BERNSTEIN–SATO IDEALS AND THE LOGARITHMIC DATA OF A DIVISOR

by

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ABSTRACT

We study a multivariate version of the Bernstein–Sato polynomial, the so-called Bernstein–Sato ideal, associated to an arbitrary factorization of an analytic germ $f = f_1 \cdots f_r$. We identify a large class of geometrically characterized germs so that the $\mathscr{D}_{X,\mathfrak{x}}[s_1, \ldots, s_r]$-annihilator of $f_1^{s_1} \cdots f_r^{s_r}$ admits the simplest possible description and, moreover, has a particularly nice associated graded object. As a consequence we are able to verify Budur's Topological Multivariable Strong Monodromy Conjecture for arbitrary factorizations of tame hyperplane arrangements by showing the zero locus of the associated Bernstein–Sato ideal contains a special hyperplane. By developing ideas of Maisonobe and Narváez-Macarro, we are able to find many more hyperplanes contained in the zero locus of this Bernstein–Sato ideal. As an example, for reduced, tame hyperplane arrangements we prove the roots of the Bernstein–Sato polynomial contained in $[-1, 0)$ are combinatorially determined; for reduced, free hyperplane arrangements we prove the roots of the Bernstein–Sato polynomial are all combinatorially determined. Finally, outside the hyperplane arrangement setting, we prove many results about a certain $\mathscr{D}_{X,\mathfrak{x}}$-map ∇_A that is expected to characterize the roots of the Bernstein–Sato ideal.

1. INTRODUCTION

1.1 Overview

For X a smooth analytic space and $f \in \mathscr{O}_{X,\mathfrak{x}}$ an analytic germ, the *Bernstein–Sato polynomial* b_f is an invariant of f that describes a very general sort of differential equation called the functional equation. Specifically, the Bernstein–Sato polynomial $b_f \in \mathbb{C}[s]$, for s some new variable, is the minimal, monic polynomial satisfying the functional equation

$$b_f f^s = P f^{s+1}.$$

Here P is some differential operator (possibly with s-terms) and the action of P on the symbols $f^{s+1}, f^s, \ldots$ is given by formally obeying the chain rule. A similar constructions applies for in the algebraic setting. That the Bernstein–Sato polynomial exists and is not zero was proved by Bernstein [1] in the algebraic case and Kashiwara [2] in the local, analytic one.

It turns out the Bernstein–Sato polynomial contains multitudes of information about the singular structure of $\operatorname{Var}(f)$. For the sake of this thesis we will restrict, primarily, to two phenomena, though the second is merely conjectural. First, the so-called *Milnor fiber* is a classical way to capture the behavior of f on $X \setminus \operatorname{Var}(f)$ near $\mathfrak{x}$ and, very loosely, arises as the fiber in the map $X \setminus \operatorname{Var}(f) \to \mathbb{C}^\star$. (Ex: when f is a homogeneous polynomial, the Milnor fiber can be identified with $\{f = 1\}$.) The fundamental group of $\mathbb{C}^\star$ lifts to an action on the Milnor fiber called the geometric monodromy; the induced action on homology is the algebraic monodromy. Malgrange [3] and Kashiwara [4] proved that $\{e^{2\pi i a} \mid a \text{ is a root of the Bernstein–Sato polynomial } b_f\} = \{\text{eigenvalues of the algebraic monodromy of Milnor fiber}\}$. Second, one can take a log resolution of $\mu : Y \to X$ of f and package the numerics of orders of vanishings of both f and the Jacobian of μ along the resultant irreducible components into, what turns out to be, a rational function. This is the *Topological Zeta Function* (see Chapter 2's Introduction for a precise definition). The *Topological Strong Monodromy Conjecture* asserts that the poles of this rational function are contained in the roots of the Bernstein–Sato polynomial of f.

We primarily study a multivariate generalization of the Bernstein–Sato polynomial, called the *Bernstein–Sato ideal*, and give particular attention to multivariate generalizations of two items from above. Namely: the relationship between the exponentials of the zeroes of the Bernstein–Sato ideal to nontrivial local systems; whether or not the poles of the Topological Multivariable Zeta Function appear as zeroes of the Bernstein–Sato ideal.

The Bernstein–Sato ideal $B_{F,\mathfrak{x}}$ is defined in an entirely similar was as the univariate version. Let $F = (f_1, \ldots, f_r)$ denote a factorization $f = f_1 \cdots f_r$. Introduce r new variables $s_1, \ldots, s_r$ and let F^S denote the symbol $f_1^{s_1} \cdots f_r^{s_r}$. Then the Bernstein–Sato ideal $B_{F,\mathfrak{x}} \subseteq \mathbb{C}[s_1, \ldots, s_r]$ consists of the polynomials $b(S)$ satisfying the functional equation

$$b(S)F^S = PF^{S+1}$$

where P is a differential operator, possibly with many s-terms, and the action of P on the symbols $F^{S+1}, F^S, \ldots$ is given by the chain and quotient rules. Sabbah [5] showed this ideal is nonzero in the local, analytic setting. Just as before, a similar construction holds for global algebraic f.

In [6], Budur revitalized the study of the Bernstein–Sato ideal with a series of conjectures. Hereafter, denoted these zeroes by $Z(B_{F,\mathfrak{x}})$. First, he conjectured a generalization of Kashiwara and Malgrange's result about eigenvalues of the algebraic monodromy by conjecturing that exponentiating $Z(B_{F,\mathfrak{x}})$ recovers the local systems on the complement of $\text{Var}(f)$, near $\mathfrak{x}$, with nontrivial cohomology. Second, he conjectured the *Topological Multivariable Strong Monodromy Conjecture*, asserting that the the *Topological Multivariable Zeta Function* has poles contained in $Z(B_{F,\mathfrak{x}})$. (This multivariable zeta function, explicitly defined in Chapter 2's Introduction, is similar to the univariate one except one also keeps track of the order of vanishings along all the factors f_k.)

1.2 Our Setting

We restrict to a class of divisors cut out by global sections f that was first considered by Walther in [7]. This class is defined in terms of geometric data, specifically data about the *logarithmic derivations* $\text{Der}_{X,x}(-\log f)$. These are the derivations on X, near $\mathfrak{x}$, that when applied to f land back in the $\mathscr{O}_{X,\mathfrak{x}}$-ideal generated by f; they also correspond to the vector fields tangent to $\mathrm{Var}(f)$. The hypotheses are as follows: f is *Saito-holonomic*, i.e. the stratification of X determined by the logarithmic derivations is locally finite; f is *strongly Euler-homogeneous*, i.e. locally everywhere f has a particular nice logarithmic derivation; f is *tame*, i.e. Saito's *logarithmic de Rham complex* [8] satisfies a sliding bound on projective dimension. These can be thought of as, respectively: a vehicle allowing induction; an assumption allowing well-defined-ness; a technical hypothesis used to obtain projective resolutions. If $\dim X \leq 3$ then f is automatically tame; if f is a hyperplane arrangement it is automatically Saito-holonomic and strongly Euler-homogeneous.

Let $\mathscr{D}_X$ be the sheaf of $\mathbb{C}$-linear differential operators on X and let $\mathscr{D}_X[S] := \mathscr{D}_X[s_1, \ldots, s_r]$ be a polynomial ring extension by central variables. Let $\mathscr{D}_{X,\mathfrak{x}}[S]F^S \subseteq \mathscr{O}_{X,\mathfrak{x}}[S, \frac{1}{f}]F^S$ be the $\mathscr{D}_{X,\mathfrak{x}}[S]$-submodule generated by the symbol F^S, where, again, the $\mathscr{D}_{X,\mathfrak{x}}[S]$-action on gF^S, $g \in \mathscr{O}_{X,\mathfrak{x}}[S, \frac{1}{f}]$, is given by the chain and quotient rules. It follows from the definitions that the Bernstein–Sato ideal satisfies

$$B_{F,\mathfrak{x}} = \mathbb{C}[S] \cap (\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} F^S + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot f)$$

So one way to understand the Bernstein–Sato ideal is to study the $\mathscr{D}_{X,\mathfrak{x}}[S]$-annihilator of F^S. Without hypothesis this is very hard: even computer packages are not able to find a generating set for this annihilator in a practical amount of time.

Our result, Theorem 2.2.1, powering this thesis's two chapters is that, under the working hypotheses of Saito-holonomic, strongly Euler-homogeneous, and tame, the $\mathscr{D}_{X,\mathfrak{x}}[S]$-annihilator of F^S has the simplest possible generating set. Every logarithmic derivation determines an element of this annihilator and, under these hypotheses, these elementary elements generate said annihilator. Walther obtained this result in the univariate, i.e. f^s, setting in [7]. We use some of his techniques but our proof is not a simple consequence of his ideas. Not only do his techniques require significant massaging to apply to our setting, but to even connect his paper to our chapter we have to develop an adequate theory of a very non-standard flat family of maps and realize one of his constructions as the special fiber. (See Section 2.6 for this development.) Not only do we obtain a nice description of

$\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}}[S]F^S$ but we also conclude the associated graded object of $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$ with respect to the total order filtration is a Cohen–Macaulay domain of dimension $\dim X + r$.

1.3 On Exponentiating the Zeroes of the Bernstein–Sato Ideal

Let $A := (a_1, \ldots, a_r) \in \mathbb{C}^r$. The connection between A being a zero, or not a zero, of the Bernstein–Sato ideal and the local system determined by $(e^{2\pi i a_1}, \ldots, e^{2\pi i a_r})$ "should" be expressible in the behavior of a certain map ∇_A. This map comes about as follows. First, let $\nabla : \mathscr{D}_{X,\mathfrak{x}}[S]F^S \to \mathscr{D}_{X,\mathfrak{x}}[S]F^S$ be the $\mathscr{D}_{X,\mathfrak{x}}$-linear map determined by sending each s_k to s_{k+1}. In particular $F^S \mapsto F^{S+1}$. There is an induced $\mathscr{D}_{X,\mathfrak{x}}$-linear map on quotient modules

$$\nabla_A : \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\sum(s_k - a_k)\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \to \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\sum(s_k - (a_k - 1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}.$$

In the univariate, i.e. f^s case, it is known ∇_a is surjective if and only if $a - 1$ is not a root of the Bernstein–Sato polynomial. Budur observed that if the same where true in the multivariate setting, i.e. the F^S case for $r > 1$, then his generalization of Malgrange and Kashiwara's result would hold. That is, if ∇_A characterized whether $A - 1$ was or was not a root of the Bernstein–Sato polynomial, then exponentiating $Z(B_{F,\mathfrak{x}})$ would compute the local systems on $X \setminus \mathrm{Var}(f)$ near $\mathfrak{x}$ with nontrivial cohomology.

In Chapter 2 we are able to show, provided that f is Sato-holonomic, strongly Euler-homogeneous, and tame that if ∇_A is injective then it is surjective. To study the reverse implication we strengthen the tame hypothesis, instead assume f is free. This means the logarithmic derivations constitute a free module. Under these strengthened hypotheses we prove ∇_A is injective if and only if it is surjective and, consequently, give a weak characterization of local systems with nontrivial cohomology in terms of the behavior of ∇_A. In this case, the key technique is understanding the $\mathscr{D}_{X,\mathfrak{x}}$-dual of ∇_A. To do so, we use duality formulas due to Narváez-Macarro [9]. In Chapter 3, section 3.7, we prove ∇_A does characterize the roots of the Bernstein–Sato ideal provided f is a free, reduced hyperplane arrangement.

The reader should note that after these results were obtained, this conjecture of Budur's was resolved in [10] by Budur, van der Veer, Wu, and Zhou. However, their proof is highly nonconstructive and the question of whether or not ∇_A characterizes the zeroes of the

Bernstein–Sato ideal remains open. We expect our results to help, eventually, unlock this behavior.

1.4 On the Topological Multivariable Strong Monodromy Conjecture

When f is a hyperplane arrangement, Budur [6] came up with the following criterion for the Topological Multivariable Strong Monodromy Conjecture: if for any central, indecomposable, hyperplane arrangement $f = f_1 \cdots f_r \in \mathbb{C}[x_1, \ldots, x_n]$

$$\{\deg(f_1)s_1 + \cdots + \deg(f_r)s_r + n = 0\} \subseteq Z(B_{F,0})$$

then the Topological Multivariable Strong Monodromy Conjecture holds for the factorization $F = (f_1, \ldots, f_r)$ of f. Since tameness is a local condition, if we can verify this criterion for central, indecomposable, and tame hyperplane arrangements then we can show this conjecture holds for any factorization of tame hyperplane arrangements.

This is exactly what we do: in Theorem 2.1.2 we prove the Topological Multivariable Strong Monodromy Conjecture holds for any factorization of a tame hyperplane arrangement. This generalizates a similar result by Walther [7] in the univariate case. Our result is a relatively easy consequence of the fact we can compute $\mathscr{D}_{X,0}[S]$-annihilator of F^S explicitly in this case: this computation lets us estimate the Bernstein–Sato ideal well enough to establish the criterion.

