*, x_0),$$

the frequency of U at x_0*, then we have that either*

$$\kappa(x_0) = 3/2 \quad or \quad \kappa(x_0) \geq 2.$$

The monotonicity of the truncated frequency follows from that of an one-parametric family of so-called Weiss-type energy functionals $\{W_\kappa^A\}_{0<\kappa<\kappa_0}$, see Section 2.7, which also play a fundamental role in the analysis of the free boundary.

The theorem above gives the following decomposition of the free boundary

$$\Gamma(U) = \Gamma_{3/2}(U) \cup \bigcup_{\kappa \geq 2} \Gamma_\kappa(U),$$

where

$$\Gamma_\kappa(U) := \{x_0 \in \Gamma(U) : \kappa(x_0) = \kappa\}.$$

The set $\Gamma_{3/2}(U)$, where the frequency is minimal is known as the *regular set* and is also denoted $\mathcal{R}(U)$.

Theorem H (Regularity of the regular set)**.** *Let U be as in Theorem G. Then $\mathcal{R}(U)$ is a relatively open subset of the free boundary $\Gamma(U)$ and is an $(n-2)$-dimensional manifold of class $C^{1,\gamma}$.*

Finally, we state our main result for the so-called *singular set*. A free boundary point $x_0 \in \Gamma(U)$ is called *singular* if the *coincidence set* $\Lambda(U) := \{x \in B_1' : U(x) = 0\}$ has H^{n-1}-density zero at x_0, i.e.,

$$\lim_{r \to 0+} \frac{H^{n-1}(\Lambda(U) \cap B_r'(x_0))}{H^{n-1}(B_r')} = 0.$$

We denote the set of all singular points by $\Sigma(U)$ and call it the *singular set*. It can be shown that if $\kappa(x_0) < \kappa_0$, then $x_0 \in \Sigma(U)$ if and only if $\kappa(x_0) = 2m$, $m \in \mathbb{N}$ (see Proposition 2.12.1). For such values of κ, we then define

$$\Sigma_\kappa(U) := \Gamma_\kappa(U).$$

Theorem I (Structure of the singular set)**.** *Let U be as in Theorem G. Then, for any $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, $\Sigma_\kappa(U)$ is contained in a countable union of $(n-2)$-dimensional manifolds of class $C^{1,\log}$.*

A more refined version of this result is given in Theorem 2.12.4.

Theorems H and I follow by establishing the uniqueness and continuous dependence of *almost homogeneous blowups* with Hölder modulus of continuity in the case of regular free boundary points and a logarithmic one in the case of the singular points. These follow from optimal growth and rotation estimates which are based on the use of Weiss-type monotonicity formulas in conjunction with so-called *epiperimetric* [20] and log-*epiperimetric* [27] inequalities for the solutions of the Signorini problem.

Proofs of Theorems F–I

While we don't give formal proofs of the theorems above in the main body of the chapter, they are contained in the following results proved there:

- ∘ Theorem F is essentially the same as Theorem 2.5.1.
- ∘ Theorem G follows by combining Theorem 2.7.2 and Corollary 2.11.2.
- ∘ The statement of Theorem H is contained in that of Theorem 2.11.5.
- ∘ The statement of Theorem I is contained in that of Theorem 2.12.4.

2.2 Coordinate transformations

In order to use the results available for almost minimizers in the case of $A \equiv I$, proved in Chapter 1, in this section we describe a "deskewing procedure" or coordinate transformations to straighten $A(x_0)$, $x_0 \in B_1$.

For the notational convenience, we will denote

$$\mathfrak{a}_{x_0} = A^{1/2}(x_0), \quad x_0 \in B_1$$

so that

$$\langle A(x_0)\xi, \xi \rangle = |\mathfrak{a}_{x_0}\xi|^2, \quad \xi \in \mathbb{R}^n.$$

Then $\mathfrak{a}_{x_0}$ is a symmetric positive definite matrix, with eigenvalues between $\lambda^{1/2}$ and $\Lambda^{1/2}$ and the mapping $x_0 \mapsto \mathfrak{a}_{x_0}$ is α-Hölder continuous for $x_0 \in B_1$. For every $x_0 \in B_1$, we define an affine transformation T_{x_0} by

$$T_{x_0}(x) = \mathfrak{a}_{x_0}^{-1}(x - x_0).$$

Note that $T_{x_0}^{-1}(y) = \mathfrak{a}_{x_0} y + x_0$. Then for the ellipsoids $E_r(x_0)$, we have

$$E_r(x_0) = T_{x_0}^{-1}(B_r) = \mathfrak{a}_{x_0} B_r + x_0, \quad T_{x_0}(E_r(x_0)) = B_r.$$

Further, we let

$$\Pi_{x_0} := T_{x_0}(\Pi).$$

Then Π_{x_0} is a hyperplane parallel to a linear subspace $\mathfrak{a}_{x_0}^{-1}\Pi$ spanned by the vectors $\mathfrak{a}_{x_0}^{-1}\mathrm{e}_1$, $\mathfrak{a}_{x_0}^{-1}\mathrm{e}_2$, ..., $\mathfrak{a}_{x_0}^{-1}\mathrm{e}_{n-1}$ and with a normal $\mathfrak{a}_{x_0}\mathrm{e}_n$. Generally, this hyperplane will be tilted with respect to Π, unless $\mathfrak{a}_{x_0}\mathrm{e}_n$ is a multiple of e_n, or equivalently that e_n is an eigenvector of the matrix $A(x_0)$, or that $a_{in}(x_0) = 0$ for $i = 1, \ldots, n-1$ for its entries. To rectify that, we construct a family of orthogonal transformations O_{x_0}, $x_0 \in B_1$, by applying the Gram-Schmidt process to the ordered basis $\{\mathfrak{a}_{x_0}^{-1}\mathrm{e}_1, \mathfrak{a}_{x_0}^{-1}\mathrm{e}_2, \ldots, \mathfrak{a}_{x_0}^{-1}\mathrm{e}_{n-1}\}$ of $\mathfrak{a}_{x_0}^{-1}\Pi$. Namely, let

$$\begin{aligned}
\mathrm{e}_1^{x_0} &:= \frac{\mathfrak{a}_{x_0}^{-1}\mathrm{e}_1}{|\mathfrak{a}_{x_0}^{-1}\mathrm{e}_1|},\\
\mathrm{e}_2^{x_0} &:= \frac{\mathfrak{a}_{x_0}^{-1}\mathrm{e}_2 - \langle \mathfrak{a}_{x_0}^{-1}\mathrm{e}_2, \mathrm{e}_1^{x_0}\rangle \mathrm{e}_1^{x_0}}{|\mathfrak{a}_{x_0}^{-1}\mathrm{e}_2 - \langle \mathfrak{a}_{x_0}^{-1}\mathrm{e}_2, \mathrm{e}_1^{x_0}\rangle \mathrm{e}_1^{x_0}|},\\
\mathrm{e}_3^{x_0} &:= \frac{\mathfrak{a}_{x_0}^{-1}\mathrm{e}_3 - \langle \mathfrak{a}_{x_0}^{-1}\mathrm{e}_3, \mathrm{e}_1^{x_0}\rangle \mathrm{e}_1^{x_0} - \langle \mathfrak{a}_{x_0}^{-1}\mathrm{e}_3, \mathrm{e}_2^{x_0}\rangle \mathrm{e}_2^{x_0}}{|\mathfrak{a}_{x_0}^{-1}\mathrm{e}_3 - \langle \mathfrak{a}_{x_0}^{-1}\mathrm{e}_3, \mathrm{e}_1^{x_0}\rangle \mathrm{e}_1^{x_0} - \langle \mathfrak{a}_{x_0}^{-1}\mathrm{e}_3, \mathrm{e}_2^{x_0}\rangle \mathrm{e}_2^{x_0}|}\\
&\ \vdots
\end{aligned}$$

Moreover, letting

$$\mathrm{e}_n^{x_0} := \frac{\mathfrak{a}_{x_0}\mathrm{e}_n}{|\mathfrak{a}_{x_0}\mathrm{e}_n|},$$

we obtain an ordered orthonormal basis $\{\mathrm{e}_1^{x_0}, \dots, \mathrm{e}_{n-1}^{x_0}, \mathrm{e}_n^{x_0}\}$ of $\mathbb{R}^n$. Then consider the rotation O_{x_0} of $\mathbb{R}^n$ that takes the standard basis $\{\mathrm{e}_1, \mathrm{e}_2, \dots, \mathrm{e}_n\}$ to the one above, i.e.,

$$O_{x_0} : \mathbb{R}^n \to \mathbb{R}^n, \quad O_{x_0}(\mathrm{e}_i) = \mathrm{e}_i^{x_0},\ i = 1, 2, \dots, n.$$

Note that the Gram-Schmidt process above guarantees that $x_0 \mapsto O_{x_0}$ is α-Hölder continuous. We also have that by construction

$$O_{x_0}^{-1} \mathfrak{a}_{x_0}^{-1} \Pi = \Pi.$$

In particular, when $x_0 \in \Pi$, we have $\Pi_{x_0} = \mathfrak{a}_{x_0}^{-1} \Pi$ and therefore

$$O_{x_0}^{-1}(\Pi_{x_0}) = \Pi.$$

Because of this property, we also define the modifications of the matrices $\mathfrak{a}_{x_0}$ and the transformations T_{x_0} as follows:

$$\bar{\mathfrak{a}}_{x_0} = \mathfrak{a}_{x_0} O_{x_0}, \quad \bar{T}_{x_0} = O_{x_0}^{-1} \circ T_{x_0},$$

so that $\bar{T}_{x_0}(x) = \bar{\mathfrak{a}}_{x_0}^{-1}(x - x_0)$. Since O_{x_0} is a rotation, we still have

$$E_r(x_0) = \bar{T}_{x_0}^{-1}(B_r), \quad \bar{T}_{x_0}(E_r(x_0)) = B_r,$$

see Fig. 2.1.

Next, for a function $U : B_1 \to \mathbb{R}$ and a point $x_0 \in B_1$, we define its "deskewed" version at x_0 by

$$u_{x_0} = U \circ \bar{T}_{x_0}^{-1}.$$

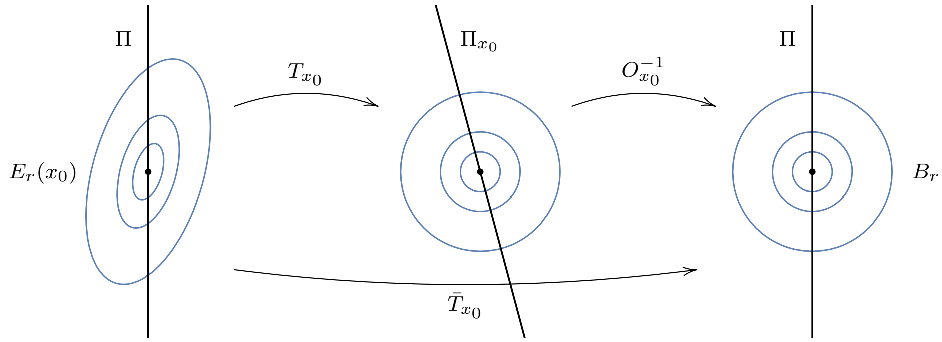

Figure 2.1. Deskewing: coordinate transformations T_{x_0}, $O_{x_0}^{-1}$, $\bar{T}_{x_0}$.

As we will see, if U is an almost minimizer, the transformed function u_{x_0} will satisfy an almost minimizing property with the identity matrix I at the origin. Before we state and prove that fact, we need the following basic change of variable formulas:

$$\int_{E_r(x_0)} U^2 = \det \mathfrak{a}_{x_0} \int_{B_r} u_{x_0}^2 \tag{2.2.1}$$

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle = \det \mathfrak{a}_{x_0} \int_{B_r} |\nabla u_{x_0}|^2 \tag{2.2.2}$$

$$\int_{\partial E_r(x_0)} U^2 \mu_{x_0}(x - x_0) = \det \mathfrak{a}_{x_0} \int_{\partial B_r} u_{x_0}^2, \tag{2.2.3}$$

with the conformal factor

$$\mu_{x_0}(z) := \frac{|\mathfrak{a}_{x_0}^{-1} z|}{|A^{-1}(x_0) z|}. \tag{2.2.4}$$

We also have the following modified version of (2.2.2).

$$\int_{E_r(x_0)} |\mathfrak{a}_{x_0}\nabla U - \langle \mathfrak{a}_{x_0}\nabla U\rangle_{E_r(x_0)}|^2 = \det \mathfrak{a}_{x_0} \int_{B_r} |\nabla u_{x_0} - \langle \nabla u_{x_0}\rangle_{B_r}|^2. \tag{2.2.5}$$

While (2.2.1)–(2.2.2) and (2.2.5) are clear, let us give more details on (2.2.3). If we let $f(x) := |\mathfrak{a}_{x_0}^{-1}(x - x_0)|$, then $\{f = t\} = \partial E_t(x_0)$, $t > 0$, and by the coarea formula

$$\int_{E_r(x_0)} U^2 dx = \int_0^r \int_{\partial E_t(x_0)} \frac{U^2}{|\nabla f(x)|} dS_x dt.$$

Using now that $1/|\nabla f(x)| = \frac{|\mathfrak{a}_{x_0}^{-1}(x - x_0)|}{|A^{-1}(x_0)(x - x_0)|} = \mu_{x_0}(x - x_0)$ and then differentiating (2.2.1), we obtain (2.2.3).

We will also need the following estimate for the conformal factor μ_{x_0}:

$$\lambda^{1/2} \le \mu_{x_0}(z) \le \Lambda^{1/2}. \tag{2.2.6}$$

Indeed, if $y = A^{-1}(x_0)z$, then

$$\mu_{x_0}(z) = \frac{|\mathfrak{a}_{x_0} y|}{|y|} \in [\lambda^{1/2}, \Lambda^{1/2}].$$

Definition 2.2.1 (Almost Signorini property at a point)**.** *We say that a function* $u \in W^{1,2}(B_R)$ *satisfies the* almost Signorini property at 0 *in* B_R *if*

$$\int_{B_r} |\nabla u|^2 \le (1+\omega(r)) \int_{B_r} |\nabla v|^2,$$

for all $0 < r < R$ *and* $v \in \mathfrak{K}_{0,u}(B_r, \Pi)$*.*

Lemma 2.2.1. *Suppose U is an almost minimizer of the A-Signorini problem in B_1. Let $x_0 \in B'_1$ be such that $E_R(x_0) \subset B_1$. Then $u_{x_0} = U \circ \bar{T}_{x_0}^{-1}$ satisfies the almost Signorini property at 0 in B_R.*

Proof. Let V be the energy minimizer of $\int_{E_r(x_0)} \langle A(x_0)\nabla V, \nabla V\rangle$ on $\mathfrak{K}_{0,U}(E_r(x_0), \Pi)$, $0 < r < R$. Then $v_{x_0} = V \circ \bar{T}_{x_0}^{-1}$ is the energy minimizer of $\int_{B_r} |\nabla v_{x_0}|^2$ on $\mathfrak{K}_{0,u_{x_0}}(B_r, \Pi)$. Moreover, by (2.2.2),

$$\begin{aligned}
\int_{B_r} |\nabla u_{x_0}|^2 &= \det \mathfrak{a}_{x_0}^{-1} \int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \\
&\le (1+\omega(r)) \det \mathfrak{a}_{x_0}^{-1} \int_{E_r(x_0)} \langle A(x_0)\nabla V, \nabla V\rangle \\
&= (1+\omega(r)) \int_{B_r} |\nabla v_{x_0}|^2.
\end{aligned}$$

This completes the proof. □

2.3 Almost A-harmonic functions

We start our analysis of almost minimizers in the absence of the thin obstacle. We call such functions almost A-harmonic functions. In this section, we establish their $C^{1,\alpha/2}$ regularity (Theorem 2.3.2). A similar result has already been proved by Anzellotti [31], but for almost minimizers over balls $\{B_r(x_0)\}$ rather than ellipsoids $\{E_r(x_0)\}$; nevertheless, the proofs are similar. The proofs in this section also illustrate how we are going to use the results available for "deskewed" functions $u_{x_0} = U \circ \bar{T}_{x_0}^{-1}$ to infer the corresponding results for almost minimizers U.

Definition 2.3.1 (Almost A-harmonic functions)**.** *We say that U is an* almost A-harmonic function *in B_1 if $U \in W^{1,2}(B_1)$ and*

$$\int_{E_r(x_0)} \langle A\nabla U, \nabla U\rangle \leq (1+\omega(r)) \int_{E_r(x_0)} \langle A\nabla V, \nabla V\rangle,$$

whenever $E_r(x_0) \Subset B_1$ and $V \in \mathfrak{K}_U(E_r(x_0)) := U + W_0^{1,2}(E_r(x_0))$.

Note that similarly to the case of A-Signorini problem, we can write the almost minimizing property above in the form with frozen coefficients

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \leq (1+\omega(r)) \int_{E_r(x_0)} \langle A(x_0)\nabla V, \nabla V\rangle.$$

Definition 2.3.2 (Almost harmonic property at a point)**.** *We say that a function $u \in W^{1,2}(B_R)$ satisfies* almost harmonic property at 0 *in B_R if*

$$\int_{B_r} |\nabla u|^2 \leq (1+\omega(r)) \int_{B_r} |\nabla v|^2,$$

for all $0 < r < R$ and $v \in \mathfrak{K}_u(B_r)$.

Lemma 2.3.1. *If U is an almost A-harmonic function in B_1 and $x_0 \in B_1$ with $E_R(x_0) \subset B_1$, then u_{x_0} satisfies the almost harmonic property at 0 in B_R.*

Proof. The proof is similar to that of Lemma 2.2.1. $\square$

Proposition 2.3.1 (cf.Proposition 1.2.2)**.** *Let U be an almost A-harmonic function in B_1. Then for any $B_r(x_0) \Subset B_1$ and $0 < \rho < r$, we have*

$$\int_{B_\rho(x_0)} |\nabla U|^2 \leq C\left[\left(\frac{\rho}{r}\right)^n + r^\alpha\right] \int_{B_r(x_0)} |\nabla U|^2, \tag{2.3.1}$$

$$\begin{aligned} \int_{B_\rho(x_0)} \left|\nabla U - \langle \nabla U\rangle_{B_\rho(x_0)}\right|^2 \leq C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} \left|\nabla U - \langle \nabla U\rangle_{B_r(x_0)}\right|^2 \\ + Cr^\alpha \int_{B_r(x_0)} |\nabla U|^2, \end{aligned} \tag{2.3.2}$$

with $C = C(n, \alpha, M)$.

Proof. Since u_{x_0} satisfies the almost harmonic property at 0, if h is the harmonic replacement of u_{x_0} in B_r (i.e., h is harmonic in B_r with $h = u_{x_0}$ on ∂B_r), then

$$\int_{B_r} |\nabla u_{x_0}|^2 \leq (1 + Mr^\alpha) \int_{B_r} |\nabla h|^2.$$

This is enough to repeat the arguments in Proposition 1.2.2, to obtain

$$\int_{B_\rho} |\nabla u_{x_0}|^2 \leq 2\left[\left(\frac{\rho}{r}\right)^n + Mr^\alpha\right] \int_{B_r} |\nabla u_{x_0}|^2,$$
$$\int_{B_\rho} |\nabla u_{x_0} - \langle \nabla u_{x_0}\rangle_{B_\rho}|^2 \leq 9\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\nabla u_{x_0} - \langle \nabla u_{x_0}\rangle_{B_r}|^2 + 24Mr^\alpha \int_{B_r} |\nabla u_{x_0}|^2.$$

Then, by the change of variables formulas (2.2.2) and (2.2.5), we have

$$\int_{E_\rho(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \leq 2\left[\left(\frac{\rho}{r}\right)^n + Mr^\alpha\right] \int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle, \tag{2.3.3}$$

$$\begin{aligned} \int_{E_\rho(x_0)} \left|\mathfrak{a}_{x_0}\nabla U - \langle \mathfrak{a}_{x_0}\nabla U\rangle_{E_\rho(x_0)}\right|^2 \leq 9\left(\frac{\rho}{r}\right)^{n+2} \int_{E_r(x_0)} \left|\mathfrak{a}_{x_0}\nabla U - \langle \mathfrak{a}_{x_0}\nabla U\rangle_{E_r(x_0)}\right|^2 \\ + 24Mr^\alpha \int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle. \end{aligned} \tag{2.3.4}$$

To show now that (2.3.3)–(2.3.4) imply (2.3.1)–(2.3.2), we first consider the case

$$0 < \rho < (\lambda/\Lambda)^{1/2} r.$$

Then, using the inclusions

$$B_\rho(x_0) \subset E_{\lambda^{-1/2}\rho}(x_0) \subset E_{\Lambda^{-1/2}r}(x_0) \subset B_r(x_0),$$

applying (2.3.3)–(2.3.4) with $\lambda^{-1/2}\rho$ and $\Lambda^{-1/2}r$ in place of ρ and r, and using the ellipticity of $A(x_0)$, we obtain (2.3.1)–(2.3.2) in this case.

In the remaining case

$$(\lambda/\Lambda)^{1/2} r \leq \rho \leq r,$$

the inequalities (2.3.1)–(2.3.2) hold readily, as

$$\int_{B_\rho(x_0)} |\nabla U|^2 \leq \left(\frac{\Lambda}{\lambda}\right)^{n/2} \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |\nabla U|^2,$$

$$\begin{aligned}\int_{B_\rho(x_0)} |\nabla U - \langle \nabla U\rangle_{B_\rho(x_0)}|^2 &\leq \int_{B_\rho(x_0)} |\nabla U - \langle \nabla U\rangle_{B_r(x_0)}|^2 \\ &\leq \left(\frac{\Lambda}{\lambda}\right)^{\frac{n+2}{2}} \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |\nabla U - \langle \nabla U\rangle_{B_r(x_0)}|^2.\end{aligned}$$

□

Theorem 2.3.2. *Let U be an almost A-harmonic function in B_1. Then $U \in C^{1,\alpha/2}(B_1)$ with*

$$\|U\|_{C^{1,\alpha/2}(K)} \leq C\|U\|_{W^{1,2}(B_1)},$$

for any $K \Subset B_1$, with $C = C(n, \alpha, M, K)$.

Proof. Let $K \Subset B_1$ and $x_0 \in \widetilde{K} := \{y \in B_1 : \operatorname{dist}(y, \partial B_1) \geq r_0\}$, where $r_0 = \frac{1}{2}\operatorname{dist}(K, \partial B_1)$. For in$(0,1)$, a direct application of Lemma 1.2.2 to (2.3.1) gives

$$\int_{B_r(x_0)} |\nabla U|^2 \leq C\|\nabla U\|^2_{L^2(B_1)} r^{n-2+2\sigma},$$

for any $0 < r < r_0$, with C depending on n, α, σ, M, K. Combining this with (2.3.2) also gives,

$$\int_{B_\rho(x_0)} \left|\nabla U - \langle \nabla U\rangle_{B_\rho(x_0)}\right|^2 \leq C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} \left|\nabla U - \langle \nabla U\rangle_{B_r(x_0)}\right|^2$$

$$+ C\|\nabla U\|^2_{L^2(B_1)} r^{n-2+2\sigma+\alpha}, \quad (2.3.5)$$

for any $0 < \rho < r < r_0$. If we take $\sigma \in (0,1)$ such that $\alpha' := \frac{-2+2\sigma+\alpha}{2} > 0$, then Lemma 1.2.2 produces

$$\int_{B_\rho(x_0)} \left|\nabla U - \langle \nabla U \rangle_{B_\rho(x_0)}\right|^2 \le C\|\nabla U\|^2_{L^2(B_1)} \rho^{n+2\alpha'}$$

and this readily implies $\nabla U \in C^{0,\alpha'}(\widetilde{K})$. Now we know that ∇U is bounded in $\widetilde{K}$, and thus $\int_{B_r(x_0)} |\nabla U|^2 \le C\|\nabla U\|^2_{L^2(B_1)} r^n$. Plugging this in the last term of (2.3.2) and repeating the arguments above, we conclude that $U \in C^{1,\alpha/2}$. □

2.4 Almost Lipschitz regularity of almost minimizers

In this section, we make the first step towards the regularity of almost minimizers for the A-Signorini problem and show that they are almost Lipschitz, i.e., $C^{0,\sigma}$ for every $0 < \sigma < 1$ (Theorem 2.4.1). The proof is based on the Morrey space embedding, similar to the case of almost A-harmonic functions, as well as the case of almost minimizers with $A = I$, treated in Chapter 1. We want to emphasize, however, that the results on almost Lipschitz and $C^{1,\beta}$ regularity of almost minimizers (in the next section) do not require any symmetry condition that was imposed in Chapter 1.

We start with an auxiliary result on the solutions of the Signorini problem.

Proposition 2.4.1. *Let h be a solution of the Signorini problem in B_1. Then*

$$\int_{B_\rho} |\nabla h|^2 \le \left(\frac{\rho}{R}\right)^n \int_{B_R} |\nabla h|^2, \quad 0 < \rho < R < 1. \quad (2.4.1)$$

Proof. The difference of this proposition from Proposition 1.3.1 is that $h(y)$ is not assumed to be even symmetric in y_n-variable. To circumvent that, we decompose h into the sum of even and odd functions in y_n, i.e.,

$$\begin{aligned} h(y', y_n) &= \frac{h(y', y_n) + h(y', -y_n)}{2} + \frac{h(y', y_n) - h(y', -y_n)}{2} \\ &=: h^*(y', y_n) + h^\sharp(y', y_n). \end{aligned} \quad (2.4.2)$$

It is easy to see that h^* is a solution of the Signorini problem, even in y_n-variable, and $h^\sharp$ is a harmonic function, odd in y_n-variable.

Then both $|\nabla h^*|^2$ and $|\nabla h^\sharp|^2$ are subharmonic functions in B_1 (see Proposition 1.3.1 for h^*), which implies that for $0<\rho<R<1$

$$\int_{B_\rho} |\nabla h^*|^2 \leq \left(\frac{\rho}{R}\right)^n \int_{B_R} |\nabla h^*|^2,$$
$$\int_{B_\rho} |\nabla h^\sharp|^2 \leq \left(\frac{\rho}{R}\right)^n \int_{B_R} |\nabla h^\sharp|^2.$$

Now observing that $\int_{B_t} |\nabla h|^2 = \int_{B_t} \left(|\nabla h^*|^2 + |\nabla h^\sharp|^2\right)$, for $0<t\leq R$, we obtain (2.4.1). □

Proposition 2.4.2 (cf. Proposition 1.3.2)**.** *Let U be an almost minimizer for the A-Signorini problem in B_1, and $B_R(x_0) \Subset B_1$. Then, there is $C_1 = C_1(n,M) > 1$ such that*

$$\int_{B_\rho(x_0)} |\nabla U|^2 \leq C_1 \left[\left(\frac{\rho}{R}\right)^n + R^\alpha\right] \int_{B_R(x_0)} |\nabla U|^2, \quad 0<\rho<R. \tag{2.4.3}$$

Proof. Case 1. Suppose $x_0 \in B_1'$. Note that u_{x_0} satisfies the Signorini property at 0 in B_r with $r = \Lambda^{-1/2}R$. If h is the Signorini replacement of u_{x_0} in B_r (that is, h solves the Signorini problem in B_r with thin obstacle 0 on Π and boundary values $h = u_{x_0}$ on ∂B_r), then h satisfies

$$\int_{B_r} \langle \nabla h, \nabla(v-h)\rangle \geq 0,$$

for any $v \in \mathfrak{K}_{0,u_{x_0}}(B_r,\Pi)$, which easily follows from the standard first variation argument. Plugging in $v = u_{x_0}$, we obtain

$$\int_{B_r} \langle \nabla h, \nabla u_{x_0}\rangle \geq \int_{B_r} |\nabla h|^2.$$

Then it follows that

$$\begin{aligned}\int_{B_r} |\nabla(u_{x_0}-h)|^2 &= \int_{B_r} \left(|\nabla u_{x_0}|^2 + |\nabla h|^2 - 2\langle \nabla u_{x_0}, \nabla h\rangle\right) \leq \int_{B_r} |\nabla u_{x_0}|^2 - \int_{B_r} |\nabla h|^2 \\ &\leq (1+Mr^\alpha)\int_{B_r} |\nabla h|^2 - \int_{B_r} |\nabla h|^2 \leq Mr^\alpha \int_{B_r} |\nabla u_{x_0}|^2,\end{aligned}$$

where in the last inequality we have used that h is the energy minimizer of the Dirichlet integral in $\mathfrak{K}_{0,u_{x_0}}(B_r,\Pi)$. Then, for $\rho \le r$, we have

$$\begin{aligned}\int_{B_\rho} |\nabla u_{x_0}|^2 &\le 2\int_{B_\rho} |\nabla h|^2 + 2\int_{B_\rho} |\nabla (u_{x_0}-h)|^2 \le 2\left(\frac{\rho}{r}\right)^n \int_{B_r} |\nabla h|^2 + 2Mr^\alpha \int_{B_r} |\nabla u_{x_0}|^2\\ &\le C\left[\left(\frac{\rho}{r}\right)^n + r^\alpha\right]\int_{B_r}|\nabla u_{x_0}|^2.\end{aligned}$$

Now, we transform back from u_{x_0} to U as we did in Proposition 2.3.1 to obtain (2.4.3) in this case.

Case 2. Now consider the case $x_0 \in B_1^+$. If $\rho \ge r/4$, then we simply have

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le 4^n \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |\nabla U|^2.$$

Thus, we may assume $\rho < r/4$. Then, let $d := \operatorname{dist}(x_0, B_1') > 0$ and choose $x_1 \in \partial B_d(x_0) \cap B_1'$.

Case 2.1. If $\rho \ge d$, then we use $B_\rho(x_0) \subset B_{2\rho}(x_1) \subset B_{r/2}(x_1) \subset B_r(x_0)$ and the result of Case 1 to write

$$\begin{aligned}\int_{B_\rho(x_0)} |\nabla U|^2 \le \int_{B_{2\rho}(x_1)} |\nabla U|^2 &\le C\left[\left(\frac{2\rho}{r/2}\right)^n + (r/2)^\alpha\right]\int_{B_{r/2}(x_1)} |\nabla U|^2\\ &\le C\left[\left(\frac{\rho}{r}\right)^n + r^\alpha\right]\int_{B_r(x_0)} |\nabla U|^2.\end{aligned}$$

Case 2.2. Suppose now $d > \rho$. If $d > r$, then $B_r(x_0) \Subset B_1^+$. Since U is almost harmonic in B_1^+, we can apply Proposition 2.3.1 to obtain

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le C\left[\left(\frac{\rho}{r}\right)^n + r^\alpha\right]\int_{B_r(x_0)} |\nabla U|^2.$$

Thus, we may assume $d \le r$. Then we note that $B_d(x_0) \subset B_1^+$ and by a limiting argument from the previous estimate, we obtain

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le C\left[\left(\frac{\rho}{d}\right)^n + r^\alpha\right]\int_{B_d(x_0)} |\nabla U|^2.$$

To estimate $\int_{B_d(x_0)} |\nabla U|^2$ in the right-hand side of the above inequality, we further consider the two subcases.

Case 2.2.1. If $r/4 \le d$, then

$$\int_{B_d(x_0)} |\nabla U|^2 \le 4^n \left(\frac{d}{r}\right)^n \int_{B_r(x_0)} |\nabla U|^2,$$

which immediately implies (2.4.3).

Case 2.2.2. It remains to consider the case $\rho < d < r/4$. Using Case 1 again, we have

$$\begin{aligned}\int_{B_d(x_0)} |\nabla U|^2 \le \int_{B_{2d}(x_1)} |\nabla U|^2 &\le C\left[\left(\frac{2d}{r/2}\right)^n + (r/2)^\alpha\right] \int_{B_{r/2}(x_1)} |\nabla U|^2 \\ &\le C\left[\left(\frac{d}{r}\right)^n + r^\alpha\right] \int_{B_r(x_0)} |\nabla U|^2,\end{aligned}$$

which also implies (2.4.3). This concludes the proof of the proposition. □

As we have seen in Chapter 1, Proposition 2.4.2 implies the almost Lipschitz regularity of almost minimizers.

Theorem 2.4.1. *Let U be an almost minimizer for the A-Signorini problem in B_1. Then $U \in C^{0,\sigma}(B_1)$ for all $0 < \sigma < 1$. Moreover, for any $K \Subset B_1$,*

$$\|U\|_{C^{0,\sigma}(K)} \le C\|U\|_{W^{1,2}(B_1)},$$

with $C = C(n, \alpha, M, \sigma, K)$.

Proof. The proof is essentially identical to that of Theorem 1.3.1. Let $K \Subset B_1$ and $x_0 \in K$. Take $r_0 = r_0(n, \alpha, M, \sigma, K) > 0$ such that $r_0 < \operatorname{dist}(K, \partial B_1)$ and $r_0^\alpha \le \varepsilon(C_1, n, n + 2\sigma - 2)$, where $\varepsilon = \varepsilon(C_1, n, n + 2\sigma - 2)$ is as in Lemma 1.2.2 and $C_1 = C_1(n, M)$ is as in Proposition 2.4.2. Then for all $0 < \rho < r < r_0$, by Proposition 2.4.2,

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le C_1\left[\left(\frac{\rho}{r}\right)^n + r^\alpha\right] \int_{B_r(x_0)} |\nabla U|^2.$$

By Lemma 1.2.2, we get

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le C(n, M, \sigma) \left(\frac{\rho}{r}\right)^{n+2\sigma-2} \int_{B_r(x_0)} |\nabla U|^2.$$

Taking $r \nearrow r_0$, we conclude that

$$\int_{B_\rho(x_0)} |\nabla U|^2 \le C(n, \alpha, M, \sigma, K) \|\nabla U\|^2_{L^2(B_1)} \rho^{n+2\sigma-2}. \tag{2.4.4}$$

By the Morrey space embedding (Corollary 3.2 in [50]), we obtain $U \in C^{0,\sigma}(K)$ with

$$\|U\|_{C^{0,\sigma}(K)} \le C(n, \alpha, M, \sigma, K) \|U\|_{W^{1,2}(B_1)}. \qquad \square \tag{2.4.5}$$

2.5 $C^{1,\beta}$ regularity of almost minimizers

In this section we prove $C^{1,\beta}$ regularity of the almost minimizers for the A-Signorini problem (Theorem 2.5.1). While we take advantage of the results available for the even symmetric almost minimizers with $A = I$ in Chapter 1, removing the symmetry condition requires new additional steps, combined with "deskewing" arguments to generalize to the variable coefficient case.

We start again with an auxiliary result for the solutions of the Signorini problem.

