ARITHMETIC BREUIL-KISIN-FARGUES MODULES AND SEVERAL TOPICS IN P-ADIC HODGE THEORY

by

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To the memory of my grandfather

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PREFACE

The thesis is greatly influenced by the work of Fargues-Fontaine, Scholze, and Fargues-Scholze. It was in the fall of 2018, after coming back from the summer school on 'Groupes algébriques et géométrisation du programme de Langlands' at Lyon, I found out there is a simple way to characterize whether an admissible modification of vector bundles with G_K-action is isomorphic to the ones constructed in the book of Fargues-Fontaine from p-adic Hodge theory. At almost the same time, I read a preprint of Sean Howe where he defined the terminology of *arithmetic* Hodge-Tate G_K-modules. Then I had the vague idea that Fargues-Fontaine's construction could help study the G_K-actions on *arithmetic* Breuil-Kisin-Fragues modules which defined from *arithmetic* Hodge-Tate modules of Sean Howe via Fargues' classification theorem. The same fall, my advisor Tong Liu told me that something I study is related to a question of Toby Gee on Breuil-Kisin-Fargues G_K-modules with descents with respect to different uniformizer and Kummer towers. A similar type of question has also been answered by Tong Liu using $(\varphi, \widehat{G})$-modules.

Most of the results on Breuil-Kisin-Fargues modules of the paper were finished in 2019, but I was struggled to find a pleasing application to it. In the beginning, I tried to use the tool I developed to study representations of finite E-height. However, one night in the summer of 2019 during the summer school on Serre conjectures and the p-adic Langlands program in Padova, I found out a critical step that I was using is not correct. At the same time, the conjecture I was working on got proved. So things have to start over.

Later in 2019, the paper of Emerton-Gee on moduli stacks of crystalline representations came out, and I find my theory has a nice explanation of some of their results, and the first version of the thesis was done at that time. In 2020, I tried the second time on the problem on representations of finite E-height based a conjecture of Kedlaya-Liu, but later I found out the conjecture of Kedlaya-Liu is not correct and was able to find a counterexample of it.

It was finally, by the end of 2020, I found out a application of my theory to the p-adic monodromy theorem inspired by a question during an online talk I give in Shen Zhen. And this is how the main body of the thesis comes from.

I have seen this theory has connections to many new results in the p-adic Hodge theory these years, and I hope it will show its importance in the future.

TABLE OF CONTENTS

ABSTRACT

Let K be a discrete valuation field with perfect residue field, we study the functor from weakly admissible filtered (φ, N, G_K)-modules over K to the isogeny category of Breuil-Kisin-Fargues G_K-modules. This functor is the composition of a functor defined by Fargues-Fontaine from weakly admissible filtered (φ, N, G_K)-modules to G_K-equivariant modifications of vector bundles over the Fargues-Fontaine curve X_{FF}, with the functor of Fargues-Scholze that between the category of admissible modifications of vector bundles over X_{FF} and the isogeny category of Breuil-Kisin-Fargues modules. We characterize the essential image of this functor and give two applications of our result. First, we give a new way of viewing the p-adic monodromy theorem of p-adic Galois representations. Also we show our theory provides a universal theory that enable us to compare many integral p-adic Hodge theories at the A_{inf} level.

1. INTRODUCTION

1.1 Review of the work of Fargues-Fontaine and Fargues-Scholze

Fargues and Fontaine in [18] construct a complete abstract curve X_{FF}, the Fargues-Fontaine curve (constructed using the perfectoid field $\mathbb{C}_p^\flat$ and p-adic field $\mathbb{Q}_p$). For any p-adic field K, they show $\mathcal{O}_X = \mathcal{O}_{X_{FF}}$ carries an action of G_K, and they define $\mathcal{O}_X$-representations of G_K as vector bundles over X_{FF} that carries a continuous $\mathcal{O}_X$-semilinear action of G_K. They can show $\mathcal{O}_X$-representations of G_K can help study p-adic representations of G_K in many aspects. For example, Fargues-Fontaine show there is a nice slope theory on X_{FF}, and prove that the category of $\mathcal{O}_X$-representations such that the underlying vector bundles over X_{FF} are semistable of slope 0 is equivalence to the category of p-adic Galois representations. Moreover, they give an explicit construction of slope 0 $\mathcal{O}_X$-representations from weakly admissible filtered (φ, N)-modules D over K. Their construction is that: first using D and the (φ, N)-structure, they construct an $\mathcal{O}_X$-representation $\mathcal{E}(D, \varphi, N)$ of G_K whose underlying vector bundle is not semistable in general, then using the filtration structure of D_K, they constructed a G_K-equivariant modification $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet)$ of $\mathcal{E}(D, \varphi, N)$ along a closed point called ∞ on X_{FF}. They can show if D is weakly admissible, then $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet)$ is of slope 0, and the $\mathbb{Q}_p$-representation corresponds to $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet)$ is nothing but the log-crystalline representation corresponding to the data $(D, \varphi, N, \mathrm{Fil}^\bullet)$. Using this construction, they give new proofs of some important theorems in p-adic Hodge theory, for instance, the result that weakly admissible implies admissible, and also the p-adic monodromy theorem of p-adic Galois representations.

The scheme X_{FF} also plays a role in Scholze's work. In his Berkeley lecture on p-adic geometry[28], Scholze defined a mixed characteristic analog of shtukas with legs. To be more precise, he introduced the functor $\mathrm{Spd}(\mathbb{Z}_p)$ which plays a similar role of a proper smooth curve in the equal characteristic story, and for any perfectoid space S in characteristic p, he was able to define shtukas over S with legs. If we restrict us to the case that when $S = \mathrm{Spa}(C)$ is just a point, with $C = \mathbb{C}_p^\flat$ the tilt of the complete algebraic closure of K, and assume there is just one leg at the point corresponds to the untilt $\mathbb{C}_p$, then he can realize shtukas over S as (admissible) modifications of vector bundles over X_{FF} along ∞. Here ∞

is the same closed point on X_{FF} as we mentioned in the work of Fargues-Fontaine. Fargues-Scholze also show that those shtukas can be realized using some commutative algebra data, called finite free Breuil-Kisin-Fargues modules, which are modules over $A_{\inf} = W(\mathcal{O}_C)$ with some additional structures.

1.2 Arithmetic Breuil-Kisin-Fargues modules and essential images of Fargues-Fontaine-Scholze functor

If we combine the construction of Fargues and Fontaine of modifications of vector bundles over X_{FF} from log-crystalline representations and the work of Fargues and Scholze that relates modifications of vector bundles over X_{FF} with local shtukas and Breuil-Kisin-Fargues modules, one can expect that if starting with a weakly admissible filtered (φ, N)-module over K, one can produce a finite free Breuil-Kisin-Fargues module (actually only up to isogeny if we do not specify an integral structure of the log-crystalline representation) using the admissible modification constructed by Fargues-Fontaine. Moreover, since the modification is G_K-equivariant and all the correspondences of Fargues-Scholze we mentioned are functorial, we have the Breuil-Kisin-Fargues module produced in this way carries a semilinear G_K-action that commutes with all other structures of it. In this paper, we will study this and call it the Fargues-Fontaine-Scholze functor. And we have the following result.

Theorem 1.2.1. (Theorem 4.1.3) The Fargues-Fontaine-Scholze functor

$$\eta_{FFS} : \mathbf{MF}_K^{wa}(\varphi, N, G_K) \to \mathbf{BKF}(G_K)^{\circ}$$

is fully faithful.

Here $\mathbf{BKF}(G_K)^{\circ}$ is the isogeny category of Breuil-Kisin-Fargues G_K-modules(cf. Definition 4.0.1) and $\mathbf{MF}_K^{wa}(\varphi, N, G_K)$ is the category of weakly admissible filtered (φ, N, G_K)-modules over K. We will see in the isogeny category, every Breuil-Kisin-Fargues module is isomorphic to a finite free Breuil-Kisin-Fargues module(cf. Remark 3.2.3), so we make the following definition:

Definition 1.2.2. A finite free Breuil-Kisin-Fargues G_K-module is called *arithmetic* if, up to isogeny, it is in the essential image of the Fargues-Fontaine-Scholze functor η_{FFS}.

The first result of this paper is that we have a characterization *arithmetic* Breuil-Kisin-Fargues modules. Moreover, we can also characterize the essential image of η_{FFS} on the subcategory $\mathbf{MF}_K^{wa}(\varphi, N)$ (resp. $\mathbf{MF}_{K,\varphi}^{wa}$) of weakly admissible filtered (φ, N)-modules (resp. weakly admissible filtered φ-modules). Recall that for a Breuil-Kisin-Fargues module $\mathfrak{M}^{\mathrm{inf}}$ it admits a de Rham realization $\mathfrak{M}^{\mathrm{inf}}_{\mathbb{C}_p} = \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}},\theta} \mathcal{O}_{\mathbb{C}_p}$ and a crystalline realization $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}} = \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \breve{K}$ where $\breve{K} = W(\bar{k})[\frac{1}{p}]$. We also define $\overline{B} = A_{\mathrm{inf}}[\frac{1}{p}]/\mathfrak{p}$ after the work of Fargues-Fontaine, here $\mathfrak{p} = \{[\varpi]a \,|\, \varpi \in \mathfrak{m}_C,\, a \in A_{\mathrm{inf}}[\frac{1}{p}]\}$.

Theorem 1.2.3. (Theorem 4.2.1) Assume the p-adic monodromy theorem for p-adic Galois representations, then we have:

(1) A Breuil-Kisin-Fargues G_K-module $\mathfrak{M}^{\mathrm{inf}}$ is arithmetic if and only if $\mathfrak{M}^{\mathrm{inf}}_{\mathbb{C}_p}[\frac{1}{p}]$ as a $\mathbb{C}_p$-representation of G_K is $\mathbb{C}_p$-admissible, i.e., it is Hodge-Tate with only 0 weight.

(2) The isogeny class of a Breuil-Kisin-Fargues G_K-module $\mathfrak{M}^{\mathrm{inf}}$ is in the essential image of $\mathbf{MF}_K^{wa}(\varphi, N)$ if and only if $\mathfrak{M}^{\mathrm{inf}}$ is arithmetic and there is a G_K fixed basis inside $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$.

(3) The isogeny class of a Breuil-Kisin-Fargues G_K-module $\mathfrak{M}^{\mathrm{inf}}$ is in the essential image of $\mathbf{MF}_{K,\varphi}^{wa}$ if and only if $(\mathfrak{M}^{\mathrm{inf}} \otimes \overline{B})^{G_K}$ as a K_0-vector space has dimension equal to the rank of $\mathfrak{M}^{\mathrm{inf}}$.

Remark 1.2.4.

(1) The p-adic monodromy theorem for p-adic Galois representations is required in (1) of the above theorem, more explicitly, we are using the fact that $\mathbf{MF}_K^{wa}(\varphi, N, G_K)$ is equivalent to the category of $\mathbb{Q}_p$-representations of G_K that are de Rham. We emphasize this in the statement of this theorem since we will show later in this paper that given a finite free Breuil-Kisin-Fargues G_K-module that satisfies (1) in the above theorem, we are able to associate it with a weakly admissible filtered (φ, N, G_K)-module, this gives another way of proving the p-adic monodromy theorem.

(2) The terminology of being *arithmetic* was first introduced in the work of Howe in [20, §4] using Hodge-Tate modules, we can show our definition are the same by (1).

There is also an integral version of the above theorem.

Theorem 1.2.5. (Theorem 4.2.9) Assume the p-adic monodromy theorem for p-adic Galois representations, and let $\mathrm{Rep}^{dR}_{\mathbb{Z}_p}(G_K)$ (resp. $\mathrm{Rep}^{lcr}_{\mathbb{Z}_p}(G_K)$, resp. $\mathrm{Rep}^{cris}_{\mathbb{Z}_p}(G_K)$) be the category of de Rham (resp. log-crystalline, resp. crystalline) representations of G_K over $\mathbb{Z}_p$-lattices, then

(1) There is an equivalence of $\mathrm{Rep}^{dR}_{\mathbb{Z}_p}(G_K)$ with the category of arithmetic Breuil-Kisin-Fargues G_K-modules.

(2) The essential image of $\mathrm{Rep}^{lcr}_{\mathbb{Z}_p}(G_K)$ of the functor in (1) are the arithmetic Breuil-Kisin-Fargues modules such that there is a G_K fixed basis inside $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$.

(3) The essential image of $\mathrm{Rep}^{cris}_{\mathbb{Z}_p}(G_K)$ of the functor in (1) are Breuil-Kisin-Fargues G_K-modules such that $(\mathfrak{M}^{\mathrm{inf}} \otimes \overline{B})^{G_K}$ as a K_0-vector space has dimension equal to the rank of $\mathfrak{M}^{\mathrm{inf}}$.

Remark 1.2.6.

(1) Using unramified descent(cf. Lemma 1.6.2), one can show in Theorem 1.2.3 and Theorem 1.2.5 one can replace G_K by I_K and K_0 by $\breve{K}$ in the statement.

(2) From the work of [6], we know there is a large class of Breuil-Kisin-Fargues G_K-modules comes from geometry: start with a proper smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$, and let $\overline{\mathfrak{X}}$ be its base change to $\mathcal{O}_{\mathbb{C}_p}$. Then there is a A_{inf}-cohomology theory attached to $\overline{\mathfrak{X}}$ which is functorial in $\overline{\mathfrak{X}}$, so all the A_{inf}-cohomology groups $H^{\mathrm{i}}_{A_{\mathrm{inf}}}(\overline{\mathfrak{X}})$ carry natural semi-linear G_K-actions that commute with all other structures. If we take the maximal free quotients of the cohomology groups, then they are all arithmetic automatically from the étale-de Rham comparison theorem. So being arithmetic is the same as asking an abstract finite free Breuil-Kisin-Fargues G_K-module to satisfy

étale-de Rham comparison theorem. We also have [6, Theorem 14.1] shows that there is a canonical isomorphism

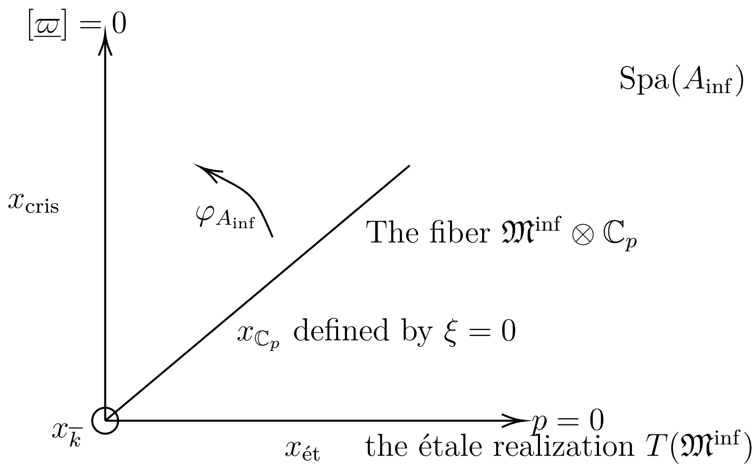
$$H^{\mathrm{i}}_{\mathrm{crys}}(\overline{\mathfrak{X}}_{\mathcal{O}_{\mathbb{C}_p}/p}/A_{\mathrm{cris}})[\frac{1}{p}] \cong H^{\mathrm{i}}_{\mathrm{crys}}(\overline{\mathfrak{X}}_{\overline{k}}/W(\overline{k})) \otimes_{W(\overline{k})} B^+_{\mathrm{cris}},$$

and we will see this is equivalent to condition (3) in Theorem 1.2.5.

(3) From Theorem 1.2.5, it is natural to ask if we can compare our theory of arithmetic BKF modules with other theories in integral p-adic Hodge theory. In Section 6, we will compare our theory with Breuil-Kisin theory (cf. [21]) and Liu's theory of $(\varphi, \hat{G})$-modules theory (cf. [25]) and a recent theory of Breuil-Kisin G_K-modules of Gao (cf. [19]). In the work of Liu and Gao, they both need an input of Kisin's theory, which relies on a choice of Kummer tower K_∞ over K, however, it was observed in [26] and [16] that there should be some compatibility of Kisin's theory for different choices of Kummer towers over K for log-crystalline representations. We will see in § 6.3, our theory of arithmetic BKF modules will give a very nice explanation of such phenomena.

(4) In the work of Bhatt-Morrow-Scholze[7] and Bhatt-Scholze[8], they show that the above A_{inf}-cohomology theory descent to a Breuil-Kisin cohomology theory. We will explore the relation of our crystalline condition (3) in Theorem 1.2.3 with the prismatic condition of Bhatt-Scholze in future work.

The Idea of proof. We use the following famous picture of $\mathrm{Spa}(A_{\mathrm{inf}})$ of Scholze:



By Fargues' classification theorem (cf. Theorem 3.2.9) of Breuil-Kisin-Fargues modules, we have in our definition of arithmetic Breuil-Kisin-Fargues modules one just need to determine the G_K-action on $\mathfrak{M}^{\mathrm{inf}}$ at $x_{\text{ét}}$ and $x_{\mathbb{C}_p}$, however, if we want to look closer to see which weakly admissible filtered (φ, N, G_K)-modules corresponds to $\mathfrak{M}^{\mathrm{inf}}$, the information is actually at x_{cris}. And the method to see how does the G_K-action expand from the two points $x_{\text{ét}}$ and $x_{\mathbb{C}_p}$ is to use Fargues-Fontaine's η_{FF} and translate their result in terms of φ-modules over certain subspaces of $\mathrm{Spa}(A_{\mathrm{inf}})$.

1.3 p-adic monodromy theorem as the inverse of Fargues-Fontaine-Scholze functor

Let $\mathfrak{M}^{\mathrm{inf}}$ be an arithmetic Breuil-Kisin-Fargues G_K-module, then if one assume the p-adic monodromy theorem for p-adic Galois representations, then we have seen the isogeny class of $\mathfrak{M}^{\mathrm{inf}}$ corresponds to a weakly admissible filtered (φ, N, G_K)-module over K via the Fargues-Fontaine-Scholze functor. We will show there is an inverse theorem in the following sense:

Theorem 1.3.1. (Theorem 5.1.15, p-adic monodromy theorem for arithmetic BKF modules) Let $\mathbf{BKF}^a(G_K)$ be the subcategory of Breuil-Kisin-Fargues G_K-modules that consists of $\mathfrak{M}^{\mathrm{inf}} \in \mathbf{BKF}(G_K)$ satisfying that $\mathfrak{M}^{\mathrm{inf}}_{\mathbb{C}_p}$ as a representation of G_K is $\mathbb{C}_p$-admissible, i.e.

arithmetic Breuil-Kisin-Fargues G_K-modules, and let $\mathbf{BKF}^a(G_K)^\circ$ be its isogeny category, then there is a functor

$$\omega_{FFS} : \mathbf{BKF}^a(G_K)^\circ \to \mathbf{MF}_K^{wa}(\varphi, N, G_K)$$

which is the quasi-inverse of η_{FFS}.

More explicitly, given $\mathfrak{M}^{\inf}$ in $\mathbf{BKF}^a(G_K)$, let $D = \mathfrak{M}^{\inf}_{\breve{K}}$, we will show using the G_K-action on $\mathfrak{M}^{\inf}$ we can restore the filtered (φ, N, G_K)-module structure on for D. We will show our p-adic monodromy theorem for arithmetic Breuil-Kisin-Fargues modules is equivalent to the p-adic monodromy theorem for p-adic Galois representations, that is every de Rham representation is potentially log-crystalline.

Remark 1.3.2.

(1) Comparing to the existing proofs of p-adic monodromy theorem for p-adic Galois representations, c.f. [5] [12] and [18], our proof is not completely new, actually in the construction of the monodromy operator N in our Proposition 5.1.10 is relied on [12, Proposition 10.11] which is also used as a key step of the proof p-adic monodromy theorem in [12] and [18].

(2) Even the proof is not completely new, we will explain why it is very natural to relate the p-adic monodromy theorem with Breuil-Kisin-Fargues modules. In particular, we will see in § 5.1.17 that ω_{FFS} can be viewed as "K-rational version" of Scholze's "?" functor considered in [28].

1.4 Comparisons of different integral p-adic Hodge theory

We will prove a rigidity result of arithmetic Breuil-Kisin-Fargues G_K-modules.

Lemma 1.4.1. *(Lemma 5.1.16) For any two arithmetic Breuil-Kisin-Fargues G_K-modules $\mathfrak{M}_1^{\inf}$ and $\mathfrak{M}_2^{\inf}$, if $T(\mathfrak{M}_1^{\inf}) \simeq T(\mathfrak{M}_2^{\inf})$, then $\mathfrak{M}_1^{\inf} \simeq \mathfrak{M}_2^{\inf}$.*

In classical integral p-adic Hodge theory has many results that show $\mathrm{Rep}^{*}_{\mathbb{Z}_p}(G_K)$ for $* \in \{\mathrm{cris}, \mathrm{lcr}, \mathrm{dR}\}$ are equivalent to φ-modules or (φ, G_K)-modules over subrings of $A_{\inf}$. For example, there are theory of Wach modules (cf. [4]), Kisin-Ren's theory (cf. [23]), Kisin modules (cf. [21]), $(\varphi, \widehat{G})$-modules (cf. [25]), and Breuil-Kisin G_K-modules (cf. [19]). We will show those φ-modules or (φ, G_K)-modules after a suitable base change to A_{inf}, give rise to arithmetic Breuil-Kisin-Fargues, and the rigidity result in Lemma 1.4.1 allows us to compare those theories at A_{inf}-level. Questions of this kind was firstly considered in [26], and recently also in [16] that discusses about the compatibility of Kisin's theory for different choices of uniformizers and Kummer towers for log-crystalline representations.

1.5 Structure of the paper

In Chapter 2, we will review the theory of Fargues-Fontaine curve and the construction of G_K-equivariant modifications of vector bundles over X_{FF} from weakly admissible filtered (φ, N, G_K)-modules. In Chapter 3, we will review various theories of φ-modules and their relations with vector bundles over the Fargues-Fontaine curve. Then we will review Fargues's classification theory for Breuil-Kisin-Fargues modules and the relation of isogeny classes of Breuil-Kisin-Fargues modules and admissible modifications of vector bundles. In Chapter 4, we will define the Fargues-Fontaine-Scholze functor η_{FFS} from $\mathbf{MF}_K^{wa}(\varphi, N, G_K)$ to the isogeny category of Breuil-Kisin-Fargues G_K-modules and prove our main result on characterization of the essential images of η_{FFS} on $\mathbf{MF}_K^{wa}(\varphi, N, G_K)$ and on some typical subcategories of $\mathbf{MF}_K^{wa}(\varphi, N, G_K)$. In Chapter 5, we will give our first application to prove the p-adic monodromy theorem of p-adic Galois representations, and we will discuss the relation of p-adic monodromy theorem and Scholze's '?' functor. In Chapter 6, we will apply our theory to show the compatibility of many existing theories in integral p-adic Hodge theory.

1.6 Notions and conventions

Throughout this paper, k will be a perfect field in characteristic p. Let $K_0 = W(k)[\frac{1}{p}]$ and $\mathcal{O}_{K_0} = W(k)$. Let K be a totally ramified finite extension of K_0, write $\mathcal{O}_K$ as the ring of integers of K and let ϖ be a uniformizer. Let $\breve{K} = W(\overline{k})[\frac{1}{p}]$ and define I_K be the inertia group inside G_K. By a compatible system of p^n-th roots of ϖ, we mean a sequence of elements $\{\varpi_n\}_{n\geq 0}$ in $\overline{K}$ with $\varpi_0 = \varpi$ and $\varpi_{n+1}^p = \varpi_n$ for all n. We normalize the p-adic valuation on K by $v_K(p) = 1$.

Define $\mathbb{C}_p$ as the p-adic completion of $\overline{K}$, there is a unique valuation $v = v_{\mathbb{C}_p}$ on $\mathbb{C}_p$ extending the p-adic valuation on K. Let $\mathcal{O}_{\mathbb{C}_p} = \{x \in \mathbb{C}_p | v(x) \geq 0\}$ and let $\mathfrak{m}_{\mathbb{C}_p} = \{x \in \mathbb{C}_p | v(x) > 0\}$. We will have $\mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}_{\mathbb{C}_p} = \overline{k}$.