1.5 Bernstein–Sato Ideals, Polynomials for Hyperplane Arrangements

In Chapter 3 we focus exclusively on tame and free hyperplane arrangements and try to determine which zeroes (resp. roots) of the Bernstein–Sato ideal (resp. polynomial) are determined by the intersection lattice of the hyperplane arrangement. Maisonobe [11] was able to, again using Narváez-Macarro's duality formulas, compute the Bernstein–Sato ideal for a reduced free arrangement factored into linear forms. However, the question of non-reduced free arrangements and different factorizations was not accessible by his methods. With significant labor, we can extend his ideas to these cases. In particular we can prove that the roots of the Bernstein–Sato polynomial of a free, reduced arrangement are comibinatorially determined and admit a reasonable combinatorial formula, cf. Theorem 3.1.4. Along the way we obtain more general versions of Narváez-Macarro's duality formulas thanks to an entirely novel approach, cf. Chapter 3, section 6.

Outside the free setting, we are also able to show, in Theorem 3.1.3, that if f is a tame arrangement, the roots of the Bernstein–Sato polynomial contained in $[-1,0)$ are combinatorially determined and admit a simple formula. It is known that these roots live in $(-2,0) \cap \mathbb{Q}$ but, even in the tame case, are not always combinatorial, cf. [7]. The "problematic" roots seem to be the ones very close to -2. In Theorem 3.1.5 we give a interesting interpretation of these "problematic" roots. We present a simple measurement for the distance between a hyperplane arrangement f and a free hyperplane arrangement containing $\mathrm{Var}(f)$ and show these roots give a lower bound for this distance.

2. BERNSTEIN–SATO IDEALS AND ANNIHILATION OF POWERS[1]

2.1 Introduction

Let X be a smooth analytic space or $\mathbb{C}$-scheme of dimension n with structure sheaf $\mathscr{O}_X$ and with the sheaf of $\mathbb{C}$-linear differential operators $\mathscr{D}_X$. Take a global function $f \in \mathscr{O}_X$. The classical construction of the Bernstein–Sato polynomial of f is as follows:

1. Consider the $\mathscr{O}_X[f^{-1}, s]$-module generated by the symbol f^s. This has a $\mathscr{D}_X[s]$-module structure induced by the formal rules of calculus.

2. The Bernstein–Sato ideal B_f of f is

$$B_f := \mathbb{C}[s] \cap \left(\mathscr{D}_X[s] \cdot f + \operatorname{ann}_{\mathscr{D}_X[s]} f^s \right).$$

3. For $X = \mathbb{C}^n$ and f a polynomial, Bernstein showed in [1] that B_f is not zero. For f local and analytic, Kashiwara [2] proved the same. Since B_f, or the local version $B_{f,\mathfrak{x}}$, is an ideal in $\mathbb{C}[s]$ it has a monic generator, the Bernstein–Sato polynomial of f.

The variety $\mathrm{V}(B_f)$ contains a lot of information about the divisor of f and its singularities. For example, if $\mathrm{Exp}(a) = e^{2\pi i a}$ and if $M_{f,\mathfrak{y}}$ is the Milnor Fiber of f at $\mathfrak{y} \in \mathrm{V}(f)$, cf. [13], then Malgrange and Kashiwara showed in [3], [4] that

$$\mathrm{Exp}(\mathrm{V}(B_{f,\mathfrak{x}})) = \bigcup_{\mathfrak{y} \in \mathrm{V}(f) \text{ near } \mathfrak{x}} \{ \text{ eigenvalues of the algebraic monodromy on } M_{f,\mathfrak{y}} \}$$

Suppose f factors as $f_1 \cdots f_r$. Let $F = (f_1, \ldots, f_r)$. Then there is a generalization of the Bernstein–Sato ideal B_f of f called the multivariate Bernstein–Sato ideal B_F of F obtained in a similar way.

1. Introduce new variables $S := s_1, \ldots, s_r$. Consider the $\mathscr{O}_X[F^{-1}, S]$-module generated by the symbol $F^S = \prod f_k^{s_k}$. Again, this is a $\mathscr{D}_X[S]$-module via formal differentiation.

[1]↑A version of this chapter has been published in the Transactions of the American Mathematical Society as [12].

2. The multivariate Bernstein–Sato ideal B_F is

$$B_F := \mathbb{C}[S] \cap \left(\mathscr{D}_X[S] \cdot f + \operatorname{ann}_{\mathscr{D}_X[S]} F^S \right).$$

3. For $X = C^n$ and $f_1, \dots, f_r$ polynomials, B_F is nonzero, see [14]. Sabbah proved in [5] the corresponding statement for $f_1, \dots, f_r$ local and analytic. However neither B_F nor $B_{F,\mathfrak{x}}$ need be principal: cf. Bahloul and Oaku [15].

The significance of $\mathrm{V}(B_F)$ or the local version $\mathrm{V}(B_{F,\mathfrak{x}})$ is less developed than the univariate counterparts. Let $f = f_1 \cdots f_r$ be a product of distinct and irreducible germs at $\mathfrak{x}$ and let $F = (f_1, \dots, f_r)$. Let $U_{F,\mathfrak{y}}$ be the intersection of a small ball about $\mathfrak{y} \in \mathrm{V}(f)$ with $X \setminus \mathrm{V}(f)$. Denote by $V(U_{F,\mathfrak{y}})$ the rank one local systems on $U_{F,\mathfrak{y}}$ with nontrivial cohomology, i.e. the set of rank one local systems L such that $H^k(U_{F,\mathfrak{y}}, L)$ is nonzero for some k. This is the cohomology support locus of f at $\mathfrak{y}$ in the language of Budur and others. Since local systems can be identified with representations $\pi_1(U_{F,\mathfrak{y}}) \to \mathbb{C}^\star$, regard $V(U_{F,\mathfrak{y}}) \subseteq (\mathbb{C}^*)^r$. In [6], Budur proposes that the relationship between the roots of the Bernstein–Sato polynomial and the eigenvalues of the algebraic monodromy is generalized by the conjecture

$$\operatorname{Exp}(\mathrm{V}(B_{F,\mathfrak{x}})) = \bigcup_{\mathfrak{y} \in \mathrm{V}(f) \text{ near } \mathfrak{x}} \operatorname{res}_{\mathfrak{y}}^{-1}(V(U_{F,\mathfrak{y}})). \tag{2.1.1}$$

where $\operatorname{res}_{\mathfrak{y}}$ restricts a local system on $U_{\mathfrak{x}}$ to a local system on $U_{\mathfrak{y}}$. (This generalization passes through the support of the Sabbah specialization complex in the same way that the proof of the univariate version uses the support of the nearby cycle functor.)

This paper follows two threads. First we study the logarithmic derivations $\operatorname{Der}_X(-\log f)$ of f inside $\operatorname{ann}_{\mathscr{D}_X[S]} F^S$. We are motivated by [7] where Walther shows that, in the univariate case and with some mild hypotheses on the divisor of f, these members generate $\operatorname{ann}_{\mathscr{D}_X[s]} f^s$.

We restrict ourselves to "nice" divisors: strongly Euler-homogeneous (possessing a particular logarithmic derivation locally everywhere); Saito-holonomic (the logarithmic stratification is locally finite); tame (a restriction on homological dimension). The main result of Section 2 is the following:

Theorem 2.1.1. *Let $F = (f_1, \dots, f_r)$ be a decomposition of $f = f_1 \cdots f_r$. If f is strongly Euler-homogeneous, Saito-holonomic, and tame then*

$$\operatorname{ann}_{\mathscr{D}_X[S]} F^S = \mathscr{D}_X[S] \cdot \{\delta - \sum_{k=1}^{r} s_k \frac{\delta \bullet f_k}{f_k} \mid \delta \in \operatorname{Der}_X(-\log f)\}.$$

The strategy is to take a filtration of $\mathscr{D}_X[S]$ and consider the associated graded object of $\operatorname{ann}_{\mathscr{D}_X[S]} F^S$. This object can be given a second filtration so its initial ideal is similar to the Liouville ideal of [7]. Section 6 provides the mild generalizations of Gröbner type arguments necessary to transfer properties from this initial ideal to the ideal itself and Section 2 proves nice things about our associated graded objects, culminating in Theorem 2.1.1. In [11], Maisonobe proves a similar statement in the more restrictive setting of free divisors where many of these methods are not needed. We crucially use one of his techniques.

Not much is known about particular elements of $\operatorname{V}(B_F)$ even when F corresponds to a factorization (not necessarily into linear forms) of a hyperplane arrangement. In [6] Budur generalized the $-\frac{n}{d}$ conjecture (see Conjecture 1.3 of [7]) as follows:

Conjecture 2.1.1. (Conjecture 3 in [6]) *Let $f = f_1 \cdots f_r$ be a central, essential, indecomposable hyperplane arrangement in $\mathbb{C}^n$. Let $F = (f_1, \dots, f_r)$ where the f_k are central hyperplane arrangements, not necessarily reduced, of degree d_k. Then*

$$\{d_1 s_1 + \cdots + d_r s_r + n = 0\} \subseteq V(B_F).$$

Using Theorem 2.1.1, we can prove Conjecture 2.1.1 in the tame case:

Theorem 2.1.2. *Let $f = f_1 \cdots f_r$ be a central, essential, indecomposable, and tame hyperplane arrangement in $\mathbb{C}^n$. Let $F = (f_1, \dots, f_r)$ where the f_k are central hyperplane arrangements, not necessarily reduced, of degree d_k. Then*

$$\{d_1 s_1 + \cdots + d_r s_r + n = 0\} \subseteq V(B_F).$$

Conjecture 2.1.1 was motivated by the formulation of the Topological Multivariable Strong Monodromy Conjecture due to Budur, see Conjecture 5 of [6]. We now state this.

First let $f = f_1 \cdots f_r$ with each $f_k \in \mathbb{C}[x_1, \ldots, x_n]$ and let $F = (f_1, \ldots, f_r)$. Given a log resolution $\mu : Y \to X$ of f, let $\{E_i\}_{i \in S}$ be the irreducible components of $f \circ \mu$, let $a_{i,j}$ be the order of vanishing of f_j along E_i, let k_i be the order of vanishing of the determinant of the Jacobian of μ along E_i, and, for $I \subseteq S$, let $E_I^\circ = \cap_{i \in I} \setminus \cup_{i \in S \setminus I} E_i$. The *topological zeta function* of F is

$$Z_F^{\mathrm{top}}(S) := \sum_{I \subseteq S} \chi(E_I^\circ) \cdot \prod_{i \in I} \frac{1}{a_{i,1}s_1 + \cdots + a_{i,r}s_r + k_i + 1}$$

and this is independent of the resolution. Conjecture 5 of [6] states:

Conjecture 2.1.2. (Topological Multivariable Strong Monodromy Conjecture) *The polar locus of $Z_F^{top}(S)$ is contained in* $\mathrm{V}(B_F)$.

By work of Budur in loc. cit., Conjecture 2.1.1 implies Conjecture 2.1.2 for hyperplane arrangements. Consequently, we conclude Section 2 with the following:

Corollary 2.1.3. *The Topological Multivariable Strong Monodromy Conjecture is true for (not necessarily reduced) tame hyperplane arrangements.*

The paper's second thread follows the link between $\mathrm{Exp}(\mathrm{V}(B_{F,\mathfrak{x}}))$ and the cohomology support loci of f near $\mathfrak{x}$. The bridge between the two is, with $A = (a_1, \ldots, a_r) \in \mathbb{C}^r$, resp. $A - 1 = (a_1 - 1, \ldots, a_r - 1) \in \mathbb{C}^r$, the $\mathscr{D}_{X,\mathfrak{x}}$-linear map

$$\nabla_A : \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \to \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}.$$

Here $(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S$, resp. $(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S$, is the submodule of $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$ generated by $s_1 - a_1, \ldots, s_r - a_r$, resp. $s_1 - (a_1 - 1), \ldots, s_r - (a_r - 1)$, and ∇_A is induced by $F^S \mapsto F^{S+1}$. In the classical, univariate case, the following are equivalent (cf. Björk, 6.3.15 of [16]): (a) $A - 1 \notin \mathrm{V}(B_{f,\mathfrak{x}})$; (b) ∇_A is injective; (c) ∇_A is surjective. Showing that (a), (b), and (c) are equivalent in the multivariate case would verify that $\mathrm{Exp}(\mathrm{V}(B_{F,\mathfrak{x}}))$ equals the cohomology suport loci of f near $\mathfrak{x}$. Moreover, under the hypotheses of Theorem 2.1.1, it would show that intersecting $V(B_{F,\mathfrak{x}})$ with appropriate hyperplanes gives $V(B_{f,\mathfrak{x}})$.