Proposition 2.5.1. *Let h be a solution of the Signorini problem in B_r, $0 < r < 1$. Define*

$$\widehat{\nabla h} := \begin{cases} \nabla h(y', y_n), & y_n \ge 0 \\ \nabla h(y', -y_n), & y_n < 0, \end{cases}$$

the even extension of ∇h from B_r^+ to B_r. Then for $0 < \alpha < 1$, there are $C_1 = C_1(n, \alpha)$, $C_2 = C_2(n, \alpha)$ such that for all $0 < \rho \le s \le (3/4)r$,

$$\int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{B_\rho}|^2 \le C_1 \left(\frac{\rho}{s}\right)^{n+\alpha} \int_{B_s} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{B_s}|^2 + C_2 \frac{s^{n+1}}{r^{n+3}} \int_{B_r} h^2. \tag{2.5.1}$$

Proof. This proposition differs from Proposition 1.4.1 only by not requiring $h(y)$ to be even in the y_n-variable. As in the proof of Proposition 2.4.1 we split h into its even and odd parts

$$h(y) = h^*(y) + h^\sharp(y), \quad y \in B_r.$$

Recall that h^* is still a solution of the Signorini problem in B_r, but now even in y_n and $h^\sharp$ is a harmonic function in B_r, odd in y_n. Then, by Proposition 2.4.1 we have

$$\int_{B_\rho} |\widehat{\nabla h^*} - \langle \widehat{\nabla h^*} \rangle_{B_\rho}|^2 \leq C_1 \left(\frac{\rho}{s}\right)^{n+\alpha} \int_{B_s} |\widehat{\nabla h^*} - \langle \widehat{\nabla h^*} \rangle_{B_s}|^2 + C_2 \frac{s^{n+1}}{r^{n+3}} \int_{B_r} (h^*)^2. \tag{2.5.2}$$

Now we need a similar estimate for $h^\sharp$. Since $h^\sharp$ is harmonic, by the standard interior estimates, we have

$$\sup_{B_{(3/4)r}} |D^2 h^\sharp| \leq \frac{C(n)}{r^2} \left(\frac{1}{r^n} \int_{B_r} (h^\sharp)^2\right)^{1/2}.$$

Thus, taking the averages on B_ρ^+, we will therefore have

$$\begin{aligned}\int_{B_\rho^+} |\nabla h^\sharp - \langle \nabla h^\sharp \rangle_{B_\rho^+}|^2 &\leq C(n) \rho^{n+2} \Big(\sup_{B_\rho} |D^2 h^\sharp|\Big)^2 \leq C(n) \frac{\rho^{n+2}}{r^{n+4}} \int_{B_r} (h^\sharp)^2 \\ &\leq C(n) \frac{s^{n+1}}{r^{n+3}} \int_{B_r} (h^\sharp)^2, \quad 0 < \rho < s \leq (3/4) r,\end{aligned}$$

which can be rewritten as

$$\int_{B_\rho} |\widehat{\nabla h^\sharp} - \langle \widehat{\nabla h^\sharp} \rangle_{B_\rho}|^2 \leq C(n) \frac{s^{n+1}}{r^{n+3}} \int_{B_r} (h^\sharp)^2. \tag{2.5.3}$$

Now using that $\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{B_\rho} = [\widehat{\nabla h^*} - \langle \widehat{\nabla h^*} \rangle_{B_\rho}] + [\widehat{\nabla h^\sharp} - \langle \widehat{\nabla h^\sharp} \rangle_{B_\rho}]$ in B_ρ, we deduce from (2.5.3) that

$$\begin{aligned}\int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{B_\rho}|^2 &\leq 2 \int_{B_\rho} |\widehat{\nabla h^*} - \langle \widehat{\nabla h^*} \rangle_{B_\rho}|^2 + 2 \int_{B_\rho} |\widehat{\nabla h^\sharp} - \langle \widehat{\nabla h^\sharp} \rangle_{B_\rho}|^2 \\ &\leq 2 \int_{B_\rho} |\widehat{\nabla h^*} - \langle \widehat{\nabla h^*} \rangle_{B_\rho}|^2 + C(n) \frac{s^{n+1}}{r^{n+3}} \int_{B_r} (h^\sharp)^2.\end{aligned} \tag{2.5.4}$$

Similarly, representing $\widehat{\nabla h^*} - \langle \widehat{\nabla h^*} \rangle_{B_s} = [\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{B_s}] - [\widehat{\nabla h^\sharp} - \langle \widehat{\nabla h^\sharp} \rangle_{B_s}]$ in B_s, we deduce from (2.5.3) (by taking $\rho = s$) that

$$\int_{B_s} |\widehat{\nabla h^*} - \langle \widehat{\nabla h^*} \rangle_{B_s}|^2 \leq 2 \int_{B_s} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{B_s}|^2 + C(n) \frac{s^{n+1}}{r^{n+3}} \int_{B_r} (h^\sharp)^2. \tag{2.5.5}$$

Hence, combining (2.5.2)–(2.5.5), and using that both $\int_{B_r} (h^*)^2$ and $\int_{B_r} (h^\sharp)^2$ cannot exceed $\int_{B_r} h^2$, we obtain the claimed estimate (2.5.1). $\square$

Theorem 2.5.1. *Let U be an almost minimizer of the A-Signorini problem in B_1. Then*

$$U \in C^{1,\beta}(B_1^\pm \cup B_1') \quad \textit{with } \beta = \frac{\alpha}{4(2n+\alpha)}.$$

Moreover, for any $K \Subset B_1^\pm \cup B_1'$, we have

$$\|U\|_{C^{1,\beta}(K)} \leq C(n, \alpha, M, K) \|U\|_{W^{1,2}(B_1)}. \tag{2.5.6}$$

Proof. Let K be a ball centered at 0. Fix a small $r_0 = r_0(n, \alpha, M, K) > 0$ to be determined later. In particular, we will ask $r_1 := r_0^{\frac{2n}{2n+\alpha}} \Lambda^{1/2} \leq (1/2) \operatorname{dist}(K, \partial B_1)$, which implies that

$$\widetilde{K} := \{y \in B_1 : \operatorname{dist}(y, K) \leq r_1\} \Subset B_1.$$

Define

$$\widehat{\nabla U}(y', y_n) := \begin{cases} \nabla U(y', y_n), & y_n \geq 0 \\ \nabla U(y', -y_n), & y_n < 0. \end{cases}$$

Our goal is to show that for $x_0 \in K$, $0 < \rho < r < r_0$,

$$\begin{aligned} \int_{B_\rho(x_0)} |\widehat{\nabla U} - \langle \widehat{\nabla U} \rangle_{B_\rho(x_0)}|^2 \leq C(n, \alpha, M) \left(\frac{\rho}{r} \right)^{n+\alpha} \int_{B_r(x_0)} |\widehat{\nabla U} - \langle \widehat{\nabla U} \rangle_{B_r(x_0)}|^2 \\ + C(n, \alpha, M, K) \|U\|_{W^{1,2}(B_1)}^2 r^{n+2\beta}. \end{aligned} \tag{2.5.7}$$

Case 1. Suppose $x_0 \in K \cap B_1'$. For given $0 < r < r_0$, we denote $\alpha' := 1 - \frac{\alpha}{8n} \in (0,1)$, $R := r^{\frac{2n}{2n+\alpha}}$. We then consider two cases:

$$\sup_{\partial E_R(x_0)} |U| \le C_3(\Lambda^{1/2}R)^{\alpha'} \quad \text{and} \quad \sup_{\partial E_R(x_0)} |U| > C_3(\Lambda^{1/2}R)^{\alpha'},$$

where $C_3 = 2[U]_{0,\alpha',\widetilde{K}} = 2\sup_{\substack{y,z\in\widetilde{K}\\ y\neq z}} \frac{|U(y)-U(z)|}{|y-z|^{\alpha'}}$.

Case 1.1. Assume that $\sup_{\partial E_R(x_0)} |U| \le C_3(\Lambda^{1/2}R)^{\alpha'}$. Then u_{x_0} satisfies almost Signorini property at 0 in B_R with

$$\sup_{\partial B_R} |u_{x_0}| \le C_3(\Lambda^{1/2}R)^{\alpha'}.$$

Let h be the Signorini replacement of u_{x_0} in B_R. If we define

$$\widehat{\nabla u_{x_0}}(y', y_n) := \begin{cases} \nabla u_{x_0}(y', y_n), & y_n \ge 0 \\ \nabla u_{x_0}(y', -y_n), & y_n < 0 \end{cases}$$

and

$$\widehat{\nabla h}(y', y_n) := \begin{cases} \nabla h(y', y_n), & y_n \ge 0 \\ \nabla h(y', -y_n), & y_n < 0, \end{cases}$$

then we have

$$\int_{B_\rho} |\widehat{\nabla u_{x_0}} - \langle \widehat{\nabla u_{x_0}} \rangle_{B_\rho}|^2 \le 3\int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h}\rangle_{B_\rho}|^2 + 6\int_{B_\rho} |\widehat{\nabla u_{x_0}} - \widehat{\nabla h}|^2, \tag{2.5.8}$$

$$\int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h}\rangle_{B_r}|^2 \le 3\int_{B_r} |\widehat{\nabla u_{x_0}} - \langle \widehat{\nabla u_{x_0}} \rangle_{B_r}|^2 + 6\int_{B_r} |\widehat{\nabla u_{x_0}} - \widehat{\nabla h}|^2. \tag{2.5.9}$$

Note that if $r_0 \le (3/4)^{\frac{2n+\alpha}{\alpha}}$, then $r < (3/4)R$, thus by Proposition 2.5.1, the Signorini replacement h satisfies, for $0 < \rho < r$,

$$\int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h}\rangle_{B_\rho}|^2 \le C(n,\alpha)\left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h}\rangle_{B_r}|^2 + C(n,\alpha)\frac{r^{n+1}}{R^3}\sup_{\partial B_R} h^2.$$

Combining the above three inequalities, we obtain

$$\int_{B_\rho} |\widehat{\nabla u_{x_0}} - \langle \widehat{\nabla u_{x_0}} \rangle_{B_\rho}|^2 \le C(n,\alpha) \left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_r} |\widehat{\nabla u_{x_0}} - \langle \widehat{\nabla u_{x_0}} \rangle_{B_r}|^2 + C(n,\alpha)\,\frac{r^{n+1}}{R^3} \sup_{\partial B_R} h^2 + C(n,\alpha) \int_{B_r} |\widehat{\nabla u_{x_0}} - \widehat{\nabla h}|^2. \quad (2.5.10)$$

Let us estimate the last term in the right-hand side of (2.5.10). Take $\delta = \delta(n,\alpha,M,K) > 0$ such that $\delta < \operatorname{dist}(K, \partial B_1)$ and $\delta^\alpha \le \varepsilon = \varepsilon(C_1, n, n+2\alpha'-2)$, where $C_1 = C_1(n,M)$ is as in Proposition 2.4.2 and ε is as in Lemma 1.2.2. If $r_0 \le \left(\Lambda^{-1/2}\delta\right)^{\frac{2n+\alpha}{2n}}$, then $\Lambda^{1/2}R < \delta$, thus, by following the proof of Theorem 2.4.1 up to (2.4.4), we have

$$\int_{B_{\Lambda^{1/2}R}(x_0)} |\nabla U|^2 \le C(n,\alpha,M,K)\|\nabla U\|^2_{L^2(B_1)} \left(\Lambda^{1/2}R\right)^{n+2\alpha'-2}.$$

It follows that

$$\int_{E_R(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \le \Lambda \int_{B_{\Lambda^{1/2}R}(x_0)} |\nabla U|^2 \le C\|\nabla U\|^2_{L^2(B_1)} R^{n+2\alpha'-2}.$$

Then by the change of variables (2.2.2), we have

$$\int_{B_R} |\nabla u_{x_0}|^2 \le C\|\nabla U\|^2_{L^2(B_1)} R^{n+2\alpha'-2}. \quad (2.5.11)$$

Now we can estimate the third term in the right-hand side of (2.5.10):

$$\begin{aligned}
\int_{B_r} |\widehat{\nabla u_{x_0}} - \widehat{\nabla h}|^2 &= 2\int_{B_r^+} |\nabla u_{x_0} - \nabla h|^2 \\
&\le 2\int_{B_R} |\nabla u_{x_0} - \nabla h|^2 \le 2\left(\int_{B_R} |\nabla u_{x_0}|^2 - \int_{B_R} |\nabla h|^2\right) \\
&\le 2MR^\alpha \int_{B_R} |\nabla h|^2 \le 2MR^\alpha \int_{B_R} |\nabla u_{x_0}|^2 \\
&\le C\|\nabla U\|^2_{L^2(B_1)} R^{n+\alpha+2\alpha'-2} = C\|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2n+\alpha}(n-\frac{1}{2})}.
\end{aligned} \quad (2.5.12)$$

To estimate the second term in the right-hand side of (2.5.10), we observe that

$$\sup_{\partial B_R} h^2 = \sup_{\partial B_R} u_{x_0}^2 = \sup_{\partial E_R(x_0)} U^2 \le C_3^2(\Lambda^{1/2}R)^{2\alpha'}.$$

Note that by (2.4.5), $C_3 \leq C(n,\alpha,M,K)\|U\|_{W^{1,2}(B_1)}$. Thus,

$$\frac{r^{n+1}}{R^3}\sup_{\partial B_R} h^2 \leq C\|U\|^2_{W^{1,2}(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}.$$

Now (2.5.10) becomes

$$\begin{aligned}\int_{B_\rho} |\widehat{\nabla u_{x_0}} - \langle\widehat{\nabla u_{x_0}}\rangle_{B_\rho}|^2 \leq C(n,\alpha)\left(\frac{\rho}{r}\right)^{n+\alpha}\int_{B_r} |\widehat{\nabla u_{x_0}} - \langle\widehat{\nabla u_{x_0}}\rangle_{B_r}|^2 \\ + C\|U\|^2_{W^{1,2}(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}. \quad (2.5.13)\end{aligned}$$

We now want to deduce (2.5.7) from (2.5.13). The complication here is that the mapping $\bar{T}_{x_0}^{-1}$ does not preserve the even symmetry with respect to the thin plane, since the conormal direction $A(x_0)\mathrm{e}_n$ might be different from the normal direction e_n to Π at x_0. To address this issue, by using the even symmetry of $\widehat{\nabla u_{x_0}}$, we rewrite (2.5.13) in terms of halfballs $B_r^+ = B_r \cap \mathbb{R}^n_+$

$$\begin{aligned}\int_{B_\rho^+} |\nabla u_{x_0} - \langle\nabla u_{x_0}\rangle_{B_\rho^+}|^2 \leq C(n,\alpha)\left(\frac{\rho}{r}\right)^{n+\alpha}\int_{B_r^+} |\nabla u_{x_0} - \langle\nabla u_{x_0}\rangle_{B_r^+}|^2 \\ + C\|U\|^2_{W^{1,2}(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}. \quad (2.5.14)\end{aligned}$$

Similarly, if we denote $E_r^+(x_0) = E_r(x_0) \cap \mathbb{R}^n_+$, then using that $\bar{T}_{x_0}(E_t^+(x_0)) = B_t^+$, $t > 0$, (2.5.14) becomes

$$\begin{aligned}\int_{E_\rho^+(x_0)} |\mathfrak{a}_{x_0}\nabla U - \langle\mathfrak{a}_{x_0}\nabla U\rangle_{E_\rho^+(x_0)}|^2 \leq C(n,\alpha)\left(\frac{\rho}{r}\right)^{n+\alpha}\int_{E_r^+(x_0)} |\mathfrak{a}_{x_0}\nabla U - \langle\mathfrak{a}_{x_0}\nabla U\rangle_{E_r^+(x_0)}|^2 \\ + C\det\mathfrak{a}_{x_0}\|U\|^2_{W^{1,2}(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}.\end{aligned}$$

Repeating the argument that (2.3.4) implies (2.3.2) in the proof of Proposition 2.3.1, we have

$$\begin{aligned}\int_{B_\rho^+(x_0)} |\nabla U - \langle\nabla U\rangle_{B_\rho^+(x_0)}|^2 \leq C\left(\frac{\rho}{r}\right)^{n+\alpha}\int_{B_r^+(x_0)} |\nabla U - \langle\nabla U\rangle_{B_r^+(x_0)}|^2 \\ + C\|U\|^2_{W^{1,2}(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}. \quad (2.5.15)\end{aligned}$$

Then by the even symmetry of $\widehat{\nabla U}$, (2.5.15) implies (2.5.7).

Case 1.2. Now we assume that $\sup_{\partial E_R(x_0)} |U| > C_3(\Lambda^{1/2}R)^{\alpha'}$. By the choice of $C_3 = 2[U]_{0,\alpha',\widetilde{K}}$, we have either

$$U \geq (C_3/2)\,(\Lambda^{1/2}R)^{\alpha'} \quad \text{in } E_R(x_0), \text{ or}$$
$$U \leq -\,(C_3/2)\,(\Lambda^{1/2}R)^{\alpha'} \quad \text{in } E_R(x_0).$$

However, from $U \geq 0$ on B_1', the only possibility is

$$U \geq (C_3/2)\,(\Lambda^{1/2}R)^{\alpha'} \quad \text{in } E_R(x_0).$$

Consequently,

$$u_{x_0} \geq (C_3/2)\,(\Lambda^{1/2}R)^{\alpha'} \quad \text{in } B_R.$$

If we let h again be the Signorini replacement of u_{x_0} in B_R, then the positivity of $h = u_{x_0} > 0$ on ∂B_R and superharmonicity of h in B_R give that $h > 0$ in B_R, and hence h is harmonic in B_R. Thus,

$$\int_{B_\rho} |\nabla h - \langle \nabla h\rangle_{B_\rho}|^2 \leq \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\nabla h - \langle \nabla h\rangle_{B_r}|^2, \quad 0 < \rho < r.$$

We next decompose $h = h^* + h^\sharp$ in B_R as in (2.4.2). Note that since both h and $h^\sharp$ are harmonic, h^* must be harmonic as well. Then we have

$$\begin{aligned}
\int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h}\rangle_{B_\rho}|^2 &\leq 3\int_{B_\rho} |\nabla h - \langle \nabla h\rangle_{B_\rho}|^2 + 6\int_{B_\rho} |\widehat{\nabla h} - \nabla h|^2 \\
&= 3\int_{B_\rho} |\nabla h - \langle \nabla h\rangle_{B_\rho}|^2 + 6\int_{B_\rho^-} \left(|2\nabla_{y'} h^\sharp|^2 + |2\partial_{y_n} h^*|^2\right) \\
&= 3\int_{B_\rho} |\nabla h - \langle \nabla h\rangle_{B_\rho}|^2 + 12\int_{B_\rho} \left(|\nabla_{y'} h^\sharp|^2 + |\partial_{y_n} h^*|^2\right),
\end{aligned}$$

and similarly,

$$\int_{B_r} |\nabla h - \langle \nabla h\rangle_{B_r}|^2 \leq 3\int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h}\rangle_{B_r}|^2 + 12\int_{B_r} \left(|\nabla_{y'} h^\sharp|^2 + |\partial_{y_n} h^*|^2\right).$$

Combining the above three inequalities, we have that for all $0 < \rho < r$

$$\int_{B_\rho} |\widehat{\nabla h} \; - \; \langle \widehat{\nabla h} \rangle_{B_\rho}|^2 \;\; \leq \;\; 3\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\widehat{\nabla h} \; - \; \langle \widehat{\nabla h} \rangle_{B_r}|^2 \; + \; 48 \int_{B_r} \left(|\nabla_{y'} h^\sharp|^2 + |\partial_{y_n} h^*|^2\right). \tag{2.5.16}$$

Now, note that if $r_0 \leq (1/2)^{\frac{2n+\alpha}{\alpha}}$, then $r \leq R/2$. By the harmonicity of both h^* and $h^\sharp$ in B_R, we have

$$\begin{aligned}
\sup_{B_{R/2}} |D^2 h^*| + \sup_{B_{R/2}} |D^2 h^\sharp| &\leq \frac{C(n)}{R} \left(\sup_{B_{(3/4)R}} |\nabla h^*| + \sup_{B_{(3/4)R}} |\nabla h^\sharp| \right) \\
&\leq \frac{C(n)}{R^{1+\frac{n}{2}}} \left(\int_{B_R} |\nabla h^*|^2 + \int_{B_R} |\nabla h^\sharp|^2 \right)^{1/2} \\
&= \frac{C(n)}{R^{1+\frac{n}{2}}} \left(\int_{B_R} |\nabla h|^2 \right)^{1/2} \leq \frac{C(n)}{R^{1+\frac{n}{2}}} \left(\int_{B_R} |\nabla u_{x_0}|^2 \right)^{1/2} \\
&\leq C(n, \alpha, M, K) \|\nabla U\|_{L^2(B_1)} R^{\alpha' - 2},
\end{aligned}$$

where the last inequality follows from (2.5.11). Also, note that $\nabla_{y'} h^\sharp = \partial_{y_n} h^* = 0$ on $B'_{R/2}$. Thus, for $y = (y', y_n) \in B_r$, we have

$$\begin{aligned}
|\nabla_{y'} h^\sharp| + |\partial_{y_n} h^*| &\leq |y_n| \left(\sup_{B_{R/2}} |D^2 h^*| + \sup_{B_{R/2}} |D^2 h^\sharp| \right) \\
&\leq C \|\nabla U\|_{L^2(B_1)} r R^{\alpha' - 2} \\
&= C \|\nabla U\|_{L^2(B_1)} r^{1 + \frac{2n}{2n+\alpha}(\alpha' - 2)},
\end{aligned}$$

with $C = (n, \alpha, M, K)$. Hence, it follows that

$$\int_{B_r} |\nabla_{y'} h^\sharp|^2 + |\partial_{y_n} h^*|^2 \leq C \|\nabla U\|^2_{L^2(B_1)} r^{n+2+\frac{4n}{2n+\alpha}(\alpha'-2)} \;\; \leq C \|\nabla U\|^2_{L^2(B_1)} r^{n + \frac{\alpha}{2(2n+\alpha)}}. \tag{2.5.17}$$

Combining (2.5.16) and (2.5.17), we obtain

$$\int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{B_\rho}|^2 \; \leq \; 3\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{B_r}|^2 + C \|\nabla U\|^2_{L^2(B_1)} r^{n + \frac{\alpha}{2(2n+\alpha)}}. \tag{2.5.18}$$

Note that (2.5.12) was induced in Case 1.1 without the use of the assumption $\sup_{\partial E_r(x_0)} |U| \le C_3 \left(\Lambda^{1/2} R\right)^{\alpha'}$, so it is also valid in this case. Finally, (2.5.8), (2.5.9), (2.5.12) and (2.5.18) give

$$
\begin{aligned}
\int_{B_\rho} |\widehat{\nabla u_{x_0}} - \langle \widehat{\nabla u_{x_0}} \rangle_{B_\rho}|^2 &\le 3 \int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{B_\rho}|^2 + 6 \int_{B_\rho} |\widehat{\nabla u_{x_0}} - \widehat{\nabla h}|^2 \\
&\le 9 \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_{B_r}|^2 + C \|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}} \\
&\quad + 6 \int_{B_\rho} |\widehat{\nabla u_{x_0}} - \widehat{\nabla h}|^2 \\
&\le 27 \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\widehat{\nabla u_{x_0}} - \langle \widehat{\nabla u_{x_0}} \rangle_{B_r}|^2 + C \|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}} \\
&\quad + 60 \int_{B_r} |\widehat{\nabla u_{x_0}} - \widehat{\nabla h}|^2 \\
&\le 27 \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\widehat{\nabla u_{x_0}} - \langle \widehat{\nabla u_{x_0}} \rangle_{B_r}|^2 + C \|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}} \\
&\quad + C \|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2n+\alpha}(n-1/2)} \\
&\le 27 \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\widehat{\nabla u_{x_0}} - \langle \widehat{\nabla u_{x_0}} \rangle_{B_r}|^2 + C \|\nabla U\|^2_{L^2(B_1)} r^{n+\frac{\alpha}{2(2n+\alpha)}}.
\end{aligned}
$$

As we have seen in Case 1.1, this implies (2.5.7). This completes the proof of (2.5.7) when $x_0 \in K \cap B_1'$.

Case 2. The extension of (2.5.7) to general $x_0 \in K$ follows from the combination of Case 1 and (2.3.5). The argument is the same as Case 2 in the proof of Theorem 1.4.1.

Thus, the estimate (2.5.7) holds in all possible cases.

To complete the proof of the theorem, we now apply Lemma 1.2.2 to the estimate (2.5.7) to obtain for $0 < \rho < r < r_0$

$$
\int_{B_\rho(x_0)} |\widehat{\nabla U} - \langle \widehat{\nabla U} \rangle_{B_\rho(x_0)}|^2 \le C \left[\left(\frac{\rho}{r}\right)^{n+2\beta} \int_{B_r(x_0)} |\widehat{\nabla U} - \langle \widehat{\nabla U} \rangle_{B_r(x_0)}|^2 + \|U\|^2_{W^{1,2}(B_1)} \rho^{n+2\beta} \right].
$$

Taking $r \nearrow r_0 = r_0(n, \alpha, M, K)$, we have

$$
\int_{B_\rho(x_0)} |\widehat{\nabla U} - \langle \widehat{\nabla U} \rangle_{B_\rho(x_0)}|^2 \le C \|U\|^2_{W^{1,2}(B_1)} \rho^{n+2\beta},
$$

with $C = C(n, \alpha, M, K)$. Then by the Campanato space embedding this readily implies that $\widehat{\nabla U} \in C^{0,\beta}(K)$ with

$$\|\widehat{\nabla U}\|_{C^{0,\beta}(K)} \leq C\|U\|_{W^{1,2}(B_1)}.$$

Since $\widehat{\nabla U} = \nabla U$ in $B_1^+ \cup B_1'$, we therefore conclude that

$$U \in C^{1,\beta}(K \cap (B_1^+ \cup B_1')),$$

and combining with the bound in Theorem 2.4.1, we also deduce that

$$\|U\|_{C^{1,\beta}(K\cap(B_1^+\cup B_1'))} \leq C(n, \alpha, M, K)\|U\|_{W^{1,2}(B_1)}.$$

To see the $C^{1,\beta}$ regularity of U in $B_1^- \cup B_1'$, we simply observe that the function $U(y', -y_n)$ is also an almost minimizer of the Signorini problem with the appropriately modified coefficient matrix A. □

2.6 Quasisymmetric almost minimizers

In the study of the free boundary in the Signorini problem, the even symmetry of the minimizer with respect to the thin space plays a crucial role. The even symmetry guarantees that the growth rate of the minimizer u over "thick" balls $B_r(x_0) \subset \mathbb{R}^n$ matches the growth rate over thin balls $B_r'(x_0) \subset \Pi$. This allows to use tools such as Almgren's monotonicity formula (see the next section) to classify the free boundary points. Without even symmetry, minimizers may have an odd component, vanishing on the thin space Π that may create a mismatch of growth rates on the thick and thin spaces.

In the case of minimizers of the Signorini problem (with $A = I$) or harmonic functions, it is easy to see that the even symmetrization

$$u^*(x) = \frac{u(x', x_n) + u(x', -x_n)}{2}$$

is still a minimizer. Unfortunately, the even symmetrization may destroy the almost minimizing property, as well as the minimizing property with variable coefficients, as can be seen from the following simple example.

Example 2.6.1. Let $u : (-1,1) \to \mathbb{R}$ be defined by $u(x) = x + x^2/4$. Then u is an almost harmonic function in $(-1,1)$ with a gauge function $\omega(r) = C(\alpha)r^\alpha$ for $0 < \alpha < 1$. In fact, u is a minimizer of the energy functional

$$\int (1 + x/2)^{-1}(v')^2$$

with a Lipschitz function $A(x) = (1 + x/2)^{-1}$ in $(-1,1)$. On the other hand, the even symmetrization

$$u^*(x) = \frac{u(x) + u(-x)}{2} = \frac{x^2}{4}$$

is not almost harmonic for any gauge function $\omega(r)$. Indeed, for any small $\delta > 0$, if we take a competitor $v = \delta^2/4$ in $(-\delta,\delta)$, then it satisfies $\int_{-\delta}^{\delta} |v'|^2 = 0$ and if u^* were almost harmonic, we would have that $\int_{-\delta}^{\delta} |(u^*)'|^2 = 0$ as well, implying that u^* is constant in $(-\delta,\delta)$, a contradiction.

To overcome this difficulty, we need to impose the A-*quasisymmetry* condition on almost minimizers U, that we have already stated in Definition 2.1.2. In this section, we give more details on quasisymmetric almost minimizers.

Recall that for each $x_0 \in B_1'$, we defined a reflection matrix P_{x_0} by

$$P_{x_0} = I - 2\frac{A(x_0)\mathrm{e}_n \otimes \mathrm{e}_n}{a_{nn}(x_0)}.$$

From the ellipticity of A, we have $a_{nn}(x_0) \geq \lambda$, thus P_{x_0} is well-defined. Note that $P_{x_0}^2 = I$. Besides, $P_{x_0}\big|_\Pi = I\big|_\Pi$ and $P_{x_0}E_r(x_0) = E_r(x_0)$. We then define the "skewed" even/odd symmetrizations of the almost minimizer U in B_1 by

$$\begin{aligned} U_{x_0}^*(x) &:= \frac{U(x) + U(P_{x_0}x)}{2}, \\ U_{x_0}^\sharp(x) &:= \frac{U(x) - U(P_{x_0}x)}{2}. \end{aligned}$$

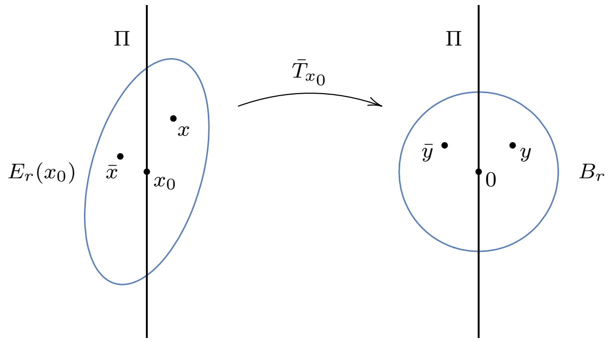

Figure 2.2. Reflection P_{x_0}: here $\bar{x} = P_{x_0}x$, $y = \bar{T}_{x_0}(x)$, and $\bar{y} = (y', -y_n) = \bar{T}_{x_0}(\bar{x})$

Note that $U^*_{x_0}$ and $U^\sharp_{x_0}$ may not be defined in all of B_1, but are defined in any ellipsoid $E_r(x_0)$ as long as it is contained in B_1. Note also that $U = U^*_{x_0}$ and $U^\sharp_{x_0} = 0$ on Π. Further, we note that transformed with $\bar{T}_{x_0}$, P_{x_0} becomes an even reflection with respect to Π, i.e.,

$$\bar{T}_{x_0} \circ P_{x_0} \circ \bar{T}^{-1}_{x_0}(y) = (y', -y_n),$$

see Fig 2.2. Therefore, denoting

$$u^*_{x_0}(y) := \frac{u_{x_0}(y', y_n) + u_{x_0}(y', -y_n)}{2},$$
$$u^\sharp_{x_0}(y) := \frac{u_{x_0}(y', y_n) - u_{x_0}(y', -y_n)}{2},$$

the even/odd symmetrizations of u_{x_0} about Π, we will have

$$U^*_{x_0} \circ \bar{T}^{-1}_{x_0} = u^*_{x_0}, \qquad U^\sharp_{x_0} \circ \bar{T}^{-1}_{x_0} = u^\sharp_{x_0}.$$

We also observe that the symmetries of $u^*_{x_0}$ and $u^\sharp_{x_0}$ imply the following decompositions

$$\int_{B_r} u^2_{x_0} = \int_{B_r} (u^*_{x_0})^2 + \int_{B_r} (u^\sharp_{x_0})^2, \tag{2.6.1}$$
$$\int_{B_r} |\nabla u_{x_0}|^2 = \int_{B_r} |\nabla u^*_{x_0}|^2 + \int_{B_r} |\nabla u^\sharp_{x_0}|^2, \tag{2.6.2}$$

which after a change of variables, can also be written as

$$\int_{E_r(x_0)} U^2 = \int_{E_r(x_0)} (U^*_{x_0})^2 + \int_{E_r(x_0)} (U^\sharp_{x_0})^2, \tag{2.6.3}$$

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle = \int_{E_r(x_0)} \langle A(x_0)\nabla U^*_{x_0}, \nabla U^*_{x_0}\rangle + \int_{E_r(x_0)} \langle A(x_0)\nabla U^\sharp_{x_0}, \nabla U^\sharp_{x_0}\rangle. \tag{2.6.4}$$

We now recall that by Definition 2.1.2, $U \in W^{1,2}(B_1)$ is called A-*quasisymmetric* if there is a constant $Q > 0$ such that

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U, \nabla U\rangle \leq Q \int_{E_r(x_0)} \langle A(x_0)\nabla U^*_{x_0}, \nabla U^*_{x_0}\rangle, \tag{2.6.5}$$

whenever $E_r(x_0) \Subset B_1$ and $x_0 \in B_1$. By the uniform ellipticity of A, (2.6.5) is equivalent to

$$\int_{E_r(x_0)} |\nabla U|^2 \leq Q \int_{E_r(x_0)} |\nabla U^*_{x_0}|^2,$$

by changing Q to $Q(\Lambda/\lambda)$, if necessary. Besides, using (2.6.4), (2.6.5) is also equivalent to

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U^\sharp_{x_0}, \nabla U^\sharp_{x_0}\rangle \leq C \int_{E_r(x_0)} \langle A(x_0)\nabla U^*_{x_0}, \nabla U^*_{x_0}\rangle, \tag{2.6.6}$$

with some $C = C(Q)$.

Lemma 2.6.2. *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1, with constant $Q > 0$. Then there are $r_1 = r_1(n, \alpha, M, Q) > 0$ and $M_1 = M_1(n, M, Q) > 0$ such that*

$$\int_{E_r(x_0)} \langle A(x_0)\nabla U^*_{x_0}, \nabla U^*_{x_0}\rangle \leq (1 + M_1 r^\alpha) \int_{E_r(x_0)} \langle A(x_0)\nabla W, \nabla W\rangle, \tag{2.6.7}$$

whenever $E_r(x_0) \Subset B_1$, $x_0 \in B_1$, $0 < r < r_1$, and $W \in \mathfrak{K}_{0,U^*_{x_0}}(E_r(x_0), \Pi)$.

Remark 2.6.3. Since we are interested in local results, in what follows, we will assume without loss of generality that $r_1 = 1$ and $M_1 = M$.