Let $C = \mathbb{C}_p^\flat$ be the tilt of $\mathbb{C}_p$, then by the theory of perfectoid fields, C is algebraically closed of characteristic p, and complete with respect to a nonarchimedean norm. Let $\mathcal{O}_C$ be the ring of the integers of C, then $\mathcal{O}_C = \mathcal{O}_{\mathbb{C}_p}^\flat = \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}$. C is also a nonarchimedean field, we will denote the valuation by v_C and normalize it by $v_C(\underline{x}) = v_{\mathbb{C}_p}(x_0)$ for $\underline{x} = (x_0, x_1, \ldots) \in \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}$. We will also write it by v if there is no confusion. We define $\mathfrak{m}_C = \{x \in C \,|\, v_C(x) > 0\}$. Define $A_{\mathrm{inf}} = W(\mathcal{O}_C)$, there is a Frobenius $\varphi_{A_{\mathrm{inf}}}$ acts on A_{inf}. A_{inf} is equipped with a surjection $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_p}$ satisfies $\theta([x]) = x^0$ for $x = (x_0, x_1, \ldots) \in \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}$. We will have the kernel of θ is principal and let ξ be a generator of $\mathrm{Ker}(\theta)$. We will write $\tilde{\xi} = \varphi(\xi)$ following the notation used in [6]. There is a G_K-action on A_{inf} from its action on $\mathcal{O}_C$, one can show θ is G_K-equivariant.

1.6.1 Notion $\mathbf{MF}_K(\varphi, N, G_K)$

A filtered (φ, N)-module over K is a finite dimensional K_0-vector space D equipped with two maps

$$\varphi, N : D \to D$$

such that

(1) φ is semi-linear with respect to the Frobenius φ_{K_0};

(2) N is K_0-linear;

(3) $N\varphi = p\varphi N$.

And a decreasing, separated and exhaustive filtration on the K-vector space $D_K = K \otimes_{K_0} D$.

Let L be a finite Galois extension of K and let $L_0 = W(k_L)[\frac{1}{p}]$. A filtered $(\varphi, N, \mathrm{Gal}(L/K))$-module over K is a filtered (φ, N)-module D over L together with a semilinear action of $\mathrm{Gal}(L/K)$ on the L_0-vector space D, such that:

(1) The action is semilinear with respect to the action of $\mathrm{Gal}(L/K)$ on L_0 via $\mathrm{Gal}(L/K) \twoheadrightarrow \mathrm{Gal}(k_L/k) = \mathrm{Gal}(L_0/K_0)$.

(2) The semilinear action of $\mathrm{Gal}(L/K)$ commutes with φ and N.

(3) The filtration on $D \otimes_{L_0} L$ is stable under the diagonal action of $\mathrm{Gal}(L/K)$ on $D \otimes_{L_0} L$, i.e., it defines a filtration on $D_K := (D \otimes_{L_0} L)^{G_K}$.

If L' is another finite Galois extension of K containing L, then one can show there is a fully faithful embedding of the category of filtered $(\varphi, N, \mathrm{Gal}(L/K))$-modules into the category of filtered $(\varphi, N, \mathrm{Gal}(L'/K))$-modules. One defines the category of filtered (φ, N, G_K)-modules

$$\mathbf{MF}_K(\varphi, N, G_K)$$

to be the limit of filtered $(\varphi, N, \mathrm{Gal}(L/K))$-modules over all finite Galois extensions L of K.

Let $\breve{K} = W(\overline{k})[\frac{1}{p}]$, and $I_K = \mathrm{Gal}_{\breve{K}}$. For any $D \in \mathbf{MF}_K(\varphi, N, G_K)$ that is define over L, we let $\overline{D} = D \otimes_{L_0} \breve{K}$, then we have there is a continuous semilinear action of G_K on $\overline{D}$ such that I_K acts on $\overline{D}$ with open kernel. We will have the inverse of this is also true:

Lemma 1.6.2. *For a finite-dimensional $\breve{K}$ vector space $\overline{D}$ with a continuous semilinear action of G_K such that the action is trivial when restricting to an open subgroup H of I_K, we will have*

$$\overline{D} = (\overline{D})^{G_L} \otimes_{L_0} \breve{K}$$

for a finite extension L of K. Moreover, L can be chosen to be a totally ramified extension of K, i.e., we can assume $L_0 = K_0$.

Proof. We can not apply Galois descent directly in this case. We give a sketch of the proof. First since the action is continuous and $W(\overline{k})$ is DVR, so fix a lattice Λ inside $\overline{D}$, we have Λ is P stable for an open subgroup P of G_K. Moreover, since G_K is compact, a standard trick will imply that there is G_K stable lattice Λ_0 in $\overline{D}$. Let $\overline{\Lambda_0} = \Lambda_0 \mod p$. Let L be the corresponding totally ramified extension of $\breve{K}$, using Krasner's lemma, we can always choose a uniformizer ϖ of L such that ϖ is algebraic over K and defines a totally ramified extension L of K. We have G_L acts on $\overline{\Lambda_0}$ via $G_L \twoheadrightarrow G_k$, and the $\overline{k}$ vector space $\overline{\Lambda_0}$ is with the discrete topology. So we can apply Galois descent to the G_k semilinear action on $\overline{\Lambda_0}$ and use the exact sequence

$$0 \to \Lambda_0 \xrightarrow{p} \Lambda_0 \to \overline{\Lambda_0} \to 0$$

one can argue via successive approximation by lifting to show $\Lambda_0^{G_L}$ is of full rank, inverting p one get

$$\overline{D} = (\overline{D})^{G_L} \otimes_{L_0} \breve{K}.$$

For details of the last part of the proof, one can refer to Lemma 3.2.6 of [9]. □

By the above lemma, for any $D \in \mathbf{MF}_K(\varphi, N, G_K)$ we can always assume the underlying φ-module is defined over $\breve{K}$ and equipped with a continuous semilinear G_K-action such that the restricted action on I_K has an open kernel. We will use this fact later in this paper.

1.6.3 Notion $\mathbf{MF}_K^{wa}(\varphi, N, G_K)$

We will let $\mathbf{MF}_K^{wa}(\varphi, N, G_K)$ be the subcategory of $\mathbf{MF}_K(\varphi, N, G_K)$ consisting of weakly admissible objects. And we define its subcategory $\mathbf{MF}_K^{wa}(\varphi, N)$ (resp. $\mathbf{MF}_{K,\varphi}^{wa}$) which is the category of weakly admissible filtered (φ, N)-modules (φ-modules).

1.6.4 Notion Linearization

Let R be a ring equipped with an endomorphism φ, for a R-module M, we write φ^*M to be $M \otimes_{R,\varphi} R$.

1.6.5 Conventions for semistable and log-crystalline

In this paper, we will use the notion of *log-crystalline representations* instead of *semistable representations* to make a difference to the semistability of vector bundles over complete regular curves.

1.6.6 Conventions for Hodge-Tate weights

We will use covariant functors when relate étale φ-modules and Galois representation, so we will assume the cyclotomic character has Hodge-Tate weight -1.

1.6.7 Conventions for modifications of vector bundles

We will see in our definition of modifications of vector bundles, we always mean the modification is admissible, i.e., a modifications such that the first vector bundle is semistable of slope 0.

2. FARGUES-FONTAINE CURVE AND THE FARGUES-FONTAINE FUNCTOR

In this chapter, we will first review the construction and properties of the Fargues-Fontaine curve over the perfectoid field C, and then we will recall the Fargues-Fontaine functor from $\mathbf{MF}_K^{wa}(\varphi, N, G_K)$ to the category of G_K-equivariant modifications of vector bundles over the Fargues-Fontaine curve.

2.1 The p-adic fundamental curve of Fargues and Fontaine

Recall C is the tilt of $\mathbb{C}_p$, to define the Fargues-Fontaine curve over C, we first review some basic period rings of Fontaine.

Let $A_{\text{inf}} = W(\mathcal{O}_C)$ and recall there is a canonical surjection

$$\theta : A_{\text{inf}} \to \mathcal{O}_{\mathbb{C}_p}$$

such that $\theta([x]) = x_0$ for $x = (x_{\mathrm{i}}) \in \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}$. Let ξ be a generator of $\mathrm{Ker}(\theta)$. Let $B_{\mathrm{dR}}^+ = \varprojlim_n A_{\text{inf}}[\frac{1}{p}]/(\xi^n A_{\text{inf}}[\frac{1}{p}])$ and recall the topology on B_{dR}^+ is the weak topology. And $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[\frac{1}{\xi}]$ equipped with a $\mathbb{Z}$-filtration $\mathrm{Fil}^n B_{\mathrm{dR}} = \xi^n B_{\mathrm{dR}}^+$. Also recall that A_{cris} is defined as the p-completed PD envelope of $A_{\text{inf}} \to \mathcal{O}_{\mathbb{C}_p}$. Let $B_{\text{cris}}^+ = A_{\text{cris}}[\frac{1}{p}]$. A_{cris} and B_{cris}^+ can be regarded as subrings of B_{dR}, and recall the element $t = \log[\epsilon] \in \mathrm{Fil}^1 B_{\mathrm{dR}}$ is well-defined where $\epsilon = (\zeta_{\mathrm{i}})_{\mathrm{i}} \in C$ is defined by a compatible system of p^n-th roots of units with $\zeta_0 = 1$. Let $B_{\text{cris}} = B_{\text{cris}}^+[\frac{1}{t}]$. Let $\underline{\varpi} = (\varpi_{\mathrm{i}}) \in C$ defined by a compatible system of p^n-th roots of ϖ with $\varpi_0 = \varpi$ a unifromizer of $\mathcal{O}_K$. Then

$$\log[\underline{\varpi}] := \sum_{\mathrm{i} \geq 1} \frac{-(1 - [\underline{\varpi}]/\varpi)^{\mathrm{i}}}{\mathrm{i}}$$

is well-defined in $\mathrm{Fil}^1 B_{\mathrm{dR}}$ and we define $B_{\mathrm{st}}^+ = B_{\text{cris}}^+[\log[\underline{\varpi}]]$ and $B_{\mathrm{st}} = B_{\text{cris}}[\log[\underline{\varpi}]]$ with the unique B_{cris}-derivation determined by $N(\log[\underline{\varpi}]) = 1$.

Remark 2.1.1. Here we use the convention that $N(\log[\underline{\varpi}]) = 1$ as [18, §10.3.2] when they define N on $B_{\log}$, this is also compatible with the convention used in [22]. We will use this fact when we compare our theory of arithmetic Breuil-Kisin-Fargues modules and Breuil-Kisin theory.

Remark 2.1.2. When we discuss potentially log-crystalline representations, there will be an issue of change of field, we might need to consider $\log[\underline{\varpi_L}]$ where $\{\underline{\varpi_L}\}$ is defined by a compatible system of p^n-th roots of a uniformizer ϖ_L of a finite extension L of K. We can always choose $\varpi_L = \varpi^{\mathrm{e}} \mod p$ for some $\mathrm{e} \in \mathbb{N}$, and we know $\underline{\varpi_L} \in C$ only depends on the classes of $\varpi_{L,n} \mod p$, so we can always choose $\underline{\varpi_L} = \underline{\varpi}^{\mathrm{e}}$. So in particular $\log[\underline{\varpi_L}] = \mathrm{e}\log[\underline{\varpi}]$ and they will define the same subring B_{st} inside B_{dR}.

For every subring A of B_{dR}, define $\operatorname{Fil}^n A = A \cap \operatorname{Fil}^n B_{\mathrm{dR}}$ for all $n \in \mathbb{Z}$. Recall that the Frobenius φ on A_{inf} extensions to A_{cris}, B^+_{cris}, B_{cris} and B_{st}. We define $B_{\mathrm{e}} = B_{\mathrm{cris}}^{\varphi=1}$, we have

Lemma 2.1.3. *The inclusion $B_{\mathrm{e}} \to B_{\mathrm{dR}}$ induces an exact sequence*

$$0 \to \mathbb{Q}_p \to B_{\mathrm{e}} \to B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \to 0.$$

Moreover, one has B_{e} is actually a PID, and this is actually one of the motivations to define the Fargues-Fontaine curve. For the story behind this, one can refer to [13].

For the Fargues-Fontaine curve $X_{FF} := X_{C,\mathbb{Q}_p}$ (here we use the notion in [18, Definition 6.5.1.], where they construct Fargues-Fontaine curves $X_{F,E}$ for pairs (F, E) where F is any perfectoid field in characteristic p and E is a discrete valuation field), an abstract definition of X_{FF} is that X_{FF} is a scheme fits into the following Cartesian diagram

$$\begin{tikzcd} X_{FF} & \mathrm{Spec}(B_{\mathrm{e}}) \arrow[l] \\ \mathrm{Spec}(B^+_{\mathrm{dR}}) \arrow[u] & \mathrm{Spec}(B_{\mathrm{dR}}) \arrow[l] \arrow[u] \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \tag{2.1.3.1}$$

In particular, we have $X_{FF} = \mathrm{Spec}(B_{\mathrm{e}}) \coprod \{\infty\}$ such that $X_{FF,\infty} = B^+_{\mathrm{dR}}$. Fargues-Fontaine give an explicit construction

$$X_{FF} = \operatorname{Proj} \oplus_{\mathrm{i}\geq 0} (B^+_{\mathrm{cris}})^{\varphi = p^{\mathrm{i}}}.$$

Recall $t = \log[\epsilon]$ satisfies $\varphi(t) = pt$, so it is a section of $\mathcal{O}(1)$. Fargues-Fontaine showed that t has a unique zero $\infty \in X_{FF}$, then we summarize some of the main results in [18]:

Theorem 2.1.4. The pointed scheme (X_{FF}, ∞) fits into the diagram 2.1.3.1, and we have:

(1) X_{FF} is a regular noetherian scheme of Krull dimension 1, or an abstract regular curve in the sense of Fargues and Fontaine.

(2) $X_{\mathrm{e}} = X_{FF} \backslash \{\infty\}$ is an affine scheme $\mathrm{Spec}(B_{\mathrm{e}})$.

(3) Vector bundles $\mathcal{E}$ over X_{FF} are equivalence to B-pairs $(M_{\mathrm{e}}, M_{\mathrm{dR}}^{+}, \iota)$, where $M_{\mathrm{e}} = \Gamma(X_{\mathrm{e}}, \mathcal{E})$ is a finite projective module over B_{e}, M_{dR}^{+} is a finite free module over B_{dR}^{+}, and ι is an isomorphism of M_{e} and M_{dR}^{+} over B_{dR}. And the functor is given by

$$\mathcal{E} \mapsto (\Gamma(\mathrm{Spec}(B_{\mathrm{e}}), \mathcal{E}), \mathcal{E}_{\infty}, \iota).$$

One advantage of having an explicit definition of X_{FF} is that one can have the following construction of vector bundles from isocrystals.

Theorem 2.1.5. (Theorem 8.2.10 in [18]) Let (D, φ) be an isocrystal over $\overline{k}$, then (D, φ) defines a vector bundle $\mathcal{E}(D, \varphi)$ over X_{FF} which is associated with the graded module

$$\oplus_{n \geq 0} (D \otimes_{\breve{K}} B_{\mathrm{cris}}^{+})^{\varphi = p^n}.$$

Moreover, this functor induces a bijection of isomorphism classes.

Definition 2.1.6.

(1) Let $\mathcal{E}$ be a vector bundle over X_{FF}, assume $\mathcal{E} \cong \mathcal{E}(D, \varphi)$ under the above theorem, let the multi-set $\{-\lambda_{\mathrm{i}}\}$ be the slope of (D, φ) under the Dieudonné-Manin classification theorem, we define the *slope* of $\mathcal{E}$ to be the multi-set $\{\lambda_{\mathrm{i}}\}$.

(2) $\mathcal{E}$ is called semistable of slope λ if and only if $\mathcal{E}$ corresponds a semisimple isocrystal of slope $-\lambda$. Rank 1 vector bundle of slope n is denoted by $\mathcal{O}(n)$ which corresponds to $(\breve{K}, p^{-n}\varphi_{\breve{K}})$.

(3) Let $\mathrm{Bun}_{X_{FF}}$ be the category of vector bundles over X_{FF} and let $\mathrm{Bun}^{\lambda}_{X_{FF}}$ be the subcategory of semistable vector bundles of slope λ.

A consequence of Theorem 2.1.5 is

Corollary 2.1.7.

(1) The category of isocrystals semisimple of slope $-\lambda$ is equivalent to $\mathrm{Bun}^{\lambda}_{X_{FF}}$.

(2) In particular, when $\lambda = 0$, the category of finite-dimensional $\mathbb{Q}_p$-vector spaces is equivalent to $\mathrm{Bun}^{0}_{X_{FF}}$*, and the functor is given by*

$$V \to V \otimes_{\mathbb{Q}_p} \mathcal{O}_X$$

with quasi inverse

$$\mathcal{E} \to H^0(X_{FF}, \mathcal{E}).$$

Proof. This is [18, Theorem 9.2.2]. □

Definition 2.1.8. Let $\mathscr{M}odif_{X_{FF}}$ be the category of triples $(\mathcal{E}_0, \mathcal{E}_1, \iota) \in \mathscr{M}odif_{X_{FF}}$, where

- $\mathcal{E}_0 \in \mathrm{Bun}^0_{X_{FF}}$;
- $\mathcal{E}_1 \in \mathrm{Bun}_{X_{FF}}$;
- $\iota : \mathcal{E}_0|_{X_{FF}-\{\infty\}} \xrightarrow{\sim} \mathcal{E}_1|_{X_{FF}-\{\infty\}}$.

We will also denote this by $\mathcal{E}_0 \dashrightarrow \mathcal{E}_1$ if no confusion arises.

Lemma 2.1.9. $\mathscr{M}odif_{X_{FF}}$ *is equivalent to the category of pairs (V, Ξ), where*

- *V is a finite dimensional vector space over $\mathbb{Q}_p$;*
- *$\Xi \in V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ is a B^+_{dR}-lattice.*

Proof. By (3) in Theorem 2.1.4, use the B-pair description of vector bundles over X_{FF}, we have for $\mathcal{E}_0 \dashrightarrow \mathcal{E}_1 \in \mathscr{M}odif_{X_{FF}}$, then $\mathcal{E}_0$ and $\mathcal{E}_1$ have the same B_{e}-part, so the modification can can be determined by $(\mathcal{E}_0, \Xi)$ where $\Xi = \mathcal{E}_{1,\infty}$ is the B^+_{dR} part of $\mathcal{E}_1$. Now by Corollary 2.1.7, this is the same as (V, Ξ) as in the statement of this lemma if we let $V = H^0(X_{FF}, \mathcal{E}_0)$. □

2.2 p-adic representations and modifications of vector bundles on the curve

We have $\mathcal{O}_X = \mathcal{O}_{X_{FF}}$ has a continuous action of G_K via its action on B^+_{cris}, and ∞ is fixed by G_K.

Definition 2.2.1.

(1) Let $\mathrm{Bun}_X(G_K)$ be the category of G_K-equivariant vector bundles over X_{FF} such that the G_K-semilinear action is continuous. Here the notion of continuity can be interpreted using the B-pair description of vector bundles over X_{FF}(c.f. (3) in Theorem 2.1.4). That is continuity is the same as the B_{e} and B^+_{dR} parts of the vector bundle both carry a continuous semilinear action of G_K. Objects in $\mathrm{Bun}_X(G_K)$ are also called $\mathcal{O}_X$-representations of G_K.

(2) Let $\mathscr{M}od\mathrm{i}f_X(G_K)$ be the category of G_K-equivariant modifications of vector bundles, i.e. triples $(\mathcal{E}_0, \mathcal{E}_1, \iota) \in \mathscr{M}od\mathrm{i}f_{X_{FF}}$ such that both $\mathcal{E}_0, \mathcal{E}_1 \in \mathrm{Bun}_X(G_K)$ and ι is G_K-equivariant.

Here is another interpretation of the G_K-action using cocycles, cf. [18, Proposition 9.1.5].

Lemma 2.2.2. *Fix $\mathcal{E} \in \mathrm{Bun}_{X_{FF}}$, given a G_K-equivariant structure on $\mathcal{E}$ is the same as choosing a continuous 1-cocycles $f = (f_g)_g \in Z^1(G_K, \mathrm{Aut}(\mathcal{E}))$. And the isomorphic classes of G_K-equivariant structure on $\mathcal{E}$ is bijective to $H^1(G_K, \mathrm{Aut}(\mathcal{E}))$.*

Definition 2.2.3. [18, Definition 9.1.6] Given $\mathcal{E} \in \mathrm{Bun}_X(G_K)$ and a continuous 1-cocycles $a = (a_g)_g$ inside $Z^1(G_K, \mathrm{Aut}(\mathcal{E}))$. And let $f = (f_g)_g \in Z^1(G_K, \mathrm{Aut}(\mathcal{E}))$ be a cocylce represents the G_K-equivariant structure on $\mathcal{E}$, and let $[f] \in H^1(G_K, \mathrm{Aut}(\mathcal{E}))$ be it cohomology class. Then one can show $a_g \circ [f_g]$ is a well-defined element in $H^1(G_K, \mathrm{Aut}(\mathcal{E}))$ and we will define the corresponded $\mathcal{O}_X$-representation by $a \wedge \mathcal{E} \in \mathrm{Bun}_X(G_K)$, i.e., it is the vector bundle with the same underlying $\mathcal{O}_X$-structure as $\mathcal{E}$ and the G_K-equivariant structure is defined by $a_g \circ [f_g]$ via Lemma 2.2.2.

In the rest of this section, we review Fargues-Fontaine's functor

$$\eta_{FF} : \mathbf{MF}^{wa}_K(\varphi, N, G_K) \to \mathscr{M}od\mathrm{i}f_X(G_K).$$

The main reference is [18, §10.3.2]. Keep the notions as in Theorem 2.1.5.

2.2.4 Construction of $\mathcal{E}(D, \varphi, N, G_K)$

Let D be a filtered (φ, N, G_K)-module, recall as we have mentioned in Lemma 1.6.2, the underlying φ-module (D, φ) can always be treated as an isocrystal over $\overline{k}$, and let

$$\mathcal{E}(D,\varphi) = \oplus_{n\geq 0}(D \widetilde{\otimes_{\breve{K}} B}^{+}_{\text{cris}})^{\varphi=p^n}$$

be the vector bundle over X_{FF} corresponds to (D, φ) under the equivalence in Theorem 2.1.5. We have the G_K-action on D defines a continuous G_K-action on $\mathcal{E}(D, \varphi)$ via the diagonal action on $D \otimes_{\breve{K}} B^+_{\text{cris}}$ and we use $\mathcal{E}(D, \varphi, G_K)$ to refer this $\mathcal{O}_X$-representation.

Note that this construction is functorial, so the relation $N\varphi = p\varphi N$ tells that N defines a G_K-equivariant map

$$N: \quad \mathcal{E}(D,\varphi,G_K) \to \mathcal{E}(D,p\varphi,G_K) = \mathcal{E}(D,\varphi,G_K) \otimes_{\mathcal{O}_X} \mathcal{O}(-1).$$

Recall we define $\log[\underline{\varpi}] := \log(\frac{[\underline{\varpi}]}{\varpi})$ and $t = \log[\epsilon]$ inside B^+_{dR}. For every $g \in G_K$, let $\log_{\underline{\varpi},g} = g(\log[\underline{\varpi}]) - \log[\underline{\varpi}]$. We will have $\log_{\underline{\varpi},g} = \log([\epsilon^{c_g}]) = c_g t$ for a 1-cocycle $c = (c_g)_g$ valued in $\mathbb{Q}_p^\times$, so $\log_{\underline{\varpi},g}$ is inside $(B^+_{\text{cris}})^{\varphi=p}$, i.e. $\log_{\underline{\varpi},g}$ defines a morphism from $\mathcal{O}(-1)$ to $\mathcal{O}$. So we know the composition:

$$\beta_g : \mathcal{E}(D,\varphi) \xrightarrow{N} \mathcal{E}(D,\varphi)\otimes\mathcal{O}(-1) \xrightarrow{\text{Id}\otimes\log_{\underline{\varpi},g}} \mathcal{E}(D,\varphi)\otimes_{\mathcal{O}_X}\mathcal{O}_X = \mathcal{E}(D,\varphi)$$

defines an element in $\text{End}(\mathcal{E}(D, \varphi))$. Moreover, since $\log_{\underline{\varpi},g}$ is a $\mathbb{Q}_p$ multiple of t, so $(\log_{\underline{\varpi},g})_g$ defines an element in $Z^1(G_K, (B^+_{\text{cris}})^{\varphi=p})$, so $\beta = (\beta_g)_g$ defined as above is an element in $Z^1(G_K, \text{End}(\mathcal{E}(D, \varphi)))$, and by the nilpotence of N, we have the image of β lies in the nilpotent elements of $\text{End}(\mathcal{E}(D, \varphi))$. So we can define

$$\alpha = (\alpha_g)_g = (-\exp(\beta_g))_g \in Z^1(G_K, \text{Aut}(\mathcal{E}(D,\varphi))). \tag{2.2.4.1}$$

Fargues-Fontaine define a new G_K-action on the vector bundle $\mathcal{E}(D,\varphi)$ by twisting, we let

$$\mathcal{E}(D,\varphi,N,G_K)=\alpha\wedge\mathcal{E}(D,\varphi,G_K).$$

i.e., $\mathcal{E}(D,\varphi,N,G_K)$ is isomorphic to $\mathcal{E}(D,\varphi)$ as vector bundle, and the G_K-action is given by twisting the G_K-action of $\mathcal{E}(D,\varphi,G_K)$ with the 1-cocycle α(cf. Definition 2.2.3).

