RESONANCE VARIETIES AND FREE RESOLUTIONS OVER AN EXTERIOR ALGEBRA

by

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To Lindsey and Geoffrey

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ABSTRACT

If E is an exterior algebra on a finite dimensional vector space and M is a graded E-module, the relationship between the resonance varieties of M and the minimal free resolution of M is studied. In the context of Orlik–Solomon algebras, we give a condition under which elements of the second resonance variety can be obtained. We show that the resonance varieties of a general M are invariant under taking syzygy modules up to a shift. As corollary, it is shown that certain points in the resonance varieties of M can be determined from minimal syzygies of a special form, and we define *syzygetic resonance varieties* to be the subvarieties consisting of such points. The (depth one) syzygetic resonance varieties of a square-free module M over E are shown to be subspace arrangements whose components correspond to graded shifts in the minimal free resolution of ${}_SM$, the square-free module over a commutative polynomial ring S corresponding to M. Using this, a lower bound for the graded Betti numbers of the square-free module M is given. As another application, it is shown that the minimality of certain syzygies of Orlik–Solomon algebras yield linear subspaces of their (syzygetic) resonance varieties and lower bounds for their graded Betti numbers.

1. INTRODUCTION

One of the central themes in the theory of hyperplane arrangements is the relationship between the topology of the complement of the arrangement and the combinatorics of the arrangement, expressible as a matroid whose ground set is the set of hyperplanes. The cohomology ring of the complement of a complex arrangement was originally computed by Brieskorn in [Bri73]. Building on this work, Orlik and Solomon give in [OS80] an explicit description of the cohomology ring in terms of the combinatorics. These *Orlik–Solomon algebras* are quotients of an exterior algebra by an ideal generated by products of linear forms. Depending on the combinatorics as they do, Orlik–Solomon algebras can be defined more generally for any matroid, not only the representable matroids associated to hyperplane arrangements.

Recently, much work has been done relating to a collection of invariants called *resonance varieties*, first defined by Falk in [Fal97]. Resonance varieties are algebraic varieties which correspond to collections of non-trivial zero divisors. While resonance varieties can be defined for any graded module over an exterior algebra, the most work has gone into their study in the context of Orlik–Solomon algebras. In this context, the first resonance variety is best understood. Building on work of Libgober and Yuzvinsky in [LY00], Falk and Yuzvinsky give in [FY07] a characterization of each of the components of the first resonance variety in terms of combinatorial structures called multinets. While there has been some work on understanding higher resonance varieties of Orlik–Solomon algebras (see, for example, Denham's paper [Den16]), a combinatorial characterization of their components remains elusive. We contribute to this pursuit in Chapter 5 by generalizing some of the results from [LY00].

While the cohomology ring (and hence resonance varieties) are determined by the combinatorics of an arrangement, not all topological invariants are so determined. It is known, for example, that the fundamental group of the complement of a complex arrangement is not determined by the combinatorics. Despite this, there is a sequence of numbers defined in terms of the fundamental group that is combinatorially determined: the sequence of Chen ranks. In addition to providing a wealth of references for further reading on the Chen ranks, Schenck and Suciu show in [SS06] that the Chen ranks coincide with the graded Betti numbers of the linear strand of the minimal free resolution of the Orlik–Solomon algebra viewed as module over the exterior algebra. Suciu conjectured a formula for the Chen ranks in terms of the first resonance variety of the Orlik–Solomon algebra in [Suc01], and this conjecture was proved by Cohen and Schenck in [CS15] making use of properties of the fundamental group of the complement. Taken together, these results give the Chen Ranks Theorem 4.4, an expression for *some* of the graded Betti numbers of Orlik–Solomon algebras in terms of linear components of the first resonance variety.

The Chen Ranks Theorem raises several questions concerning the relationship between the graded Betti numbers and the resonance varieties:

(1) Are *all* of the graded Betti numbers expressible in terms of (higher) resonance varieties?

(2) To what extent is the restriction to Orlik-Solomon algebras necessary? Does the relationship extend to arbitrary graded modules over an exterior algebra?

(3) Are there other ways in which resonance varieties relate to minimal free resolutions?

Our approach to exploring these questions takes the graded Betti numbers as basic and then explores what resonance results from them. More precisely, the graded Betti numbers count minimal syzygies of a module, and we explore which points in the resonance varieties are directly involved in creating a minimal syzygy. We create a new definition (Definition 6.4) calling the (sometimes proper!) subvarieties of such points *syzygetic resonance varieties*. We establish several basic properties, and notably we prove in Theorem 6.1 and Remark 6.5 that both resonance varieties and syzygetic resonance varieties are invariant under passing to syzygy modules.

Due to their definition being closely tied to the minimal free resolution of a module, studying syzygetic resonance varieties requires in-depth knowledge of resolutions. Yanagawa defined in [Yan00] a class of modules called *square-free modules* in order to give a unifying framework for studying quotients by square-free monomial ideals. In [EPY03], Eisenbud, Popescu, and Yuzvinsky describe the minimal free resolutions of square-free modules over an exterior algebra. Making use of this resolution, we compute the syzygetic resonance varieties for the class of square-free modules in Theorem 8.1. We then proceed to give in Corollary 8.3 a lower bound for the graded Betti numbers of a square-free module in terms of linear components of the syzygetic resonance varieties in the style of the Chen Ranks Theorem. We close by using Theorem 8.1 to obtain some information about the syzygetic resonance varieties and graded Betti numbers of Orlik-Solomon algebras.

1.1 Outline

This document is organized as follows: In Chapter 2, we give introductory definitions regarding exterior and symmetric algebras, resolutions, syzygies, and matroids. We also establish much of the notation that will be used throughout the course of the document. In Chapter 3, we review the definition of resonance varieties and prove that resonance varieties are Zariski closed. In Chapter 4, we define Orlik–Solomon algebras and review some of the historical results concerning their resonance varieties. In Chapter 5, we generalize some results originally used to determine the first resonance variety of Orlik–Solomon algebras, and we use these generalizations to give a condition under which one can find elements of the second resonance variety. In Chapter 6, we prove that resonance varieties are invariant under passing to syzygy modules, and we use this to define syzygetic resonance varieties. In Chapter 7, we review the definition of square-free modules and describe their free resolutions given in [EPY03]. In Chapter 8, we compute the syzygetic resonance varieties of square-free modules and give a bound on their graded Betti numbers. Finally in Chapter 9, we describe what implications the computation of syzygetic resonance varieties from the previous chapter has for Orlik–Solomon algebras.

2. PRELIMINARIES

Throughout this chapter, we review the necessary foundational material supporting the main results presented in the dissertation. We also take the opportunity to establish notational standards which will be used throughout the document. The chapter is split into three sections. The first two sections discuss the basic algebraic objects we will use throughout this document, while the third section reviews the necessary combinatorial objects. In the first section, we present a discussion of exterior and symmetric algebras, drawing special attention to conventions concerning degrees, monomials, and modules related to these algebras. In the second section, we review free resolutions and Betti numbers, and we prove a basic result concerning the minimality of certain syzygies. Finally in the third section, we review the basic terminolgy associated to matroids.

Throughout this document, $n \in \mathbb{N}$ will be a natural number, $\Bbbk$ will be a fixed field, and V will be an n-dimensional $\Bbbk$-vector space with fixed basis $x_1, \ldots, x_n$.

2.1 Exterior and Symmetric Algebras

In this section, we discuss the basic definitions, gradings, and conventions for modules over exterior and symmetric algebras. Because both of these algebras are defined as a quotient of a tensor algebra, we focus much of our attention on this setting, and we allow the product structure and gradings to be inherited via the quotient construction.

2.1.1 Definitions

The *tensor algebra* of V is the $\Bbbk$-vector space

$$\begin{aligned} \bigotimes V &= \bigoplus_{i=0}^{\infty} \underbrace{V \otimes \cdots \otimes V}_{i \text{ times}} \\ &= \Bbbk \oplus V \oplus (V \otimes V) \oplus \cdots \end{aligned}$$

with product structure given by linearly extending the product of basis elements given by

$$(x_{i_1} \otimes \cdots \otimes x_{i_k}) \cdot (x_{j_1} \otimes \cdots \otimes x_{j_\ell}) = x_{i_1} \otimes \cdots \otimes x_{i_k} \otimes x_{j_1} \otimes \cdots \otimes x_{j_\ell}.$$

The *exterior algebra* $\bigwedge V$ of V, denoted by E throughout this document, is the quotient of the tensor algebra $\bigotimes V$ by the two-sided ideal generated by the elements $x_i \otimes x_i$ and $x_i \otimes x_j + x_j \otimes x_i$ for each pair i, j. The *symmetric algebra* $\operatorname{Sym} V$ of V, denoted by S throughout this document, is the quotient of the tensor algebra $\bigotimes V$ by the two-sided ideal generated by the elements $x_i \otimes x_j - x_j \otimes x_i$ for each pair i, j. For both E and S, we continue to use the basis elements x_i to denote the image of x_i under the relevant quotient. In all cases, we omit the $\otimes$ symbol when expressing elements so that, for example, $x_1 x_2$ means (the image of) $x_1 \otimes x_2$. Further information regarding these algebras can be found in [Eis95, Appendix A2.3]

2.1.2 Degrees and Gradings

We now discuss two natural gradings which can be assigned to the tensor, exterior, and symmetric algebras, and we lay out notational conventions for degrees.

Throughout, we use boldface letters $\mathbf{a}$ to denote multidegrees $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, and we let $\mathbf{e}_i$ denote the i-th standard basis vector of $\mathbb{Z}^n$. Define the *total degree* of a multidegree $\mathbf{a}$ to be $|\mathbf{a}| = a_1 + \cdots + a_n$, the sum of its components. Define the *support of a multidegree* to be $\operatorname{supp}(\mathbf{a}) = \{i \mid a_i \neq 0\}$, the collection of indices i such that the i-th component of $\mathbf{a}$ is non-zero.

In the tensor, exterior, or symmetric algebras of V, we denote by $x^{\mathbf{a}}$ the monomial $x_1^{a_1} \cdots x_n^{a_n}$ whenever $\mathbf{a} \in \mathbb{N}^n$. For non-zero monomials, we define the *support of the monomial* $x^{\mathbf{a}}$ to be $\operatorname{supp}(x^{\mathbf{a}}) = \{x_i \mid a_i \neq 0\}$, the collection of basis elements x_i for which the i-th component of $\mathbf{a}$ is non-zero. For any linear combination of monomials

$$\eta = \sum_{\mathbf{a} \in T} \eta_{\mathbf{a}} x^{\mathbf{a}},$$

with $T \subset \mathbb{N}^n$ a finite set and each $\eta_{\mathbf{a}}$ a non-zero element of $\Bbbk$, define the *support of an element* η to be $\operatorname{supp}(\eta) = \bigcup_{\mathbf{a}\in T} \operatorname{supp}(x^{\mathbf{a}})$.

A multidegree $\mathbf{a}$ or a monomial $x^{\mathbf{a}}$ is said to be *square-free* if $a_i \in \{0, 1\}$ for all i. If $T = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, n\}$ is a set of indices, let $x_T = x_{i_1} \cdots x_{ik}$ be the square-free monomial whose multidegree has support T. For exterior and symmetric algebras, we define the *quotient of square-free monomials with nested support* as follows: if $\operatorname{supp}(\mathbf{a}) \subset \operatorname{supp}(\mathbf{b})$ for square-free multidegrees $\mathbf{a}$ and $\mathbf{b}$, we define $x^{\mathbf{b}}/x^{\mathbf{a}}$ to be the signed monomial such that $(x^{\mathbf{b}}/x^{\mathbf{a}})x^{\mathbf{a}} = x^{\mathbf{b}}$. Thus, for example, $x^{\mathbf{e}_1+\mathbf{e}_2}/x^{\mathbf{e}_2} = x_1$ in $E = \bigwedge V$ and in $S = \operatorname{Sym} V$. On the other hand, $x^{\mathbf{e}_1+\mathbf{e}_2}/x^{\mathbf{e}_1}$ is equal to $-x_2$ in E and is equal to x_2 in S.

We give the tensor algebra of V a $\mathbb{Z}^n$*-grading* by setting the degree of $x^{\mathbf{a}}$ to be $\mathbf{a}$. We give the tensor algebra a $\mathbb{Z}$*-grading by total degree* by setting the degree of $x^{\mathbf{a}}$ to be $|\mathbf{a}|$. Under both gradings, the ideals defining the exterior and symmetric algebras of V are homogeneous, hence both E and S inherit the $\mathbb{Z}^n$-grading and the $\mathbb{Z}$-grading by total degree.

2.1.3 Modules

We here establish conventions regarding modules over the exterior algebra E and symmetric algebra S on V. In particular, we set conventions for homomorphisms, and we fix notation for homogeneous components and shifts of graded modules.

Throughout this document, all E-modules and all S-modules will be finitely generated and $\mathbb{Z}$-graded. All E-modules M we consider will be graded E-bimodules with $\tau\eta = (-1)^{\deg(\tau)\deg(\eta)}\eta\tau$ for homogeneous elements $\tau \in M$ and $\eta \in E$. All module homomorphisms will be graded (or where applicable, $\mathbb{Z}^n$-graded) homomorphisms as *right* modules. In particular, this means that maps between free modules can be represented by *left multiplication* with a matrix.

If M is a graded E or S module and $i, j \in \mathbb{Z}$, we let M_i be the i-th graded component of M and define the shift $M(-j)$ so that $M(-j)_i = M_{i-j}$. Similarly, if M is $\mathbb{Z}^n$-graded and $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, then $M_{\mathbf{a}}$ is the graded component of M corresponding to $\mathbf{a}$ and we define the shift $M(-\mathbf{b})$ so that $M(-\mathbf{b})_{\mathbf{a}} = M_{\mathbf{a}-\mathbf{b}}$.

2.2 Resolutions, Syzygies, and Betti Numbers

In this section, we review the definitions of minimal free resolutions, syzygies, and Betti numbers. We then prove an elementary result showing that certain syzygies are necessarily minimal, a fact which will be needed in subsequent chapters. Further information on the topics discussed herein can be found in [Eis95, Chapters 19 and 20], [Eis05, Chapter 1], and [HH11, Appendix A.2].

2.2.1 Definitions

Let M be a $\mathbb{Z}$-graded or $\mathbb{Z}^n$-graded module over E or S. A *graded free resolution* of M is a complex

$$(F_\bullet, d)\colon \; \cdots \xrightarrow{\;d\;} F_2 \xrightarrow{\;d\;} F_1 \xrightarrow{\;d\;} F_0 \longrightarrow 0,$$

where each F_i is a graded free module, the maps $d\colon F_i \to F_{i-1}$ are homogeneous, and

$$\begin{aligned} H_i(F_\bullet, d) &= \frac{\ker(d\colon F_i \to F_{i-1})}{\operatorname{im}(d\colon F_{i+1} \to F_i)} \\ &\cong \begin{cases} M & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases} \end{aligned}$$

A graded free resolution of M is said to be *minimal* if the image of each map $d\colon F_i \to F_{i-1}$ is in the submodule $(x_1, \ldots, x_n)\, F_{i-1}$ or, equivalently, if each map $d\colon F_i \to F_{i-1}$ can be represented by a matrix with entries in $(x_1, \ldots, x_n)$. A well-known consequence of the graded version of Nakayama's Lemma is that minimal free resolutions of modules over E and S exist and are unique up to isomorphism.

If $(F_{\bullet,d})$ is a minimal free resolution of M, then we define $\operatorname{im}(d\colon F_i \to F_{i-1}) = \ker(F_{i-1} \to F_{i-2})$ to be the *i-th syzygy module of* M, and denote it $\operatorname{Syz}_i^E(M)$ or $\operatorname{Syz}_i^S(M)$. Elements of a syzygy module are called *syzygies*, and elements which are part of a minimal set of generators of a syzygy moduleare called *minimal syzygies.* If M is a $\mathbb{Z}$-graded E-module, then each F_i has an expression

$$F_i = \bigoplus_{j \in \mathbb{Z}} E(-j)^{\beta^E_{i,j}(M)},$$

and we call the numbers $\beta^E_{i,j}(M)$ the *graded Betti numbers of M*. The number $\beta^E_{i,j}(M)$ is the number of elements of degree j in a minimal generating set of $\mathrm{Syz}^E_i(M)$. Similarly, if M is a $\mathbb{Z}^n$-graded E-module, then each F_i has an expression

$$F_i = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} E(-\mathbf{a})^{\beta^E_{i,\mathbf{a}}(M)},$$

and we call the numbers $\beta^E_{i,\mathbf{a}}(M)$ the *$\mathbb{Z}^n$-graded Betti numbers of M*. The number $\beta^E_{i,\mathbf{a}}(M)$ is the number of elements of degree $\mathbf{a}$ in a minimal generating set of $\mathrm{Syz}^E_i(M)$. We use the same terminology and analogous notation for $\mathbb{Z}$-graded and $\mathbb{Z}^n$-graded modules over S.

2.2.2 A Condition for a Syzygy to be Minimal

We prove here a basic result regarding the minimality of syzygies of a certain form, those which are "one degree higher" than the previous minimal syzygies. While we prove the result over E, this is a general fact about standard-graded $\Bbbk$-algebras.

Proposition 2.1. *Let F and G be graded free E-modules, and let $\varphi\colon F \oplus E \to G$ be a graded E-linear map such that* $\operatorname{im}\varphi \subset (x_1, \dots, x_n)\, G$*. If $(\eta, \lambda) \in F \oplus E$ is in* $\ker\varphi$ *for some nonzero $\lambda \in E_1$, then (η, λ) is part of a minimal generating set of* $\ker\varphi$*.*

Proof. Let $(\epsilon_1, \rho_1), \dots, (\epsilon_k, \rho_k)$ be a minimal generating set for $\ker\varphi$ consisting of homogeneous elements. Then there exist $\alpha_1, \dots, \alpha_k \in E$ such that

$$(\eta, \lambda) = \alpha_1(\epsilon_1, \rho_1) + \cdots + \alpha_k(\epsilon_k, \rho_k),$$

whence

$$\lambda = \alpha_1\rho_1 + \cdots + \alpha_k\rho_k.$$

Since $\operatorname{im}\varphi \subset (x_1, \ldots, x_n)\, G$, it follows that each ρ_i is zero or is an element of E_p for some $p \geq 1$. Since $\lambda \in E_1$ and each nonzero ρ_i is of positive degree, it follows that for some $j \in \{1, \ldots, k\}$ the element $\rho_j \in E_1$ and $\alpha_j \in \Bbbk \setminus \{0\}$. It follows that

$$(\epsilon_j, \rho_j) = \frac{1}{\alpha_j}\left((\eta, \lambda) - \sum_{i \neq j}(\epsilon_i, \rho_i)\right),$$

whence (η, λ) together with the elements (ϵ_i, ρ_i) with $i \neq j$ minimally generate $\ker\varphi$. $\square$

2.3 Combinatorics

In this section, we define the basic combinatorial object used throughout this document, and we provide a readily available class of examples from which we will draw on in later chapters. Material in this section is adapted from [Oxl03], with further reading available in [Oxl11].

Throughout this document, for $n \in \mathbb{N}$ we let $[n]$ denote the collection $\{1, \ldots, n\} \subset \mathbb{N}$. For any subset $X \subset [n]$, we let $|X|$ denote the number of elements in X.

2.3.1 Definition of Matroid

A *matroid* M on $[n]$ is an abstraction of the basic properties of linear dependence and independence on a set of n vectors. The matroid M consists of a collection $\mathcal{I}$ of subsets of $[n]$ called *independent* sets, subject to the following conditions:

- The collection $\mathcal{I}$ is not empty.
- If $X \in \mathcal{I}$ is independent, every subset $Y \subset X$ is also in $\mathcal{I}$.
- If $X, Y \in \mathcal{I}$ with $|X| \gneq |Y|$, then there exists $i \in X \setminus Y$ such that $Y \cup \{i\} \in \mathcal{I}$.

Subsets $X \notin \mathcal{I}$ are called *dependent sets*, and minimal dependent sets are called *circuits*. A matroid is called *loop-free* if it contains no dependent sets containing 1 element, and it is called *simple* if it contains no dependent sets consisting of 1 or 2 elements. The *rank* of a subset $X \subset [n]$ is the number of elements in a maximal independent subset of X. The *rank* of M is the rank of $[n]$. The *closure* $\overline{X}$ of $X \subset [n]$ is the largest subset of $[n]$ containing X and having the same rank as X. A *flat* of M is a subset of $[n]$ equal to its own closure. The collection of all flats of M is called the *lattice of flats*, and it is denoted by $L(\mathsf{M})$. The collection of all flats of rank $p \in \mathbb{N}$ is denoted $L_p(\mathsf{M})$.

We now give some classes of examples of matroids. The first class of examples are those which inspired the definition of a matroid, the class of *representable matroids*. While all explicit examples of matroids we give are representable, it is worth noting that the vast majority of matroids are non-representable. See [Nel18] for details on this point.

Example 2.2. Let $v_1, \dots, v_n$ be a collection of vectors in a $\Bbbk$-vector space. Declaring a subset $X \subset [n]$ to be independent if the corresponding set of vectors $\{v_i \mid i \in X\}$ is a linearly independent set defines a matroid on $[n]$. Matroids which arise in this manner are called $\Bbbk$-representable matroids.

Next, we describe a subcollection of representable matroids which will act as a convenient source of examples for use in later chapters.