Proof. Let V be the energy minimizer of

$$\int_{E_r(x_0)} \langle A(x_0)\nabla V, \nabla V\rangle \quad \text{on } \mathfrak{K}_{0,U}(E_r(x_0),\Pi).$$

Then $v_{x_0} = V \circ \bar{T}_{x_0}^{-1}$ is the energy minimizer of

$$\int_{B_r} |\nabla v_{x_0}|^2 \quad \text{on } \mathfrak{K}_{0,u_{x_0}}(B_r,\Pi).$$

Note that $v^*_{x_0}$ is a solution of the Signorini problem, even in y_n, with $v^*_{x_0} = u^*_{x_0}$ on ∂B_r. Similarly, $v^\sharp_{x_0}$ is a harmonic function, odd in y_n, with $v^\sharp_{x_0} = u^\sharp_{x_0}$ on ∂B_r. Thus, $v^*_{x_0}$ is the energy minimizer of

$$\int_{B_r} |\nabla v^*_{x_0}|^2 \quad \text{on } \mathfrak{K}_{0,u^*_{x_0}}(B_r,\Pi),$$

and so $V^*_{x_0}$ is the energy minimizer of

$$\int_{E_r(x_0)} \langle A(x_0)\nabla V^*_{x_0}, \nabla V^*_{x_0}\rangle \quad \text{on } \mathfrak{K}_{0,U^*_{x_0}}(E_r(x_0),\Pi).$$

Thus, to show (2.6.7), it is enough to show

$$\int_{B_r} |\nabla u^*_{x_0}|^2 \le (1+M_1 r^\alpha)\int_{B_r} |\nabla v^*_{x_0}|^2.$$

To this end, we first observe that the quasisymmetry of U implies the quasisymmetry of u_{x_0}:

$$\int_{B_r} |\nabla u^\sharp_{x_0}|^2 \le C\int_{B_r} |\nabla u^*_{x_0}|^2.$$

Using this, together with the symmetry of $u^*_{x_0}$, $u^\sharp_{x_0}$, $v^*_{x_0}$ and $v^\sharp_{x_0}$, we have

$$\begin{aligned}
\int_{B_r} |\nabla u^*_{x_0}|^2 &= \int_{B_r} |\nabla u_{x_0}|^2 - \int_{B_r} |\nabla u^\sharp_{x_0}|^2 \\
&\le (1+Mr^\alpha)\int_{B_r} |\nabla v_{x_0}|^2 - \int_{B_r} |\nabla u^\sharp_{x_0}|^2 \\
&= (1+Mr^\alpha)\int_{B_r} |\nabla v^*_{x_0}|^2 + (1+Mr^\alpha)\int_{B_r} |\nabla v^\sharp_{x_0}|^2 - \int_{B_r} |\nabla u^\sharp_{x_0}|^2 \\
&\le (1+Mr^\alpha)\int_{B_r} |\nabla v^*_{x_0}|^2 + Mr^\alpha\int_{B_r} |\nabla u^\sharp_{x_0}|^2
\end{aligned}$$

$$\leq (1+Mr^{\alpha})\int_{B_r}|\nabla v_{x_0}^*|^2 + CMr^{\alpha}\int_{B_r}|\nabla u_{x_0}^*|^2.$$

Therefore,

$$\int_{B_r}|\nabla u_{x_0}^*|^2 \leq \frac{1+Mr^{\alpha}}{1-CMr^{\alpha}}\int_{B_r}|\nabla v_{x_0}^*|^2 \leq (1+M_1r^{\alpha})\int_{B_r}|\nabla v_{x_0}^*|^2,$$

for $0<r<r_1=(2CM)^{-1/\alpha}$, as desired. $\square$

Remark 2.6.4. If U satisfies the following *weak quasisymmetry* with order $-\gamma$:

$$\int_{E_r(x_0)}|\nabla U|^2 \leq Q\,r^{-\gamma}\int_{E_r(x_0)}|\nabla U_{x_0}^*|^2,$$

whenever $E_r(x_0)\Subset B_1$, $x_0\in B_1'$ for some $0<\gamma<\alpha$, then it is easy to see from the proof of Lemma 2.6.2 that $U_{x_0}^*$ satisfies (2.6.7), but with $\alpha-\gamma>0$ instead of α.

Theorem 2.6.5. *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. Then for $x_0\in B_{1/2}'$ and $0<r\leq(1/2)\Lambda^{-1/2}$, we have $U_{x_0}^*\in C^{1,\beta}(E_r^{\pm}(x_0)\cup E_r'(x_0))$ with $\beta=\frac{\alpha}{4(2n+\alpha)}$. Moreover,*

$$\|U_{x_0}^*\|_{C^{1,\beta}(K)}\leq C(n,\alpha,M,K,r)\|U_{x_0}^*\|_{W^{1,2}(E_r(x_0))},$$

for any $K\Subset E_r^{\pm}(x_0)\cup E_r'(x_0)$. Similarly, $u_{x_0}^*\in C^{1,\beta}(B_r^{\pm}\cup B_r')$ with

$$\|u_{x_0}^*\|_{C^{1,\beta}(K)}\leq C(n,\alpha,M,K,r)\|u_{x_0}^*\|_{W^{1,2}(B_r)},$$

for any $K\Subset B_r^{\pm}\cup B_r'$.

Proof. From Theorem 2.5.1, we have $U\in C^{1,\beta}(B_1^{\pm}\cup B_1')$, which immediately gives $U_{x_0}^*\in C^{1,\beta}(E_r^{\pm}(x_0)\cup E_r'(x_0))$, by using the inclusion $E_r(x_0)\subset B_{\Lambda^{1/2}r}(x_0)\subset B_1$. Thus, for

$$\widehat{\nabla U_{x_0}^*}(x',x_n):=\begin{cases}\nabla U_{x_0}^*(x',x_n), & x_n\geq 0\\ \nabla U_{x_0}^*(x',-x_n), & x_n<0,\end{cases}$$

we have $\widehat{\nabla U^*_{x_0}} \in C^{0,\beta}(E_r(x_0))$ with

$$\|\widehat{\nabla U^*_{x_0}}\|_{C^{0,\beta}(K)} \leq C(n, \alpha, M, K, r)\|U\|_{W^{1,2}(E_r(x_0))},$$

for any $K \Subset E_r(x_0)$. Hence, it is enough to show that

$$\|U\|_{W^{1,2}(E_r(x_0))} \leq C\|U^*_{x_0}\|_{W^{1,2}(E_r(x_0))}.$$

Now, note that by (2.6.3)–(2.6.4), we readily have

$$\|U\|_{W^{1,2}(E_r(x_0))} \leq C\left(\|U^*_{x_0}\|_{W^{1,2}(E_r(x_0))} + \|U^\sharp_{x_0}\|_{W^{1,2}(E_r(x_0))}\right),$$

and thus, it will suffice to show that

$$\|U^\sharp_{x_0}\|_{W^{1,2}(E_r(x_0))} \leq C\|U^*_{x_0}\|_{W^{1,2}(E_r(x_0))}.$$

By the symmetry again,

$$\langle U^\sharp_{x_0}\rangle_{E_r(x_0)} = \langle u^\sharp_{x_0}\rangle_{B_r} = 0,$$

thus by Poincare's inequality,

$$\|U^\sharp_{x_0}\|_{L^2(E_r(x_0))} \leq C(n, M)r\|\nabla U^\sharp_{x_0}\|_{L^2(E_r(x_0))}. \tag{2.6.8}$$

Finally, by the quasisymmetry of U, we have

$$\|\nabla U^\sharp_{x_0}\|_{L^2(E_r(x_0))} \leq C\|\nabla U^*_{x_0}\|_{L^2(E_r(x_0))},$$

see (2.6.6). This completes the proof of the theorem for $U^*_{x_0}$.

Applying now the affine transformation $\bar{T}_{x_0}$, we obtain the part of the theorem for $u^*_{x_0}$. $\square$

We complete this section with a version of Signorini's complementarity condition that will play an important role in the analysis of the free boundary.

Lemma 2.6.6 (Complementarity condition)**.** *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1, and $x_0 \in B'_{1/2}$. Then $u^*_{x_0}$ satisfies the following* complementarity condition

$$u^*_{x_0}(\partial^+_{y_n} u^*_{x_0}) = 0 \quad \text{on } B'_{R_0}, \quad R_0 = (1/2)\Lambda^{-1/2},$$

where $\partial^+_{y_n} u^*_{x_0}$ on B'_{R_0} is computed as the limit from inside $B^+_{R_0}$. Moreover, if $x_0 \in \Gamma(U)$, then

$$u^*_{x_0}(0) = 0 \quad \text{and} \quad |\widehat{\nabla u^*_{x_0}}(0)| = 0.$$

Proof. Let $y_0 \in B'_{R_0}$ be such that $u^*_{x_0}(y_0) > 0$. Then we need to show that $\partial^+_{y_n} u^*_{x_0}(y_0) = 0$. Since $u_{x_0} = u^*_{x_0}$ on Π, we have $u_{x_0}(y_0) > 0$ and by continuity $u_{x_0} > 0$ in a small ball $B_\delta(y_0)$. Then $U > 0$ in $\Omega = \bar{T}^{-1}_{x_0}(B_\delta(y_0))$. We claim now that U is almost A-harmonic in Ω. Indeed, if $E_r(y) \Subset \Omega$ (not necessarily with $y \in B'_1$) and V is $A(y)$-harmonic replacement of U on $E_r(y)$ (i.e. $\operatorname{div}(A(y)\nabla V) = 0$ in $E_r(y)$ with $V = U$ on $\partial E_r(y)$), then since $V = U > 0$ on $\partial E_r(y)$, by the minimum principle $V > 0$ on $\overline{E_r(y)}$. This means that $V \in \mathfrak{K}_{0,U}(E_r(y), \Pi)$ and therefore we must have

$$\int_{E_r(y)} \langle A(y)\nabla U, \nabla U\rangle \leq (1+\omega(r)) \int_{E_r(y)} \langle A(y)\nabla V, \nabla V\rangle,$$

which also implies that U is an almost A-harmonic function in Ω. Hence, $U \in C^{1,\alpha/2}(\Omega)$ by Theorem 2.3.2, implying also that $u_{x_0} \in C^{1,\alpha/2}(B_\delta(y_0))$. Consequently, also $u^*_{x_0} \in C^{1,\alpha/2}(B_\delta(y_0))$ and by even symmetry in the y_n-variable, we therefore conclude that $\partial^+_{y_n} u^*_{x_0}(y_0) = 0$.

The second part of the lemma now follows by the $C^{1,\beta}$ regularity and the complementarity condition. □

2.7 Weiss- and Almgren-type monotonicity formulas

In this section we introduce two technical tools: Weiss- and Almgren-type monotonicity formulas, that will play a fundamental role in the analysis of the free boundary. In fact, the proofs of these formulas follow immediately from the case $A \equiv I$, following the deskewing procedure.

To proceed, we fix a constant $\kappa_0 > 0$. We can take it as large as we want, however, some constants in what follows, will depend on κ_0. Then for $0 < \kappa < \kappa_0$, we consider the Weiss-type energy functional introduced in Chapter 1:

$$W_\kappa(t, v, x_0) := \frac{e^{at^\alpha}}{t^{n+2\kappa-2}} \left[\int_{B_t(x_0)} |\nabla v|^2 - \kappa \frac{1 - bt^\alpha}{t} \int_{\partial B_t(x_0)} v^2 \right],$$

with

$$a = a_\kappa = \frac{M(n + 2\kappa - 2)}{\alpha}, \quad b = \frac{M(n + 2\kappa_0)}{\alpha}.$$

(The formula in Chapter 1 corresponds to the case $M = 1$.) Based on that, we define an appropriate version of Weiss's functional for our problem. For a function V in $E_r(x_0)$, let

$$W^A_\kappa(t, V, x_0) \quad := \quad \frac{e^{at^\alpha}}{t^{n+2\kappa-2}} \left[\int_{E_t(x_0)} \langle A(x_0) \nabla V, \nabla V \rangle \; - \; \kappa \; \frac{1 - bt^\alpha}{t} \int_{\partial E_t(x_0)} V^2 \mu_{x_0}(x - x_0) \right], \tag{2.7.1}$$

for $0 < t < r$, with a, b same as above, where the weight μ_{x_0} is as in (2.2.4). Note that by the change of variables formulas (2.2.1)–(2.2.3), we have

$$W^A_\kappa(t, V, x_0) := \det \mathfrak{a}_{x_0} W_\kappa(t, v_{x_0}, 0), \quad v_{x_0} = V \circ \bar{T}^{-1}_{x_0}. \tag{2.7.2}$$

Let now U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1 and $x_0 \in B_{1/2}$. By Lemma 2.6.2, $U^*_{x_0}$ satisfies the almost A-Signorini property at x_0 in $E_{(1/2)\Lambda^{-1/2}}(x_0)$. Thus $u^*_{x_0}$ also satisfies the almost Signorini property at 0 in $B_{(1/2)\Lambda^{-1/2}}$. By using this observation, we then have the following Weiss-type monotonicity formulas for $U^*_{x_0}$ and $u^*_{x_0}$.

Theorem 2.7.1 (Weiss-type monotonicity formula)**.** *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. Suppose $x_0 \in B'_{1/2}$. Let $0 < \kappa < \kappa_0$ with a fixed $\kappa_0 > 0$. Then, for $0 < t < t_0 = t_0(n, \alpha, \kappa_0, M)$,*

$$\frac{d}{dt} W_\kappa(t, u^*_{x_0}, 0) \geq \frac{e^{at^\alpha}}{t^{n+2\kappa-2}} \int_{\partial B_t} \left(\partial_\nu u^*_{x_0} - \frac{\kappa(1 - bt^\alpha)}{t} u^*_{x_0} \right)^2,$$

$$\frac{d}{dt} W^A_\kappa(t, U^*_{x_0}, x_0) \geq \frac{e^{at^\alpha}}{t^{n+2\kappa-2}} \int_{\partial E_t(x_0)} \left(\langle \mathfrak{a}_{x_0} \nabla U^*_{x_0}, \nu \rangle - \frac{\kappa(1 - bt^\alpha)}{t} U^*_{x_0} \right)^2 \mu_{x_0}(x - x_0).$$

In particular, $W_\kappa(t, u^*_{x_0}, 0)$ and $W^A_\kappa(t, U^*_{x_0}, x_0)$ are nondecreasing in t for $0 < t < t_0$.

Proof. We note that the proof of Theorem 1.5.1 for the monotonicity of $W_\kappa(t, v, x_0)$ requires the function v to be an almost minimizer for the Signorini problem for the monotonicity of its energy. However, it is not hard to see that the almost minimizing property of v is used only when it is compared with the κ-homogeneous replacement w of v on balls centered at the given point x_0 to obtain

$$\int_{B_t(x_0)} |\nabla w|^2 \geq \frac{1}{1 + t^\alpha} \int_{B_t(x_0)} |\nabla v|^2,$$

see (1.5.2). This means that the argument in the proof of Theorem 1.5.1 also works in our case and implies the part of the theorem for $u^*_{x_0}$. We note that the constants a_κ and b in our case will have an additional factor of M, as we work with $\omega(r) = Mr^\alpha$ rather than $\omega(r) = r^\alpha$ in our case, but this change of the constants can be easily traced.

The part of the theorem for $U^*_{x_0}$ follows by a change of variables. □

The families of monotonicity formulas $\{W_\kappa\}_{0<\kappa<\kappa_0}$ and $\{W^A_\kappa\}_{0<\kappa<\kappa_0}$ have an important feature that their intervals of monotonicity and the constant b can be taken the same for all $0 < \kappa < \kappa_0$. Because of that, their monotonicity indirectly implies that of another important quantity that we describe below. Namely, recall that for a function v in $B_r(x_0)$, *Almgren's frequency* of v at x_0 is defined as

$$N(t, v, x_0) := \frac{t \int_{B_t(x_0)} |\nabla v|^2}{\int_{\partial B_t(x_0)} v^2}, \quad 0 < t < r.$$

Note that this quantity is well-defined when v has an almost Signorini property at x_0 and $x_0 \in \Gamma(v)$, since vanishing of $\int_{\partial B_t(x_0)} v^2$ for any $t > 0$, would imply vanishing of v in $B_t(x_0)$ by taking 0 as a competitor and consequently that $x_0 \notin \Gamma(v)$.

Next consider a modification of N, which we call the *truncated frequency*:

$$\widehat{N}_{\kappa_0}(t, v, x_0) := \min\left\{\frac{1}{1 - bt^\alpha} N(t, v, x_0), \kappa_0\right\},$$

where b is as in Weiss-type monotonicity formulas for $\kappa < \kappa_0$. We next define the appropriate version of N, $\widehat{N}_{\kappa_0}$ in our setting. For a function V in $E_r(x_0)$, we define

$$\begin{aligned} N^A(t, V, x_0) &:= N(t, v_{x_0}, 0), \\ \widehat{N}^A_{\kappa_0}(t, V, x_0) &:= \widehat{N}_{\kappa_0}(t, v_{x_0}, 0), \end{aligned}$$

for $0 < t < r$, where $v_{x_0} = V \circ \bar{T}_{x_0}^{-1}$. More explicitly, we have

$$\begin{aligned} N^A(t, V, x_0) &:= \frac{t \int_{E_t(x_0)} \langle A(x_0) \nabla V, \nabla V \rangle}{\int_{\partial E_t(x_0)} V^2 \mu_{x_0}(x - x_0)}, \\ \widehat{N}^A_{\kappa_0}(t, V, x_0) &:= \min\left\{\frac{1}{1 - bt^\alpha} N^A(t, V, x_0), \kappa_0\right\}. \end{aligned}$$

As observed in Theorem 1.5.4, the Weiss-type monotonicity formula implies the following monotonicity of $\widehat{N}^A_{\kappa_0}$.

Theorem 2.7.2 (Almgren-type monotonicity formula)**.** *Let U, κ_0, and t_0 be as in Theorem 2.7.1, and $x_0 \in B'_{1/2}$ a free boundary point. Then*

$$t \mapsto \widehat{N}^A_{\kappa_0}(t, U^*_{x_0}, x_0) = \widehat{N}_{\kappa_0}(t, u^*_{x_0}, 0)$$

is nondecreasing for $0 < t < t_0$.

Definition 2.7.1 (Almgren's frequency at free boundary point)**.** *For an A-quasisymmetric almost minimizer U of the A-Signorini problem in B_1 and $x_0 \in \Gamma(U)$ let*

$$\kappa(x_0) := \widehat{N}^A_{\kappa_0}(0+, U^*_{x_0}, x_0) = \widehat{N}_{\kappa_0}(0+, u^*_{x_0}, 0).$$

We call $\kappa(x_0)$ Almgren's frequency *at* x_0.

Remark 2.7.3. Note that even though the monotonicity of the truncated frequency is stated in Theorem 2.7.2 only for $x_0 \in B'_{1/2} \cap \Gamma(U)$, by a simple recentering and a scaling argument, it will be monotone also at all $x_0 \in \Gamma(U)$, but for a possibly shorter interval of values $0 < t < t_0(x_0)$ depending on x_0. Thus, $\kappa(x_0)$ exists at all $x_0 \in \Gamma(U)$.

Further note that when $\kappa(x_0) < \kappa_0$, then $\widehat{N}^A_{\kappa_0}(t, U^*_{x_0}, x_0) = \frac{1}{1-bt^\alpha} N^A(t, U^*_{x_0}, x_0)$ for small t and therefore

$$\kappa(x_0) = N^A(0+, U^*_{x_0}, x_0),$$

which means that it will not change if we replace κ_0 with a larger value.

2.8 Almgren rescalings and blowups

Our analysis of the free boundary is based on the analysis of blowups, which are the limits of rescalings of the solutions at free boundary points. In Signorini problem, there are a few types of rescalings that use different normalizations. In this section, we look at so-called Almgren rescalings and blowups that play well with the Almgren frequency formula.

Let $V \in W^{1,2}(B_1)$ and $x_0 \in B'_{1/2}$ be a free boundary point. For small $r > 0$ define the *Almgren rescaling* of V at x_0 by

$$V^A_{x_0,r}(x) := \frac{V(rx + x_0)}{\left(\frac{1}{r^{n-1}} \int_{\partial E_r(x_0)} V^2 \mu_{x_0}(x - x_0)\right)^{1/2}}.$$

The Almgren rescalings have the following normalization and scaling properties

$$\|V^A_{x_0,r}\|_{L^2(\mathfrak{a}_{x_0}\partial B_1)} = 1$$
$$N^{A(x_0)}(\rho, V^A_{x_0,r}, 0) = N^A(\rho r, V, x_0).$$

Here $N^{A(x_0)}$ denotes Almgren's frequency for a constant matrix $A(x_0)$. Thus, we also have $N^A(r,V,x_0) = N^{A(x_0)}(r,V,x_0)$. Note that when $A = I$, then

$$V^I_{x_0,r} = \frac{V(rx+x_0)}{\left(\frac{1}{r^{n-1}}\int_{\partial B_r(x_0)} V^2\right)^{1/2}}$$

is same as the Almgren rescaling in Chapter 1, and satisfies

$$\|V^I_{x_0,r}\|_{L^2(\partial B_1)} = 1$$
$$N(\rho, V^I_{x_0,r}, 0) = N(\rho r, V, x_0).$$

We will call the limits of $V^A_{x_0,r}$ over any subsequence $r = r_{\mathrm{j}} \to 0+$ *Almgren blowups* of V at x_0 and denote them by $V^A_{x_0,0}$.

By using a change of variables, we can express Almgren rescalings of V in terms of those of $v_{x_0} = V \circ \bar{T}^{-1}_{x_0}$ and vice versa. Namely, we have

$$(v_{x_0})^I_r(y) = (\det \mathfrak{a}_{x_0})^{1/2} V^A_{x_0,r}(\bar{\mathfrak{a}}_{x_0} y),$$

wherever they are defined. Applied to the particular case $V = U^*_{x_0}$, we have

$$(u^*_{x_0})^I_r(y) = (\det \mathfrak{a}_{x_0})^{1/2} (U^*_{x_0})^A_{x_0,r}(\bar{\mathfrak{a}}_{x_0} y).$$

Proposition 2.8.1 (Existence of Almgren blowups)**.** *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1, and $x_0 \in B'_{1/2} \cap \Gamma(U)$ be such that $\kappa(x_0) < \kappa_0$. Then, every sequence of Almgren rescalings $(U^*_{x_0})^A_{x_0,t_{\mathrm{j}}}$, with $t_{\mathrm{j}} \to 0+$, contains a subsequence, sill denoted t_{j} such that for a function $(U^*_{x_0})^A_{x_0,0} \in C^1_{\mathrm{loc}}(\mathfrak{a}_{x_0}(B^{\pm}_1 \cup B'_1))$*

$$(U^*_{x_0})^A_{x_0,t_{\mathrm{j}}} \to (U^*_{x_0})^A_{x_0,0} \quad \text{in } C^1_{\mathrm{loc}}(\mathfrak{a}_{x_0}(B^{\pm}_1 \cup B'_1)).$$

Moreover, $(U^*_{x_0})^A_{x_0,0}$ extends to a nonzero solution of the $A(x_0)$-Signorini problem in $\mathbb{R}^n$, $(U^*_{x_0})^A_{x_0,0}(x) = (U^*_{x_0})^A_{x_0,0}(P_{x_0}x)$, and it is homogeneous of degree $\kappa(x_0)$ in $\mathbb{R}^n$.

Similarly, every sequence of Almgren rescalings $(u^*_{x_0})^I_{t_{\rm j}}$, with $t_{\rm j} \to 0+$ contains a subsequence, sill denoted $t_{\rm j}$ such that for a function $(u^*_{x_0})^I_0 \in C^1_{\rm loc}(B_1^\pm \cup B_1')$

$$(u^*_{x_0})^I_{t_{\rm j}} \to (u^*_{x_0})^I_0 \quad \text{in } C^1_{\rm loc}(B_1^\pm \cup B_1').$$

Moreover, $(u^*_{x_0})^I_0$ extends to a nonzero solution of the Signorini problem in $\mathbb{R}^n$, even in y_n, and it is homogeneous of degree $\kappa(x_0)$ in $\mathbb{R}^n$.

Proof. Step 1. Since $\kappa(x_0) < \kappa_0$, we must have $N(t, u^*_{x_0}, 0) < \kappa_0$ for small $t > 0$. Then, for such t

$$\int_{B_1} |\nabla (u^*_{x_0})^I_t|^2 = N(1, (u^*_{x_0})^I_t, 0) = N(t, u^*_{x_0}, 0) \le \kappa_0,$$

and combined with the normalization $\fint_{\partial B_1} \left((u^*_{x_0})^I_t\right)^2 = 1$, we see that the family $(u^*_{x_0})^I_t$ is bounded in $W^{1,2}(B_1)$, for small $t > 0$. Hence, for any sequence $t_{\rm j} \to 0+$, there is a function $(u^*_{x_0})^I_0 \in W^{1,2}(B_1)$ such that, over a subsequence,

$$\begin{aligned} (u^*_{x_0})^I_{t_{\rm j}} &\to (u^*_{x_0})^I_0 && \text{weakly in } W^{1,2}(B_1),\\ (u^*_{x_0})^I_{t_{\rm j}} &\to (u^*_{x_0})^I_0 && \text{strongly in } L^2(\partial B_1). \end{aligned}$$

In particular, $\fint_{\partial B_1} \left((u^*_{x_0})^I_0\right)^2 = 1$, implying that $(u^*_{x_0})^I_0 \not\equiv 0$ in B_1.

Step 2. For $0 < t < 1$ and $x \in B_{1/(2t)}(x_0)$, let

$$U_{x_0,t}(x) = U(x_0 + t(x - x_0)), \quad A_{x_0,t}(x) = A(x_0 + t(x - x_0)).$$

Then by a simple scaling argument, we have that $U_{x_0,t}$ is an almost minimizer of the $A_{x_0,t}$-Signorini problem in $B_{1/(2t)}(x_0)$ with a gauge function $\mu_t(r) = (tr)^\alpha \le r^\alpha$. In particular, for any $R > 0$, we will have that $U_{x_0,t} \in C^{1,\beta}(E_R^\pm(x_0) \cup E_R'(x_0))$ for $0 < t < t(R, M)$ with

$$\|U_{x_0,t}\|_{C^{1,\beta}(K)} \le C \|U_{x_0,t}\|_{W^{1,2}(E_R(x_0))},$$

with $C = C(n, \alpha, M, R, K)$, for any $K \Subset E_R^{\pm}(x_0) \cup E_R'(x_0)$. Then, arguing as in the proof of Theorem 2.6.5, by using the quasisymmetry of U, we obtain that

$$\|(U_{x_0,t})^*_{x_0}\|_{C^{1,\beta}(K)} \leq C\|(U_{x_0,t})^*_{x_0}\|_{W^{1,2}(E_R(x_0))},$$

where

$$(U_{x_0,t})^*_{x_0}(x) = \frac{U_{x_0,t}(x) + U_{x_0,t}(P_{x_0}x)}{2}.$$

Next, observing that $(u^*_{x_0})^I_t$ is a positive constant multiple of $(U_{x_0,t})^*_{x_0} \circ \bar{T}^{-1}_{x_0}$, we obtain that

$$\|(u^*_{x_0})^I_t\|_{C^{1,\beta}(K)} \leq C\|(u^*_{x_0})^I_t\|_{W^{1,2}(B_R)},$$

for any $K \Subset B_R^{\pm} \cup B_R'$. Taking $R = 1$, combined with the boundedness of $(u^*_{x_0})^I_t$ in $W^{1,2}(B_1)$ for small $t > 0$, it follows that up to a subsequence,

$$(u^*_{x_0})^I_{t_j} \to (u^*_{x_0})^I_0 \quad \text{in } C^1_{\mathrm{loc}}(B_1^{\pm} \cup B_1').$$

Step 3. Next, we claim that the blowup $(u^*_{x_0})^I_0$ is a solution of the Signorini problem in B_1. Indeed, fix $0 < R < 1$, and for each t_j let h_{t_j} be the Signorini replacement of $(u^*_{x_0})^I_{t_j}$ in B_R. Then a first variation argument gives (see (1.3.2))

$$\int_{B_R} \langle \nabla h_{t_j}, \nabla((u^*_{x_0})^I_{t_j} - h_{t_j})\rangle \geq 0.$$

Since $(u^*_{x_0})^I_{t_j}$ has an almost Signorini property at 0 with a gauge function $r \mapsto C(t_j r)^\alpha$, it follows that

$$\int_{B_R} |\nabla((u^*_{x_0})^I_{t_j} - h_{t_j})|^2 \leq C(Rt_j)^\alpha \int_{B_R} |\nabla(u^*_{x_0})^I_{t_j}|^2.$$

This implies that $h_{t_j} \to (u^*_{x_0})^I_0$ weakly in $W^{1,2}(B_R)$. On the other hand, by the boundedness of the sequence h_{t_j} in $W^{1,2}(B_R)$, we have also boundedness in $C^{1,1/2}$ norm locally in $(B_R^{\pm} \cup B_R')$

and hence, over a subsequence, $h_{t_j} \to (u^*_{x_0})^I_0$ in $C^1_{\mathrm{loc}}(B^\pm_R \cup B'_R)$. By this convergence, we then conclude that $(u^*_{x_0})^I_0$ satisfies

$$\begin{aligned} \Delta(u^*_{x_0})^I_0 = 0 \quad &\text{in } B_R \setminus B'_R \\ (u^*_{x_0})^I_0 \geq 0, \quad -\partial^+_{y_n}(u^*_{x_0})^I_0 \geq 0, \quad (u^*_{x_0})^I_0 \partial^+_{y_n}(u^*_{x_0})^I_0 = 0 \quad &\text{on } B'_R, \end{aligned}$$

and hence, by letting $R \to 1$, $(u^*_{x_0})^I_0$ itself solves the Signorini problem in B_1.

Step 4. Recall now that the blowup $(u^*_{x_0})^I_0$ is nonzero in B_1. In particular, $\int_{\partial B_r}((u^*_{x_0})^I_0)^2 > 0$ for any $0 < r < 1$, otherwise we would have that $(u^*_{x_0})^I_0$ is identically zero on ∂B_r and consequently also on B_r. Using this fact, combined with C^1_{loc} convergence in $B^\pm_1 \cup B'_1$, we have that for any $0 < r < 1$

$$\begin{aligned} N(r, (u^*_{x_0})^I_0, 0) &= \lim_{t_j \to 0} N(r, (u^*_{x_0})^I_{t_j}, 0) = \lim_{t_j \to 0} N(rt_j, u^*_{x_0}, 0) \\ &= N(0+, u^*_{x_0}, 0) = \kappa(x_0). \end{aligned}$$

Thus, Almgren's frequency of $(u^*_{x_0})^I_0$ is constant $\kappa(x_0)$ on $0 < r < 1$ which is possible only if $(u^*_{x_0})^I_0$ is a $\kappa(x_0)$-homogeneous solution of the Signorini problem in B_1, see Theorem 9.4 in [48]. Finally, by using the homogeneity, we readily extend $(u^*_{x_0})^I_0$ to a solution of the Signorini problem in all of $\mathbb{R}^n$. This completes the proof for $(u^*_{x_0})^I_0$.

The corresponding result for $(U^*_{x_0})^A_{x_0,t_j}$ follows now by a change of variables. □

With Proposition 2.8.1 at hand, we can repeat the argument in the proof of Lemma 1.6.1 with $u^*_{x_0}$ to obtain the following, which is possible since $u^*_{x_0}$ satisfies the complementarity condition and an Almgren-type monotonicity formula with a blowup as a nonzero solution of the Signorini problem.

Lemma 2.8.1 (Minimal frequency)**.** *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. If $x_0 \in B'_{1/2} \cap \Gamma(U)$, then*

$$\kappa(x_0) \geq \frac{3}{2}.$$

Consequently, we also have

$$\widehat{N}^A_{\kappa_0}(t, U^*_{x_0}, x_0) = \widehat{N}_{\kappa_0}(t, u^*_{x_0}, 0) \geq 3/2 \quad \textit{for } 0 < t < t_0.$$

Lemma 2.8.1 readily gives the following. (see Corollary 2.8.2)

Corollary 2.8.2. *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1 and x_0 a free boundary point. Then*

$$W^A_{3/2}(t, U^*_{x_0}, x_0) = \det \mathfrak{a}_{x_0} W_{3/2}(t, u^*_{x_0}, 0) \geq 0, \quad \textit{for } 0 < t < t_0.$$

2.9 Growth estimates

The first result in this section (Lemma 2.9.1) provides growth estimates for the quasisymmetric almost minimizers near free boundary points x_0 with $\kappa(x_0) \geq \kappa$. Such estimates were obtained in Lemma 1.7.1 in the case $A \equiv I$ as a consequence of Weiss-type monotonicity formulas. However, they contain an unwanted logarithmic term that creates difficulties in the blowup analysis of the problem.

The next two results (Lemmas 2.9.2 and 2.9.3) remove the logarithmic term from these estimates for $\kappa = 3/2$, by establishing first a growth rate for $W_{3/2}$. (Recall that $\kappa(x_0) \geq 3/2$ at every free boundary point x_0, by Lemma 2.8.1.) These are analogous to Lemmas 1.7.3, 1.7.4 in the case $A \equiv I$ and follow from the so-called epiperimetric inequality for $\kappa = 3/2$ (see e.g. Themrem 1.7.2). Later, in Section 2.12, we remove the logarithmic term also in the case $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, see Lemma 2.12.1.

The results in this section are stated in terms of both $u^*_{x_0}$ and $U^*_{x_0}$, as we need both forms in the subsequent arguments. We note that the estimates for $u^*_{x_0}$ follow directly from Lemmas 1.7.1, 1.7.3, 1.7.4 and the ones for $U^*_{x_0}$ are obtained by using the deskewing procedure and therefore we skip all proofs in this section.

In the estimates below, as well in the rest of the chapter, we use the notation

$$R_0 := (1/2)\Lambda^{-1/2},$$

which is the radius of the largest ball B_{R_0}, where $u^*_{x_0}$ is guaranteed to exists for any $x_0 \in B_{1/2}$ for an almost minimizer U in B_1.

Lemma 2.9.1 (Weak growth estimate)**.** *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1 and $x_0 \in B'_{1/2} \cap \Gamma(U)$. If*

$$\kappa(x_0) \geq \kappa$$

for some $\kappa \leq \kappa_0$, then

$$\begin{aligned}
\int_{\partial B_t} (u^*_{x_0})^2 &\leq C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} \left(\log \frac{1}{t}\right) t^{n+2\kappa-1}, \\
\int_{B_t} |\nabla u^*_{x_0}|^2 &\leq C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} \left(\log \frac{1}{t}\right) t^{n+2\kappa-2}, \\
\int_{\partial E_t(x_0)} (U^*_{x_0})^2 &\leq C \|U\|^2_{W^{1,2}(B_1)} \left(\log \frac{1}{t}\right) t^{n+2\kappa-1}, \\
\int_{E_t(x_0)} |\nabla U^*_{x_0}|^2 &\leq C \|U\|^2_{W^{1,2}(B_1)} \left(\log \frac{1}{t}\right) t^{n+2\kappa-2},
\end{aligned}$$

for $0 < t < t_0 = t_0(n, \alpha, M, \kappa_0)$ and $C = C(n, \alpha, M, \kappa_0)$.