In any case, (a) implies (b) and (c). Under the hypotheses of Theorem 2.1.1, we prove that $s_1 - a_1, \ldots, s_r - a_r$ behaves like a regular sequence on $\mathscr{D}_X[S]F^S$. This allows us to recreate a picture similar to Björk's and prove, using different methods, the main result of Section 3:

Theorem 2.1.3. *Let $f = f_1 \cdots f_r$ be strongly Euler-homogeneous, Saito-holonomic, and tame and let $F = (f_1, \ldots, f_r)$. If ∇_A is injective then ∇_A is surjective.*

In Section 4 we strengthen the hypotheses of Theorem 2.1.1 and assume f is reduced and free, that is, we assume $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$ is a free $\mathscr{O}_{X,\mathfrak{x}}$-module. In [9] Narváez–Macarro computed the $\mathscr{D}_{X,\mathfrak{x}}[s]$-dual of $\mathscr{D}_{X,\mathfrak{x}}[s]f^s$ for certain free divisors; in [11], Maisonobe shows that this computation easily applies to $\mathscr{D}_{X,\mathfrak{x}}[S]$-dual of $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$. For our free divisors we compute the $\mathscr{D}_{X,\mathfrak{x}}$-dual of $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S}$ and lift ∇_A to this dual. Consequently, we prove:

Theorem 2.1.4. *Let $f = f_1 \cdots f_r$ be reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let $F = (f_1, \ldots, f_r)$. Then ∇_A is injective if and only if ∇_A surjective.*

In Section 5 we summarize the relationship between the cohomology support loci of f near $\mathfrak{x}$, $\mathrm{Exp}(V(B_{F,\mathfrak{x}}))$, and ∇_A. In [17], the authors characterize membership in the cohomology support loci of f near $\mathfrak{x}$ in terms of the simplicity of certain perverse sheaves. When f is reduced, strongly Euler-homogeneous, Saito-holonomic, and free, we show this characterization can be stated in terms of the simplicity of the $\mathscr{D}_{X,\mathfrak{x}}$-module $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S}$.

2.2 The $\mathscr{D}_X[S]$-Annihilator of F^S

As in the introduction, let X be a smooth analytic space or $\mathbb{C}$-scheme of dimension n and with structure sheaf $\mathscr{O}_X$. Let $f \in \mathscr{O}_X$ be regular with divisor $Y = \mathrm{Div}(f)$ and corresponding ideal sheaf $\mathscr{I}_Y$. Throughout, $Y = \mathrm{Div}(f)$ will not necessarily be reduced. Let $\mathscr{D}_X$ be the sheaf of $\mathbb{C}$-linear differential operators with $\mathscr{O}_X$-coefficients and let $\mathscr{D}_X[s]$ and $\mathscr{D}_X[S] = \mathscr{D}_X[s_1, \ldots, s_r]$ be polynomials rings over $\mathscr{D}_X$.

Recall the *order filtration* $F_{(0,1)}$ on $\mathscr{D}_X$ induced, in local coordinates, by making every ∂_{x_k} weight one and every element of $\mathscr{O}_X$ weight zero. Denote the differential operators of order at most k as $F^k_{(0,1)}$ and the associated graded object as $\mathrm{gr}_{(0,1)}(\mathscr{D}_X)$.

Definition 2.2.1. *Let* $\mathrm{Der}_X(-\log f) = \mathrm{Der}_X(-\log(Y))$*, be the sheaf of* logarithmic derivations*, i.e. the* $\mathscr{O}_X$*-module with local generators on* U *the set*

$$\mathrm{Der}_X(-\log f) := \{\delta \textit{ a vector field in } \mathscr{D}_X(U) \mid \delta \bullet \mathscr{I} \subseteq \mathscr{I}\}.$$

We also put

$$\mathrm{Der}_X(-\log_0 f) := \{\delta \in \mathrm{Der}_X(-\log f) \mid \delta \bullet f = 0\}.$$

Note that $\mathrm{Der}_X(-\log_0 f)$ *may depend on the choice of defining equation for f, which is why we have fixed a global* f*.*

Definition 2.2.2. *For* $\mathfrak{x} \in X$*, we say that* $f \in \mathscr{O}_{X,\mathfrak{x}}$ *is* Euler-homogeneous *at* $\mathfrak{x}$ *if there exists* $E_\mathfrak{x} \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f)$ *such that* $E_\mathfrak{x} \bullet f = f$*. If* $E_\mathfrak{x}$ *vanishes at* $\mathfrak{x}$ *then* f *is* strongly Euler-homogeneous *at* $\mathfrak{x}$*.*

Finally, a divisor Y *is (strongly) Euler-homogeneous if there is a defining equation* f *at each* $\mathfrak{x}$ *such that* f *is (strongly) Euler-homogeneous at* $\mathfrak{x}$*.*

Example 2.2.1. Let $f = x(2x^2 + yz)$. Note that $\mathrm{Sing}(f) = \{z - \text{axis}\} \cup \{y - \text{axis}\}$. Along the z-axis there is the strong Euler-homogeneity induced by $\frac{1}{3}x(\partial_x \bullet f) + \frac{2}{3}y(\partial_y \bullet f)$; along the y-axis there is the strong Euler-homogeneity induced by $\frac{1}{3}x(\partial_x \bullet f) + \frac{2}{3}z(\partial_z \bullet f)$. Since f is automatically strongly Euler-homogeneous on the smooth locus, f is strongly Euler-homogeneous everywhere.

Example 2.2.2. Let f be a central hyperplane arrangement. Then the Euler vector field $\sum x_i \partial_{x_i}$ shows that f is strongly Euler-homogeneous at the origin. A coordinate change argument implies f is strongly Euler-homogeneous.

Definition 2.2.3. *Define the* total order filtration $F_{(0,1,1)}$ *as the filtration on* $\mathscr{D}_X[S]$ *induced by the* $(0,1,1)$*-weight assignment that, in local coordinates, gives elements of the form* $\mathscr{O}_U \partial^{\boldsymbol{u}} S^{\boldsymbol{v}}$*,* $\boldsymbol{u}$*,* $\boldsymbol{v}$ *non-negative integral vectors, weight* $\sum u_i + \sum v_i$*. Let* $F^k_{(0,1,1)}$ *be the homogeneous operators of weight at most* k *with respect to the total order filtration. When the context is clear, we will use* $F^k_{(0,1,1)}$ *to refer to the similarly defined filtration on* $\mathscr{D}_X[s]$ *(the classical case). Denote the associated graded object associated to* $F_{(0,1,1)}$ *as* $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_X[S])$*.*

Our principal objective is to study the annihilator of F^S—the left $\mathscr{D}_X[S]$-ideal $\operatorname{ann}_{\mathscr{D}_X[S]} F^S$. Take the $\mathscr{O}_X[f_1^{-1}, \dots, f_r^{-1}, S]$-module generated freely by the symbol $F^S = \prod f_k^{s_k}$. To make this a $\mathscr{D}_X[S]$-module define, for a derivation δ and $h \in \mathscr{O}_X$,

$$\delta \bullet \frac{hS^{\mathbf{v}}}{f^j} F^S = \delta \bullet (\frac{h}{f^j}) S^{\mathbf{v}} F^S + \sum_k s_k \frac{(\delta \bullet f_k) h S^{\mathbf{v}}}{f_k f^j} F^S.$$

In most cases $\operatorname{ann}_{\mathscr{D}_X[S]} F^S$ is very hard to compute. In the classical setting, there is a natural identification between the $(0,1,1)$-homogeneous elements of $\operatorname{ann}_{\mathscr{D}_X[s]} f^s$ and $\operatorname{Der}_X(-\log f)$. We will establish a similar correspondence.

Definition 2.2.4. *The* annihilating derivations *of F^S are the elements of the $\mathscr{O}_X$-module*

$$\theta_F := \operatorname{ann}_{\mathscr{D}_X[S]} F^S \cap F^1_{(0,1,1)}.$$

We say $\operatorname{ann}_{\mathscr{D}_X[S]} F^S$ *is* generated by derivations *when* $\operatorname{ann}_{\mathscr{D}_X[S]} F^S = \mathscr{D}_X[S] \cdot \theta_F$.

Proposition 2.2.3. *For $f = f_1 \cdots f_r$, let $F = (f_1, \dots, f_r)$. Then as $\mathscr{O}_X$-modules,*

$$\psi_F : \operatorname{Der}_X(-\log f) \xrightarrow{\simeq} \theta_F$$

where ψ_F is given by

$$\delta \mapsto \delta - \sum_{k=1}^{r} s_k \frac{\delta \bullet f_k}{f_k}.$$

Proof. We first prove the claim locally. By Lemma 3.4 of [18], $\operatorname{Der}_X(-\log f) = \bigcap \operatorname{Der}_X(-\log f_k)$; in particular, $\delta - \sum_k s_k \frac{\delta \bullet f_k}{f_k}$ lies in $\mathscr{D}_{X,\mathfrak{x}}[S]$.

Fix a coordinate system. Take $P \in \theta_{F,\mathfrak{x}}$, $P = \delta + p(S)$, where $\delta \in \mathscr{D}_{X,\mathfrak{x}}$ is a derivation and $p(S) = \sum_k b_k s_k \in \mathscr{O}_{X,\mathfrak{x}}[S]$ is necessarily S-homogeneous of S-degree 1. Keep the notation F^s and the f_k for the local versions at $\mathfrak{x}$. By definition,

$$0 = \left(\delta - \sum_k b_k s_k \right) \bullet F^S = \sum_k \left(s_k \frac{\delta \bullet f_k}{f_k} - b_k s_k \right) F^S.$$

Because $\mathscr{D}_{X,\mathfrak{x}} F^S$ is a free $\mathscr{O}_{X,\mathfrak{x}}[f^{-1}, S]$-module $\sum_k (s_k \frac{\delta \bullet f_k}{f_k} - b_k s_k) = 0$. Thus for each k, $\frac{\delta \bullet f_k}{f_k} s_k - b_k s_k = 0$. So $\delta \bullet f_k \in \mathscr{O}_{X,\mathfrak{x}} \cdot f_k$; moreover, $\delta \bullet f_k = b_k f_k$.

We have shown $\delta \in \bigcap \mathrm{Der}_{X,\mathfrak{x}}(-\log f_k)$ and, in fact,

$$\theta_{F,\mathfrak{x}} = \{\delta - \sum_k b_k s_k \mid \delta \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f),\, \delta \bullet f_k = b_k f_k\}.$$

Thus the map $\psi_F : \mathrm{Der}_{X,\mathfrak{x}}(-\log f) \to \theta_{F,\mathfrak{x}}$ given by $\delta \mapsto \delta - \sum_k \frac{\delta \bullet f_k}{f_k} s_k$ is a well-defined $\mathscr{O}_{X,\mathfrak{x}}$-linear isomorphism for a fixed coordinate system. Showing that $\theta_{F,\mathfrak{x}}$ commutes with coordinate change is routine and is effectively shown in Remark 2.2.7 (b).

Since $\delta \in \mathrm{Der}(-\log f)$ precisely when $\delta \bullet f = 0$ in $\mathscr{O}_X/(f)$, membership in $\mathrm{Der}(-\log f)$ is a local condition. The above shows that $\psi_{F,\mathfrak{x}}^{-1}$ is an $\mathscr{O}_{X,\mathfrak{x}}$-isomorphism at all $\mathfrak{x}$; hence ψ_F^{-1} is an isomorphism. □

2.2.1 Hypotheses on Y and F.

In this subsection we introduce many of the standard hypothesis on Y and F we use throughout the paper.

Definition 2.2.5. *Let $U \subseteq X$ be open and $f \in \mathscr{O}_X(U)$. We will say $F = (f_1, \ldots, f_r)$ is a* decomposition *of f when $f = f_1 \cdots f_r$.*

We will also restrict to divisors Y such that $\mathrm{Der}_X(-\log Y)$ has a light constraint on its dimension.

Definition 2.2.6. *Consider the sheaf of differential forms of degree k: $\Omega_X^k = \bigwedge^k \Omega_X^1$ and the differential $d : \Omega_X^k \to \Omega_X^{k+1}$. We define the subsheaf of logarithmic differential forms $\Omega_X^k(\log f)$ by*

$$\Omega_X^k(\log f) := \{w \in \frac{1}{f}\Omega_X^k \mid d(w) \in \frac{1}{f}\Omega_X^{k+1}\}.$$

See 1.1 and 1.2 in [8] for more details.

We say $f \in \mathscr{O}_X(U)$, $U \subseteq X$ open, is tame if the projective dimension of $\Omega_U^k(\log f)$ is at most k at each $\mathfrak{x} \in U$. A divisor Y is tame if it admits tame defining equations locally everywhere. See Defintion 3.8 and the surrounding text in [7] for more details on tame divisors. In particular, if $n \leq 3$ then Y is automatically tame.

We will also use a stratification of X that respects the logarithmic data of Y.