Example 2.3. Let Γ be a simple graph with n edges and ℓ vertices, with the edges labeled 1 through n. Labeling the vertices 1 through ℓ, we associate to the k-th edge $\{i, j\}$ the linear form $y_i - y_j$ in a polynomial ring $\Bbbk[y_1, \dots, y_\ell]$. The *graphic matroid* associated to Γ, denoted $\mathsf{M}(\Gamma)$, is the realizable matroid on $[n]$ where a collection of edge labels is declared independent if the corresponding collection of linear forms is $\Bbbk$-linearly independent. Equivalently, a collection of edge labels is independent if the corresponding collection of edges do not contain a cycle.

Finally, we describe one way in which one can produce a matroid from another matroid.

Example 2.4. Let M be a matroid on $[n]$, and let $\mathcal{I}$ be the collection of independent sets of M. If $X \subset [n]$, defining a subset Y of X to be independent if and only if $Y \in \mathcal{I}$ defines a matroid $\mathsf{M}{\restriction_X}$ on X. We call $\mathsf{M}{\restriction_X}$ the *restriction* of M to X.

3. RESONANCE VARIETIES

In this chapter, we define some of the primary objects of study throughout the rest of the document, the *resonance varieties*. As is noted in the historical overview provided by Yuzvinsky in [Yuz12], resonance varieties were first introduced by Falk in [Fal97] to study the cohomology ring of the complement of a complex hyperplane arrangement. The study of these cohomology jump loci in the more general setting of modules over an exterior algebra was first carried out in [AAH00] under the name *rank variety*. More recently, resonance varieties have been defined in [Suc16] for general commutative differential graded algebras. Our study focuses on modules over an exterior algebra.

In the first section, we give the definition and several examples. Next, we show that resonance varieties are in fact algebraic varieties by describing a defining ideal for the varieties.

3.1 Definition and Examples

We give here the definition of resonance varieties. As usual, $E = \bigwedge V$ is the exterior algebra on the $\Bbbk$-vector space V with basis $x_1, \ldots, x_n$.

Definition 3.1. For a graded E-module M and $\lambda \in E_1$, let $(M^\bullet, \lambda)$ be the cochain complex with M_i in cohomological degree i and differential given by left multiplication with λ. The *i-th resonance variety of M of depth j* is

$$\mathcal{R}^i_j(M) = \{(\lambda_1, \ldots, \lambda_n) \in \Bbbk^n \mid \dim_{\Bbbk} H^i(M^\bullet, \lambda_1 x_1 + \cdots + \lambda_n x_n) \geq j\},$$

and we write $\mathcal{R}^i(M)$ for $\mathcal{R}^i_1(M)$. If $\lambda = \lambda_1 x_1 + \cdots + \lambda_n x_n \in E_1$, we write $\lambda \in \mathcal{R}^i_j(M)$ to mean $(\lambda_1, \ldots, \lambda_n) \in \mathcal{R}^i_j(M)$, and tacitly make use of this correspondence throughout the document. We call an element of $\mathcal{R}^i_j(M)$ a *resonant element* or *resonant weight*.

We give two basic examples which will be useful later:

Example 3.2. If F is a finite free E-module, then for all i and j

$$\mathcal{R}^i_j(F) = \begin{cases} \{0\} & \text{if } \dim_{\Bbbk} F_i \geq j \\ \emptyset & \text{otherwise.} \end{cases}$$

Indeed, it is clear from the definition that $0 \in \mathcal{R}^i_j(F)$ if and only if $\dim_{\Bbbk} F_i \geq j$, hence it remains only to show that no element other than 0 is resonant. To this end, let $f \in F$ and $\lambda \in E_1 \setminus \{0\}$ be given such that $\lambda f = 0$. We will argue that $f = \lambda g$ for some $g \in F$. After fixing an isomorphism of F with a direct sum of shifted copies of E, notice that the conditions $\lambda f = 0$ and $f = \lambda g$ are satisfied if and only if they are satisfied componentwise. Replacing F with E and f with one of its components allows us to assume that $F = E$. Since E is the exterior algebra on the $\Bbbk$-vector space V, every vector space isomorphism of V onto itself induces a ring isomorphism of E onto itself. Choosing an isomorphism of V sending λ to x_1, it suffices to show the result when $\lambda = x_1$ and $F = E$. In this setting, we may write $f = \sum_{\mathbf{a} \in T} f_{\mathbf{a}} x^{\mathbf{a}}$ with $T \subset \{0,1\}^n$ a finite set and each $f_{\mathbf{a}} \in \Bbbk$. Then

$$x_1 f = \sum_{\substack{\mathbf{a} \in T \\ x_1 \notin \operatorname{supp}(x^{\mathbf{a}})}} f_{\mathbf{a}} x_1 x^{\mathbf{a}}$$

is zero if and only if $f_{\mathbf{a}} = 0$ for each $\mathbf{a} \in T$ such that $x_1 \notin \operatorname{supp}(x^{\mathbf{a}})$. It follows that

$$f = x_1 \sum_{\substack{\mathbf{a} \in T \\ x_1 \in \operatorname{supp}(x^{\mathbf{a}})}} f_{\mathbf{a}} x^{\mathbf{a} - \mathbf{e}_1}.$$

Thus $\lambda f = 0$ implies that f is a multiple of λ, whence λ is not resonant.

Example 3.3. If $\mathbf{a} \in \{0,1\}^n$ is a square-free multidegree, then

$$\mathcal{R}^i_j(E/\operatorname{supp}(x^{\mathbf{a}})) = \begin{cases} \operatorname{span}_{\Bbbk} \operatorname{supp}(x^{\mathbf{a}}) & \text{if } 0 \leq i \leq n - |\mathbf{a}| \text{ and } 1 \leq j \leq \binom{n-|\mathbf{a}|}{i} \\ \emptyset & \text{otherwise.} \end{cases}$$

Indeed, notice that $E/\operatorname{supp}(x^{\mathbf{a}})$ is isomorphic to the exterior algebra E' on the $\Bbbk$-vector space with basis $\{x_1, \ldots, x_n\} \setminus \operatorname{supp}(x^{\mathbf{a}})$, viewed as E-module by letting $x_i \in \operatorname{supp}(x^{\mathbf{a}})$ act as zero. It follows from this that the span of $\operatorname{supp}(x^{\mathbf{a}})$ is contained in the cited resonance varieties. That these are the only non-zero resonant weights follows from Example 3.2 which shows that all resonance weights of E' come from multiplication by 0.

3.2 Resonance Varieties are Zariski Closed

Because we will have occasion to use the fact that resonance varieties are algebraic varieties, we include a proof of this fact for completeness. While other proofs of this fact can be found throughout the literature, we adapt the approach presented in [BW15].

Let M be a finitely generated E-module and let $R = \Bbbk[s_1, \ldots, s_n]$ be a polynomial ring with $s_1, \ldots, s_n$ indeterminates. In the tensor algebra $R \otimes_{\Bbbk} E$, let $\Lambda = \sum_{i=1}^{n} s_i \otimes x_i$. Left multiplication by Λ makes $R \otimes_{\Bbbk} M$ a cochain complex of free R-modules:

$$(R \otimes_{\Bbbk} M^{\bullet}, \Lambda\cdot)\colon \quad \cdots \longrightarrow R \otimes_{\Bbbk} M_{i-1} \xrightarrow{\Lambda} R \otimes_{\Bbbk} M_i \xrightarrow{\Lambda} R \otimes_{\Bbbk} M_{i+1} \longrightarrow \cdots.$$

Let $(\lambda_1, \ldots, \lambda_n)$ be a point in $\Bbbk^n$, let $m = (s_1 - \lambda_1, \ldots, s_n - \lambda_n)$ be the corresponding R-ideal, and let $\lambda = \sum_{i=1}^{n} \lambda_i x_i \in E_1$. Notice that specializing s_i to λ_i by applying the functor $R/m \otimes_R -$ to $(R \otimes_{\Bbbk} M^{\bullet}, \Lambda\cdot)$ reproduces the cochain complex $(M^{\bullet}, \lambda\cdot)$. Thus, $\lambda \in \mathcal{R}^i_j(M)$ if and only if $\dim_{R/m} H^i(R/m \otimes_R R \otimes_{\Bbbk} M^{\bullet}, id_{R/m} \otimes \Lambda\cdot) \geq j$.

Definition 3.4 ([BW15], Definition-Proposition 2.2)**.** Let M be a finitely generated E-module and let $R = \Bbbk[s_1, \ldots, s_n]$ a polynomial ring. Let $(F^{\bullet}, d)$ be a complex of finite free R-modules quasi-isomorphic to $(R \otimes_{\Bbbk} M^{\bullet}, \Lambda\cdot)$. Independent of the choice of $(F^{\bullet}, d)$, we define the *cohomology jump ideals* of M to be the R-ideals

$$J^i_j(M) = I_{(\operatorname{rank} F^i)-j+1}\left(d^{i-1} \oplus d^i\right).$$

Here $I_t(d^{i-1} \oplus d^i)$ denotes the ideal generated by the $t \times t$-minors of the $k \times \ell$ matrix $d^{i-1} \oplus d^i$, where $I_t(d^{i-1} \oplus d^i) = 0$ for $t > \min\{k, \ell\}$ and $I_t(d^{i-1} \oplus d^i) = R$ for $t \leq 0$.

The cohomology jump ideals of M give defining equations for the resonance varieties of M. This is the content of the next proposition.

Proposition 3.5 (cf. [BW15], Corollary 2.5)**.** *Let M be a finitely generated E-module and let $R = \Bbbk[s_1, \ldots, s_n]$ a polynomial ring. If m is a maximal ideal of R, then $J^i_j(M) \subset m$ if and only if*

$$\dim_{R/m} H^i(R/m \otimes_R R \otimes_\Bbbk M^\bullet, id_{R/m} \otimes \Lambda\cdot) \geq j.$$

In particular, $\mathcal{R}^i_j(M)$ is the Zariski closed subset of $\Bbbk^n$ given by

$$\mathcal{R}^i_j(M) = \mathbb{V}(J^i_j(M)),$$

where $\mathbb{V}$ denotes the vanishing set in $\Bbbk^n$.

Proof. Let m be a maximal ideal of R, and let $(F^\bullet, d)$ be a complex of finte free R-modules quasi-isomorphic to $(R \otimes_\Bbbk M^\bullet, \Lambda\cdot)$. Because $(F^\bullet, d)$ is a complex of finte free R-modules, each map $d^i \colon F^i \to F^{i+1}$ can be represented by a matrix. Applying the functor $R/m \otimes_R -$ to $(F^\bullet, d)$ amounts to replacing elements in the matrix representations of the maps d^k with their images in R/m. It follows that

$$\begin{aligned} R/m \otimes_R J^i_j(M) &= R/m \otimes_R I_{(\operatorname{rank} F^i)-j+1}\left(d^{i-1} \oplus d^i\right) \\ &= I_{(\operatorname{rank} F^i)-j+1}\left(\left(id_{R/m} \otimes d^{i-1}\right) \oplus \left(id_{R/m} \otimes d^i\right)\right), \end{aligned}$$

where the maps $id_{R/m} \otimes d^{i-1}$ and $id_{R/m} \otimes d^i$ are maps between (R/m)-vector spaces. From this we see that $J^i_j(M) \subset m$ if and only if

$$\operatorname{rank}\left(\left(id_{R/m} \otimes d^{i-1}\right) \oplus \left(id_{R/m} \otimes d^i\right)\right) \leq (\operatorname{rank} F^i) - j. \tag{3.6}$$

Since

$$\operatorname{rank}\left(id_{R/m} \otimes d^i\right) = (\operatorname{rank} F^i) - \operatorname{nullity}\left(id_{R/m} \otimes d^i\right)$$

and

$$\operatorname{rank}\left(id_{R/m} \otimes d^{i-1}\right) - \operatorname{nullity}\left(id_{R/m} \otimes d^i\right) = -\dim_{R/m} H^i(R/m \otimes_R F^\bullet, id_{R/m} \otimes d),$$

the inequality (3.6) becomes

$$(\operatorname{rank} F^i) - \dim_{R/m} H^i(R/m \otimes_R F^\bullet, id_{R/m} \otimes d) \leq (\operatorname{rank} F^i) - j.$$

Thus, $J^i_j(M) \subset m$ if and only if

$$\dim_{R/m} H^i(R/m \otimes_R F^\bullet, id_{R/m} \otimes d) \geq j.$$

Because $(F^\bullet, d)$ is a complex of finite free R-modules quasi-isomorphic to $(M^\bullet, \Lambda\cdot)$, we obtain the desired inequality. The note about the resonance variety $\mathcal{R}^i_j(M)$ follows from the discussion preceding Definition 3.4. □

4. RESONANCE VARIETIES OF ORLIK–SOLOMON ALGEBRAS

In this chapter, we discuss the resonance varieties of Orlik–Solomon algebras. Orlik–Solomon algebras were originally defined as an expression of the cohomology ring of the complement of a complex hyperplane arrangement, but they can be defined in a more general setting. It is in the context of these algebras that resonance varieties were first defined, and it is this context which has been most studied. Because of this, the the historical results on Orlik–Solomon algebras act as guideposts for further studies on resonance varieties, and Orlik–Solomon algebras act as a touchstone against which to test new results.

This chapter is split into two sections. In the first section, we give the basic definitions of complex hyperplane arrangements and Orlik–Solomon algebras. The second section presents historical results on the resonance varieties of Orlik–solomon algebras.

4.1 Hyperplane Arrangements and Orlik–Solomon Algebras

In this section we provide a preliminary discussion of hyperplane arrangements. We refer the reader to [OT92] for further background on the topics herein discussed.

4.1.1 The Data of a Complex Hyperplane Arrangement

A *central hyperplane arrangement* $\mathcal{A}$ in $\mathbb{C}^\ell$ is the zero set $\mathbb{V}(Q)$ of a polynomial $Q = \prod h_i$, where each h_i is a polynomial in $\mathbb{C}[y_1, \ldots, y_\ell]$ which is homogeneous of degree 1; the zero set of each linear form h_i is a hyperplane H_i, and $\mathbb{V}(Q)$ is the union of the H_i. The *combinatorial data* associated to an arrangement $\mathcal{A}$ of n hyperplanes can be expressed in the form of the $\mathbb{C}$-representable matroid $\mathsf{M}(\mathcal{A})$ on $[n]$, where a subset $X \subset [n]$ is independent if the corresponding collection $\{h_i \mid i \in X\}$ is a $\mathbb{C}$-linearly independent set. The *topological data* associated to a hyperplane arrangement is the complement $\mathcal{M}(\mathcal{A}) = \mathbb{C}^\ell \setminus \mathbb{V}(Q)$ of the arrangement equipped with the subspace topology induced from $\mathbb{C}^\ell$.

Just as for matroids, one readily available class of hyperplane arrangements comes from graphs.

Example 4.1. Let Γ be a simple graph with n edges and ℓ vertices, with the edges labeled 1 through n. Labelling the vertices 1 through ℓ, associate to the k-th edge $\{i, j\}$ with $i < j$ the linear form $h_{i,j} = y_i - y_j$ in a polynomial ring $\Bbbk[y_1, \ldots, y_\ell]$. The *graphic arrangement* $\mathcal{A}$ associated to Γ is defined by the polynomial $Q = \prod_{i<j} h_{i,j}$. By definition, the combinatorial data $\mathsf{M}(\mathcal{A})$ is the same as the graphic matroid $\mathsf{M}(\Gamma)$.

One of the central themes in the theory of hyperplane arrangements is the interplay of combinatorial and topological information associated to the arrangement. A foundational result in this direction gives a description of the cohomology ring of the complement of an arrangement in $\mathbb{C}^\ell$ purely in terms of its combinatorial data. We discuss this description of the cohomology ring next.

4.1.2 Orlik–Solomon Algebras and Arrangement Complements

The cohomology ring of the complement of a complex hyperplane arrangement was studied by Brieskorn in [Bri73]. In this work, Brieskorn identified the cohomology ring with an algebra generated by certain holomorphic differential forms, an isomorphism originally conjectured by Arnol′d in [Arn69]. Building on Brieskorn's work, Orlik and Solomon give in [OS80] an explicit description of the cohomology ring in terms of generators and relations, a description which only relies on the combinatorial data associated to the arrangement.

In order to define the algebra given by Orlik and Solomon, we consider the vector space V with basis $x_1, \ldots, x_n$, and we let $E = \bigwedge V$ be the exterior algebra on V. Define a derivation ∂ on E by setting $\partial(x_i) = 1$ and extend ∂ to a map on E by linearity and the graded Leibniz rule (so that $\partial(\eta\tau) = \partial(\eta)\tau + (-1)^{\deg \eta}\eta\partial(\tau)$ for η and τ homogeneous).

Definition 4.2. Let V be an n-dimensional vector space with basis $x_1, \ldots, x_n$, let $E = \bigwedge V$ be the exterior algebra on V, and let M be a matroid on $[n]$. The *Orlik–Solomon ideal* of M over E is the ideal

$$\begin{aligned} I(\mathsf{M}) &= (\partial(x_T) \mid \mathbf{a} \in \{0,1\}^n, \operatorname{supp}(\mathbf{a}) \text{ a circuit of } \mathsf{M}) \\ &= \Big((x_{i_2} - x_{i_1}) \cdots (x_{i_k} - x_{i_{k-1}}) \mid \{i_1 < \cdots < i_k\} \text{ a circuit of } \mathsf{M}\Big). \end{aligned}$$

The *Orlik–Solomon algebra* of M over E is the quotient E/I of E by the Orlik-Solomon ideal $I(\mathsf{M})$.

Defined in this manner, Orlik and Solomon's result can be stated as follows:

Theorem 4.3 ([OS80])**.** *Let $\mathcal{A}$ be an arrangement of n hyperplanes in $\mathbb{C}^\ell$, and let $\mathcal{M}(\mathcal{A})$ be the complement of the arrangement. Then the cohomology ring of $\mathcal{M}(\mathcal{A})$ with $\mathbb{C}$-coefficients is isomorphic to the Orlik–Solomon algebra of the $\mathbb{C}$-representable matroid $\mathsf{M}(\mathcal{A})$.*

While the cohomology ring of the complement of a complex hyperplane arrangement can be expressed solely in terms of the combinatorial data of the arrangement, the combinatorics cannot recover all of the topological data. Indeed, Rybnikov gives in [Ryb11] two hyperplane arrangements with the same combinatorial data but whose complements have non-isomorphic fundamental groups.

4.2 Historical Results

In this section, we discuss some of the prominent historical results concerning resonance varieties of Orlik–Solomon algebras. We discuss two types of results: (1) results which are special to $\mathbb{C}$-representable matroids by virtue of their appeal to the topological data of a complex hyperplane arrangement; (2) results which hold for for general Orlik–Solomon algebras. We refer the reader to [Den16], [SS06], and [Suc14] for a more in-depth discussion of several of the results mentioned here.

Throughout this section and the rest of the document, for a matroid M we write $\mathcal{R}^i_j(\mathsf{M})$ in place of $\mathcal{R}^i_j(E/I(\mathsf{M}))$, where $E/I(\mathsf{M})$ is the Orlik–Solomon algebra of M.

4.2.1 Resonance for Complex Hyperplane Arrangements

In this subsection, we discuss two results pertaining to resonance varieties of Orlik–Solomon algebras associated to complex hyperplane arrangements. The first of the two results describes what the resonance varieties of Orlik–Solomon algebras look like. The second result uses the first in order to express a relationship between the resonance varieties and graded Betti numbers of an Orlik–Solomon algebra. This second result acts as motivation for the definition introduced in Chapter 6.

Throughout this subsection, let $\mathcal{A}$ be a central hyperplane arrangement of n hyperplanes in $\mathbb{C}^\ell$, and let $\mathsf{M} = \mathsf{M}(\mathcal{A})$ be the $\mathbb{C}$-representable matroid associated to $\mathcal{A}$. We add the extra assumption that the base field $\Bbbk$ of V has characteristic zero.

The Shape of Resonance Varieties

Let $E = \bigwedge V$, and let $E/I(\mathsf{M})$ be the Orlik–Solomon algebra of M. Cohen and Suciu in [CS99] and Cohen and Orlik in [CO00] show that all resonance varieties $\mathcal{R}^i_j(\mathsf{M})$ are unions of linear subspaces of $\Bbbk^n$. Denham points out in [Den16] that when $\Bbbk$ has characteristic zero, it is an open question whether or not the resonance varieties of Orlik–Solomon algebras of non-representable matroids are unions of linear subspaces. In contrast to the characteristic zero case, Falk gives an example in [Fal07] showing that if $\Bbbk$ has positive characteristic then the resonance varieties associated to the Orlik–Solomon algebra $E/I(\mathsf{M})$ need not be unions of linear subspaces.

The Chen Ranks Theorem

Our discussion here focuses on a sequence of numbers defined from the topological data associated to the hyperplane arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$. We refer the reader to [SS06] for further references and discussion on the history of the results we cite below.

Let G be the fundamental group of $\mathcal{M}(\mathcal{A})$, the complement of $\mathcal{A}$. Let $G' = [G, G]$ be the commutator subgroup of G, and let G'' to be the commutator subgroup of G'. The *Chen ranks* of G, first introduced in [Che51], are the lower central series ranks of $H = G/G''$. Explicitly, setting $H_1 = H$ and $H_i = [H_{i-1}, H]$, the Chen ranks of G are the sequence of ranks

$$\theta_i(G) = \operatorname{rank} H_i/H_{i+1}.$$

It was shown in [PS04] that the Chen ranks of G are determined solely by the combinatorial data $\mathsf{M} = \mathsf{M}(\mathcal{A})$ of the arrangement $\mathcal{A}$. This result laid the groundwork for the task of finding a combinatorial formula this sequence of numbers defined from the topological data of the arrangement.