Lemma 2.9.2. *Let U and x_0 be as above. Then, there exists $\delta = \delta(n, \alpha) > 0$ such that*

$$\begin{aligned}
0 \leq W_{3/2}(t, u^*_{x_0}, 0) &\leq C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} t^\delta, \\
0 \leq W^A_{3/2}(t, U^*_{x_0}, x_0) &\leq C \|U\|^2_{W^{1,2}(B_1)} t^\delta,
\end{aligned}$$

for $0 < t < t_0 = t_0(n, \alpha, M)$ and $C = C(n, \alpha, M)$.

Lemma 2.9.3 (Optimal growth estimate)**.** *Let U and x_0 be as above. Then,*

$$\begin{aligned}
\int_{\partial B_t} (u^*_{x_0})^2 &\leq C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} t^{n+2}, \\
\int_{B_t} |\nabla u^*_{x_0}|^2 &\leq C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} t^{n+1}, \\
\int_{\partial E_t(x_0)} (U^*_{x_0})^2 &\leq C \|U\|^2_{W^{1,2}(B_1)} t^{n+2}, \\
\int_{E_t(x_0)} |\nabla U^*_{x_0}|^2 &\leq C \|U\|^2_{W^{1,2}(B_1)} t^{n+1},
\end{aligned}$$

for $0 < t < t_0 = t_0(n, \alpha, M)$ *and* $C = C(n, \alpha, M)$.

2.10 3/2-almost homogeneous rescalings and blowups

In this section we study another kind of rescalings and blowups that will play a fundamental role in the analysis of regular free boundary points where $\kappa(x_0) = 3/2$ (see the next section), namely 3/2-almost homogeneous blowups. The main result that we prove in this section is the uniqueness and Hölder continuous dependence of such blowups at a free boundary point x_0 (Lemma 2.10.3).

For a function v in B_1 and $x_0 \in B'_{1/2}$, we define the 3/2-*almost homogeneous rescalings* of v at x_0 by

$$v^{\phi}_{x_0,t}(x) = \frac{v(tx + x_0)}{\phi(t)}, \quad \phi(t) = \mathrm{e}^{-\left(\frac{3b}{2\alpha}\right)t^{\alpha}} t^{3/2},$$

with b as in the Weiss-type monotonicity formulas $W^A_{3/2}$ and $W_{3/2}$. When $x_0 = 0$, we simply write $v^{\phi}_{0,t} = v^{\phi}_t$.

The name is explained by the fact that

$$\lim_{t\to 0} \frac{\phi(t)}{t^{3/2}} = 1,$$

and the reason to look at such rescalings instead of 3/2-homogeneous rescalings (that would correspond to $\phi(t) = t^{3/2}$) is how they play well with the Weiss-type monotonicity formulas $W^A_{3/2}$ and $W_{3/2}$.

Now, if U is an A-quasisymmetric almost minimizer and $x_0 \in B'_{1/2} \cap \Gamma(U)$, then for any fixed $R > 1$, if $t = t_{\mathrm{j}} > 0$ is small, then by Lemma 2.9.3,

$$\int_{B_R} |\nabla (u^*_{x_0})^{\phi}_t|^2 = \frac{\mathrm{e}^{\frac{3b}{\alpha}t^{\alpha}}}{t^{n+1}} \int_{B_{Rt}} |\nabla u^*_{x_0}|^2 \le C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} R^{n+1}, \tag{2.10.1}$$

$$\int_{\partial B_R} ((u^*_{x_0})^{\phi}_t)^2 = \frac{\mathrm{e}^{\frac{3b}{\alpha}t^{\alpha}}}{t^{n+2}} \int_{\partial B_{Rt}} (u^*_{x_0})^2 \le C \|u^*_{x_0}\|^2_{W^{1,2}(B_{R_0})} R^{n+2}, \tag{2.10.2}$$

with $C = C(n, \alpha, M)$, $R_0 = (1/2)\Lambda^{-1/2}$. Hence, $(u^*_{x_0})^\phi_{t_\mathrm{j}}$ is a bounded sequence in $W^{1,2}(B_R)$. Next, arguing as in the proof of Proposition 2.8.1, we will have that

$$\|\widehat{\nabla (u^*_{x_0})}^\phi_t\|_{C^{0,\beta}(K)} \le C\|(u^*_{x_0})^\phi_t\|_{W^{1,2}(B_R)}, \tag{2.10.3}$$

with $C = C(n, \alpha, M, R, K)$ for $K \Subset B_R$. Thus, by letting $R \to \infty$ and using Cantor's diagonal argument, we can conclude that over a subsequence $t = t_\mathrm{j} \to 0+$,

$$(u^*_{x_0})^\phi_{t_\mathrm{j}} \to (u^*_{x_0})^\phi_0 \quad \text{in} \quad C^1_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1}).$$

We call such $(u^*_{x_0})^\phi_0$ a 3/2-*homogeneous blowup* of $u^*_{x_0}$ at 0. (We may skip the "almost" modifier here as the limit is the same as for 3/2-homogeneous rescalings.) Furthermore, from the relation

$$(u^*_{x_0})^\phi_t(y) = (U^*_{x_0})^\phi_{x_0,t}(\bar{\mathfrak{a}}_{x_0} y),$$

we also conclude that for any sequence $t_\mathrm{j} \to 0+$, there is a subsequence, still denoted by t_j, such that

$$(U^*_{x_0})^\phi_{x_0,t_\mathrm{j}} \to (U^*_{x_0})^\phi_{x_0,0} \quad \text{in} \quad C^1_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1}).$$

Apriori, the blowups $(u^*_{x_0})^\phi_0$ and $(U^*_{x_0})^\phi_{x_0,0}$ may depend on the sequence $t_\mathrm{j} \to 0+$. However, this does not happen in the case of 3/2-homogeneous blowups. We start with what we call a rotation estimate for rescalings.

Lemma 2.10.1 (Rotation estimate)**.** *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1, $x_0 \in B'_{1/2}$ a free boundary point, and δ as in Lemma 2.9.2. Then,*

$$\int_{\partial B_1} |(u^*_{x_0})^\phi_t - (u^*_{x_0})^\phi_s| \le C\|u^*_{x_0}\|_{W^{1,2}(B_{R_0})} t^{\delta/2},$$
$$\int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,t} - (U^*_{x_0})^\phi_{x_0,s}| \le C\|U\|_{W^{1,2}(B_1)} t^{\delta/2},$$

for $s < t < t_0 = t_0(n, \alpha, M)$ and $C = C(n, \alpha, M)$.

Proof. This is an analogue of Lemma 1.8.2, which follows from the computation done in the proof of Lemma 1.7.1, the growth estimate for $W_{3/2}$ in Lemma 1.7.3 and a dyadic argument. The analogues of those results in our case are stated in Lemma 2.9.1 and 2.9.2. This proves the lemma for $u^*_{x_0}$. The estimate for $(U^*_{x_0})^\phi_{x_0,t}$ then follows from the equality

$$(u^*_{x_0})^\phi_t(y) = (U^*_{x_0})^\phi_{x_0,t}(\bar{\mathfrak{a}}_{x_0} y), \quad y \in B_{R_0/t}. \qquad \square$$

The uniqueness of 3/2-homogeneous blowup now follows.

Lemma 2.10.2. *Let $(U^*_{x_0})^\phi_{x_0,0}$ and $(u^*_{x_0})^\phi_0$ be blowups of $(U^*_{x_0})^\phi_{x_0,t}$ and $(u^*_{x_0})^\phi_t$, respectively, at a free boundary point $x_0 \in B'_{1/2}$. Then,*

$$\int_{\partial B_1} |(u^*_{x_0})^\phi_t - (u^*_{x_0})^\phi_0| \le C\|u^*_{x_0}\|_{W^{1,2}(B_{R_0})} t^{\delta/2},$$
$$\int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,t} - (U^*_{x_0})^\phi_{x_0,0}| \le C\|U\|_{W^{1,2}(B_1)} t^{\delta/2},$$

for $0 < t < t_0(n, \alpha, M)$ and $C = C(n, \alpha, M)$, where $\delta = \delta(n, \alpha) > 0$ is as in Lemma 2.10.1. In particular, the blowups $(u^*_{x_0})^\phi_0$ and $(U^*_{x_0})^\phi_{x_0,0}$ are unique.

Proof. If $(u^*_{x_0})^\phi_0$ is the limit of $(u^*_{x_0})^\phi_{t_j}$ for $t_j \to 0$, then the first part of the lemma follows immediately from Lemma 2.10.1, by taking $s = t_j \to 0$ and passing to the limit.

To see the uniqueness of blowups, we observe that $(u^*_{x_0})^\phi_0$ is a solution of the Signorini problem in B_1, by arguing as in the proof of Proposition 2.8.1 for Almgren blowups. Now, if v_0 is another blowup, over a possibly different sequence $t_j \to 0$, then passing to the limit in the first part of the lemma we will have

$$\int_{\partial B_1} |v_0 - (u^*_{x_0})^\phi_0|^2 = 0,$$

implying that both v_0 and $(u^*_{x_0})^\phi_0$ are solutions of the Signorini problem in B_1 with the same boundary values on ∂B_1. By the uniqueness of such solutions, we have $v_0 = (u^*_{x_0})^\phi_0$ in B_1.

The equality propagates to all of $\mathbb{R}^n$ by the unique continuation of harmonic functions in $\mathbb{R}^n_\pm$. This completes the proof for $u^*_{x_0}$. An analogous argument holds for $U^*_{x_0}$ using the equalities

$$(u^*_{x_0})^\phi_t(y) = (U^*_{x_0})^\phi_{x_0,t}(\bar{\mathfrak{a}}_{x_0}y), \quad y \in B_{R_0/t},$$
$$(u^*_{x_0})^\phi_0(y) = (U^*_{x_0})^\phi_{x_0,0}(\bar{\mathfrak{a}}_{x_0}y), \quad y \in \mathbb{R}^n.$$

$\square$

The rotation estimate for rescalings implies not only the uniqueness of blowups and the convergence rate to blowups, but also the continuous dependence of blowups on a free boundary point.

Lemma 2.10.3 (Continuous dependence of blowups)**.** *There exists $\rho = \rho(n, \alpha, M) > 0$ such that if $x_0, y_0 \in B_\rho$ are free boundary points of U, then*

$$\int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,0} - (U^*_{y_0})^\phi_{y_0,0}| \le C|x_0 - y_0|^\gamma, \tag{2.10.4}$$
$$\int_{\partial B_1} |(u^*_{x_0})^\phi_0 - (u^*_{y_0})^\phi_0| \le C|x_0 - y_0|^\gamma, \tag{2.10.5}$$
$$\int_{\partial B_1} |(u^*_{x_0})^\phi_0 - (u^*_{y_0})^\phi_0| \le C|x_0 - y_0|^\gamma, \tag{2.10.6}$$

with $C = C(n, \alpha, M, \|U\|_{W^{1,2}(B_1)})$, $\gamma = \gamma(n, \alpha, M) > 0$.

Proof. Step 1. Let $d = |x_0 - y_0|$ and $d^\tau \le r \le 2d^\tau$ with $\tau = \tau(\alpha) \in (0, 1)$ to be determined later.

Next note that we can incorporate the weight $\mu_{x_0}/\det \mathfrak{a}_{x_0}$ with μ_{x_0} as in (2.2.4) in the integral on the left hand side of (2.10.4) because of the bounds

$$\left(\frac{\lambda}{\Lambda}\right)^{1/2} \le \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}} \le \left(\frac{\Lambda}{\lambda}\right)^{1/2}.$$

Then, by using Lemma 2.10.2, we have

$$
\begin{aligned}
\int_{\mathfrak{a}_{x_0}\partial B_1} &|(U^*_{x_0})^\phi_{x_0,0} - (U^*_{y_0})^\phi_{y_0,0}| \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}} \\
&\le \int_{\mathfrak{a}_{x_0}\partial B_1} \Big(|(U^*_{x_0})^\phi_{x_0,0} - (U^*_{x_0})^\phi_{x_0,r}| + |(U^*_{x_0})^\phi_{x_0,r} - (U^*_{x_0})^\phi_{y_0,r}| \\
&\quad + |(U^*_{x_0})^\phi_{y_0,r} - (U^*_{y_0})^\phi_{y_0,r}| + |(U^*_{y_0})^\phi_{y_0,r} - (U^*_{y_0})^\phi_{y_0,0}| \Big) \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}} \\
&\quad + \int_{\mathfrak{a}_{y_0}\partial B_1} |(U^*_{y_0})^\phi_{y_0,r} - (U^*_{y_0})^\phi_{y_0,0}| \frac{\mu_{y_0}}{\det \mathfrak{a}_{y_0}} \\
&\quad - \int_{\mathfrak{a}_{y_0}\partial B_1} |(U^*_{y_0})^\phi_{y_0,r} - (U^*_{y_0})^\phi_{y_0,0}| \frac{\mu_{y_0}}{\det \mathfrak{a}_{y_0}} \\
&\le 2Cr^{\delta/2} + I_r + II_r + III_r \\
&\le Cd^{\tau\delta/2} + I_r + II_r + III_r,
\end{aligned} \tag{2.10.7}
$$

where

$$
\begin{aligned}
I_r &= \int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,r} - (U^*_{x_0})^\phi_{y_0,r}| \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}}, \\
II_r &= \int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{y_0,r} - (U^*_{y_0})^\phi_{y_0,r}| \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}}, \\
III_r &= \int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{y_0})^\phi_{y_0,r} - (U^*_{y_0})^\phi_{y_0,0}| \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}} - \int_{\mathfrak{a}_{y_0}\partial B_1} |(U^*_{y_0})^\phi_{y_0,r} - (U^*_{y_0})^\phi_{y_0,0}| \frac{\mu_{y_0}}{\det \mathfrak{a}_{y_0}}.
\end{aligned}
$$

Step 2. By the definition of the almost homogeneous rescalings, we have

$$
I_r \le \frac{C}{d^{\tau(n+1/2)}} \int_{\mathfrak{a}_{x_0}\partial B_r} |U^*_{x_0}(z+x_0) - U^*_{x_0}(z+y_0)| dS_z.
$$

This gives

$$
\begin{aligned}
\frac{1}{d^\tau} \int_{d^\tau}^{2d^\tau} I_r \, dr &\le \frac{C}{d^{\tau(n+3/2)}} \int_{d^\tau}^{2d^\tau} \int_{\mathfrak{a}_{x_0}\partial B_r} |U^*_{x_0}(z+x_0) - U^*_{x_0}(z+y_0)| dS_z dr \\
&\le \frac{C}{d^{\tau(n+3/2)}} \int_{\mathfrak{a}_{x_0}(B_{2d^\tau}\setminus B_{d^\tau})} |U^*_{x_0}(z+x_0) - U^*_{x_0}(z+y_0)| dz \\
&= \frac{C}{d^{\tau(n+3/2)}} \int_{\mathfrak{a}_{x_0}(B_{2d^\tau}\setminus B_{d^\tau})} \left| \int_0^1 \frac{d}{ds} \left[U^*_{x_0}(z + x_0(1-s) + y_0 s) \right] ds \right| dz \\
&\le \frac{C}{d^{\tau(n+3/2)}} |x_0 - y_0| \int_0^1 \int_{\mathfrak{a}_{x_0}(B_{2d^\tau}\setminus B_{d^\tau})} |\nabla U^*_{x_0}(z + x_0(1-s) + y_0 s)| dz ds \\
&\le \frac{C}{d^{\tau(n+3/2)-1}} \int_0^1 \int_{\mathfrak{a}_{x_0} B_{2d^\tau} + [x_0(1-s)+y_0 s]} |\nabla U^*_{x_0}| dz ds.
\end{aligned}
$$

Notice that the last integral is taken over

$$\begin{aligned}\mathfrak{a}_{x_0}B_{2d^\tau}+[x_0(1-s)+y_0s]&=\mathfrak{a}_{x_0}[B_{2d^\tau}+s\mathfrak{a}_{x_0}^{-1}(y_0-x_0)]+x_0\\&\subset\mathfrak{a}_{x_0}B_{2d^\tau+\lambda^{-1/2}d}+x_0\subset E_{3d^\tau}(x_0),\end{aligned}$$

if $\rho=\rho(n,\alpha,M)$ is small so that $(2\rho)^{1-\tau}\le\lambda^{1/2}$ which readily implies $d^{1-\tau}\le\lambda^{1/2}$. Thus,

$$\begin{aligned}\frac{1}{d^\tau}\int_{d^\tau}^{2d^\tau}I_r\,dr&\le\frac{C}{d^{\tau(n+3/2)-1}}\int_0^1\int_{E_{3d^\tau}(x_0)}|\nabla U_{x_0}^*|dzds\\&\le\frac{C}{d^{\tau(n/2+3/2)-1}}\left(\int_{E_{3d^\tau}(x_0)}|\nabla U_{x_0}^*|^2\right)^{1/2}\\&\le C\|U\|_{W^{1,2}(B_1)}d^{1-\tau},\end{aligned}$$

where the third inequality follows from Lemma 2.9.3.

Step 3. By the definition of rescalings and symmetrizations, we have

$$\begin{aligned}II_r&\le\frac{C}{d^{\tau(n+1/2)}}\int_{\mathfrak{a}_{x_0}\partial B_r+y_0}|U_{x_0}^*(z)-U_{y_0}^*(z)|dS_z\\&\le\frac{C}{d^{\tau(n+1/2)}}\int_{\mathfrak{a}_{x_0}\partial B_r+y_0}|U(P_{x_0}z)-U(P_{y_0}z)|dS_z.\end{aligned}$$

This gives

$$\begin{aligned}\frac{1}{d^\tau}\int_{d^\tau}^{2d^\tau}II_r\,dr&\le\frac{C}{d^{\tau(n+3/2)}}\int_{\mathfrak{a}_{x_0}(B_{2d^\tau}\setminus B_{d^\tau})+y_0}|U(P_{x_0}z)-U(P_{y_0}z)|dz\\&\le\frac{C}{d^{\tau(n+3/2)}}\int_{\mathfrak{a}_{x_0}(B_{2d^\tau}\setminus B_{d^\tau})+y_0}\int_0^1\left|\frac{d}{ds}[U([(1-s)P_{x_0}+sP_{y_0}]z)]\right|dsdz\\&\le\frac{C|P_{x_0}-P_{y_0}|}{d^{\tau(n+3/2)}}\int_0^1\int_{\mathfrak{a}_{x_0}(B_{2d^\tau}\setminus B_{d^\tau})+y_0}|\nabla U([(1-s)P_{x_0}+sP_{y_0}]z)|dzds.\end{aligned}$$

Now we do the change of variables

$$y=[(1-s)P_{x_0}+sP_{y_0}]z.$$

Since P_{x_0} and P_{y_0} are upper-triangular matrices with diagonal entries $1, 1, \dots, 1, -1$, so is $(1-s)P_{x_0} + sP_{y_0}$. Thus

$$|\det[(1-s)P_{x_0} + sP_{y_0}]| = 1.$$

Moreover, $y \in [(1-s)P_{x_0} + sP_{y_0}](\mathfrak{a}_{x_0} B_{2d^\tau} + y_0)$. Since

$$\mathfrak{a}_{x_0} B_{2d^\tau} + y_0 \subset \mathfrak{a}_{y_0} B_{2(\Lambda/\lambda)^{1/2} d^\tau} + y_0 = E_{2(\Lambda/\lambda)^{1/2} d^\tau}(y_0),$$

we have

$$P_{y_0}(\mathfrak{a}_{x_0} B_{2d^\tau} + y_0) \subset P_{y_0} E_{2(\Lambda/\lambda)^{1/2} d^\tau}(y_0) = E_{2(\Lambda/\lambda)^{1/2} d^\tau}(y_0).$$

Similarly, since

$$\begin{aligned}
\mathfrak{a}_{x_0} B_{2d^\tau} + y_0 = E_{2d^\tau}(x_0) + (y_0 - x_0) &\subset B_{2\Lambda^{1/2} d^\tau}(x_0) + (y_0 - x_0) \\
&\subset B_{4\Lambda^{1/2} d^\tau}(x_0) \subset E_{4(\Lambda/\lambda)^{1/2} d^\tau}(x_0),
\end{aligned}$$

we have

$$P_{x_0}(\mathfrak{a}_{x_0} B_{2d^\tau} + y_0) \subset E_{4(\Lambda/\lambda)^{1/2} d^\tau}(x_0).$$

Thus

$$\begin{aligned}
y &\in (1-s)P_{x_0}(\mathfrak{a}_{x_0} B_{2d^\tau} + y_0) + sP_{y_0}(\mathfrak{a}_{x_0} B_{2d^\tau} + y_0) \\
&\subset (1-s)E_{4(\Lambda/\lambda)^{1/2} d^\tau}(x_0) + sE_{2(\Lambda/\lambda)^{1/2} d^\tau}(y_0) \\
&\subset B_{6(\Lambda/\lambda^{1/2}) d^\tau} + x_0 + s(y_0 - x_0) \\
&\subset B_{7(\Lambda/\lambda^{1/2}) d^\tau} + x_0 \subset E_{7(\Lambda/\lambda) d^\tau}(x_0).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{d^\tau} \int_{d^\tau}^{2d^\tau} II_r \, dr &\le \frac{C}{d^{\tau(n+3/2)-\alpha}} \int_0^1 \int_{E_{7(\Lambda/\lambda) d^\tau}(x_0)} |\nabla U| dz ds \\
&\le \frac{C}{d^{\tau(n/2+3/2)-\alpha}} \left(\int_{E_{7(\Lambda/\lambda) d^\tau}(x_0)} |\nabla U|^2 \right)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C}{d^{\tau(n/2+3/2)-\alpha}}\left(\int_{E_{7(\Lambda/\lambda)d^\tau}(x_0)}|\nabla U_{x_0}^*|^2\right)^{1/2}\\
&\leq C\|U\|_{W^{1,2}(B_1)}d^{\alpha-\tau},
\end{aligned}
$$

for small ρ, where the third inequality follows from the quasisymmetry property and the last inequality from Lemma 2.9.3.

Step 4. By the change of variables, we have

$$
\begin{aligned}
III_r &= \int_{\partial B_1}|(U_{y_0}^*)^\phi_{y_0,r}(\mathfrak{a}_{x_0}z)-(U_{y_0}^*)^\phi_{y_0,0}(\mathfrak{a}_{x_0}z)| - \int_{\partial B_1}|(U_{y_0}^*)^\phi_{y_0,r}(\mathfrak{a}_{y_0}z)-(U_{y_0}^*)^\phi_{y_0,0}(\mathfrak{a}_{y_0}z)|\\
&\leq \int_{\partial B_1}|(U_{y_0}^*)^\phi_{y_0,r}(\mathfrak{a}_{x_0}z)-(U_{y_0}^*)^\phi_{y_0,r}(\mathfrak{a}_{y_0}z)| + \int_{\partial B_1}|(U_{y_0}^*)^\phi_{y_0,0}(\mathfrak{a}_{x_0}z)-(U_{y_0}^*)^\phi_{y_0,0}(\mathfrak{a}_{y_0}z)|\\
&\leq C\left(\|\nabla(U_{y_0}^*)^\phi_{y_0,r}\|_{L^\infty(B_{\Lambda^{1/2}})}+\|\nabla(U_{y_0}^*)^\phi_{y_0,0}\|_{L^\infty(B_{\Lambda^{1/2}})}\right)|\mathfrak{a}_{x_0}-\mathfrak{a}_{y_0}|,
\end{aligned}
$$

where we have used the fact that both $\mathfrak{a}_{x_0}z$ and $\mathfrak{a}_{y_0}z$ are contained in $\overline{B_{\Lambda^{1/2}}}$ for $z\in\partial B_1$. To estimate the gradients of rescalings we first observe that by the inclusion $B_{r\Lambda^{1/2}}(y_0)\subset E_{r(\Lambda/\lambda)^{1/2}}(y_0)\subset B_{r\Lambda/\lambda^{1/2}}(y_0)$, we have

$$
\|\nabla(U_{y_0}^*)^\phi_{y_0,r}\|_{L^\infty(B_{\Lambda^{1/2}})}\leq \frac{C}{r^{1/2}}\|\nabla U_{y_0}^*\|_{L^\infty(B_{r\Lambda^{1/2}}(y_0))}\leq \frac{C}{r^{1/2}}\|\nabla U\|_{L^\infty(B_{r\Lambda/\lambda^{1/2}}(y_0))}.
$$

Let $U_{y_0,r}(x):=U(r(x-y_0)+y_0)$. Then, arguing as in the proof of Proposition 2.8.1, we have

$$
\|\nabla U_{y_0,r}\|_{L^\infty(B_{\Lambda/\lambda^{1/2}}(y_0))}\leq C(n,\alpha,M)\|U_{y_0,r}\|_{W^{1,2}(B_{2\Lambda/\lambda^{1/2}}(y_0))}.
$$

Thus

$$
\begin{aligned}
\|\nabla U\|_{L^\infty(B_{r\Lambda/\lambda^{1/2}}(y_0))} &= \frac{1}{r}\|\nabla U_{y_0,r}\|_{L^\infty(B_{\Lambda/\lambda^{1/2}}(y_0))}\\
&\leq \frac{C}{r}\|U_{y_0,r}\|_{W^{1,2}(B_{2\Lambda/\lambda^{1/2}}(y_0))}\\
&\leq \frac{C}{r^{n/2+1}}\|U\|_{L^2(B_{2r\Lambda/\lambda^{1/2}}(y_0))}+\frac{C}{r^{n/2}}\|\nabla U\|_{L^2(B_{2r\Lambda/\lambda^{1/2}}(y_0))}\\
&\leq \frac{C}{r^{n/2+1}}\|U_{y_0}^*\|_{L^2(E_{2r\Lambda/\lambda}(y_0))}+\frac{C}{r^{n/2}}\|\nabla U_{y_0}^*\|_{L^2(E_{2r\Lambda/\lambda}(y_0))}\\
&\leq Cr^{1/2}\|U\|_{W^{1,2}(B_1)},
\end{aligned}
$$

where we have used the inclusion $B_{2r\Lambda/\lambda^{1/2}}(y_0) \subset E_{2r\Lambda/\lambda}(y_0)$ and the quasisymmetry property in the third inequality and Lemma 2.9.3 in the forth. Therefore,

$$\|\nabla (U^*_{y_0})^\phi_{y_0,r}\|_{L^\infty(B_{\Lambda^{1/2}})} \le \frac{C}{r^{1/2}}\|\nabla U\|_{L^\infty(B_{r\Lambda/\lambda^{1/2}}(y_0))} \le C\|U\|_{W^{1,2}(B_1)}.$$

Moreover, by C^1_{loc} convergence of $(U^*_{y_0})^\phi_{y_0,r}$ to $(U^*_{y_0})^\phi_{y_0,0}$, we also have

$$\|\nabla (U^*_{y_0})^\phi_{y_0,0}\|_{L^\infty(B_{\Lambda^{1/2}})} = \lim_{r_j\to 0+}\|\nabla (U^*_{y_0})^\phi_{y_0,r_j}\|_{L^\infty(B_{\Lambda^{1/2}})} \le C\|U\|_{W^{1,2}(B_1)}. \tag{2.10.8}$$

Therefore,

$$III_r \le C|\mathfrak{a}_{x_0} - \mathfrak{a}_{y_0}|\|U\|_{W^{1,2}(B_1)} \le C\|U\|_{W^{1,2}(B_1)} d^\alpha.$$

Step 5. Now we are ready to prove (2.10.4). Using the estimates in Steps 2–4 and taking the average over $d^\tau \le r \le 2d^\tau$, we have

$$\int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,0} - (U^*_{y_0})^\phi_{y_0,0}| \le C\|U\|_{W^{1,2}(B_1)}(d^{\tau\delta/2} + d^{1-\tau} + d^{\alpha-\tau} + d^\alpha).$$

If we simply take $\tau = \alpha/2$, then we conclude

$$\int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,0} - (U^*_{y_0})^\phi_{y_0,0}| \le C|x_0 - y_0|^\gamma,$$

with $\gamma = \alpha\delta/4$ and $C = C(n, \alpha, M, \|U\|_{W^{1,2}(B_1)})$.

Step 6. To prove (2.10.5), we first observe that from (2.10.4),

$$\begin{aligned}
\int_{\partial B_1} |(u^*_{x_0})^\phi_0(z) - (u^*_{y_0})^\phi_0(\bar{\mathfrak{a}}_{y_0}^{-1}\bar{\mathfrak{a}}_{x_0} z)| &= \int_{\partial B_1} |(U^*_{x_0})^\phi_{x_0,0}(\bar{\mathfrak{a}}_{x_0} z) - (U^*_{y_0})^\phi_{y_0,0}(\bar{\mathfrak{a}}_{x_0} z)| \\
&= \int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,0} - (U^*_{y_0})^\phi_{y_0,0}| \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}} \\
&\le C|x_0 - y_0|^\gamma.
\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\partial B_1} |(u^*_{y_0})^\phi_0(z) - (u^*_{y_0})^\phi_0(\bar{\mathfrak{a}}_{y_0}^{-1}\bar{\mathfrak{a}}_{x_0} z)| &= \int_{\mathfrak{a}_{x_0}\partial B_1} |(u^*_{y_0})^\phi_0(\bar{\mathfrak{a}}_{x_0}^{-1} z) - (u^*_{y_0})^\phi_0(\bar{\mathfrak{a}}_{y_0}^{-1} z)| \frac{\mu_{x_0}}{\det \mathfrak{a}_{x_0}}\\ &\leq C\|\nabla (u^*_{y_0})^\phi_0\|_{L^\infty(B_{(\Lambda/\lambda)^{1/2}})} |\bar{\mathfrak{a}}_{x_0}^{-1} - \bar{\mathfrak{a}}_{y_0}^{-1}|\\ &\leq C\|\nabla (U^*_{y_0})^\phi_{y_0,0}\|_{L^\infty(B_{\Lambda/\lambda^{1/2}})} |x_0 - y_0|^\alpha\\ &\leq C\|U\|_{W^{1,2}(B_1)} |x_0 - y_0|^\alpha,\end{aligned}$$

where the last inequality follows from (2.10.8). (It is easy to see that we can enlarge the domain in (2.10.8).) Therefore, combining the preceding two estimates, we conclude that

$$\int_{\partial B_1} |(u^*_{x_0})^\phi_0 - (u^*_{y_0})^\phi_0| \leq C|x_0 - y_0|^\gamma.$$

Step 7. Finally, (2.10.5) implies (2.10.6), by arguing precisely as in Proposition 7.4 in [20]. $\square$

2.11 Regularity of the regular set

In this section we combine the uniqueness and Hölder continuous dependence of 3/2-homogeneous blowups of the symmetrized almost minimizers $(U^*_{x_0})^\phi_{x_0,0}$ (Lemma 2.10.3) with a classification of such blowups at so-called regular points (Proposition 2.11.1) to prove one of the main results of this chapter, the $C^{1,\gamma}$ regularity of the regular set (Theorem 2.11.5). While some arguments follow directly from those in the case $A \equiv I$ by a coordinate transformation $\bar{T}_{x_0}$, the dependence of these transformations on x_0 creates an additional difficulty.

We start by defining the regular set.

Definition 2.11.1 (Regular points)**.** *For an A-quasisymmetric almost minimizer U for the A-Signorini problem in B_1, we say that a free boundary point x_0 of U is* regular *if*

$$\kappa(x_0) = 3/2.$$

We denote the set of all regular points of U by $\mathcal{R}(U)$ and call it the regular set*.*

We explicitly observe here that $3/2 < 2 \leq \kappa_0$, so the fact $x_0 \in \mathcal{R}(U)$ is independent of the choice of $\kappa_0 \geq 2$, see Remark 2.7.3.

The proofs of the following two results (Lemma 2.11.1 and Proposition 2.11.1) are established precisely as in Lemma 1.9.1 and Proposition 1.9.1 for the transformed functions $u^*_{x_0}$. The equivalent statements for $U^*_{x_0}$ are obtained by changing back to the original variables.

Lemma 2.11.1 (Nondegeneracy at regular points). *Let $x_0 \in B'_{1/2} \cap \mathcal{R}(U)$ for an A-quasisymmetric almost minimizer U for the A-Signorini problem in B_1. Then, for $\kappa = 3/2$,*

$$\liminf_{t\to 0} \int_{\mathfrak{a}_{x_0}\partial B_1} ((U^*_{x_0})^{\phi}_{x_0,t})^2 \mu_{x_0} = \det \mathfrak{a}_{x_0} \liminf_{t\to 0} \int_{\partial B_1} ((u^*_{x_0})^{\phi}_t)^2 > 0.$$

Proposition 2.11.1. *If $\kappa(x_0) < 2$, then necessarily $\kappa(x_0) = 3/2$ and*

$$\begin{aligned}
(u^*_{x_0})^{\phi}_0(z) &= a_{x_0} \operatorname{Re}(z' \cdot \nu_{x_0} + i|z_n|)^{3/2}, \\
(U^*_{x_0})^{\phi}_{x_0,0}(x) &= a_{x_0} \operatorname{Re}((\bar{\mathfrak{a}}^{-1}_{x_0} x)' \cdot \nu_{x_0} + i|(\bar{\mathfrak{a}}^{-1}_{x_0} x)_n|)^{3/2},
\end{aligned}$$

for some $a_{x_0} > 0$, $\nu_{x_0} \in \partial B'_1$.

The next two corollaries are obtained by repeating the same arguments as in Corollaries 1.9.2 and 1.9.3.

Corollary 2.11.2 (Almgren's frequency gap). *Let U and x_0 be as in Lemma 2.11.1. Then either*

$$\kappa(x_0) = 3/2 \quad \text{or} \quad \kappa(x_0) \geq 2.$$

Corollary 2.11.3. *The regular set $\mathcal{R}(U)$ is a relatively open subset of the free boundary.*

The combination of Proposition 2.11.1 and Lemma 2.10.3 implies the following lemma.