There is another way of thinking about this new action.

Lemma 2.2.5. *Let D be a (φ,N)-module. There is an φ-equivariant isomorphic*

$$\begin{array}{ccc} D\otimes_{\breve{K}} B^+_{\text{cris}} & \xrightarrow{\sim} & (D\otimes_{\breve{K}} B^+_{\text{st}})^{N=0} \\ y & \mapsto & \hat{y}:=\sum_{\mathrm{i}\geq 0}\frac{(-1)^{\mathrm{i}}}{\mathrm{i}!}N^{\mathrm{i}}(v)\otimes\log([\varpi])^{\mathrm{i}}, \end{array}$$

for every $y\in D$ and extends linearly to $D\otimes_{\breve{K}} B^+_{\text{cris}}$.

Proof. Here we will use a similar computation in [25, §7.2]. Write $X=\log([\varpi])$ and $\gamma_{\mathrm{i}}(X)=\frac{X^{\mathrm{i}}}{\mathrm{i}!}$. We have $N(\gamma_{\mathrm{i}}(X))=\gamma_{\mathrm{i}-1}(X)$ by our convention $N(X)=1$. And we have

$$N(\hat{y})=\sum_{\mathrm{i}\geq 0}(-1)^{\mathrm{i}}(N^{\mathrm{i}+1}(v)\otimes\gamma_{\mathrm{i}}(X)+N^{\mathrm{i}}(v)\otimes\gamma_{\mathrm{i}-1}(X))=0$$

i.e., $\hat{y}\in(D\otimes_{\breve{K}} B^+_{\text{st}})^{N=0}$. A direct computation also shows that $y\to\hat{y}$ is φ-equivariant. And we define

$$\widehat{D}=\{\hat{v}\,|\,v\in D\}.$$

One can check $\widehat{D}$ has a structure of isocrystal over $\overline{k}$ via

$$\breve{K}\hookrightarrow(B^+_{\text{st}})^{N=0}\hookrightarrow B^+_{\text{st}}$$

and it is isomorphic to D. Moreover via $B^+_{\text{cris}}=(B^+_{\text{st}})^{N=0}\hookrightarrow B^+_{\text{st}}$, one have $(D\otimes_{\breve{K}} B^+_{\text{st}})^{N=0}$ is a B^+_{cris}-module which is isomorphic to $\widehat{D}\otimes_{\breve{K}} B^+_{\text{cris}}$. $\square$

If $D\in\mathbf{MF}_K(\varphi,N,G_K)$, given $D\otimes B^+_{\text{st}}$ the diagonal action of G_K, then we can view $D\otimes B^+_{\text{cris}}$ as a G_K-stable subspace of $D\otimes B^+_{\text{st}}$ via the the above lemma.

Lemma 2.2.6. *With the notions in Lemma 2.2.5, we have G_K acts on $(D \otimes_{\breve{K}} B_{\mathrm{st}}^+)^{N=0}$ by*

$$g(\hat{y}) = \exp(-\log_{\underline{\varpi},g} N))(\widehat{g(y)}).$$

Proof. We can write $\hat{y} = \exp(-\log([\underline{\varpi}])N)(y)$ for $y \in D$. Use the fact that the G_K-action commutes with N, we have

$$\begin{aligned} g(\hat{y}) &= g(\exp(-\log([\underline{\varpi}])N)(y)) \\ &= \exp(-g(\log([\underline{\varpi}]))N)(g(y)) \\ &= \exp((-\log_{\underline{\varpi},g} - \log([\underline{\varpi}]))N)(g(y)) \\ &= \exp(-\log_{\underline{\varpi},g} N)\exp((-\log([\underline{\varpi}])N)(g(y)) \\ &= \exp(-\log_{\underline{\varpi},g} N)(\widehat{g(y)}). \end{aligned}$$

□

In terms of vector bundles, the map $y \mapsto \hat{y}$ in Lemma 2.2.5 defines an isomorphism of graded modules:

$$\hat{h} : \oplus_{\mathrm{i}\geq 0}(D \otimes B_{\mathrm{cris}}^+)^{\varphi=p^{\mathrm{i}}} \xrightarrow{\sim} \oplus_{\mathrm{i}\geq 0}(D \otimes B_{\mathrm{st}}^+)^{N=0,\varphi=p^{\mathrm{i}}},$$

So it defines an automorphism $\hat{h}_{\mathcal{E}}$ of $\mathcal{E} = \mathcal{E}(D)$. Given $\oplus_{\mathrm{i}\geq 0}(D \otimes B_{\mathrm{st}}^+)^{N=0,\varphi=p^{\mathrm{i}}}$ the G_K-action defined in Lemma 2.2.6, and Lemma 2.2.6 implies this action, in terms of cocylces, is given by

$$(\alpha_g \circ \hat{h}_{\mathcal{E}} \circ f_g \circ \hat{h}_{\mathcal{E}}^{-1})_g,$$

where $(f_g)_g$ is a coclyce defines the G_K-structure of $\mathcal{E}(D, \varphi, G_K)$ and α_g is defined in 2.2.4.1. In other words, this action is exactly $\alpha \wedge \mathcal{E}(D, \varphi, G_K)$ by Definition 2.2.3.

2.2.7 B_{e} and B_{dR}^+-representations of G_K

From (3) in Theorem 2.1.4, one can also describe an $\mathcal{O}_X$-representation in term of B-pairs, we want to note that the terminology of B-pairs was first appeared in the work of Berger [3].

Proposition 2.2.8. An $\mathcal{O}_X$-representation $\mathcal{E}$ is equivalence to a G_K-representation on a B-pair $(M_{\mathrm{e}}, M^+_{\mathrm{dR}}, \iota)$. Here $M_{\mathrm{e}} = \Gamma(X_{\mathrm{e}}, \mathcal{E})$ (resp. M^+_{dR}) is a finite free B_{e}-module (resp. B^+_{dR}-module) with a continuous G_K-semilinear action, and ι is a G_K-equivariant isomorphism of M_{e} and M^+_{dR} over B_{dR}.

We want to describe $\mathcal{E}(D, \varphi, N, G_K)$ in terms of B-pairs. For the B^+_{dR} part, we want to mention the following results. They play a critical role in our theory.

Definition 2.2.9. A B_{dR}(resp. B^+_{dR})-representation of G_K is a finite free B_{dR} (resp. B^+_{dR}) module together with a continuous G_K-semilinear action. We say a B_{dR} (resp. B^+_{dR})-representation W of G_K is flat (resp. generically flat) if $\dim_K(W^{G_K}) = \dim_{B_{\mathrm{dR}}} W$ (resp. $\dim_K(W[\frac{1}{t}]^{G_K}) = \dim_{B_{\mathrm{dR}}} W[\frac{1}{t}]$).

Proposition 2.2.10.

(1) $-\otimes_K B_{\mathrm{dR}}$ induces an equivalence of finite dimensional K vector space and flat B_{dR}-representations of G_K, and the quasi-inverse is given by $W \mapsto W^{G_K}$.

(2) $(V, \mathrm{Fil}^\bullet V) \mapsto \mathrm{Fil}^0(V \otimes_K B_{\mathrm{dR}})$ induces an equivalence of filtered K vector spaces and generically flat B^+_{dR}-representations of G_K, and the quasi-inverse is given by $W \mapsto (W[\frac{1}{t}]^{G_K}, (t^\bullet W)^{G_K})$.

(3) A rank d B^+_{dR}-representation W of G_K is generically flat if and only if there exist $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ such that

$$W \simeq \oplus_{\mathrm{i}=1}^d t^{a_{\mathrm{i}}} B^+_{\mathrm{dR}}.$$

Proof. This is §10.4 [18]. □

We also make the following definition of B^+_{dR}-flat representations. They are one of the key ingredients in our theory.

Definition 2.2.11. A B^+_{dR}-representation W of G_K of rank d is flat if there is a G_K-equivariant isomorphism

$$W \simeq \oplus_{\mathrm{i}=1}^d B^+_{\mathrm{dR}}.$$

Proposition 2.2.12.

(1) The B_{dR}^{+}-representation W of G_K is B_{dR}^{+}-flat if and only if $W \otimes_{B_{\mathrm{dR}}^{+}} \mathbb{C}_p$ as a $\mathbb{C}_p$-representation is $\mathbb{C}_p$-admissible, i.e., it is Hodge-Tate with all weight equal to 0.

(2) There is an equivalence of B_{dR}^{+}-flat representations of G_K and finite dimensional K-vector space, and the functor is given by $W \mapsto (W[\frac{1}{t}])^{G_K}$ with quasi-inverse given by $V \mapsto V \otimes_K B_{\mathrm{dR}}^{+}$.

Proof. For (1), the author's proof first appeared in [16, Proposition F.13.]. Assume that $W \otimes_{B_{\mathrm{dR}}^{+}} \mathbb{C}_p$ has a G_K-stable basis, we need to construct a G_K-stable basis of W. Recall that B_{dR}^{+} has the weak topology such that the quotients $B_{\mathrm{dR}}^{+}/(\xi^n) \simeq \mathbb{C}_p^n$ are equipped with the p-adic topology, so we will argue via constructing a successive lifting of G_K-fixed basis $\{\mathrm{e}_{\mathrm{i}}^{(n)}\}$ of $W/t^{\mathrm{i}}W$ satisfying $\mathrm{e}_{\mathrm{i}}^{(n)} = \mathrm{e}_{\mathrm{i}}^{(n+1)} \mod (\xi^n)$ for all $n \geq 1$.

For $n = 1$, this is given by the assumption. And assume we have already constructed $\{\mathrm{e}_{\mathrm{i}}^{(n-1)}\}$, we choose an arbitrary lifting $\{\tilde{\mathrm{e}}_{\mathrm{i}}^{(n)}\}$ of $\{\mathrm{e}_{\mathrm{i}}^{(n-1)}\}$ to $W \otimes_{B_{\mathrm{dR}}^{+}} B_{\mathrm{dR}}^{+}/(\xi^n)$. For $g \in G_K$, assume g acts by

$$g.(\tilde{\mathrm{e}}_1^{(n)}, \tilde{\mathrm{e}}_2^{(n)}, ..., \tilde{\mathrm{e}}_d^{(n)}) = (\tilde{\mathrm{e}}_1^{(n)}, \tilde{\mathrm{e}}_2^{(n)}, ..., \tilde{\mathrm{e}}_d^{(n)}).A_g^{(n)}$$

we have the following diagram:

$$\begin{tikzcd}
g.(\tilde{\mathrm{e}}_1^{(n)}, \tilde{\mathrm{e}}_2^{(n)}, ..., \tilde{\mathrm{e}}_d^{(n)}) \arrow[r, equal] \arrow[d, two heads, "\mathrm{mod}\ (\xi^{n-1})"] & (\tilde{\mathrm{e}}_1^{(n)}, \tilde{\mathrm{e}}_2^{(n)}, ..., \tilde{\mathrm{e}}_d^{(n)})A_g^{(n)} \arrow[d, two heads, "\mathrm{mod}\ (\xi^{n-1})"] \\
g.(\mathrm{e}_1^{(n-1)}, \mathrm{e}_2^{(n-1)}, ..., \mathrm{e}_d^{(n-1)}) \arrow[r, equal] & (\mathrm{e}_1^{(n-1)}, \mathrm{e}_2^{(n-1)}, ..., \mathrm{e}_d^{(n-1)})
\end{tikzcd}$$

Let $d = \mathrm{rank}(\mathcal{M})$, we have $A_g^{(n)}$ is a $d \times d$ matrix with coefficients in $B_{\mathrm{dR}}^{+}/(\xi^n)$ satisfies $A_g^{(n)} \equiv I_d \mod (\xi^{n-1})$. Recall, t is also a generator of (ξ) in B_{dR}^{+}, so one can write

$$A_g^{(n)} = I_d + t^{n-1}B_g^{(n)}$$

for some $B_g^{(n)} \in M_d(B_{\mathrm{dR}}^{+}/(\xi^n))$ and this expression is uniquely determined by the class of $B_g^{(n)}$ in $M_d(B_{\mathrm{dR}}^{+}/(\xi))$.

The map $g \mapsto A_g^{(n)}$ defines a 1-cocycle of G_K in $\mathrm{GL}_d(B_{\mathrm{dR}}^+/(\xi^n))$. And by a simple computation, we will have $B_{hg}^{(n)} = B_h^{(n)} + \chi^{n-1}(h)h(B_g^{(n)})$, i.e., the map $g \mapsto B_g^{(n)}$ defines a 1-cocycle of G_K in $M_d(B_{\mathrm{dR}}^+/(\xi)(n-1))$, that is, a 1-cocycle of G_K in $M_{d\times d}(\mathbb{C}_p(n-1))$. Since $n > 1$ by our assumption, by Tate-Sen, we have such a 1-cocycle is a 1-coboundary, i.e., there is a $\alpha^{(n)} \in M_d(\mathbb{C}_p)$ such that $B_g^{(n)} = \chi^{n-1}(g)g(\alpha^{(n)}) - \alpha^{(n)}$.

Then let $(\mathrm{e}_1^{(n)}, \mathrm{e}_2^{(n)}, ..., \mathrm{e}_d^{(n)}) = (\tilde{\mathrm{e}}_1^{(n)}, \tilde{\mathrm{e}}_2^{(n)}, ..., \tilde{\mathrm{e}}^{(n)})(I_d - t^{n-1}\alpha^{(n)})$, we have:

$$\begin{aligned}
g.(\mathrm{e}_1^{(n)}, ..., \mathrm{e}_d^{(n)}) &= (\tilde{\mathrm{e}}_1^{(n)}, ..., \tilde{\mathrm{e}}_d^{(n)})(I_d + t^{n-1}B_g^{(n)})g.(I_d - t^{n-1}\alpha^{(n)}) \\
&= (\tilde{\mathrm{e}}_1^{(n)}, ..., \tilde{\mathrm{e}}_d^{(n)})(I_d + t^{n-1}B_g^{(n)} - g.t^{n-1}g.(\alpha^{(n)})) \\
&= (\tilde{\mathrm{e}}_1^{(n)}, ..., \tilde{\mathrm{e}}_d^{(n)})(I_d + t^{n-1}(\chi^{n-1}(g)g(\alpha^{(n)}) - \alpha^{(n)}) - \chi^{n-1}(g)t^{n-1}g.(\alpha^{(n)})) \\
&= (\tilde{\mathrm{e}}_1^{(n)}, ..., \tilde{\mathrm{e}}_d^{(n)})(I_d - t^{n-1}\alpha^{(n)})
\end{aligned}$$

and satisfies $\mathrm{e}_{\mathrm{i}}^{(n-1)} \equiv \mathrm{e}_{\mathrm{i}}^{(n)} \mod (\xi^{n-1})$.

For (2), it is a consequence of (2) in Proposition 2.2.10. $\square$

Remark 2.2.13.

(1) Given a p-adic de Rham representations V of G_K, there are two ways to assign a G_K-stable B_{dR}^+-lattice in $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$. Using (2) of Proposition 2.2.10, we know there is a lattice Ξ_0 that corresponds to $D_{\mathrm{dR}}(V)$ with the Hodge filtration. On the other hand, we can also give $D_{\mathrm{dR}}(V)$ the trivial filtration, and produce another G_K-stable lattice $\Xi_1 = \mathbf{D}_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}^+$.

(2) B_{dR}^+ is not $(\mathbb{Q}_p, G_K)$-regular since tB_{dR}^+ is G_K-stable but t is not a unit. However one can define for a $\mathbb{Q}_p$-representation V it is called B_{dR}^+-admissible if the following holds

$$(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{G_K} \otimes_K B_{\mathrm{dR}}^+ \xrightarrow{\sim} V \otimes_K B_{\mathrm{dR}}^+.$$

Then one can show by Proposition 2.2.10, B_{dR}^+-admissibility is equivalent to B_{dR}-admissibility plus the condition that the Hodge-Tate weights are non-negative (one can also refer to [9, Exercise 15.5.5]). In [16, Proposition F.13.], the proof actually shows that the representation T is B_{dR}^+-admissibility, and that is the reason why they require the Frobenius on Breuil-Kisin-Fargues modules is an endomorphism. We will come back to this issue.

Recall the following result of Berger on B_{e}-representations and (φ, N)-modules.

Proposition 2.2.14. [18, p. 10.3.20] There are functors

$$\begin{aligned} \mathscr{D}_{\log} : \mathrm{Rep}_{B_{\mathrm{e}}}(G_K) \to & \quad (\varphi, N)\text{-modules} \\ M \mapsto & \quad (M \otimes_{B_{\mathrm{e}}} B_{\mathrm{st}})^{G_K} \end{aligned}$$

and

$$\begin{aligned} \mathscr{V}_{\log} : (\varphi, N)\text{-modules} \to & \quad \mathrm{Rep}_{B_{\mathrm{e}}}(G_K) \\ D \mapsto & \quad (D \otimes_{\breve{K}} B_{\mathrm{st}})^{\varphi=1, N=0} \end{aligned}$$

We have $\mathrm{Id} \xrightarrow{\sim} \mathscr{D}_{\log} \circ \mathscr{V}_{\log}$, and $\mathscr{V}_{\log} \circ \mathscr{D}_{\log}(M) = M$ if and only if M is in the essential image of $\mathscr{V}_{\log}$. Similar result holds when replacing (φ, N)-modules by φ-modules and B_{st} by B_{cris}.

2.2.15 B-pairs for $\mathcal{E}(D, \varphi, N, G_K)$ and $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)$

Fargues-Fontaine give a nice description of the $\mathcal{O}_X$-representation $\mathcal{E}(D, \varphi, N, G_K)$ in terms of B-pair.

Proposition 2.2.16. Let $(M_{\mathrm{e}}, M_{\mathrm{dR}}^+, \iota)$ be the G_K-representation on a B-pair corresponds to the $\mathcal{O}_X$-representation $\mathcal{E}(D, \varphi, N, G_K)$, then

$$M_{\mathrm{e}} \xrightarrow{\sim} \mathscr{V}_{\log}(D) = (D \otimes_{\breve{K}} B_{\mathrm{st}})^{\varphi=1, N=0}$$

and

$$M_{\mathrm{dR}}^+ \xrightarrow{\sim} D_K \otimes_K B_{\mathrm{dR}}^+$$

as G_K-modules. Here $D \otimes_{\breve{K}} B_{\mathrm{st}}$ is given by the diagonal action of G_K and $D_K = (D \otimes_{\breve{K}} B_{\mathrm{dR}})^{G_K}$.

Proof. For the B_{e} part, it follows from [18, §10.3.5]. For the B_{dR}^+ part, one can follow the argument of [18, Proposition 10.3.18]. Or we can use Proposition 2.2.12, it is equivalence to show it is B_{dR}^+-flat. But the latter can be easily seen from the fact

$$\exp(-\log_{\underline{\varpi},g} N) \equiv 1 \mod \mathrm{Ker}(\theta)$$

since $\log_{\underline{\varpi},g} = c_g t$ is a multiply of t. □

Now let us construct a modification of $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)$. By Proposition 2.2.10 a filtration on $D_K = (D \otimes_{\breve{K}} B_{\mathrm{dR}})^{G_K}$ is equivalent to a G_K stable B_{dR}^+ lattice in $D \otimes_{\breve{K}} B_{\mathrm{dR}}$. Fargues-Fontaine define the $\mathcal{O}_X$-representation $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)$ by letting:

$$\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)|_{X_{FF}\backslash\infty} = \mathcal{E}(D, \varphi, N, G_K)|_{X_{FF}\backslash\infty}$$

and

$$\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)_\infty = \mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}}).$$

To show $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K) \dashrightarrow \mathcal{E}(D, \varphi, N, G_K)$ defines a G_K-equivariant modification, it remains to show the admissibility, i.e., $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)$ is semistable of slope 0. This follows from the Proposition below, which is one of the key result of Fargues-Fontaine explaining the relation of weakly admissibility and the slope of vector bundles.

Proposition 2.2.17. The filtered (φ, N, G_K)-module D is weakly admissible if and only if $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)$ is semistable of slope 0. Moreover, let V be the potentially log-crystalline representation of G_K corresponding to D, then there is a G_K-equivariant isomorphism

$$V = H^0(X_{FF}, \mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)),$$

where V is the potentially log-crystalline representation corresponding to the data $(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)$.

Proof. For the first part, it is stated in [18, §10.5.3, Remark 10.5.8] for filtered φ-modules, and for filtered (φ, N)-modules, it is [13, Proposition 5.6]. For the second part of the proof, let V be the potentially log-crystalline representation corresponding to $(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)$ and let $\mathcal{E}_V = V \otimes_{\mathbb{Q}_p} \mathcal{O}_X$ be the corresponding slope 0 $\mathcal{O}_X$-representation of G_K. It is equivalent to show

$$\mathcal{E}_V = \mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)$$

and we can prove it by comparing the B-pairs of $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)$ and $\mathcal{E}_V$ by of Proposition 2.2.8. While we have the B_e-part of the $\mathcal{O}_X$-representation $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)$ is the same as the B_e-part of the $\mathcal{O}_X$-representation $\mathcal{E}(D, \varphi, N, G_K)$ by construction, and which is equal to

$$\mathscr{V}_{\log}(D) = (D \otimes B_{st})^{\varphi=1, N=0}$$

by Proposition 2.2.16. The B_{dR}^+-part of $\mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K)$ is the B_{dR}^+-representation

$$\mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}})$$

by definition.

On the other hand, the B-pair correspond to $\mathcal{E}_V$ is

$$(V \otimes_{\mathbb{Q}_p} B_\mathrm{e}, V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+).$$

Since V is potentially log-crystalline, so we have $V \otimes_{\mathbb{Q}_p} B_\mathrm{e}$ is potentially log-crystalline as a B_e-representation in the sense that there is a G_K-equivariant isomorphism

$$V \otimes_{\mathbb{Q}_p} B_\mathrm{e} = \left(\left((V \otimes_{\mathbb{Q}_p} B_\mathrm{e}) \otimes_{B_\mathrm{e}} B_{\mathrm{st}} \right)^{G_L} \otimes_{L_0} B_{\mathrm{st}} \right)^{\varphi=1, N=0}$$

for a finite Galois extension L of K by Proposition 10.3.20 of [18], and $\left((V \otimes_{\mathbb{Q}_p} B_\mathrm{e}) \otimes_{B_\mathrm{e}} B_{\mathrm{st}}\right)^{G_L} \simeq D$ as (φ, N, G_K)-modules. For the B_{dR}^+-part, since V is de Rham, so the B_{dR}^+-representation $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+$ is *generically flat*, so Proposition 2.2.10 shows that there is a G_K-equivariant isomorphism

$$V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+ = \mathrm{Fil}^0(D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}).$$

$\square$

2.2.18 The Fargues-Fontaine functor

Definition 2.2.19. We define

$$\begin{aligned} \eta_{FF} : \mathbf{MF}_K^{wa}(\varphi, N, G_K) &\to \mathscr{M}odif_X(G_K) \\ D &\mapsto \mathcal{E}(D, \varphi, N, \mathrm{Fil}^\bullet, G_K) \dashrightarrow \mathcal{E}(D, \varphi, N, G_K). \end{aligned}$$

Let's show η_{FF} is fully faithful.

Lemma 2.2.20. *$\mathscr{M}odif_X(G_K)$ is equivalent to the category of pairs (V, Ξ) where V is a representation of G_K over a $\mathbb{Q}_p$-vector space and Ξ is a G_K stable B_{dR}^+-lattice in $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+$.*

Proof. Directly from Lemma 2.1.9. □

Theorem 2.2.21. η_{FF} is fully faithful, and the essential image of η_{FF} in terms of (V, Ξ) such that V is a de Rham representation of G_K and Ξ is a B_{dR}^+ stable lattice in $V \otimes B_{\mathrm{dR}}$ such that as B_{dR}^+-representations of G_K, it is flat.