Noting that in this context $\mathcal{R}^1(\mathsf{M})$ is a union of linear subspaces by the results from the preceding discussion, Suciu conjectured the following formula in [Suc01] expressing the Chen ranks in terms of the first resonance variety:

$$\theta_{i+1}(G) = i\sum_{r\geq 1} h_r \binom{r+i-1}{i+1}, \text{ for } i \text{ sufficiently large,}$$

where h_r is the number of r-dimensional subspaces in $\mathcal{R}^1(\mathsf{M})$. In their paper [SS06], Schenck and Suciu prove one inequality of the conjectured formula and identify the Chen ranks with some of the graded Betti numbers of the Orlik–Solomon algebra:

$$\theta_{i+1}(G) = \beta_{i,i+1}(E/I(\mathsf{M})).$$

Finally in [CS15], Cohen and Schenck provide the other inequality of the conjectured Chen ranks formula. Putting these pieces together yields the following:

Theorem 4.4 (Chen Ranks Theorem, [SS06] and [CS15])**.** *Let M be a $\mathbb{C}$-representable matroid, let V be a $\Bbbk$-vector space with $\Bbbk$ of characteristic zero, and let $E = \bigwedge V$. For sufficiently large i, the graded Betti numbers of the Orlik–Solomon algebra of M are given by the formula*

$$\beta^E_{i,i+1}(E/I(\mathsf{M})) = i\sum_{r\geq 1} h_r \binom{r+i-1}{i+1},$$

where h_r is the number of r-dimensional linear components of the first resonance variety $\mathcal{R}^1_1(E/I(\mathsf{M}))$.

Despite relating objects solely definable for an arbitrary graded E-module, namely graded Betti numbers and resonance varieties, the proof of this result makes use of knowledge about the fundamental group of the complement of a complex hyperplane arrangement, information derived from the topological data of the arrangement. This raises the question of what relationship unmediated by topology might exist between graded Betti numbers and resonance varieties. We explore this question in the latter half of this document, beginning in Chapter 6.

4.2.2 Resonance for General Matroids

One of the most celebrated results in the theory of resonance varieties of Orlik–Solomon algebras of general (not necessarily $\mathbb{C}$-representable) matroids gives a characterization of the components of the first resonance variety. Building on work by Libgober and Yuzvinsky in [LY00], Falk and Yuzvinsky in [FY07] give a correspondence between components of the first resonance variety and combinatorial structures called *multinets*. Because the authors of the aforementioned articles work in terms of arrangements but their results hold for Orlik–Solomon algebras of general matroids, we follow the presentation in [Den16].

Definition 4.5. Let M be a matroid on $[n]$. A (k, d) *multinet* on M consists of:

- a partition of $[n]$ into some number $k \geq 3$ of disjoint subsets $\mathcal{L} = \{\mathcal{L}_1, \ldots, \mathcal{L}_k\}$;
- the *base locus*, a subset $\mathcal{X} \subseteq L_2(\mathsf{M})$ of rank 2 flats of M

satisfying the following properties:

- every flat $X \in \mathcal{X}$ has at least three elements
- $|\mathcal{L}_\ell| = d$ for all ℓ
- if $i, j \in [n]$ are in different parts of the partition, then the closure $\overline{\{i, j\}}$ is in the base locus $\mathcal{X}$
- for any $X \in \mathcal{X}$, the number $|X \cap \mathcal{L}_\ell|$ is independent of ℓ
- for any ℓ and any $i, j \in \mathcal{L}_\ell$, there is a sequence $i = i_0, i_1, \ldots, i_s = j$ such that the closure $\overline{\{i_{r-1}, i_r\}}$ is not in the base locus $\mathcal{X}$

Given a multinet $(\mathcal{L}, \mathcal{X})$ with k parts on a matroid M, we associate a subspace of E_1 as follows: For each ℓ, define

$$u_\ell = \sum_{i \in \mathcal{L}_\ell} x_i,$$

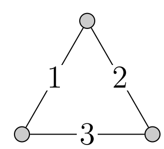
and let $Q_{(\mathcal{L},\mathcal{X})} = \mathrm{span}_{\Bbbk}\{u_\ell - u_r \mid 1 \leq \ell, r \leq k\}$. The importance of these subspaces associated to multinets comes from the following:

Theorem 4.6 ([FY07], Theorems 2.4 and 2.5)**.** *Let M be a matroid on $[n]$, let V be an n-dimensional $\Bbbk$-vector space where $\Bbbk$ has characteristic zero, and let $E = \bigwedge V$. Then the irreducible components of $\mathcal{R}^1(\mathsf{M})$ are in one-to-one correspondence with the multinets on submatroids of $\mathsf{M}\restriction_X$ with $X \subseteq [n]$, and*

$$\mathcal{R}^1(\mathsf{M}) = \{0\} \cup \bigcup_{X \subseteq [n]} \bigcup_{\substack{(\mathcal{L},\mathcal{X}) \\ \text{multinet on} \\ \mathsf{M}\restriction_X}} Q_{(\mathcal{L},\mathcal{X})}.$$

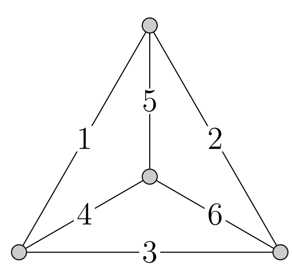
We provide several examples of hyperplane arrangements supporting multinets to illustrate the preceding results.

Example 4.7. Let M be the graphic matroid on $[3]$ associated to the following graph:



Setting $\mathcal{L}_1 = \{1\}$, $\mathcal{L}_2 = \{2\}$, $\mathcal{L}_3 = \{3\}$, and letting $\mathcal{X}$ consist of the flat $\{1, 2, 3\}$ defines a multinet on M. Because this is the only multinet on M, it follows from Theorem 4.6 that $\mathcal{R}^1(\mathsf{M}) = \mathrm{span}_{\Bbbk}\{x_1 - x_2, x_2 - x_3\}$.

Example 4.8. Let M be the graphic matroid on $[6]$ associated to the following graph:



Setting $\mathcal{L}_1 = \{1,6\}$, $\mathcal{L}_2 = \{2,4\}$, $\mathcal{L}_3 = \{3,5\}$, and $\mathcal{X} = \{\{1,4,5\},\{2,5,6\},\{3,4,6\},\{1,2,3\}\}$ defines a multinet on M. It follows from Theorem 4.6 that

$$\mathrm{span}_{\Bbbk}\{x_1 + x_6 - x_2 - x_4, x_2 + x_4 - x_3 - x_5\} \subseteq \mathcal{R}^1(\mathsf{M}).$$

It is shown in Section 3.5 of [SS06] that the only multinets on graphic arrangements are those given in the preceding examples. It follows from Theorem 4.6 that the first resonance variety of a graphic arrangement is a union of components, one for each complete subgraph on 3 or 4 vertices.

Finally, we give one particularly notable example of a non-graphic matroid which admits a multinet.

Example 4.9. The *Hessian arrangement* is a complex hyperplane arrangement in $\mathbb{C}^3$ with defining polynomial $Q = xyz(x^3 - y^3)(x^3 - z^3)(y^3 - z^3)$. The matroid corresponding to this arrangement supports a multinet with 4 parts. It was recently proved in [BK20] that this is the only example of a hyperplane arrangement supporting a multinet with more than 3 parts, confirming a conjecture of Yuzvinsky in [Yuz12].

Theorem 4.6 has become the standard toward which one strives to reach when exploring resonance varieties $\mathcal{R}^i(\mathsf{M})$ with $i > 1$. While there has been some work on understanding there higher resonance varieties (see, for example, Denham's paper [Den16]), a combinatorial characterization of their components remains elusive. We offer some first steps toward a generalization of the multinet Theorem 4.6 in the next chapter.

5. FIRST STEPS TOWARD GENERALIZING MULTINETS

The multinet Theorem 4.6 has its origins in [LY00]. In this article, Libgober and Yuzvinsky give a system of equations characterizing, for given λ and η in E_1, whether or not $\lambda\eta$ is in the Orlik–Solomon ideal. In the first section, we generalize several of the results from [LY00] and produce such a system of equations for higher degree elements. In the second section, we demonstrate how the result interacts with the Multinet Theorem 4.6 and discuss roadblocks toward generalizing the Multinet Theorem for higher resonance varieties. In the third section, we use the system of equations we created to give a sufficient condition for when elements of the second resonance variety of a submatroid are contained in the second resonance variety of the over-matroid.

Throughout this chapter, M is a loop-free matroid on $[n]$. For a fixed $X \in L(\mathsf{M})$, let $\mathsf{M}\restriction_X$ be the restriction of M to X, and let $E\restriction_X$ be the subalgebra of E generated in degree one by the elements $x_i \in X$. The Orlik-Solomon ideal $I(\mathsf{M}\restriction_X)$ of $\mathsf{M}\restriction_X$ is an ideal of $E\restriction_X$ contained in $I(\mathsf{M})$. For $\eta = \sum_{T \subset [n]} \eta_T x_T$ with $\eta_T \in \Bbbk$, recall that we define the *restriction* of η to X to be $\eta\restriction_X = \sum_{T \subset X} \eta_T x_T$.

5.1 A System of Equations

We begin by introducing some notation and two technical lemmas from Chapter 3 of [OT92], results which are also discussed in Section 2 of [Fal97]. For a flat $Y \in L(\mathsf{M})$, let $E_Y = \mathrm{span}_\Bbbk\{x_T \mid \overline{T} = Y\}$, and notice that $E = \bigoplus_{Y \in L(\mathsf{M})} E_Y$.

Lemma 5.1 ([OT92], Proposition 3.24)**.** *For a matroid* M*, the Orlik–Solomon ideal of* M *has the following decomposition:*

$$I(\mathsf{M}) = \bigoplus_{Y \in L(\mathsf{M})} I(\mathsf{M}) \cap E_Y.$$

Lemma 5.2 ([OT92], Lemma 3.28)**.** *For a matroid* M *and a flat* $Y \in L(\mathsf{M})$*, the following equality of Orlik–Solomon algebras holds:*

$$I(\mathsf{M}\restriction_X) = I(\mathsf{M}) \cap E\restriction_X.$$

We are now ready to begin the process of obtaining our desired system of equations. We first prove a result that shows an element of E is in an Orlik–Solomon ideal if and only if it is "locally" in the Orlik–Solomon ideal.

Lemma 5.3 (cf. [LY00], Lemma 3.2)**.** *If $\eta \in E_p$, then $\eta \in I(\mathsf{M})$ if and only if $\eta\restriction_X \in I(\mathsf{M}\restriction_X)$ for all $X \in L_p(\mathsf{M})$.*

Proof. Suppose first that $\eta \in I(\mathsf{M})$, and we will show that for a given $X \in L_p(\mathsf{M})$ we have that $\eta\restriction_X \in I(M\restriction_X)$. By Lemma 5.1, we may assume that $\eta \in E_Y$ for some flat $Y \in L(\mathsf{M})$. Thus, η has an expression

$$\eta = \sum_{\substack{T\subset[n]\\ \overline{T}=Y}} \eta_T x_T,$$

where each $\eta_T \in \Bbbk$. If $Y \subseteq X$, then each T with $\overline{T} = Y$ is also contained in X, whence $\eta\restriction_X = \eta$ is in $E\restriction_X \cap I(\mathsf{M}) = I(\mathsf{M}\restriction_X)$ by Lemma 5.2. On the other hand, if $Y \not\subseteq X$, then no set T with $\overline{T} = Y$ can be contained in X by virtue of the fact that X is closed and closures respect containment. In this case, $\eta\restriction_X = 0$ is also contained in $I(\mathsf{M}\restriction_X)$.

Conversely, suppose that $\eta \in E_p$ is such that $\eta\restriction_X \in I(\mathsf{M}\restriction_X)$ for all $X \in L_p(\mathsf{M})$. Write

$$\eta = \sum_{\substack{T\subset[n]\\ |T|=p}} \eta_T x_T$$

with $\eta_T \in \Bbbk$. We prove that $\eta \in I(\mathsf{M})$ by induction on the number of nonzero $\eta_T \in \Bbbk$. If all η_T are zero, then $\eta = 0 \in I(\mathsf{M})$. If not all η_T are zero, fix a subset $R \subset [n]$ with $|R| = p$ such that $\eta_R \neq 0$ and $X = \overline{R}^{\mathsf{M}}$ has minimal rank over the closures of all such subsets. If X has rank $< p$, then every subset $T \subset X$ of size p is dependent in $\mathsf{M}\restriction_X$. It follows that for such T, $x_T \in I(\mathsf{M}\restriction_X)$, therefore $\eta\restriction_X \in I(\mathsf{M}\restriction_X) \subseteq I(\mathsf{M})$. If X has rank p, then by assumption $\eta\restriction_X \in I(\mathsf{M}\restriction_X) \subseteq I(\mathsf{M})$. In any case, it follows that showing $\eta \in I(\mathsf{M})$ is equivalent to showing that $\tau = \eta - \eta\restriction_X \in I(\mathsf{M})$. Since τ has fewer nonzero summands than η, the result will follow by the induction hypothesis if we can show that $\tau\restriction_Y \in I(\mathsf{M}\restriction_Y)$ for all flats $Y \in L_p(\mathsf{M})$. To this end, let $Y \in L_p(\mathsf{M})$ be given. Notice that $\eta\restriction_Y \in I(\mathsf{M}\restriction_Y)$ assumption, hence showing that $\tau \in I(\mathsf{M}\restriction_Y)$ is equivalent to showing that $(\eta\restriction_X)\restriction_Y \in I(\mathsf{M}\restriction_Y)$. If $(\eta\restriction_X)\restriction_Y = 0$, we are done. If $(\eta\restriction_X)\restriction_Y \neq 0$, then there exists $T \subset X \cap Y$ with $|T| = p$ and $\eta_T \neq 0$. Since X has minimal rank and $\overline{T}^{\mathsf{M}} \subseteq X \cap Y$, it follows that $X = \overline{T}^{\mathsf{M}} \subseteq Y$. We therefore have that

$$(\eta\restriction_X)\restriction_Y = \eta\restriction_X \in I(\mathsf{M}\restriction_X) \subseteq I(\mathsf{M}\restriction_Y).$$

This concludes the induction and the proof. □

Next, we give another condition for an element to be contained within an Orlik–Solomon ideal. The primary piece of information used is that Orlik–Solomon ideals are generated by images under the derivation ∂ with $\partial(x_i) = 1$. The key point is that an application of ∂ can be "undone" by multiplication by $x_1 + \cdots + x_n$.

Lemma 5.4. *For any $\eta \in E$, $\eta \in I(\mathsf{M})$ implies that $(\sum_i x_i)\eta \in I(\mathsf{M})$ and $\partial(\eta) \in I(\mathsf{M})$. The converse holds if the characteristic of $\Bbbk$ does not divide n.*

Proof. Suppose first that $\eta \in I(\mathsf{M})$. Since $I(\mathsf{M})$ is an ideal, $(\sum x_i)\eta \in I(\mathsf{M})$ as well. Since I is generated by elements of the form $\partial(x_R)$, we may write $\eta = \sum w_R \partial(x_R)$ with $w_R \in E$ and $\partial(x_R) \in I(\mathsf{M})$. Since $\partial^2 = 0$, $\partial(\eta) = \sum \partial(w_R)\partial(x_R) \in I(\mathsf{M})$.

Conversely, suppose that the characteristic of $\Bbbk$ does not divide n. By the graded product rule, $\partial((\sum x_i)\eta) = n\eta - (\sum x_i)\partial(\eta)$, it follows that $\eta = (1/n)(\partial((\sum x_i)\eta) + (\sum x_i)\partial(\eta))$. Applying the forward implication to $(\sum x_i)\eta$ and $\partial(\eta)$, it follows that $\partial((\sum x_i)\eta)$ and $(\sum x_i)\partial(\eta)$ are in $I(\mathsf{M})$. Since η is a linear combination of these two elements, it follows that $\eta \in I(\mathsf{M})$ as desired. □

Combining the previous two lemmas together gives a necessary and sufficient condition for when a homogeneous element of E is an element of $I(\mathsf{M})$:

Proposition 5.5 (cf. [LY00], Lemma 3.3)**.** *Let M be a loop-free matroid on $[n]$, let $I(\mathsf{M})$ be the Orlik–Solomon ideal of M, and let $\eta \in E_p$. If the characteristic of $\Bbbk$ does not divide n, then $\eta \in I(\mathsf{M})$ if and only if for every sequence of flats $X_1 \subset X_2 \subset \cdots \subset X_p$ with $X_i \in L_i(\mathsf{M})$ the following equality holds:*

$$\partial((\cdots\partial(\eta\restriction_{X_p})\cdots)\restriction_{X_1}) = 0.$$

Proof. We prove the result by induction on the degree p of the homogeneous element. The case when $p = 0$ is clear since $E_0 \cap I(\mathsf{M}) = 0$ because M is loop-free. Suppose now that the result holds for elements of degree $p-1$, and let $\eta \in E_p$ be given. By Lemma 5.3, $\eta \in I(\mathsf{M})$ if and only if for all $X_p \in L_p(\mathsf{M})$ we have that $\eta\restriction_{X_p} \in I(\mathsf{M}\restriction_{X_p})$. For a fixed $X_p \in L_p(\mathsf{M})$, Lemma 5.4 implies that $\eta\restriction_{X_p} \in I(\mathsf{M}\restriction_{X_p})$ if and only if $(\sum_{i\in X_p} x_i)\eta\restriction_{X_p} \in I(\mathsf{M}\restriction_{X_p})$ and $\partial(\eta\restriction_{X_p}) \in I(\mathsf{M}\restriction_{X_p})$. Since X_p has rank p, it follows that $\mathsf{M}\restriction_{X_p}$ is a matroid of rank p and $E_{p+1} \subset I(\mathsf{M}\restriction_{X_p})$. It therefore follows that $(\sum_{i\in X_p} x_i)\eta\restriction_{X_p} \in I(\mathsf{M}\restriction_{X_p})$. This shows that $\eta\restriction_{X_p} \in I(\mathsf{M}\restriction_{X_p})$ if and only if $\partial(\eta\restriction_{X_p}) \in I(\mathsf{M}\restriction_{X_p})$. Since $\partial(\eta\restriction_{X_p})$ is homogeneous of degree $p-1$, the induction hypothesis implies that $\partial(\eta\restriction_{X_p}) \in I(\mathsf{M}\restriction_{X_p})$ if and only if for every sequence of flats $X_1 \subset \cdots X_{p-1}$ with $X_i \in L_i(\mathsf{M}\restriction_{X_p})$ we have that

$$\partial((\cdots\partial((\partial(\eta\restriction_{X_p}))\restriction_{X_{p-1}})\cdots)\restriction_{X_1}) = 0.$$

Since the flats in $L_i(\mathsf{M}\restriction_{X_p})$ are precisely the flats in $L_i(\mathsf{M})$ contained in X_p, the result follows. □

5.2 Relationship to the Multinet Theorem

In this section, we discuss how the system of equations produced in Proposition 5.5 relates to the Multinet Theorem 4.6. We then discuss some of the roadblocks toward achieving an analogue to the Multinet Theorem.

5.2.1 Multinets Yield Resonance

Let M be a loop-free matroid on $[n]$, let V be a $\Bbbk$-vector space on $x_1, \ldots, x_n$ with $\Bbbk$ characteristic zero, and let $E = \bigwedge V$. Suppose that M admits a multinet $(\mathcal{L}, \mathcal{X})$ with $k > 3$ parts $\mathcal{L}_1, \ldots, \mathcal{L}_k$ as in Definition 4.5. For each $1 \leq \ell \leq k$, let

$$u_\ell = \sum_{i\in\mathcal{L}_\ell} x_i,$$

and let $Q_{(\mathcal{L},\mathcal{X})} = \operatorname{span}_{\Bbbk}\{u_\ell - u_r \mid 1 \leq \ell, r \leq k\}$. The Multinet Theorem 4.6 implies that $Q_{(\mathcal{L},\mathcal{X})} \subseteq \mathcal{R}^1(\mathsf{M})$. We will show how to obtain this by making use of our system of equations given in Proposition 5.5.

We will show that $\lambda = u_1 - u_2 \in \mathcal{R}^1(\mathsf{M})$, the case for a general element in $Q_{(\mathcal{L},\mathcal{X})}$ being analogous. Since $k \geq 3$, there is an element $\eta = u_1 - u_3$ in $Q_{(\mathcal{L},\mathcal{X})} \subseteq E_1$ which is not in $I(\mathsf{M}) + (\lambda)$. It follows that if $\lambda\eta \in I(\mathsf{M})$, then $\lambda \in \mathcal{R}^1(\mathsf{M})$. By Proposition 5.5, $\lambda\eta \in I(\mathsf{M})$ if and only if for every sequence $X_1 \subset X_2$ of flats with $X_i \in L_i(\mathsf{M})$, we have that

$$\partial((\partial(\lambda\restriction_{X_2}\eta\restriction_{X_2}))\restriction_{X_1}) = 0. \tag{5.6}$$

Let $X_1 \subset X_2$ be given flats with $X_i \in L_i(\mathsf{M})$. If $X_2 \in \mathcal{X}$, the fact that ∂ is a derivation implies that

$$\begin{aligned}
\partial(\lambda\restriction_{X_2}\eta\restriction_{X_2}) &= \partial(\lambda\restriction_{X_2})\eta\restriction_{X_2} - \lambda\restriction_{X_2}\partial(\eta\restriction_{X_2}) \\
&= (|X_2 \cap \mathcal{L}_1| - |X_2 \cap \mathcal{L}_2|)\eta\restriction_{X_2} - \lambda\restriction_{X_2}(|X_2 \cap \mathcal{L}_1| - |X_2 \cap \mathcal{L}_3|) \\
&= 0,
\end{aligned}$$

where the last equation holds since $|X_2 \cap \mathcal{L}_\ell|$ is independent of ℓ by the definition of a multinet. Thus (5.6) holds in the case where $X_2 \in \mathcal{X}$. If $X_2 \notin \mathcal{X}$, then the definition of a multinet implies X_2 cannot meet more than one part of the partition. Hence $X_2 \subset \mathcal{L}_\ell$ for some ℓ. If either $\lambda\restriction_{X_2}$ or $\eta\restriction_{X_2}$ are zero, then (5.6) holds and we are done. In the case where both are nonzero, the fact that λ and η are both in $Q_{(\mathcal{L},\mathcal{X})} = \operatorname{span}_{\Bbbk}\{u_\ell - u_r \mid 1 \leq \ell, r \leq k\}$ means that $\lambda\restriction_{X_2} = \eta\restriction_{X_2}$, whence their product is zero and (5.6) holds.