Lemma 2.11.4. *Let U and x_0 be as in Lemma 2.11.1. Then there exists $\rho > 0$, depending on x_0 such that $B'_\rho(x_0) \cap \Gamma(U) \subset \mathcal{R}(U)$ and if*

$$(u^*_{\bar{x}})^{\phi}_0(z) = a_{\bar{x}} \operatorname{Re}(z' \cdot \nu_{\bar{x}} + i|z_n|)^{3/2}$$

is the unique 3/2-homogeneous blowup of $u^*_{\bar{x}}$ at $\bar{x} \in B'_\rho(x_0) \cap \Gamma(u)$, then

$$|a_{\bar{x}} - a_{\bar{y}}| \le C_0 |\bar{x} - \bar{y}|^\gamma,$$
$$|\nu_{\bar{x}} - \nu_{\bar{y}}| \le C_0 |\bar{x} - \bar{y}|^\gamma,$$

for any $\bar{x}, \bar{y} \in B'_\rho(x_0) \cap \Gamma(u)$ with a constant C_0 depending on x_0.

Proof. The proof follows by repeating the argument in Lemma 7.5 in [20] with $(u^*_{\bar{x}})^\phi_0$, $(u^*_{\bar{y}})^\phi_0$. □

Now we are ready to prove the main result on the regularity of the regular set.

Theorem 2.11.5 ($C^{1,\gamma}$ regularity of the regular set)**.** *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. Then, if $x_0 \in B'_{1/2} \cap \mathcal{R}(U)$, there exists $\rho > 0$, depending on x_0 such that, after a possible rotation of coordinate axes in $\mathbb{R}^{n-1}$, one has $B'_\rho(x_0) \cap \Gamma(U) \subset \mathcal{R}(U)$, and*

$$B'_\rho(x_0) \cap \Gamma(U) = B'_\rho(x_0) \cap \{x_{n-1} = g(x_1, \dots, x_{n-2})\},$$

for $g \in C^{1,\gamma}(\mathbb{R}^{n-2})$ with an exponent $\gamma = \gamma(n, \alpha, M) \in (0,1)$.

Proof. The proof of the theorem is similar to those of in Theorem 1.2 in [20] and Theorem 1.9.5. However, we provide full details since there are technical differences.

Step 1. By relative openness of $\mathcal{R}(U)$ in $\Gamma(U)$, for small $\rho > 0$ we have $B'_{2\rho}(x_0) \cap \Gamma(U) \subset \mathcal{R}(U)$. We then claim that for any $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that for $\bar{x} \in B'_\rho(x_0) \cap \Gamma(U)$, $r < r_\varepsilon$, we have that

$$\|(u^*_{\bar{x}})^\phi_r - (u^*_{\bar{x}})^\phi_0\|_{C^1(\overline{B^\pm_1})} < \varepsilon.$$

Assuming the contrary, there is a sequence of points $\bar{x}_j \in B'_\rho(x_0) \cap \Gamma(U)$ and radii $r_j \to 0$ such that

$$\|(u^*_{\bar{x}_j})^\phi_{r_j} - (u^*_{\bar{x}_j})^\phi_0\|_{C^1(\overline{B^\pm_1})} \ge \varepsilon_0,$$

for some $\varepsilon_0 > 0$. Taking a subsequence if necessary, we may assume $\bar{x}_j \to \bar{x}_0 \in \overline{B'_\rho(x_0)} \cap \Gamma(U)$. Using estimates (2.10.1)–(2.10.3), we can see that $\nabla(u^*_{\bar{x}_j})^\phi_{r_j}$ are uniformly bounded in $C^{0,\beta}(B_2^{\pm} \cup B_2')$. Since $(u_{\bar{x}_j}^*)_{r_j}^{\phi}(0) = 0$, we also have that $(u_{\bar{x}_j}^*)_{r_j}^{\phi}$ is uniformly bounded in $C^{1,\beta}(B_2^{\pm} \cup B_2')$. Thus, we may assume that for some w

$$(u_{\bar{x}_j}^*)_{r_j}^{\phi} \to w \quad \text{in } C^1(\overline{B_1^{\pm}}).$$

By arguing as in the proof of Proposition 2.8.1, we see that the limit w is a solution of the Signorini problem in B_1. Further, by Lemma 2.10.2, we have

$$\|(u_{\bar{x}_j}^*)_{r_j}^{\phi} - (u_{\bar{x}_j}^*)_0^{\phi}\|_{L^1(\partial B_1)} \to 0.$$

On the other hand, by Lemma 2.11.4, we have

$$(u_{\bar{x}_j}^*)_0^{\phi} \to (u_{\bar{x}_0}^*)_0^{\phi} \quad \text{in } C^1(\overline{B_1^{\pm}}),$$

and thus

$$w = (u_{\bar{x}_0}^*)_0^{\phi} \quad \text{on } \partial B_1.$$

Since both w and $(u_{\bar{x}_0}^*)_0^{\phi}$ are solutions of the Signorini problem, they must coincide also in B_1. Therefore

$$(u_{\bar{x}_j}^*)_{r_j}^{\phi} \to (u_{\bar{x}_0}^*)_0^{\phi} \quad \text{in } C^1(\overline{B_1^{\pm}}),$$

implying also that

$$\|(u_{\bar{x}_j}^*)_{r_j}^{\phi} - (u_{\bar{x}_j}^*)_0^{\phi}\|_{C^1(\overline{B_1^{\pm}})} \to 0,$$

which contradicts our assumption.

Step 2. For a given $\varepsilon > 0$ and a unit vector $\nu \in \mathbb{R}^{n-1}$ define the cone

$$\mathcal{C}_\varepsilon(\nu) = \{x' \in \mathbb{R}^{n-1} : x' \cdot \nu > \varepsilon|x'|\}.$$

By Lemma 2.11.4, we may assume $a_{\bar{x}} \geq \frac{a_{x_0}}{2}$ for $\bar{x} \in B'_\rho(x_0) \cap \Gamma(U)$ by taking ρ small. For such ρ, we then claim that for any $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that for any $\bar{x} \in B'_\rho(x_0) \cap \Gamma(U)$, we have

$$\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon} \subset \{u^*_{\bar{x}}(\cdot, 0) > 0\}.$$

Indeed, denoting $\mathcal{K}_\varepsilon(\nu) = \mathcal{C}_\varepsilon \cap \partial B'_{1/2}$, we have for some universal $C_\varepsilon > 0$

$$\mathcal{K}_\varepsilon(\nu_{\bar{x}}) \Subset \{(u^*_{\bar{x}})^\phi_0(\cdot, 0) > 0\} \cap B'_1 \quad \text{and} \quad (u^*_{\bar{x}})^\phi_0(\cdot, 0) \geq a_{\bar{x}} C_\varepsilon \geq \frac{a_{x_0}}{2} C_\varepsilon \quad \text{on } \mathcal{K}_\varepsilon(\nu_{\bar{x}}).$$

Since $\frac{a_{x_0}}{2} C_\varepsilon$ is independent of $\bar{x}$, by Step 1 we can find $r_\varepsilon > 0$ such that for $r < 2r_\varepsilon$,

$$(u^*_{\bar{x}})^\phi_r(\cdot, 0) > 0 \quad \text{on } \mathcal{K}_\varepsilon(\nu_{\bar{x}}).$$

This implies that for $r < 2r_\varepsilon$,

$$u^*_{\bar{x}}(\cdot, 0) > 0 \quad \text{on} \quad r\mathcal{K}_\varepsilon(\nu_{\bar{x}}) = \mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap \partial B'_{r/2}.$$

Taking the union over all $r < 2r_\varepsilon$, we obtain

$$u^*_{\bar{x}}(\cdot, 0) > 0 \quad \text{on } \mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}.$$

Step 3. We claim that for given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for any $\bar{x} \in B'_\rho(x_0) \cap \Gamma(U)$, we have $-\left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right) \subset \{u^*_{\bar{x}}(\cdot, 0) = 0\}$.

Indeed, we first note that

$$-\partial^+_{x_n}(u^*_{\bar{x}})^\phi_0 \geq a_{\bar{x}} C_\varepsilon > \left(\frac{a_{x_0}}{2}\right) C_\varepsilon \quad \text{on } -\mathcal{K}_\varepsilon(\nu_{\bar{x}}),$$

for a universal constant $C_\varepsilon > 0$. From Step 1, there exists $r_\varepsilon > 0$ such that for $r < 2r_\varepsilon$,

$$-\partial^+_{x_n}(u^*_{\bar{x}})^\phi_r(\cdot, 0) > 0 \quad \text{on } -\mathcal{K}_\varepsilon(\nu_{\bar{x}}).$$

By arguing as in Step 2, we obtain

$$-\partial^+_{x_n} u^*_{\bar{x}}(\cdot, 0) > 0 \quad \text{on} \ -\left(\mathcal{C}(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right).$$

By the complementarity condition in Lemma 2.6.6, we therefore conclude that

$$-\left(\mathcal{C}(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right) \subset \{-\partial^+_{x_n} u^*_{\bar{x}}(\cdot, 0) > 0\} \subset \{u^*_{\bar{x}}(\cdot, 0) = 0\}.$$

Step 4. By direct computation, we have

$$\mathcal{C}_{\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\nu^A_{\bar{x}}) \cap B'_{\lambda^{1/2}r_\varepsilon} \subset \bar{\mathfrak{a}}_{\bar{x}} \left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right),$$

where

$$\nu^A_{\bar{x}} := \frac{(\bar{\mathfrak{a}}_{\bar{x}}^{-1})^{\mathrm{tr}} \nu_{\bar{x}}}{|(\bar{\mathfrak{a}}_{\bar{x}}^{-1})^{\mathrm{tr}} \nu_{\bar{x}}|}.$$

(Here $(\cdot)^{\mathrm{tr}}$ stands for the transpose of the matrix.) Indeed, if $y' \in \mathcal{C}_{\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\nu^A_{\bar{x}}) \cap B'_{\lambda^{1/2}r_\varepsilon}$, then

$$y' \in B'_{\lambda^{1/2}r_\varepsilon} = \bar{\mathfrak{a}}_{\bar{x}} \left(\bar{\mathfrak{a}}_{\bar{x}}^{-1} B'_{\lambda^{1/2}r_\varepsilon}\right) \subset \bar{\mathfrak{a}}_{\bar{x}} B'_{r_\varepsilon},$$

and

$$\begin{aligned}
\langle \bar{\mathfrak{a}}_{\bar{x}}^{-1} y', \nu_{\bar{x}} \rangle &= \langle y', (\bar{\mathfrak{a}}_{\bar{x}}^{-1})^{\mathrm{tr}} \nu_{\bar{x}} \rangle = \langle y', \nu^A_{\bar{x}} \rangle |(\bar{\mathfrak{a}}_{\bar{x}}^{-1})^{\mathrm{tr}} \nu_{\bar{x}}| \\
&\geq (\Lambda^{1/2}\lambda^{-1/2}\varepsilon |y'|)(\Lambda^{-1/2}) \\
&= \lambda^{-1/2}\varepsilon |y'| \geq \varepsilon |\bar{\mathfrak{a}}_{\bar{x}}^{-1} y'|.
\end{aligned}$$

Combining this with Step 2 and Step 3, for $\bar{x} \in B'_\rho(x_0) \cap \Gamma(U)$,

$$\begin{aligned}
\bar{x} + \left(\mathcal{C}_{\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\nu^A_{\bar{x}}) \cap B'_{\lambda^{1/2}r_\varepsilon}\right) &\subset \bar{x} + \bar{\mathfrak{a}}_{\bar{x}} \left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}\right) \\
&\subset \{U^*_{\bar{x}}(\cdot, 0) > 0\}, \\
\bar{x} - \left(\mathcal{C}_{\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\nu^A_{\bar{x}}) \cap B'_{\lambda^{1/2}r_\varepsilon}\right) &\subset \{U^*_{\bar{x}}(\cdot, 0) = 0\}.
\end{aligned}$$

Step 5. By rotation in $\mathbb{R}^{n-1}$ we may assume $\nu^A_{x_0} = \mathrm{e}_{n-1}$. For any $\varepsilon > 0$, by Lemma 2.11.4 and the Hölder continuity of A, we can take $\rho_\varepsilon = \rho(x_0, \varepsilon, M)$, possibly smaller than ρ in the previous steps, such that

$$\mathcal{C}_{2\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\mathrm{e}_{n-1}) \cap B'_{\lambda^{1/2}r_\varepsilon} \subset \mathcal{C}_{\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\nu^A_{\bar{x}}) \cap B'_{\lambda^{1/2}r_\varepsilon},$$

for $\bar{x} \in B'_{\rho_\varepsilon}(x_0) \cap \Gamma(U)$. By Step 4, we also have

$$\begin{aligned}
\bar{x} + \left(\mathcal{C}_{2\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\mathrm{e}_{n-1}) \cap B'_{\lambda^{1/2}r_\varepsilon}\right) &\subset \{U(\cdot, 0) > 0\},\\
\bar{x} - \left(\mathcal{C}_{2\Lambda^{1/2}\lambda^{-1/2}\varepsilon}(\mathrm{e}_{n-1}) \cap B'_{\lambda^{1/2}r_\varepsilon}\right) &\subset \{U(\cdot, 0) = 0\}.
\end{aligned}$$

Now, fixing $\varepsilon = \varepsilon_0$, by the standard arguments, we conclude that there exists a Lipschitz function $g : \mathbb{R}^{n-2} \to \mathbb{R}$ with $|\nabla g| \le C_{n,M}/\varepsilon_0$ such that

$$\begin{aligned}
B'_{\rho_{\varepsilon_0}}(x_0) \cap \{U(\cdot, 0) = 0\} &= B'_{\rho_{\varepsilon_0}}(x_0) \cap \{x_{n-1} \le g(x'')\},\\
B'_{\rho_{\varepsilon_0}}(x_0) \cap \{U(\cdot, 0) > 0\} &= B'_{\rho_{\varepsilon_0}}(x_0) \cap \{x_{n-1} > g(x'')\}.
\end{aligned}$$

Step 6. Taking $\varepsilon \to 0$ in Step 5, $\Gamma(U)$ is differentiable at x_0 with normal $\nu^A_{x_0}$. Recentering at any $\bar{x} \in B'_{\rho_{\mathrm{e}_0}}(x_0) \cap \Gamma(U)$, we see that $\Gamma(U)$ has a normal $\nu^A_{\bar{x}}$ at $\bar{x}$. By noticing that $\bar{x} \mapsto \nu^A_{\bar{x}}$ is $C^{0,\gamma}$, we conclude that the function g in Step 5 is $C^{1,\gamma}$. This completes the proof. $\square$

2.12 Singular points

In this section we study another type of free boundary points for almost minimizers, the so-called singular set $\Sigma(U)$. Because of the machinery developed in the earlier sections, we are able to prove a stratification type result for $\Sigma(U)$ (Theorem 2.12.4), following a similar approach for the minimizers and almost minimizers with $A = I$.

Definition 2.12.1 (Singular points). *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. We say that a free boundary point x_0 is* singular *if the coincidence set $\Lambda(U)=\{U(\cdot,0)=0\}\subset B_1'$ has zero H^{n-1}-density at x_0, i.e.,*

$$\lim_{r\to 0+}\frac{H^{n-1}\left(\Lambda(U)\cap B_r'(x_0)\right)}{H^{n-1}(B_r')}=0.$$

We denote the set of all singular points by $\Sigma(U)$ and call it the singular set*.*

Denote by $\bar{\mathfrak{a}}_{x_0}'$ the $(n-1)\times(n-1)$ submatrix of $\bar{\mathfrak{a}}_{x_0}$ formed by the first $(n-1)$ rows and columns. We then claim that there are constants $C,c>0$ depending only on n, λ, and Λ such that

$$c\le|\det\bar{\mathfrak{a}}_{x_0}'|\le C. \tag{2.12.1}$$

Indeed, this follows from the ellipticity of $\mathfrak{a}_{x_0}$ and the invariance of both $\mathbb{R}^{n-1}\times\{0\}$ and $\{0\}\times\mathbb{R}$ under $\bar{\mathfrak{a}}_{x_0}$, since we have

$$|\det\bar{\mathfrak{a}}_{x_0}'(\bar{\mathfrak{a}}_{x_0})_{nn}|=|\det\bar{\mathfrak{a}}_{x_0}|=|\det\mathfrak{a}_{x_0}|$$

and

$$|(\bar{\mathfrak{a}}_{x_0})_{nn}|=|\langle\bar{\mathfrak{a}}_{x_0}\mathrm{e}_n,\mathrm{e}_n\rangle|=|\bar{\mathfrak{a}}_{x_0}\mathrm{e}_n|\in[\lambda^{1/2},\Lambda^{1/2}].$$

Recall now that for $x_0\in\Gamma(u)$, $u_{x_0}(y)=U(\bar{\mathfrak{a}}_{x_0}y+x_0)$ and note that $\bar{\mathfrak{a}}_{x_0}'B_r'+x_0=E_r'(x_0)$. Thus,

$$H^{n-1}(\Lambda(U)\cap E_r'(x_0))=|\det\bar{\mathfrak{a}}_{x_0}'|H^{n-1}(\Lambda(u_{x_0}^*)\cap B_r'). \tag{2.12.2}$$

Now, by (2.12.2) and (2.12.1), together with $B_{\lambda^{1/2}r}(x_0)\subset E_r(x_0)\subset B_{\Lambda^{1/2}r}(x_0)$, we have

$$\begin{aligned}\lim_{r\to 0+}\frac{H^{n-1}\left(\Lambda(U)\cap B_r'(x_0)\right)}{H^{n-1}(B_r')}=0 &\iff \lim_{r\to 0+}\frac{H^{n-1}\left(\Lambda(U)\cap E_r'(x_0)\right)}{H^{n-1}(E_r'(x_0))}=0\\ &\iff \lim_{r\to 0+}\frac{H^{n-1}\left(\Lambda(u_{x_0}^*)\cap B_r'\right)}{H^{n-1}(B_r')}=0.\end{aligned}$$

In terms of Almgren rescalings $(u^*_{x_0})^I_r$, we can rewrite the condition above as

$$\lim_{r\to 0+} H^{n-1}\left(\Lambda((u^*_{x_0})^I_r)\cap B'_1\right)=0.$$

We then have the following characterization of singular points.

Proposition 2.12.1 (Characterization of singular points)**.** *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1, and $x_0\in B'_{1/2}\cap\Gamma(U)$ be such that $\kappa(x_0)=\kappa<\kappa_0$. Then the following statements are equivalent.*

(i) $x_0\in\Sigma(U)$.

(ii) *any Almgren blowup $(u^*_{x_0})^I_0$ of $u^*_{x_0}$ at 0 is a nonzero polynomial from the class*

$$\mathcal{Q}_\kappa=\{q : q \text{ is homogeneous polynomial of degree } \kappa \text{ such that}$$
$$\Delta q=0,\ q(y',0)\geq 0,\ q(y',y_n)=q(y',-y_n)\}.$$

(iii) *any Almgren blowup $(U^*_{x_0})^A_{x_0,0}$ of $U^*_{x_0}$ at x_0 is a nonzero polynomial from the class*

$$\mathcal{Q}^{A,x_0}_\kappa=\{p : p \text{ is homogeneous polynomial of degree } \kappa \text{ such that}$$
$$\operatorname{div}(A(x_0)\nabla p)=0,\ p(x',0)\geq 0,\ p(x)=p(P_{x_0}x)\}.$$

(iv) $\kappa(x_0)=2m$ *for some* $m\in\mathbb{N}$.

Proof. This is the analogue of Proposition 1.10.1 in the case $A\equiv I$.

Clearly, (ii) and (iii) are equivalent. By Proposition 2.8.1, any Almgren blowup $(u^*_{x_0})^I_0$ of $u^*_{x_0}$ at 0 is a nonzero global solution of the Signorini problem, homogeneous of degree κ. Moreover, $(u^*_{x_0})^I_0$ is a C^1_{loc} limit of Almgren rescalings $(u^*_{x_0})^I_{t_j}$ in $\mathbb{R}^n_\pm\cup\mathbb{R}^{n-1}$. Since $u^*_{x_0}$ also satisfies the complementarity condition in Lemma 2.6.6, the equivalence among (i), (ii) and (iv) follows by repeating the arguments in Proposition 2.8.1. $\square$

In order to proceed with the blowup analysis at singular points, we need to remove the logarithmic term from the growth estimates in Lemma 2.9.1. This was achieved in

Lemma 1.10.6 in the case $A \equiv I$ by using a bootstrapping argument Lemmas 1.10.2–1.10.4, Corollary 1.10.5, based on the log-epiperimetric inequality of [27]. All the arguments above work directly for $u^*_{x_0}$ (and then for $U^*_{x_0}$, by deskewing) and we obtain the following optimal growth estimate.

Lemma 2.12.1 (Optimal growth estimate at singular points)**.** *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. If $x_0 \in B'_{1/2} \cap \Gamma(U)$ and $\kappa(x_0) = \kappa < \kappa_0$, $\kappa = 2m$, $m \in \mathbb{N}$, then there are t_0 and C, depending on n, α, M, κ, κ_0, $\|U\|_{W^{1,2}(B_1)}$, such that for $0 < t < t_0$,*

$$\int_{\partial B_t} (u^*_{x_0})^2 \le Ct^{n+2\kappa-1}, \qquad \int_{B_t} |\nabla u^*_{x_0}|^2 \le Ct^{n+2\kappa-2},$$
$$\int_{\partial E_t(x_0)} (U^*_{x_0})^2 \le Ct^{n+2\kappa-1}, \quad \int_{E_t(x_0)} |\nabla U^*_{x_0}|^2 \le Ct^{n+2\kappa-2}.$$

With this growth estimate at hand, we now proceed as in the beginning of Section 2.10 but with $\kappa = 2m < \kappa_0$ in place of $\kappa = 3/2$. Namely, for such κ, let

$$\phi(r) = \phi_\kappa(r) := \mathrm{e}^{-\left(\frac{\kappa b}{\alpha}\right)r^\alpha} r^\kappa, \quad 0 < r < t_0,$$

where $b = \frac{M(n+2\kappa_0)}{\alpha}$ is as in Weiss-type monotonicity formula. Then, define the *κ-almost homogeneous rescalings* of a function v at x_0 by

$$v^\phi_{x_0,r}(x) := \frac{v(rx + x_0)}{\phi(r)}.$$

Again, when $x_0 = 0$, we simply write $v^\phi_{0,r} = v^\phi_r$.

The growth estimates in Lemma 2.12.1 enable us to consider *κ-homogeneous blowups*

$$(u^*_{x_0})^\phi_{t_j} \to (u^*_{x_0})^\phi_0 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1}),$$
$$(U^*_{x_0})^\phi_{x_0,t_j} \to (U^*_{x_0})^\phi_{x_0,0} \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1}),$$

for $t = t_j \to 0+$, similar to $3/2$-homogeneous blowups in Section 2.10.

Furthermore, the arguments in Proposition 1.10.2 also go through for $u^*_{x_0}$ (and then for $U^*_{x_0}$, by deskewing), and we obtain the following rotation estimate for almost homogeneous rescalings.

Proposition 2.12.2 (Rotation estimate)**.** *For U and x_0 as in Lemma 2.12.1, there exist $C > 0$ and $t_0 > 0$ such that*

$$\int_{\partial B_1} |(u^*_{x_0})^\phi_t - (u^*_{x_0})^\phi_s| \leq C\left(\log\frac{1}{t}\right)^{-\frac{1}{n-2}},$$

$$\int_{\mathfrak{a}_{x_0}\partial B_1} |(U^*_{x_0})^\phi_{x_0,t} - (U^*_{x_0})^\phi_{x_0,s}| \leq C\left(\log\frac{1}{t}\right)^{-\frac{1}{n-2}},$$

for $0 < s < t < t_0$. In particular, the blowups $(u^*_{x_0})^\phi_0$ and $(U^*_{x_0})^\phi_{x_0,0}$ are unique.

We next show that the rotation estimate as above holds uniformly for $u^*_{x_0}$ replaced with its Almgren rescalings $(u^*_{x_0})^I_r$, $0 < r < 1$. (Note that the objects $\left[(u^*_{x_0})^I_r\right]^\phi_t$ in the proposition below are κ-almost homogeneous rescalings of Almgren rescalings.)

Proposition 2.12.3. *For U and x_0 as in Lemma 2.12.1 and $0 < r < 1$, there are $C > 0$ and $t_0 > 0$, independent of r such that*

$$\int_{\partial B_1} \left|\left[(u^*_{x_0})^I_r\right]^\phi_t - \left[(u^*_{x_0})^I_r\right]^\phi_s\right| \leq C\left(\log\frac{1}{t}\right)^{-\frac{1}{n-2}},$$

for $0 < s < t < t_0$. In particular, the κ-homogeneous blowup $\left[(u^*_{x_0})^I_r\right]^\phi_0$ is unique.

Proof. We first observe that since $u^*_{x_0}$ has the almost Signorini property at 0, $(u^*_{x_0})^I_r$ also has the almost Signorini property at 0. This implies that $W_\kappa(\rho, (u^*_{x_0})^I_r, 0)$ and $\widehat{N}_{\kappa_0}(\rho, (u^*_{x_0})^I_r, 0)$ are monotone nondecreasing on ρ. Thus

$$\widehat{N}_{\kappa_0}(0+, (u^*_{x_0})^I_r, 0) = \lim_{\rho\to 0}\widehat{N}_{\kappa_0}(\rho, (u^*_{x_0})^I_r, 0) = \lim_{\rho\to 0}\widehat{N}_{\kappa_0}(\rho r, u^*_{x_0}, 0) = \kappa(x_0) = \kappa.$$

Fix $R > 1$. If t is small, then we can argue as in the proof of Proposition 2.8.1 to obtain that for any $K \Subset B^\pm_R \cup B'_R$,

$$\left\|\left[(u^*_{x_0})^I_r\right]^\phi_t\right\|_{C^{1,\beta}(K)} \leq C(n, \alpha, M, R, K)\left\|\left[(u^*_{x_0})^I_r\right]^\phi_t\right\|_{W^{1,2}(B_R)}.$$

Those are all we need to proceed all the arguments with $(u^*_{x_0})^I_r$ as in Lemmas 1.10.2–1.10.4, Corollary 1.10.5, Lemma 1.10.6, and Proposition 1.10.2. This completes the proof. $\square$

Once we have Proposition 2.12.3, we can argue as in Lemma 1.10.8 to obtain the nondegeneracy for $u^*_{x_0}$, and also for $U^*_{x_0}$.

Lemma 2.12.2 (Nondegeneracy at singular points)**.** *Let U and x_0 be as in Lemma 2.12.1. Then*

$$\liminf_{t\to 0}\int_{\partial B_1}((u^*_{x_0})^\phi_t)^2 = \liminf_{t\to 0}\frac{1}{t^{n+2\kappa-1}}\int_{\partial B_t}(u^*_{x_0})^2 > 0,$$
$$\liminf_{t\to 0}\int_{\mathfrak{a}_{x_0}\partial B_1}((U^*_{x_0})^\phi_{x_0,t})^2 = \liminf_{t\to 0}\frac{1}{t^{n+2\kappa-1}}\int_{\partial E_t(x_0)}(U^*_{x_0})^2 > 0.$$

To state our main result on the singular set, we need to introduce certain subsets of $\Sigma(U)$. For $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, let

$$\Sigma_\kappa(U) := \{x_0 \in \Sigma(U) : \kappa(x_0) = \kappa\} = \Gamma_\kappa(U).$$

Note that the last equality follows from the implication (iv) $\Rightarrow$ (i) in Proposition 2.12.1.

Lemma 2.12.3. *The set $\Sigma_\kappa(U)$ is of topological type F_σ; i.e., it is a countable union of closed sets.*

Proof. For $\mathrm{j} \in \mathbb{N}$, $\mathrm{j} \geq 2$, let

$$F_\mathrm{j} := \left\{x_0 \in \Sigma_\kappa(U) \cap \overline{B_{1-1/\mathrm{j}}} : \frac{1}{\mathrm{j}} \leq \frac{1}{\rho^{n+2\kappa-1}}\int_{\partial E_\rho(x_0)}(U^*_{x_0})^2 \leq \mathrm{j} \text{ for } 0 < \rho < \frac{1}{2\mathrm{j}}\right\}.$$

Note that if $x_\mathrm{j} \to x_0$, then by the local uniform continuity of U and A,

$$\int_{\partial E_\rho(x_\mathrm{i})}(U^*_{x_\mathrm{i}})^2 \to \int_{\partial E_\rho(x_0)}(U^*_{x_0})^2.$$

Using this, together with Lemma 2.12.1, Lemma 2.12.2 and Lemma 2.9.1, we can argue as in Lemma 1.10.9 to prove that $\Sigma_\kappa(U) = \cup_{\mathrm{j}=2}^\infty F_\mathrm{j}$ and each F_j is closed. $\square$

Next, for $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$ and $x_0 \in \Sigma_\kappa(U)$, we define

$$d_{x_0}^{(\kappa)} := \dim\{\xi \in \mathbb{R}^{n-1} : \xi \cdot \nabla_{y'}(u_{x_0}^*)_0^\phi(y',0) \equiv 0 \text{ on } \mathbb{R}^{n-1}\},$$

which has the meaning of the dimension of $\Sigma_\kappa(u_{x_0}^*)$ at 0, and where $(u_{x_0}^*)_0^\phi$ is the unique κ-homogeneous blowup of $u_{x_0}^*$ at 0. We note here that $d_{x_0}^{(\kappa)}$ can only take the values $0, 1, \dots, n-2$. Indeed, otherwise $(u_{x_0}^*)_0^\phi$ would vanish identically on Π and consequently on $\mathbb{R}^n$, since it is a solution of the Signorini problem, even symmetric with respect to Π (see [14]). However, that would contradict the nondegeneracy Lemma 2.12.2. Then, for $d = 0, 1, \dots, n-2$, let

$$\Sigma_\kappa^d(U) := \{x_0 \in \Sigma_\kappa(U) : d_{x_0}^{(\kappa)} = d\}.$$

Theorem 2.12.4 (Structure of the singular set)**.** *Let U be an A-quasisymmetric almost minimizer for the A-Signorini problem in B_1. Then for every $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, and $d = 0, 1, \dots, n-2$, the set $\Sigma_\kappa^d(U)$ is contained in the union of countably many submanifolds of dimension d and class $C^{1,\log}$.*

Proof. We follow the idea in Theorem 1.10.10. For $x_0 \in \Sigma_\kappa(U) \cap B'_{1/2}$, let $q_{x_0} \in \mathcal{Q}_\kappa$ denote the unique κ-homogeneous blowup of $u_{x_0}^*$ at 0. By the optimal growth (Lemma 2.12.1) and the nondegeneracy (Lemma 2.12.2), we can write

$$q_{x_0} = \eta_{x_0} q_{x_0}^I, \quad \eta_{x_0} > 0, \quad \|q_{x_0}^I\|_{L^2(\partial B_1)} = 1,$$

where $q_{x_0}^I \in \mathcal{Q}_\kappa$ is the corresponding Almgren blowup. If x_1, $x_2 \in \Sigma_\kappa(U) \cap B'_{1/2}$, for $t > 0$, to be chosen below, we can write

$$\begin{aligned}\|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} &\le \|q_{x_1} - (u_{x_1}^*)_t^\phi\|_{L^1(\partial B_1)} + \|(u_{x_1}^*)_t^\phi - (u_{x_2}^*)_t^\phi\|_{L^1(\partial B_1)} + \|q_{x_2} - (u_{x_2}^*)_t^\phi\|_{L^1(\partial B_1)} \\ &\le C\left(\log\frac{1}{t}\right)^{-\frac{1}{n-2}} + \|(u_{x_1}^*)_t^\phi - (u_{x_2}^*)_t^\phi\|_{L^1(\partial B_1)},\end{aligned} \tag{2.12.3}$$

where we have used Proposition 2.12.2 in the second inequality. Moreover, we have

$$
\begin{aligned}
\|(u^*_{x_1})^\phi_t - (u^*_{x_2})^\phi_t\|_{L^1(\partial B_1)} &= \frac{1}{2\phi(t)}\int_{\partial B_1} |U(t\bar{\mathfrak{a}}_{x_1}y + x_1) + U(P_{x_1}(t\bar{\mathfrak{a}}_{x_1}y + x_1)) \\
&\qquad - U(t\bar{\mathfrak{a}}_{x_2}y + x_2) - U(P_{x_2}(t\bar{\mathfrak{a}}_{x_2}y + x_2))|\, dS_y \\
&\le \frac{C}{t^\kappa}\int_{\partial B_1}\Big(|U(t\bar{\mathfrak{a}}_{x_1}y + x_1) - U(t\bar{\mathfrak{a}}_{x_2}y + x_2)| \\
&\qquad + |U(P_{x_1}(t\bar{\mathfrak{a}}_{x_1}y + x_1)) - U(P_{x_1}(t\bar{\mathfrak{a}}_{x_2}y + x_2))| \\
&\qquad + |U(P_{x_1}(t\bar{\mathfrak{a}}_{x_2}y + x_2)) - U(P_{x_2}(t\bar{\mathfrak{a}}_{x_2}y + x_2))|\Big)\, dS_y \\
&\le \frac{C}{t^\kappa}\|\nabla U\|_{L^\infty(B_1)}\left(|\bar{\mathfrak{a}}_{x_1} - \bar{\mathfrak{a}}_{x_2}| + |x_1 - x_2| + |P_{x_1} - P_{x_2}|\right) \\
&\le C\frac{|x_1 - x_2|^\alpha}{t^\kappa} = C|x_1 - x_2|^{\alpha/2},
\end{aligned}
\tag{2.12.4}
$$

if we choose $t = |x_1 - x_2|^{\frac{\alpha}{2\kappa}}$ and have $|x_1 - x_2| < (1/4\Lambda^{-1}\lambda^{1/2})^{\frac{2\kappa}{\alpha}}$. Combining (2.12.3) and (2.12.4), we obtain

$$
\|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} \le C\left(\log\frac{1}{|x_1 - x_2|}\right)^{-\frac{1}{n-2}}.
$$

After this, we can repeat the argument in the proof of Theorem 1.10.10 to obtain the estimates that for $x_0 \in \Sigma_\kappa(U) \cap B'_{1/2}$, there is $\delta = \delta(x_0) > 0$ such that

$$
\begin{aligned}
|\eta_{x_1} - \eta_{x_2}| &\le C\left(\log\frac{1}{|x_1 - x_2|}\right)^{-\frac{1}{2(n-2)}}, \\
\|q^I_{x_1} - q^I_{x_2}\|_{L^\infty(B_1)} &\le C\left(\log\frac{1}{|x_1 - x_2|}\right)^{-\frac{1}{2(n-2)}}, \quad x_1, x_2 \in \Sigma_\kappa(U) \cap B'_\delta(x_0).
\end{aligned}
$$

Now, we also have the similar result for $U^*_{x_0}$. For $x_0 \in \Sigma_\kappa(U) \cap B_{1/2}$, where $\kappa = 2m$, $m \in \mathbb{N}$, let $p_{x_0} \in \mathcal{Q}^{A,x_0}_\kappa$ be the unique κ-homogeneous blowup of $U^*_{x_0}$ at x_0. Then we can write

$$
p_{x_0} = \eta^A_{x_0} p^A_{x_0}, \quad \eta^A_{x_0} > 0, \quad \|p^A_{x_0}\|_{L^2(\partial B_1)} = 1,
$$

where $p^A_{x_0} \in \mathcal{Q}^{A,x_0}_\kappa$ is the corresponding Almgren blowup of $U^*_{x_0}$. Using that

$$q^I_{x_0}(z) = (\det \mathfrak{a}_{x_0})^{1/2}\, p^A_{x_0}(\mathfrak{a}_{x_0} z), \quad q_{x_0}(z) = p_{x_0}(\mathfrak{a}_{x_0} z),$$

together with the ellipticity and Hölder continuity of $\mathfrak{a}_{x_0}$ and the homogeneity of blowups, we easily conclude that for $x_0 \in \Sigma_\kappa(U) \cap B'_{1/2}$, there is $\delta = \delta(x_0) > 0$ such that

$$\begin{aligned} |\eta^A_{x_1} - \eta^A_{x_2}| &\leq C \left(\log \frac{1}{|x_1 - x_2|}\right)^{-\frac{1}{2(n-2)}}, \\ \|p^A_{x_1} - p^A_{x_2}\|_{L^\infty(B_1)} &\leq C \left(\log \frac{1}{|x_1 - x_2|}\right)^{-\frac{1}{2(n-2)}}, \quad x_1, x_2 \in \Sigma_\kappa(U) \cap B'_\delta(x_0). \end{aligned}$$

Once we have these estimates, as well as Lemma 2.12.3, we can apply the Whitney Extension Theorem of Fefferman [56], to complete the proof, similar to that of Theorem 1.7 in [27]. □

2.A Example of almost minimizers

Example 2.A.1. Let U be a solution of the A-Signorini problem in B_1 with velocity field $b \in L^p(B_1)$, $p > n$:

$$\begin{aligned} &-\operatorname{div}(A\nabla U) + \langle b(x), \nabla U\rangle = 0 \quad \text{in } B_1^\pm, \\ &U \geq 0, \quad \langle A\nabla U, \nu^+\rangle + \langle A\nabla U, \nu^-\rangle \geq 0, \\ &U(\langle A\nabla U, \nu^+\rangle + \langle A\nabla U, \nu^-\rangle) = 0 \quad \text{on } B'_1, \end{aligned}$$

where $\nu^\pm = \mp e_n$ and $\langle A\nabla U, \nu^\pm\rangle$ on B'_1 are understood as the limits from inside $B_1^\pm$. We interpret this in the weak sense that U satisfies the variational inequality

$$\int_{B_1} \langle A\nabla U, \nabla(W - U)\rangle + \langle b, \nabla U\rangle(W - U) \geq 0,$$

for any competitor $W \in \mathfrak{K}_{0,U}(B_1, \Pi)$. Then U is an almost minimizer of the A-Signorini problem in B_1 with thin obstacle $\psi = 0$ on $\Pi = \mathbb{R}^{n-1} \times \{0\}$ and a gauge function $\omega(r) = Cr^{1-n/p}$, $C = C(n, p, \lambda, \Lambda)\|b\|^2_{L^p(B_1)}$.