Definition 2.2.7. *(Compare with 3.8 in [8]) Define a relation on X by identifying two points $\mathfrak{x}$ and $\mathfrak{y}$ if there exists an open $U \subseteq X$, $\mathfrak{x}, \mathfrak{y} \in U$ and a derivation $\delta \in \operatorname{Der}_U(-\log(Y \cap U))$ such that (i) δ is nowhere vanishing on U and (ii) the integral flow of δ passes through $\mathfrak{x}$ and $\mathfrak{y}$. The transitive closure of this relation stratifies X into equivalence classes. The irreducible components of the equivalence classes are called the logarithmic strata; the collection of all strata the* logarithmic stratification.

We say Y is Saito-holonomic if the logarithmic stratification is locally finite, i.e. at every $\mathfrak{x} \in X$ there is an open $U \subseteq X$, $\mathfrak{x} \in U$, such that U intersects finitely many logarithmic strata. Equivalently, Y is Saito-holonomic if the dimension of $\{\mathfrak{x} \in X \mid rk_{\mathbb{C}}(\operatorname{Der}_X(-\log Y) \otimes \mathscr{O}_{X,\mathfrak{x}}/\mathfrak{m}_{X,\mathfrak{x}} = i\}$ is at most i.

Remark 2.2.4.(a) Pick $\mathfrak{x} \in X$ and consider its log stratum D with respect to $f = f_1 \cdots f_r$. We can find logarithmic derivations $\delta_1, \ldots, \delta_m$ at $\mathfrak{x}$, $m = \dim D$, that are $\mathbb{C}$-independent at $\mathfrak{x}$. Each δ_i also lies in $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_i)$. By Proposition 3.6 of [8] there exists a coordinate system $(x_1, \ldots, x_n)$ so that these generators can be written as $\delta_k = \frac{\partial}{\partial x_{n-m+k}} + \sum_{1 \leq j \leq n-m} g_{jk}(x) \frac{\partial}{\partial x_j}$, with the g_{jk} analytic functions defined near $\mathfrak{x}$.

(b) By Lemma 3.5 and Proposition 3.6 of [8], the same change of coordinates ϕ_F from 2.2.4.(a) fixes the last m coordinates and satisfies $\phi_F(x_1, \ldots, x_{n-m}, 0) = (x_1, \ldots, x_m, 0)$. Moreover, it simultaneously satisfies $f_i(\phi_F(x_1, \ldots, x_m)) = u_i(x_1, \ldots, x_m) f_i(x_1, \ldots, x_{n-m}, 0)$ where $u_i(x_1, \ldots, x_m)$ is a unit for $1 \leq i \leq m$.

(c) Now assume the logarithmic stratification is locally finite and the log stratum D of $\mathfrak{x}$ has dimension 0. So $D = \{\mathfrak{x}\}$. Since every other zero dimensional strata is disjoint from D, there exists an open $U \ni \mathfrak{x}$ such that $U \setminus \mathfrak{x}$ consists only of points whose logarithmic stratum are of positive dimension.

(d) By Lemma 3.4 of [8], for a divisor Y connected components of $X \setminus Y$ and $Y \setminus \operatorname{Sing}(Y)$ are logarithmic strata.

Example 2.2.5. Let $f = x(2x^2 + yz)$ and note that $\operatorname{Sing}(f) = \{z - \text{axis}\} \cup \{y - \text{axis}\}$. Since the Euler derivation $x\partial_x + y\partial_y + z\partial_z$ is a logarithmic derivation, the z-axis $\setminus\{0\}$ and the y-axis $\setminus\{0\}$ are logarithmic strata. Therefore f is Saito-holonomic.

Example 2.2.6. By Example 3.14 of [8], hyperplane arrangements are Saito-holonomic.

2.2.2 Generalized Liouville Ideals.

In Section 3 of [7], Walther defines the *Liouville ideal* L_f as the ideal in $\operatorname{gr}_{(0,1)}(\mathscr{D}_X)$ generated by the symbols $\operatorname{gr}_{(0,1)}(\operatorname{Der}_X(-\log_0 f)$. As $\operatorname{Der}_X(-\log_0 f) \subseteq \operatorname{ann}_{\mathscr{D}_X} f^s$, L_f represents the contribution of $\operatorname{Der}_X(-\log_0 f)$ to $\operatorname{gr}_{(0,1)}(\operatorname{ann}_{\mathscr{D}_X} f^s)$. When f is strongly Euler-homogeneous with strong Euler-homogeneity $E_{\mathfrak{x}}$, L_f is coordinate independent (see Remark 3.2 [7]) and $\operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[s]) \cdot L_{f,\mathfrak{x}}$ and $\operatorname{gr}_{(0,1)}(E_{\mathfrak{x}}) - s$ generate the simplest degree one elements of $\operatorname{gr}_{(0,1,1)}(\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[s]} f^s)$.

If we want to generalize this to F^S, there is no obvious inclusion between $\operatorname{Der}_X(-\log_0 f)$ and $\operatorname{ann}_{\mathscr{D}_X} F^S$. In fact, $\delta \in \operatorname{Der}_X(-\log_0 f)$ is in $\operatorname{ann}_{\mathscr{D}_X} F^S$ preciscely when $\delta \in \bigcap \operatorname{Der}_X(-\log_0 f_i)$. Trying to define a generalized Liouville ideal using $\bigcap \operatorname{Der}_X(-\log_0 f_i)$ would lose too many elements of $\operatorname{Der}_X(-\log_0 f)$.

Definition 2.2.8. *Recall the isomorphism of $\mathscr{O}_X$-modules from Proposition 2.2.3*

$$\psi_F : \operatorname{Der}_X(-\log f) \xrightarrow{\simeq} \theta_F,$$

which is given by

$$\psi_F(\delta) = \delta - \sum s_k \frac{\delta \bullet f_k}{f_k}.$$

This restricts to a map of sheaves of $\mathscr{O}_X$-modules:

$$\psi_F : \operatorname{Der}_X(-\log_0 f) \hookrightarrow \theta_F.$$

Let the generalized Liouville ideal L_F *by the ideal in* $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_X[S])$ *generated by the symbols of* $\psi_F(\mathrm{Der}_X(-\log_0 f))$ *in the associated graded ring:*

$$L_F := \mathrm{gr}_{(0,1,1)}(\mathscr{D}_X[S]) \cdot \mathrm{gr}_{(0,1,1)}(\psi_F(\mathrm{Der}_X(-\log_0 f))).$$

We also define

$$\begin{aligned}\widetilde{L_F} :&= \mathrm{gr}_{(0,1,1)}(D_X[S]) \cdot \mathrm{gr}_{(0,1,1)}(\theta_F) \\ &= \mathrm{gr}_{(0,1,1)}(D_X[S]) \cdot \mathrm{gr}_{(0,1,1)}(\psi_F(\mathrm{Der}_X(-\log(f))).\end{aligned}$$

Remark 2.2.7. With E_x a Euler-homogeneity for f at $\mathfrak{x}$, the $\mathscr{O}_{X,\mathfrak{x}}$-module direct sum $\mathrm{Der}_{X,\mathfrak{x}}(-\log f) \simeq \mathrm{Der}_{X,\mathfrak{x}}(-\log_0 f) \oplus \mathscr{O}_{X,\mathfrak{x}} \cdot E_x\ L_{F,\mathfrak{x}}$, depends on the choice of defining equation for f. Following Remark 3.2 of [7], if the the divisor of f is strongly Euler-homogeneous, then the algebraic properties of $L_{F,\mathfrak{x}}$ and $\widetilde{L_{F,\mathfrak{x}}}$ are independent of the choice of local equation of $\mathrm{Div}(f)$.

(a) To this end, let x and x' denote two coordinates systems, $J = (\frac{\partial x'_j}{\partial x_i})$ the Jacobian matrix with rows i, columns j, ∂ and ∂' column vectors of partial differentials in the x and x' coordinates, respectively. Let $\nabla(g)$, $\nabla'(g)$ be the gradient, as a column vector, of g in the two coordinate systems. Finally, express a derivation δ in terms of the two coordinate systems as $\delta = c_\delta^T \partial = c_\delta'^T \partial'$, where c_δ, c'_δ, are column vectors of $\mathscr{O}_X$ functions representing the coefficients of the partials in the x and x' coordinates. Note that in x'-coordinates $c'_\delta = J^T c_\delta$.

(b) In x-coordinates $\psi_F(\delta) = c_\delta^T \partial - \sum_k s_k \frac{c_\delta^T \nabla(f_k)}{f_k}$. In x'-coordinates $\delta = c_\delta^T J \partial'$ and $\psi_F(\delta) = c_\delta^T J \partial' - \sum_k s_k \frac{c_\delta^T J \nabla'(f_k)}{f_k}$. Thus ψ_F commutes with coordinate change. (Note that strongly Euler-homogeneous is not needed here.)

(c) Suppose $E_\mathfrak{x}$ is a strong Euler-homogeneity at $\mathfrak{x} \in X$ for f. Recall from Remark 3.2 of [7] that for a unit $u \in \mathscr{O}_{X,\mathfrak{x}}$, the map $\alpha_u : \mathrm{Der}_{X,\mathfrak{x}}(-\log_0 f) \to \mathrm{Der}_{X,\mathfrak{x}}(-\log_0 uf)$ given by $\alpha_u(\delta) = \delta - \frac{\delta \bullet u}{u + E_\mathfrak{x} \bullet u} E_\mathfrak{x}$ is an $\mathscr{O}_{X,\mathfrak{x}}$-isomorphism that commutes with coordinate change. In particular, let $u = \prod_{1\leq i\leq r} u_i$ be a product of units and let $uF = (u_1 f_1, \ldots, u_r f_r)$. Then we have an $\mathscr{O}_{X,\mathfrak{x}}$-isomorphism

$$\psi_F \circ \alpha_u \circ \psi_F^{-1} : \psi_F(\mathrm{Der}_{X,\mathfrak{x}}(-\log_0 f)) \to \psi_F(\mathrm{Der}_{X,\mathfrak{x}}(-\log_0 uf))$$

that commutes with coordinate change.

(d) To be precise,

$$\begin{aligned}\psi_F \circ \alpha_u \circ \psi_F^{-1}(\delta - \sum_k s_k \frac{\delta \bullet f_k}{f_k}) = \delta - \frac{\delta \bullet u}{u + E_{\mathfrak{x}} \bullet u} E_{\mathfrak{x}} - \sum_k s_k \frac{\delta \bullet u_k}{u_k} \\ - \sum_k s_k \frac{\delta \bullet f_k}{f_k} + \sum_k s_k \frac{(\delta \bullet u)(E_{\mathfrak{x}} \bullet (u_k f_k))}{u_k f_k (u + E_{\mathfrak{x}} \bullet u)}.\end{aligned}$$

Note that $E_{\mathfrak{x}} \bullet (u_k f_k)$ is a multiple of f_k and $\delta \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f_k)$ so all these fractions make sense.

(e) Inspection reveals that the morphism of graded objectes induced by $\psi_F \circ \alpha_u \circ \psi_F^{-1}$ is an $\mathscr{O}_{X,\mathfrak{x}}[S]$-linear endomorphism β_u on $\mathrm{gr}_{(0,1,1)} \mathscr{D}_{X,\mathfrak{x}}[S]$, where

$$\begin{aligned}\beta_u(\mathrm{gr}_{(0,1,1)}(\partial)) = \mathrm{gr}_{(0,1,1)}(\partial) - \frac{\partial \bullet u}{u + E_{\mathfrak{x}} \bullet u} \mathrm{gr}_{(0,1,1)}(E_{\mathfrak{x}}) - \sum_k s_k \frac{\partial \bullet u_k}{u_k} \\ + \sum_k s_k \frac{(\partial \bullet u)(E_{\mathfrak{x}} \bullet (u_k f_k))}{u_k f_k (u + E_{\mathfrak{x}} \bullet u)}.\end{aligned}$$

Since the $\mathscr{O}_{X,\mathfrak{x}}$-linear endomorphism of $\mathrm{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}})$ given by $\mathrm{gr}_{(0,1)}(\partial) \to \mathrm{gr}_{(0,1)}(\partial) - \frac{\partial \bullet u}{u + E_{\mathfrak{x}} \bullet u} \mathrm{gr}_{(0,1)}(E_{\mathfrak{x}})$ is surjective and injective, β_u is as well. So $\beta_u(L_{F,\mathfrak{x}}) = L_{uF,\mathfrak{x}}$. It is clear by (d) that β_u commutes with coordinate change.

(f) Therefore for strongly-Euler-homogeneous f, the local algebraic properties of $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_X[S])/L_F$ are independent of the choice of local equations for the $f_1, \ldots, f_r$.