5.2.2 Obstacles in Generalization

For a matroid M, the Multinet Theorem accomplishes producing resonance weights in $\mathcal{R}^1(\mathsf{M})$ by describing a set of conditions under which one has in hand elements which solve the system of equations from Proposition 5.5. More precisely, a multinet produces an element $\lambda \in E_1$ and guarantees the existence of an $\eta \in E_1$ such that $\lambda\eta \in I(\mathsf{M})$ but $\eta \notin (\lambda) + I(\mathsf{M})$. Finding conditions under which such λ and η can be described is made easier by two facts:

(1) Because $\eta \in E_1$ and $I(\mathsf{M})$ in degree one is small, $\eta \notin (\lambda) + I(\mathsf{M})$ essentially amounts to η not being a multiple of λ. Indeed, this is exactly the condition in the case when M is simple. For higher resonance varieties, η will be in degree $p > 1$ and $I(\mathsf{M})$ in degree p contains more elements.

(2) Because we require $\lambda\eta \in I(\mathsf{M})$ and both λ and η are in E_1, the system of equation produced from Proposition 5.5 is relatively small. For higher resonance varieties, η will be in degree $p > 1$ and the system of equations is more complicated.

With the previous two points in mind, the next easiest case to consider is how to find elements in $\mathcal{R}^2(\mathsf{M})$. In order to simplify (1), assume that M contains no dependent sets of size 1, 2, or 3 so that $I(\mathsf{M}) = 0$ in degree 2. In this situation, finding elements of $\mathcal{R}^2(\mathsf{M})$ amounts to finding $\lambda \in E_1$ and $\eta \in E_2$ satisfying the system of equations from Proposition 5.5. We leave this as an open problem:

Problem 5.7. Let M be a matroid which contains no dependent sets of size 1, 2, or 3. Find conditions on M so that elements $\lambda \in E_1$ and $\eta \in E_2$ can be produced so that $\lambda\eta$ satisfies the system of equations from Proposition 5.5.

5.3 Application: Adding Edges to Graphs

Let $\mathsf{M}' = M\restriction_{[n-1]}$ be the restriction of M to $[n-1]$. Here, we use the preceding result to give a new condition for determining when resonant weights in $\mathcal{R}^2(\mathsf{M}')$ yield resonant weights of $\mathcal{R}^2(\mathsf{M})$. This partially addresses a point brought up by Denham in Example 4.11 of [Den16] where it is shown that for graphic matroids M' and M, there need not be a containment $\mathcal{R}^2(\mathsf{M}') \subseteq \mathcal{R}^2(\mathsf{M})$. After the proof, we give a concrete example demonstrating its use for graphic arrangements.

To help with the statement, we introduce a piece of notation. Given $\lambda = \lambda_1 x_1 + \cdots + \lambda_n x_n$ in $\mathcal{R}^2(\mathsf{M}')$ we define the *base locus* of λ with respect to M' to be the collection of flats

$$\begin{aligned}\mathcal{X}_{\mathsf{M}'}(\lambda) &= \left\{ X \in L(\mathsf{M}') \mid \sum_{i \in X} \lambda_i = 0 \right\} \\ &= \left\{ X \in L(\mathsf{M}') \mid \partial(\lambda\restriction_X) = 0 \right\}.\end{aligned}$$

Theorem 5.8. *Let* M *be a simple matroid on* $[n]$*, and let* $\mathsf{M}' = M\restriction_{[n-1]}$ *be the submatroid of* M *on* $[n-1]$*. Let* E *be the exterior algebra on a vector space with basis* $x_1, \ldots, x_n$ *over a field whose characteristic does not divide* n*, and let* $E' = E\restriction_{[n-1]}$ *be the subalgebra on* $x_1, \ldots, x_{n-1}$*. Let* $\lambda \in \mathcal{R}^2(\mathsf{M}')$*. If there exists* $X \in L_2(\mathsf{M})$ *with* $n \in X$ *and* $X \setminus \{n\} \notin \mathcal{X}_{\mathsf{M}'}(\lambda)$*, then* $\lambda \in \mathcal{R}^2(\mathsf{M})$*.*

Proof. Using the notation of the statement of the theorem, the fact that $\lambda \in \mathcal{R}^2(\mathsf{M}')$ means that $\lambda \in E'_1$ and there exists $\eta \in E'_2$ such that $\lambda\eta \in I(\mathsf{M}')$ and $\eta \notin (\lambda)\, E' + I(\mathsf{M}')$. Since $E' \subseteq E$ and $I(\mathsf{M}') \subseteq I(\mathsf{M})$, it follows that $\lambda\eta \in I(\mathsf{M})$. To show that $\lambda \in \mathcal{R}^2(\mathsf{M})$, it suffices to show that $\eta \notin (\lambda)\, E + I(\mathsf{M})$. Equivalently, it suffices to show that for all $\sigma \in E_1$, the element $\eta + \lambda\sigma$ is not in $I(\mathsf{M})$.

Suppose toward contradiction that there exists $\sigma \in E_1$ such that $\eta + \lambda\sigma \in I(\mathsf{M})$. Because E' is the subalgebra of E generated by $x_1, \ldots, x_{n-1}$, we may write $\sigma = \rho + \alpha x_n$ with $\rho \in E'_1$ and $\alpha \in \Bbbk$. By Proposition 5.5, for each sequence of flats $X_1 \subset X_2$ with $X_i \in L_i(\mathsf{M})$ we have that

$$\partial(\partial(\eta\restriction_{X_2} + \lambda\restriction_{X_2}(\rho\restriction_{X_2} + \alpha x_n\restriction_{X_2}))\restriction_{X_1}) = 0. \tag{5.9}$$

We first treat the case when $\alpha = 0$. Let $Y_1 \subset Y_2$ be a given sequence of flats with $Y_i \in L_i(\mathsf{M}')$. Because the independent sets of M' are the independent sets of M contained in $[n-1]$, it follows that the closures $X_i = \overline{Y_i}^{\mathsf{M}}$ of Y_i in M are flats in $L_i(\mathsf{M})$. Moreover, $X_i = \overline{Y_i}^{\mathsf{M}}$ is equal to Y_i or $Y_i \cup \{n\}$. In either case, the fact that η, λ, and ρ are in E' implies that their restrictions to X_1 or X_2 equal their restrictions to Y_1 and Y_2. By (5.9) and the fact $\alpha = 0$, it follows that

$$\partial(\partial(\eta\restriction_{Y_2} + \lambda\restriction_{Y_2}\rho\restriction_{Y_2})\restriction_{Y_1}) = 0. \tag{5.10}$$

Since Y_1 and Y_2 were arbitrary, it follows that (5.10) holds for all $Y_1 \subset Y_2$ with $Y_i \in L_i(\mathsf{M}')$. Proposition 5.5 implies $\eta + \lambda\rho \in I(\mathsf{M}')$, whence $\eta \in (\lambda)\, E' + I(\mathsf{M}')$, a contradiction.

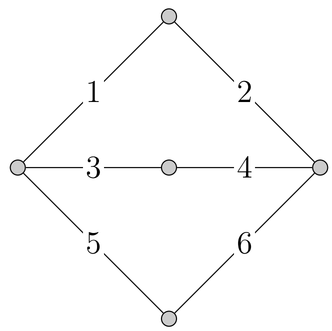
Suppose now that $\alpha \neq 0$. Let $X_1 = \{n\}$ and $X_2 = X$, the flat in $L_2(\mathsf{M})$ guaranteed by the hypothesis of the theorem. Then (5.9) becomes

$$\partial(\lambda\restriction_{X_2})\alpha = 0,$$

whence $\partial(\lambda\restriction_{X_2}) = 0$. Since $\lambda \in E'$, it follows that $0 = \partial(\lambda\restriction_{X_2}) = \partial(\lambda\restriction_{X_2\setminus\{n\}})$, whence $X_2 \setminus \{n\} \in \mathcal{X}_{\mathsf{M}'}(\lambda)$, a contradiction. This concludes the proof. $\square$

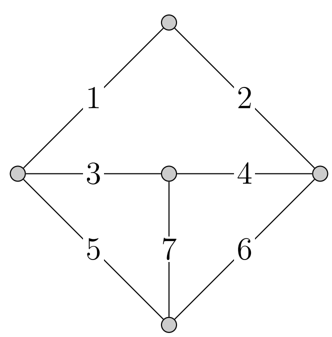
We now give an example illustrating how Theorem 5.8 can be used for tracing resonance when adding edges to graphs.

Example 5.11. Let M' be the graphic matroid on $[6]$ associated to the following graph:



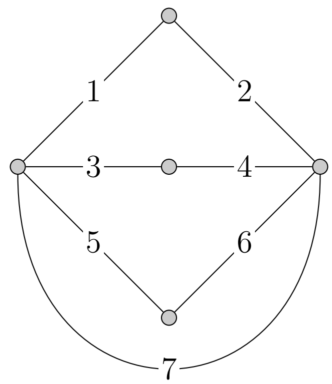
For $\lambda = \lambda_1 x_1 + \cdots + \lambda_6 x_6$ with $(\lambda_1, \dots, \lambda_6)$ a general element of $\mathbb{V}(y_1 + y_2, y_3 + y_4, y_5 + y_6)$, it can be checked that for $\eta = (\lambda_1 - \lambda_2)x_1x_2 + (\lambda_3 - \lambda_4)x_3x_4 + (\lambda_5 - \lambda_6)x_5x_6$ we have that $\lambda\eta \in I(\mathsf{M}')$ but $\eta \notin (\lambda) + I(\mathsf{M}')$. Since λ was general and resonance varieties are Zariski closed, it follows that $\mathbb{V}(y_1 + y_2, y_3 + y_4, y_5 + y_6) \subseteq \mathcal{R}^2(\mathsf{M}')$.

Now let M_1 be the graphic matroid on $[7]$ associated to the following graph



Note that $X = \{2, 7\}$ is a rank 2 flat of M_1 such that $X \setminus \{7\} \notin \mathcal{X}(\lambda)$ by virtue of the fact that λ is general in $\mathbb{V}(y_1 + y_2, y_3 + y_4, y_5 + y_6)$. It follows from Theorem 5.8 that $\lambda \in \mathcal{R}^2(\mathsf{M}_1)$, whence $\mathbb{V}(y_1 + y_2, y_3 + y_4, y_5 + y_6, y_7) \subseteq \mathcal{R}^2(\mathsf{M}_1)$ since λ is general and resonance varieties are Zariski closed.

On the other hand, let M_2 be the graphic matroid on [7] associated to the following graph



There are exactly three rank 2 flats containing 7, namely $\{1,2,7\}$, $\{3,4,7\}$, and $\{5,6,7\}$. Since $\{1,2\}$, $\{3,4\}$, and $\{5,6\}$ are all in $\mathcal{X}_{\mathsf{M}'}(\lambda)$, Theorem 5.8 does not apply in this situation. Additionally, it can be checked that $\mathbb{V}(y_1+y_2, y_3+y_4, y_5+y_6, y_7) \not\subseteq \mathcal{R}^2(\mathsf{M}_2)$.

In light of the previous example, we close with the following:

Question 5.12. Does the converse of Theorem 5.8 hold?

6. SYZYGETIC RESONANCE VARIETIES

One notable problem with which one must grapple when studying free resolutions over an exterior algebra is the fact that these resolutions are nearly always infinite. Because these resolutions can not be written down end-to-end, one often seeks a finite means to express the resolution's infinite behavior. Through this lens, the Chen Ranks Theorem 4.4 is enticing: it suggests that resonance varieties may be intimately connected to the asymptotic behavior of the graded Betti numbers of a module.

Despite the intimations of a deep relationship, known results exploring resonance and Betti numbers leave something to be desired. To begin, the Chen Ranks Theorem applies only to special types of E-modules, and it only yields expressions for *some* of the graded Betti numbers (those corresponding to the linear strand). While Thieu shows in [Thi11] that the analogue of the Chen Ranks Theorem 4.4 holds for some, but not all, quotients of an exterior algebra by a monomial ideal, the Betti numbers described are still limited to the linear strand. We undertake the task of trying to understand the relationship between resonance and graded Betti numbers from first principles.

In this chapter, we define a new object which we call a *syzygetic resonance variety*. These varieties consist of the resonant weights that pertain to the graded Betti numbers. These new varieties will be the main objects of study throughout the remainder of the document. We begin, however, with a pair of results which bring forth the relationship between resonance varieties and syzygy modules which will play a crucial role in the definition of syzygetic resonance varieties.

6.1 Syzygetic Manifestations of Resonance

We show here that up to a shift, the resonance varieties of M are invariant under taking syzygy modules. Though the proof is elementary, the author is unaware of the result appearing elsewhere in the literature. Upon seeing the result, Graham Denham suggested a much shorter proof making use of the long exact sequence in cohomology coming from a short exact sequence of complexes. Despite the brevity of this alternate proof, the author's original proof is presented here because it introduces a technique for constructing syzygies which will be of importance for proving subsequent results. As a corollary of the result, we show that minimal syzygies of M of a special form give resonant weights of M.

Theorem 6.1. *Let M be a graded E-module, $(F_\bullet, d)$ the minimal graded free resolution of M, and let $\mathrm{Syz}_p^E(M) = \mathrm{im}(d\colon F_p \to F_{p-1})$ be the p-th syzygy module of M. Then for all $p, j \geq 1$ and all $i \in \mathbb{Z}$,*

$$\mathcal{R}_j^i(M) \setminus \{0\} = \mathcal{R}_j^{i+p}\left(\mathrm{Syz}_p^E(M)\right) \setminus \{0\}.$$

Proof. By iteratively taking syzygies, it suffices to prove the result for $p = 1$. Let $j \geq 1$ be given. We first show that $\mathcal{R}_j^i(M) \setminus \{0\} \subset \mathcal{R}_j^{i+1}(\mathrm{Syz}_1^E(M)) \setminus \{0\}$. To this end, let $\lambda \in \mathcal{R}_j^i(M) \setminus \{0\}$ be given. There exist $\eta^1, \dots, \eta^j \in M_i$ whose images are $\Bbbk$-linearly independent in $M_i / (\lambda) M_{i-1}$ and are such that $\lambda\eta^1 = \cdots = \lambda\eta^j = 0$. Since $(F_\bullet, d)$ is a graded free resolution of M, there is a graded surjection $\epsilon\colon F_0 \to M$ whose kernel is $\mathrm{Syz}_1^E(M)$. Let $\eta_0^1, \dots, \eta_0^j \in (F_0)_i$ be such that $\epsilon(\eta_0^k) = \eta^k$ for all $1 \leq k \leq j$. Since $\epsilon(\lambda\eta_0^k) = \lambda\eta^k = 0$, it follows that $\lambda\eta_0^k \in (\mathrm{Syz}_1^E(M))_{i+1}$ for all $1 \leq k \leq j$. Since $\lambda^2\eta_0^k = 0$, to show that $\lambda \in \mathcal{R}_j^{i+1}(\mathrm{Syz}_1^E(M)) \setminus \{0\}$, it suffices to show that the images of $\lambda\eta_0^1, \dots, \lambda\eta_0^j$ are $\Bbbk$-linearly independent in $\mathrm{Syz}_1^E(M) / (\lambda) \, \mathrm{Syz}_1^E(M)$. To this end, let $a_1, \dots, a_j \in \Bbbk$ and $m \in \mathrm{Syz}_1^E(M)$ be such that

$$a_1\lambda\eta_0^1 + \cdots + a_j\lambda\eta_0^j + \lambda m = 0.$$

By Example 3.2,

$$a_1\eta_0^1 + \cdots + a_j\eta_0^j + m = \lambda n$$

for some $n \in F_0$, whence

$$\begin{aligned}
\lambda\epsilon(n) &= \epsilon(\lambda n) \\
&= \epsilon(a_1\eta_0^1 + \cdots + a_j\eta_0^j + m) \\
&= a_1\epsilon(\eta_0^1) + \cdots + a_j\epsilon(\eta_0^j) + \epsilon(m) \\
&= a_1\eta^1 + \cdots + a_j\eta^j.
\end{aligned}$$

This shows that $a_1\eta^1 + \cdots + a_j\eta^j \in (\lambda)\, M$. Since the images of $\eta^1, \ldots, \eta^j$ are $\Bbbk$-linearly independent elements of $M/(\lambda)\, M$, it follows that $a_1 = \cdots = a_j = 0$. Thus, the images of $\lambda\eta_0^1, \ldots, \lambda\eta_0^j$ are $\Bbbk$-linearly independent in $\mathrm{Syz}_1^E(M)/(\lambda)\,\mathrm{Syz}_1^E(M)$. Therefore $\lambda \in \mathcal{R}_j^{i+1}(\mathrm{Syz}_1^E(M)) \setminus \{0\}$.

Conversely, we show that $\mathcal{R}_j^{i+1}(\mathrm{Syz}_1^E(M)) \setminus \{0\} \subset \mathcal{R}_j^i(M) \setminus \{0\}$. Let $\lambda \in \mathcal{R}_j^{i+1}(\mathrm{Syz}_1^E(M)) \setminus \{0\}$ be given. There exist $\eta_1^1, \ldots, \eta_1^j \in (\mathrm{Syz}_1^E(M))_{i+1}$ such that $\lambda\eta_1^1 = \cdots = \lambda\eta_1^j = 0$ and whose images in $\mathrm{Syz}_1^E(M)/(\lambda)\,\mathrm{Syz}_1^E(M)$ are linearly independent. By Example 3.2, there exist elements $\eta_0^1, \ldots, \eta_0^j \in (F_0)_i$ such that $\eta_1^k = \lambda\eta_0^k$ for each $1 \leq k \leq j$, whence $0 = \epsilon(\eta_1^k) = \lambda\epsilon(\eta_0^k)$ for each k. To show that $\lambda \in \mathcal{R}_j^i(M) \setminus \{0\}$, it therefore suffices to show that the images of $\epsilon(\eta_0^1), \ldots, \epsilon(\eta_0^j)$ in $M/(\lambda)\, M$ are $\Bbbk$-linearly independent. To this end, let $a_1, \ldots, a_j \in \Bbbk$ and $m \in M_{i-1}$ be such that

$$a_1\epsilon(\eta_0^1) + \cdots + a_j\epsilon(\eta_0^j) + \lambda m = 0.$$

Choose $m_0 \in (F_0)_{i-1}$ such that $\epsilon(m_0) = m$. Then

$$\begin{aligned} a_1\eta_1^1 + \cdots + a_j\eta_1^j &= a_1\lambda\eta_0^1 + \cdots + a_j\lambda\eta_0^j \\ &= \lambda(a_1\eta_0^1 + \cdots + a_j\eta_0^j + \lambda m_0), \end{aligned}$$

and $a_1\eta_0^1 + \cdots + a_j\eta_0^j + \lambda m_0 \in \mathrm{Syz}_1^E(M)$. By the independence of the images of $\eta_1^1, \ldots, \eta_1^j$ in $\mathrm{Syz}_1^E(M)/(\lambda)\,\mathrm{Syz}_1^E(M)$, it follows that $a_1 = \cdots = a_j = 0$. This establishes the linear independence of the images of $\epsilon(\eta_0^1), \ldots \epsilon(\eta_0^j)$ in $M/(\lambda)\, M$, concluding the proof. □

We give an example showing that the element 0 may not pass between resonance varieties when passing to a syzygy module.

Example 6.2. Let E be the exterior algebra on a three-dimensional vector space with basis x_1, x_2, x_3. Let $M = E/(x_1x_2, x_1x_3, x_2x_3)$ so that $\mathrm{Syz}_1^E(M) = (x_1x_2, x_1x_3, x_2x_3)$. Notice that

$$0 \in \mathcal{R}_1^0(M) \not\subset \mathcal{R}_1^1\left(\mathrm{Syz}_1^E(M)\right) = \emptyset$$

and

$$\emptyset = \mathcal{R}_1^2(M) \not\supset \mathcal{R}_1^3\left(\mathrm{Syz}_1^E(M)\right) \ni 0.$$

Finally, we close this section by noting that minimal syzygies of M with a special form correspond to resonant weights, and we show that syzygies of this form perpetuate through the resolution of M. More precisely, we have the following:

Corollary 6.3. *Let M be a graded E-module, and let $(F_\bullet, d)$ be a graded minimal E-free resolution of M. If $\lambda \in E_1$ and $\eta^1, \ldots, \eta^j \in (F_p)_{i+p}$ are such that $\{\lambda\eta^1, \ldots, \lambda\eta^j\}$ is part of a minimal generating set of $\mathrm{Syz}_{p+1}^E(M)$, then:*

(i) $\lambda \in \mathcal{R}_j^i(M)$.