Proof. Using Lemma 2.2.20 and Proposition 2.2.8, we can write down η_{FF} in terms of (V, Ξ)-pairs:

$$\begin{aligned} \tilde{\eta}_{FF} : \mathbf{MF}_K^{wa}(\varphi, N, G_K) &\to \{(V, \Xi)\} \\ D &\mapsto (V(D), D_{\mathrm{dR}}(V(D)) \otimes_K B_{\mathrm{dR}}^+) \end{aligned}$$

where $V(D)$ is the potentially log-crystalline representation of G_K corresponds to D. So it is obvious that η_{FF} is fully faithful and the essential image lies inside the category defined in the theorem.

And if a pair (V, Ξ) such that V is a de Rham representation of G_K and Ξ a flat B_{dR}^+-representations of G_K inside $V \otimes B_{\mathrm{dR}}$, then by the equivalence in Proposition 2.2.10, we will have $\Xi = D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}^+$, by the p-adic monodromy theorem of p-adic Galois representations, we have V is potentially log-crystalline, so corresponds to some $D \in \mathbf{MF}_K(\varphi, N, G_K)$. And again by the computations in Proposition 2.2.17, we have the the B-pair corresponds to $\eta_{FF}(D)$ is (V, Ξ). □

3. Φ-MODULES AND BREUIL-KISIN-FARGUES MODULES

This chapter will discuss φ-modules over various base rings and their relation with vector bundles over X_{FF}. And we will also discuss Breuil-Kisin-Fargues modules and recall their relations with modifications of vector bundles over the Fargues-Fontaine curve.

3.1 φ-modules and vector bundles

Recall we have the rings in characteristic p: k, $\overline{k}$, $\mathcal{O}_C$, and C on which the Frobenius endomorphism is defined. Moreover, since we have

$$\mathcal{O}_C = \varprojlim_{x \to x^p} \mathcal{O}_{\overline{K}}/p = \varprojlim_{x \to x^p} \mathcal{O}_{\overline{K}_0}/p.$$

So there is a canonical map $k \to \mathcal{O}_C$ comes from $k \simeq \mathcal{O}_{K_0}/p \to \mathcal{O}_{\overline{K}_0}/p$ which has a uniquely lifting to a section $\overline{k} \to \mathcal{O}_C$ of $\mathcal{O}_C \to \overline{k}$ by étaleness. This defines a canonical φ-equivariant section

$$W(\overline{k}) \to A_{\mathrm{inf}}.$$

Let's recall the following constructions of rings with φ-endomorphisms, the main reference of this part is [18, §11]. Define $B^+ = \cap_n \varphi^n(B^+_{\mathrm{cris}})$ and let B be the completion of $B^b := A_{\mathrm{inf}}[\frac{1}{[\varpi]}, \frac{1}{p}]$ with respect to $\{|\cdot|_\rho\}_{\rho \in (0,1)}$, here for $\rho \in (0,1)$, define

$$|\sum_{\mathrm{i} \gg -\infty} [x_{\mathrm{i}}] p^{\mathrm{i}}|_\rho = \sup\{|x_{\mathrm{i}}| \rho^{\mathrm{i}}\}$$

where $\sum_{\mathrm{i} \gg -\infty} [x_{\mathrm{i}}] p^{\mathrm{i}} \in B^b$. For $\rho \in (0,1)$, let B^+_ρ be the completion of $A_{\mathrm{inf}}[\frac{1}{p}]$ with respect to $|\cdot|_\rho$, then $B^+ = \cap_{\rho \in (0,1)} B^+_\rho$ and $B^+ \hookrightarrow B$. Consider the ideal $\mathfrak{p} = \cup_n (\varphi^{-n}[\underline{\varpi}]) \subset A_{\mathrm{inf}}[\frac{1}{p}]$ where ϖ is any pseudo uniformizer in C, and define $\tilde{\mathfrak{p}} = \mathfrak{p} B^+$, we will have $A_{\mathrm{inf}}[\frac{1}{p}] \to B^+$ induces an isomorphism $A_{\mathrm{inf}}[\frac{1}{p}]/\mathfrak{p} = B^+/\tilde{\mathfrak{p}}$ by [18, p. 11.1.1], and let $\overline{B}$ denote this quotient. Then we have $\overline{B}$ is a local domain with maximal ideal $\mathfrak{m}_{\overline{B}} := W(\mathfrak{m}_C)[\frac{1}{p}]/\mathfrak{p}$ and residue field $\breve{K}$. Let's recall the following properties of B^+:

Example 3.1.1. For $\rho \in (0,1)$, we have $B^+_\rho = \widehat{A_{\mathrm{inf}}[\frac{[a]}{p}]}[\frac{1}{p}]$, where $|a| = \rho$ and the completion is for the $(p, [\varpi])$-adic topology for any $\varpi \in \mathfrak{m}_C$, but we have $[a]$ is divisible by p in $\widehat{A_{\mathrm{inf}}[\frac{[a]}{p}]}$, so the topology is the same as the p-adic topology. In particular, we have $\widehat{A_{\inf}[\frac{[a]}{p}]}$ is also completed under the $[\varpi]$-adic topology.

Definition 3.1.2. Let R be a ring that is equipped with a Frobenius endomorphism φ, a φ-module over R is a finite projective module M together with

$$\varphi^* M := M \otimes_{R,\varphi} R \xrightarrow{\sim} M.$$

And we use $\varphi\text{-Mod}_R$ to denote the category of φ-modules over R. For a φ-module M, we define its global section $H^0(M)$ by

$$H^0(M) = M^{\varphi=1}.$$

Proposition 3.1.3.

(1) $M \mapsto M \otimes_{B^+} \overline{B}$ induces an equivalence of $\varphi\text{-Mod}_{B^+}$ and $\varphi\text{-Mod}_{\overline{B}}$.

(2) For $(d, h) \in \mathbb{N} \times \mathbb{N}_{>0}$ with $(h, d) = 1$, and $R \in \{\breve{K}, B, B^+, B^+_{\mathrm{cris}}, \overline{B}\}$, let $R(\frac{d}{h})$ be the φ-module free on the bases $\{\mathrm{e_i}\}_{\mathrm{i}=0}^{h-1}$ and $\varphi(\mathrm{e_i}) = \mathrm{e_{i+1}}$ for $\mathrm{i} \in [0, h-2]$ and $\varphi(\mathrm{e}_{h-1}) = p^d \mathrm{e}_0$. Then any $M \in \varphi\text{-Mod}_R$ is a direct sum of $R(\frac{d}{h})$ over pairs (d, h).

(3) By (2), for any $M \in \varphi\text{-Mod}_{B^+}$ (resp. $M_{\overline{B}} \in \varphi\text{-Mod}_{\overline{B}}$), let $\breve{M} = M \otimes_{B^+} \breve{K}$ (resp. $\breve{M} = M_{\overline{B}} \otimes_{\overline{B}} \breve{K}$), then there is a φ-equivariant section $s : \breve{M} \to M$ (resp. $s : \breve{M} \to M_{\overline{B}}$) reducing to the identity over $\breve{K}$.

Proof. This is [18, Theorem 11.1.7]. $\square$

Lemma 3.1.4.

(1) For all $h \in \mathbb{N}_{>0}$ and $d \in \mathbb{Z}$, the natural maps between B, B^+, B^+_{cris} and $\overline{B}$ induces

$$B^{\varphi^h = p^d} = (B^+)^{\varphi^h = p^d} = (B^+_{\mathrm{cris}})^{\varphi^h = p^d} = \overline{B}^{\varphi^h = p^d}.$$

(2) If $d < 0$, then we will have

$$B^{\varphi^h = p^d} = 0.$$

Proof. (2) is from [18, Proposition 4.1.2]. For (1), in *loc.cit.*, they show $B^{\varphi^h=p^d} = (B^+)^{\varphi^h=p^d}$ and $(B^+)^{\varphi^h=p^d} = \overline{B}^{\varphi^h=p^d}$ when $d > 0$. We give a different approach to prove this for all d.

Let $R(\frac{d}{h})$ be as in (2) in in Proposition 3.1.3 and for $R \in \{\overline{B}, B^+, B, B^+_{\text{cris}}\}$, we have

$$R^{\varphi^h=p^d} = \mathrm{Hom}_\varphi(R(\frac{d}{h}), R).$$

Since the categories of φ-modules over R are all equivalent to each other by [18, §11.1]. So the above Hom are all equal with each other. □

Proposition 3.1.5. Let $M^+ \in \varphi\text{-Mod}_{B^+}$ and define $M = M^+ \otimes_{B^+} B$ (resp. $M_{\text{cris}} = M^+ \otimes_{B^+} B^+_{\text{cris}}$, resp. $M_{\overline{B}} = M^+ \otimes_{B^+} \overline{B}$). Then we have for all $h \in \mathbb{N}_{>0}$ and $d \in \mathbb{Z}$.

$$(M^+)^{\varphi^h=p^d} = M^{\varphi^h=p^d} = (M_{\text{cris}})^{\varphi^h=p^d} = M_{\overline{B}}^{\varphi^h=p^d}.$$

Proof. Use the same trick as in Lemma 3.1.4, we have $M^{\varphi^h=p^d} = \mathrm{Hom}_\varphi(B(\frac{d}{h}), M)$, and similar formula for M^+ M_{cris}, and $M_{\overline{B}}$. So the fully faithfulness of the base change functors implies the result. □

Theorem 3.1.6. [18] Let the category $\varphi\text{-Mod}_{B^+}$, $\varphi\text{-Mod}_B$, $\varphi\text{-Mod}_{B^+_{\text{cris}}}$, and $\varphi\text{-Mod}_{\overline{B}}$ are all equivalent to the category of $\mathrm{Bun}_{X_{FF}}$. And one direction of the functor is

$$(M, \varphi) \mapsto \mathcal{E}(M) = \widetilde{\oplus_{n\geq 0} M^{\varphi=p^n}}$$

for $(M, \varphi) \in \varphi\text{-Mod}_{B^+}$ (resp. $\varphi\text{-Mod}_B$, resp. $\varphi\text{-Mod}_{B^+_{\text{cris}}}$, resp. $\varphi\text{-Mod}_{\overline{B}}$).

Proof. The equivalence of $\varphi\text{-Mod}_{B^+}$, $\varphi\text{-Mod}_B$ and $\mathrm{Bun}_{X_{FF}}$ is from [18, §11.4]. And $M \mapsto M \otimes_{B^+} B^+_{\text{cris}}$ induces an equivalence of $\varphi\text{-Mod}_{B^+}$ and $\varphi\text{-Mod}_{B^+_{\text{cris}}}$ and with quasi-inverse functor given by $M \mapsto \cap_n \varphi^n(M)$. The equivalence of $\varphi\text{-Mod}_{B^+}$ and $\varphi\text{-Mod}_{\overline{B}}$ was mentioned in [18, §11].

For the last statement for the consistency of the functor from different φ-modules to $\mathrm{Bun}_{X_{FF}}$, first we have by [18, Theorem 11.1.9]. The functor

$$(M, \varphi) \mapsto \widetilde{\oplus_{n\geq 0} M^{\varphi=p^n}}$$

induces an equivalence of φ-Mod_{B^+} and $\mathrm{Bun}_{X_{FF}}$. We want to emphasize that for M^+, M, M_{cris}, and $M_{\overline{B}}$ are as in Proposition 3.1.5, we have

$$(M^+)^{\varphi=p^n} \simeq M^{\varphi=p^n} \simeq (M_{\mathrm{cris}})^{\varphi=p^n} \simeq M_{\overline{B}}^{\varphi=p^n}$$

for all n, so define same graded modules. And this is precisely what we have proved in Proposition 3.1.5. $\square$

Definition 3.1.7. With the notions in Proposition 3.1.3, and let $\lambda = \frac{d}{h}$. We define the category φ-Mod_R^λ of semisimple φ-modules of slope λ over R to be the subcategory of φ-Mod_R consisting M that is finite a direct sum of $R(\lambda)$.

From Theorem 3.1.6 and Corollary 2.1.7, we have the following lemma:

Lemma 3.1.8. *For $R \in \{\breve{K}, B, B^+, B^+_{\mathrm{cris}}, \overline{B}\}$, φ-Mod_R^λ are canonically equivalent to each other under the natural base changes. And they are all equivalent to the category of vector bundles semistable of slope $-\lambda$ over X_{FF}.*

In particular, let $M \in \varphi$-Mod_R^λ with $R \in \{B, B^+, B^+_{\mathrm{cris}}, \overline{B}\}$ and let $M_{\breve{K}}$ be its base change to $\breve{K}$, then there is a canonical φ-equivariant section $s : M_{\breve{K}} \to M$ that reduces to identity over $\breve{K}$.

The ring $\overline{B}$ plays an important role in our theory, so we give a brief discussion of it here.

Definition 3.1.9. Let $B^+_{(1,1]} = \cup_{\rho\in(0,1)} B^+_\rho$, and define $\mathfrak{p}_1 = \mathfrak{p}B^+_{(1,1]}$ and $\mathfrak{m}_1 = W(\mathfrak{m}_C)B^+_{(1,1]}$. It is easy to see from Example 3.1.1 that $B^+_{(1,1]}/\mathfrak{p}_1 = \overline{B}$ and $B^+_{(1,1]}/\mathfrak{m}_1 = \breve{K}$. This $B^+_{(1,1]}$ carries a Frobenius endomorphism induces from the Frobenius endomorphisms on B^+_ρ, so we can define φ-$\mathrm{Mod}_{B^+_{(1,1]}}$. We give $B^+_{(1,1]}$ the final topology induced by the inclusions $B^+_\rho \to B^+_{(1,1]}$.

Lemma 3.1.10. *The bases change functor along the natural map $B^+ \to B^+_{(1,1]}$ induces an equivalence of φ-Mod_{B^+} and φ-$\mathrm{Mod}_{B^+_{(1,1]}}$.*

Proof. The proof is similar to [28, Proposition 12.3.5]. Let $(M, \varphi_M) \in \varphi$-$\mathrm{Mod}_{B^+_{(1,1]}}$, then there is $\rho \in (0,1)$, such that (M, φ_M) is defined over B^+_ρ. Then we can use the fact $\varphi : B^+_\rho \xrightarrow{\sim} B^+_{\rho^p}$ to extend this φ-module over B^+_ρ to a φ-module over B^+ using φ_M. $\square$

Lemma 3.1.11. *We have* $(B^+_{(1,1]}, \mathfrak{p}_1)$ *is a henselian pair in the sense of [31, Tag 09XI].*

Proof. Here we are using Gabber's definition of henselian pair following [31, Tag 09XI], we will show:

$\mathfrak{p}_1$ is contained in the Jacobson radical of $B^+_{(1,1]}$ and every monic polynomial $f(T) \in B^+_{(1,1]}[T]$ of the form

$$f(T) = T^n(T-1) + a_nT^n + \ldots + a_1T + a_0$$

with $a_n, \ldots, a_0 \in \mathfrak{p}_1$ and $n \geq 1$ has a root in $1 + \mathfrak{p}_1$.

Let $x \in \mathfrak{p}_1$ and $b \in B^+_{(1,1]}$, then $xb = [\varpi]\frac{y}{p^n}$ for a pseudo uniformizer $\varpi \in \mathfrak{m}_C$ and $y \in \widehat{A_{\inf}[\frac{[a]}{p}]}$ from the discussion in Example 3.1.4. Then we have $\frac{y}{p^n}$ is inside $\widehat{A_{\inf}[\frac{[a]}{p^{n+1}}]} = \widehat{A_{\inf}[\frac{[\tilde{a}]}{p}]}$. Since $\widehat{A_{\inf}[\frac{[\tilde{a}]}{p}]}$ is $[\varpi]$-adically complete, so $1 + xb$ is a unit, in particular, x is in the Jacobson radical of $B^+_{(1,1]}$.

Similarly, for the monic polynomial

$$f(T) = T^n(T-1) + a_nT^n + \ldots + a_1T + a_0$$

we can find $\varpi_0, a_0 \in \mathfrak{m}_C$ such that $a_i \in [\varpi_0]\widehat{A_{\inf}[\frac{[a_0]}{p}]}$ for all i. Again, using the fact that $\widehat{A_{\inf}[\frac{[a_0]}{p}]}$ is $[\varpi_0]$-adically complete. We have the pair $(\widehat{A_{\inf}[\frac{[a_0]}{p}]}, [\varpi_0])$ is henselian by [31, Tag 0ALJ]. So $f(T)$ has a root in $1 + [\varpi_0]\widehat{A_{\inf}[\frac{[a_0]}{p}]} \subset 1 + \mathfrak{p}_1$. □

Corollary 3.1.12. *The reduction from* $B^+_{(1,1]}$ *to* $\overline{B}$ *induces an equivalence of* $\varphi\text{-}\mathrm{Mod}_{B^+_{(1,1]}}$ *and* $\varphi\text{-}\mathrm{Mod}_{\overline{B}}$.

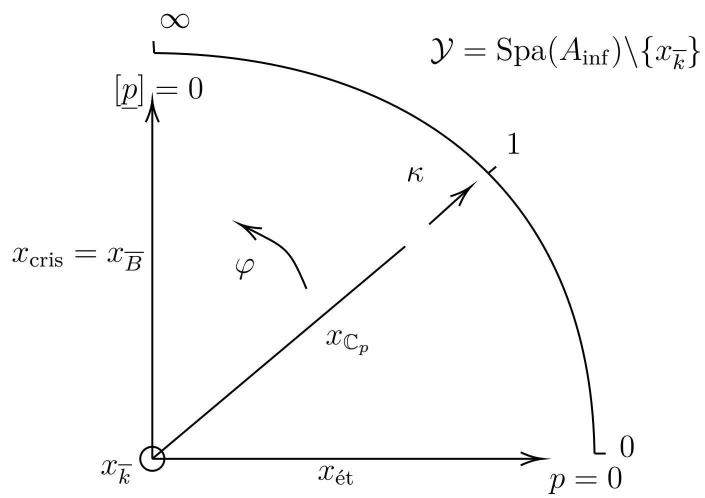
Proof. This is from of [2, Lemma 4.1.26]. First we want to point out the prove of Lemma 4.1.26 in *loc.cit.* does not use the prism structure in their statement. To apply their lemma, we just need to show φ is topologically nilpotent on $\mathfrak{p}_1$. For $x \in \mathfrak{p}_1$, $x = [\varpi]\frac{a}{p^n}$ for some $\varpi \in \mathfrak{m}_C$ and $a \in B^+_\rho$ satisfies $|a|_\rho \leq 1$, so

$$|\varphi^n([\varpi]\frac{a}{p^n})|_\rho = \frac{1}{p^n}|[\varpi]^{p^n}|_\rho|\varphi^n(a)|_\rho$$

converge to 0. □

Remark 3.1.13. The above argument gives an different proof of (1) in Proposition 3.1.3 by Lemma 3.1.10.

Remark 3.1.14. Let's also gives a geometric interpretation of the ring $\overline{B}$. Recall we have the following picture



Here $\mathcal{Y}$ is defined by the locus in $\mathrm{Spa}(A_{\inf})$ such that $p(x) \neq 0$ or $[\underline{p}](x) \neq 0$, there is a continuous map $\kappa : \mathcal{Y} \to [0, \infty]$ given by the relative position of x to $[\underline{p}]$ and p. And for $x \in \mathcal{Y}$, $\kappa(x) = \infty$ implies x factor through $A_{\inf} \to A_{\inf}[\frac{1}{p}] \to A_{\inf}[\frac{1}{p}]/\mathfrak{p} = \overline{B}$. Here $A_{\inf} \to A_{\inf}[\frac{1}{p}]$ comes from the condition $p(x) \neq 0$ (since we require $[\underline{p}](x) = 0$) and $\mathfrak{p}$ is the ideal subject to the condition $\{[\underline{p}](x) = 0\}$. In particular, x_{cris} in this picture should be considered not just a point but the space $\overline{B}$. The space $\overline{B}$ is actually very huge in the sense that it has infinite Krull dimension (cf. [24]). Moreover, one can show there is an uncountable chain of prime ideals inside $\overline{B}$ that is fixed by φ (cf. [15]).

We will see φ-modules over $\overline{B}$ plays an important role when studying the crystalline Galois representations.

3.2 Breuil-Kisin-Fargues modules and modifications of vector bundles

In this section, we will review the theory of Breuil-Kisin-Fargues modules and their relation with admissible modifications of vector bundles over X_{FF}. The main reference of this section is [6] and [1].

Definition 3.2.1.

(1) A Breuil-Kisin-Fargues module is a finite presented module $\mathfrak{M}^{\inf}$ over $A_{\inf}$ with an isomorphism
$$\varphi_{\mathfrak{M}^{\inf}} : \mathfrak{M}^{\inf} \otimes_{A_{\inf},\varphi} A_{\inf}[\frac{1}{\tilde{\xi}}] \simeq \mathfrak{M}^{\inf}[\frac{1}{\tilde{\xi}}]$$
such that $\mathfrak{M}^{\inf}[\frac{1}{p}]$ is a finite free $A_{\inf}[\frac{1}{p}]$-module. Recall here $\tilde{\xi} = \varphi(\xi)$ as we defined in § 2.1.

(2) A Breuil-Kisin-Fargues module is a finite free if the underlying module is finite free over $A_{\inf}$. In this paper, we define **BKF** to be the category of **finite free** Breuil-Kisin-Fargues modules. And we let $\mathbf{BKF}^{\circ}$ to be the isogeny category of **BKF**. We will also use BKF modules to refer objects in **BKF**.

(3) A Breuil-Kisin-Fargues module $\mathfrak{M}^{\inf}$ is called effective if $\varphi_{\mathfrak{M}^{\inf}}(\mathfrak{M}^{\inf} \otimes_{A_{\inf},\varphi} A_{\inf}) \subset \mathfrak{M}^{\inf}$.

Example 3.2.2. There are free rank 1 Breuil-Kisin-Fargues modules $A_{\inf}\{n\}$ defined for all $n \in \mathbb{Z}$(cf. [6, Example 4.24]). Just note in this paper, our $A_{\inf}\{1\}$ is the $A_{\inf}\{-1\}$ in [6, Example 4.24] due to our conventions on Hodge-Tate weights 1.6.6, by this convention we will have $A_{\inf}\{n\} \otimes_{A_{\inf}} \overline{B} = \overline{B}(n)$ defined in (2) Proposition 3.1.3. For any $\mathfrak{M}^{\inf} \in \mathbf{BKF}$, we define
$$\mathfrak{M}^{\inf}\{n\} = \mathfrak{M}^{\inf} \otimes_{A_{\inf}} A_{\inf}\{n\}.$$
It can be shown that $\mathfrak{M}^{\inf}\{n\}$ is effective for $n \gg 0$(cf. [14, §2.3.6.]).

Remark 3.2.3. In the proof of [1, Lemma 3.9], any Breuil-Kisin-Fargues module is isogeny to a finite free one. In particular, we have $\mathbf{BKF}^{\circ}$ is also the isogeny category of **all** Breuil-Kisin-Fargues modules.

Definition 3.2.4. Let $\mathbf{HT}_{\mathbb{Z}_p}$ be the category of pairs (T, Ξ), where T is a finite free $\mathbb{Z}_p$-lattice and Ξ is a B_{dR}^{+}-lattice in $T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$. And let **HT** be the isogeny category of $\mathbf{HT}_{\mathbb{Z}_p}$, i.e., **HT** consists of pairs (V, Ξ) where V is a finite $\mathbb{Q}_p$-vector space and Ξ is a B_{dR}^{+}-lattice in $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$.

Definition 3.2.5. Given $\mathfrak{M}^{\text{inf}} \in \mathbf{BKF}$, we define:

(1) $T(\mathfrak{M}^{\text{inf}}) = (\mathfrak{M}^{\text{inf}} \otimes_{A_{\text{inf}}} W(C))^{\varphi=1}$;

(2) $V(\mathfrak{M}^{\text{inf}}) = (\mathfrak{M}^{\text{inf}} \otimes_{A_{\text{inf}}} W(C))^{\varphi=1}[\frac{1}{p}]$;

(3) $\mathfrak{M}^{\text{inf}}_{\mathbb{C}_p} = \mathfrak{M}^{\text{inf}} \otimes_{A_{\text{inf}},\theta} \mathbb{C}_p$;

(4) $\mathfrak{M}^{\text{inf}}_{\text{cris}} = \mathfrak{M}^{\text{inf}} \otimes_{A_{\text{inf}}} B^+_{\text{cris}}$;

(5) $\mathfrak{M}^{\text{inf}}_{\breve{K}} = \mathfrak{M}^{\text{inf}} \otimes_{A_{\text{inf}}} W(\overline{k})[\frac{1}{p}]$.