(g) It is also clear that α_u sends $E_{\mathfrak{x}}$, a strong Euler-homogeneity for f, to a strong Euler-homogeneity for uf and so $\beta_u(\widetilde{L_{F,\mathfrak{x}}}) = \widetilde{L_{uF,\mathfrak{x}}}$. Hence, if f is strongly Euler-homogeneous then the local properties of $\widetilde{L_F}$ do not depend on the defining equations of the f_k.

At the smooth points of f, L_F and $\widetilde{L_F}$ are well understood. First, a lemma:

Lemma 2.2.8. *Suppose $f = f_1 \cdots f_r$ has the Euler-homogeneity $E_{\mathfrak{x}}$ at $\mathfrak{x} \in X$. Let $F = (f_1, \ldots, f_r)$. Then*

$$\operatorname{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{x}})) \notin \mathfrak{m}_{\mathfrak{x}} \mathscr{O}_{X,\mathfrak{x}}[Y][S] \subseteq \mathscr{O}_{X,\mathfrak{x}}[Y][S] \simeq \operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]).$$

Proof. Working at $\mathfrak{x} \in X$ and letting $\widehat{f_k} = \prod_{j \neq k} f_j$:

$$f = E_{\mathfrak{x}} \bullet f = \sum_k (E_{\mathfrak{x}} \bullet f_k) \widehat{f_k} = \left(\sum_k \frac{E_{\mathfrak{x}} \bullet f_k}{f_k} \right) f.$$

So $1 = \sum \frac{E_{\mathfrak{x}} \bullet f_k}{f_k}$ in $\mathscr{O}_{X,\mathfrak{x}}$; thus there exists a j such that $\frac{E_{\mathfrak{x}} \bullet f_j}{f_j} \notin \mathfrak{m}_{\mathfrak{x}}$. As $\psi_F(E_{\mathfrak{x}}) = E_{\mathfrak{x}} + \sum s_k \frac{E_{\mathfrak{x}} \bullet f_k}{f_k}$ the claim follows after looking at the symbol $\operatorname{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{x}}))$. $\square$

Proposition 2.2.9. *Let $f = f_1 \cdots f_r$ be strongly Euler-homogeneous and let $F = (f_1, \ldots, f_r)$. Then locally at smooth points, L_F and $\widetilde{L_F}$ are prime ideals of dimension $n+r+1$ and $n + r$ respectively. Moreover, for any $\mathfrak{x} \in X$:*

$$\dim \operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) / L_{F,\mathfrak{x}} \geq n + r + 1;$$

$$\dim \operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) / \widetilde{L_{F,\mathfrak{x}}} \geq n + r.$$

Proof. Let $\mathfrak{x} \in X$ be a part of the smooth locus of f; fix coordinates and choose ∂_{x_i} such that $\partial_{x_i} \bullet f$ is a unit in $\mathscr{O}_{X,\mathfrak{x}}$. Then $\Gamma = \{\partial_{x_k} - \frac{\partial_{x_k} \bullet f}{\partial_{x_i} \bullet f} \partial_{x_i}\}_{k=1, k \neq i}^{n} \subseteq \operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f)$ is a set of $n-1$ linearly independent elements. Saito's Criterion (cf. page 270 of [8]) implies that Γ together with $E_{\mathfrak{x}}$, the strong Euler derivation, gives a free basis for $\operatorname{Der}_{X,\mathfrak{x}}(-\log f)$. Hence, Γ generates $\operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f)$ freely. As $\mathscr{O}_{X,\mathfrak{x}}[Y][S]/L_{F,\mathfrak{x}} \simeq \mathscr{O}_{X,\mathfrak{x}}[y_i][S]$, $L_{F,\mathfrak{x}}$ is a prime ideal of dimension $n + r + 1$.

By Lemma 2.2.8, and the choice of j outlined in its proof, there is a ring map

$$\mathscr{O}_{X,\mathfrak{x}}[Y][S] / \operatorname{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{x}})) \simeq \mathscr{O}_{X,\mathfrak{x}}[Y][s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_r].$$

Consider the image of $\mathrm{gr}_{(0,1,1)}(\psi_F(y_k - \frac{\partial_{x_k}\bullet f}{\partial_{x_i}\bullet f} y_i))$ (hereafter denoted with $\overline{(-)}$) in $\mathscr{O}_{X,\mathfrak{x}}[Y][S]/\,\mathrm{gr}_{(0,1,1)}(\psi_F(E_\mathfrak{x}))$. Since $E_\mathfrak{x}$ is a strong Euler-homogeneity, the coefficient of each y_k in $\mathrm{gr}_{(0,1,1)}(\psi_F(E_\mathfrak{x}))$ lies in $\mathfrak{m}_\mathfrak{x}$. Thus the coefficient of y_k in

$$\begin{aligned}\overline{\mathrm{gr}_{(0,1,1)}(\psi_F(y_k - \frac{\partial_{x_k}\bullet f}{\partial_{x_i}\bullet f} y_i))} &\in \mathscr{O}_{X,\mathfrak{x}}[Y][S]/\,\mathrm{gr}_{(0,1,1)}(\psi_F(E_\mathfrak{x})) \\ &\simeq \mathscr{O}_{X,\mathfrak{x}}[Y][s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_r].\end{aligned}$$

belongs to $\mathscr{O}_{X,\mathfrak{x}} \setminus \mathfrak{m}_\mathfrak{x}$. So as rings, $\mathscr{O}_{X,\mathfrak{x}}[Y]/\widetilde{L_{F,\mathfrak{x}}} \simeq \mathscr{O}_{X,\mathfrak{x}}[y_i][s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_r]$ and $\widetilde{L_{F,\mathfrak{x}}}$ is a prime ideal of dimension $n+r$.

Since the smooth points are dense, we get the desired inequalities. $\square$

Take a generator $\mathrm{gr}_{(0,1,1)}\left(\delta - \sum s_k \frac{\delta\bullet f_k}{f_k}\right)$, $\delta \in \mathrm{Der}_X(-\log_0 f)$, of $L_{F,\mathfrak{x}}$. Erasing the s_k-terms results in $\mathrm{gr}_{(0,1,1)}(\delta) = \mathrm{gr}_{(0,1)}(\delta) \in L_{f,\mathfrak{x}}$. This process is formalized by filtering $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ in such a way that the s_k-terms have degree 0 and then taking the initial ideal of $L_{F,\mathfrak{x}}$.

Definition 2.2.9. *It is well known that for an open $U \subseteq X$ with a fixed coordinate system $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_X(U)[S]) \simeq \mathscr{O}_X(U)[Y][S]$, where $y_i = \mathrm{gr}_{(0,1,1)}(\partial_{x_i})$. Grade this by the integral vector $(0,1,0) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^r$. For example the element $gY^{\boldsymbol{u}}S^{\boldsymbol{v}}$, where $\boldsymbol{u}, \boldsymbol{v}$ are nonnegative integral vectors and $g \in \mathscr{O}_U$, will have $(0,1,0)$-degree $\sum_j u_j$. Changing coordinate systems does not effect the number of y-terms so this extends to a grading on $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_X(U)[S])$.*

Define $\mathrm{in}_{(0,1,0)} L_F$ to be the initial ideal of the generalized Liouville ideal with respect to the $(0,1,0)$-grading. See Section 6 for details about initial ideals.

We now have three ideals: L_F, $\mathrm{in}_{(0,1,0)} L_F$, and $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_X[S]) \cdot L_f$, the ideal extension of L_f to $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_X[S])$. Proposition 2.6.6 shows how some nice properties of $\mathrm{in}_{(0,1,0)} L_F$ transfer to nice properties of L_F. The following construction will let us transfer nice properties of L_f, and consequently of $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_X[S]) \cdot L_f$, to nice properties of $\mathrm{in}_{(0,1,1)} L_F$.

Proposition 2.2.10. *Assume $f = f_1 \cdots f_r$ is strongly Euler-homogeneous and let $F = (f_1, \ldots, f_r)$. Consider $\operatorname{gr}_{(0,1,1)}(\mathscr{D}_X[S]) \cdot L_f$, the extension of the Liouville ideal to $\operatorname{gr}_{(0,1,1)}(\mathscr{D}_X[S])$. Then there is a surjection of rings:*

$$\frac{\operatorname{gr}_{(0,1,1)}(\mathscr{D}_X[S])}{\operatorname{gr}_{(0,1,1)}(\mathscr{D}_X[S]) \cdot L_f} \twoheadrightarrow \frac{\operatorname{gr}_{(0,1,1)}(\mathscr{D}_X[S])}{\operatorname{in}_{(0,1,0)} L_F}. \tag{2.2.1}$$

Proof. L_f is generated by the symbols of $\delta \in \operatorname{Der}_X(-\log_0 f)$ in $\operatorname{gr}_{(0,1)}(\mathscr{D}_X)$. Thinking of $\operatorname{gr}_{(0,1)}(\mathscr{D}_X) \subseteq \operatorname{gr}_{(0,1,1)}(\mathscr{D}_X[S])$, $\operatorname{gr}_{(0,1,1)}(\mathscr{D}_X[S]) \cdot L_f$ will have the generators $\operatorname{gr}_{(0,1,1)}(\delta)$. On the other hand L_F is locally generated by $\operatorname{gr}_{(0,1,1)}\left(\delta - \sum s_k \frac{\delta \bullet f_k}{f_k}\right)$ for $\delta \in \operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f)$. Each such generator has $(0,1,0)$-initial term $\operatorname{gr}_{(0,1,1)}(\delta)$. So $\operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot L_{f,\mathfrak{x}} \subseteq \operatorname{in}_{(0,1,0)} L_{F,\mathfrak{x}}$. □

Proposition 2.2.11. *Suppose $f = f_1 \cdots f_r$ is a strongly Euler-homogeneous divisor and let $F = (f_1, \ldots, f_r)$. Then the following data transfer from the Liouville ideal to the initial ideal of the generalized Liouville ideal:*

(a) If $\dim \operatorname{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}})/L_{f,\mathfrak{x}} = n + 1$, then

$$\dim \operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])/L_{F,\mathfrak{x}} = n + r + 1;$$

(b) If L_f is locally a prime ideal, then there is an isomorphism of rings

$$\frac{\operatorname{gr}_{(0,1,1)}(\mathscr{D}_X[S])}{\operatorname{gr}_{(0,1,1)}(\mathscr{D}_X[S]) \cdot L_f} \simeq \frac{\operatorname{gr}_{(0,1,1)}(\mathscr{D}_X[S])}{\operatorname{in}_{(0,1,0)} L_F};$$

(c) If L_f is locally Cohen–Macaulay and prime, then L_F is locally Cohen–Macaulay.

Proof. Because $\operatorname{gr}_{(0,1,1)}(\mathscr{D}_X[S]) \cdot L_f$ is the extension of L_f into a ring with new variables S, there are ring isomorphisms

$$\frac{\operatorname{gr}_{(0,1,1)}(D_X[S])}{\operatorname{gr}_{(0,1,1)}(D_X[S]) \cdot L_f} \simeq \frac{\mathscr{O}_X[Y][S]}{\mathscr{O}_X[Y][S] \cdot L_f} \simeq \frac{\mathscr{O}_X[Y]}{L_f}[S] \simeq \frac{\operatorname{gr}_{(0,1)}(\mathscr{D}_X)}{L_f}[S].$$

So if $\dim \operatorname{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}})/L_{f,\mathfrak{x}} = n + 1$, $\dim \operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])/\operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot L_f = n + r + 1$. Similarly if $L_{f,\mathfrak{x}}$ is prime, then $\operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot L_{f,\mathfrak{x}}$ is prime.

The map (2.2.1) gives $n+r+1 \geq \dim \mathrm{in}_{(0,1,0)} L_{F,\mathfrak{x}}$. By Proposition 2.6.6 and Remark 2.6.7, $\dim \mathrm{in}_{(0,1,0)} L_{F,\mathfrak{x}} \geq \dim L_{F,\mathfrak{x}}$. Proposition 2.2.9 gives $\dim L_{F,\mathfrak{x}} \geq n+r+1$, proving (a). As for (b), the hypotheses guarantee that the map (2.2.1) is locally a surjection from a domain to a ring of the same dimension and hence an isomorphism. To prove (c), recall Proposition 2.6.6 and Remark 2.6.7 show that if $\mathrm{in}_{(0,1,0)} L_F$ is locally Cohen–Macaulay, then L_F is locally Cohen–Macaulay. So (b) implies (c). $\square$

2.2.3 Primality of $L_{F,\mathfrak{x}}$ and $\widetilde{L_{F,\mathfrak{x}}}$.

Now we show that when f is strongly Euler-homogeneous and Saito-holonomic and F a decomposition of f, that the conclusions of Proposition 2.2.11 imply L_F and $\widetilde{L_F}$ are locally prime. The method of argument relies on the Saito-holonomic condition: we use the coordinate transformation of Remark 2.2.4 to reduce the dimension of logarithmic stratum.