(ii) Every syzygy module $\mathrm{Syz}_{k+1}^E(M)$ with $k \geq p$ has a minimal generating set containing $\{\lambda\eta_k^1, \ldots, \lambda\eta_k^j\}$ for some $\eta_k^1, \ldots, \eta_k^j \in (F_k)_{i+k}$.

Proof. Regarding (i), note that since $\{\lambda\eta^1, \ldots, \lambda\eta^j\}$ is part of a minimal generating set for $\mathrm{Syz}_{p+1}^E(M)$, the images of these elements in $\mathrm{Syz}_{p+1}^E(M)/\left(\lambda\right)\mathrm{Syz}_{p+1}^E(M)$ are $\Bbbk$-linearly independent. Since $\lambda(\lambda\eta^\ell) = 0$ for each $1 \leq \ell \leq j$, it follows that $\lambda \in \mathcal{R}_j^{p+i+1}(\mathrm{Syz}_{p+1}^E(M))$. By the previous theorem, $\lambda \in \mathcal{R}_j^i(M)$.

Regarding (ii), the fact that $\{\lambda\eta^1, \ldots, \lambda\eta^j\}$ is part of a minimal generating set for $\mathrm{Syz}_{p+1}^E(M)$ implies that for each $1 \leq \ell \leq j$, there exist $\eta_{p+1}^\ell \in (F_{p+1})_{i+p+1}$ such that $d(\eta_{p+1}^\ell) = \lambda\eta^\ell$. It follows from Proposition 2.1 that $\{\lambda\eta_{p+1}^1, \ldots, \lambda\eta_{p+1}^j\}$ is part of a minimal generating set for $\mathrm{Syz}_{p+2}^E(M)$. Inductively defining η_k^ℓ to map to $\lambda\eta_{k-1}^\ell$ for each $1 \leq \ell \leq j$ and $k > p+1$ establishes the result. $\square$

The resonant weights associated to minimal syzygies as in Corollary 6.3 can be interpreted as those resonant weights directly relating to the graded Betti numbers of the module M. We give these resonant weights a name in the next section.

6.2 Definition and Basic Properties

In this section we give the definition of syzygetic resonance varieties as a means to explore the relationship between graded Betti numbers and resonance varieties. Our approach takes the graded Betti numbers as basic, and we identify resonant weights which contribute to the Betti numbers. Since graded Betti numbers count numbers of minimal generators of a syzygy module, we single out points in the resonance varieties which are directly involved in creating a minimal syzygy as described in Corollary 6.3.

Definition 6.4. Let M be a graded E-module with graded minimal free resolution $(F_\bullet, d)$. The i-th *syzygetic resonance variety of M of depth j*, denoted $\mathcal{SR}^i_j(M)$, is the Zariski closure of the collection of points $(\lambda_1, \ldots, \lambda_n) \in \Bbbk^n$ such that, setting $\lambda = \lambda_1 x_1 + \cdots + \lambda_n x_n$, there exists $p \in \mathbb{Z}$ and $\eta^1, \ldots, \eta^j \in (F_p)_{i+p}$ so that $\lambda\eta^1, \ldots, \lambda\eta^j$ form part of a minimal generating set of $\mathrm{Syz}^E_{p+1}(M)$. As for resonance varieties, we let $\mathcal{SR}^i(M) = \mathcal{SR}^i_1(M)$.

Remark 6.5. Corollary 6.3 (ii) shows that syzygetic resonance varieties $\mathcal{SR}^i_j(M)$ are invariant under taking syzygies up to a shift in i. Explicitly, for all $p > 0$ we have that

$$\mathcal{SR}^i_j(M) = \mathcal{SR}^{i+p}_j(\mathrm{Syz}^E_p(M)).$$

Remark 6.6. From Corollary 6.3 (i) and the fact that resonance varieties are Zariski-closed (Proposition 3.5), it follows that

$$\mathcal{SR}^i_j(M) \subseteq \mathcal{R}^i_j(M).$$

We will show in Example 7.14 that this may be a proper containment.

While the syzygetic resonance varieties may be proper subvarieties of the resonance varieties, we always have some equalities. The following result is of particular note because it shows that the resonance varieties involved in the Chen Ranks Theorem 4.4 are actually syzygetic resonance varieties.

Proposition 6.7. *If M is an E-module generated in degrees at least k, then for all $j > 0$ there is an equality $\mathcal{SR}^k_j(M) \setminus \{0\} = \mathcal{R}^k_j(M) \setminus \{0\}$. In particular, if M is a simple matroid, then $\mathcal{SR}^1_j(\mathsf{M}) \setminus \{0\} = \mathcal{R}^1_j(\mathsf{M}) \setminus \{0\}$.*

Proof. Suppose that M is generated in degrees at least k, and let $\lambda \in \mathcal{R}^k_j(M) \setminus \{0\}$. It follows that there exists $\eta^1, \ldots, \eta^j \in M_k$ such that $\lambda\eta^\ell = 0$ for all $1 \leq \ell \leq j$ and the image of the collection $\{\eta^1, \ldots, \eta^j\}$ in $M/(\lambda)M$ is $\Bbbk$-linearly independent. Let $(F_\bullet, d)$ be a minimal graded free resolution of M. Since M is generated in degrees at least k, there are minimal generators $\eta_0^1, \ldots \eta_0^j$ of F_0 such that η_0^ℓ maps to η^ℓ. Since $\lambda\eta^\ell = 0$ for each ℓ, it follows that for each $1 \leq \ell \leq j$, the element $\lambda\eta_0^\ell$ is in $\mathrm{Syz}_1^E(M)$ and is a minimal syzygy by Proposition 2.1. As in Theorem 6.1, the collection $\{\lambda\eta_0^1, \ldots, \lambda\eta_0^j\}$ is $\Bbbk$-linearly independent, whence part of a minimal generating set for $\mathrm{Syz}_1^E(M)$. By defintion, $\lambda \in \mathcal{SR}_j^k(M)$.

Finally, the point about a simple matroid M follows because

$$\begin{aligned}
\mathcal{SR}_j^1(\mathsf{M}) \setminus \{0\} &= \mathcal{SR}_j^2(\mathrm{Syz}_1^E(E/I(\mathsf{M}))) \setminus \{0\} \\
&= \mathcal{SR}_j^2(I(\mathsf{M})) \setminus \{0\} \\
&= \mathcal{R}_j^2(I(\mathsf{M})) \setminus \{0\} \\
&= \mathcal{R}_j^2(\mathrm{Syz}_1^E(E/I(\mathsf{M}))) \setminus \{0\} \\
&= \mathcal{R}_j^1(\mathsf{M}) \setminus \{0\},
\end{aligned}$$

where the first equality uses Remark 6.5, the third equality uses the fact that $I(\mathsf{M})$ is generated in degrees at least 2 since M is simple, and the last equality uses Theorem 6.1. $\square$

The next example shows that a given syzygy module need not realize all syzygetic resonance by showing that more may be realized by a higher syzygy module. In particular, this shows that the syzygetic resonance varieties are an asymptotic feature of the minimal free resolution.

Example 6.8. Let E be the exterior algebra on a $\Bbbk$-vector space with basis x_1, x_2, x_3, x_4. Let $M = E/(x_1x_2, x_3x_4)$. The first two maps in the minimal free resolution of M are

$$E(-4) \oplus E(-3)^4 \xrightarrow{\begin{pmatrix} x_3x_4 & x_1 & x_2 & 0 & 0 \\ -x_1x_2 & 0 & 0 & x_3 & x_4 \end{pmatrix}} E(-2)^2 \xrightarrow{\begin{pmatrix} x_1x_2 & x_3x_4 \end{pmatrix}} E$$

Let $\lambda = x_1 + x_3 \in E_1$. Then $\lambda \in \mathcal{R}^2(M)$ since $\lambda x_2x_4 = 0$ in M while $x_2x_4 \notin (\lambda)M$. The first map in the minimal free resolution of M shows that all minimal first syzygies of M are degree 2 so that no minimal first syzygy is of the form $\lambda\eta_0$ with $\eta_0 \in E_2$. Therefore, no element of $\mathrm{Syz}_1^E(M)$ enables us to conclude that $\lambda \in \mathcal{SR}^2(M)$. On the other hand, notice that $\mathrm{Syz}_2^E(M)$ is the E-submodule of $E(-2)^2$ minimally generated by the columns of the matrix corresponding to the second map in the resolution. Since

$$\lambda \begin{pmatrix} x_4 \\ -x_2 \end{pmatrix} = \begin{pmatrix} x_3x_4 \\ -x_1x_2 \end{pmatrix} - x_4 \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ x_3 \end{pmatrix},$$

is a minimal second syzygy, $\lambda \in \mathcal{SR}^2(M)$.

In light of this example, one natural question is if there is a single syzygy module that realizes all syzygetic resonance:

Question 6.9. Let M be a graded E-module. Does there exist k such that for general $\lambda \in \mathcal{SR}^i(M)$ there is a minimal k-th syzygy of degree $i+k$ of the form $\lambda\eta$?

We show in Remark 8.19 that the answer is 'yes' for the class of square-free modules. We introduce these modules in the next chapter, and then compute their syzygetic resonance varieties in Chapter 8.

7. SQUARE-FREE MODULES AND THEIR RESOLUTIONS

Square-free modules over a polynomial ring were first defined by Yanagawa in [Yan00] as a means to study Alexander duality. In this context, part of the utility of the class lies in the fact that Stanley–Reisner rings, their syzygy modules, and certain associated Ext modules are all square-free, thus providing a unified framework under which one can treat all of these objects. Our interest in square-free modules over an exterior algebra stems from the fact that the structure of their minimal free resolution is well understood, thus making them an ideal case study for syzygetic resonance varieties. With respect to their resolutions, Römer in [Röm01] and Aramova, Avramov, and Herzog in [AAH00] describe the resolution of square-free modules over an exterior algebra. In [EPY03], Eisenbud, Popescu, and Yuzvinsky give a conceptual presentation of the resolutions produced by the preceding authors, and it is this approach we follow throughout the chapter.

In the first section of the chapter, we give the definition of square-free modules. Next, we review the definition of the basic building blocks of resolutions of square-free modules, the Cartan resolutions. Finally, we describe in the last section how several Cartan resolutions are pieced together into a double complex, the total complex of which resolves the square-free module. The structure of this double complex is crucial to our determination of the syzygetic resonant weights of square-free modules in the next chapter.

7.1 Definition of Square-Free Modules

Given the $\Bbbk$-vector space V with basis $x_1, \dots, x_n$, we let $E = \bigwedge V$ be the exterior algebra on V and we let $S = \operatorname{Sym} V$ be the symmetric algebra on V. Throughout this chapter, we consider $\mathbb{Z}^n$-graded modules over E and S with x_i having degree $\mathbf{e}_i$ in both settings.

Definition 7.1. Let F and G be finite free $\mathbb{Z}^n$-graded modules over E or S with generators in square-free degree, and let $\varphi\colon F \to G$ be a $\mathbb{Z}^n$-graded map. A $\mathbb{Z}^n$-graded module M over E or S is *square-free* if $M \cong \operatorname{coker} \varphi$.

Choosing a matrix representation of the map φ from Defintion 7.1 yields a matrix whose entries are scalar multiples of square-free monomials. Interpreting the entries of this matrix as elements of E or elements of S gives a correspondence between square-free E-modules and square-free S-modules. Following [EPY03], we introduce notation for this correspondence.

Definition 7.2. If M is a square-free E-module, we denote by ${}_SM$ the square-free S-module to which M corresponds in the sense described above.

We give several examples to help clarify the boundaries of the class of square-free modules. The first example will be important to the the application of our results to Orlik-Solomon algebras in Chapter 9.

Example 7.3. Let I be the ideal in E generated by the square-free monomials $x^{\mathbf{a}_1}, \dots, x^{\mathbf{a}_k}$. Then there is an exact sequence

$$\bigoplus_{i=1}^k E(-\mathbf{a}_i), \xrightarrow{\begin{pmatrix} x^{\mathbf{a}_1} & \cdots & x^{\mathbf{a}_k} \end{pmatrix}} E \longrightarrow E/I \longrightarrow 0$$

whence E/I is a square-free module. In this situation, the S-module associated to E/I is ${}_S(E/I) = S/I$, where we abuse notation by understanding I to be either the E-ideal or S-ideal generated by $x^{\mathbf{a}_1}, \dots, x^{\mathbf{a}_k}$, depending on the context.

Next, we give an example to show that the class of square-free modules is strictly larger than the class of quotients by square-free monomial ideals.

Example 7.4. The cokernel of the map

$$\begin{matrix} E(-1,-1,-1,0) \\ \oplus \\ E(-1,-1,0,-1) \end{matrix} \xrightarrow{\begin{pmatrix} x_2x_3 & x_2x_4 \\ x_1x_3 & x_1x_4 \end{pmatrix}} \begin{matrix} E(-1,0,0,0) \\ \oplus \\ E(0,-1,0,0) \end{matrix}$$

is a square-free module.

Finally, we give an example showing that not every matrix with square-free monomial entries corresponds to a square-free module.

Example 7.5. For free E-modules F_1, F_2, G_1, G_2 of rank one, the cokernel of the map

$$F_1 \oplus F_2 \xrightarrow{\begin{pmatrix} x_1 & x_1 \\ x_2 & x_1 \end{pmatrix}} G_1 \oplus G_2$$

is *not* square-free. Indeed, if the cokernel were to be square-free, then the first column of the matrix would force the basis elements of F_1 and thus G_2 to have multidegrees with support containing 1. On the other hand, the second column of the matrix would force the basis element of G_2 to have multidegree with support not containing 1, a contradiction.

7.2 Cartan Resolutions

In this section, we describe the basic building blocks of the minimal free resolutions of square-free modules over E, the Cartan resolutions of the $\mathbb{Z}^n$-graded modules $E_{\mathbf{a}} := (E/\operatorname{supp}(x^{\mathbf{a}}))(-\mathbf{a})$ with $\mathbf{a}$ square-free. Direct sums of these Cartan resolutions form the columns of the double complex resolving square-free E-modules. In anticipation of its use in the next section, we note that the modules $E_{\mathbf{a}}$ are shifts of the modules from Example 3.3.

Again following the notation from [EPY03], for each square-free degree $\mathbf{a}$ in E let $L_{\mathbf{a}}$ be the $\mathbb{Z}^n$-graded $\Bbbk$-vector space with basis $\operatorname{supp}(x^{\mathbf{a}})$ and $x_i \in \operatorname{supp}(x^{\mathbf{a}})$ of degree $\mathbf{e}_i$. The *ℓ-th divided power of* $L_{\mathbf{a}}$ is defined to be the module $D_\ell = ((\operatorname{Sym}_\ell(L_{\mathbf{a}}^*))^*$, where $-^*$ denotes the $\Bbbk$-dual. There are *diagonal maps* $D_{\ell+1}(L_{\mathbf{a}}) \to D_\ell \otimes L_{\mathbf{a}}$ dual to the multiplication maps $\operatorname{Sym}_\ell(L_{\mathbf{a}}^*) \otimes L_{\mathbf{a}}^* \to \operatorname{Sym}_{\ell+1}(L_{\mathbf{a}}^*)$. Explicitly, if $\operatorname{supp}(x^{\mathbf{a}}) = \{x_{i_1}, \ldots, x_{i_m}\}$, if $X_{i_1}, \ldots, X_{i_m}$ is the dual basis of $L_{\mathbf{a}}^*$, and if $a_1 + \cdots + a_m = \ell + 1$, then the diagonal map from $D_{\ell+1}(L_{\mathbf{a}})$ to $D_\ell \otimes L_{\mathbf{a}}$ is the $\Bbbk$-linear map of $\mathbb{Z}^n$-graded vector spaces such that

$$(X_{i_1}^{a_1} \cdots X_{i_m}^{a_m})^* \mapsto \sum_{a_j > 0} (X_{i_1}^{a_1} \cdots X_{i_j}^{a_j - 1} \cdots X_{i_m}^{a_m})^* \otimes x_i,$$

where the degree of $(X_{i_1}^{a_1} \cdots X_{i_m}^{a_m})^*$ is $a_1 \mathbf{e}_{i_1} + \cdots + a_m \mathbf{e}_{i_m}$.

Definition 7.6. The *Cartan resolution* $(\Phi_\bullet(E_{\mathbf{a}}), d_v) := (D_\bullet(L_{\mathbf{a}}) \otimes E(-\mathbf{a}), d_v)$ of $E_{\mathbf{a}}$ is the $\mathbb{Z}^n$-graded minimal free resolution

$$\cdots \xrightarrow{d_v} D_2(L_{\mathbf{a}}) \otimes E(-\mathbf{a}) \xrightarrow{d_v} D_1(L_{\mathbf{a}}) \otimes E(-\mathbf{a}) \xrightarrow{d_v} E(-\mathbf{a})$$

where the $\mathbb{Z}^n$-graded maps d_v are given by the diagonal maps followed by multiplication in $E(-\mathbf{a})$. The $\mathbb{Z}^n$-grading is defined so that the degree of a tensor $\eta_1 \otimes \eta_2$ in $D_\ell(L_{\mathbf{a}}) \otimes E(-\mathbf{a})$ is the sum of the degrees of η_1 and η_2. In particular, the basis elements of $D_\ell(L_{\mathbf{a}}) \otimes E(-\mathbf{a})$ viewed as a $\mathbb{Z}$-graded module have total degree $\ell + |\mathbf{a}|$.

We give an explicit example to illustrate the notation.

Example 7.7. Let V be the vector space with basis x_1, x_2, x_3, let $E = \bigwedge V$, and let $\mathbf{b} = (1,1,0)$. Then $L_{\mathbf{b}} = \Bbbk(-1,0,0) \oplus \Bbbk(0,-1,0)$ is the $\mathbb{Z}^n$-graded vector space with basis x_1, x_2. Let X_1, X_2 be the corresponding dual basis of $L_{\mathbf{b}}^*$. Letting basis elements of $L_{\mathbf{b}}^*$ stand for their $\Bbbk$-span, the Cartan resolution $(\Phi_\bullet(E_{\mathbf{b}}), d_v)$ of $E_{\mathbf{b}} = (E/(x_1, x_2))(-1,-1,0)$ begins:

$$\cdots \longrightarrow \begin{matrix} (X_1^2)^* \otimes E(-\mathbf{b}) \\ \oplus \\ (X_1X_2)^* \otimes E(-\mathbf{b}) \\ \oplus \\ (X_2^2)^* \otimes E(-\mathbf{b}) \end{matrix} \xrightarrow{d_v = \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_1 & x_2 \end{pmatrix}} \begin{matrix} X_1^* \otimes E(-\mathbf{b}) \\ \oplus \\ X_2^* \otimes E(-\mathbf{b}) \end{matrix} \xrightarrow{d_v = \begin{pmatrix} x_1 & x_2 \end{pmatrix}} E(-\mathbf{b}).$$

The degrees are defined so that, for example, $(X_1^2)^* \otimes 1$ in $(X_1^2)^* \otimes E(-\mathbf{b})$ has multidegree $(2,0,0) + (1,1,0) = (3,1,0)$ and total degree 4.

7.2.1 Maps Between Cartan Resolutions

In this subsection we describe maps d_h between two Cartan resolutions $\Phi_\bullet(E_{\mathbf{b}})$ and $\Phi_\bullet(E_{\mathbf{a}})$ defined for compatible square-free degrees $\mathbf{b}$ and $\mathbf{a}$. These maps d_h will be used in defining the horizontal maps of the double complex used in the resolution of square-free E-modules.

If $x^{\mathbf{a}}$ and $x^{\mathbf{b}}$ are monomials such that $\operatorname{supp}(\mathbf{a}) \subset \operatorname{supp}(\mathbf{b})$, the map $E_{\mathbf{b}} \to E_{\mathbf{a}}$ given by left multiplication by $x^{\mathbf{b}}/x^{\mathbf{a}}$ induces a $\mathbb{Z}^n$-graded map $d_h \colon \Phi_\bullet(E_{\mathbf{b}}) \to \Phi_\bullet(E_{\mathbf{a}})$ on the Cartan resolutions described as follows: if $\operatorname{supp}(x^{\mathbf{b}}) = \{x_{i_1}, \dots, x_{i_m}\}$, if $X_{i_1}, \dots, X_{i_m}$ is the dual basis of $L_{\mathbf{b}}$, and if $a_1 + \cdots + a_m = j$, then the map

$$d_h \colon \Phi_j(E_{\mathbf{b}}) = D_j(L_{\mathbf{b}}) \otimes E(-\mathbf{b}) \to D_j(L_{\mathbf{a}}) \otimes E(-\mathbf{a}) = \Phi_j(E_{\mathbf{a}})$$

is determined by

$$(X_{i_1}^{a_1} \cdots X_{i_m}^{a_m})^* \otimes 1 \mapsto (X_{i_1}^{a_1} \cdots X_{i_m}^{a_m})^* \otimes (-1)^{j(1+\left|\deg(x^{\mathbf{b}}/x^{\mathbf{a}})\right|)} x^{\mathbf{b}}/x^{\mathbf{a}}$$

if $(X_{i_1}^{a_1} \cdots X_{i_m}^{a_m})^* \in D_j(L_{\mathbf{a}})$, and mapping it to zero otherwise. With this definition, the map d_h anti-commutes with the differentials d_v of the Cartan resolutions: $d_v d_h = -d_h d_v$.

Again, we illustrate the notation with an example.