Remark 3.2.6. It is easy to see that $V(\mathfrak{M}^{\text{inf}})$, $\mathfrak{M}^{\text{inf}}_{\mathbb{C}_p}$, $\mathfrak{M}^{\text{inf}}_{\text{cris}}$ and $\mathfrak{M}^{\text{inf}}_{\breve{K}}$ are well-defined on the isogeny classes of BKF modules.

Lemma 3.2.7. *For $\mathfrak{M}^{\text{inf}} \in \mathbf{BKF}$, we have $(T(\mathfrak{M}^{\text{inf}}), \mathfrak{M}^{\text{inf}} \otimes_{A_{\text{inf}}} B^+_{\text{dR}}) \in \mathbf{HT}_{\mathbb{Z}_p}$.*

Proof. By [6, Lemma 4.26], $T(\mathfrak{M}^{\text{inf}})$ is a $\mathbb{Z}_p$-lattice of the same rank equal to $\mathfrak{M}^{\text{inf}}$, and $T(\mathfrak{M}^{\text{inf}}) \otimes_{\mathbb{Z}_p} B_{\text{dR}} = \mathfrak{M}^{\text{inf}} \otimes B_{\text{dR}}$, so $(T(\mathfrak{M}^{\text{inf}}), \mathfrak{M}^{\text{inf}} \otimes_{A_{\text{inf}}} B^+_{\text{dR}}) \in \mathbf{HT}$. □

Lemma 3.2.8. *For $\mathfrak{M}^{\text{inf}} \in \mathbf{BKF}$, we have $\mathfrak{M}^{\text{inf}}_{\text{cris}} \in \varphi\text{-}\mathrm{Mod}_{B^+_{\text{cris}}}$*

Proof. It is enough to show $\tilde{\xi}$ is a unit in B^+_{cris}. There are many way to see this, for readers familiar with the language of δ-rings [8], we give a short proof as follow, we have $A_{\text{cris}} = A_{\text{inf}}\{\frac{\tilde{\xi}}{p}\}^{\wedge}_{\delta}$, and both $\tilde{\xi}$ and p are distinguished, so $\frac{\tilde{\xi}}{p}$ is a unit in A_{cris}. □

Theorem 3.2.9. (Fargues' classification theorem, cf. [6, Theorem 4.28]) The functor

$$\mathfrak{M}^{\text{inf}} \mapsto (T(\mathfrak{M}^{\text{inf}}), \mathfrak{M}^{\text{inf}} \otimes_{A_{\text{inf}}} B^+_{\text{dR}})$$

induces an equivalence of $\mathbf{BKF}$ and $\mathbf{HT}_{\mathbb{Z}_p}$.

Corollary 3.2.10. *There is an equivalence of $\mathscr{M}\mathrm{odif}_{X_{FF}}$ and $\mathbf{BKF}^{\circ}$.*

Proof. By Lemma 2.1.9 and Theorem 3.2.9, both category are equivalent to $\mathbf{HT}$. □

Remark 3.2.11. One can regard a modification $\mathcal{E}_0 \dashrightarrow \mathcal{E}_1$ as a modification of $\mathcal{E}_0$ by a B^+_{dR}-lattice at a formal neighborhood at ∞, and this gives the equivalence in Lemma 2.1.9. On the other hand, one can also view it as a modification of $\mathcal{E}_1$ by a lattice in $\mathcal{E}_{1,\infty} \otimes_{B^+_{\text{dR}}} B_{\text{dR}}$. And this gives the following result.

Proposition 3.2.12. The functor

$$\mathfrak{M}^{\mathrm{inf}} \to (\mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris}}, V(\mathfrak{M}^{\mathrm{inf}}) \otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}}).$$

defines a fully faithfully functor from $\mathbf{BKF}^{\circ}$ to $\{(M_{B^+_{\mathrm{cris}}}, \Xi)\}$, where the latter is the category consisting of pairs $(M_{B^+_{\mathrm{cris}}}, \Xi)$, with $M_{B^+_{\mathrm{cris}}} \in \varphi\text{-}\mathrm{Mod}_{B^+_{\mathrm{cris}}}$ and Ξ is a B^+_{dR} lattice in $M \otimes_{B^+_{\mathrm{cris}}} B_{\mathrm{dR}}$.

Proof. If $\mathfrak{M}^{\mathrm{inf}}$ corresponds to $\mathcal{E}_0 \dashrightarrow \mathcal{E}_1$, then if one carefully tracks the functor defined by Scholze in [28, Theorem 14.1.1], one has $\mathcal{E}_1$ corresponds to the φ-module $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris}}$. And we have $\mathcal{E}_0$ is a semistable slope 0 vector bundle with global section equals to $V(\mathfrak{M}^{\mathrm{inf}})$, in particular, the B^+_{dR}-lattice defined by $\mathcal{E}_{0,\infty}$ is $V(\mathfrak{M}^{\mathrm{inf}}) \otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}}$. By the B-pair description of vector bundles over X_{FF} we have the functor is fully faithful. □

We also want to discuss about sections to the natural reduction of $\mathfrak{M}^{\mathrm{inf}}$ to $\mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \breve{K}$.

Lemma 3.2.13. *For $\mathfrak{M}^{\mathrm{inf}} \in \mathbf{BKF}$, let $M_{\mathrm{cris}} = \mathfrak{M}^{\mathrm{inf}} \otimes B^+_{\mathrm{cris}}$, and $M_{\overline{B}} = \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \overline{B}$. For any perfect subfield $l \subset \overline{k}$ and let $L_0 = W(l)[\frac{1}{p}]$. For any φ-stable L_0-vector space $V \subset M_{\overline{B}}$ such that $V \otimes_{L_0} \overline{B} = M_{\overline{B}}$, there is an unique φ-equivariant section $s : V \to M_{B^+_{\mathrm{cris}}}$ such that modulo $\mathfrak{p}B^+_{\mathrm{cris}}$ we get identity on V.*

Proof. The proof is combination of [10, Lemma 4.5.6] and [17, Proposition 4.26]. Since we will use the formula of the section s, let us give an explicit construction of the section following the idea in *loc.cit.*.

First, we can always reduce to the case that $\mathfrak{M}^{\mathrm{inf}}$ is effective, since we can always twist $\mathfrak{M}^{\mathrm{inf}}$ by $A_{\mathrm{inf}}\{n\}$ and V by $L_0(n)$ simultaneously by some $n \gg 0$ to ensure that $\mathfrak{M}^{\mathrm{inf}}$ and V being effective by Example 3.2.2. So we can assume there is a φ-stable $\mathcal{O}_{L_0}$-lattice Λ inside V. Since we have $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}} = (\Lambda \otimes_{\mathcal{O}_{L_0}} \overline{B}) \otimes_{\overline{B}} \breve{K}$, we can view Λ as a sub-φ-module of $\mathfrak{M}^{\mathrm{inf}}[\frac{1}{p}] \otimes_{A_{\mathrm{inf}}[\frac{1}{p}]} \overline{B}$.

Let $\{\bar{\mathrm{e}}_{\mathrm{i}}\}_{\mathrm{i}=1}^{d}$ be a basis of Λ and choose an arbitrary lifting $\{\mathrm{e}_1, \ldots, \mathrm{e}_d\}$ as a basis of $\mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[\frac{1}{p}]$, we can always replace $\{\bar{\mathrm{e}}_{\mathrm{i}}\}_{\mathrm{i}=1}^{d}$ by $\{p^k \bar{\mathrm{e}}_{\mathrm{i}}\}_{\mathrm{i}=1}^{d}$ and $\{\mathrm{e}_1, \ldots, \mathrm{e}_d\}$ by $\{p^k \mathrm{e}_1, \ldots, p^k \mathrm{e}_d\}$ to ensure $\{\mathrm{e}_1, \ldots, \mathrm{e}_d\}$ is inside $\mathfrak{M}^{\mathrm{inf}}$. We define $A_0 \in M_d(O_{L_0}) \subset \overline{B}$ be the matrix defined by

$$\varphi(\bar{\mathrm{e}}_1, \ldots, \bar{\mathrm{e}}_d) = (\bar{\mathrm{e}}_1, \ldots, \bar{\mathrm{e}}_d) A_0.$$

let $A \in M_d(A_{\text{inf}})$ be the matrix defined by

$$\varphi(\mathrm{e}_1, \ldots, \mathrm{e}_d) = (\mathrm{e}_1, \ldots, \mathrm{e}_d)A.$$

Then to find s it is enough to find $Y \in M_d(B^+_{\text{cris}})$ such that $Y \equiv I \mod \mathfrak{p}$ and

$$YA_0 = A\varphi(Y),$$

and the section is defined by

$$s(\bar{\mathrm{e}}_1, \ldots, \bar{\mathrm{e}}_d) = (\mathrm{e}_1, \ldots, \mathrm{e}_d)Y.$$

For $n \in \mathbb{N}_{>0}$, define

$$Y_n = A\varphi(A)\cdots\varphi^{n-1}(A)\varphi^{n-1}(A_0^{-1})\cdots\varphi(A_0^{-1})A_0^{-1}.$$

It is enough to show Y_n converges to $Y \in M_d(B^+_{\text{cris}})$. Since we have $A \equiv A_0 \mod \mathfrak{p}$ and there is $a \in \mathbb{N}$ such that $A_0^{-1} \in \frac{1}{p^a}M_d(O_{L_0})$, we have there is a $B_0 \in M_d(\mathcal{O}_{L_0})$ such that $AB_0 = p^aI + [\varpi]Z$, where ϖ is a pseudo uniformizer in C and $Z \in M_d(A_{\text{inf}})$. We will have

$$Y_n - Y_{n-1} = A\varphi(A)\cdots\varphi^{n-2}(A)\frac{[\varpi]^{p^{(n-1)}}}{p^{an-a}}\varphi^{n-1}(Z)\varphi^{n-2}(B_0)\cdots\varphi(B_0)B_0.$$

It is easy to show that $\frac{[\varpi]^{p^{(n-1)}}}{p^{an-a}}$ converge to 0 in B^+_{cris} (actually in B^+). So we have Y_n converges to some Y. The uniqueness is also the same as in [10, Lemma 4.5.6] and we skip it here. $\square$

Apply the above lemma to the case $l = \bar{k}$, i.e., $L_0 = \breve{K}$, we have the following Corollary on the section s define in (3) of Proposition 3.1.3.

Corollary 3.2.14. *For $\mathfrak{M}^{\text{inf}} \in$ **BKF**, let $M_{\text{cris}} = \mathfrak{M}^{\text{inf}} \otimes B^+_{\text{cris}}$, $M_{\overline{B}} = \mathfrak{M}^{\text{inf}} \otimes_{A_{\text{inf}}} \overline{B}$, and $\mathfrak{M}^{\text{inf}}_{\breve{K}} = \mathfrak{M}^{\text{inf}} \otimes_{A_{\text{inf}}} \breve{K}$. For any section*

$$\bar{s} : \mathfrak{M}^{\text{inf}}_{\breve{K}} \to M_{\overline{B}}$$

reducing to the identity over $\breve{K}$. There is a unique section

$$s : M_{\breve{K}} \to M_{\mathrm{cris}}$$

such that reducing $\mathfrak{p}B^+_{\mathrm{cris}}$ is identify to $\overline{s}$.

Proof. Just apply Lemma 3.2.13 to $\overline{s}(M_{\breve{K}})$. $\square$

There is a definition related to the above results.

Definition 3.2.15. (Rigidifications c.f. [1]) For any $\mathfrak{M}^{\mathrm{inf}} \in \mathbf{BKF}$ define $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris}} = \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{cris}}$, $\mathfrak{M}^{\mathrm{inf}}_{\overline{B}} = \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \overline{B}$, and $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}} = \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \breve{K}$. A rigidification of $\mathfrak{M}^{\mathrm{inf}}$ over B^+_{cris} (resp. $\overline{B}$) is a φ-equivariant section

$$s : \mathfrak{M}^{\mathrm{inf}}_{\breve{K}} \to M_{\mathrm{cris}} \quad (\text{resp. } \ \overline{s} : \mathfrak{M}^{\mathrm{inf}}_{\breve{K}} \to M_{\overline{B}})$$

reducing to the identity over $\breve{K}$.

By Corollary 3.2.14, we have.

Corollary 3.2.16. *For any $\mathfrak{M}^{\mathrm{inf}} \in \mathbf{BKF}$, a rigidification of $\mathfrak{M}^{\mathrm{inf}}$ over $\overline{B}$ is the same as a rigidification over B^+_{cris}.*

4. FARGUES-FONTAINE-SCHOLZE FUNCTOR AND ARITHMETIC BREUIL-KISIN-FARGUES MODULES

Definition 4.0.1. Let $\mathbf{BKF}(G_K)$ be the category of finite free Breuil-Kisin-Fargues G_K-modules consists of Breuil-Kisin-Fargues module equipped with a continuous semilinear G_K-action that commutes with $\varphi_{\mathfrak{M}^{\inf}}$. And let $\mathbf{BKF}(G_K)^{\circ}$ be the corresponded isogeny category.

One easily deduce the following "G_K-version" of the Fargues' classification theorem.

Proposition 4.0.2. There is an equivalence of $\mathscr{M}od\mathrm{i}f_X(G_K)$ and $\mathbf{BKF}(G_K)^{\circ}$.

Definition 4.0.3. We define the Fargues-Fontaine-Scholze functor

$$\eta_{FFS} : \mathbf{MF}_K^{wa}(\varphi, N, G_K) \to \mathbf{BKF}(G_K)^{\circ}$$

to be the composition of the Fargues-Fontaine functor η_{FF} with the functor in Proposition 4.0.2.

In this chapter, we will show the essential image of η_{FFS} is characterized by what we call arithmatic Breuil-Kisin-Fargues G_K-modules; moreover, we will also characterize the essential images corresponds to the subcategory $\mathbf{MF}_K^{wa}(\varphi, N)$ (resp. $\mathbf{MF}_{K,\varphi}^{wa}$).

4.1 Arithmetic Breuil-Kisin-Fargues G_K modules

Definition 4.1.1. Let $\mathfrak{M}^{\inf} \in \mathbf{BKF}(G_K)$, $\mathfrak{M}^{\inf}$ is called arithmetic if and only if $\mathfrak{M}^{\inf}_{\mathbb{C}_p}$ as a $\mathbb{C}_p$-representation of G_K is $\mathbb{C}_p$-admissible, i.e., it is Hodge-Tate with only 0 weight. We let $\mathbf{BKF}^a(G_K)$ to be the category of arithmetic Breuil-Kisin-Fargues modules, and $\mathbf{BKF}^a(G_K)^{\circ}$ be the corresponded isogeny category.

Remark 4.1.2. The condition of being arithmetic is well defined on the isogeny class of $\mathfrak{M}^{\inf}$.

Theorem 4.1.3. The Fargues-Fontaine-Scholze functor is fully faithful. The essential image of η_{FFS} consists of arithmetic Breuil-Kisin-Fargues modules.

Proof. The fully faithfulness comes from the fully faithfullness of η_{FF} and the equivalence in Proposition 4.0.2. By Theorem 2.2.21, we know the essential image of η_{FF} consist of Hodge-Tate modules (V, Ξ) such that V is a de Rham representation and $\Xi = D_{\mathrm{dR}}(V) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+$. So it is enough to show that if $\mathfrak{M}^{\mathrm{inf}}$ is an arithmetic Breuil-Kisin-Fargues G_K-modules, then $V = V(\mathfrak{M}^{\mathrm{inf}})$ is de Rham and

$$\mathfrak{M}^{\mathrm{inf}} \otimes B_{\mathrm{dR}}^+ \xrightarrow{\sim} D_{\mathrm{dR}}(V) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+.$$

First, we have $\mathfrak{M}^{\mathrm{inf}}_{\mathbb{C}_p}$ being $\mathbb{C}_p$-admissible, is equivalent to $\mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^+$ is B_{dR}^+-flat by Proposition 2.2.12. In particular, $\mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^+$ is generically flat, that is $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}}$ is B_{dR}-flat, in other word, V is de Rham. By (2) in Proposition 2.2.12, $\mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^+$ corresponds to $D_{\mathrm{dR}}(V) = (V \otimes B_{\mathrm{dR}})^{G_K}$ with the trivial filtration, so we have

$$\mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^+ \simeq D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}^+.$$

□

Remark 4.1.4.

(1) In [16], the definition of BKF modules is slightly different from our definition. They use ξ instead of $\tilde{\xi}$ in the definition. So the our definition is differed by a Frobenius twist. Moreover, they require the $\varphi^{\mathrm{inf}}_{\mathfrak{M}}$ is an endomorphism on $\mathfrak{M}^{\mathrm{inf}}$ this will result in the Hodge-Tate weights of $T(\mathfrak{M}^{\mathrm{inf}})$ being non-negative.

(2) In [16, Propostion F.13], they actually show that T is B_{dR}^+-admissible in the sense of (2) in Remark 2.2.13, which requires the non-negativity of Hodge-Tate weights of $T(\mathfrak{M}^{\mathrm{inf}})$.

Remark 4.1.5. As we have mentioned in Remark 2.2.13, for a de Rham representation T, a natural way to define a G_K-stable B_{dR}^+-lattice in $D_{\mathrm{dR}}(T) \otimes_K B_{\mathrm{dR}}$ is given by

$$\Xi = \mathrm{Fil}^0(D_{\mathrm{dR}}(T) \otimes_K B_{\mathrm{dR}}).$$

Here if we give $D_{\mathrm{dR}}(T)$ the Hodge filtration, we get a lattice Ξ_0 and if we give $D_{\mathrm{dR}}(T)$ the trivial filtration, we let Ξ_1 the the resulting lattice. One can check the pair (T, Ξ_0) corresponds to the trivial BKF G_K-module $T \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$ and (T, Ξ_1) corresponds to the arithmetic Breuil-Kisin-Fargues modules in Theorem 4.1.3.

4.2 Conditions for log-crystalline and crystalline representations.

Let $\mathbf{MF}_K^{wa}(\varphi, N)$ (resp. $\mathbf{MF}_{K,\varphi}^{wa}$) be the subcategory of $\mathbf{MF}_K^{wa}(\varphi, N, G_K)$ consists of weakly admissible filtered (φ, N)-modules (resp. weakly admissible filtered φ-modules). In this section, we will characterize of the essential image of η_{FFS} of $\mathbf{MF}_K^{wa}(\varphi, N)$ and $\mathbf{MF}_{K,\varphi}^{wa}$.

Theorem 4.2.1.

(1) An arithmetic Breuil-Kisin-Fargues G_K-module $\mathfrak{M}^{\mathrm{inf}}$ is in the essential image of $\mathbf{MF}_K^{wa}(\varphi, N)$ if and only if there is a G_K-fixed basis inside $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$.

(1') An arithmetic Breuil-Kisin-Fargues G_K-module $\mathfrak{M}^{\mathrm{inf}}$ is in the essential image of $\mathbf{MF}_K^{wa}(\varphi, N)$ if and only if the initial group I_K acts trivially on $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}}[\frac{1}{p}]$.

(2) A Breuil-Kisin-Fargues G_K-module $\mathfrak{M}^{\mathrm{inf}}$ is in the essential image of $\mathbf{MF}_{K,\varphi}^{wa}$ if and only if $(\mathfrak{M}^{\mathrm{inf}} \otimes \overline{B})^{G_K}$ as a K_0-vector space has dimension equal to the rank of $\mathfrak{M}^{\mathrm{inf}}$.

(2') A Breuil-Kisin-Fargues G_K-module $\mathfrak{M}^{\mathrm{inf}}$ is in the essential image of $\mathbf{MF}_{K,\varphi}^{wa}$ if and only if $(\mathfrak{M}^{\mathrm{inf}} \otimes \overline{B})^{I_K}$ as a $\breve{K}$-vector space has dimension equal to the rank of $\mathfrak{M}^{\mathrm{inf}}$.

Remark 4.2.2. In Theorem 4.2.1, we have (1) and $(1')$ (resp. (2) and $(2')$) are equivalent due to unramified descent(cf. proof in Lemma 1.6.2). Or this is from the well-known fact that a p-adic representation of G_K is crystalline (resp. log-crystalline) if and only if when restricted on I_K it is crystalline (resp. log-crystalline).

For (2) in Theorem 4.2.1, we need the following lemma.

Lemma 4.2.3. *Taking the G_K-invariant of the exact sequence*

$$0 \to \mathfrak{p} \to A_{\mathrm{inf}}[\frac{1}{p}] \to \overline{B} \to 0,$$

We will have $K_0 \hookrightarrow \overline{B}^{G_K}$. Let $\overline{B}_{\mathrm{fin}}^{G_K}$ be the sub-algebra consists of $v \in \overline{B}^{G_K}$ such that $\{\varphi^n(v)\}_{n\geq 0}$ generates a finite dimensional K_0 vector space. We have $\overline{B}_{\mathrm{fin}}^{G_K} = K_0$.

Proof. We have $A_{\mathrm{inf}}[\frac{1}{p}]^{G_K} = K_0$, i.e., $A_{\mathrm{inf}}[\frac{1}{p}]^{G_K} = W(\overline{k})[\frac{1}{p}]^{G_K}$. By the Teichmüller expansion of elements in $\mathfrak{p}$, we will have

$$\mathfrak{p}^{G_K} = \mathfrak{p} \cap W(\overline{k})[\frac{1}{p}]^{G_K} = 0.$$

So we have $K_0 \hookrightarrow \overline{B}^{G_K}$.

Now $\overline{B}_{\mathrm{fin}}^{G_K}$ decomposes into isocrystals over k, in particular, it decomposes into subspace $(\overline{B}^{\varphi^h = p^d})^{G_K}$ for pairs (h, d) with $d \in \mathbb{Z}$ and $h \in \mathbb{N}_{>0}$. Then by Lemma 3.1.4, we have

$$(\overline{B}^{\varphi^h = p^d}) = (B_{\mathrm{cris}}^+)^{\varphi^h = p^d},$$

which is also G_K-equivariant so the result follows from $(B_{\mathrm{cris}}^+)^{G_K} = K_0$. $\square$

Remark 4.2.4. We don't know whether $\overline{B}^{G_K} = K_0$ at this time, but as in the statement of (2) in Theorem 4.2.1, $(\mathfrak{M}^{\mathrm{inf}} \otimes \overline{B})^{G_K}$ can be always viewed as a module over $\overline{B}_{\mathrm{fin}}^{G_K} = K_0$.

Proof. (of (1') Theorem 4.2.1) If (the isogeny class of) an arithemtic BKF G_K-modules $\mathfrak{M}^{\mathrm{inf}}$ corresponds to $D \in \mathbf{MF}_K^{wa}(\varphi, N, G_K)$ that also corresponds to a G_K-equivariant modification $\mathcal{E}_0 \dashrightarrow \mathcal{E}_1$, then the recall that as in §2.2, we show the G_K-action on $\mathcal{E}_1$ is given by $\alpha \wedge \mathcal{E}(D, \varphi, G_K))$. And recall α corresponds to the cocycle

$$(\exp(-\log_{\underline{\varpi}, g}))_g. \tag{4.2.4.1}$$

On the other hand, we know by Proposition 3.2.12, $\mathcal{E}_1$ corresponds to $\mathfrak{M}_{\mathrm{cris}}^{\mathrm{inf}}$. So there is a G_K-equivariant isomorphism

$$D = (D \otimes B_{\mathrm{cris}}^+) \otimes_{B_{\mathrm{cris}}^+} \breve{K} = \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \breve{K}$$

and $D \in \mathbf{MF}_K^{wa}(\varphi, N)$ if and only if I_K acts trivially on D. So by equation 4.2.4.1, it is enough to show

$$\exp(-\log_{\underline{\varpi},g} N)) \equiv 1 \mod W(\mathfrak{m}^\flat)B_{\mathrm{cris}}^+.$$

Recall we have $\log_{\underline{\varpi},g}$ is a multiple of t, so the result follows from $t \in W(\mathfrak{m}^\flat)B_{\mathrm{cris}}^+$, which is directly from the definition of t. $\square$

Remark 4.2.5. If we assume an arithmetic BKF G_K-module $\mathfrak{M}^{\mathrm{inf}}$ satisfies the condition (2) in Theorem 4.2.1, then a similar argument will show that $\mathfrak{M}^{\mathrm{inf}}$ corresponds to an object in $\mathbf{MF}_{K,\varphi}^{wa}$. However, we will prove something more substantial, i.e., condition (2) in Theorem 4.2.1 implies $\mathfrak{M}^{\mathrm{inf}}$ being arithmetic!