Our first proof mirrors the proof of Theorem 3.17 in [7]. Because our situation is a little more technical and because we end up using this argument again in Theorem 2.2.2, we give full details.

Theorem 2.2.1. *Suppose $f = f_1 \cdots f_r$ is strongly Euler-homogeneous and Saito-holonomic and let $F = (f_1, \ldots, f_r)$. If L_f is locally Cohen–Macaulay and prime of dimension $n+1$, then L_F is locally Cohen–Macaulay and prime of dimension $n+r+1$. In particular, this happens when f is strongly Euler-homogeneous, Saito-holonomic, and tame.*

Proof. If we prove the second sentence, the third will follow by Theorem 3.17 and Remark 3.18 of [7]. By Proposition 2.2.11, the only thing to prove in the second sentence is that L_F is locally prime. To do this we induce on the dimension of X. If $\dim X$ is 1, then $L_{F,\mathfrak{x}} = 0$ and the claim is trivially true.

So we may assume the claim holds for all X with dimension less than n. Suppose $\mathfrak{x}$ belong to a logarithmic stratum σ of dimension k. If $k = n$, then by Proposition 2.2.9 and Remark 2.2.4, $L_{F,\mathfrak{x}}$ is prime. Now assume $0 < k < n$. By Remark 2.2.4, we can find a coordinate transformation near $\mathfrak{x}$ such that each $f_i = u_i g_i$, where u_i is a unit near $\mathfrak{x}$ and $g_i(x_1, \ldots, x_n) = f_i(x_1, \ldots, x_{n-k}, 0, \ldots, 0)$, cf. 3.6 of [8]. By Remark 2.2.7, $L_{F,\mathfrak{x}}$ is well-behaved under coordinate transformations and multiplication by units, so we may instead prove the claim for $L_{G,\mathfrak{x}}$, where $g = \prod g_i$ and $G = (g_1, \ldots, g_r)$. Let X' be the space of the first $n-k$ coordinates and $\mathfrak{x}\prime$ the first $n-k$ coordinates of $\mathfrak{x}$. When viewing g_i' as an element of $\mathscr{O}_{X',\mathfrak{x}\prime}$, call it g_i'. Let $g' = \prod g_i'$ and $G' = (g_1', \ldots, g_r')$. Because strongly Euler-homogeneous descends from X to X', see Remark 2.8 in [7], local properties $L_{G'}$ do not depend on the choice of the defining equations for the g_i. Now

$$\operatorname{Der}_{X,\mathfrak{x}}(-\log g) = \mathscr{O}_{X,\mathfrak{x}} \cdot \operatorname{Der}_{X',\mathfrak{x}\prime}(-\log g') + \sum_{1 \leq j \leq k} \mathscr{O}_{X,\mathfrak{x}} \cdot \partial_{x_{n-k+j}},$$

where $\partial_{x_{n-k+j}} \in \operatorname{Der}_{X,\mathfrak{x}}(-\log_0 g_i)$ for each $1 \leq j \leq k$ and $1 \leq i \leq r$. Therefore $\mathscr{O}_{X,\mathfrak{x}}[y_1, \ldots, y_n][S]/L_{G,\mathfrak{x}} \simeq \mathscr{O}_{X,\mathfrak{x}}[y_1, \ldots, y_{n-k}][S]/L_{G',\mathfrak{x}\prime}$. Since Saito-holonomicity descends to g', see 3.5 and 3.6 of [8] and Remark 2.6 of [7], we may instead prove the claim for X' and $L_{G',\mathfrak{x}\prime}$. Since $\dim X' < \dim X$, the induction hypothesis proves the claim.

So we may assume σ has dimension 0. By Remark 2.2.4, there is some open $U \ni \mathfrak{x}$, such that $\mathfrak{x} = U \cap \sigma$ and $U \setminus \mathfrak{x}$ consists of points whose logarithmic strata are of strictly positive dimension. The discussion above implies L_F is prime at all points of $U \setminus \mathfrak{x}$.

Let $\pi : \operatorname{Spec} \mathscr{O}_X[Y][S] \twoheadrightarrow \operatorname{Spec} \mathscr{O}_X$ be the map induced by $\mathscr{O}_X \hookrightarrow \mathscr{O}_X[Y][S]$. If L_F is not prime at $\mathfrak{x}$, it must have more than one irreducible component that intersects $\pi^{-1}(\mathfrak{x})$. As L_F is prime at points of $U \setminus \mathfrak{x}$, if $L_{F,x}$ is not prime it must have an "extra" irreducible component $\mathrm{V}(\mathfrak{q})$ lying inside $\pi^{-1}(\mathfrak{x})$. By assumption, $L_{F,\mathfrak{x}}$ is Cohen–Macaulay of dimension $n+r+1$ and so $\mathrm{V}(\mathfrak{q})$ has dimension $n+r+1$. But $\pi^{-1}(\mathfrak{x})$ has dimension $n+r$. Thus $L_{F,\mathfrak{x}}$ is prime completing the induction. □

Proposition 2.2.12. *Suppose $f = f_1 \cdots f_r$ is a strongly Euler-homogeneous divisor and let $F = (f_1, \ldots, f_r)$. If L_F is locally prime, Cohen–Macaulay, and of dimension $n+r+1$, then $\widetilde{L_F}$ is locally Cohen–Macaulay of dimension $n+r$. In particular, this happens when f is strongly Euler-homogeneous, Saito-holonomic, and tame.*

Proof. Let $E_{\mathfrak{x}}$ be a strong Euler-homogeneity and consider $\operatorname{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{x}}))$, which is $(0,1,1)$-homogeneous of degree 1. The generalized Liouville ideal is generated by the elements

$\psi_F(\mathrm{Der}_{X,\mathfrak{x}}(-\log_0 f)$. If $\mathrm{gr}_{(0,1,1)}(\psi_F(E_\mathfrak{x})) \in L_{F,\mathfrak{x}}$, then $\psi_F(E_\mathfrak{x}) \in \psi_F(\mathrm{Der}_{X,\mathfrak{x}}(-\log_0 f)$. This is impossible since $E_\mathfrak{x} \notin \mathrm{Der}_{X,\mathfrak{x}}(-\log_0 f)$.

Locally, $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])/\widetilde{L_{F,\mathfrak{x}}}$ is obtained from $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])/L_{F,\mathfrak{x}}$ by modding out by a non-zero element, which must be regular. So $\widetilde{L_F}$ is locally Cohen–Macaulay of dimension at most $n+r$. That locally the dimension $\widetilde{L_F}$ is $n+r$ follows from the dimension inequality in Proposition 2.2.9.

The final sentence is true by Theorem 2.2.1. □

This section's first main result is that $\widetilde{L_{F,\mathfrak{x}}}$ is locally prime when f is strongly Euler-homogeneous, Saito-holonomic, and tame. The strategy is the same as in Theorem 2.2.1. Under much stricter hypotheses, and in his language, Maisonobe shows in Proposition 3 of [11], that $\widetilde{L_F}$ is locally prime. Experts will note that we recycle the part of his argument where he reduced dimension in our proof.

Theorem 2.2.2. *Assume that $f = f_1 \cdots f_r$ is strongly Euler-homogeneous and Saito-holonomic and let $F = (f_1, \ldots, f_r)$. If $\widetilde{L_F}$ is locally Cohen–Macaulay of dimension $n+r$, then $\widetilde{L_F}$ is locally prime. In particular, $\widetilde{L_F}$ is locally prime, Cohen–Macaulay, and of dimension $n+r$ when f is strongly Euler-homogeneous, Saito-holonomic, and tame.*

Proof. By Proposition 2.2.12, it suffices to prove the first claim. The proof follows the inductive argument of Theorem 2.2.1 with a slight modification at the end.

If $\dim X$ is 1, then $\widetilde{L_{F,\mathfrak{x}}}$ is generated by $\psi_F(E_\mathfrak{x})$, $E_\mathfrak{x}$ a strong Euler-homogeneity. By Lemma 2.2.8, $\mathscr{O}_{X,\mathfrak{x}}[Y][S]/\widetilde{L_{F,\mathfrak{x}}} \simeq \mathscr{O}_{X,\mathfrak{x}}[Y][s_1, \ldots s_{j-1}, s_{j+1}, \ldots s_r]$.

Now assume the claim holds for all X with dimension less than n and $\mathfrak{x}$ belongs to a logarithmic stratum σ of dimension k. If $k = n$, then $\widetilde{L_{F,\mathfrak{x}}}$ is prime by Proposition 2.2.9. If $0 < k < n$ we can make the same coordinate transformation as in Theorem 2.2.1 and instead prove $\widetilde{L_G}$ is locally prime where $g_i(x) = f_i(x_1, \ldots, x_{n-k}, 0, \ldots, 0)$. Using the notation of Theorem 2.2.1, X' is strongly Euler-homogeneous and Saito-holonomic and

$$\mathrm{Der}_{X,\mathfrak{x}}(-\log g) = \mathscr{O}_{X,\mathfrak{x}} \cdot \mathrm{Der}_{X',\mathfrak{x}'}(-\log g') + \sum_{1 \le j \le k} \mathscr{O}_{X,\mathfrak{x}} \cdot \partial_{x_{n-k+j}},$$

where $\partial_{x_{n-k+j}} \in \mathrm{Der}_{X,\mathfrak{x}}(-\log_0 g_i)$ for each $1 \leq j \leq k$ and $1 \leq i \leq r$. Moreover, the strong Euler-homogeneity $E_{\mathfrak{x}'}$ for g' at $\mathfrak{x}' \in X'$ can be viewed as a strong Euler-homogeneity for g at $\mathfrak{x} \in X$. Therefore $\mathscr{O}_{X,\mathfrak{x}}[y_1, \ldots, y_n][S]/\widetilde{L_{G,\mathfrak{x}}} \simeq \mathscr{O}_{X,\mathfrak{x}}[y_1, \ldots, y_{n-k}][S]/\widetilde{L_{G',\mathfrak{x}'}}$. Since $\dim X' < \dim X$, the induction hypothesis shows that $\widetilde{L_{G',\mathfrak{x}'}}$ is prime.

So we may assume σ has dimension 0. Let $\pi : \operatorname{Spec} \mathscr{O}_{X,\mathfrak{x}}[Y][S] \twoheadrightarrow \operatorname{Spec} \mathscr{O}_X$ be the map induced by $\mathscr{O}_X \hookrightarrow \mathscr{O}_{X,\mathfrak{x}}[Y][S]$. Reasoning as in Theorem 2.2.1, we deduce that if $\widetilde{L_{F,\mathfrak{x}}}$ is not prime then there exists a irreducible component $\mathrm{V}(\mathfrak{q})$ of $\widetilde{L_{F,\mathfrak{x}}}$ contained entirely in $\pi^{-1}(\mathfrak{x})$.

By assumption, $\widetilde{L_{F,\mathfrak{x}}}$ is Cohen–Macaulay of dimension $n+r$ and $\mathrm{V}(\mathfrak{q})$ has dimension $n+r$. Let $E_{\mathfrak{x}}$ be the strong Euler-homogeneity at $\mathfrak{x}$. Then $\mathrm{V}(\mathfrak{q}) \subseteq \pi^{-1}(\mathfrak{x}) \cap \mathrm{V}(\mathrm{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{x}})))$. We will show that the intersection of $\mathrm{V}(\mathrm{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{x}})))$ and $\pi^{-1}(\mathfrak{x})$ is proper; since the dimension of $\pi^{-1}(\mathfrak{x})$ is $n+r$ this will show that $\mathrm{V}(\mathfrak{q})$, which we know is of dimension $n+r$, is contained in a closed set of strictly smaller dimension. Therefore no such $\mathfrak{q}$ exists and $\widetilde{L_{F,\mathfrak{x}}}$ is prime.

Recall $\mathrm{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{x}})) = \mathrm{gr}_{(0,1,1)}(E_{\mathfrak{x}}) - \sum \frac{E_{\mathfrak{x}} \bullet f_k}{f_k} s_k$. Lemma 2.2.8 proves that there exists an index j such that $\frac{E_{\mathfrak{x}} \bullet f_j}{f_j} \notin \mathfrak{m}_{\mathfrak{x}}$. So there is a closed point in $\pi^{-1}(\mathfrak{x})$ that does not lie in $\mathrm{V}(\mathrm{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{x}})))$. In particular, the intersection of $\mathrm{V}(\mathrm{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{x}})))$ and $\pi^{-1}(\mathfrak{x})$ is proper and the inductive step is complete. $\square$

2.2.4 The $\mathscr{D}_X[S]$-annihilator of F^S.

Let $\mathrm{Jac}(f)$ be the Jacobian ideal of f. In a given coordinate system, there is a natural $\mathscr{O}_{X,\mathfrak{x}}$-linear map

$$\phi_f : \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[s]) \twoheadrightarrow \mathrm{Sym}_{\mathscr{O}_{X,\mathfrak{x}}}(\mathrm{Jac}(f)) \twoheadrightarrow R(\mathrm{Jac}(f))$$

given by

$$s \mapsto ft \text{ and } \mathrm{gr}_{(0,1,1)}(\partial_{x_k}) \mapsto (\partial_{x_k} \bullet f)t.$$

Its kernel contains $\mathrm{gr}_{(0,1)}(\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[s]} f^s)$. (See Section 1.3 in [19] for details.) So we have the containments

$$L_{f,\mathfrak{x}} + \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[s]) \cdot \mathrm{gr}_{(0,1,1)}(E_{\mathfrak{x}} - s) \subseteq \mathrm{gr}_{(0,1)}(\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[s]} f^s) \subseteq \ker(\phi_f)$$

and equality will hold throughout if $L_{f,\mathfrak{x}}+\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,x}[s])\cdot\mathrm{gr}_{(0,1,1)}(E_{\mathfrak{x}}-s)$ agrees with $\ker(\phi_f)$.