Example 7.8. Let V be the vector space with basis x_1, x_2, x_3, let $E = \bigwedge V$, and let X_1, X_2 be basis elements dual to x_1, x_2. For multidegrees $\mathbf{b} = (1,1,0)$ and $\mathbf{a} = (1,0,0)$, notice that $x^{\mathbf{b}}/x^{\mathbf{a}} = -x_2$. The map $d_h \colon (\Phi_\bullet(E_{\mathbf{b}}), d_v) \to (\Phi_\bullet(E_{\mathbf{a}}, d_v))$ between Cartan resolutions begins:

$$\begin{tikzcd}
\vdots \arrow[d] & & \vdots \arrow[d] \\
(X_1^2)^* \otimes E(-\mathbf{a}) \arrow[d, "d_v"] & & \begin{matrix} (X_1^2)^* \otimes E(-\mathbf{b}) \\ \oplus \\ (X_1X_2)^* \otimes E(-\mathbf{b}) \\ \oplus \\ (X_2^2)^* \otimes E(-\mathbf{b}) \end{matrix} \arrow[ll, "d_h = \begin{pmatrix} -x_2 & 0 & 0 \end{pmatrix}"'] \arrow[d, "d_v"] \\
X_1^* \otimes E(-\mathbf{a}) \arrow[d, "d_v"] & & X_1^* \otimes E(-\mathbf{b}) \oplus X_2^* \otimes E(-\mathbf{b}) \arrow[ll, "d_h = \begin{pmatrix} -x_2 & 0 \end{pmatrix}"'] \arrow[d, "d_v"] \\
E(-\mathbf{a}) & & E(-\mathbf{b}). \arrow[ll, "d_h = \begin{pmatrix} -x_2 \end{pmatrix}"']
\end{tikzcd}$$

7.3 Resolutions of Square-Free Modules

In this section we give the structure of minimal free resolutions of square-free modules over E. As an immediate application, we then give an example showing that the syzygetic resonance varieties may be properly contained in the resonance varieties as was noted after Remark 6.6.

Theorem 7.9 ([EPY03], Propsositions 5.5 and 5.6,)**.** *Let M be a square-free module over E, and let ${}_SM$ be the corresponding square-free module over S. Suppose that ${}_SM$ has a $\mathbb{Z}^n$-graded minimal free resolution*

$$0 \longrightarrow \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} S(-\mathbf{a})^{\beta^S_{p,\mathbf{a}}({}_SM)} \longrightarrow \cdots \longrightarrow \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} S(-\mathbf{a})^{\beta^S_{0,\mathbf{a}}({}_SM)} \longrightarrow 0,$$

where the symbols $\beta^S_{p,\mathbf{a}}({}_SM) := \dim_{\Bbbk}\left(\operatorname{Tor}^S_p({}_SM,\Bbbk)\right)_{\mathbf{a}}$ denote the $\mathbb{Z}^n$-graded Betti numbers of ${}_SM$. Then the $\mathbb{Z}^n$-graded minimal free resolution of M is given by the total complex of the double complex

$$0 \longrightarrow \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} \Phi_\bullet(E_\mathbf{a})^{\beta^S_{p,\mathbf{a}}({}_SM)} \longrightarrow \cdots \longrightarrow \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} \Phi_\bullet(E_\mathbf{a})^{\beta^S_{0,\mathbf{a}}({}_SM)} \longrightarrow 0. \tag{7.10}$$

The vertical maps in the double complex are given by the differentials d_v in the Cartan resolutions. Each horizontal map $\Phi_\bullet(E_\mathbf{b}) \to \Phi_\bullet(E_\mathbf{a})$ is given by d_h when $\operatorname{supp}(\mathbf{a}) \subsetneq \operatorname{supp}(\mathbf{b})$ and the corresponding map $S(-\mathbf{b}) \to S(-\mathbf{a})$ is nonzero, and the horizontal map is zero otherwise.

The structure of the resolution gives the following relationship between the graded Betti numbers of M and ${}_SM$:

Corollary 7.11 (Cor 5.7, [EPY03]. See also [AAH00] and [Röm01].)**.** *If M is a square-free E-module, the $\mathbb{Z}^n$-graded Betti numbers are determined by the $\mathbb{Z}^n$-graded Betti numbers of ${}_SM$. In particular, the $\mathbb{Z}^n$-graded Poincaré series satisfy the equality:*

$$\sum_{i=0}^{\infty}\sum_{\mathbf{a}\in\mathbb{N}^n} \beta^E_{i,\mathbf{a}}(M)t^i\mathbf{u}^\mathbf{a} = \sum_{i=0}^{\infty}\sum_{\mathbf{a}\in\mathbb{N}^n} \beta^S_{i,\mathbf{a}}({}_SM)\frac{t^i\mathbf{u}^\mathbf{a}}{\prod_{j\in\operatorname{supp}(\mathbf{a})}(1-tu_j)}.$$

With the aim to improve clarity when utilizing the resolution from Theorem 7.9 in the next section, we introduce the following:

Notation 7.12. Let M be a square-free E-module with minimal free resolution given by the total complex of (7.10). We view the referenced double complex as a first-quadrant double complex $F_{\bullet,\bullet}$ by setting

$$F_{i,j} = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} (\Phi_j(E_{\mathbf{a}}))^{\beta^S_{i,\mathbf{a}}({}_SM)}.$$

for $i, j \geq 0$, and setting $F_{i,j} = 0$ otherwise. If $\eta_{i,j} \in F_{i,j}$ and $\mathbf{a} \in \mathbb{Z}^n$ is a multi-degree, we say that $\eta_{i,j}$ *is suppported on* $\mathbf{a}$ if the $\mathbf{a}$-component of $\eta_{i,j}$ is nonzero. We let $d_v \colon F_{i,j} \to F_{i,j-1}$ denote the vertical differential, $d_h \colon F_{i,j} \to F_{i-1,j}$ denote the horizontal differential, and $d = d_v + d_h$ denote the total differential. With this notation, the minimal free resolution of M is the total complex of the double complex:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \iddots \\
0 & F_{0,2} \arrow[l] \arrow[d, "d_v"] & F_{1,2} \arrow[l, "d_h"] \arrow[d, "d_v"] & F_{2,2} \arrow[l, "d_h"] \arrow[d, "d_v"] & \cdots \\
0 & F_{0,1} \arrow[l] \arrow[d, "d_v"] & F_{1,1} \arrow[l, "d_h"] \arrow[d, "d_v"] & F_{2,1} \arrow[l, "d_h"] \arrow[d, "d_v"] & \cdots \\
0 & F_{0,0} \arrow[l] \arrow[d] & F_{1,0} \arrow[l, "d_h"] \arrow[d] & F_{2,0} \arrow[l, "d_h"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & \cdots
\end{tikzcd} \tag{7.13}$$

To close this section, we give an example showing that not every resonant weight is a syzygetic resonance weight, as was mentioned after Remark 6.6.

Example 7.14. Let V be a $\Bbbk$-vector space with basis $x_1, \ldots, x_5$, let E be the exterior algebra on V, and let S be the symmetric algebra on V. Let I be the E-ideal $(x_1x_2, x_1x_3, x_2x_3x_4x_5)\,E$ and let $M = E/I$. Notice that $x_1 \in \mathcal{R}^2(E/I)$ because $x_1 \cdot x_2x_3 \in I$ but $x_2x_3 \notin I + (x_1)$.

On the other hand, note that the $\mathbb{Z}^5$-graded minimal free resolution of the module ${}_SM = S/\left(x_1x_2, x_1x_3, x_2x_3x_4x_5\right)S$ has the form

$$0 \longrightarrow \begin{matrix} S(-1,-1,-1,0,0) \\ \oplus \\ S(-1,-1,-1,-1,-1) \end{matrix} \longrightarrow \begin{matrix} S(-1,-1,0,0,0) \\ \oplus \\ S(-1,0,-1,0,0) \\ \oplus \\ S(0,-1,-1,-1,-1) \end{matrix} \longrightarrow S \longrightarrow 0\ .$$

Let

$$T_1 = \{(-1,-1,0,0,0),(-1,0,-1,0,0),(0,-1,-1,-1,-1)\} \subset \mathbb{Z}^5$$

and

$$T_2 = \{(-1,-1,-1,0,0),(-1,-1,-1,-1,-1)\} \subset \mathbb{Z}^5$$

be the collections of $\mathbb{Z}^5$-graded shifts appearing in homological degrees one and two of the minimal free resolution of ${}_SM$. Using Notation 7.12, it follows that

$$\operatorname{Tot}(F_{\bullet,\bullet})_1 = \bigoplus_{\mathbf{a}\in T_1} \Phi_0(E_{\mathbf{a}})$$

and for each $p \geq 2$,

$$\operatorname{Tot}(F_{\bullet,\bullet})_p = \left(\bigoplus_{\mathbf{a}\in T_1} \Phi_{p-1}(E_{\mathbf{a}})\right) \oplus \left(\bigoplus_{\mathbf{a}\in T_2} \Phi_{p-2}(E_{\mathbf{a}})\right).$$

Viewing $\operatorname{Tot}(F_{\bullet,\bullet})$ as $\mathbb{Z}$-graded (as opposed to $\mathbb{Z}^n$-graded) by taking total degrees, we see that for all $p \geq 1$

$$\operatorname{Tot}(F_{\bullet,\bullet})_p = E(-(p+1))^{2\binom{p}{1}+\binom{p}{2}} \oplus E(-(p+3))^{\binom{p+2}{3}+\binom{p+2}{4}}.$$

Because the total complex of $F_{\bullet,\bullet}$ is the minimal free resolution of M by Theorem 7.9, it follows that all minimal p-th syzygies of M have total degrees $p+1$ or $p+3$. It follows that $\mathcal{SR}^2(M) = \emptyset$ by Definition 6.4. Thus, $\mathcal{SR}^2(M) \subsetneq \mathcal{R}^2(M)$.

8. SYZYGETIC RESONANCE VARIETIES OF SQUARE-FREE MODULES

In this chapter, we use the structure of the minimal free resolution of a square-free module over an exterior algebra to compute the syzygetic resonance varieties. As a corollary, we derive lower bounds for the graded Betti numbers of square-free modules in terms of the syzygetic resonance varieties in the style of the Chen Ranks Theorem 4.4. For quotients by square-free monomial ideals, we use Hochster's formula for the $\mathbb{Z}^n$-graded Betti numbers of a Stanley–Resiner ring to give another expression for the syzygetic resonance varieties in the style used for a similar expression of the resonance varieties.

8.1 Statement and Immediate Consequences

As usual, let V be a $\Bbbk$-vector space, $E = \bigwedge V$ the exterior algebra on V and $S = \operatorname{Sym} V$ be the symmetric algebra on V.

Theorem 8.1. *Let M be a square-free module over E, and let ${}_SM$ be the corresponding square-free module over S. Then*

$$\mathcal{SR}^i(M) = \bigcup_{\substack{\beta^S_{q,\mathbf{a}}({}_SM)\neq 0 \\ |\mathbf{a}|-q=i}} \operatorname{span}_{\Bbbk} \operatorname{supp}(x^{\mathbf{a}}). \tag{8.2}$$

Before proceeding to the proof of Theorem 8.1 in the second section, we provide some discussion of immediate consequences. In the first subsection we look at the implications this theorem has for the graded Betti numbers of square-free modules. In the second subsection, we rephrase the result for quotients by square-free monomial ideals.

8.1.1 Syzygetic Resonance and Betti Numbers

In this subsection, we give an immediate consequence of Theorem 8.1 as it pertains to the relationship between graded Betti numbers and syzygetic resonance. We then provide some discussion on possible future directions toward better understanding this relationship.

Corollary 8.3. *Let M be a square-free E-module. Then for sufficiently large i and j, the graded Betti numbers of M satisfy*

$$\beta_{i,j}(M) \geq \sum_{r>0} h_r \binom{j-1}{j-r},$$

where h_r is the number of r-dimensional components of $\mathcal{SR}^{j-i}(M)$.

Proof. By Theorem 8.1, $\mathcal{SR}^{j-i}(M)$ contains the r-dimensional component $\operatorname{span}_{\Bbbk} \operatorname{supp} x^{\mathbf{a}}$ with $|\mathbf{a}| = r$ if and only if $\beta^S_{i+r-j,\mathbf{a}}({}_SM) \neq 0$. The result follows from the relationship between the $\mathbb{Z}^n$-graded Betti numbers of M and ${}_SM$ given in Corollary 7.11. □

While this result provides information about square-free modules and the Chen Ranks Theorem 4.4 provides information about Orlik–Solomon algebras of hyperplane arrangements, it is interesting to compare the two results. Taking $j = i+1$ in Corollary 8.3, notice that if M is such that $\mathcal{SR}^1(M)$ only consists of 2-dimensional components then the bound provided by the corollary agrees with the number given in the Chen Ranks Theorem. This is relevant because we remarked that the Hessian arrangement in Example 4.9 is the only arrangement whose first resonance variety has a component of dimension larger than 2, hence the Chen Ranks Theorem largely talks about modules whose first syzygetic resonance variety only consists of 2-dimensional components. On the other hand, the values of the Betti numbers given by the Chen Ranks Theorem are larger than the bound provided by Corollary 8.3 once 3-dimensional components exist.

The preceding discussion suggests that $\mathcal{SR}^i(M)$ contains some but not all of the graded Betti information. We give a simple example to demonstrate that this is the case:

Example 8.4. Let V be a vector space on x_1, x_2, x_3, and let $M = E/(x_1x_2, x_1x_3, x_2x_3)$. It follows that ${}_SM = S/(x_1x_2, x_1x_3, x_2x_3)$ has minimal free resolution

$$0 \longrightarrow S(-1,-1,-1)^2 \longrightarrow \begin{matrix} S(-1,-1,0) \\ \oplus \\ S(-1,0,-1) \\ \oplus \\ S(0,-1,-1) \end{matrix} \longrightarrow S \longrightarrow 0.$$

By Theorem 8.1, it follows that $\mathcal{SR}^1(M) = \operatorname{span}_{\Bbbk}\{x_1, x_2, x_3\}$. The bound on the graded Betti numbers of M given by Corolloary 8.3 are strictly less than the actual values given by Corollary 7.11.

Notably, the difference between the actual graded Betti numbers and the lower bound given by Corollary 8.3 can be traced to the following two facts: (1) $\mathcal{SR}^2(M)$ does not detect all non-zero graded Betti numbers. For example, it does not detect that $\beta^S_{1,(1,1,0)}({}_SM) \neq 0$; (2) $\mathcal{SR}^2(M)$ only detects that $\beta^S_{2,(1,1,1)}({}_SM) \neq 0$, not that its value is 2.

This example shows that there are two relevant pieces of information that remain obscured when looking only at $\mathcal{SR}^i(M)$: (1) the syzygetic resonance variety can not detect all non-zero Betti numbers of ${}_SM$; (2) the syzygetic resonance variety can not determine the multiplicity of the Betti numbers it can detect. Both of these obstacles seem to require a refinement of the syzygetic resonance varieties, and we suggest lines along which one might be able to recover these two pieces of information.

Addressing obstacle (1) seems to require knowledge of which syzygy module first realizes a syzygetic resonance weight. One way to address this is to note that the syzygetic resonance varieties $\mathcal{SR}^i(M)$ admit an increasing filtration by the subvarieties of resonant weights associated to minimal p-th syzygyies. Explicitily, if $F_\bullet$ is the minimal free resolution of M, let $\mathcal{S}_p\mathcal{R}^i(M)$ be the Zariski closure of the collection of points $(\lambda_1, \ldots, \lambda_n) \in \Bbbk^n$ such that there exists $\eta \in (F_{p-1})_{i+p-1}$ so that $\lambda\eta$ is a minimal generator of $\operatorname{Syz}^E_p(M)$, where $\lambda = \lambda_1 x_1 + \cdots + \lambda_n x_n$. It follows from Corollary 6.3 that

$$\mathcal{S}_1\mathcal{R}^i(M) \subseteq \mathcal{S}_2\mathcal{R}^i(M) \subseteq \cdots \subseteq \mathcal{SR}^i(M).$$

This filtration preserves the homological information of when a given syzygetic resonant weight occurs, and leads to the following:

Question 8.5. Can the filtration $\mathcal{S}_p\mathcal{R}^i(M)$ on $\mathcal{SR}^i(M)$ be used to refine Theorem 8.1 in such a way that all non-zero Betti numbers can can be identified?

Regarding obstacle (2), the need to keep track of multiplicity, recall that we defined the syzygetic resonance varieties $\mathcal{SR}^i_j(M)$ with depth j for a bimodule M in analogy with the resonance varieties of depth j. Our proof of Theorem 8.1 only tracks the depth one syzygetic resonance, but it seems probable that with more care one might produce a version of the theorem for $\mathcal{SR}^i_j(M)$. These syzygetic resonance varieties carry multiplicity information, and therefore we have the following:

Question 8.6. Can Theorem 8.1 be refined to compute $\mathcal{SR}^i_j(M)$ for each square-free module M and all indices i and j? Does such a theorem allow one to recover the multiplicity of the nonzero graded Betti numbers of ${}_SM$?

8.1.2 Syzygetic Resonance of Quotients by Square-free Monomial Ideals

In this subsection, we translate our Theorem 8.1 so that it is amenable to comparison with a result expressing the resonance varieties of quotients of exterior algebras by square-free monomial ideals. Specifically, in [PS09] Papadima and Suciu establish the following:

Theorem 8.7 ([PS09], Theorem 3.8)**.** *Let V be a $\Bbbk$-vector space on $x_1, \dots, x_n$, let $E = \bigwedge V$ be the exterior algebra, and let I be a square-free monomial ideal in E. For $W \subset [n]$, define $\lambda_W = \sum_{i \in W} x_i$. Then*

$$\mathcal{R}^i_j(E/I) = \bigcup_{\substack{W \subset [n] \\ \dim H^i(E/I, \lambda_W \cdot) \geq j}} \operatorname{span}_{\Bbbk}\{x_i \mid i \in W\},$$

where $H^i(E/I, \lambda_W \cdot)$ denotes the cohomology of the complex with differential given by multiplication with λ_W.

In the setting where I is a square-free monomial ideal in E, to the square-free module $M = E/I$ the associated square-free S-module ${}_SM$ is S/I, a Stanley–Reisner ring. Using Hochster's formula for the graded Betti numbers of S/I (see, for example, [HH11]), Theorem 8.1 takes the form

Corollary 8.8. *Let V be a $\Bbbk$-vector space on $x_1, \ldots, x_n$, let $E = \bigwedge V$ be the exterior algebra, and let I be a square-free monomial ideal in E. For $W \subset [n]$, define $\lambda_W = \sum_{i \in W} x_i$ and $J_W = I + (x_i \mid i \notin W)$. Then*

$$\mathcal{SR}_1^i(E/I) = \bigcup_{\substack{W \subset [n] \\ \dim H^i(E/J_W, \lambda_W \cdot) \geq j}} \operatorname{span}_{\Bbbk}\{x_i \mid i \in W\},$$

For quotients by square-free monomial ideals, this gives us a condition for when syzygetic resonance varieties coincide with the usual resonance varieties:

Corollary 8.9. *Let V be a $\Bbbk$-vector space on $x_1, \ldots, x_n$, let $E = \bigwedge V$ be the exterior algebra, and let I be a square-free monomial ideal in E. For $W \subset [n]$, define $\lambda_W = \sum_{i \in W} x_i$. Then $\mathcal{SR}_1^i(E/I) = \mathcal{R}_1^i(E/I)$ if and only if the cohomology $H^i(E/I, \lambda_W \cdot)$ vanishes whenever $H^i(E/(I + (x_i \mid i \notin W)), \lambda_W \cdot)$ vanishes.*

We close this subsection with an example demonstrating this this last corollary:

Example 8.10. Let V be a $\Bbbk$-vector space on x_1, x_2, x_3, let $E = \bigwedge V$ be the exterior algebra, and let $I = (x_1x_2, x_1x_3)$. Then taking $W = \{1\}$ in the corollary, we have that $\dim H^2(E/I, x_1 \cdot) = 1$ while $H^2(E/(I + (x_2, x_3)), x_1 \cdot) = 0$. It follows that x_1 is in $\mathcal{R}_1^2(E/I)$ but not it $\mathcal{SR}_1^2(E/I)$.

8.2 Proof of Theorem

In this section we give the proof of Theorem 8.1. We will prove that each of the two sets referenced in (8.2) is contained in the other. Both containments make crucial use of the fact that the vertical strands of the double complex defining the minimal free resolution of a square-free E-module are direct sums of Cartan resolutions. Using Notation 7.12, two key points to note from this are

(1) the vertical differential d_v does not cross strands, so applying d_v to an element in $\Phi_\ell(E_{\mathbf{a}}) \subseteq F_{i,\ell}$ produces an element in $\Phi_{\ell-1}(E_{\mathbf{a}}) \subseteq F_{i,\ell-1}$ *in the same summand* $\Phi_\bullet(E_{\mathbf{a}})$ of $F_{i,\bullet}$;

(2) the Cartan resolutions are minimal free resolutions of the modules $E_{\mathbf{a}} := E/\operatorname{supp}(x^{\mathbf{a}})(-\mathbf{a})$, and we showed in Example 3.3 that the only resonant weights of such modules are those in $\operatorname{span}_{\Bbbk} \operatorname{supp}(x^{\mathbf{a}})$.

We now prove that the right-hand side of (8.2) is contained in the left-hand side by constructing a minimal syzygy which is a multiple of a general element $\lambda \in \operatorname{span}_{\Bbbk} \operatorname{supp}(x^{\mathbf{a}})$. A non-zero Betti number $\beta^S_{q,\mathbf{a}}({}_SM)$ implies that a vertical Cartan strand $\Phi_\bullet(E_{\mathbf{a}})$ begins at $F_{q,0}$, and hence for this general element λ there is an element from this strand mapping to it under d_v. The image of this element under d_h need not be a multiple of λ, but by adding to it the image of an element under d_v we can produce a multiple of λ. This results in a new element one column to the left whose image under d_h is not a multiple of λ, and we repeat the modification process. This inductive process must end because the double complex $F_{\bullet,\bullet}$ is nonzero only in the first quadrant, and we arrive at a syzygy which is a multiple of λ.