Lemma 4.2.6. *For $\mathfrak{M}^{\mathrm{inf}} \in \mathbf{BKF}(G_K)$, let $M_{\mathrm{cris}} = \mathfrak{M}^{\mathrm{inf}} \otimes B_{\mathrm{cris}}^+$, and $M_{\overline{B}} = \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \overline{B}$. Let $L_0 = W(l)[\frac{1}{p}]$ with $k \subset l \subset \overline{k}$. For any φ-stable L_0-vector space $V \subset M_{\overline{B}}$ such that $V \otimes_{L_0} \overline{B} = M_{\overline{B}}$. Assume V is also G_K-stable, then the φ-equivariant section $s : V \to M_{\mathrm{cris}}$ defined in Lemma 3.2.13 is G_K-equivariant.*

Proof. This is a slight generalization of [27, Lemma 3.15]. We use the same reduction as in Lemma 3.2.13 and keep the notions. Fix $g \in G_K$ and let $D \in M_d(A_{\mathrm{inf}})$ and $D_0 \in M_d(\mathcal{O}_{L_0})$ be the matrix defined by

$$g(\bar{\mathrm{e}}_1, \ldots, \bar{\mathrm{e}}_d) = (\bar{\mathrm{e}}_1, \ldots, \bar{\mathrm{e}}_d)D_0$$

and

$$g(\mathrm{e}_1, \ldots, \mathrm{e}_d) = (\mathrm{e}_1, \ldots, \mathrm{e}_d)D.$$

We have $D - D_0 = [\varpi]W$ with pseudo uniformizer $\varpi \in C$ and $W \in M_d(A_{\mathrm{inf}})$. It is enough to check

$$Dg(Y) = YD_0$$

Since the G_K-action commutes with φ, we will have

$$Dg(A) = A\varphi(D) \quad \text{and} \quad D_0 g(A_0) = A_0\varphi(D_0).$$

For $n \in \mathbb{N}_{>0}$, we have

$$\begin{aligned}
Dg(Y_n) &= Dg(A\varphi(A)\cdots\varphi^{n-1}(A)\varphi^{n-1}(A_0^{-1})\cdots\varphi(A_0^{-1})A_0^{-1}) \\
&= Dg(A)g(\varphi(A))\cdots g(\varphi^{n-1}(A))g(\varphi^{n-1}(A_0^{-1}))\cdots g(\varphi(A_0^{-1}))g(A_0^{-1}) \\
&= A\varphi(A)\cdots\varphi^{n-1}(A)\varphi^{n}(D)g(\varphi^{n-1}(A_0^{-1}))\cdots g(\varphi(A_0^{-1}))g(A_0^{-1}) \\
&= A\varphi(A)\cdots\varphi^{n-1}(A)\varphi^{n}(D_0+[\varpi]W)g(\varphi^{n-1}(A_0^{-1}))\cdots g(\varphi(A_0^{-1}))g(A_0^{-1}) \\
&= A\varphi(A)\cdots\varphi^{n-1}(A)g(\varphi^{n-1}(A_0^{-1}))\cdots g(\varphi(A_0^{-1}))g(A_0^{-1})D_0 \\
&\quad + \frac{[\varpi]^{p^n}}{p^{na}}A\varphi(A)\cdots\varphi^{n-1}(A)\varphi^{n}(W)g(\varphi^{n-1}(B_0))\cdots g(\varphi(B_0))g(B_0).
\end{aligned}$$

And the result follows from that $\frac{[\varpi]^{p^n}}{p^{na}}$ converges to 0 in B^+. $\square$

Proof. (of (2) in Theorem 4.2.1) By Lemma 4.2.6, we have the G_K-invariant section inside $(\mathfrak{M}^{\mathrm{inf}} \otimes \overline{B})^{G_K}$ lifts to a G_K-invariant section inside $\mathfrak{M}^{\mathrm{inf}} \otimes B^+_{\mathrm{cris}}$, so (3) implies there is a G_K-fixed basis inside $\mathfrak{M}^{\mathrm{inf}} \otimes B^+_{\mathrm{cris}}$. Base change along $\theta : B^+_{\mathrm{cris}} \to \mathbb{C}_p$, we have $\mathfrak{M}^{\mathrm{inf}} \otimes \mathbb{C}_p$ has a G_K-invariant basis, i.e., $\mathfrak{M}^{\mathrm{inf}}$ is arithmetic.

Now as in Remark 4.2.5, the rest is similar to the proof of (1). We give another way of proving by showing the B_{e}-part of $\mathcal{E}_1$ in the G_K-equivariant modification $\mathcal{E}_0 \dashrightarrow \mathcal{E}_1$ corresponds to $\mathfrak{M}^{\mathrm{inf}}$ is crystalline in the sense that it is equal to $\mathscr{V}_{\log}(D)$ for some φ-module D under the functor $\mathscr{V}_{\log}$ we defined in Proposition 2.2.14, see also Definition 10.2.13 in [18] for the notion of B_{e}-representations being crystalline. We have the B_{e}-representation corresponds to $\mathcal{E}_1$ is

$$M_{\mathrm{e}} = (\mathfrak{M}^{\mathrm{inf}} \otimes B_{\mathrm{cris}})^{\varphi=1}.$$

And by Proposition 10.2.12, *loc.cit.*, it is crystalline if and only if

$$\dim_{K_0}(M_{\mathrm{e}} \otimes_{B_{\mathrm{e}}} B_{\mathrm{cris}})^{G_K} = \mathrm{rank}_{A_{\mathrm{inf}}}\mathfrak{M}^{\mathrm{inf}}.$$

But we have $D = (\mathfrak{M}^{\mathrm{inf}} \otimes \overline{B})^{G_K} \subset (M_{\mathrm{e}} \otimes_{B_{\mathrm{e}}} B_{\mathrm{cris}})^{G_K}$, so M_{e} is crystalline.

On the other hand, from the construction of the Fargues-Fontaine functor, when $N = 0$, we have a G_K-equivariant isomorphism $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris}} \simeq D \otimes B^+_{\mathrm{cris}}$ and $D \otimes B^+_{\mathrm{cris}}$ is equipped with the diagonal action. When $\mathfrak{M}^{\mathrm{inf}}$ corresponds to a crystalline representation, $D = D_{\mathrm{cris}}$ has a K_0-basis fixed by G_K. $\square$

Remark 4.2.7. A key idea of Berger's B_{e}-representation theory is that for an $\mathcal{O}_X$-representation $\mathcal{E}$ of G_K, its B_{e} part will determine the (φ, N)-module structure. This translates into to our theory says that for arithmetic Breuil-Kisin-Fargues modules $\mathfrak{M}^{\mathrm{inf}}$, G_K-action on $\mathfrak{M}^{\mathrm{inf}} \otimes B_{\mathrm{cris}}^{+}$ determines the (φ, N)-module structure, in particular, it tells p-adic Hodge property of $T(\mathfrak{M}^{\mathrm{inf}})$ being log-crystalline or crystalline.

Remark 4.2.8. Since $T(\mathfrak{M}^{\mathrm{inf}})$ being log-crystalline or crystalline is just determined by $V(\mathfrak{M}^{\mathrm{inf}})$, it is easy to deduce the following "integral version" of Theorem 4.2.1.

Theorem 4.2.9. Let $\mathrm{Rep}_{\mathbb{Z}_p}^{dR}(G_K)$ (resp. $\mathrm{Rep}_{\mathbb{Z}_p}^{lcr}(G_K)$, resp. $\mathrm{Rep}_{\mathbb{Z}_p}^{cris}(G_K)$) be the category of de Rham (resp. log-crystalline, resp. crystalline) representations of G_K over a $\mathbb{Z}_p$ lattice, then

(1) There is a equivalence of $\mathrm{Rep}_{\mathbb{Z}_p}^{dR}(G_K)$ with the category of arithmetic BKF G_K-modules.

(2) The essential image of $\mathrm{Rep}_{\mathbb{Z}_p}^{lcr}(G_K)$ of the functor in (1) are the arithmetic BKF modules such that there is a G_K-fixed basis in $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$.

(2') The essential image of $\mathrm{Rep}_{\mathbb{Z}_p}^{lcr}(G_K)$ of the functor in (1) are the arithmetic BKF modules such that the I_K-action on $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$ is trivial.

(3) The essential image of $\mathrm{Rep}_{\mathbb{Z}_p}^{cris}(G_K)$ of the functor in (1) are BKF G_K-modules such that $(\mathfrak{M}^{\mathrm{inf}} \otimes \overline{B})^{G_K}$ as a K_0-vector space has dimension equal to the rank of $\mathfrak{M}^{\mathrm{inf}}$.

(3') The essential image of $\mathrm{Rep}_{\mathbb{Z}_p}^{cris}(G_K)$ of the functor in (1) are BKF G_K-modules such that $(\mathfrak{M}^{\mathrm{inf}} \otimes \overline{B})^{I_K}$ as a $\breve{K}$-vector space has dimension equal to the rank of $\mathfrak{M}^{\mathrm{inf}}$.

5. THE INVERSE OF FARGUES-FONTAINE-SCHOLZE FUNCTOR

When we prove Theorem 2.2.21, we need to use the p-adic monodromy theorem for p-adic Galois representations, i.e., we use the fact that for any de Rham representation of G_K, it corresponds to an object in $\mathbf{MF}_K^{wa}(\varphi, N, G_K)$. We have the following diagram:

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{dR}}(G_K) \arrow[r, "\sim"] \arrow[d, dashed, bend left, "\text{monodromy thm}"] & \mathbf{HT}_{\mathbb{Z}_p}^{dR}(G_K) \arrow[r, "\sim"] & \mathbf{BKF}^a(G_K) \arrow[dll, dashed] \\
\mathrm{Rep}_{\mathbb{Z}_p}^{\text{p-lcr}}(G_K) \arrow[u, hook] & &
\end{tikzcd}$$

Here $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{dR}}(G_K)$ (resp. $\mathrm{Rep}_{\mathbb{Z}_p}^{\text{p-lcr}}(G_K)$) is the category of de Rham representations (resp. potential log-crystalline representations) of G_K over a $\mathbb{Z}_p$-lattice, we let $\mathbf{HT}_{\mathbb{Z}_p}^{dR}(G_K)$ be the category of pairs (T, Ξ) such that $T \in \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{dR}}(G_K)$, and Ξ is a G_K-stable B_{dR}^+-lattice in $T \otimes B_{\mathrm{dR}}$ which is also B_{dR}^+-flat. The equivalence of $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{dR}}(G_K)$ and $\mathbf{HT}_{\mathbb{Z}_p}^{dR}(G_K)$ is given by

$$T \mapsto (T, D_{\mathrm{dR}}(T) \otimes_K B_{\mathrm{dR}}^+).$$

And the equivalence of $\mathbf{HT}_{\mathbb{Z}_p}^{dR}(G_K)$ and $\mathbf{BKF}^a(G_K)$ is by Theorem 4.2.9. In this chapter, we will construct the functor

$$\mathbf{BKF}^a(G_K) \to \mathbf{MF}_K^{wa}(\varphi, N, G_K)$$

which will induce a functor on isogeny classes

$$\omega_{FFS} : \mathbf{BKF}^a(G_K)^\circ \to \mathbf{MF}_K^{wa}(\varphi, N, G_K)$$

which will be shown to be the inverse of the Fargues-Fontaine-Scholze functor η_{FFS}.

5.1 p-adic monodromy theorem for arithmetic BKF G_K-modules

For $\mathfrak{M}^{\mathrm{inf}} \in \mathbf{BKF}^a(G_K)$, let $D = D(\mathfrak{M}^{\mathrm{inf}}) = \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \breve{K}$ with the induced φ-structure. As we have discussed in Lemma 1.6.2, unramified descent enables us to assume $K_0 = \breve{K}$. In this section, we will endow D a filtered (φ, N, G_K)-module structure and prove it is weakly admissible.

5.1.1 Finiteness of the G_K-action

We first show the I_K-action on D has an open kernel. We will first prove it in the semisimple case:

Corollary 5.1.2. *(to Lemma 3.1.8) let $(M, \varphi_M) \in \varphi\text{-}\mathrm{Mod}_R^\lambda$ with $R \in \{B, B^+, B^+_{\mathrm{cris}}, \overline{B}\}$ and let $M_{\breve{K}}$ be its base change to $\breve{K}$, assume M equipped with a semilinear action of G_K commutes with φ_M, then the a canonical φ-equivariant section $s : M_{\breve{K}} \to M$ defined in Lemma 3.1.8 is also G_K-equivariant.*

Lemma 5.1.3. *Let $(M, \varphi_M) \in \varphi\text{-}\mathrm{Mod}_R^\lambda$ with $R \in \{B, B^+, B^+_{\mathrm{cris}}\}$ and let $M_{\breve{K}}$ be its base change to $\breve{K}$, assume M equipped with a semilinear action of G_K commutes with φ_M, also assume that $M_{\mathbb{C}_p} := M \otimes_{R,\theta} \mathbb{C}_p$ as a representation of G_K is $\mathbb{C}_p$ admissible. Then the G_K-action on $M_{\breve{K}}$ is potentially unramified, i.e., the I_K-action on $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$ has an open kernel.*

Proof. Let $D = \mathrm{Hom}_{\breve{K}}(\breve{K}(\lambda), M_{\breve{K}})$, we will have D is a φ-module over $\breve{K}$ with the φ-structure given by $\varphi(f) = \varphi \circ f \circ \varphi^{-1}$. Define $W = H^0(D) = \mathrm{Hom}_\varphi(\breve{K}(\lambda), M_{\breve{K}})$. Under the equivalence in Corollary 2.1.7 and use the fact that $D \simeq M_{\breve{K}} \otimes_{\breve{K}} \breve{K}(-\lambda)$, we have D corresponds to a slope 0 vector bundle

$$\mathcal{D} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}(-\lambda), \mathcal{E}(M)) \simeq \mathcal{E}(M) \otimes_{\mathcal{O}_X} \mathcal{O}(\lambda),$$

here we use $\mathcal{H}om$ to denote the the sheaf Hom and $\mathcal{E}(M)$ is the vector bundle corresponds to M. By (2) of Corollary 2.1.7, the global section of $\mathcal{D}$ is W and $\mathcal{D}$ is canonically isomorphic to $W \otimes_{\mathbb{Q}_p} \mathcal{O}_X$. Under this isomorphism, we will have

$$W \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \mathcal{D} \otimes_{\mathcal{O}_X} \mathbb{C}_p \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}(-\lambda), \mathcal{E}(M))_\infty \otimes \mathbb{C}_p \simeq \mathrm{Hom}_{\mathbb{C}_p}(\mathcal{O}(-\lambda)_{\mathbb{C}_p}, \mathcal{E}(M)_{\mathbb{C}_p})$$

here $\mathcal{O}(-\lambda)_{\mathbb{C}_p}$ and $\mathcal{E}(M)_{\mathbb{C}_p}$ are their fibers over ∞. So by Corollary 5.1.2, we have there is a I_K-equivariant isomorphism

$$W \otimes_{\mathbb{Q}_p} \mathbb{C}_p \simeq \mathrm{Hom}_{\mathbb{C}_p}(R(\lambda) \otimes_R \mathbb{C}_p, M_{\mathbb{C}_p})$$

where $R(\lambda) \otimes_R \mathbb{C}_p$ has the trivial I_K-action. Then W as I_K-representation is $\mathbb{C}_p$-admissible by the assumption that $M_{\mathbb{C}_p}$ is $\mathbb{C}_p$-admissible. Using a Theorem of Sen [30], one has I_K acts via a finite quotient on W.

On the other hand, $M_{\breve{K}}$ is a finite direct sum of $\breve{K}(\lambda)$, and there is an equivalence of the category of $\varphi\text{-Mod}_{\breve{K}}^{\lambda}$ and $\mathrm{Vect}_{D_\lambda}$, where $\mathrm{Vect}_{D_\lambda}$ is the category of finite free modules over $D_\lambda := \mathrm{End}_\varphi(\breve{K}(\lambda), \breve{K}(\lambda))$. And the functors are given by $E \mapsto \mathrm{Hom}_\varphi(\breve{K}(\lambda), E)$ and $V \mapsto V \otimes_{D_\lambda} \breve{K}(\lambda)$. In particular, $M_{\breve{K}}$ is canonically isomorphic to $W \otimes_{D_\lambda} \breve{K}(\lambda)$, so the I_K-action on $M_{\breve{K}}$ also has an open kernel. $\square$

Now let $(M, \varphi_M) \in \varphi\text{-Mod}_R$ with $R \in \{B, B^+, B^+_{\mathrm{cris}}, \overline{B}\}$, assume M equipped with a semilinear action of G_K commutes with φ_M. First, we have a decomposition of φ-module:

$$M \simeq \oplus_{\mathrm{i}=1}^{n} M(\lambda_{\mathrm{i}})$$

for semisimple submodules $M(\lambda_{\mathrm{i}})$. We assume $\lambda_{\mathrm{i}+1} > \lambda_{\mathrm{i}}$ for all i, and choose a bases $\{\mathrm{e}_{\mathrm{i}}\}_{\mathrm{i}=1}^{d}$ of M such that

(1) e_{i} is in $s(M_{\breve{K}}(\lambda_a))$ for some λ_a appear in the slope of $M_{\breve{K}}$, and s is given in Corollary 4.2.6.

(2) If $\mathrm{i} > \mathrm{j}$ and $\mathrm{e}_{\mathrm{i}} \in M(\lambda_a)$, $\mathrm{e}_{\mathrm{j}} \in M(\lambda_b)$, then $\lambda_a \geq \lambda_b$.

For $g \in G_K$, and let $A_g \in M_d(R)$ be the matrix defined by

$$g(\mathrm{e}_1, \ldots, \mathrm{e}_d) = (\mathrm{e}_1, \ldots, \mathrm{e}_d) A_g.$$

Lemma 5.1.4. *Under the assumptions of the basis* $\{\mathrm{e_i}\}_{\mathrm{i}=1}^d$ *as above, we will have* A_g *is an upper triangle block matrix of the form*

$$A_g = \begin{pmatrix} * & H^0(R(\lambda_2-\lambda_1)) & H^0(R(\lambda_3-\lambda_1)) & \dots & H^0(R(\lambda_n-\lambda_1)) \\ 0 & * & H^0(R(\lambda_3-\lambda_2)) & \dots & H^0(R(\lambda_n-\lambda_2)) \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the blocks are define by the decomposition $M \simeq \oplus_{\mathrm{i}}^n M(\lambda_{\mathrm{i}})$*. In particular, if we define a flag*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

by $M_m = \oplus_{\mathrm{i}=1}^m M(\lambda_{\mathrm{i}})$ *for* $m \in [1,n]$*, and* $M_0 = 0$*. Then the* G_K*-action stabilize this flag.*

Proof. For (i,j) with $\mathrm{i},\mathrm{j} \in [1,n]$, then the block at the entry (i,j) defines a φ-equivariant map from $M(\lambda_{\mathrm{i}})$ to $M(\lambda_{\mathrm{j}})$. So we have its must have entries in $\mathrm{Hom}_\varphi(M(\lambda_{\mathrm{i}}), M(\lambda_{\mathrm{j}}))$. Using the fact $\mathrm{Hom}_\varphi(R(\lambda), R(\mu))$ is some copies of $H^0(R(\mu-\lambda))$, then the result follows from $H^0(R(\lambda)) = R^{\varphi^h = p^d} \neq 0$ if and only if $\lambda \geq 0$. $\square$

Now let $\mathfrak{M}^{\mathrm{inf}}$ be an arithmetic Breuil-Kisin-Fargues G_K-modules, and let $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris}} = \mathfrak{M}^{\mathrm{inf}} \otimes B^+_{\mathrm{cris}}$. Then we have by the above lemma, we have the $\{\mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris,i}}\}$ inside $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris}} = \mathfrak{M}^{\mathrm{inf}} \otimes B^+_{\mathrm{cris}}$ is stabilized by G_K and satisfies $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris,i+1}}/\mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris,i}}$ is semisimple for all i. Also we have that Hodge-Tate representations with weight 0 are stable under taking subquotients. So we can prove by induction using Lemma 5.1.3 and the fact $H^0(\breve{K}(\lambda)) = 0$ when $\lambda \neq 0$, that we have:

Proposition 5.1.5. Let $\mathfrak{M}^{\mathrm{inf}}$ be an arithmetic Breuil-Kisin-Fargues G_K-modules, and let $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}} = \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \breve{K}$, then the I_K-action on $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$ has an open kernel.

5.1.6 Construction of the monodromy operator

In § 5.1.6, we assume $K_0 = \breve{K}$. From § 5.1.1, we know that if $\mathfrak{M}^{\mathrm{inf}}$ is an arithmetic Breuil-Kisin-Fargues G_K-modules, then there is a finite extension L of K such that for any basis

$\vec{\mathrm{e}} = \{\mathrm{e}_1, \ldots, \mathrm{e}_d\}$ of $\mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris}}$ comes from a basis of $s(\mathfrak{M}^{\mathrm{inf}}_{K_0})$ satisfies the condition in Lemma 5.1.4, G_L acts on $\vec{\mathrm{e}}$ via an upper triangle block matrix $A_g = A_{g,\vec{\mathrm{e}}}$ of the shape:

$$\begin{pmatrix} I & H^0(B^+_{\mathrm{cris}}(\lambda_2 - \lambda_1)) & H^0(B^+_{\mathrm{cris}}(\lambda_3 - \lambda_1)) & \ldots & H^0(B^+_{\mathrm{cris}}(\lambda_n - \lambda_1)) \\ 0 & I & H^0(B^+_{\mathrm{cris}}(\lambda_3 - \lambda_2)) & \ldots & H^0(B^+_{\mathrm{cris}}(\lambda_n - \lambda_2)) \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Following mainly the idea in [12] and also [18, Proposition 10.6.17], we make the following definition.

Definition 5.1.7. For $(h, d) \in \mathbb{N}_{>0} \times \mathbb{N}$ with $(h, d) = 1$. Define

$$u : (B^+_{\mathrm{cris}})^{\varphi^h = p^d} \to \mathbb{C}_p^h$$

by $u \mapsto (\theta(u), \theta(\varphi(u)), \ldots, \theta(\varphi^{h-1}(u)))$ and denote

$$H^1_g(G_K, (B^+_{\mathrm{cris}})^{\varphi^h = p^d}) = \ker(H^1(G_K, (B^+_{\mathrm{cris}})^{\varphi^h = p^d}) \xrightarrow{u_*} H^1(G_K, \mathbb{C}_p^h)).$$

And recall the following proposition of Colmez.

Proposition 5.1.8. ([12, Proposition 10.11] in the form of [18, Proposition 10.6.17]) When $K = \breve{K}$, for $(h, d) \in \mathbb{N}_{>0} \times \mathbb{N}$ with $(h, d) = 1$,

$$H^1_g(G_K, (B^+_{\mathrm{cris}})^{\varphi^h = p^d}) \neq 0$$

if and only if $h = d = 1$, and

$$H^1_g(G_K, (B^+_{\mathrm{cris}})^{\varphi = p}) = \mathbb{Q}_p \cdot \log_{\underline{\varpi}, g}$$

where the 1-cocycle $\log_{\underline{\varpi}, g}$ is defined in § 2.1 by $\log_{\underline{\varpi}, g} = g(\log[\underline{\varpi}]) - \log[\underline{\varpi}]$.

Definition 5.1.9. Fix a basis $\vec{\mathrm{e}}_0 = \{\mathrm{e_i}\}$ of $s(\mathfrak{M}_K^{\inf})$ as in Lemma 5.1.4, for any unipotent matrix Y of the shape

$$\begin{pmatrix} I & H^0(B^+_{\mathrm{cris}}(\lambda_2-\lambda_1)) & H^0(B^+_{\mathrm{cris}}(\lambda_3-\lambda_1)) & \dots & H^0(B^+_{\mathrm{cris}}(\lambda_n-\lambda_1)) \\ 0 & I & H^0(B^+_{\mathrm{cris}}(\lambda_3-\lambda_2)) & \dots & H^0(B^+_{\mathrm{cris}}(\lambda_n-\lambda_2)) \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

we define $\vec{\mathrm{e}}_Y = \{\mathrm{e_i}\cdot Y\}$. In other words, all basis of the flag $\{M_{\mathrm{i}}\}$ define in § 5.1.1 such that it coincides with $\vec{\mathrm{e}}_0$ on all semisimple quotient are of this form. We call basis of the form $\vec{\mathrm{e}}_Y$ an admissible base change of $\vec{\mathrm{e}}_0$.