This motivates our analysis of $\mathrm{ann}_{\mathscr{D}_X[S]} F^S$: we will construct a map ϕ_F from $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_X[S])$ into a Rees-algebra like object and squeeze $\mathrm{gr}_{(0,1,1)}(\mathrm{ann}_{\mathscr{D}_X[S]} F^S)$ between $\widetilde{L_F}$ and $\ker(\phi_F)$.

Definition 2.2.10. *Let* $\mathrm{Jac}(f_i)$ *be the Jacobian ideal of* f_i *and consider the multi-Rees algebra* $R(\mathrm{Jac}(f_1),\ldots,\mathrm{Jac}(f_r))$ *associated to these* r *Jacobian ideals. Consider the* $\mathscr{O}_{X,\mathfrak{x}}$*-linear map*

$$\phi_F : \mathrm{gr}_{(0,1,1)}(\mathscr{D}_X[S]) \to R(\mathrm{Jac}(f_1),\ldots,\mathrm{Jac}(f_r)) \subseteq \mathscr{O}_X[S]$$

given, having fixed local coordinates on U*, by*

$$y_i \mapsto \sum_k \frac{f}{f_k}(\partial_{x_i} \bullet f_k)s_k \text{ and } s_k \mapsto f s_k.$$

Proposition 2.2.13. *Let* $f = f_1 \cdots f_r$ *and* $F = (f_1,\ldots,f_r)$*. Then*

$$\mathrm{gr}_{(0,1,1)}(\mathrm{ann}_{\mathscr{D}_X[S]} F^S) \subseteq \ker(\phi_F).$$

Proof. It is enough to show this locally, so take $P \in \mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} F^S$ of order ℓ under the $(0,1,1)$-filtration. For any Q of order ℓ it is always true that $f^\ell Q \bullet F^S \in \mathscr{O}_{X,\mathfrak{x}}[S]F^S$. Any time a partial is applied to gF^S, a s-term only comes out of the product rule when the partial is applied to F^S. A straightforward calculation shows that the S-lead term of $f^\ell PF^S$ is exactly $\phi_F(\mathrm{gr}_{(0,1,1)}(P))F^S$. Since $f^\ell P$ annihilates F^S, we conclude $\mathrm{gr}_{(0,1,1)}(P) \in \ker(\phi_F)$. ∎

Proposition 2.2.14. $\ker(\phi_F)$ *is a prime ideal of dimension* $n+r$*.*

Proof. It is prime. Since Rees rings are domains, to count dimension we squeeze $\phi_F(\mathrm{gr}_{(0,1,1)}(\mathscr{D}_X[S]))$ between two well-behaved multi-Rees algebras: $R((f),\ldots,(f))$ and

$R(\mathrm{Jac}(f_1), \ldots, \mathrm{Jac}(f_r))$ (the first multi-Rees algebra is built using r copies of (f)). As the latter is the target of ϕ_F and $\phi_F(s_i) = f s_i$ this is easy:

$$R((f), \ldots, (f)) \subseteq \phi_F(\mathrm{gr}_{(0,1,1)} \mathscr{D}_X[S]) \subseteq R(\mathrm{Jac}(f_1), \ldots, \mathrm{Jac}(f_r))$$

Moreover, the dimension of a multi-Rees algebra is well known: $R(I_1, \ldots, I_r) = r+$ the dimension of the ground ring.

So $\phi_F(\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]))$ is a domain squeezed between subrings of $\mathscr{O}_{X,\mathfrak{x}}[S]$ of dimension $n + r$. The result then follows by the following lemma:

Lemma 2.2.15. *Let $R \subseteq A \subseteq B \subseteq C \subseteq R[X]$ be finitely generated, graded R-algebras, whose gradings are inherited from the standard grading on $R[X]$. Assume that R is a universally caternary Notherian domain. If $\dim A = \dim C$, then $\dim A = \dim B = \dim C$.*

Proof. Claim: if $\mathfrak{m}^\star$ is a graded maximal ideal of A, then $\mathfrak{m}^\star B \neq B$. We prove the contrapositive. So assume $\mathfrak{m}^\star B = B$. Then $\mathfrak{m}^\star R[X] = R[X]$. Write $\mathfrak{m}^\star = (a_1, \ldots, a_\ell)$ in terms of homogeneous generators $a_i \in A$ and find $r_1, \ldots, r_n$ in $R[X]$ such that $1 = \sum r_i a_i$. Since the degree of 1 is zero, we can assume either r_i and a_i are both degree 0 or $r_i = 0$. Thus $1 = \sum r_i a_i$ occurs in $\mathfrak{m}^\star \cap R$ and so $\mathfrak{m}^\star = A$, a contradiction.

Now we argue using a version of Nagata's Altitude Formula (see [20] Theorem 13.8): $\dim(B_\mathfrak{q}) = \dim(A_\mathfrak{p}) + \dim(Q(A) \otimes_A B)$, for $\mathfrak{q} \in \operatorname{Spec} B$ maximal with respect to the property $\mathfrak{q} \cap A = \mathfrak{p}$. Since B is a finitely generated A-algebra, and tensors are right exact, $Q(A) \otimes_A B$ is a finitely generated $Q(A)$-algebra. Thus $\dim(Q(A) \otimes_A B) = \mathrm{trdeg}_{Q(A)}(Q(A) \otimes_A B) = \mathrm{trdeg}_{Q(A)} Q(Q(A) \otimes_A B) = \mathrm{trdeg}_{Q(A)} Q(B) = \mathrm{trdeg}_A B$. Similar statements hold for the other pairs $A \subseteq C$ and $B \subseteq C$.

Let $\mathfrak{m} \in \operatorname{Spec} A$ such that $\dim(A_\mathfrak{m}) = \dim(A)$. By the claim in the first paragraph (so assuming $\mathfrak{m}$ is graded if necessary), we can find $\mathfrak{q} \in \operatorname{Spec} C$ maximal with respect to the property $\mathfrak{q} \cap A = \mathfrak{m}$. So $\dim(C_\mathfrak{q}) = \dim(A_\mathfrak{m}) + \mathrm{trdeg}_A C$. Therefore $\dim(C) \geq \dim(A) + \mathrm{trdeg}_A C$ and hence $\mathrm{trdeg}_A C = 0$. Since we are looking at algebras finitely generated over the appropriate subring, transcendence degree is additive. So $0 = \mathrm{trdeg}_A B$ and $0 = \mathrm{trdeg}_B C$.

Let $\mathfrak{m} \in \operatorname{Spec} A$ with $\dim(A_{\mathfrak{m}}) = \dim(A)$, as before. Again, using the claim, select $\mathfrak{p} \in \operatorname{Spec} B$ maximal with respect to the property $\mathfrak{p} \cap A = \mathfrak{m}$. So $\dim(B_{\mathfrak{p}}) = \dim(A_{\mathfrak{m}}) + \operatorname{trdeg}_A B$; hence $\dim(B) \geq \dim(A)$. Argue similarly for $B \subseteq C$ to determine $\dim(C) \geq \dim(B)$. This ends the proof. $\square$

The following is an analogous statement to Corollary 3.23 in [7]:

Corollary 2.2.16. *There is the containment*

$$\widetilde{L_F} \subseteq \operatorname{gr}_{(0,1,1)}(\operatorname{ann}_{\mathscr{D}_X[S]} F^S) \subseteq \ker(\phi_F).$$

If f is strongly Euler-homogeneous, Saito-holonomic, and tame then all three ideals are equal.

Proof. The containments follow from the construction of $\widetilde{L_F}$ and Proposition 2.2.13. They are equalities when f is suitably nice because, by Theorem 2.2.2 and Proposition 2.2.14, at each $\mathfrak{x} \in X$ the outer ideals are prime of the same dimension. $\square$

Because $\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_F \subseteq \operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} F^S$, we can use Corollary 2.2.16 and a type of Gröbner basis argument to prove:

Theorem 2.2.3. *If $f = f_1 \cdots f_r$ is strongly Euler-homogeneous, Saito-holonomic, and tame and if $F = (f_1, \ldots, f_r)$, then the $\mathscr{D}_X[S]$-annihilator of F^S is generated by derivations, that is*

$$\operatorname{ann}_{\mathscr{D}_X[S]} F^S = \mathscr{D}_X[S] \cdot \theta_F.$$

Proof. Take $P \in \operatorname{ann}_{\mathscr{D}_X[S]} F^S$ of order k under the total order filtration. By Corollary 2.2.16, there exist $L_1, \ldots, L_k \in \theta_F$, $n_1, \ldots, n_k \in \mathscr{O}_X[Y][S]$ such that

$$\operatorname{gr}_{(0,1,1)}(P) = \sum n_i \cdot \operatorname{gr}_{(0,1,1)}(L_i).$$

Since $\operatorname{gr}_{(0,1,1)}(P)$ is homogeneous of degree k and $\operatorname{gr}_{(0,1,1)}(L_i)$ is homogeneous of degree 1, we may assume the n_i are homogeneous. For each i select $N_i \in \mathscr{D}_X[S]$ such that $n_i = \operatorname{gr}_{(0,1,1)}(N_i)$. Consequently, $P - \sum N_i \cdot L_i$ has order (under the total order filtration) less than k and lies in $\operatorname{ann}_{\mathscr{D}_X[S]} F^S$. Since $\mathscr{O}_X[S] \cap \operatorname{ann}_{\mathscr{D}_X[S]} F^S = 0$, an induction argument shows $P \in \mathscr{D}_X[S] \cdot \theta_F$. $\square$

Corollary 2.2.17. *Let $f = f_1 \cdots f_r \in \mathbb{C}[x_1, \ldots, x_n]$, where each $f_k \in \mathbb{C}[x_1, \ldots, x_n]$, and let $F = (f_1, \ldots, f_r)$. If f is strongly Euler-homogeneous, Saito-holonomic, and tame, then the $\mathscr{D}_X[S]$-annihilator of F^S is generated by derivations, that is*

$$\operatorname{ann}_{\mathscr{D}_X[S]} F^S = \mathscr{D}_X[S] \cdot \theta_F.$$

More generally, if X is the analytic space associated to a smooth $\mathbb{C}$-scheme and if f and $F = (f_1, \ldots, f_r)$ are algebraic, then the conclusion of Theorem 2.2.3 holds in the algebraic category.

Proof. This follows from Theorem 2.2.3 and the fact algebraic functions have algebraic derivatives and hence algebraic syzygies. See Theorem 3.26 and Remark 2.11 in [7] for more details. $\square$

2.2.5 Comparing Different Factorizations of f

Definition 2.2.11. *Consider the functional equation*

$$b_{f,\mathfrak{x}}(s) f^s = P f^{s+1}$$

where $b_{f,\mathfrak{x}}(s) \in \mathbb{C}[s]$ and $P \in \mathscr{D}_{X,\mathfrak{x}}[s]$. Let $B_{f,\mathfrak{x}}$ be the ideal in $\mathbb{C}[s]$ generated by all such $b_{f,\mathfrak{x}}(s)$, that is the ideal generated by the Bernstein–Sato polynomial*. We may write $B_{f,\mathfrak{x}} = (\mathscr{D}_{X,x}[s] \cdot f + \operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[s]} f^s) \cap \mathbb{C}[s]$. Then the variety $\mathrm{V}(B_{f,\mathfrak{x}})$ consists of the roots of the Bernstein–Sato polynomial.*

In the multivariate situation we may consider functional equations of the form

$$b_{F,\mathfrak{x}}(S) F^S = P F^{S+1}$$

where $b_{F,\mathfrak{x}}(S) \in \mathbb{C}[S]$ and $P \in \mathscr{D}_{X,\mathfrak{x}}[S]$. Just as above, the set of all such $b_{F,\mathfrak{x}}(S)$ form an ideal $B_{F,\mathfrak{x}} = (\mathscr{D}_{X,x}[S] \cdot f + \operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} F^S) \cap \mathbb{C}[S]$. The variety $\mathrm{V}(B_{F,\mathfrak{x}})$ is called the Bernstein–Sato variety *of F.*

It would be interesting to compare $\mathrm{V}(B_{F,\mathfrak{x}})$ and $\mathrm{V}(B_{G,\mathfrak{x}})$ where F and G correspond to two different factorizations of f. The following is a particular case of Lemma 4.20 of [6]:

Proposition 2.2.18. (Lemma 4.20 of [6]) Suppose that $f = f_1 \cdots f_r$ is strongly Euler-homogeneous, Saito-holonomic, and tame. Let $\mathbb{C}[S] = \mathbb{C}[s_1, \ldots, s_r]$, $F = (f_1, \ldots, f_r)$, and $G = (f_1, \ldots, f_{r-2}, f_{r-1}f_r)$. Then

$$B_{F,\mathfrak{x}} + \mathbb{C}[S] \cdot (s_{r-1} - s_r) \subseteq \mathbb{C}[S] \cdot B_{G,\mathfrak{x}} + \mathbb{C}[S] \cdot (s_{r-1} - s_r).$$

In particular, let $\Delta : \mathbb{C} \mapsto \mathbb{C}^r$ be the diagonal embedding. Then $\Delta(\mathrm{V}(B_{f,\mathfrak{x}})) \subseteq \mathrm{V}(B_{F,\mathfrak{x}})$.