Lemma 8.11. *In the setting of Theorem 8.1,*

$$\mathcal{SR}^i(M) \supseteq \bigcup_{\substack{\beta^S_{q,\mathbf{a}}({}_SM)\neq 0\\ |\mathbf{a}|-q=i}} \operatorname{span}_{\Bbbk} \operatorname{supp}(x^{\mathbf{a}}).$$

Proof. Throughout, we use Notation 7.12 for the double complex $F_{\bullet,\bullet}$ whose total complex gives the minimal $\mathbb{Z}^n$-graded free resolution of M.

Let $\beta^S_{q,\mathbf{a}}({}_SM) \neq 0$ and let $\lambda \in \operatorname{span}_{\Bbbk} \operatorname{supp}(x^{\mathbf{a}})$ be a general element. By Theorem 7.9, $F_{q,0}$ has a nonzero summand of the form $\Phi_0(E_{\mathbf{a}})$. Define $\eta_{q,0}$ to be the basis element 1 in this summand $\Phi_0(E_{\mathbf{a}})$, and note that $\lambda\eta_{q,0}$ has total degree $|\mathbf{a}|+1$. Since the vertical strand $\Phi_\bullet(E_{\mathbf{a}})$ is a $\mathbb{Z}^n$-graded free resolution of $E_{\mathbf{a}}$, there exists an element $\tau_{q,1} \in \Phi_1(E_{\mathbf{a}})$ of total degree $|\mathbf{a}|+1$ in this strand such that $d_v(\tau_{q,1}) = \lambda\eta_{q,0}$. We therefore have the following diagram:

$$\begin{tikzcd} d_h(\tau_{q,1}) & \tau_{q,1} \arrow[l, mapsto, "d_h"'] \arrow[d, mapsto, "d_v"] \\ & \lambda\eta_{q,0} \end{tikzcd} \tag{8.12}$$

We induct on $0 \le k \le q$ to define elements $\tau_{q-k,k+1} \in F_{q-k,k+1}$ of total degree $|\mathbf{a}|+1$ and $\eta_{q-k,k} \in F_{q-k,k}$ of total degree $|\mathbf{a}|$ such that:

(i) if $d_h(\tau_{q-k,k+1})$ is supported on a multi-degree $\mathbf{b}$, then $\operatorname{supp}(\lambda) \not\subset \operatorname{supp}(x^{\mathbf{b}})$;

(ii) the following equation holds:

$$d(\tau_{q,1} + \tau_{q-1,2} + \cdots + \tau_{q-k,k+1}) = \lambda(\eta_{q,0} + \eta_{q-1,1} + \cdots + \eta_{q-k,k}) + d_h(\tau_{q-k,k+1}).$$

Recalling the definition from Notation 7.12, condition (i) restricts on which summands of

$$F_{q-k,k+1} := \bigoplus_{\mathbf{b}\in\mathbb{Z}^n} \Phi_{k+1}(E_{\mathbf{b}})^{\beta^S_{i,\mathbf{b}}({}_S M)}$$

the element $d_h(\tau_{q-k,k+1})$ is nonzero. This condition is needed to guarantee that the vertical strands of Cartan resolutions on which $d_h(\tau_{q-k,k+1})$ lives only resolve modules $E_{\mathbf{b}}$ for which λ is *not* resonant. Expressing condition (ii) pictorially by situating elements at their location in the double complex $F_{\bullet,\bullet}$, for each k we require the following diagram:

$$\begin{tikzcd}
d_h(\tau_{q-k,k+1}) & \tau_{q-k,k+1} \arrow[l, mapsto, "d_h"'] \arrow[d, mapsto, "d_v"] & & & & \\
& \lambda\eta_{q-k,k} & {} \arrow[l, mapsto, "d_h"'] & & & \\
& & & \ddots & & \\
& & & {} & \tau_{q-1,2} \arrow[l, mapsto, "d_h"'] \arrow[d, mapsto, "d_v"] & \\
& & & & \lambda\eta_{q-1,1} & \tau_{q,1} \arrow[l, mapsto, "d_h"'] \arrow[d, mapsto, "d_v"] \\
& & & & & \lambda\eta_{q,0}
\end{tikzcd} \tag{8.13}$$

Notice that the result will be established once $\tau_{q-k,k+1} \in F_{q-k,k+1}$ and $\eta_{q-k,k} \in F_{q-k,k}$ have been defined for all $0 \leq k \leq q$ such that properties (i), (ii) hold. Indeed, since $F_{\bullet,\bullet}$ is a first-quadrant double complex, $d_h(\tau_{0,q+1}) = 0$ and the equation in (ii) for $k = q$ then reduces to

$$d(\tau_{q,1} + \tau_{q-1,2} + \cdots + \tau_{0,q+1}) = \lambda(\eta_{q,0} + \eta_{q-1,1} + \cdots + \eta_{0,q}.)$$

By the exactness of $\mathrm{Tot}(F_{\bullet,\bullet})$, it follows that $\lambda(\eta_{q,0} + \eta_{q-1,1} + \cdots + \eta_{0,q})$ is a $(q+1)$-st syzygy of total degree $|\mathbf{a}| + 1$. The minimality of the syzygy is guaranteed by Proposition 2.1 and the fact that $\eta_{q,0}$ is a basis element and $\lambda \in E_1$. By the Definition 6.4 of syzygetic resonance varieties, it follows that $\lambda \in \mathcal{SR}^{|\mathbf{a}|-q}(M)$. Since $\lambda \in \operatorname{span}_{\Bbbk} \operatorname{supp}(x^{\mathbf{a}})$ was general and $\mathcal{SR}^{|\mathbf{a}|-q}(M)$ is Zariski-closed by definition, it follows that $\operatorname{span}_{\Bbbk} \operatorname{supp}(x^{\mathbf{a}}) \subseteq \mathcal{SR}^{|\mathbf{a}|-q}(M)$, as desired.

It remains to inductively define for each $0 \leq k \leq q$ the elements $\tau_{q-k,k+1} \in F_{q-k,k+1}$ of total degree $|\mathbf{a}|+1$ and $\eta_{q-k,k} \in F_{q-k,k}$ of total degree $|\mathbf{a}|$ satisfying (i) and (ii). Regarding the basis of the induction $k = 0$, we use $\eta_{q,0}$ and $\tau_{q,1}$ defined at the beginning of the proof. To see that property (i) is satisfied, notice first that $\operatorname{supp}(\lambda) = \operatorname{supp}(x^{\mathbf{a}})$ since $\lambda \in \operatorname{span}_{\Bbbk} \operatorname{supp}(x^{\mathbf{a}})$ is general. For any $\mathbf{b}$ supporting $d_h(\tau_{q,1})$, we have by Theorem 7.9 that $\operatorname{supp}(\mathbf{b}) \subsetneq \operatorname{supp}(\mathbf{a})$, thus establishing (i). Note that (ii) is satisfied because of (8.12). For the inductive step, suppose that we have defined elements through $\tau_{q-k,k+1}$ and $\eta_{q-k,k}$ satisfying (i) and (ii), and we wish to define $\tau_{q-k-1,k+2}$ and $\eta_{q-k-1,k+1}$.

We begin by showing that the left-most element in the diagram (8.13) maps to a multiple of λ under d_v, whence multiplying this left-most element by λ produces an element that maps to zero under d_v. To this end, notice that $d^2 = 0$ implies that $d_v d_h = -d_h d_v$ and $d_h^2 = 0$. The diagram

$$\begin{tikzcd}
d_h(\tau_{q-k,k+1}) \arrow[d, mapsto, "d_v"] & \tau_{q-k,k+1} \arrow[l, mapsto, "d_h"'] \arrow[d, mapsto, "d_v"] & \\
0 & \lambda\eta_{q-k,k} \arrow[l, mapsto, "d_h"'] & \tau_{q-k+1,k} \arrow[l, mapsto, "d_h"']
\end{tikzcd}$$

then shows that

$$\begin{aligned} d_v(d_h(\tau_{q-k,k+1})) &= -d_h(d_v(\tau_{q-k,k+1})) \\ &= -d_h(d_v(\tau_{q-k,k+1})) - d_h^2(\tau_{q-k+1,k}) \\ &= -d_h(d_v(\tau_{q-k,k+1}) + d_h(\tau_{q-k+1,k})) \\ &= -d_h(\lambda\eta_{q-k,k}) \\ &= -\lambda d_h(\eta_{q-k,k}) \end{aligned}$$

is a multiple of λ. It follows that

$$d_v(\lambda d_h(\tau_{q-k,k+1})) = 0. \tag{8.14}$$

We define $\tau_{q-k-1,k+2} \in F_{q-k-1,k+2}$ of total degree $|\mathbf{a}| + 1$ and $\eta_{q-k-1,k+1} \in F_{q-k-1,k+1}$ of total degree $|\mathbf{a}|$ locally on the strands of Cartan resolutions on which $d_h(\tau_{q-k,k+1})$ is supported so that

$$d_v(\tau_{q-k-1,k+2}) + d_h(\tau_{q-k,k+1}) = \lambda\eta_{q-k-1,k+1}. \tag{8.15}$$

Note that we may give such a local definition because the vertical differential d_v does not cross strands. Once we define such $\tau_{q-k-1,k+2}$ and $\eta_{q-k-1,k+1}$, notice that by (ii) of the induction hypothesis,

$$\begin{aligned} &d(\tau_{q,1} + \tau_{q-1,2} + \cdots + \tau_{q-k,k+1} + \tau_{q-k-1,k+2}) \\ =&d(\tau_{q,1} + \tau_{q-1,2} + \cdots + \tau_{q-k,k+1}) + d(\tau_{q-k-1,k+2}) \\ =&\lambda(\eta_{q,0} + \eta_{q-1,1} + \cdots + \eta_{q-k,k}) + d_h(\tau_{q-k,k+1}) + d_v(\tau_{q-k-1,k+2}) + d_h(\tau_{q-k-1,k+2}) \\ =&\lambda(\eta_{q,0} + \eta_{q-1,1} + \cdots + \eta_{q-k,k} + \eta_{q-k-1,k+1}) + d_h(\tau_{q-k-1,k+2}). \end{aligned}$$

thus satisfying property (ii) for the inductive step. Moreover, since $\tau_{q-k-1,k+2}$ will be supported on the same multidegrees $\mathbf{b}$ as $d_h(\tau_{q-k,k+1})$, property (i) of the induction hypothesis implies that $\operatorname{supp}(\lambda) \not\subset \operatorname{supp}(x^{\mathbf{b}})$ for any $\mathbf{b}$ supporting $\tau_{q-k-1,k+2}$. By Theorem 7.9, all multidegrees $\mathbf{c}$ supporting $d_h(\tau_{q-k-1,k+2})$ are contained in some multidegree $\mathbf{b}$ supporting

$\tau_{q-k-1,k+2}$. It follows that $\mathrm{supp}(\lambda)$ is not contained in $\mathrm{supp}(x^{\mathbf{c}})$ for any multidegree $\mathbf{c}$ supporting $d_h(\tau_{q-k-1,k+2})$. This establishes (i) for the inductive step. It therefore suffices to define $\tau_{q-k-1,k+2}$ and $\eta_{q-k-1,k+1}$ locally on a vertical Cartan resolution so that (8.15) holds.

Fix a multi-degree $\mathbf{b}$ and Cartan resolution $\Phi_\bullet(E_{\mathbf{b}})$ on which $d_h(\tau_{q-k,k+1})$ is supported. Replacing $d_h(\tau_{q-k,k+1})$ with its component in this vertical strand, we may suppose that $d_h(\tau_{q-k,k+1}) \in \Phi_{k+1}(E_{\mathbf{b}})$. Moreover, since the vertical differentials d_v do not cross strands, we may suppose that (8.14) still holds. Since $\Phi_\bullet(E_{\mathbf{b}})$ is a minimal free resolution of $E_{\mathbf{b}}$, this equation implies that $\lambda d_h(\tau_{q-k,k+1}) \in \mathrm{Syz}^E_{k+2}(E_{\mathbf{b}})$. Since resonance varieties are invariant under passing to syzygy modules by Theorem 6.1 and since $\mathrm{supp}(\lambda) \not\subset \mathrm{supp}(x^{\mathbf{b}})$ by (ii) of the induction hypothesis, it follows from Example 3.3 that $\lambda \notin \mathcal{R}^{|\mathbf{a}|+1}(E_{\mathbf{b}})$. Since $\lambda d_h(\tau_{q-k,k+1})$ has total degree $|\mathbf{a}|+1$ and $\lambda \cdot \lambda d_h(\tau_{q-k,k+1}) = 0$, this last fact implies that $\lambda d_h(\tau_{q-k,k+1}) \in \lambda \, \mathrm{Syz}^E_{k+2}(E_{\mathbf{b}})$. By the fact that the Cartan resolution is a $\mathbb{Z}^n$-graded free resolution, $\mathrm{Syz}^E_{k+2}(E_{\mathbf{b}}) = d_v\left(\Phi_{k+2}(E_{\mathbf{b}})\right)$, whence there exists $\tau_{q-k-1,k+2} \in \Phi_{k+2}(E_{\mathbf{b}})$ of total degree $|\mathbf{a}|+1$ so that

$$\lambda d_h(\tau_{q-k,k+1}) = -\lambda d_v(\tau_{q-k-1,k+2}),$$

whence

$$\lambda(d_h(\tau_{q-k,k+1}) + d_v(\tau_{q-k-1,k+2})) = 0.$$

Finally, by Example 3.2, it follows that there exists $\eta_{q-k-1,k+1} \in \Phi_{k+1}(E_{\mathbf{b}})$ of total degree $|\mathbf{a}|$ so that

$$d_h(\tau_{q-k,k+1}) + d_v(\tau_{q-k-1,k+2}) = \lambda \eta_{q-k-1,k+1}.$$

This concludes the local definitions of $\tau_{q-k-1,k+2}$ and $\eta_{q-k-1,k+1}$, establishing the result. $\square$

We now prove that the left-hand side of (8.2) is contained in the right-hand side. The proof proceeds by taking a minimal syzygy $\lambda\eta$ which is a multiple of a given syzygetic resonant weight λ. By exactness, there is an element τ which maps to $\lambda\eta$, the minimality of which forces τ to involve basis elements. We then focus on the right-most vertical Cartan resolution $\Phi_\bullet(E_{\mathbf{a}})$ on which τ is given by basis elements, and show that for degree reasons the component of $d(\tau)$ in this Cartan resolution comes only from the vertical differential

d_v. Since $d(\tau)$ is a multiple of λ, the definition of the Cartan resolution then implies that $\operatorname{supp}(\lambda) \subseteq \operatorname{supp}(x^{\mathbf{a}})$, establishing the result.

Lemma 8.16. *In the setting of Theorem 8.1,*

$$\mathcal{SR}^i(M) \subseteq \bigcup_{\substack{\beta^S_{q,\mathbf{a}}({}_SM)\neq 0 \\ |\mathbf{a}|-q=i}} \operatorname{span}_{\Bbbk} \operatorname{supp}(x^{\mathbf{a}}).$$

Proof. Throughout, we use Notation 7.12 to express the minimal $\mathbb{Z}^n$-graded free resolution of M as $\operatorname{Tot}(F_{\bullet,\bullet})$. Let general $\lambda \in \mathcal{SR}^i(M)$ with $i \geq 1$ be given. Since λ is general, by Definition 6.4 there exists $\eta \in \operatorname{Tot}_j(F_{\bullet,\bullet})$ of total degree $i+j$ such that $\lambda\eta \in \operatorname{Syz}^E_{j+1}(M)$ is a minimal syzygy. By Corollary 6.3 (ii), we may suppose that $j \geq \operatorname{projdim}({}_SM) := p$. Since the total complex $\operatorname{Tot}(F_{\bullet,\bullet})$ is exact at position j, there is an element $\tau \in \operatorname{Tot}_{j+1}(F_{\bullet,\bullet})$ of total degree $i+j+1$ so that $d(\tau) = \lambda\eta$. Letting $\eta_{\ell,k}$ and $\tau_{\ell,k}$ denote the components of η and τ in each of the $F_{\ell,k}$ and recalling that $F_{\ell,k} = 0$ for all $\ell < 0$ and $\ell > p = \operatorname{projdim}({}_SM)$ (by Theorem 7.9), we have the following diagram:

$$\begin{tikzcd}
\tau_{0,j+1} \arrow[d, mapsto, "d_v"] & & & & \\
\lambda\eta_{0,j} & \tau_{1,j} \arrow[l, mapsto, "d_h"'] \arrow[d, mapsto, "d_v"] & & & \\
& \lambda\eta_{1,j-1} & {} \arrow[l, mapsto, "d_h"'] & & \\
& & \ddots & & \\
& & {} & \tau_{p-1,j-p+2} \arrow[l, mapsto, "d_h"'] \arrow[d, mapsto, "d_v"] & \\
& & & \lambda\eta_{p-1,j-p+1} & \tau_{p,j-p+1} \arrow[l, mapsto, "d_h"'] \arrow[d, mapsto, "d_v"] \\
& & & & \lambda\eta_{p,j-p}
\end{tikzcd}$$

The fact that $\lambda\eta$ is a minimal syzygy implies that there is some vertical strand $\Phi_\bullet(E_{\mathbf{a}})$ on which the component of τ is a $\Bbbk$-linear combination of basis elements. Fix one such strand $\Phi_\bullet(E_{\mathbf{a}})$ in the rightmost column $F_{q,\bullet}$ containing a strand with this property, taking note that this implies that

$$\beta^S_{q,\mathbf{a}}({}_S M) \neq 0. \tag{8.17}$$

Let $\hat{\tau}$ denote the component of τ in this vertical strand. Since $\tau \in \mathrm{Tot}_{j+1}(F_{\bullet,\bullet})$ has total degree $i+j+1$, it follows that

$$\hat{\tau} \in \Phi_{j+1-q}(E_{\mathbf{a}}) \subseteq F_{q,j+1-q}$$

also has total degree $i+j+1$. Since $\hat{\tau}$ is a $\Bbbk$-linear combination of basis elements of $\Phi_{j+1-q}(E_{\mathbf{a}})$, it follows that $\hat{\tau}$ has total degree $|\mathbf{a}|+j+1-q$. Combining these two degree computations for $\hat{\tau}$ shows that

$$|\mathbf{a}| - q = i. \tag{8.18}$$

Taking (8.17) and (8.18) together, we see that the result will follow if we can show that $\lambda \in \mathrm{span}_{\Bbbk} \mathrm{supp}(x^{\mathbf{a}})$.

We argue now that the component of $d(\tau) = \lambda\eta$ in the chosen strand $\Phi_\bullet(E_{\mathbf{a}}) \subseteq F_{q,\bullet}$ is given by $d_v(\hat{\tau})$. Supposing this is true, it will follow that $d_v(\lambda\hat{\tau}) = 0$. Since $\hat{\tau}$ consists of basis elements, Proposition 2.1 implies that $\lambda\hat{\tau}$ is a minimal syzygy, thus λ is a syzygetic resonant weight of $E_{\mathbf{a}}$ by Definition 6.4. Since the syzygetic resonance varieties are contained in the resonance varieties by Remark 6.6, it follows that λ is a resonant weight of $E_{\mathbf{a}}$. Example 3.3 shows that all resonant weights of $E_{\mathbf{a}}$ are contained in $\mathrm{span}_{\Bbbk} \mathrm{supp}(x^{\mathbf{a}})$, whence λ is in this collection, as desired. It therefore suffices to show that the component of $d(\tau) = \lambda\eta$ in the chosen strand $\Phi_\bullet(E_{\mathbf{a}}) \subseteq F_{q,\bullet}$ is given by $d_v(\hat{\tau})$.

Notice that since the vertical differential does not cross strands of Cartan resolutions, the component of $d_v(\tau)$ in the chosen strand $\Phi_\bullet(E_{\mathbf{a}})$ is $d_v(\hat{\tau})$. It therefore remains to show that the horizontal component $d_h(\tau)$ in the strand is zero. By Theorem 7.9, every vertical strand $\Phi_\bullet(E_{\mathbf{b}}) \subseteq F_{q+1,\bullet}$ mapping to $\Phi_\bullet(E_{\mathbf{a}})$ under d_h has $|\mathbf{b}| \geq |\mathbf{a}| + 1$. In particular, the basis elements of

$$\Phi_{j-q}(E_{\mathbf{b}}) \subseteq F_{q+1,j-q} \subseteq \mathrm{Tot}_{j+1}(F_{\bullet,\bullet})$$

have total degree at least

$$|\mathbf{b}| + j - q \geq |\mathbf{a}| + 1 + j - q = i + j + 1,$$

where the last equality follows from (8.18). Since τ has total degree $i + j + 1$ and since no component of τ in $F_{q+1,\bullet}$ consists of basis elements by choice of q, it follows that τ is not supported on any vertical strands of Cartan resolutions $\Phi_\bullet(E_{\mathbf{b}})$ which map to our fixed strand $\Phi_\bullet(E_{\mathbf{a}})$. It follows that the component of $d(\tau) = \lambda\eta$ in the chosen strand $\Phi_\bullet(E_{\mathbf{a}})$ is $d_v(\hat{\tau})$, as was required. $\square$

To close out this subsection, we give an answer to Question 6.9 for square-free modules.