If $\vec{\mathrm{e}}$ is an admissible base change of $\vec{\mathrm{e}}_0$ defined by Y, then let $A_{g,\vec{\mathrm{e}}}$ be the matrix defined by

$$g\vec{\mathrm{e}} = \vec{\mathrm{e}}A_{g,\vec{\mathrm{e}}},$$

therefore, $A_{g,\vec{\mathrm{e}}} = Y^{-1}A_{g,\vec{\mathrm{e}}_0}g(Y)$, in particular, use the fact if $a \in H^0(B^+_{\mathrm{cris}}(\lambda))$ and $b \in H^0(B^+_{\mathrm{cris}}(\mu))$, then $a\cdot b \in H^0(B^+_{\mathrm{cris}}(\lambda+\mu))$, we can check $A_{g,\vec{\mathrm{e}}}$ is unipotent of the shape

$$\begin{pmatrix} I & H^0(B^+_{\mathrm{cris}}(\lambda_2-\lambda_1)) & H^0(B^+_{\mathrm{cris}}(\lambda_3-\lambda_1)) & \dots & H^0(B^+_{\mathrm{cris}}(\lambda_n-\lambda_1)) \\ 0 & I & H^0(B^+_{\mathrm{cris}}(\lambda_3-\lambda_2)) & \dots & H^0(B^+_{\mathrm{cris}}(\lambda_n-\lambda_2)) \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with the same blocking as $A_{g,\vec{\mathrm{e}}_0}$. With these notions, we have the following construction of monodromy operator:

Proposition 5.1.10. Let $\mathfrak{M}^{\inf}$ be an arithmetic Breuil-Kisin-Fargues G_K-modules, such that G_K acts trivially on $\mathfrak{M}_K^{\inf}$, fix a basis $\vec{\mathrm{e}}_0$ of $s(\mathfrak{M}_K^{\inf})$ as in Lemma 5.1.4. For any admissible base change $\vec{\mathrm{e}}$ of $\vec{\mathrm{e}}_0$, use the fact that G_K acts on $\vec{\mathrm{e}}$ via a unipotent matrix $A_{g,\vec{\mathrm{e}}}$, $B_{g,\vec{\mathrm{e}}} = \log A_{g,\vec{\mathrm{e}}}$ is well-defined and nilpotent. Then there is an admissible base change $\vec{\mathrm{e}}$ of $\vec{\mathrm{e}}_0$ such that

$$N = \frac{B_{g,\vec{\mathrm{e}}}}{-\log_{\varpi,g}}$$

is independent of the choice of $g \in G_K$ and defines a nilpotent matrix in $M_d(\mathbb{Q}_p)$.

Proof. With the notation as in Lemma 5.1.4, use the fact if $a \in H^0(B^+_{\mathrm{cris}}(\lambda))$ and $b \in H^0(B^+_{\mathrm{cris}}(\mu))$, then $a \cdot b \in H^0(B^+_{\mathrm{cris}}(\lambda+\mu))$, we will have $B_{g,\vec{\mathrm{e}}}$ is inside:

$$\begin{pmatrix} 0 & H^0(B^+_{\mathrm{cris}}(\lambda_2-\lambda_1)) & H^0(B^+_{\mathrm{cris}}(\lambda_3-\lambda_1)) & \dots & H^0(B^+_{\mathrm{cris}}(\lambda_n-\lambda_1)) \\ 0 & 0 & H^0(B^+_{\mathrm{cris}}(\lambda_3-\lambda_2)) & \dots & H^0(B^+_{\mathrm{cris}}(\lambda_n-\lambda_2)) \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

And all matrix of such shape forms an abelian group P (in addition) which is isomorphic to copies of $H^0(B^+_{\mathrm{cris}}(\lambda_{\mathrm{i}}-\lambda_{\mathrm{j}}))$. Now since we have for $h, g \in G_K$, we have

$$A_{gh,\vec{\mathrm{e}}} = A_{g,\vec{\mathrm{e}}}\, g(A_{h,\vec{\mathrm{e}}}).$$

Taking logarithm, we have

$$B_{gh,\vec{\mathrm{e}}} = B_{g,\vec{\mathrm{e}}} + g(B_{h,\vec{\mathrm{e}}}).$$

In particular, we have $g \mapsto B_{g,\vec{\mathrm{e}}}$ defines a cocycle with value in P. And if we have an admissible base change $\vec{\mathrm{e}}_Y$ of $\vec{\mathrm{e}}$ by a unipotent matrix Y, let $X = \log Y$, we have $B_{g,\vec{\mathrm{e}}_Y} - B_{g,\vec{\mathrm{e}}} = -X + g(X)$. On contrast, if two 1-cocycles are cohomologous, i.e., $B_{g,\vec{\mathrm{e_1}}} - B_{g,\vec{\mathrm{e_2}}} = -X + g(X)$ by some element X in P, in particular, X is nilpotent, and $\exp(X)$ defines an admissible base change.

Similarly, let h be the least common multiple of the denominators of $\{\lambda_{\mathrm{i}} - \lambda_{\mathrm{j}}\}_{\mathrm{i}>\mathrm{j}}$. And we have a map

$$u = (\theta, \theta \circ \varphi, \dots, \theta \circ \varphi^{h-1}) : P \to P^h_{\mathbb{C}_p}.$$

where $P_{\mathbb{C}_p}$ is the additive group defined by the block matrix

$$\begin{pmatrix} 0 & \mathbb{C}_p & \mathbb{C}_p & \dots & \mathbb{C}_p \\ 0 & 0 & \mathbb{C}_p & \dots & \mathbb{C}_p \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

and the blocking is the same as in the definition of P. We can repeat all discussions after the base change under u, where we should get 1-cocycles from the Galois action. However, since $\mathfrak{M}^{\mathrm{inf}}$ is arithmetic, so we have $B_{g,\vec{\mathrm{e}}}$ is trivial in $H^1(G_K, P_{\mathbb{C}_p})$. Assume $B_{g,\vec{\mathrm{e}}_Y}$ is 0 under θ for the admissible base change $\vec{\mathrm{e}}_Y$ of $\vec{\mathrm{e}}$. We have for all $r \in \mathbb{Z}$, we have $\varphi^r_{\mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris}}}$ induces a G_K-equivariant isomorphism:

$$\mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris}} \otimes_\theta \mathbb{C}_p \simeq \mathfrak{M}^{\mathrm{inf}}_{\mathrm{cris}} \otimes_{\theta\circ\varphi^r} \mathbb{C}_p.$$

In particular, $B_{g,\vec{\mathrm{e}}_Y}$ is also trivial under $\theta \circ \varphi^r$ for all $r \in \mathbb{Z}$, i.e., trivial in $H^1(G_K, P^h_{\mathbb{C}_p})$.

Finally, we have P decompose into $(B^+_{\mathrm{cris}})^{\frac{d}{h}}$ for different (d,h) defined by $\{\lambda_{\mathrm{i}} - \lambda_{\mathrm{j}}\}_{\mathrm{i}>\mathrm{j}}$. Proposition 5.1.8 implies there is an admissible bases $\vec{\mathrm{e}}_Y$ such that

$$N = \frac{B_{g,\vec{\mathrm{e}}_Y}}{-\log_{\underline{\varpi},g}}$$

is independent of the choice of $g \in G_K$ and defines a nilpotent matrix in $M_d(\mathbb{Q}_p)$. □

Remark 5.1.11. We have when $\lambda_{\mathrm{i}} - \lambda_{\mathrm{j}} = 1$, we have the block of N at the (i,j)-entry defines an element in $\mathrm{Hom}(M_{\breve{K}}(\lambda_{\mathrm{j}}), M_{\breve{K}}(\lambda_{\mathrm{i}}))^{\varphi=p^{-1}}$, on the other hand, $\mathrm{Hom}(M_{\breve{K}}(\lambda_{\mathrm{j}}), M_{\breve{K}}(\lambda_{\mathrm{i}}))$ decomposes into direct sum of $(\breve{K}, p^{-1}\varphi_{\breve{K}})$, so we have we can always make entry of N inside $\mathbb{Q}_p$.

Remark 5.1.12. Compare to § 2.2, For the $\mathcal{O}_X$-representation $\mathcal{E}(D,\varphi,N)$ defined by a filtered (φ,N)-modules D, it corresponds to the φ-module $D \otimes B^+_{\mathrm{cris}}$ with the G_K-action given by

$$g(y) = \exp(\log_{\underline{\varpi},g} N)y.$$

So what we have done is just reverse engineering of on the construction of G_K-action defined in § 2.2.

Remark 5.1.13. In Proposition 5.1.8 is key step in [12] that Colmez developed to prove the p-adic monodromy theorem. Also in [18], Proposition 5.1.8 is also used to compare extensions classes of $\mathcal{O}_X$-representations that are trivial at ∞ and (φ,N)-modules. Our proof is motivated by their idea, while we could give a more explicit construct of the (φ,N)-modules. We will apply this computation in a further work.

5.1.14 The filtration structure and weakly admissibility

Let $\mathfrak{M}^{\text{inf}}$ be an arithmetic BKF G_K-module, and define $\mathfrak{M}^{\text{inf}}_{dR} = \mathfrak{M}^{\text{inf}} \otimes_{A_{\text{inf}}} B^+_{\text{dR}}$. Then as in Proposition 2.2.10, the arithmetic condition is equivalent to $\mathfrak{M}^{\text{inf}}_{dR}$ as a B^+_{dR}-representation of G_K is generically flat and

$$\mathfrak{M}^{\text{inf}}_{dR} = (\mathfrak{M}^{\text{inf}}_{dR}[\frac{1}{t}])^{G_K} \otimes_K B^+_{\text{dR}}.$$

Recall we have the G_K-equivariant section

$$s : D = \mathfrak{M}^{\text{inf}}_{\breve{K}} \to \mathfrak{M}^{\text{inf}}_{\text{cris}}$$

So we have $D_K := (s(D) \otimes_{\breve{K}} B_{\text{dR}})^{G_K} = (\mathfrak{M}^{\text{inf}}_{dR}[\frac{1}{t}])^{G_K}$. By Proposition 2.2.10, to define a filtration on D_K is the same as giving a G_K-stable B^+_{dR}-lattice in $\mathfrak{M}^{\text{inf}}_{dR}[\frac{1}{t}]$. And we just choose the B^+_{dR}-lattice $T(\mathfrak{M}^{\text{inf}}) \otimes_{\mathbb{Z}_p} B^+_{\text{dR}}$.

To check the filtered (φ, N, G_K)-module D is weakly admissible, first we can restrict to the open subgroup I_L of I_K that acts trivially on D, i.e., we can ignore the G_K-action on D. Let $\mathcal{E}_0(D, \varphi, N, \text{Fil}^\bullet) \dashrightarrow \mathcal{E}_1(D, \varphi, N)$ be the I_L-equivariant modification of vector bundles over X_{FF} defined by D. It is enough to check $\mathcal{E}_0(D, \varphi, N, \text{Fil}^\bullet)$ is of slope 0.

First as in Remark 5.1.12, $\mathcal{E}_1(D, \varphi, N)$ together with the I_L-action is exactly corresponds to the φ-module $\mathfrak{M}^{\text{inf}}_{\text{cris}}$ with the twisted G_K-action. To check $\mathcal{E}_0(D, \varphi, N, \text{Fil}^\bullet) \dashrightarrow \mathcal{E}_1(D, \varphi, N)$ corresponds to the isogeny class of $\mathfrak{M}^{\text{inf}}$, by Proposition 3.2.12. it is enough to check

$$\mathfrak{M}^{\text{inf}} \otimes B^+_{\text{dR}} \simeq T(\mathfrak{M}^{\text{inf}}) \otimes B^+_{\text{dR}} \simeq \text{Fil}^0(D_K \otimes B_{\text{dR}}) \simeq \mathcal{E}_{0,\infty}.$$

But this just follows from the definition of $\mathcal{E}_0(D, \varphi, N, \text{Fil}^\bullet)$ in § 2.2.

By Proposition 2.2.14 and Remark 4.2.7, the I_L-action on $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$ is trivial implies $T(\mathfrak{M}^{\mathrm{inf}})$ as a representation of I_L is log-crystalline, i.e., $T(\mathfrak{M}^{\mathrm{inf}})$ is potentially log-crystalline as a representation of G_K. So we have completed the diagram:

$$\begin{tikzcd}
\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Z}_p}(G_K) \arrow[r, "\sim"] & \mathbf{HT}^{\mathrm{dR}}_{\mathbb{Z}_p}(G_K) \arrow[r, "\sim"] & \mathbf{BKF}^a(G_K) \arrow[dll, "\mathfrak{M}^{\mathrm{inf}} \mapsto T(\mathfrak{M}^{\mathrm{inf}})"] \\
\mathrm{Rep}^{\mathrm{p\text{-}lcr}}_{\mathbb{Z}_p}(G_K) \arrow[u, hook] & &
\end{tikzcd}$$

Taking the isogeny categories in the above diagram, we have:

Theorem 5.1.15. There is a functor

$$\omega_{FFS} : \mathbf{BKF}^a(G_K)^\circ \to \mathbf{MF}^{wa}_K(\varphi, N, G_K)$$

given by $\mathfrak{M}^{\mathrm{inf}} \to \mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$ and equips $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$ with the filtered (φ, N, G_K)-modules structure constructed in § 5. The existence of ω_{FFS} implies the p-adic monodromy theorem for p-adic Galois representations, i.e., all de Rham representations are potentially log-crystalline. And we have ω_{FFS} is a quasi-inverse of η_{FFS}.

Proof. The only thing left to prove is $\eta_{FFS} \circ \omega_{FFS} \simeq \mathrm{Id}$. And it is by the following lemma. □

Lemma 5.1.16. *(rigidity of arithmetic Breuil-Kisin-Fargues G_K-modules) For any two arithmetic Breuil-Kisin-Fargues G_K-modules $\mathfrak{M}^{\mathrm{inf}}_1$ and $\mathfrak{M}^{\mathrm{inf}}_2$, if $T(\mathfrak{M}^{\mathrm{inf}}_1) \simeq T(\mathfrak{M}^{\mathrm{inf}}_2)$, then $\mathfrak{M}^{\mathrm{inf}}_1 \simeq \mathfrak{M}^{\mathrm{inf}}_2$.*

Proof. By Fargues classification theorem (cf. Theorem 3.2.9), it is enough to show there is a G_K-equivariant isomorphism

$$\mathfrak{M}^{\mathrm{inf}}_1 \otimes B^+_{\mathrm{dR}} \simeq \mathfrak{M}^{\mathrm{inf}}_2 \otimes B^+_{\mathrm{dR}}.$$

But we have known by definition, both $\mathfrak{M}^{\mathrm{inf}}_1 \otimes B^+_{\mathrm{dR}}$ and $\mathfrak{M}^{\mathrm{inf}}_2 \otimes B^+_{\mathrm{dR}}$ are B^+_{dR}-flat representations inside $T(\mathfrak{M}^{\mathrm{inf}}_1) \otimes B_{\mathrm{dR}}$, so both of them are isomorphic to $D_{\mathrm{dR}}(T(\mathfrak{M}^{\mathrm{inf}})) \otimes_K B^+_{\mathrm{dR}}$. □

5.1.17 Relation with Scholze's 'question mark functor'

In [28], Scholze wanted to have an explicit description of the ? functor in the following diagram:

$$\begin{tikzcd}
\{\, p\text{-divisible groups over } \mathcal{O}_{\mathbb{C}_p}\} \arrow[r, "\sim"] \arrow[d] & \{\, (T,W)\} \arrow[d, "?"] \\
\{\, p\text{-divisible groups over } \overline{k}\} \arrow[r, "\sim"] & \{\, \text{Dieudonné modules}\}
\end{tikzcd}$$

Here $\{(T,W)\}$ is the category consists of pairs (T,W), where T is a finite free $\mathbb{Z}_p$-lattice and W is a sub-$\mathbb{C}_p$-space of $T\otimes_{\mathbb{Z}_p}\mathbb{C}_p$. And

$$\{\, p\text{-divisible groups over } \mathcal{O}_{\mathbb{C}_p}\} \to \{(T,W)\}$$

is given by $G \mapsto (T_pG, \mathrm{Lie}G \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p(1))$, which is an equivalence by [29]. And the answer to this question is that using the Fargues' classification theorem of BKF modules, cf. Theorem 3.2.9, one can extend the diagram with

$$\begin{tikzcd}
\{\, (T,W)\} \arrow[r, "\sim"] \arrow[d, "?"] & \{\, \text{minuscule BKF modules}\} \arrow[dl, "\mathfrak{M}^{\mathrm{inf}}\mapsto\mathfrak{M}^{\mathrm{inf}}\otimes_{A_{\mathrm{inf}}}W(\overline{k})"] \\
\{\, \text{Dieudonné modules}\} &
\end{tikzcd}$$

and "?" functor becomes a base-change functor of modules. We have a similar diagram for the p-adic monodromy theorem:

$$\begin{tikzcd}
\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K) \arrow[r, "\sim"] \arrow[d, dashed, "p\text{-adic monodromy theorem}"'] & \mathbf{HT}_{dR}(G_K) \\
\mathbf{MF}^{wa}_K(\varphi, N, G_K) &
\end{tikzcd}$$

Here $\mathbf{HT}_{dR}(G_K)$ is the category consists of pairs (V, Ξ), where V is p-adic de Rham representation of G_K and $\Xi = D_{dR}(T) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+$ is a G_K-stable B_{dR}^+-lattice of $T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ that is flat as a B_{dR}^+-representation. And the equivalence of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and $\mathbf{HT}_{dR}(G_K)$ is given by

$$V \mapsto (V, D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}^+).$$

So if we try to play the same trick of Scholze, one can use Fargues' classification theorem and extends the diagram by adding the category of *arithmetic* Breuil-Kisin-Fargues modules:

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K) \arrow[r, "\sim"] \arrow[d, dashed] & \mathbf{HT}^{\mathrm{dR}}(G_K) \arrow[r, "\sim"] & \mathbf{BKF}^a(G_K)^\circ \arrow[dll, dashed, "\omega_{FFS}"] \\
\mathbf{MF}_K^{wa}(\varphi, N, G_K) & &
\end{tikzcd}$$

And we have shown in this chapter that ω_{FFS} is a base change functor $\mathfrak{M}^{\mathrm{inf}} \to \mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$, then upgrade $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$ with the "K-rational structure" of $\mathfrak{M}^{\mathrm{inf}}$, that is we give $\mathfrak{M}^{\mathrm{inf}}_{\breve{K}}$ a structure of weakly admissible filtered (φ, N, G_K)-modules structure.

Remark 5.1.18. For a p-divisible group G over $\mathcal{O}_K$, and let G_k be its special fiber, the "?" functor relates to

$$G \mapsto D(G_k)[\frac{1}{p}].$$

where $D(G_k)$ is the (covariant) Dieudonné module of G_k, and is closely related to Grothendieck-Messing period morphism defined by Rapoport-Zink. We know p-divisible groups G over $\mathcal{O}_K$ are equivalent to crystalline G_K-representations over $\mathbb{Z}_p$ lattice with Hodge-Tate weights in $[0, 1]$, so a more reasonable "higher weights" generalization of this is to consider only crystalline or potentially crystalline representations, i.e., we should use the diagram:

$$\begin{tikzcd}
\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K) \arrow[r, "\sim"] \arrow[d, dashed] & \mathbf{BKF}^{\mathrm{cris}}(G_K)^\circ \arrow[dl, dashed, "\omega_{FFS}"] \\
\mathbf{MF}_{K,\varphi}^{wa} &
\end{tikzcd}$$

where $\mathbf{BKF}^{\mathrm{cris}}(G_K)^{\circ}$ is the isogeny category of Breuil-Kisin-Fargues G_K-modules satisfies condition (2) in Theorem 4.2.1. In terms of modification of vector bundles, $\mathbf{BKF}^{\mathrm{cris}}(G_K)^{\circ}$ corresponds to a G_K-equivariant modification $\mathcal{E}_0 \dashrightarrow \mathcal{E}_1$ plus an trivialization of $\mathcal{E}_1$. We recall that a trivialization is the same as a rigidification of the corresponded φ-module defined in 3.2.15, and we showed that it is the same as a rigidification over $\overline{B}$.

In [28], Scholze defines a moduli space $\mathrm{Sht}^{\mathrm{int}}_{GL_n,b,\mu}$ which is defined over $\breve{K}$ for a finite extension K of $\mathbb{Q}_p$, we will not recall the full definition but just mention that the $C = \mathbb{C}_p^{\flat}$ point of $\mathrm{Sht}_{GL_n,b,\mu}$ parametrizes Breuil-Kisin-Fargues modules with conditions on the kernel and cokernel of $\varphi^{\mathrm{inf}}_{\mathfrak{M}}$ defined by a cocharacter μ of GL_n and a rigidification by an isocrystal over $\mathbb{F}_p$ defined by b. We wonder if there is any applications of our result to study the rational points under the Grothendieck-Messing period morphism of Scholze define in §23.3 of *loc.cit*.

6. RELATIONS WITH WACH AND BREUIL-KISIN THEORY

In this section we will compare our theory with some other existing theories in integral p-adic Hodge theory and prove certain compatibility results using the rigidity of arithmetic Breuil-Kisin-Fargues modules (cf. Lemma 5.1.16).

6.1 Wach modules and crystalline representations

Throughout this section, we assume that $K = K_0$ is unramified for simplicity. We will discuss general K in Remark 6.1.5. We will construct a functor from Wach modules over K to $\mathbf{BKF}^a(G_K)$. First recall the definition of Wach modules after Berger. Let $\{\epsilon_{\mathrm{i}}\}_{\mathrm{i}\geq 0}$ be a compatible system of p^n-th roots of 1, i.e., $\epsilon_0 = 1$ and $\epsilon_{\mathrm{i}+1}^p = \epsilon_{\mathrm{i}}$ for all i. $\{\epsilon_{\mathrm{i}}\}_{\mathrm{i}\geq 0}$ defines an element $\underline{\epsilon} \in \mathcal{O}_C$, let $\mu = [\underline{\epsilon}] - 1 \in A_{\mathrm{inf}}$. Let $\mathfrak{S}_\mu = W(k)[[\mu]]$ as a subring of A_{inf} which is stable under φ on A_{inf}. Let $q = \frac{\varphi(\mu)}{\mu} \in \mathfrak{S}_\mu$. By [6, §3.3], we have q generates the kernel of $\theta \circ \varphi^{-1}$, i.e., we have $(q) = (\tilde{\xi})$ in A_{inf}. Let $K_{p^\infty} = \cup_{\mathrm{i}} K(\epsilon_{\mathrm{i}})$, and let $\Gamma = \mathrm{Gal}(K_{p^\infty}/K)$, note that Γ acts on $\mathfrak{S}_\mu$.

Definition 6.1.1. A (finite free) Wach module $\mathfrak{M}$ over K is a finite free $\mathfrak{S}_\mu$ module together with

$$\varphi_{\mathfrak{M}_\mu} : \mathfrak{S}_\mu \otimes_{\varphi, \mathfrak{S}_\mu} \mathfrak{M}_\mu \to \mathfrak{M}_\mu$$

such that the cokernal is killed by a power of q. $\mathfrak{M}_\mu$ also equipped with a semilinear Γ-action on $\mathfrak{M}_\mu$ commutes with $\varphi_{\mathfrak{M}_\mu}$, satisfies Γ acts trivially on $\overline{\mathfrak{M}_\mu} := \mathfrak{M}_\mu \otimes_{\mathfrak{S}_\mu} W(k)$.

And we have the following theorem of Berger.

Theorem 6.1.2. ([4]) There is an equivalence of G_K-stable $\mathbb{Z}_p$-lattices in crystalline representations with non-negative Hodge–Tate weights and the category of finite free Wach modules. And satisfies if $\mathfrak{M}_\mu$ corresponds to a crystalline representation T, then

$$(\mathfrak{M}_\mu \otimes_{\mathfrak{S}_\mu} W(C))^{\varphi=1} \simeq T.$$

Remark 6.1.3. Here we use the covariant version of the theorem, note that we still get non-negative Hodge–Tate weights, since in our convention 1.6.6 we let the cyclotomic character to have Hodge-Tate weight -1.