Proof. For any $E_r(x_0) \Subset B_1$ and $W \in \mathfrak{K}_{0,U}(E_r(x_0), \Pi)$, we extend W as equal to U in $B_1 \setminus E_r(x_0)$ to obtain

$$\int_{E_r(x_0)} \langle A\nabla U, \nabla(W-U)\rangle + \langle b, \nabla U\rangle(W-U) \geq 0. \tag{2.A.1}$$

Let V be the minimizer of the energy functional

$$\int_{E_r(x_0)} \langle A\nabla V, \nabla V\rangle \quad \text{on } \mathfrak{K}_{0,U}(E_r(x_0), \Pi).$$

Then it follows from a standard variation argument that V satisfies the variational inequality

$$\int_{E_r(x_0)} \langle A\nabla V, \nabla(W-V)\rangle \geq 0 \quad \text{for any } W \in \mathfrak{K}_{0,U}(E_r(x_0), \Pi). \tag{2.A.2}$$

Taking $W = U \pm (U-V)^+$ in (2.A.1) and $W = V + (U-V)^+$ in (2.A.2), we obtain

$$\int_{E_r(x_0)} \langle A\nabla(U-V)^+, \nabla(U-V)^+\rangle \leq -\int_{E_r(x_0)} \langle b, \nabla U\rangle(U-V)^+.$$

Similarly, taking $W = U + (V-U)^+$ in (2.A.1) and $W = V \pm (V-U)^+$ in (2.A.2), we get

$$\int_{E_r(x_0)} \langle A\nabla(V-U)^+, \nabla(V-U)^+\rangle \leq \int_{E_r(x_0)} \langle b, \nabla U\rangle(V-U)^+.$$

These two inequalities give

$$\int_{E_r(x_0)} \langle A\nabla(U-V), \nabla(U-V)\rangle \leq \int_{E_r(x_0)} |b||\nabla U||U-V|.$$

Applying Hölder's inequality,

$$\begin{aligned}\int_{E_r(x_0)} |\nabla(U-V)|^2 &\leq \lambda^{-1} \int_{E_r(x_0)} \langle A\nabla(U-V), \nabla(U-V)\rangle \\ &\leq \lambda^{-1} \|b\|_{L^p(E_r(x_0))} \|\nabla U\|_{L^2(E_r(x_0))} \|U-V\|_{L^{p^*}(E_r(x_0))},\end{aligned}$$

with $p^* = 2p/(p-2)$. Since $U - V \in W_0^{1,2}(E_r(x_0))$ and $\operatorname{diam}(E_r(x_0)) \leq 2\Lambda^{1/2}r$, from the Sobolev's inequality,

$$\|U - V\|_{L^{p^*}(E_r(x_0))} \leq C(n, p, \lambda, \Lambda) r^{1-n/p} \|\nabla(U - V)\|_{L^2(E_r(x_0))}.$$

Now we have

$$\int_{E_r(x_0)} |\nabla(U - V)|^2 \leq C r^{2(1-n/p)} \int_{E_r(x_0)} |\nabla U|^2, \tag{2.A.3}$$

with $C = C(n, p, \lambda, \Lambda)\|b\|^2_{L^p(B_1)}$. Thus,

$$\begin{aligned}
\int_{E_r(x_0)} \langle A\nabla U, \nabla U\rangle - \int_{E_r(x_0)} \langle A\nabla V, \nabla V\rangle &= \int_{E_r(x_0)} \langle A\nabla(U+V), \nabla(U-V)\rangle \\
&\leq C \int_{E_r(x_0)} |\nabla(U+V)||\nabla(U-V)| \\
&\leq C r^{\gamma} \int_{E_r(x_0)} \left(|\nabla U|^2 + |\nabla V|^2\right) + C r^{-\gamma} \int_{E_r(x_0)} |\nabla(U-V)|^2 \\
&\leq C r^{\gamma} \int_{E_r(x_0)} \langle A\nabla U, \nabla U\rangle + C r^{\gamma} \int_{E_r(x_0)} \langle A\nabla V, \nabla V\rangle \\
&\qquad + C r^{2(1-n/p)-\gamma} \int_{E_r(x_0)} \langle A\nabla U, \nabla U\rangle,
\end{aligned}$$

where we applied Young's inequality and used (2.A.3) at the end. We choose $\gamma = 1 - n/p$ to complete the proof. $\square$

3. ALMOST MINIMIZERS FOR CERTAIN FRACTIONAL VARIATIONAL PROBLEMS

3.1 Introduction and Main Results

3.1.1 Fractional harmonic functions

Given $0 < s < 1$, we say that a function $u \in \mathcal{L}_s(\mathbb{R}^n) := L^1(\mathbb{R}^n, (1+|x|^{n+2s})^{-1})$ is *s-fractional harmonic* in an open set $\Omega \subset \mathbb{R}^n$ if

$$(-\Delta_x)^s u(x) := C_{n,s}\,\text{p.v.}\int_{\mathbb{R}^n} \frac{u(x)-u(x+z)}{|z|^{n+2s}} = 0 \quad \text{in } \Omega, \tag{3.1.1}$$

where p.v. stands for Cauchy's principal value and $C_{n,s}$ is a normalization constant. The formula above is just one of many equivalent definitions of the fractional Laplacian $(-\Delta_x)^s$, another one being a pseudo-differential operator with Fourier symbol $|\xi|^{2s}$. We refer to a recent review of Garofalo [60] for basic properties of $(-\Delta_x)^s$, as well as many historical remarks concerning that operator.

In recent years, there has been a surge of interest in nonlocal problems involving the fractional Laplacian, when it was discovered that the problems can be localized by the use of the so-called Caffarelli-Silvestre extension procedure [10]. Namely, for $a = 1-2s \in (-1,1)$, let

$$P(x,y) := C_{n,a}\frac{|y|^{1-a}}{(|x|^2+|y|^2)^{\frac{n+1-a}{2}}}, \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+ = \mathbb{R}^{n+1}_+,$$

(to be called the Poisson kernel for the extension operator L_a) and consider the convolution, still denoted by u,

$$u(x,y) := u * P(\cdot,y) = \int_{\mathbb{R}^n} u(z)P(x-z,y)dz, \quad (x,y) \in \mathbb{R}^{n+1}_+.$$

Note that $u(x,y)$ solves the Cauchy problem

$$L_a u := \operatorname{div}(|y|^a \nabla u) = 0 \quad \text{in } \mathbb{R}^{n+1}_+,$$
$$u(x,0) = u(x) \quad \text{on } \mathbb{R}^n,$$

where $\nabla = \nabla_{x,y}$ is the full gradient in x and y variables. L_a is known as the Caffarelli-Silvestre *extension operator*. Then, one can recover $(-\Delta_x)^s u$ as the fractional normal derivative on $\mathbb{R}^n$

$$(-\Delta_x)^s u(x) = -C_{n,a} \lim_{y\to 0+} y^a \partial_y u(x,y), \quad x \in \mathbb{R}^n$$

to be understood in the appropriate sense of traces. Now, going back to the definition (3.1.1), if we consider the even reflection of u in y-variable to all of $\mathbb{R}^{n+1}$, i.e.,

$$u(x,y) = u(x,-y), \quad x \in \mathbb{R}^n,\ y < 0,$$

then the following fact holds: $u(x)$ is s-fractional harmonic in Ω if and only if $u(x,y)$ satisfies

$$L_a u = 0 \quad \text{in } \widetilde{\Omega} := \mathbb{R}^{n+1}_- \cup (\Omega \times \{0\}) \cup \mathbb{R}^{n+1}_+. \tag{3.1.2}$$

(We will refer to solutions of $L_a u = 0$ as L_a*-harmonic functions.*) This is essentially Lemma 4.1 in [10]. Since $L_a u = 0$ in $\mathbb{R}^n_\pm$ by definition, the condition (3.1.2) is equivalent to asking

$$L_a u = 0 \quad \text{in } \mathbb{B}_r(x_0),$$

for any ball $\mathbb{B}_r(x_0)$ centered at $x_0 \in \Omega$ such that $\mathbb{B}_r(x_0) \Subset \widetilde{\Omega}$, or equivalently $B_r(x_0) \Subset \Omega$. Now, observing that the solutions of the above equation are minimizers of the weighted Dirichlet energy $\int_{\mathbb{B}_r(x_0)} |\nabla v|^2 |y|^a$, we obtain the following fact.

Proposition 3.1.1. *A function $u \in \mathcal{L}_s(\mathbb{R}^n)$ is s-fractional harmonic in Ω if and only if its reflected Caffarelli-Silvestre extension $u(x,y)$ is in $W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega}, |y|^a)$ and for any ball $\mathbb{B}_r(x_0)$ with $x_0 \in \Omega$ such that $B_r(x_0) \Subset \Omega$, we have*

$$\int_{\mathbb{B}_r(x_0)} |\nabla u|^2 |y|^a \le \int_{\mathbb{B}_r(x_0)} |\nabla v|^2 |y|^a,$$

for any $v \in u + W^{1,2}_0(\mathbb{B}_r(x_0), |y|^a)$.

We take this proposition as the starting point for the definition of almost s-fractional harmonic functions, in the spirit of Anzellotti [31].

Definition 3.1.1 (Almost s-fractional harmonic functions). *Let $r_0 > 0$ and $\omega : (0, r_0) \to [0, \infty)$ be a modulus of continuity*[1]. *We say that a function $u \in \mathcal{L}_s(\mathbb{R}^n)$ is* almost s-fractional harmonic *in an open set $\Omega \subset \mathbb{R}^n$, with a gauge function ω, if its reflected Caffarelli-Silvestre extension $u(x, y)$ is in $W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega}, |y|^a)$ and for any ball $\mathbb{B}_r(x_0)$ with $x_0 \in \Omega$ and $0 < r < r_0$ such that $B_r(x_0) \Subset \Omega$, we have*

$$\int_{\mathbb{B}_r(x_0)} |\nabla u|^2 |y|^a \leq (1 + \omega(r)) \int_{\mathbb{B}_r(x_0)} |\nabla v|^2 |y|^a, \tag{3.1.3}$$

for any $v \in u + W^{1,2}_0(\mathbb{B}_r(x_0), |y|^a)$.

3.1.2 Fractional obstacle problem

A function $u \in \mathcal{L}_s(\mathbb{R}^n)$ is said to solve the s-fractional obstacle problem with obstacle ψ in an open set $\Omega \subset \mathbb{R}^n$, if

$$\min\{(-\Delta_x)^s u, u - \psi\} = 0 \quad \text{in } \Omega. \tag{3.1.4}$$

We refer to [11], [13], [61] for general introduction and basic results on this problem. With the help of the reflected Caffarelli-Silvestre extension, we can rewrite the problem as a Signorini-type problem for the operator L_a:

$$\begin{aligned} L_a u &= 0 \quad \text{in } \mathbb{R}^{n+1}_{\pm} \\ \min\{-\partial^a_y u, u - \psi\} &= 0 \quad \text{in } \Omega, \end{aligned}$$

where

$$\partial^a_y u(x, 0) := \lim_{y \to 0+} y^a \partial_y u(x, y).$$

This, in turn, can be written in the following variational form, see [13].

[1] i.e., a nondecreasing function with $\omega(0+) = 0$

Proposition 3.1.2. *A function $u \in \mathcal{L}_s(\mathbb{R}^n)$ solves (3.1.4) if and only if its reflected Caffarelli-Silvestre extension $u(x,y)$ is in $W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega})$ and for any ball $\mathbb{B}_r(x_0)$ with $x_0 \in \Omega$ such that $B_r(x_0) \Subset \Omega$, we have*

$$\int_{\mathbb{B}_r(x_0)} |\nabla u|^2 |y|^a \le \int_{\mathbb{B}_r(x_0)} |\nabla v|^2 |y|^a,$$

for any $v \in \mathfrak{K}_{\psi,u}(\mathbb{B}_r(x_0), |y|^a) := \{v \in u + W^{1,2}_0(\mathbb{B}_r(x_0), |y|^a) : v \ge \psi \text{ on } B_r(x_0)\}$.

Definition 3.1.2 (Almost minimizers for s-fractional obstacle problem)**.** *Let $r_0 > 0$ and $\omega : (0, r_0) \to [0, \infty)$ be a modulus of continuity. We say that a function $u \in \mathcal{L}_s(\mathbb{R}^n)$ is an* almost minimizer for the s-fractional obstacle problem *in an open set $\Omega \subset \mathbb{R}^n$, with a gauge function ω, if its reflected Caffarelli-Silvestre extension $u(x,y)$ is in $W^{1,2}_{\mathrm{loc}}(\widetilde{\Omega}, |y|^a)$ and for any ball $\mathbb{B}_r(x_0)$ with $x_0 \in \Omega$ and $0 < r < r_0$ such that $B_r(x_0) \Subset \Omega$, we have*

$$\int_{\mathbb{B}_r(x_0)} |\nabla u|^2 |y|^a \le (1 + \omega(r)) \int_{\mathbb{B}_r(x_0)} |\nabla v|^2 |y|^a, \tag{3.1.5}$$

for any $v \in \mathfrak{K}_{\psi,u}(\mathbb{B}_r(x_0), |y|^a)$.

The notion of almost minimizers above is related to the one for the thin obstacle problem ($s = 1/2$) studied in Chapter 1, but there are certain important differences. In Definition 3.1.2, we ask the almost minimizing property (3.1.5) to hold only for balls centered on the "thin space" $\mathbb{R}^n$, while in Chapter 1, we ask that property for balls centered at any point in an open set in the "thick space" $\mathbb{R}^{n+1}$. In a sense, this means that here we think of the perturbation from minimizers as living on the thin space, while in Chapter 1 they live in the thick space.

3.1.3 Main results and structure

In this chapter, our main concern is the regularity of almost minimizers in their original variables.

We start with examples of almost minimizers in Section 3.2. We then proceed to prove the following results, echoing those in [31] and Chapter 1.

Theorem J. *Let $u \in \mathcal{L}_s(\mathbb{R}^n)$ be almost s-fractional harmonic in Ω. Then*

(i) *u is almost Lipschitz in Ω, i.e, $u \in C^{0,\sigma}(\Omega)$ for any $0 < \sigma < 1$.*

(ii) *If $\omega(r) = r^\alpha$, then $u \in C^{1,\beta}(\Omega)$ for some $\beta = \beta_{n,a,\alpha} > 0$.*

(iii) *If $0 < s < 1/2$ or $s = 1/2$ and $\omega(r) = r^\alpha$ for some $\alpha > 0$, then u is actually s-fractional harmonic in Ω.*

In the case of the s-fractional obstacle problem, our results are obtained under the assumption that $1/2 \leq s < 1$. Also, because of the technical nature of the problem, we restrict ourselves to the case $\psi = 0$.

Theorem K. *Let $u \in \mathcal{L}_s(\mathbb{R}^n)$ be an almost minimizer for the s-fractional obstacle problem with obstacle $\psi = 0$ in Ω.*

(i) *If $1/2 \leq s < 1$, then $u \in C^{0,\sigma}(\Omega)$ for any $0 < \sigma < 1$.*

(ii) *If $1/2 \leq s < 1$ and $\omega(r) = r^\alpha$ for some $\alpha > 0$, then $u \in C^{1,\beta}(\Omega)$ for some $\beta = \beta_{n,a,\alpha} > 0$.*

The proofs follow the general approach in [31] and Chapter 1 by first obtaining growth estimates for minimizers (see Section 3.3) and then deriving their perturbed versions for almost minimizers (Section 3.4 for s-fractional harmonic functions and Section 3.5 for the s-fractional obstacle problem). The regularity then follows by an embedding theorem of a Morrey-Campanato-type space into the Hölder space, which we included in Appendix 3.A. Finally, Appendix 3.B contains the proof of orthogonal polynomial expansion of L_a-harmonic functions, that we rely on in deriving the growth estimates in Section 3.3. The polynomial expansion has other interesting corollaries such as the (known) real-analyticity of s-fractional harmonic functions, which are of independent interest.

3.1.4 Notation

Throughout this chapter we use the following notation. $\mathbb{R}^n$ is the n-dimensional Euclidean space. The points of $\mathbb{R}^{n+1}$ are denoted by $X = (x, y)$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y \in \mathbb{R}$. We routinely identify $x \in \mathbb{R}^n$ with $(x, 0) \in \mathbb{R}^n \times \{0\}$. $\mathbb{R}^{n+1}_\pm$ stands for open halfspaces $\{X = (x, y) \in \mathbb{R}^{n+1} : \pm y > 0\}$.

We use the following notations for balls of radius r in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$

$$\begin{aligned}
\mathbb{B}_r(X) &= \{Z \in \mathbb{R}^{n+1} : |X - Z| < r\}, && \text{(Euclidean) ball in } \mathbb{R}^{n+1},\\
\mathbb{B}_r^{\pm}(x) &= \mathbb{B}_r(x,0) \cap \{\pm y > 0\}, && \text{half-ball in } \mathbb{R}^{n+1},\\
B_r(x) &= \mathbb{B}_r(x,0) \cap \{y = 0\}, && \text{ball in } \mathbb{R}^n.
\end{aligned}$$

We typically drop the center from the notation if it is the origin. Thus, $\mathbb{B}_r = \mathbb{B}_r(0)$, $B_r = B_r(0)$, etc.

Next, $\nabla u = \nabla_X u = (\partial_{x_1} u, \ldots, \partial_{x_n} u, \partial_y u)$ stands for the full gradient, while $\nabla_x u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)$. We also use the standard notations for partial derivatives, such as $\partial_{x_i} u$, u_{x_i}, u_y etc.

In integrals, we often drop the variable and the measure of integration if it is with respect to the Lebesgue measure or the surface measure. Thus,

$$\int_{\mathbb{B}_r} u|y|^a = \int_{\mathbb{B}_r} u(X)|y|^a dX, \quad \int_{\partial\mathbb{B}_r} u|y|^a = \int_{\partial\mathbb{B}_r} u(X)|y|^a dS_X,$$

where S_X stands for the surface measure.

By $L^2(\mathbb{B}_R, |y|^a)$ and $L^2(\partial\mathbb{B}_R, |y|^a)$ we indicate the weighted Lebesgue spaces of functions with the norms

$$\begin{aligned}
\|u\|^2_{L^2(\mathbb{B}_R,|y|^a)} &= \int_{\mathbb{B}_R} u^2|y|^a\\
\|u\|^2_{L^2(\partial\mathbb{B}_R,|y|^a)} &= \int_{\partial\mathbb{B}_R} u^2|y|^a.
\end{aligned}$$

$W^{1,2}(\mathbb{B}_R, |y|^a)$ is the corresponding weighted Sobolev space of functions with the norm

$$\|u\|^2_{W^{1,2}(\mathbb{B}_R,|y|^a)} = \|u\|^2_{L^2(\mathbb{B}_R,|y|^a)} + \|\nabla u\|^2_{L^2(\mathbb{B}_R,|y|^a)}.$$

We also use other typical notations for Sobolev spaces. Thus, $W_0^{1,2}(\mathbb{B}_R, |y|^a)$ stands for the closure of $C_0^\infty(\mathbb{B}_R)$ in $W^{1,2}(\mathbb{B}_R, |y|^a)$.

For $x \in \mathbb{R}^n$ and $r > 0$, we indicate by $\langle u \rangle_{x,r}$ the $|y|^a$-weighted integral mean value of a function u over $\mathbb{B}_r(x)$. That is,

$$\langle u \rangle_{x,r} = \fint_{\mathbb{B}_r(x)} u|y|^a := \frac{1}{\omega_{n+1+a} r^{n+1+a}} \int_{\mathbb{B}_r(x)} u|y|^a,$$

where $\omega_{n+1+a} = \int_{\mathbb{B}_1} |y|^a$ is the $|y|^a$-weighted volume of the unit ball $\mathbb{B}_1$ in $\mathbb{R}^{n+1}$. (Note that here and throughout the thesis, the sign $\fint$ denotes the integral mean value with respect to the weighted measure $|y|^a dX$.) Finally, similarly to the other notations, we drop the origin if it is 0 and write $\langle u \rangle_r$ for $\langle u \rangle_{0,r}$.

3.2 Examples of almost minimizers

Before we proceed with the proofs of the main results, we would like to give some examples of almost minimizers.

Example 3.2.1. Let $u \in \mathcal{L}_s(\mathbb{R}^n)$ be a solution of

$$(-\Delta_x)^s u + b(x) \cdot \nabla_x u = 0 \quad \text{in } \Omega,$$

where $b = (b^1, b^2, \dots, b^n) \in W^{1,\infty}(\Omega)$ and $1/2 < s < 1$ (or $-1 < a < 0$). Then u is almost s-fractional harmonic with a gauge function $\omega(r) = Cr^{-a}$ (note that $-a > 0$).

Proof. Consider a ball $\mathbb{B}_r(x_0)$ centered at $x_0 \in \Omega$ such that $B_r(x_0) \Subset \Omega$. Without loss of generality assume that $x_0 = 0$. Let v be the minimizer of

$$\int_{\mathbb{B}_r} |\nabla v|^2 |y|^a$$

on $u + W_0^{1,2}(\mathbb{B}_r, |y|^a)$. Then

$$\int_{\mathbb{B}_r} \nabla v \nabla (u - v) |y|^a = 0,$$

and as a consequence,

$$\int_{\mathbb{B}_r} (|\nabla u|^2 - |\nabla v|^2)|y|^a = \int_{\mathbb{B}_r} |\nabla (u - v)|^2 |y|^a.$$

Then, we have

$$
\begin{aligned}
\int_{\mathbb{B}_r}(|\nabla u|^2-|\nabla v|^2)|y|^a &= 2\int_{\mathbb{B}_r^+}|\nabla(u-v)|^2|y|^a \\
&= 2\int_{\mathbb{B}_r^+}|\nabla(u-v)|^2|y|^a+\operatorname{div}(|y|^a\nabla(u-v))\,(u-v) \\
&= 2\int_{\mathbb{B}_r^+}\operatorname{div}\left(|y|^a\nabla\left(\frac{(u-v)^2}{2}\right)\right) \\
&= 2\int_{(\partial\mathbb{B}_r)^+}|y|^a(u-v)(u_\nu-v_\nu)-2\int_{\mathbb{B}_r}(u-v)(\partial_y^a u-\partial_y^a v) \\
&= C\int_{\mathbb{B}_r}(u-v)(-\Delta_x)^s u \\
&= -C\int_{\mathbb{B}_r}(u-v)b^i u_{x_i}
\end{aligned}
$$

with $C=C_{n,a}$. Next, extending b^i to $\mathbb{R}^{n+1}$ by $b^i(x,y):=b^i(x)$, we have

$$
\begin{aligned}
\int_{\mathbb{B}_r}(|\nabla u|^2-|\nabla v|^2)|y|^a &= -C\int_{\mathbb{B}_r'}(u-v)b^i u_{x_i} \\
&= C\int_{\mathbb{B}_r^+}\partial_y\left((u-v)b^i u_{x_i}\right) \\
&= C\int_{\mathbb{B}_r^+}(u_y-v_y)b^i u_{x_i}+(u-v)b^i u_{x_i y} \\
&\le C\|b\|_{W^{1,\infty}(\Omega)}\int_{\mathbb{B}_r^+}|\nabla u|^2+|\nabla v|^2 \\
&\qquad + C\int_{\partial(\mathbb{B}_r^+)}(u-v)b^i u_y \nu_{x_i}-C\int_{\mathbb{B}_r^+}\partial_{x_i}((u-v)b^i)u_y \\
&= C\|b\|_{W^{1,\infty}(\Omega)}\int_{\mathbb{B}_r^+}|\nabla u|^2+|\nabla v|^2 \\
&\qquad - C\int_{\mathbb{B}_r^+}((u_{x_i}-v_{x_i})b^i+(u-v)b^i_{x_i})u_y \\
&\le C\|b\|_{W^{1,\infty}(\Omega)}\int_{\mathbb{B}_r^+}|\nabla u|^2+|\nabla v|^2+|u-v|^2.
\end{aligned}
$$

Using Poincare's inequality, it follows that

$$
\begin{aligned}
\int_{\mathbb{B}_r}(|\nabla u|^2-|\nabla v|^2)|y|^a \le C\int_{\mathbb{B}_r}|\nabla u|^2+|\nabla v|^2 &\le Cr^{-a}\int_{\mathbb{B}_r}(|\nabla u|^2+|\nabla v|^2)|y|^a \\
&\le Cr^{-a}\int_{\mathbb{B}_r}|\nabla u|^2|y|^a.
\end{aligned}
$$

Hence,

$$\int_{\mathbb{B}_r(x_0)} |\nabla u|^2|y|^a \leq (1+Cr^{-a}) \int_{\mathbb{B}_r(x_0)} |\nabla v|^2|y|^a,$$

for $0<r<r_0$, with C and r_0 depending on n, a, and $\|b\|_{W^{1,\infty}(\Omega)}$. $\square$

Example 3.2.2. Let $u \in \mathcal{L}_s(\mathbb{R}^n)$ be a solution of the obstacle problem for fractional Laplacian with drift

$$\min\{(-\Delta_x)^s u + b(x)\cdot\nabla_x u, u\} = 0 \quad \text{in } \Omega,$$

where $b=(b^1,b^2,\ldots,b^n)\in W^{1,\infty}(\Omega)$ and $1/2<s<1$ (or $-1<a<0$). Then u is an almost minimizer for s-fractional obstacle problem in Ω with an obstacle $\psi=0$ and a gauge function $\omega(r)=Cr^{-a}$.

The obstacle problem above has been studied earlier in [17] and [57].

Proof. We argue similarly to Example 3.2.1. Let $\mathbb{B}_r(x_0)$ centered at $x_0\in\Omega$ such that $B_r(x_0)\Subset\Omega$. Without loss of generality assume that $x_0=0$. Let v be the minimizer of

$$\int_{\mathbb{B}_r} |\nabla v|^2|y|^a$$

on $\mathfrak{K}_{0,u}(\mathbb{B}_r,|y|^a)=\{v\in u+W^{1,2}_0(\mathbb{B}_r,|y|^a): v\geq 0 \text{ on } B_r\}$. Next, we write

$$\begin{aligned}
\int_{\mathbb{B}_r}(|\nabla u|^2-|\nabla v|^2)|y|^a &= 2\int_{\mathbb{B}_r}\nabla u\nabla(u-v)|y|^a - \int_{\mathbb{B}_r}|\nabla(u-v)|^2|y|^a\\
&\leq 2\int_{\mathbb{B}_r}\nabla u\nabla(u-v)|y|^a\\
&= 4\int_{\mathbb{B}_r^+}\nabla u\nabla(u-v)|y|^a + \operatorname{div}(|y|^a\nabla u)(u-v)\\
&= -4\int_{B_r}(u-v)\partial_y^a u\\
&= C\int_{B_r}(u-v)(-\Delta_x)^s u\\
&= C\left[-\int_{B_r\cap\{u>0\}}(u-v)b^{\mathrm{i}}u_{x_i} + \int_{B_r\cap\{u=0\}}(-v)\,(-\Delta_x)^s u\right]\\
&\leq C\left[-\int_{B_r\cap\{u>0\}}(u-v)b^i u_{x_i} - \int_{B_r\cap\{u=0\}}(-v)b^i u_{x_i}\right]
\end{aligned}$$

$$= -C \int_{B_r} (u-v) b^i u_{x_i},$$

where we used that $(-\Delta)^s u + b^i u_{x_i} \geq 0$ and $-v \leq 0$ on $B_r \cap \{u = 0\}$ in the last inequality.

Then we complete the proof as in Example 3.2.1. $\square$

3.3 Growth estimates for minimizers

In this section we prove growth estimates for L_a-harmonic functions and solutions of the Signorini problem for L_a, i.e., minimizers v of the weighted Dirichlet integral

$$\int_{\mathbb{B}_r} |\nabla v|^2 |y|^a$$

on $v + W_0^{1,2}(\mathbb{B}_r, |y|^a)$ or on the thin obstacle constraint set $\mathfrak{K}_{0,v}(\mathbb{B}_r, |y|^a)$.

The idea is that these estimates will extend to almost minimizers and will ultimately imply their regularity with the help of Morrey-Campanato-type space embedding.

The proofs in this section are akin to those in Chapter 1 for almost minimizers of the thin obstacle problem. Yet, one has to be careful with different growth rates for tangential and normal derivatives.

3.3.1 Growth estimates for L_a-harmonic functions

Lemma 3.3.1. *Let $v \in W^{1,2}(\mathbb{B}_R, |y|^a)$ be a solution of $L_a v = 0$ in $\mathbb{B}_R$. If v is even in y, then for $0 < \rho < R$*

$$\int_{\mathbb{B}_\rho} |\nabla_x v|^2 |y|^a \leq \left(\frac{\rho}{R}\right)^{n+1+a} \int_{\mathbb{B}_R} |\nabla_x v|^2 |y|^a$$
$$\int_{\mathbb{B}_\rho} |v_y|^2 |y|^a \leq \left(\frac{\rho}{R}\right)^{n+3+a} \int_{\mathbb{B}_R} |v_y|^2 |y|^a.$$

Proof. Note that we can write

$$v(x,y) = \sum_{k=0}^{\infty} p_k(x,y),$$

where p_k's are L_a-harmonic homogeneous polynomials of degree k (see Appendix 3.B). Then $\{\partial_{x_i} p_k\}_{k=1}^{\infty}$ are L_a-harmonic homogeneous polynomials of degree $k-1$, and thus orthogonal in $L^2(\partial\mathbb{B}_1, |y|^a)$. Thus,

$$\begin{aligned}\int_{\mathbb{B}_\rho} |\nabla_x v|^2 |y|^a &= \sum_{k=1}^{\infty} \int_{\mathbb{B}_\rho} |\nabla_x p_k|^2 |y|^a = \sum_{k=1}^{\infty} \left(\frac{\rho}{R}\right)^{n+1+a+2(k-1)} \int_{\mathbb{B}_R} |\nabla_x p_k|^2 |y|^a \\ &\le \left(\frac{\rho}{R}\right)^{n+1+a} \sum_{k=1}^{\infty} \int_{\mathbb{B}_R} |\nabla_x p_k|^2 |y|^a = \left(\frac{\rho}{R}\right)^{n+1+a} \int_{\mathbb{B}_R} |\nabla_x v|^2 |y|^a.\end{aligned}$$

Similarly, $\{|y|^a \partial_y p_k\}_{k=1}^{\infty}$ are L_{-a}-harmonic homogeneous functions of degree $k-1+a$, and thus orthogonal in $L^2(\partial\mathbb{B}_1, |y|^{-a})$. Notice that since $p_1(x,y) = p_1(x)$ is independent of y variable by the even symmetry, we have $|y|^a \partial_y p_1 = 0$. Thus,

$$\begin{aligned}\int_{\mathbb{B}_\rho} |v_y|^2 |y|^a &= \int_{\mathbb{B}_\rho} \left||y|^a v_y\right|^2 |y|^{-a} = \sum_{k=2}^{\infty} \int_{\mathbb{B}_\rho} \left||y|^a \partial_y p_k\right|^2 |y|^{-a} \\ &= \sum_{k=2}^{\infty} \left(\frac{\rho}{R}\right)^{n+1-a+2(k-1+a)} \int_{\mathbb{B}_R} ||y|^a \partial_y p_k|^2 |y|^{-a} \le \left(\frac{\rho}{R}\right)^{n+3+a} \int_{\mathbb{B}_R} |v_y|^2 |y|^a.\end{aligned}$$

□

Lemma 3.3.2. *Let v be a solution of $L_a v = 0$ in $\mathbb{B}_R$, even in y. Then, for $0 < \rho < R$,*

$$\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v \rangle_\rho|^2 |y|^a \le \left(\frac{\rho}{R}\right)^{n+a+3} \int_{\mathbb{B}_R} |\nabla_x v - \langle \nabla_x v \rangle_R|^2 |y|^a. \tag{3.3.1}$$

Proof. First note that since $L_a(\nabla_x v) = 0$ in $\mathbb{B}_R$, $\langle \nabla_x v \rangle = \nabla_x v(0)$ by the mean value theorem for L_a-harmonic functions, see Lemma 2.9 in [13]. If we use the expansion $v = \sum_{k=0}^{\infty} p_k(x,y)$ in $\mathbb{B}_R$ as in the proof of Lemma 3.3.1, then $\nabla_x v - \nabla_x v(0) = \sum_{k=2}^{\infty} \nabla_x p_k$ and consequently

$$\begin{aligned}\int_{\mathbb{B}_\rho} |\nabla_x v - \nabla_x v(0)|^2 |y|^a &= \sum_{k=2}^{\infty} \int_{\mathbb{B}_\rho} |\nabla_x p_k|^2 |y|^a = \sum_{k=2}^{\infty} \left(\frac{\rho}{R}\right)^{n+a+2k-1} \int_{\mathbb{B}_R} |\nabla_x p_k|^2 |y|^a \\ &\le \left(\frac{\rho}{R}\right)^{n+a+3} \sum_{k=2}^{\infty} \int_{\mathbb{B}_R} |\nabla_x p_k|^2 |y|^a \\ &= \left(\frac{\rho}{R}\right)^{n+a+3} \int_{\mathbb{B}_R} |\nabla_x v - \nabla_x v(0)|^2 |y|^a.\end{aligned}$$

□

3.3.2 Growth estimates for the solutions of the Signorini problem for L_a

Our estimates for the solutions of the Signorini problem will require an assumption that $1/2 \le s < 1$, or $a \le 0$. Also, unless stated otherwise, the obstacle ψ is assumed to be zero.