Remark 8.19. Let M be a square-free module with $\mathbb{Z}$-graded minimal free resolution $F_\bullet$. Notice that the proof of Theorem 8.1 shows that if $\lambda \in \mathcal{SR}^i(M)$, then there is a minimal k-th syzygy of the form $\lambda\eta$ for some $\eta \in F_{k-1}$ and some k at most $\operatorname{projdim}_S M + 1$. By Corollary 6.3 (ii), it follows that for every $\lambda \in \mathcal{SR}^i(M)$ there is such a minimal syzygy in the $\operatorname{projdim}_S M + 1$ syzygy module of M.

9. SYZYGETIC RESONANCE VARIETIES OF ORLIK–SOLOMON ALGEBRAS

In this chapter, we explore the implications of Theorem 8.1 for Orlik–Solomon algebras. The theorem does not immediately apply in this situation because the theorem identifies the syzygetic resonance varieties of square-free modules, but Orlik–Solomon algebras are in general not even $\mathbb{Z}^n$-graded. However, the fact that the definition of syzygetic resonance varieties depend only on a $\mathbb{Z}$-grading allows us to pass information through $\mathbb{Z}$-graded maps. The strategy in this chapter is to construct a $\mathbb{Z}$-graded map from a square-free module onto an Orlik–Solomon algebra, and then use Theorem 8.1 to obtain information which can be passed via the surjection.

9.1 Application to Orlik-Solomon Algebras

We give here an application of Theorem 8.1 to Orlik-Solomon algebras. We begin by constructing a square-free module which surjects onto a given Orlik-Solomon algebra. We then use this to produce a chain map comparing the free resolutions of the square-free module and the Orlik-Solomon algebra. With this chain map in hand, we give a condition under which facts about the syzygetic resonance and graded Betti numbers of the square-free module can be transferred to the Orlik-Solomon algebra.

We close by giving two examples. The first example falls into the case in which the constructed surjection from a square-free module is an isomorphism, in which case we may apply Theorem 8.1 to completely determine the syzygetic resonance varieties. We then consider an example in which the surjection is not an isomorphism, and we draw weaker conclusions than in the preceding example.

We now proceed with the construction of the surjection. The idea behind the construction is that each Orlik-Solomon ideal has a generating set expressible as a product of linear forms, and associating each linear form to a variable allows one to view the Orlik-Solomon algebra as the image of a quotient by a square-free monomial ideal.

Construction 9.1. Let M be a matroid on $[n]$, and let $I = I(\mathsf{M})$ be its Orlik-Solomon ideal in E. Choose a generating set for I so that each generator $\eta_1, \dots, \eta_k$ of I admits an expression as a product of linear forms. For each generator η_i, fix one such factorization $\eta_i = \lambda_{i,1} \cdots \lambda_{i,m_i}$ with each $\lambda_{i,j} \in E_1$. Noting that the span of the elements $\lambda_{i,j}$ is a subspace of $E_1 = V$, choose elements $\sigma_1, \dots, \sigma_p$ so that the elements $\lambda_{i,j}$ together with the elements σ_q span E_1. Let $\tilde{V}$ be a $\Bbbk$-vector space with one basis element $y_{i,j}$ for each distinct $\lambda_{i,j}$ and one basis element z_q for each distinct σ_q. Let $\tilde{E}$ be the exterior algebra on $\tilde{V}$, and let $\tilde{I}$ be the ideal of $\tilde{E}$ corresponding to I, namely the ideal generated by the elements $\tilde{\eta}_i = y_{i,1} \cdots y_{i,m_i}$. Notice that $\tilde{I}$ is a square-free monomial ideal of $\tilde{E}$, whence by Example 7.3, the quotient $\tilde{E}/\tilde{I}$ is a square-free $\tilde{E}$-module. Let $\epsilon \colon \tilde{E} \to E$ be the surjection defined by sending each $y_{i,j}$ to $\lambda_{i,j}$ and sending each z_q to σ_q. Then ϵ induces a $\mathbb{Z}$-graded surjection $\epsilon \colon \tilde{E}/\tilde{I} \to E/I$ of $\tilde{E}$-modules, where E/I is viewed as $\tilde{E}$-modules via restriction of scalars using $\epsilon \colon \tilde{E} \to E$.

Writing $\tilde{S} = \mathrm{Sym}(\tilde{V})$ for the symmetric algebra on $\tilde{V}$, let

$$0 \longrightarrow \textstyle\bigoplus_{\mathbf{a} \in \mathbb{Z}^n} \Phi_\bullet(E_{\mathbf{a}})^{\beta^{\tilde{S}}_{p,\mathbf{a}}(_{\tilde{S}}\tilde{E}/\tilde{I})} \longrightarrow \cdots \longrightarrow \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} \Phi_\bullet(E_{\mathbf{a}})^{\beta^{\tilde{S}}_{0,\mathbf{a}}(_{\tilde{S}}\tilde{M})} \longrightarrow 0.$$

be the double complex $F_{\bullet,\bullet}$ whose total complex $(\mathrm{Tot}(F_{\bullet,\bullet}), d)$ gives the minimal $\mathbb{Z}^n$-graded $\tilde{E}$-free resolution of $\tilde{E}/\tilde{I}$, guaranteed to exist by Theorem 7.9. Let $G_\bullet$ be the minimal $\mathbb{Z}$-graded E-free resolution of E/I. Using restriction of scalars via $\epsilon \colon \tilde{E} \to E$, we can view $G_\bullet$ is an acyclic complex of $\tilde{E}$-modules. Since $\mathrm{Tot}(F_{\bullet,\bullet})$ is a free resolution, it follows from the Comparison Theorem that the map $\epsilon \colon \tilde{E}/\tilde{I} \to E/I$ lifts to a chain map $\rho \colon \mathrm{Tot}(F_{\bullet,\bullet}) \to G_\bullet$.

We now give a condition under which information about the syzygetic resonance varieties and graded Betti numbers of the square-free module $\tilde{E}/\tilde{I}$ can be transferred to the Orlik-Solomon algebra E/I. The idea behind the theorem is that the structure of the minimal free resolution of the square-free module $\tilde{E}/\tilde{I}$ is built from Cartan resolutions, and these Cartan resolutions are linear. If the syzygy of E/I corresponding to the beginning of one such Cartan resolution in $\mathrm{Tot}(F_{\bullet,\bullet})$ is minimal, then for degree reasons the image under ρ of every syzygy in the Cartan resolution is minimal, thus giving the condition on the graded Betti numbers. The statement about syzygetic resonance follows from the fact that we constructed minimal syzygies with a scalar of λ out of these Cartan syzygies in Lemma 8.11, whence the images under ρ of these syzygies will also be minimal.

Theorem 9.2. *As in Construction 9.1, let E/I be an Orlik-Solomon algebra with minimal $\mathbb{Z}$-graded E-free resolution $G_\bullet$, let $\tilde{E}/\tilde{I}$ be a square-free module with minimal $\mathbb{Z}^n$-graded $\tilde{E}$-free resolution $(\operatorname{Tot}(F_{\bullet,\bullet}), d)$, and let $\rho\colon \operatorname{Tot}(F_{\bullet,\bullet}) \to G_\bullet$ be the chain map of $\tilde{E}$-modules lifting the surjection $\epsilon\colon \tilde{E}/\tilde{I} \to E/I$. If for some $q \in \mathbb{Z}$ and $\mathbf{a} \in \mathbb{Z}^n$ the image of the minimal q-th syzygy*

$$d(1) \in d\left(\Phi_0(E_{\mathbf{a}})\right) \subseteq d\left(F_{q,0}\right) \subseteq \operatorname{Syz}_q^{\tilde{E}}\left(\tilde{E}/\tilde{I}\right)$$

under the chain map ρ is a minimal q-th syzygy of E/I, then

$$\epsilon\left(\operatorname{span}_{\Bbbk} \operatorname{supp}(y^{\mathbf{a}})\right) \subseteq \mathcal{SR}^{|\mathbf{a}|-q}(E/I),$$

and for all $i \geq 0$,

$$\beta^E_{q+i,|\mathbf{a}|+i}(E/I) \geq \binom{i+|\mathbf{a}|-1}{|\mathbf{a}|-1}.$$

Moreover, if $\epsilon\colon \tilde{E}/\tilde{I} \to E/I$ is an isomorphism of $\mathbb{Z}$-graded $\tilde{E}$-modules, then for each i, j

$$\epsilon\left(\mathcal{SR}^i\left(\tilde{E}/\tilde{I}\right)\right) = \mathcal{SR}^i(E/I)$$

and

$$\beta^{\tilde{E}}_{i,j}\left(\tilde{E}/\tilde{I}\right) = \beta^E_{i,j}(E/I).$$

Proof. With the notation as in the statement, suppose that $\rho(d(1))$ is a minimal q-th syzygy of E/I. Let general $\lambda \in \operatorname{span}_{\Bbbk} \operatorname{supp}(y^{\mathbf{a}})$ be given. By the proof of Lemma 8.11, there is a minimal $(q+1)$-st syzygy of total degree $|\mathbf{a}|+1$ of the form $\lambda\eta$, where the component of η in the given copy of $\Phi_0(E_{\mathbf{a}}) \subset F_{q,0}$ is 1. It follows from Proposition 2.1 that $\rho(\lambda\eta) = \lambda\cdot\rho(\eta) \in G_q$ is a minimal $(q+1)$-st syzygy of E/I of total degree $\mathbf{a}+1$. Since the $\tilde{E}$-module structure on G_q is given by $\epsilon\colon \tilde{E} \to E$, it follows that $\epsilon(\lambda)\rho(\eta)$ is a minimal $(q+1)$-st syzygy of E/I of degree $|\mathbf{a}|+1$. It follows from Definition 6.4 that $\epsilon(\lambda) \in \mathcal{SR}^{|\mathbf{a}|-q}(E/I)$. Since λ was a general element of $\operatorname{span}_{\Bbbk} \operatorname{supp}(y^{\mathbf{a}})$ and since syzygetic resonance varieties are Zariski closed (by definition), the desired containment follows.

Regarding the statement about the graded Betti numbers, we provide proof by induction on $i \geq 0$. The case $i = 0$ holds because

$$\operatorname{span}_{\Bbbk} \rho(d(1)) = \rho\left(d\left(D_0(E_{\mathbf{a}})\right)\right) \subset \left(\operatorname{Syz}_q^E(E/I)\right)_{|\mathbf{a}|}$$

is a one-dimensional space of minimal syzygies. Suppose now that

$$\rho\left(d\left(D_{i-1}(E_{\mathbf{a}})\right)\right) \subset \left(\operatorname{Syz}_{q+i-1}^E(E/I)\right)_{|\mathbf{a}|+i-1}$$

is a subspace of minimal syzygies of dimension $\binom{i+|\mathbf{a}|-2}{|\mathbf{a}|-1}$. Note that by the definition of the Cartan resolution,

$$d\left(D_i(E_{\mathbf{a}})\right) \subseteq \left(\operatorname{Syz}_{q+i}^{\tilde{E}}\left(\tilde{E}/\tilde{I}\right)\right)_{|\mathbf{a}|+i}$$

is a subspace of minimal syzygies of dimension $\binom{i+|\mathbf{a}|-1}{|\mathbf{a}|-1}$. Moreover, by the structure $\operatorname{Tot}(F_{\bullet,\bullet})$, each such syzygy is of the form $\lambda\eta + \tau$ for some $\lambda \in E_1$ and η such that $d(\eta) \in D_{i-1}(E_{\mathbf{a}})$. It follows that for each such minimal syzygy $\lambda\eta + \tau$, the image

$$\rho(\lambda\eta + \tau) = \lambda\rho(\eta) + \rho(\tau)$$

under ρ is a syzygy. Since $d(\rho(\eta)) = \rho(d(\eta))$ is in the collection of syzygies $\rho\left(d\left(D_{i-1}(E_{\mathbf{a}})\right)\right)$ which are minimal by the induction hypothesis, it follows from Proposition 2.1 that the syzygies of the form $\rho(\lambda\eta + \tau)$ are all minimal. Since there is a subspace of these syzygies of dimension $\binom{i+|\mathbf{a}|-1}{|\mathbf{a}|-1}$, this concludes the induction and establishes the result.

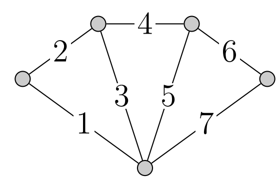
Finally, we consider the result in the situation where $\epsilon\colon \tilde{E} \to E$ is an isomorphism. In this situation, $\tilde{E}/\tilde{I}$ and E/I are isomorphic as $\mathbb{Z}$-graded (but not $\mathbb{Z}^n$-graded!) $\tilde{E}$-modules. Since $E \cong \tilde{E}$ as $\tilde{E}$-modules, the minimal free E-resolution $G_\bullet$ of E/I is a minimal free $\tilde{E}$-resolution as well. These two facts combined show that

$$\beta_{i,j}^E(E/I) = \beta_{i,j}^{\tilde{E}}(E/I) = \beta_{i,j}^{\tilde{E}}(\tilde{E}/\tilde{I}).$$

To show the equality involving the syzygetic resonance varieties, let a general $\lambda \in \mathcal{SR}^i(\tilde{E}/\tilde{I}) \subseteq \tilde{E}_1$ be given. Since $E/I \cong \tilde{E}/\tilde{I}$ as $\tilde{E}$-module, there exists a minimal $(q+1)$-st syzygy of E/I (viewed as $\tilde{E}$-module) of the form $\lambda \cdot \eta$ of total degree $q+i+1$. Since $\tilde{E} \cong E$ via ϵ, this is a minimal syzygy of E/I (viewed as E-module) of the form $\epsilon(\lambda)\cdot\epsilon(\eta)$, whence $\lambda \in \mathcal{SR}^i(E/I)$. Since $\mathcal{SR}^i(E/I)$ is Zariski closed, it follows that $\epsilon\left(\mathcal{SR}^i(\tilde{E}/\tilde{I})\right) \subseteq \mathcal{SR}^i(E/I)$. The other containment follows *mutatis mutandis*. $\square$

We now give two examples showing some conclusions one can draw from Theorem 9.2 in a concrete setting. We begin with an example in which the map ϵ is an isomorphism, allowing us to completely characterize the graded Betti numbers and the syzygetic resonance varieties.

Example 9.3. Let M be the matroid corresponding to the following graph:



The Orlik–Solomon ideal of $I = I(\mathsf{M})$ of M is generated by the three elements

- $\eta_1 = (x_2 - x_1)(x_3 - x_2) := \lambda_1\lambda_2$,
- $\eta_2 = (x_4 - x_3)(x_5 - x_4) := \lambda_3\lambda_4$,
- $\eta_3 = (x_6 - x_5)(x_7 - x_6) := \lambda_5\lambda_6$.

Notice that the λ_i together with $\sigma_1 := x_1 \in E_1$ form a $\Bbbk$-basis for E_1. Let $\tilde{E}$ be the exterior algebra on a $\Bbbk$-vector space $\tilde{V}$ with basis $y_1, \ldots, y_6$ and z_1, and let $\tilde{I}$ be the ideal in $\tilde{E}$ generated by $\tilde{\eta}_1 = y_1y_2$, $\tilde{\eta}_2 = y_3y_4$, and $\tilde{\eta}_3 = y_5y_6$. Define $\epsilon\colon \tilde{E} \to E$ by mapping y_i to λ_i and mapping z_1 to σ_1. It follows that $\epsilon\colon \tilde{E} \to E$ and whence $\epsilon\colon \tilde{E}/\tilde{I} \to E/I$ are isomorphisms. It follows from Theorem 9.2 that the $\mathbb{Z}$-graded Betti numbers and the syzygetic resonance varieties of E/I are determined by those of $\tilde{E}/\tilde{I}$.

Letting $\tilde{S}$ be the symmetric algebra on $\tilde{V}$, Theorem 8.1 and Corollary 7.11 imply that the $\mathbb{Z}^n$-graded Betti numbers and syzygetic resonance varieties of $\tilde{E}/\tilde{I}$ are determined by the

$\mathbb{Z}^n$-graded Betti numbers of ${}_{\tilde{S}}(\tilde{E}/\tilde{I})$. Since the resolution of this module is just the Koszul complex on y_1y_2, y_3y_4, y_5y_6, the $\mathbb{Z}^n$-graded Betti numbers are:

- $\beta^{\tilde{S}}_{0,(0,0,0,0,0,0,0)}\left({}_{\tilde{S}}(\tilde{E}/\tilde{I})\right) = 1$
- $\beta^{\tilde{S}}_{1,(1,1,0,0,0,0,0)}\left({}_{\tilde{S}}(\tilde{E}/\tilde{I})\right) = 1$
- $\beta^{\tilde{S}}_{1,(0,0,1,1,0,0,0)}\left({}_{\tilde{S}}(\tilde{E}/\tilde{I})\right) = 1$
- $\beta^{\tilde{S}}_{1,(0,0,0,0,1,1,0)}\left({}_{\tilde{S}}(\tilde{E}/\tilde{I})\right) = 1$
- $\beta^{\tilde{S}}_{2,(1,1,1,1,0,0,0)}\left({}_{\tilde{S}}(\tilde{E}/\tilde{I})\right) = 1$
- $\beta^{\tilde{S}}_{2,(1,1,0,0,1,1,0)}\left({}_{\tilde{S}}(\tilde{E}/\tilde{I})\right) = 1$
- $\beta^{\tilde{S}}_{2,(0,0,1,1,1,1,0)}\left({}_{\tilde{S}}(\tilde{E}/\tilde{I})\right) = 1$
- $\beta^{\tilde{S}}_{3,(1,1,1,1,1,1,0)}\left({}_{\tilde{S}}(\tilde{E}/\tilde{I})\right) = 1$

Putting everything together, we have that for all $i \geq 3$ the graded Betti numbers of E/I are:

$$\beta^E_{i,i+1}(E/I) = 3\binom{i}{1},$$

$$\beta^E_{i,i+2}(E/I) = 3\binom{i+1}{3},$$

and

$$\beta^E_{i,i+3}(E/I) = \binom{i+2}{5}.$$

The syzygetic resonance varieties of E/I are:

$$\mathcal{SR}^1(E/I) = \operatorname{span}_{\Bbbk}\{\lambda_1,\lambda_2\} \cup \operatorname{span}_{\Bbbk}\{\lambda_3,\lambda_4\} \cup \operatorname{span}_{\Bbbk}\{\lambda_5,\lambda_6\}$$

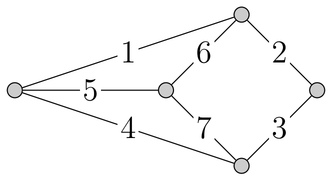
$$\mathcal{SR}^2(E/I) = \operatorname{span}_{\Bbbk}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4\} \cup \operatorname{span}_{\Bbbk}\{\lambda_1,\lambda_2,\lambda_5,\lambda_6\} \cup \operatorname{span}_{\Bbbk}\{\lambda_3,\lambda_4,\lambda_5,\lambda_6\}$$

and

$$\mathcal{SR}^3(E/I) = \operatorname{span}_{\Bbbk}\{\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6\}.$$

Finally, we give an example in which Theorem 9.2 yields only a portion of the information about the syzygetic resonance and graded Betti numbers.

Example 9.4. Let M be the matroid corresponding to the following graph:



Then the Orlik–Solomon ideal $I = I(\mathsf{M})$ of M is generated by the following four elements:

- $\eta_1 = (x_5 - x_1)(x_6 - x_5) := \lambda_1\lambda_2$,
- $\eta_2 = (x_5 - x_4)(x_7 - x_5) := \lambda_3\lambda_4$,
- $\eta_3 = (x_3 - x_2)(x_6 - x_3)(x_7 - x_6) := \lambda_5\lambda_6\lambda_7$,
- $\eta_4 = (x_2 - x_1)(x_3 - x_2)(x_4 - x_3) := \lambda_8\lambda_5\lambda_9$.

Notice that the λ_i together with $\sigma_1 := x_1 \in E_1$ form a $\Bbbk$-basis for E_1. Let $\tilde{E}$ be the exterior algebra on a $\Bbbk$-vector space $\tilde{V}$ with basis $y_1, \ldots, y_9$ and z_1, and let $\tilde{I}$ be the ideal in $\tilde{E}$ generated by $\tilde{\eta}_1 = y_1y_2$, $\tilde{\eta}_2 = y_3y_4$, $\tilde{\eta}_3 = y_5y_6y_7$, and $\tilde{\eta}_4 = y_8y_5y_9$. Define $\epsilon\colon \tilde{E} \to E$ by mapping y_i to λ_i and mapping z_1 to σ_1, and notice that ϵ is not an isomorphism.

Letting $\tilde{S}$ be the symmetric algebra on $\tilde{V}$, one can check that

$$\beta^{\tilde{S}}_{2,(1,1,1,1,0,0,0,0,0,0)}\left({}_{\tilde{S}}(\tilde{E}/\tilde{I})\right) = 1.$$

(This comes from the Koszul syzygy on y_1y_2, y_3y_4.) Further, one can check that the syzygy of E/I corresponding to $d(1) \in d\left(\Phi_0(E_{(1,1,1,1,0,0,0,0,0,0)})\right)$ is also minimal. (This is the Koszul syzygy on $\lambda_1\lambda_2, \lambda_3\lambda_4$.) It follows from Theorem 9.2 that for all $i \geq 0$,

$$\beta^E_{2+i,4+i}(E/I) \geq \binom{i+3}{3}$$

and

$$\operatorname{span}_{\Bbbk}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \subseteq \mathcal{SR}^2(E/I).$$

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