If $\mathfrak{M}$ is a Wach module, let define $\mathfrak{M}_\mu^{\mathrm{inf}} = \mathfrak{M}_\mu \otimes_{\mathfrak{S}_\mu} A_{\mathrm{inf}}$, $\mathfrak{M}_\mu^{\mathrm{inf}}$ has a φ-structure defined by $\varphi_{\mathfrak{M}_\mu} \otimes \varphi_{A_{\mathrm{inf}}}$ and a semilinear G_K-action.

Lemma 6.1.4. $\mathfrak{M}_\mu^{\mathrm{inf}}$ *with the* G_K*-action defined as above is an arithmetic Breuil-Kisin-Fargues* G_K*-module, moreover, it is actually crystalline in the sense that it satisfies the condition (3) in Theorem 4.2.9.*

Proof. Firstly, we have $(q) = (\tilde{\xi})$, so $\mathfrak{M}_\mu^{\mathrm{inf}}$ together with G_K-action is a Breuil-Kisin-Fargues G_K-module. We claim $(\mathfrak{M}_\mu^{\mathrm{inf}} \otimes B_{\mathrm{cris}}^+)^{G_K}$ has a basis over K_0 of full rank. This is from the fact there is a G_K-invariant section of $\overline{\mathfrak{M}}$ inside $\mathfrak{M}_\mu^{\mathrm{inf}} \otimes B_{\mathrm{cris}}^+$ by [23, Lemma 2.2.2]. □

Remark 6.1.5. In the proof of Lemma 6.1.4, when $K = K_0$, we should also have $\mathfrak{M}_\mu^{\mathrm{inf}}$ is arithmetic for Kisin-Ren's generalization of Wach modules (c.f. [23]), where the cyclotomic tower is replaced by Lubin-Tate tower. For general ramified K, there should be a paralleled theory for arithmetic Breuil-Kisin-Fargues G_K-modules over $A_{\mathrm{inf},K} := A_{\mathrm{inf}} \otimes_{W(k)} \mathcal{O}_K$ that relates to modifications of vector bundles over the Fargues-Fontaine curve defined using perfectoid field C and the field K (rather than $\mathbb{Q}_p$). We will discuss this in further work.

6.2 Kisin modules, $(\varphi, \hat{G})$-modules

For general ramified K. We will use $\vec{\varpi} := \{\varpi_n\}$ to denote a compatible system of p^n-th roots of a uniformizer ϖ of $\mathcal{O}_K$, i.e., $\varpi_0 = \varpi$ and $\varpi_{\mathrm{i}} = \varpi_{\mathrm{i}+1}^p$ for all i. We define $K_\infty = \cup_{n=1}^\infty K(\varpi_n)$, we will also write it as $K_{\infty,\vec{\varpi}}$ when we want to emphasize the choice of $\vec{\varpi}$. The compatible system $\{\varpi_n\}$ also defines an element $\underline{\varpi}$ in $\mathcal{O}_C$. View $\mathfrak{S} = W(k)[[u]]$ as a sub-$W(k)$-algebra of A_{inf} determined by $u \mapsto [\underline{\varpi}]$. We will also use $\mathfrak{S}_{\vec{\varpi}}$ to emphasize the choice of $\vec{\varpi}$. One can check $\varphi_{A_{\mathrm{inf}}}(u) = u^p$, so in particular $\mathfrak{S}$ in stable under $\varphi_{A_{\mathrm{inf}}}$, let $\varphi_{\mathfrak{S}} = \varphi_{A_{\mathrm{inf}}}|_{\mathfrak{S}}$. We also have G_{K_∞} fix u so G_{K_∞} acts trivially on $\mathfrak{S}$. Let $E(u) \in W(k)[u]$ be a minimal polynomial of ϖ over K_0. Also let $K_{\infty,p^\infty} = \cup_{\mathrm{i}} K(\varpi_{\mathrm{i}}, \epsilon_{\mathrm{i}})$, i.e., L is the normalization of K_∞, define $\widehat{G} = \mathrm{Gal}(K_{\infty,p^\infty}/K)$.

Definition 6.2.1. A (finite free) Kisin module is finite free $\mathfrak{S}$ module $\mathfrak{M}$ together with

$$\varphi_{\mathfrak{M}} : \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$$

such that the cokernel of $\varphi_{\mathfrak{M}}$ is killed by a power of $E(u)$.

One can check $E(u)$ generates kernel of θ, i.e., $(\xi) = (E(u))$ in A_{inf}, so we have

Lemma 6.2.2. *For a Kisin module $\mathfrak{M}$, $\mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{inf}}$ is a Breuil-Kisin-Fargues module.*

Let T be a log-crystalline representation of G_K over $\mathbb{Z}_p$ with non-negative Hodge-Tate weights, then Kisin in [21] can show T is of finite $E(u)$-height in the sense that there is Kisin module

$$\mathfrak{M} \subset (T \otimes_{\mathbb{Z}_p} W(C))^{G_{K_\infty}}$$

that is φ-stable and spans $T \otimes_{\mathbb{Z}_p} W(C)$ as a $W(C)$-module. This Kisin module is also uniquely determined by $T = (\mathfrak{M} \otimes_{\mathfrak{S}} W(C))^{\varphi=1}$ as a representation of G_{K_∞}. So we can write $\mathfrak{M} = \mathfrak{M}(T)$ and let $\mathfrak{M}^{\mathrm{inf}}(T) = \mathfrak{M}(T) \otimes_{\mathfrak{S},\varphi} A_{\mathrm{inf}}$ be the corresponded Breuil-Kisin-Fargues module which carries a natural G_{K_∞}-semilinear action that commutes with $\varphi_{\mathfrak{M}^{\mathrm{inf}}}$. We claim that there is a unique way to extend this to a G_K-semilinear action so that $\mathfrak{M}^{\mathrm{inf}}(T)$ is an arithmetic Breuil-Kisin-Fargues G_K-module. Actually, we will prove a more general result for T being potentially log-crystalline representation of G_K over $\mathbb{Z}_p$ with non-negative Hodge-Tate weights, we choose any finite Galois extension L/K such that $T|_{G_L}$ is log-crystalline. Then as we have discussed, $T_L := T|_{G_L}$ is of finite E-height for a choice of p^n-th roots of a uniformizer ϖ_L of $\mathcal{O}_L$ and let $\mathfrak{M}(T_L)$ (resp. $\mathfrak{M}^{\mathrm{inf}}(T_L)$) be the corresponded Kisin module (resp. Breuil-Kisin-Fargues module with G_{L_∞}-action).

Theorem 6.2.3. Up to isomorphic, there is a unique way to extends the G_{L_∞}-semilinear action on $\mathfrak{M}^{\mathrm{inf}}(T) := \mathfrak{M}^{\mathrm{inf}}(T_L)$ to an action of G_K so that $\mathfrak{M}^{\mathrm{inf}}(T)$ is an arithmetic Breuil-Kisin-Fargues G_K-module and satisfies a G_K-equivariant isomorphism

$$(\mathfrak{M}^{\mathrm{inf}}(T) \otimes W(C))^{\varphi=1} = T.$$

Proof. The proof is an explicit comparing the construction of Kisin module and our construction of arithmetic Breuil-Kisin-Fargues modules. We need to compute (T, Ξ) and its G_{L_∞}-action for $\mathfrak{M}^{\inf}(T_L)$.

We give a brief review of the construction of Kisin module from log-crystalline representation: let T is a potentially log-crystalline representation of G_K over $\mathbb{Z}_p$, L/K be a finite Galois extension such that $T|_{G_L}$ becomes log-crystalline, and let $L_0 = W(k_L)[\frac{1}{p}]$ and define $D = (T \otimes B_{\mathrm{st}})^{G_L}$ as the filtered (φ, N, G_K)-module associated with $T \otimes \mathbb{Q}_p$. Then we obtain a filtered (φ, N)-module D over L or $(D, \varphi, N, \mathrm{Fil}^\bullet)$ by forgetting the G_K-action. D corresponds to the log-crystalline representation $T \otimes \mathbb{Q}_p|_{G_L}$. Now let $\mathcal{O}$ be the ring of rigid analytic functions over the open unit disc over L_0 in the variable u. Let $\mathfrak{S} = W(k_L)[[u]]$, then one has $\mathfrak{S}[\frac{1}{p}] \subset \mathcal{O}$ and there is a $\varphi_{\mathcal{O}}$ extending $\varphi_{\mathfrak{S}}$. Fix $(\varpi_{L,n})$ any choice of compatible system of p^n-th roots of a uniformizer $\varpi_{L,0}$ of L, then one can easily show that the the inclusion $\mathfrak{S}[\frac{1}{p}] \to A_{\inf}[\frac{1}{p}]$ with $u \mapsto [(\overline{\varpi_{L,n}})]$ extends to an inclusion $\mathcal{O} \to B^+$. Geometrically, $\mathcal{O}$ (resp. B^+) is the locus $\{p \neq 0\}$ of $\mathrm{Spa}(\mathfrak{S})$ (resp. $\mathrm{Spa}(A_{\inf})$), and restrict the covering map $\mathrm{Spa}(A_{\inf}) \to \mathrm{Spa}(\mathfrak{S})$ to these loci will give $\mathcal{O} \to B^+$.

Roughly speaking, Kisin defined the $\mathfrak{S}$ module by descending a φ-module $\mathcal{M}(D)$ over $\mathcal{O}$ using the theory of slope of Kedlaya. In particular, we have $\mathfrak{M} \otimes \mathcal{O} = \mathcal{M}(D)$. And a theorem of Fontaine says that the ways of descent $\mathcal{M}(D)$ to $\mathfrak{M}$ are canonically corresponded with G_{L_∞}-stable $\mathbb{Z}_p$-lattices in $T \otimes \mathbb{Q}_p$, where $L_\infty = \cup_{n=1}^{\infty} L(\varpi_{L,n})$. Then Kisin define $\mathfrak{M}$ to be the $\mathfrak{S}$-module descents $\mathcal{M}(D)$ using the lattice $T|_{G_{L_\infty}}$. So we have

$$T(\mathfrak{M}^{\inf}(T_L)) = (\mathfrak{M} \otimes_{\mathfrak{S}} W(C))^{\varphi=1} = T|_{G_{L_\infty}}.$$

For $\Xi = \mathfrak{M}^{\inf}(T_L) \otimes B_{\mathrm{dR}}^+$, we need to review the construction of $\mathcal{M}(D)$ we mentioned above. For all $n \in \mathbb{Z}$ consider the composition:

$$\theta_n : \mathcal{O} \longrightarrow B^+ \xrightarrow{\varphi^{-n}} B^+ \xrightarrow{\theta} \mathbb{C}_p$$

and let x_n be the closed points on the rigid open unit disc defined by θ_n. And define $\mathcal{O}_{st} = \mathcal{O}[l_u]$, the $\mathcal{O}$-algebra generated by $l_u = \underline{\varpi_L}]$ inside B_{dR}^+. And extend the φ-action to l_u by $\varphi(l_u) = pl_u$ and define a $\mathcal{O}$-derivation N on $\mathcal{O}_{st}^+$ by letting $N(l_u) = 1$. Given $(D, \varphi, N, \mathrm{Fil}^\bullet)$, Kisin defines $\mathcal{M}(D)$ as certain modification of the vector bundle

$$(\mathcal{O}[l_u] \otimes_{L_0} D)^{N=0}$$

over $\mathcal{O}$ along stalks at x_n for $n \geq 0$. However to compute Ξ, we only need to the stalk at x_{-1}. and there is a natural isomorphism[21, Proposition 1.2.8.]:

$$(\mathcal{O}[l_u] \otimes_{L_0} D)^{N=0} = (L_0[l_u] \otimes_{L_0} D)^{N=0} \otimes_{L_0} \mathcal{O} \xrightarrow{\eta \otimes \mathrm{id}} D \otimes_{L_0} \mathcal{O}.$$

This tells us $\Xi = (\mathcal{M}(D) \otimes_{\mathcal{O}} B^+) \otimes_{B^+, \varphi} B_{\mathrm{dR}}^+$ is isomorphic to

$$(D \otimes_{L_0, \varphi} L_0) \otimes_{L_0} B_{\mathrm{dR}}^+ \cong D \otimes_{L_0} B_{\mathrm{dR}}^+$$

with G_{L_∞} acts trivially on D by construction.

To finish the proof, we just need to show there is a unique way to extend the G_{L_∞}-action on $(T_{L_\infty}, D \otimes B_{\mathrm{dR}}^+)$ to G_K such that on T it is the original potentially log-crystalline representation, and on $\Xi = D \otimes B_{\mathrm{dR}}^+$ is B_{dR}^+-flat. But we have if we want $\Xi \in T \otimes B_{\mathrm{dR}}$ is G_K stable and flat, by Proposition 2.2.12, it has to equal to

$$D_{\mathrm{dR}}(T) \otimes_K B_{\mathrm{dR}}^+ = (D \otimes L)^{G_K} \otimes B_{\mathrm{dR}}^+.$$

□

We also recall the following definition of $(\varphi, \widehat{G})$-modules of Liu.

Definition 6.2.4. Let $\mathrm{Mod}_{\mathfrak{S}, \widehat{R}}^{\varphi, \widehat{G}}$ be the category of triples $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \widehat{G})$ called $(\varphi, \widehat{G})$-modules, where

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a Kisin module;

(2) $\widehat{G}$ is a continuous $\widehat{R}$-semilinear $\widehat{R}$-action on $\widehat{\mathfrak{M}} := \mathfrak{M} \otimes_{\mathfrak{S}, \varphi} \widehat{R}$;

(3) $\widehat{G}$ commutes with $\varphi_{\widehat{\mathfrak{M}}}$;

(4) regarding $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule of $\widehat{\mathfrak{M}}$, we have $\mathfrak{M} \subset \widehat{\mathfrak{M}}^{\mathrm{Gal}(K_{\infty,p^\infty}/K_\infty)}$;

(5) $\widehat{G}$ acts trivially on $\widehat{\mathfrak{M}} \otimes_{\widehat{R}} W(k)$.

Here $\widehat{R}$ is a subring of A_{inf}. We will not give the explicit definition of $\widehat{R}$, we just list two properties we need in our applications:

(1) $\widehat{R} \subset A_{\mathrm{inf}}^{G_{K_{\infty,p^\infty}}}$;

(2) the image of $\widehat{R} \hookrightarrow A_{\mathrm{inf}} \xrightarrow{\theta}$ is K.

The main result in [25] is

Theorem 6.2.5. There is an equivalence of log-crystalline representations of non-negative Hodge-Tate weights with the category of $(\varphi, \hat{G})$-modules. And satisfies if $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \widehat{G})$ corresponds to a log-crystalline representation T, then

$$(\widehat{\mathfrak{M}} \otimes_{\mathfrak{S}} W(C))^{\varphi=1} \simeq T.$$

Lemma 6.2.6. *$(\mathfrak{M}, \varphi_{\mathfrak{M}}, \widehat{G})$ be a $(\varphi, \widehat{G})$-module, let $\mathfrak{M}^{\mathrm{inf}} = \widehat{\mathfrak{M}} \otimes_{\widehat{R}} A_{\mathrm{inf}}$, then $\mathfrak{M}^{\mathrm{inf}}$ is a Breuil-Kisin-Fargues G_K-module where the G_K-action comes from the $\hat{G}$-action on $\widehat{\mathfrak{M}}$, moreover, $\mathfrak{M}^{\mathrm{inf}}$ is arithmetic.*

Proof. We have $\mathfrak{M}^{\mathrm{inf}} = \widehat{\mathfrak{M}} \otimes_{\widehat{R}} A_{\mathrm{inf}} = \mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{inf}}$ is a Breuil-Kisin-Fargues module. And we have $\widehat{\mathfrak{M}} \otimes_{\widehat{R},\theta} K$ is a K vector space and

$$\mathfrak{M}^{\mathrm{inf}} \otimes \mathbb{C}_p = (\widehat{\mathfrak{M}} \otimes_{\widehat{R},\theta} K) \otimes_K \mathbb{C}_p.$$

Moreover, use the definition, we have $\widehat{\mathfrak{M}} \otimes_{\widehat{R},\theta} K$ has a G_K linear action and there is a basis fixed by G_{K_∞} coming from a basis of $\mathfrak{M}$. The result is from the following lemma on Kummer theory and Galois descent. ☐

Lemma 6.2.7. *All closed normal subgroup of G_K containing G_{K_∞} are open.*

Proof. This is from the fact K can only contain finitely mainly p^n-th roots of 1. ☐

So now as we mentioned in the beginning of Chapter 6, we have the following compatibility results for Kisin modules defined using different choice of uniformizer and Kummer tower, and the compatibility of Kisin modules, Wach modules and Kisin-Ren's modules when $K = K_0$ and T is crystalline.

Theorem 6.2.8. Let T be a log-crystalline representations with non-negative Hodge-Tate weights, for different choice of $\vec{\varpi}$, the arithmetic Breuil-Kisin-Fargues G_K-modules $\mathfrak{M}^{\mathrm{inf}}$ from Lemma 6.2.6 are all isomorphic to the arithmetic Breuil-Kisin-Fargues modules $\mathfrak{M}^{\mathrm{inf}}(T)$ defined in Theorem 4.2.9, in particular, they are all isomorphic to each other.

Moreover, if T is crystalline then $\mathfrak{M}^{\mathrm{inf}}(T)$ is isomorphic to the arithmetic Breuil-Kisin-Fargues G_K-modules defined from Wach modules as in Lemma 6.1.4.

Remark 6.2.9. We want to note that the above result also been proved in [26].

6.3 Breuil-Kisin-Fargues G_K-modules admit all descents

Definition 6.3.1. [16, F.7. Definition] Let $\mathfrak{M}^{\mathrm{inf}}$ be a Breuil-Kisin-Fargues G_K-module. Then we say that $\mathfrak{M}^{\mathrm{inf}}$ admits all descents over K if the following conditions hold.

(1) For any choice ϖ of uniformaizer of $\mathcal{O}_K$ and any compatible system $\vec{\varpi} = (\varpi_n)$ of p^n-th roots of ϖ, there is a Breuil-Kisin module $\mathfrak{M}_{\vec{\varpi}}$ defined using $\vec{\varpi}$ such that $\mathfrak{M}_{\vec{\varpi}} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{inf}}$ is isomorphic to $\mathfrak{M}^{\mathrm{inf}}$ and $\mathfrak{M}_{\vec{\varpi}}$ is fixed by $G_{K_{\vec{\varpi},\infty}}$ under the above isomorphism, where $K_{\vec{\varpi},\infty} = \cup_n K(\varpi_n)$;

(2) $\mathfrak{M}_{\vec{\varpi}} \otimes_{\mathfrak{S},\varphi} (\mathfrak{S}/E(u_{\vec{\varpi}})\mathfrak{S})$ is independent of the choice of $\vec{\varpi}$ as a $\mathcal{O}_K$-submodule of $\mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \mathcal{O}_{\mathbb{C}_p}$;

(3) Let $u_{\vec{\varpi}} = [(\overline{\varpi_n})]$, then $\mathfrak{M}_{\vec{\varpi}} \otimes_{\mathfrak{S},\varphi} (\mathfrak{S}/u_{\vec{\varpi}}\mathfrak{S})$ is independent of the choice of $\vec{\varpi}$ as a $W(k)$-submodule of $\mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} W(\overline{k})$.

Remark 6.3.2. Theorem 6.2.8 will implies if T is a log-crystalline representation of G_K, and let $\mathfrak{M}^{\mathrm{inf}}(T)$ be arithmetic Breuil-Kisin-Fargues G_K-module corresponds to T under Theorem 4.2.9, then $\mathfrak{M}^{\mathrm{inf}}(T)$ admits all descents over K.

The following result is first proved by Gee-Liu in [16], which can be regarded as an inverse of Theorem 6.2.8.

Proposition 6.3.3. Let $\mathfrak{M}^{\mathrm{inf}}$ be a Breuil-Kisin-Fargues G_K-module, and assume $\mathfrak{M}^{\mathrm{inf}}$ admits all descents over K, then $\mathfrak{M}^{\mathrm{inf}}$ is arithmetic and satisfies the condition (2) in Theorem 4.2.9, i.e., the inertia subgroup I_K of G_K acts trivially on $\overline{\mathfrak{M}^{\mathrm{inf}}} = \mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} W(\overline{k})$. In particular, $T(\mathfrak{M}^{\mathrm{inf}})$ is log-crystalline.

Proof. The proof is bases on [16, F.15] on Kummer theory, which will implies the closed subgroup generated by $\{K_{\vec{\varpi},\infty}\}_{\vec{\varpi}}$ is G_K.

So (2) in Definition 6.3.1 will imply $\mathfrak{M}^{\mathrm{inf}} \otimes \mathbb{C}_p$ has a G_K fixed basis, i.e., $\mathfrak{M}^{\mathrm{inf}}$ is arithmetic. And (3) in Definition 6.3.1 will imply that $\mathfrak{M}^{\mathrm{inf}} \otimes \breve{K}$ also has a G_K fixed basis, so the the inertia subgroup I_K acts trivially on on this basis. □

Remark 6.3.4. In a recent work [19], he was able to show that if an arithmetic Breuil-Kisin-Fargues G_K-module $\mathfrak{M}^{\mathrm{inf}}$ admits one descent, that is for one choice of compatible system $\vec{\varpi} = (\varpi_n)$ of p^n-th roots of a uniformizer ϖ of $\mathcal{O}_K$, there is a Breuil-Kisin module $\mathfrak{M}_{\vec{\varpi}}$ defined using $\vec{\varpi}$ such that $\mathfrak{M}_{\vec{\varpi}} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{inf}}$ is isomorphic to $\mathfrak{M}^{\mathrm{inf}}$ and $\mathfrak{M}_{\vec{\varpi}}$ is fixed by $G_{K_{\vec{\varpi},\infty}}$ under the above isomorphism, then $\mathfrak{M}^{\mathrm{inf}}$ is arithmetic. Actually, his work corrects a mistake in [11] and shows that if a p-adic representation is of finite E-height, then the representation is de Rham. We want to note this result is deep since for a representation with non-negative Hodge-Tate weights is of finite E-height that defined by a Kisin module

$$\mathfrak{M} \hookrightarrow T \otimes_{\mathbb{Z}_p} W(C)$$

then we don't know a priori that $\mathfrak{M} \otimes_{\mathfrak{S},\varphi} A_{\mathrm{inf}} \hookrightarrow T \otimes_{\mathbb{Z}_p} W(C)$ is stable G_K.

6.4 Crystalline condition

At last, we show how the ring $\overline{B}$ helps define a crystalline condition for arithmetic Breuil-Kisin-Fargues G_K-module $\mathfrak{M}^{\mathrm{inf}}$ with descents to Kisin modules. We observe that for Kisin's $\mathfrak{S}$

$$\mathfrak{S} \xrightarrow{\varphi} A_{\mathrm{inf}} \to \overline{B}$$

uniquely factor through $\mathfrak{S} \to W(k)$ since $u \in p$. So we slightly modify the (3) in Definition 6.3.1

Definition 6.4.1. Let $\mathfrak{M}^{\mathrm{inf}}$ be a Breuil-Kisin-Fargues G_K-module, and $\mathfrak{M}^{\mathrm{inf}}$ admits all descents over K, we say it is crystalline if it also satisfies

(3') Let $u_{\vec{\varpi}} = [(\overline{\varpi_n})]$, then $\mathfrak{M}_{\vec{\varpi}} \otimes_{\mathfrak{S},\varphi} (\mathfrak{S}/u_{\vec{\varpi}}\mathfrak{S})$ is independent of the choice of $\vec{\varpi}$ as a $W(k)$-submodule of $\mathfrak{M}^{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} \overline{B}$.

A straightforward consequence from (3) in Theorem 4.2.9 is

Proposition 6.4.2. Let $\mathfrak{M}^{\mathrm{inf}}$ be a Breuil-Kisin-Fargues G_K-module and assume $\mathfrak{M}^{\mathrm{inf}}$ satisfies (3) in Definition 6.4.1, then $\mathfrak{M}^{\mathrm{inf}}$ is arithmetic and satisfies the condition (3) in Theorem 4.2.9. In particular, $T(\mathfrak{M}^{\mathrm{inf}})$ is crystalline.

Remark 6.4.3. The crystalline conditions used in [16] and [19] will automatically fit into our Proposition 6.4.2, for example, in [16], we have for $u = [\underline{\varpi}]$, $gu - u \in \mathfrak{p}$ for all $g \in G_K$. Very recently, Bhatt-Scholze announced another crystalline condition using certain descent conditions on prismatic site, we expect that will be also related to our condition using $\overline{B}$.

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