The first estimate is the analogue of Lemma 3.3.1, but with less information of the growth of v_y.

Lemma 3.3.3. *Let v be a solution of the Signorini problem for L_a in $\mathbb{B}_R$, even in y, with $a \le 0$. Then, for $0 < \rho < R$*

$$\int_{\mathbb{B}_\rho} |\nabla v|^2 |y|^a \le \left(\frac{\rho}{R}\right)^{n+1+a} \int_{\mathbb{B}_R} |\nabla v|^2 |y|^a. \tag{3.3.2}$$

Proof. We use the following property: if v is as in the statement of the lemma, then v_{x_i}, $i = 1, \dots, n$, and $y|y|^{a-1} v_y$ are Hölder continuous in $\mathbb{B}_R$, see [13]. Moreover, we have that

$$L_a(v_{x_i}^{\pm}) \ge 0, \quad L_{-a}((y|y|^{a-1} v_y)^{\pm}) \ge 0 \quad \text{in } \mathbb{B}_R.$$

This follows from the fact that $L_a v_{x_i} = 0$ in $\{\pm v_{x_i} > 0\}$ and $L_{-a}(y|y|^{a-1} v_y) = 0$ in $\{\pm y|y|^{a-1} v_y > 0\}$, by the complementarity condition $v_y v = 0$ on B_R, as well as an argument in Exercise 2.6 or Exercise 9.5 in [48]. As a consequence, we have

$$L_a(|\nabla_x v|^2) \ge 0, \quad L_{-a}(||y|^a v_y|^2) \ge 0 \quad \text{in } \mathbb{B}_R.$$

We next use the following $|y|^a$-weighted sub-mean value property for L_a-subharmonic functions: If $L_a w \ge 0$ weakly in $\mathbb{B}_R$, $-1 < a < 1$, then

$$\rho \mapsto \frac{1}{\rho^{n+1+a}} \int_{B_\rho} w|y|^a$$

is nondecreasing. This follows by integration from the spherical sub-mean value property, see Lemma 2.9 in [13]. Thus, we have that

$$\rho \mapsto \frac{1}{\rho^{n+1+a}} \int_{\mathbb{B}_\rho} |\nabla_x v|^2 |y|^a$$

$$\rho \mapsto \frac{1}{\rho^{n+1-a}} \int_{\mathbb{B}_\rho} |y|^a u_y^2$$

are monotone nondecreasing for $0 < \rho < R$. This implies

$$\int_{\mathbb{B}_\rho} |\nabla_x v|^2 |y|^a \leq \left(\frac{\rho}{R}\right)^{n+1+a} \int_{\mathbb{B}_R} |\nabla_x v|^2 |y|^a$$
$$\int_{\mathbb{B}_\rho} v_y^2 |y|^a \leq \left(\frac{\rho}{R}\right)^{n+1-a} \int_{\mathbb{B}_R} v_y^2 |y|^a.$$

In the case $a \leq 0$, we therefore conclude that the bound (3.3.2) holds. □

Lemma 3.3.4. *Let v be a solution of the Signorini problem for L_a in $\mathbb{B}_R$, even in y, with $a \leq 0$. If $v(0) = 0$, then there exists $C = C_{n,\alpha}$ such that for $0 < \rho < r < (3/4)R$,*

$$\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v \rangle_\rho|^2 |y|^a \leq \left(\frac{\rho}{r}\right)^{n+a+3} \int_{\mathbb{B}_r} |\nabla_x v - \langle \nabla_x v \rangle_r|^2 |y|^a + C\, \frac{\rho^{n+2}}{R^{2+2s}} \|v\|^2_{L^\infty(\mathbb{B}_R)}$$

Proof. Define

$$\varphi(r) := \frac{1}{r^{n+a+3}} \int_{\mathbb{B}_r} |\nabla_x v - \langle \nabla_x v \rangle_r|^2 |y|^a.$$

Then,

$$\begin{aligned}
\varphi(r) &= \frac{1}{r^{n+a+3}} \left[\int_{\mathbb{B}_r} |\nabla_x v|^2 |y|^a - 2\langle \nabla_x v \rangle_r \int_{\mathbb{B}_r} \nabla_x v |y|^a + \langle \nabla_x v \rangle_r^2 \int_{\mathbb{B}_r} |y|^a\right] \\
&= \frac{1}{r^{n+a+3}} \left[\int_{\mathbb{B}_r} |\nabla_x v|^2 |y|^a - \frac{1}{\omega_{n+1+a} r^{n+1+a}} \left(\int_{\mathbb{B}_r} \nabla_x v |y|^a\right)^2\right].
\end{aligned}$$

Thus, using the Cauchy-Schwarz and Young's inequality, we obtain

$$\begin{aligned}
\varphi'(r) = \frac{1}{r^{n+a+3}} \Bigg[&- \frac{n+a+3}{r} \int_{\mathbb{B}_r} |\nabla_x v|^2 |y|^a + \int_{\partial \mathbb{B}_r} |\nabla_x v|^2 |y|^a \\
&+ \frac{n+a+3}{\omega_{n+1+a} r^{n+2+a}} \left(\int_{\mathbb{B}_r} \nabla_x v |y|^a\right)^2 + \frac{n+1+a}{\omega_{n+1+a} r^{n+2+a}} \left(\int_{\mathbb{B}_r} \nabla_x v |y|^a\right)^2 \\
&- \frac{2}{\omega_{n+1+a} r^{n+1+a}} \left(\int_{\mathbb{B}_r} \nabla_x v |y|^a\right) \left(\int_{\partial \mathbb{B}_r} \nabla_x v |y|^a\right) \Bigg] \\
\geq -\frac{C}{r^{n+a+3}} &\left[\frac{1}{r} \int_{\mathbb{B}_r} |\nabla_x v|^2 |y|^a + \left(\frac{1}{r} \int_{\mathbb{B}_r} |\nabla_x v|^2 |y|^a\right)^{1/2} \left(\int_{\partial \mathbb{B}_r} |\nabla_x v|^2 |y|^a\right)^{1/2}\right] \\
\geq -\frac{C}{r^{n+a+3}} &\left[\frac{1}{r} \int_{\mathbb{B}_r} |\nabla_x v|^2 |y|^a + \int_{\partial \mathbb{B}_r} |\nabla_x v|^2 |y|^a\right].
\end{aligned}$$

Next, we note that
$$[\nabla_x v]_{C^{0,s}\left(\mathbb{B}_{3/4R}\right)} \le \frac{C_{n,s}}{R^{1+s}}\|v\|_{L^\infty(\mathbb{B}_R)}.$$

Indeed, this follows from the known interior regularity for solutions of the Signorini problem for L_a in $\mathbb{B}_1$ in the case $R=1$, see e.g. [13], and a simple scaling argument for all $R>0$. Noting also that $\nabla_x v(0)=0$, since v attains its minimum on B_r at 0, we have that for $X\in\overline{\mathbb{B}_r}$ with $r<(3/4)R$

$$|\nabla_x v(X)| = |\nabla_x v(X) - \nabla_x v(0)| \le \frac{C}{R^{1+s}} r^s \|v\|_{L^\infty(\mathbb{B}_R)}$$

and so
$$\frac{1}{r}\int_{\mathbb{B}_r} |\nabla_x v|^2|y|^a + \int_{\partial\mathbb{B}_r} |\nabla_x v|^2|y|^a \le C\,\frac{r^{n+1}}{R^{2+2s}}\|v\|^2_{L^\infty(\mathbb{B}_R)}.$$

This gives
$$\varphi'(r) \ge -\frac{C}{r^{a+2}}\frac{1}{R^{2+2s}}\|v\|^2_{L^\infty(\mathbb{B}_R)}.$$

Thus, for $0<\rho<r<(3/4)R$,

$$\varphi(r)-\varphi(\rho) = \int_\rho^r \varphi'(t)\,dt \ge -C\,\frac{\rho^{-1-a}-r^{-1-a}}{R^{2+2s}}\|v\|^2_{L^\infty(\mathbb{B}_R)}.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2|y|^a &= \rho^{n+a+3}\varphi(\rho)\\
&\le \rho^{n+a+3}\left(\varphi(r) + C\,\frac{\rho^{-1-a}-r^{-1-a}}{R^{2+2s}}\|v\|^2_{L^\infty(\mathbb{B}_R)}\right)\\
&\le \left(\frac{\rho}{r}\right)^{n+a+3}\int_{\mathbb{B}_r} |\nabla_x v - \langle \nabla_x v\rangle_r|^2|y|^a + C\,\frac{\rho^{n+2}}{R^{2+2s}}\|v\|^2_{L^\infty(\mathbb{B}_R)}.
\end{aligned}$$

□

Lemma 3.3.5. *Let v be a solution of the Signorini problem for L_a in $\mathbb{B}_R$, even in y. Then there are $C_1=C_{n,a}$, $C_2=C_{n,a}$ such that for all $0<\rho<S<(3/8)R$,*

$$\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2|y|^a \le C_1\left(\frac{\rho}{S}\right)^{n+a+3}\int_{\mathbb{B}_S} |\nabla_x v - \langle \nabla_x v\rangle_S|^2|y|^a + C_2\,\frac{S^{n+2}}{R^{2+2s}}\|v\|^2_{L^\infty(\mathbb{B}_R)}.$$

Proof. If $\rho \geq S/8$, then we immediately have

$$\begin{aligned}
\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v \rangle_\rho|^2 |y|^a &\leq C \left(\frac{8\rho}{S}\right)^{n+a+3} \int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v \rangle_\rho|^2 |y|^a \\
&\leq C \left(\frac{\rho}{S}\right)^{n+a+3} \int_{\mathbb{B}_S} |\nabla_x v - \langle \nabla_x v \rangle_S|^2 |y|^a.
\end{aligned}$$

Thus we may assume $\rho < S/8$. Due to Lemma 3.3.4, we may assume $v(0) > 0$. Let $d := \mathrm{dist}\,(0, \{v(\cdot, 0) = 0\}) > 0$. Then $L_a v = 0$ in $\mathbb{B}_d$. Thus, if $d \geq S$, we may use Lemma 3.3.2 to obtain

$$\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v \rangle_\rho|^2 |y|^a \leq \left(\frac{\rho}{S}\right)^{n+a+3} \int_{\mathbb{B}_S} |\nabla_x v - \langle \nabla_x v \rangle_S|^2 |y|^a.$$

Thus we may also assume $d < S$.

Case 1. $S/4 \leq d\,(< S)$.

Case 1.1. Suppose $0 < \rho < d\,(< S)$. Then using $L_a(\nabla_x v) = 0$ in $\mathbb{B}_d$ again,

$$\begin{aligned}
\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v \rangle_\rho|^2 |y|^a &\leq \left(\frac{\rho}{d}\right)^{n+a+3} \int_{\mathbb{B}_d} |\nabla_x v - \langle \nabla_x v \rangle_d|^2 |y|^a \\
&\leq C \left(\frac{\rho}{S}\right)^{n+a+3} \int_{\mathbb{B}_S} |\nabla_x v - \langle \nabla_x v \rangle_S|^2 |y|^a.
\end{aligned}$$

Case 1.2. Suppose $\rho \geq d\,(\geq S/4)$. Then

$$\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v \rangle_\rho|^2 |y|^a \leq \left(\frac{4\rho}{S}\right)^{n+a+3} \int_{\mathbb{B}_S} |\nabla_x v - \langle \nabla_x v \rangle_S|^2 |y|^a.$$

Case 2. $0 < d < S/4$.

Case 2.1. Suppose $\rho < d/2$. Take $x_1 \in \partial(B_d)$ such that $v(x_1) = 0$. Then using inclusions $\mathbb{B}_\rho \subset \mathbb{B}_{d/2} \subset \mathbb{B}_{(3/2)d}(x_1) \subset \mathbb{B}_{S/2}(x_1) \subset \mathbb{B}_{R/2}(x_1)$, $L_a v = 0$ in $\mathbb{B}_d$ and the preceding Lemma 3.3.4, we obtain

$$\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v \rangle_\rho|^2 |y|^a$$

$$
\begin{aligned}
&\le \left(\frac{2\rho}{d}\right)^{n+a+3} \int_{\mathbb{B}_{d/2}} |\nabla_x v - \langle \nabla_x v\rangle_{d/2}|^2 |y|^a \\
&\le \left(\frac{2\rho}{d}\right)^{n+a+3} \int_{\mathbb{B}_{(3/2)d}(x_1)} |\nabla_x v - \langle \nabla_x v\rangle_{x_1,(3/2)d}|^2 |y|^a \\
&\le \left(\frac{2\rho}{d}\right)^{n+a+3} \left[\left(\frac{3d}{S}\right)^{n+a+3} \int_{\mathbb{B}_{S/2}(x_1)} |\nabla_x v - \langle \nabla_x v\rangle_{x_1,S/2}|^s |y|^a + C\,\frac{S^{n+2}}{R^{2+2s}}\|v\|^2_{L^\infty(\mathbb{B}_{R/2}(x_1))}\right] \\
&\le C\left(\frac{\rho}{S}\right)^{n+a+3} \int_{\mathbb{B}_S} |\nabla_x v - \langle \nabla_x v\rangle_S|^2 |y|^a + C\,\frac{S^{n+2}}{R^{2+2s}}\|v\|^2_{L^\infty(\mathbb{B}_R)}.
\end{aligned}
$$

Case 2.2. Suppose $d/2 \le \rho$. Then we see that $\mathbb{B}_\rho \subset \mathbb{B}_{3\rho}(x_1) \subset \mathbb{B}_{S/2}(x_1) \subset \mathbb{B}_S$. As we did in Case 2.1, we have

$$
\begin{aligned}
&\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2 |y|^a \\
&\qquad \le \int_{\mathbb{B}_{3\rho}(x_1)} |\nabla_x v - \langle \nabla_x v\rangle_{x_1,3\rho}|^2 |y|^a \\
&\qquad \le C\left(\frac{\rho}{S}\right)^{n+a+3} \int_{\mathbb{B}_{S/2}(x_1)} |\nabla_x v - \langle \nabla_x v\rangle_{x_1,S/2}|^2 |y|^a + C\,\frac{S^{n+2}}{R^{2+2s}}\|v\|^2_{L^\infty(\mathbb{B}_{R/2}(x_1))} \\
&\qquad \le C\left(\frac{\rho}{S}\right)^{n+a+3} \int_{\mathbb{B}_S} |\nabla_x v - \langle \nabla_x v\rangle_S|^2 |y|^a + C\,\frac{S^{n+2}}{R^{2+2s}}\|v\|^2_{L^\infty(\mathbb{B}_R)}.
\end{aligned}
$$

□

Corollary 3.3.6. *Let v be a solution of the Signorini problem for L_a in $\mathbb{B}_R$, even in y. Then there are $C_1 = C_{n,a}$, $C_2 = C_{n,a}$ such that for all $0 < \rho < S < (3/16)R$,*

$$
\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2 |y|^a \le C_1\left(\frac{\rho}{S}\right)^{n+a+3} \int_{\mathbb{B}_S} |\nabla_x v - \langle \nabla_x v\rangle_S|^2 |y|^2 + C_2\,\frac{S^{n+2}}{R^{2+2s}}\langle v^2\rangle_R.
$$

Proof. Since $v^\pm = \max(\pm v, 0) \ge 0$ and $L_a(v^\pm) = 0$ in $\{v^\pm > 0\}$, we have $L_a(v^\pm) \ge 0$ in $\mathbb{B}_R$. (For this, one may follow the argument in Exercise 2.6 or Exercise 9.5 in [48].) Thus, we have by Theorem 2.3.1 in [62]

$$
\sup_{\mathbb{B}_{R/2}} v^\pm \le C\left(\frac{1}{\omega_{n+1+a}R^{n+1+a}} \int_{\mathbb{B}_R} \left(v^\pm\right)^2 |y|^a\right)^{1/2}.
$$

Hence,

$$
\|v\|^2_{L^\infty(\mathbb{B}_{R/2})} \le C\langle v^2\rangle_R,
$$

which completes the proof. $\square$

3.4 Almost s-fractional harmonic functions

In this section we prove Theorem J, by deducing growth estimates for almost s-fractional harmonic functions from that of s-fractional harmonic functions and then applying the Morrey-Campanato space embedding to deduce the regularity of almost s-fractional harmonic functions.

Theorem 3.4.1 (Almost Lipschitz regularity)**.** *If u is an almost s-fractional harmonic function in B_1, $0 < s < 1$, then $u \in C^{0,\sigma}(B_1)$ for any $0 < \sigma < 1$.*

Proof. Let K be a compact subset of B_1 containing 0. Take $\delta = \delta_{n,\omega,\sigma,K} > 0$ such that $\delta < \operatorname{dist}(K, \partial B_1)$ and $\omega(\delta) \le \varepsilon$, where $\varepsilon = \varepsilon_{2,n+1+a,n-1+a+2\sigma}$ is as Lemma 1.2.2. For $0 < R < \delta$, let v be a minimizer of

$$\int_{\mathbb{B}_R} |\nabla v|^2 |y|^a$$

on $u + W_0^{1,2}(\mathbb{B}_R, |y|^a)$. Then $L_a v = 0$ in $\mathbb{B}_R$. In particular,

$$\int_{\mathbb{B}_R} \nabla v \cdot \nabla(u - v) |y|^a = 0,$$

and hence

$$\begin{aligned} \int_{\mathbb{B}_R} |\nabla(u-v)|^2 |y|^a &= \int_{\mathbb{B}_R} |\nabla u|^2 |y|^a - \int_{\mathbb{B}_R} |\nabla v|^2 |y|^a - 2\int_{\mathbb{B}_R} \nabla v \cdot \nabla(u-v)|y|^a \\ &\le \omega(R) \int_{\mathbb{B}_R} |\nabla v|^2 |y|^a. \end{aligned}$$

Moreover, by Lemma 3.3.1, for $0 < \rho < R$ we have

$$\int_{\mathbb{B}_\rho} |\nabla v|^2 |y|^a \le \left(\frac{\rho}{R}\right)^{n+1+a} \int_{\mathbb{B}_R} |\nabla v|^2 |y|^a.$$

Thus

$$\int_{\mathbb{B}_\rho} |\nabla u|^2 |y|^a \le 2\int_{\mathbb{B}_\rho} |\nabla v|^2 |y|^a + 2\int_{\mathbb{B}_\rho} |\nabla(u-v)|^2 |y|^a$$

$$
\begin{aligned}
&\leq 2\left(\frac{\rho}{R}\right)^{n+1+a}\int_{\mathbb{B}_R}|\nabla v|^2|y|^a + 2\int_{\mathbb{B}_\rho}|\nabla(u-v)|^2|y|^a\\
&\leq 2\left(\frac{\rho}{R}\right)^{n+1+a}\int_{\mathbb{B}_R}|\nabla v|^2|y|^a + 2\omega(R)\int_{\mathbb{B}_R}|\nabla v|^2|y|^a\\
&\leq 2\left[\left(\frac{\rho}{R}\right)^{n+1+a}+\varepsilon\right]\int_{\mathbb{B}_R}|\nabla u|^2|y|^a.
\end{aligned}
$$

By Lemma 1.2.2,

$$
\int_{\mathbb{B}_\rho}|\nabla u|^2|y|^a \leq C_{n,a,\sigma}\left(\frac{\rho}{R}\right)^{n-1+a+2\sigma}\int_{\mathbb{B}_R}|\nabla u|^2|y|^a,
$$

for any $0<\sigma<1$. Taking $R \nearrow \delta$ we have

$$
\int_{\mathbb{B}_\rho}|\nabla u|^2|y|^a \leq C_{n,a,\sigma,\delta}\|\nabla u\|^2_{L^2(\mathbb{B}_1,|y|^a)}\rho^{n-1+a+2\sigma}. \tag{3.4.1}
$$

By weighted Poincaré inequality (Theorem 1.5 in [62])

$$
\int_{\mathbb{B}_\rho}|u-\langle u\rangle_\rho|^2|y|^a \leq C_{n,a,\sigma,\delta}\|\nabla u\|^2_{L^2(\mathbb{B}_1,|y|^a)}\rho^{n+1+a+2\sigma}.
$$

Now, a similar estimates holds at all point $x_0 \in K$, which implies the Hölder continuity of u (see Theorem 3.A.1) with

$$
\|u\|_{C^{0,\sigma}(K)} \leq C_{n,a,\omega,\sigma,K}\|u\|_{W^{1,2}(\mathbb{B}_1,|y|^a)}. \qquad \square
$$

Theorem 3.4.2 ($C^{1,\beta}$ regularity)**.** *If u is an almost s-fractional harmonic function in B_1, $0<s<1$, with gauge function $\omega(r)=r^\alpha$, $\alpha>0$, then $\nabla_x u \in C^{0,\beta}(B_1)$ for some $\beta = \beta(n,s,\alpha)$.*

Proof. Let $K \Subset B_1$ be a ball and take $0<\delta<\operatorname{dist}(K,\partial B_1)$. Let $B_R(x_0)\Subset B_1$ with $0<R<\delta$, for $x_0\in K$. For simplicity write $x_0=0$, and let v be the L_a-harmonic function in $\mathbb{B}_R$ with $v=u$ on $\partial\mathbb{B}_R$. Then, by Jensen's inequality we have

$$
\int_{\mathbb{B}_\rho}|\langle\nabla_x u\rangle_\rho - \langle\nabla_x v\rangle_\rho|^2|y|^a \leq \int_{\mathbb{B}_\rho}|\nabla_x u-\nabla_x v|^2|y|^a,
$$

and hence

$$\begin{aligned}\int_{\mathbb{B}_\rho} |\nabla_x u - \langle \nabla_x u\rangle_\rho|^2 |y|^a &\le 3\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2 |y|^a + 3\int_{\mathbb{B}_\rho} |\nabla_x u - \nabla_x v|^2 |y|^a \\ &\quad + 3\int_{\mathbb{B}_\rho} |\langle \nabla_x u\rangle_\rho - \langle \nabla_x v\rangle_\rho|^2 |y|^a \\ &\le 3\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2 |y|^a + 6\int_{\mathbb{B}_\rho} |\nabla_x u - \nabla_x v|^2 |y|^a.\end{aligned}$$

Similarly,

$$\int_{\mathbb{B}_R} |\nabla_x v - \langle \nabla_x v\rangle_R|^2 |y|^a \le 3\int_{\mathbb{B}_R} |\nabla_x u - \langle \nabla_x u\rangle_R|^2 |y|^a + 6\int_{\mathbb{B}_R} |\nabla_x u - \nabla_x v|^2 |y|^a.$$

Next let $\beta \in (0, \alpha/2)$. Then using the estimate (3.4.1) in the proof of Theorem 3.4.1 with $\sigma = 1 + \beta - \frac{\alpha}{2}$, we have

$$\begin{aligned}\int_{\mathbb{B}_R} |\nabla u - \nabla v|^2 |y|^a = \int_{\mathbb{B}_R} |\nabla u|^2 |y|^a - \int_{\mathbb{B}_R} |\nabla v|^2 |y|^a \le R^\alpha \int_{\mathbb{B}_R} |\nabla u|^2 |y|^a \\ \le C\|\nabla u\|^2_{L^2(\mathbb{B}_1, |y|^a)} R^{n+1+a+2\beta}.\end{aligned}$$

Then, with the help of Lemma 3.3.2, we have that for $\rho < R$

$$\begin{aligned}&\int_{\mathbb{B}_\rho} |\nabla_x u - \langle \nabla_x u\rangle_\rho|^2 |y|^a \\ &\quad\le C\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2 |y|^a + C\int_{\mathbb{B}_\rho} |\nabla_x u - \nabla_x v|^2 |y|^a \\ &\quad\le C\left(\frac{\rho}{R}\right)^{n+a+3} \int_{\mathbb{B}_R} |\nabla_x v - \langle \nabla_x v\rangle_R|^2 |y|^a + C\int_{\mathbb{B}_\rho} |\nabla_x u - \nabla_x v|^2 |y|^a \\ &\quad\le C\left(\frac{\rho}{R}\right)^{n+a+3} \int_{\mathbb{B}_R} |\nabla_x u - \langle \nabla_x u\rangle_R|^2 |y|^a + C\int_{\mathbb{B}_R} |\nabla_x u - \nabla_x v|^2 |y|^a \\ &\quad\le C\left(\frac{\rho}{R}\right)^{n+a+3} \int_{\mathbb{B}_R} |\nabla_x u - \langle \nabla_x u\rangle_R|^2 |y|^a + C\|\nabla u\|^2_{L^2(\mathbb{B}_1, |y|^a)} R^{n+1+a+2\beta}.\end{aligned}$$

Hence, by Lemma 1.2.2, we obtain that for $\rho < R$

$$\int_{\mathbb{B}_\rho} |\nabla_x u - \langle \nabla_x u\rangle_\rho|^2 |y|^a$$

$$\leq C\left[\left(\frac{\rho}{R}\right)^{n+1+a+2\beta}\int_{\mathbb{B}_R}|\nabla_x u-\langle\nabla_x u\rangle_R|^2|y|^a+\|\nabla u\|^2_{L^2(\mathbb{B}_1,|y|^a)}\rho^{n+1+a+2\beta}\right].$$

Taking $R\nearrow\delta$, we have

$$\int_{\mathbb{B}_\rho}|\nabla_x u-\langle\nabla_x u\rangle_\rho|^2|y|^a\leq C_{n,a,\alpha,\beta,K}\|\nabla u\|^2_{L^2(\mathbb{B}_1,|y|^a)}\rho^{n+1+a+2\beta}.$$

Now, a similar estimate holds for any $x_0\in K$. Fixing β and applying Theorem 3.A.1, we have

$$\|\nabla_x u\|_{C^{0,\beta}(K)}\leq C_{n,a,\alpha,K}\|u\|_{W^{1,2}(\mathbb{B}_1,|y|^a)}.$$

□

Remark 3.4.3. From the assumption for almost minimizers that the Caffarelli-Silvestre extension $u\in W^{1,2}_{\mathrm{loc}}$ we know only that $\nabla_x u\in L^2_{\mathrm{loc}}$, which is not sufficient to deduce the existence of the trace of $\nabla_x u$ on B_1. However, in the proof of Theorem 3.4.2 we showed that $\nabla_x u$ is in a Morrey-Campanato space, which implies the existence of the trace as the limit of averages

$$T(\nabla_x u)(x_0)=\lim_{r\to 0+}\langle\nabla_x u\rangle_{x_0,r}.$$

It is not hard to see that $T(\nabla_x u)$ is the distributional derivative $\nabla_x u$ on B_1. Indeed, if $\eta\in C^\infty_0(B_1)$, then extending it to $\mathbb{R}^{n+1}$ by $\eta(x,y)=\eta(x)$, we have

$$\begin{aligned}\int_{B_1}T(\partial_{x_i}u)\eta&=\lim_{r\to 0+}\int_{B_1}\langle\partial_{x_i}u\rangle_{x,r}\eta=\lim_{r\to 0+}\int_{B_1}\partial_{x_i}u\langle\eta\rangle_{x,r}\\&=\lim_{r\to 0+}-\int_{B_1}u\langle\partial_{x_i}\eta\rangle_{x,r}=-\int_{B_1}u\partial_{x_i}\eta.\end{aligned}$$

Theorem 3.4.4. *Let u be an almost s-fractional harmonic function in B_1 for $0<s<1/2$ or $s=1/2$ and a gauge function $\omega(r)=r^\alpha$ for some $\alpha>0$. Then u is actually s-fractional harmonic in B_1.*

Proof. We argue as in the proof Theorem 3.4.1. Let K, δ, R, v be as in the proof of that theorem. Then, by Lemma 3.3.1, for $0<\rho<R$

$$\int_{\mathbb{B}_\rho}|v_y|^2|y|^a\leq\left(\frac{\rho}{R}\right)^{n+3+a}\int_{\mathbb{B}_R}|v_y|^2|y|^a.$$

Thus, for any $0 < \sigma < 1$, we have

$$\begin{aligned}
\int_{\mathbb{B}_\rho} ||y|^a u_y|^2 |y|^{-a} &\le 2\int_{\mathbb{B}_\rho} |v_y|^2|y|^a + 2\int_{\mathbb{B}_\rho} |u_y - v_y|^2|y|^a \\
&\le 2\left(\frac{\rho}{R}\right)^{n+3+a} \int_{\mathbb{B}_R} |v_y|^2|y|^a + 2\int_{\mathbb{B}_\rho} |u_y - v_y|^2|y|^a \\
&\le 4\left(\frac{\rho}{R}\right)^{n+3+a} \int_{\mathbb{B}_R} |u_y|^2|y|^a + 6\int_{\mathbb{B}_R} |u_y - v_y|^2|y|^a \\
&\le 4\left(\frac{\rho}{R}\right)^{n+3+a} \int_{\mathbb{B}_R} ||y|^a u_y|^2|y|^{-a} + 6\omega(R)\int_{\mathbb{B}_R} |\nabla u|^2|y|^a \\
&\le 4\left(\frac{\rho}{R}\right)^{n+3+a} \int_{\mathbb{B}_R} ||y|^a u_y|^2|y|^{-a} + C_{n,a,\sigma,\delta}\,\omega(R)\|\nabla u\|^2_{L^2(\mathbb{B}_1,|y|^a)} R^{n-1+a+2\sigma},
\end{aligned}$$

where we used (3.4.1) in the last inequality.

Consider now the two cases in statement of the theorem.

Case 1. $0 < s < 1/2$ (or $a > 0$). In this case by Lemma 1.2.2,

$$\begin{aligned}
&\int_{\mathbb{B}_\rho} ||y|^a u_y|^2|y|^{-a} \\
&\quad\le C\left[\left(\frac{\rho}{R}\right)^{n-1+a+2\sigma}\int_{\mathbb{B}_R} ||y|^a u_y|^2|y|^{-a} + \omega(\delta)\|\nabla u\|^2_{L^2(\mathbb{B}_1,|y|^a)}\rho^{n-1+a+2\sigma}\right] \\
&\quad\le C\|\nabla u\|^2_{L^2(\mathbb{B}_1,|y|^a)}\rho^{n+1-a+(-2+2a+2\sigma)}.
\end{aligned}$$

Now we take $\sigma = 1 - a/2 \in (0,1)$ to have $-2+2a+2\sigma = a > 0$. Varying the center, we have a similar bound at every $x \in K$. Then, by Theorem 3.A.1, we obtain that the limit of the averages $T(y|y|^{a-1}u_y) = 0$ on B_1. This implies that $(-\Delta_x)^s u = 0$ on B_1. Indeed, arguing as in Remark 3.4.3, by considering the mollifications u_ε in x-variable, we note that

$$\int_{\mathbb{B}_\rho} ||y|^a (u_\varepsilon)_y|^2|y|^{-a} \le C\rho^{n+1-a+a}$$

which implies that $T(y|y|^{a-1}(u_\varepsilon)_y) = 0$ on $K \Subset B_1$. On the other hand, $u_\varepsilon \in C^2 \cap \mathcal{L}_s(\mathbb{R}^n)$, which implies that $y|y|^{a-1}(u_\varepsilon)_y$ is continuous up to $y = 0$, since we can explicitly write, for $y > 0$, the symmetrized formula

$$y^a(u_\varepsilon)_y(x,y) = \int_{\mathbb{R}^n} \frac{u_\varepsilon(x+z) + u_\varepsilon(x-z) - 2u_\varepsilon(x)}{|z|^2}|z|^2 y^a \partial_y P(z,y)dz$$

with locally integrable kernel $|z|^2|y^a\partial_y P(z,y)| \le C/|z|^{n-1-a}$. Hence, we obtain that $(-\Delta_x)^s u_\varepsilon = \partial_y^a u_\varepsilon = 0$ on the ball $K \Subset B_1$. Then, passing to the limit as $\varepsilon \to 0$, this implies that $(-\Delta_x)^s u = 0$ in B_1.

Case 2. $s = 1/2$ (or $a = 0$) and $\omega(r) = r^\alpha$. In this case, we have a bound

$$\int_{\mathbb{B}_\rho} |u_y|^2 \le 4\left(\frac{\rho}{R}\right)^{n+3} \int_{\mathbb{B}_R} |u_y|^2 + C\|\nabla u\|^2_{L^2(\mathbb{B}_1)} R^{n-1+2\sigma+\alpha}.$$

Then, by Lemma 1.2.2, we have

$$\begin{aligned}\int_{\mathbb{B}_\rho} |u_y|^2 &\le C\left[\left(\frac{\rho}{R}\right)^{n-1+2\sigma+\alpha} \int_{\mathbb{B}_R} |u_y|^2 + \|\nabla u\|^2_{L^2(\mathbb{B}_1)}\rho^{n-1+2\sigma+\alpha}\right] \\ &\le C\|\nabla u\|^2_{L^2(\mathbb{B}_1)}\rho^{n+1+(\alpha-2+2\sigma)}.\end{aligned}$$

Taking $1-\alpha/4 < \sigma < 1$, we can guarantee that $\alpha - 2 + 2\sigma > \alpha/2 > 0$, which implies that $T(y|y|^{-1}u_y) = 0$ on B_1. Then, arguing as at the end of Case 1, we conclude that $(-\Delta_x)^{1/2}u = 0$ in B_1. □

We finish this section with formal proof of Theorem J.

Proof of Theorem J. Parts (i), (ii), and (iii) are proved in Theorems 3.4.1, 3.4.2, and 3.4.4, respectively. □

3.5 Almost minimizers for s-fractional obstacle problem

In this section we investigate the regularity of almost minimizers for the s-fractional obstacle problem with zero obstacle and give a proof of Theorem K. All results in this section are proved under the assumption $1/2 \le s < 1$, or $-1 < a \le 0$.

Theorem 3.5.1 (Almost Lipschitz regularity)**.** *Let u be an almost minimizer for s-fractional obstacle problem with zero obstacle in B_1, for $1/2 \le s < 1$. Then $u \in C^{0,\sigma}(B_1)$ for any $0 < \sigma < 1$ with*

$$\|u\|_{C^{0,\sigma}(K)} \le C_{n,a,\omega,\sigma,K}\|u\|_{W^{1,2}(\mathbb{B}_1,|y|^a)}$$

for any $K \Subset B_1$.

Proof. Let $K \Subset B_1$ with $0 \in K$. Take $\delta = \delta_{n,a,\omega,\sigma,K} > 0$ such that $\delta < \operatorname{dist}(K, \partial B_1)$ and $\omega(\delta) \le \varepsilon$, where $\varepsilon = \varepsilon_{2,n+1+a,n-1+a+2\sigma}$ as in Lemma 1.2.2. For $0 < R < \delta$, let v be the minimizer of

$$\int_{\mathbb{B}_R} |\nabla v|^2 |y|^a$$

on $\mathfrak{K}_{0,u}(\mathbb{B}_R, |y|^a)$. Then v satisfies the variational inequality

$$\int_{\mathbb{B}_R} \nabla v \nabla (w - v) |y|^a \ge 0$$

for any $w \in \mathfrak{K}_{0,u}(\mathbb{B}_R, |y|^a)$. Particularly, taking $w = u$, we have

$$\int_{\mathbb{B}_R} \nabla v \nabla (u - v) |y|^a \ge 0.$$

As a consequence,

$$\begin{aligned} \int_{\mathbb{B}_R} |\nabla (u-v)|^2 |y|^a &= \int_{\mathbb{B}_R} |\nabla u|^2 |y|^a - \int_{\mathbb{B}_R} |\nabla v|^2 |y|^a - 2 \int_{\mathbb{B}_R} \nabla v \cdot \nabla (u-v) |y|^a \\ &\le \omega(R) \int_{\mathbb{B}_R} |\nabla v|^2 |y|^a. \end{aligned}$$

Next, we use (3.3.2) to derive a similar estimate for u. We have,

$$\begin{aligned} \int_{\mathbb{B}_\rho} |\nabla u|^2 |y|^a &\le 2 \int_{\mathbb{B}_\rho} |\nabla v|^2 |y|^a + 2 \int_{\mathbb{B}_\rho} |\nabla (u-v)|^2 |y|^a \\ &\le 2 \left(\frac{\rho}{R}\right)^{n+1+a} \int_{\mathbb{B}_R} |\nabla v|^2 |y|^a + 2\omega(R) \int_{\mathbb{B}_R} |\nabla v|^2 |y|^a \\ &\le 2 \left[\left(\frac{\rho}{R}\right)^{n+1+a} + \varepsilon \right] \int_{\mathbb{B}_R} |\nabla u|^2 |y|^a. \end{aligned}$$

Hence, by Lemma 1.2.2,

$$\int_{\mathbb{B}_\rho} |\nabla u|^2 |y|^a \le C_{n,a,\sigma} \left(\frac{\rho}{R}\right)^{n-1+a+2\sigma} \int_{\mathbb{B}_R} |\nabla u|^2 |y|^a.$$

As we have seen in Theorem 3.4.1, this implies

$$\int_{\mathbb{B}_\rho} |\nabla u|^2 |y|^a \le C_{n,a,\sigma,\delta} \|\nabla u\|^2_{L^2(\mathbb{B}_1,|y|^a)} \rho^{n-1+a+2\sigma} \tag{3.5.1}$$

then

$$\int_{\mathbb{B}_\rho} |u - \langle u \rangle_\rho|^2 |y|^a \le C_{n,a,\sigma,\delta} \|\nabla u\|^2_{L^2(\mathbb{B}_1,|y|^a)} \rho^{n+1+a+2\sigma}$$

and ultimately

$$\|u\|_{C^{0,\sigma}(K)} \le C_{n,a,\omega,\sigma,K} \|u\|_{W^{1,2}(\mathbb{B}_1,|y|^a)}.$$

□

Theorem 3.5.2 ($C^{1,\beta}$ regularity)**.** *Let u be an almost minimizer for the s-fractional obstacle problem with zero obstacle in B_1, $1/2 \le s < 1$, and a gauge function $\omega(r) = r^\alpha$. Then $\nabla_x u \in C^{0,\beta}(B_1)$ for $\beta < \frac{\alpha s}{8(n+1+a+\alpha/2)}$ and for any $K \Subset B_1$ there holds*

$$\|\nabla_x u\|_{C^{0,\beta}(K)} \le C_{n,a,\alpha,\beta,K} \|u\|_{W^{1,2}(\mathbb{B}_1,|y|^a)}.$$

Proof. Let K be a thin ball centered at 0 such that $K \Subset \mathbb{B}_1$. Let $\varepsilon := \frac{\alpha}{4(n+1+a+\alpha/2)}$ and $\gamma := 1 - \frac{s\varepsilon}{2(1-\varepsilon)}$. We fix $R_0 = R_0(n,a,\alpha,K) > 0$ small so that $R_0^{1-\varepsilon} \le d/2$, where $d := \operatorname{dist}(K, \partial B_1)$ and $R_0 < \left(\frac{3}{16}\right)^{1/\varepsilon}$. Then $\widetilde{K} := \{x \in B_1 : \operatorname{dist}(x,K) \le R_0^{1-\varepsilon}\} \Subset B_1$. We claim that for $x_0 \in K$ and $0 < \rho < R < R_0$,

$$\begin{aligned}\int_{\mathbb{B}_\rho(x_0)} |\nabla_x u - \langle \nabla_x u \rangle_{x_0,\rho}|^2 |y|^a \le C_{n,a} \left(\frac{\rho}{R}\right)^{n+a+3} \int_{\mathbb{B}_R(x_0)} |\nabla_x u - \langle \nabla_x u \rangle_{x_0,R}|^2 |y|^a \\ + C_{n,a,\alpha,K} \|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)} R^{n+1+a+s\varepsilon}.\end{aligned} \tag{3.5.2}$$

Note that once we have this bound, the proof will follow by the application of Lemma 1.2.2 and Theorem 3.A.1.

For simplicity we may assume $x_0 = 0$, and fix $0 < R < R_0$. Let $\overline{R} := R^{1-\varepsilon}$. Let v be the minimizer of

$$\int_{\mathbb{B}_{\overline{R}}} |\nabla v|^2 |y|^a$$

on $\mathfrak{K}_{0,u}(\mathbb{B}_{\overline{R}}, |y|^a)$. Then by (3.3.2) and (3.5.1) with $\sigma = \gamma$, for $0 < \rho \le \overline{R}$

$$\begin{aligned}\int_{\mathbb{B}_\rho} |\nabla v|^2 |y|^a &\le \left(\frac{\rho}{\overline{R}}\right)^{n+1+a} \int_{\mathbb{B}_{\overline{R}}} |\nabla v|^2 |y|^a \le \left(\frac{\rho}{\overline{R}}\right)^{n+1+a} \int_{\mathbb{B}_{\overline{R}}} |\nabla u|^2 |y|^a \\ &\le C_{n,a,\alpha,K} \left(\frac{\rho}{\overline{R}}\right)^{n+1+a} \|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)} \overline{R}^{n-1+a+2\gamma} \\ &\le C_{n,a,\alpha,K} \|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)} \rho^{n-1+a+2\gamma}.\end{aligned} \tag{3.5.3}$$

This gives

$$⨍_{\mathbb{B}_\rho} |v - v_\rho|^2 |y|^a \le C_1 \|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)} \rho^{2\gamma}, \quad C_1 = C_{n,a,\alpha,K}. \tag{3.5.4}$$

Since this estimate holds for any $0 < \rho < \overline{R}$, the standard dyadic argument gives

$$|v(0) - \langle v \rangle_{\overline{R}}| \le C_2 \|u\|_{W^{1,2}(\mathbb{B}_1,|y|^a)} \overline{R}^\gamma, \quad C_2 = C_{n,a,\alpha,K}. \tag{3.5.5}$$

Moreover, using (3.3.2) and (3.5.1) again, we have for any $x_1 \in B_{\overline{R}/2}$, $0 < \rho < \overline{R}/2$,

$$\begin{aligned}\int_{\mathbb{B}_\rho(x_1)} |\nabla v|^2 |y|^a &\le \left(\frac{2\rho}{\overline{R}}\right)^{n+1+a} \int_{\mathbb{B}_{\overline{R}/2}(x_1)} |\nabla v|^2 |y|^a \le \left(\frac{2\rho}{\overline{R}}\right)^{n+1+a} \int_{\mathbb{B}_{\overline{R}}} |\nabla u|^2 |y|^a \\ &\le C_{n,a,\alpha,K} \|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)} \rho^{n-1+a+2\gamma},\end{aligned} \tag{3.5.6}$$

which implies

$$[v]_{C^{0,\gamma}(\overline{B_{R/2}})} \le C_3 \|u\|_{W^{1,2}(\mathbb{B}_1,|y|^a)}, \quad C_3 = C_{n,a,\alpha,K}. \tag{3.5.7}$$

Now we define

$$C_4 := C_1 + C_2^2 + C_3^2.$$

Our analysis then distinguishes the following two cases

$$\langle v^2 \rangle_{\overline{R}} \le 6\, C_4 \|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)} \overline{R}^{2\gamma} \qquad \text{or} \qquad \langle v^2 \rangle_{\overline{R}} > 6\, C_4 \|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)} \overline{R}^{2\gamma}.$$

Case 1. Suppose first that

$$\langle v^2 \rangle_{\overline{R}} \le 6 C_4 \|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)} \overline{R}^{2\gamma}.$$

Note that $R_0 < \left(\frac{3}{16}\right)^{1/\varepsilon}$ implies $R < \frac{3}{16}\overline{R}$. Then, using Corollary 3.3.6, we see that for $0 < \rho < R$,

$$\begin{aligned}
\int_{\mathbb{B}_\rho} |\nabla_x u - \langle \nabla_x u\rangle_\rho|^2 |y|^a &\le 3\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2 |y|^a + 6\int_{\mathbb{B}_\rho} |\nabla_x u - \nabla_x v|^2 |y|^a\, dx \\
&\le C_{n,a}\left(\frac{\rho}{R}\right)^{n+a+3} \int_{\mathbb{B}_R} |\nabla_x v - \langle \nabla_x v\rangle_R|^2 |y|^a \\
&\qquad + C_{n,a}\,\frac{R^{n+2}}{\overline{R}^{2+2s}}\langle v^2\rangle_{\overline{R}} + 6\int_{\mathbb{B}_\rho} |\nabla_x u - \nabla_x v|^2 |y|^a \\
&\le C\left(\frac{\rho}{R}\right)^{n+a+3} \int_{\mathbb{B}_R} |\nabla_x u - \langle \nabla_x u\rangle_R|^2 |y|^a \\
&\qquad + C\,\frac{R^{n+2}}{\overline{R}^{2+2s}}\langle v^2\rangle_{\overline{R}} + C\int_{\mathbb{B}_R} |\nabla_x u - \nabla_x v|^2 |y|^a.
\end{aligned}$$

Note that for $\sigma := 1 - \alpha/4$

$$\begin{aligned}
\int_{\mathbb{B}_R} |\nabla_x u - \nabla_x v|^2 |y|^a &\le \int_{\mathbb{B}_{\overline{R}}} |\nabla_x u - \nabla_x v|^2 |y|^a \le \overline{R}^\alpha \int_{\mathbb{B}_{\overline{R}}} |\nabla v|^2 |y|^a \\
&\le \overline{R}^\alpha \int_{\mathbb{B}_{\overline{R}}} |\nabla u|^2 |y|^a \le C_{n,a,\alpha,K}\overline{R}^\alpha \|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)}\overline{R}^{n-1+a+2\sigma} \\
&= C\|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)} R^{n+1+a+\alpha/4}.
\end{aligned}$$

Moreover by the assumption

$$C\,\frac{R^{n+2}}{\overline{R}^{2+2s}}\langle v^2\rangle_{\overline{R}} \le C_{n,a,\alpha,K}\|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)} R^{n+2}\overline{R}^{2\gamma-2-2s} = C\|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)} R^{n+1+a+s\varepsilon}.$$

Hence, we obtain (3.5.2) in this case.

Case 2. Now we assume

$$\langle v^2\rangle_{\overline{R}} > 6\, C_4 \|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)}\overline{R}^{2\gamma}.$$

Then, by (3.5.4) and (3.5.5) we obtain

$$⨍_{\mathbb{B}_{\overline{R}}} |v - v(0)|^2 |y|^a \le 2⨍_{\mathbb{B}_{\overline{R}}} |v - v_{\overline{R}}|^2 |y|^a + 2⨍_{\mathbb{B}_{\overline{R}}} |v_{\overline{R}} - v(0)|^2 |y|^a \le 2C_4\|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)}\overline{R}^{2\gamma}.$$

Combining the latter bound and the assumption,

$$
\begin{aligned}
v(0)^2 &= \fint_{\mathbb{B}_{\overline{R}}} |v(0)|^2|y|^a \geq \frac{1}{2}\fint_{\mathbb{B}_{\overline{R}}} |v(X)|^2|y|^a - \fint_{\mathbb{B}_{\overline{R}}} |v(X)-v(0)|^2|y|^a \\
&\geq C_4\|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)}\overline{R}^{2\gamma}.
\end{aligned}
$$

Since $C_4 \geq C_3^2$, we have $v > 0$ on $B_{\overline{R}/2}$ by (3.5.7). Thus, $L_a v = 0$ in $\mathbb{B}_{\overline{R}/2}$, and by Lemma 3.3.2 we have for $0 < \rho < R$

$$
\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2|y|^a \leq \left(\frac{\rho}{R}\right)^{n+a+3}\int_{\mathbb{B}_R} |\nabla_x v - \langle \nabla_x v\rangle_R|^2|y|^a.
$$

Thus,

$$
\begin{aligned}
&\int_{\mathbb{B}_\rho} |\nabla_x u - \langle \nabla_x u\rangle_\rho|^2|y|^a \\
&\quad\leq 3\int_{\mathbb{B}_\rho} |\nabla_x v - \langle \nabla_x v\rangle_\rho|^2|y|^a + 6\int_{\mathbb{B}_\rho} |\nabla_x u - \nabla_x v|^2|y|^a \\
&\quad\leq 3\left(\frac{\rho}{R}\right)^{n+a+3}\int_{\mathbb{B}_R} |\nabla_x v - \langle \nabla_x v\rangle_R|^2|y|^a + 6\int_{\mathbb{B}_\rho} |\nabla_x u - \nabla_x v|^2|y|^a \\
&\quad\leq C\left(\frac{\rho}{R}\right)^{n+a+3}\int_{\mathbb{B}_R} |\nabla_x u - \langle \nabla_x u\rangle_R|^2|y|^a + C\int_{\mathbb{B}_R} |\nabla_x u - \nabla_x v|^2|y|^a \\
&\quad\leq C\left(\frac{\rho}{R}\right)^{n+a+3}\int_{\mathbb{B}_R} |\nabla_x u - \langle \nabla_x u\rangle_R|^2|y|^a + C\|u\|^2_{W^{1,2}(\mathbb{B}_1,|y|^a)}R^{n+1+a+\alpha/4}.
\end{aligned}
$$

This implies (3.5.2) and completes the proof. □

Proof of Theorem K. Parts (i) and (ii) are contained in Theorems 3.5.1 and 3.5.2, respectively. □

3.A Morrey-Campanato-type Space

Theorem 3.A.1. *Let $u \in L^2(\mathbb{B}_1, |y|^a)$ and M be such that $\|u\|_{L^2(\mathbb{B}_1,|y|^a)} \leq M$ and for some $\sigma \in (0,1)$*

$$
\int_{\mathbb{B}_r(x)} |u - \langle u\rangle_{x,r}|^2|y|^a \leq M^2 r^{n+1+a+2\sigma}, \quad \langle u\rangle_{x,r} = \frac{1}{\omega_{n+1+a}r^{n+1+a}}\int_{\mathbb{B}_r(x)} u\,|y|^a
$$

for any ball $\mathbb{B}_r(x)$ *centered at* $x = (x, 0) \in B_{1/2}$ *and radius* $0 < r < r_0 \leq 1/2$. *Then for any* $x \in B_{1/2}$ *there exists the limit of averages*

$$Tu(x) := \lim_{r \to 0} \langle u \rangle_{x,r},$$

which will also satisfy

$$\int_{\mathbb{B}_r(x)} |u - Tu(x)|^2 |y|^a \leq C_{n,a,\sigma} M^2 r^{n+1+a+2\sigma}.$$

Moreover, $Tu \in C^{0,\sigma}(B_{1/2})$ *with*

$$\|Tu\|_{C^{0,\sigma}(B_{1/2})} \leq C_{n,a,\sigma,r_0} M.$$

Remark 3.A.2. Note, we can redefine $u(x,0) = Tu(x)$ for any $x \in B_{1/2}$, making $(x, 0)$ a Lebesgue point for u.

Proof. Let $x, z \in B_{1/2}$ and $0 < \rho < r < r_0$ be such that $\mathbb{B}_\rho(x) \subset \mathbb{B}_r(z)$. Then

$$\begin{aligned}
|\langle u \rangle_{x,\rho} - \langle u \rangle_{z,r}| &\leq \fint_{\mathbb{B}_\rho(x)} |u - \langle u \rangle_{z,r}||y|^a \leq \left(\frac{r}{\rho}\right)^{n+1+a} \fint_{\mathbb{B}_r(z)} |u - \langle u \rangle_{z,r}||y|^a \\
&\leq \left(\frac{r}{\rho}\right)^{n+1+a} \left(\fint_{\mathbb{B}_r(z)} |u - \langle u \rangle_{z,r}|^2 |y|^a\right)^{1/2} \left(\fint_{\mathbb{B}_r(z)} |y|^a\right)^{1/2} \\
&\leq C_{n,a} \left(\frac{r}{\rho}\right)^{n+1+a} M r^\sigma.
\end{aligned}$$

Now, taking $x = z$ and using a dyadic argument, we can conclude that

$$|\langle u \rangle_{x,\rho} - \langle u \rangle_{x,r}| \leq C_{n,a,\sigma} M r^\sigma, \quad \text{for any } 0 < s = \rho < r < r_0.$$

Indeed, let $k = 0, 1, 2, \ldots$ be such that $r/2^{k+1} \leq \rho < r/2^k$. Then

$$|\langle u \rangle_{x,\rho} - \langle u \rangle_{x,r}| \leq \sum_{\mathrm{j}=1}^{k} |\langle u \rangle_{x,r/2^{\mathrm{j}-1}} - \langle u \rangle_{x,r/2^{\mathrm{j}}}| + |\langle u \rangle_{x,r/2^k} - \langle u \rangle_{x,\rho}|$$

$$\leq C_{n,a} M \sum_{\mathrm{j}=1}^{k+1} (r/2^{\mathrm{j}-1})^\sigma \leq C_{n,a,\sigma} M r^\sigma.$$

This implies that the limit

$$Tu(x) = \lim_{r \to 0} \langle u \rangle_{x,r}$$

exists and

$$|Tu(x) - \langle u \rangle_{x,r}| \leq C_{n,a,\sigma} M r^\sigma.$$

Hence, we also have the Hölder integral bound

$$\int_{\mathbb{B}_r(x)} |u - Tu(x)|^2 |y|^a \leq C_{n,a,\sigma} M^2 r^{n+1+a+2\sigma}.$$

Besides, we have

$$|Tu(x)| \leq \langle u \rangle_{x,r_0} + C_{n,a,\sigma} M r_0^\sigma \leq C_{n,a,\sigma,r_0} M.$$

It remains to estimate the Hölder seminorm of Tu on $B_{1/2}$. Let $x, z \in B_{1/2}$ and consider two cases.

Case 1. If $|x - z| < r_0/4$, let $r = 2|x - z|$. Then note that $\mathbb{B}_{r/2}(x) \subset \mathbb{B}_r(z)$ and therefore we can write

$$\begin{aligned} |Tu(x) - Tu(z)| &\leq |Tu(x) - \langle u \rangle_{x,r/2}| + |Tu(z) - \langle u \rangle_{z,r}| + |\langle u \rangle_{x,r/2} - \langle u \rangle_{z,r}| \\ &\leq C_{n,a,\sigma} M r^\sigma = C_{n,a,\sigma} M |x - z|^\sigma. \end{aligned}$$

Case 2. If $|x - z| \geq r_0/4$, then

$$|Tu(x) - Tu(z)| \leq |Tu(x)| + |Tu(z)| \leq C_{n,a,\sigma,r_0} M \leq C_{n,a,\sigma,r_0} M |x - z|^\sigma.$$

Thus, we conclude

$$\|Tu\|_{C^{0,\sigma}(B_{1/2})} \leq C_{n,a,\sigma,r_0} M. \qquad \square$$

3.B Polynomial expansion for Caffarelli-Silvestre extension

Some of the results in Section 3.3 rely on polynomial expansion theorem for L_a-harmonic functions given below.

Theorem 3.B.1. *Let $u \in W^{1,2}(\mathbb{B}_1, |y|^a)$, $-1 < a < 1$, be a weak solution of the equation $L_a u = 0$ in $\mathbb{B}_1$, even in y. Then we have the following polynomial expansion:*

$$u(x,y) = \sum_{k=0}^{\infty} p_k(x,y)$$

locally uniformly in $\mathbb{B}_1$, where $p_k(x,y)$ are L_a-harmonic polynomials, homogeneous of degree k and even in y. Moreover, the polynomials p_k above are orthogonal in $L^2(\partial\mathbb{B}_1, |y|^a)$, i.e.,

$$\int_{\partial\mathbb{B}_1} p_k p_m |y|^a = 0, \quad k \neq m.$$

In particular, u is real analytic in $\mathbb{B}_1$.

This theorem has the following immediate corollaries, which are of independent interest and are likely known in the literature. We state them here for reader's convenience and for possible future reference.

Corollary 3.B.2. *Let $u \in W^{1,2}(\mathbb{B}_1, |y|^a)$, $-1 < a < 1$, be a weak solution of the equation $L_a u = 0$ in $\mathbb{B}_1$. Then, we have a representation*

$$u(x,y) = \varphi(x,y) + y|y|^{-a}\psi(x,y), \quad (x,y) \in \mathbb{B}_1,$$

where $\varphi(x,y)$ and $\psi(x,y)$ are real analytic functions, even in y.

Corollary 3.B.3. *Let $u \in \mathcal{L}_s(\mathbb{R}^n)$ satisfies $(-\Delta)^s u = 0$ in the unit ball $B_1 \subset \mathbb{R}^n$. Then u is real analytic in B_1.*

Corollary 3.B.4. *Let $u \in W^{1,2}(\mathbb{B}_1, |y|^a)$, $-1 < a < 1$, be a weak solution of the equation $L_a u = 0$ in $\mathbb{B}_1$, even in y. If $u(\cdot, 0) \equiv 0$ in B_1, then $u \equiv 0$ in $\mathbb{B}_1$.*

The proof of Theorem 3.B.1 and subsequently those of Corollaries 3.B.2, 3.B.3, and 3.B.4 are based on the following lemmas. We follow the approach of [63] for harmonic functions.

Let $\mathcal{P}_m^* = \{p : p(x,y) \text{ polynomial of degree} \leq m, \text{ even in } y\}$.

Lemma 3.B.5. *Let $p \in \mathcal{P}_m^*$. Then there exists $\tilde{p} \in \mathcal{P}_m^*$ such that*

$$L_a \tilde{p} = 0 \quad \text{in } \mathbb{B}_1, \quad \tilde{p} = p \quad \text{on } \partial\mathbb{B}_1.$$

In other words, the solution of the Dirichlet problem for L_a in $\mathbb{B}_1$ with boundary values in $\mathcal{P}_m^*$ on $\partial\mathbb{B}_1$ is itself in $\mathcal{P}_m^*$.

Proof. For $m = 0, 1$, we simply have $\tilde{p} = p$. For $m \geq 2$, we proceed as follows.

For $q \in \mathcal{P}_{m-2}^*$ define $Tq \in \mathcal{P}_{m-2}^*$ by

$$(Tq)(x,y) = |y|^{-a} L_a((1 - x^2 - y^2)q(x,y)).$$

(It is straightforward to verify that Tq is indeed in $\mathcal{P}_{m-2}^*$). We now claim that the mapping $T : \mathcal{P}_{m-2}^* \to \mathcal{P}_{m-2}^*$ is bijective. Since T is clearly linear and $\mathcal{P}_{m-2}^*$ is finite dimensional it is equivalent to showing that T is injective. To this end, suppose that $Tq = 0$ for some $q \in \mathcal{P}_{m-2}^*$. This means that $Q(x,y) = (1 - x^2 - y^2)q(x,y)$ is L_a-harmonic in $\mathbb{B}_1$:

$$L_a Q = 0 \quad \text{in } \mathbb{B}_1.$$

On the other hand $Q = 0$ on $\partial\mathbb{B}_1$ and therefore, by the maximum principle $Q = 0$ in $\mathbb{B}_1$. But this implies that $q = 0$ in $\mathbb{B}_1$, or that $q \equiv 0$. Hence, the mapping T is injective, and consequently bijective. It is now easy to see that

$$\tilde{p} = p - (1 - x^2 - y^2)T^{-1}(|y|^{-a} L_a(p)) \in \mathcal{P}_m^*$$

satisfies the required properties. ∎

Lemma 3.B.6. *Polynomials, even in y, are dense in the subspace of functions in $L^2(\partial\mathbb{B}_1, |y|^a)$, even in y.*

Proof. Polynomials, even in y are dense in the space of continuous functions in $C(\partial\mathbb{B}_1)$, even in y, with the uniform norm. The claim now follows from the observation that the embedding $C(\partial\mathbb{B}_1) \hookrightarrow L^2(\partial\mathbb{B}_1, |y|^a)$ is continuous:

$$\|v\|_{L^2(\partial\mathbb{B}_1,|y|^a)} \leq \|v\|_{L^\infty(\partial\mathbb{B}_1)} \left(\int_{\partial\mathbb{B}_1} |y|^a\right)^{1/2} \leq C\|v\|_{L^\infty(\partial\mathbb{B}_1)}. \qquad \square$$

Lemma 3.B.7. *The subspace of functions in $L^2(\partial\mathbb{B}_1, |y|^a)$, even in y, has an orthonormal basis $\{p_k\}_{k=0}^\infty$ consisting of homogeneous L_a-harmonic polynomials p_k, even in y.*

Proof. If p is a polynomial, even in y, then restricted to $\partial\mathbb{B}_1$ it can be replaced with an L_a-harmonic polynomial $\tilde{p}$. On the other hand, if we decompose

$$\tilde{p} = \sum_{i=0}^{m} q_i$$

where q_i is a homogeneous polynomial of order i, even in y, then

$$|y|^{-a} L_a \tilde{p} = \sum_{i=2}^{m} |y|^{-a} L_a q_i$$

where $|y|^{-a} L_a q_i$ is a homogeneous polynomial of order $i-2$, $i = 2, \ldots, m$. Hence, $L_a \tilde{p} = 0$ iff $L_a q_i = 0$, for all $i = 0, \ldots, m$ (for $i = 0, 1$ this holds automatically).

We further note that if q_i and q_j are two homogeneous L_a-harmonic polynomials of degrees $i \neq j$, then they are orthogonal in $L^2(\partial\mathbb{B}_1, |y|^a)$. Indeed,

$$0 = \int_{\mathbb{B}_1} q_i \operatorname{div}(|y|^a \nabla q_j) - \operatorname{div}(|y|^a \nabla q_i) q_j = \int_{\partial\mathbb{B}_1} (q_i \partial_\nu q_j - q_j \partial_\nu q_i)|y|^a = (j-i) \int_{\partial\mathbb{B}_1} q_i q_j |y|^a.$$

Using this and following the standard orthogonalization process, we can construct a basis consisting of homogeneous L_a-harmonic polynomials. $\square$

Lemma 3.B.8. *Let $u \in W^{1,2}(\mathbb{B}_1, |y|^a) \cap C(\overline{\mathbb{B}_1})$ is a weak solution of $L_a u = 0$ in $\mathbb{B}_1$. Then*

$$\|u\|_{L^\infty(K)} \leq C_{n,a,K} \|u\|_{L^2(\partial\mathbb{B}_1,|y|^a)}.$$

for any $K \Subset \mathbb{B}_1$.

Proof. First, we note that by [64]

$$\|u\|_{L^\infty(K)} \leq C_{n,a,K}\|u\|_{L^2(\mathbb{B}_1,|y|^a)}.$$

So we just need to show that

$$\|u\|_{L^2(\mathbb{B}_1,|y|^a)} \leq C_{n,a}\|u\|_{L^2(\partial\mathbb{B}_1,|y|^a)}.$$

This follows from the fact that u^2 is a subsolution: $L_a(u^2) \geq 0$ in $\mathbb{B}_1$ and therefore the weighted spherical averages

$$r \mapsto \frac{1}{\omega_{n,a} r^{n+a}} \int_{\partial\mathbb{B}_r} u^2|y|^a, \quad 0 < r < 1$$

are increasing. Integrating, we easily obtain that

$$\|u\|_{L^2(\mathbb{B}_1,|y|^a)} \leq C_{n,a}\|u\|_{L^2(\partial\mathbb{B}_1,|y|^a)}. \qquad \square$$

We are now ready to prove Theorem 3.B.1.

Proof of Theorem 3.B.1. Without loss of generality we may assume $u \in W^{1,2}(\mathbb{B}_1, |y|^a) \cap C(\overline{\mathbb{B}_1})$, otherwise we can consider a slightly smaller ball. Now, using the orthonormal basis $\{p_k\}_{k=0}^\infty$ from Lemma 3.B.7 we represent

$$u = \sum_{k=0}^\infty a_k p_k \quad \text{in } L^2(\partial\mathbb{B}_1, |y|^a).$$

We then claim that

$$u(x,y) = \sum_{k=0}^\infty a_k p_k(x,y) \quad \text{uniformly on any } K \Subset \mathbb{B}_1.$$

Indeed, if $u_m(x,y) = \sum_{k=0}^m a_k p_k(x,y)$, then $\|u - u_m\|_{L^2(\partial\mathbb{B}_1,|y|^2)} \to 0$ as $m \to \infty$ and therefore by Lemma 3.B.8

$$\|u - u_m\|_{L^\infty(K)} \leq C_{n,a,K}\|u - u_m\|_{L^2(\partial\mathbb{B}_1,|y|^a)} \to 0. \qquad \square$$

We now give the proofs of the corollaries.

Proof of Corollary 3.B.2. Write $u(x,y)$ in the form

$$u(x,y) = u_{\text{even}}(x,y) + u_{\text{odd}}(x,y),$$

where u_{even} and u_{odd} are even and odd in y, respectively. Clearly, both functions are L_a-harmonic. Moreover, by Theorem 3.B.1, u_{even} is real analytic and we take $\varphi = u_{\text{even}}$. On the other hand, consider

$$v(x,y) = |y|^a \partial_y u_{\text{odd}}(x,y).$$

Then, v is L_{-a}-harmonic in $\mathbb{B}_1$ and again by Theorem 3.B.1, v is real analytic. We can now represent

$$u_{\text{odd}}(x,y) = y|y|^{-a}\psi(x,y), \quad \psi(x,y) = y^{-1}|y|^a \int_0^y |s|^{-a} v(x,s)ds.$$

It is not hard to see that $\psi(x,y)$ is real analytic, which completes our proof. □

Proof of Corollary 3.B.3. The proof follows immediately from Theorem 3.B.1 by considering the Caffarelli-Silvestre extension

$$u(x,y) = u * P(\cdot, y) = \int_{\mathbb{R}^n} P(x-z,y)u(z)dz, \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}_+$$

where $P(x,y) = C_{n,a} \frac{y^{1-a}}{(|x|^2+y^2)^{(n+1-a)/2}}$ is the Poisson kernel for L_a, and noting that its extension to $\mathbb{R}^{n+1}$ by even symmetry in y (still denoted u) satisfies $L_a u = 0$ in $\mathbb{B}_1$. □

Proof of Corollary 3.B.4. Represent $u(x,y)$ as a locally uniformly convergent in $\mathbb{B}_1$ series

$$u(x,y) = \sum_{k=0}^{\infty} q_k(x,y),$$

where $q_k(x,y)$ is a homogeneous of degree k L_a-harmonic polynomial, even in y. We have

$$u(x,0) = \sum_{k=0}^{\infty} q_k(x,0) \equiv 0$$

from which we conclude that $q_k(x,0) \equiv 0$. We now want to show that $q_k \equiv 0$. To this end represent

$$q_k(x) = \sum_{j=0}^{[k/2]} p_{k-2j}(x) y^{2j},$$

where $p_{k-2j}(x)$ is a homogeneous polynomial of order $k-2j$ in x. Clearly $p_k(x) \equiv 0$. Taking partial derivatives $\partial_x^\alpha q_k(x)$ of order $|\alpha| = k-2$, we see that

$$\partial_x^\alpha q_k(x) = c_\alpha y^2, \quad c_\alpha = \partial_x^\alpha p_{k-2}$$

is L_a-harmonic, which can happen only when $c_\alpha = 0$. Hence $D_x^{k-2} p_{k-2}(x) \equiv 0$ and therefore $p_{k-2} \equiv 0$. Then taking consequently derivatives of orders $k-2j$, $j = 2, \ldots$, we conclude that $p_{k-2\mathrm{j}}(x) \equiv 0$ for all $j = 0, \ldots, [k/2]$ and hence $q_k(x,y) \equiv 0$. □

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