Under the hypotheses of Theorem 2.2.3, on the level of annihilators we obtain a more precise statement:

Proposition 2.2.19. *Suppose $f = f_1 \cdots f_r$ is strongly Euler-homogeneous, Saito-holonomic, and tame. Let $F = (f_1, \ldots, f_r)$ and $G = (f_1, \ldots, f_{r-2}, f_{r-1}f_r)$. Then there is an isomorphism of rings:*

$$\frac{\mathscr{D}_X[s_1, \ldots, s_r]}{\operatorname{ann}_{\mathscr{D}_X[s_1,\ldots,s_r]} \mathscr{D}_X F^S + (s_{r-1} - s_r)} \simeq \frac{\mathscr{D}_X[s_1, \ldots, s_{r-1}]}{\operatorname{ann}_{\mathscr{D}_X[s_1,\ldots,s_{r-1}]} \mathscr{D}_X[s_1, \ldots, s_{r-1}]G^S}.$$

Proof. This follows from Theorem 2.2.3, the definition of $\psi_{F,\mathfrak{x}}(\delta)$ for δ a logarithmic derivation, and a straightforward computation using the product rule. ∎

Remark 2.2.20. Let $F = (f_1, \ldots, f_r)$ correspond to a factorization of f where f is strongly Euler-homogeneous, Saito-holonomic, and tame. For $a \in \mathbb{C}$, $\mathscr{D}_{X,\mathfrak{x}}[S] \cdot (s_1 - a, \ldots, s_r - a) = \mathscr{D}_{X,\mathfrak{x}}[S] \cdot (s_1 - s_2, \ldots, s_{r-1} - s_r, s_r - a)$. By Proposition 2.2.19, there is a ring isomorphism $\mathscr{D}_{X,\mathfrak{x}}[S]F^S/(s_1 - a, \ldots, s_r - a) \cdot \mathscr{D}_{X,\mathfrak{x}}[S]F^S \simeq \mathscr{D}_{X,\mathfrak{x}}[s]f^s/(s - a) \cdot \mathscr{D}_{X,\mathfrak{x}}[s]f^s$. Using this fact we propose in Remark 2.3.2 a more precise way to analyze the diagonal embedding of Proposition 2.2.18.

2.2.6 Hyperplane Arrangements.

Finally let us turn to the algebraic setting and particular to *central hyperplane arrangements* $\mathcal{A} \subseteq \mathbb{C}^n = X$ whose defining equations are given by $f_{\mathcal{A}} = \prod L_i$, where the $L_i \in \mathbb{C}[x_1, \ldots, x_n]$ are homogeneous polynomials of degree 1. A central hyperplane arrangement is *indecomposable* if there is no choice of coordinates $t_1 \sqcup t_2$, t_1 and t_2 disjoint, such that $f_{\mathcal{A}} = g_1(t_1)g_2(t_2)$. Central hyperplane arrangements are strongly Euler-homogeneous and Saito-holonomic, cf. examples 2.2.2, 2.2.6.

Write D_n for the n^{th} Weyl Algebra $\mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$. Let $F = (f_1, \ldots, f_r)$ be some decomposition of $f_{\mathcal{A}}$ into factors. Construct the $D_n[s]$-module ($D_n[S]$-module) generated by the symbol f^s (F^S) in an entirely similar way as in the analytic setting. Furthermore, define the roots of the Bernstein–Sato polynomial B_f and the Bernstein–Sato variety B_F just as before. For an algebraic f equipped with an algebraic decomposition F, B_f and B_F agree with the analytic versions because algebraic functions have algebraic derivatives and syzygies.

In [6], Budur makes the following conjecture:

Conjecture 2.2.21. (Conjecture 3 in [6]) *Let $\mathcal{A}$ be a central, essential, indecomposable hyperplane arrangement. Factor $f_{\mathcal{A}} = f_1 \cdots f_r$, where each factor f_k is of degree d_k and the f_k are not necessarily reduced, and let $F = (f_1, \ldots, f_r)$. Then*

$$\{d_1 s_1 + \cdots + d_r s_r + n = 0\} \subseteq \mathrm{V}(B_F).$$

This conjecture is related to the Topological Multivariable Strong Monodromy Conjecture, see Conjecture 2.1.2, for hyperplane arrangements, which claims that the polar locus of the topological zeta function of $F = (f_1, \ldots, f_r)$ is contained in $\mathrm{V}(B_{F,0})$. In Theorem 8 of loc. cit. Budur proves Conjecture 2.2.21 implies the Topological Multivariable Strong Monodromy Conjecture for hyperplane arrangements. See [6], in particular subsection 1.3 and Theorem 8, for details.

Walther proves in Theorem 5.13 of [7] the $r = 1$ version of this conjecture: if f is a tame and indecomposable central hyperplane arrangement of degree d, then $-n/d \in \mathrm{V}(B_f)$. Analogously, we prove Conjecture 2.2.21 in the tame case:

Theorem 2.2.4. *Suppose $f_{\mathcal{A}}$ is a central, essential, indecomposable, and tame hyperplane arrangement. Let $F = (f_1, \ldots, f_r)$ be a decomposition of $f_{\mathcal{A}}$ where f_k has degree d_k and the f_k are not necessarily reduced. Then*

$$\{d_1 s_1 + \cdots + d_r s_r + n = 0\} \subseteq \mathrm{V}(B_F).$$

Proof. Since $f_{\mathcal{A}}$ is homogeneous, $\mathrm{Der}_X(-\log f)$ is a graded $\mathbb{C}[X]$-module after giving each x_i degree one and each ∂_i degree -1. In the proof of Theorem 5.13 of [7], Walther shows that the indecomposablity hypothesis implies there exists a system of coordinates such that $\delta \in \mathrm{Der}_X(-\log f)$ is homogeneous of positive total degree or $\delta = w \sum x_i \partial_i$, $w \in \mathbb{C}$. Fix this system of coordinates and $E = \sum x_i \partial_i$ for the rest of the proof.

By Corollary 2.2.17, $\mathrm{ann}_{D_n[S]} F^S = D_n[S] \cdot \psi_F(\mathrm{Der}_X(-\log f))$. Recall $\psi_F(\delta) = \delta - \sum \frac{\delta \bullet f_k}{f_k} s_k$. If δ is of positive $(1, -1)$ total degree, then the coefficient of each s_k is either 0 or of positive degree as polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. This shows $\psi_F(\delta) \in D_n[S] \cdot (X)$, where $D_n[S] \cdot (X)$ is the left ideal generated by $x_1, \ldots, x_n$. Because $E + n \in D_n \cdot (X)$,

$$\begin{aligned}
\mathrm{ann}_{D_n[S]} F^S + D_n[S] \cdot f &\subseteq D_n[S] \cdot (X) + D_n[S] \cdot \psi_F(E) \\
&= D_n[S] \cdot (X) + D_n[S] \cdot (E - \sum d_k s_k) \\
&= D_n[S] \cdot (X) + D_n[S] \cdot (-n - \sum d_k s_k).
\end{aligned}$$

Suppose $P(S)$ is in the intersection of $D_n[S] \cdot (X) + D_n[S] \cdot (-n - \sum d_k s_k)$ and $\mathbb{C}[S]$. For each root α of $-n - \sum d_k s_k$ there is a natural evaluation map $D_n[S] \mapsto D_n$ sending $P \mapsto P(\alpha) \in D_n \cdot (X)$. Since $D_n \cdot (X)$ is a proper ideal of D_n, $P(\alpha) = 0$ for all such α. Therefore $\mathrm{V}(P(S)) \supseteq \mathrm{V}(\mathbb{C}[S] \cdot (-n - \sum d_k s_k))$ and we have shown

$$\mathrm{V}(B_F) = \mathrm{V}((\mathrm{ann}_{D_n[S]} F^S + D_n[S] \cdot f) \cap \mathbb{C}[S]) \supseteq \mathrm{V}(-n - \sum d_k s_k).$$

□

As outlined in the introduction, Theorem 2.2.4 is related to the Topological Multivariable Strong Monodromy Conjecture, that is, to Conjecture 2.1.2.

Corollary 2.2.22. *The Topological Multivariable Strong Monodromy Conjecture is true for (not necessarily reduced) tame hyperplane arrangements.*

Proof. This follows by Theorem 8 of [6] since tameness is a local condition. □

Remark 2.2.23. Not all arrangements are tame. For example, the $\mathbb{C}^4$-arrangement $\prod_{(a_1,\dots,a_4)\in\{0,1\}^4}(a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4)$ is not tame. If an arrangement has rank at most 3, then it is automatically tame.

2.3 The Map ∇_A

In this section we analyze the injectivity of $\mathscr{D}_{X,\mathfrak{x}}$-map

$$\nabla_A : \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \to \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}$$

under the nice hypotheses of the previous section. This will, see Section 5, let us better understand the relationship between $\mathrm{V}(B_{F,\mathfrak{x}})$ and the cohomology support loci of f near $\mathfrak{x}$. The section has two parts: a brief discussion of Koszul complexes associated to central elements over certain non-commutative rings with an application to $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S}$; a detailed proof that under nice hypotheses, if ∇_A is injective then it is surjective.

Let's first give a precise definition of ∇_A.

Definition 2.3.1. *(Compare to 5.5 and 5.10, in particular ρ_α, in [6]) Define*

$$\nabla : \mathscr{D}_{X,\mathfrak{x}}[S]F^S \to \mathscr{D}_{X,\mathfrak{x}}[S]F^S$$

by sending $s_i \mapsto s_i + 1$ for all i. To be precise, in local coordinates declare $\partial^{\boldsymbol{u}} = \prod_t \partial_{x_t}^{u_t}$, $S^{\boldsymbol{v}} = \prod_k s_k^{v_k}$, and let $S+1$ be shorthand for replacing each s_i with a s_i+1. Then ∇ is given by the assignment

$$\sum_{\boldsymbol{u},\boldsymbol{v}} Q_{\boldsymbol{u},\boldsymbol{v}} \partial^{\boldsymbol{u}} S^{\boldsymbol{v}} \bullet F^S \mapsto \sum_{\boldsymbol{u},\boldsymbol{v}} Q_{\boldsymbol{u},\boldsymbol{v}} \partial^{\boldsymbol{u}} (S+1)^{\boldsymbol{v}} \bullet F^{S+1}.$$

This is a homomorphism of $\mathscr{D}_{X,\mathfrak{x}}$-modules but is not $\mathbb{C}[S]$-linear.

Denote the ideal of $\mathscr{D}_{X,\mathfrak{x}}[S]$ generated by $s_1 - a_1, \dots, s_r - a_r$, for $a_1, \dots, a_r \in \mathbb{C}$ by $(S - A)\mathscr{D}_{X,\mathfrak{x}}[S]$. Then ∇ is injective and sends $(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S$ onto $(S+1-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1} = (S-(A-1)\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1} \subseteq (S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S$. Let ∇_A be the induced homomorphism of $\mathscr{D}_{X,\mathfrak{x}}$-modules:

$$\nabla_A : \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \longrightarrow \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}.$$

As mentioned in the introduction, a source of our motivation is investigating the three statements that show up in the following proposition.

Proposition 2.3.1. *Consider the following three statements, where $A-1$ denotes the tuple $(a_1 - 1, \dots, a_r - 1) \in \mathbb{C}^r$:*

(a) $A - 1 \notin \mathrm{V}(B_{F,\mathfrak{x}})$;

(b) ∇_A is injective;

(c) ∇_A is surjective.

Then in any case (a) implies (b) and (c).

Proof. Choose a functional equation $B(S)F^S = P(S)F^{S+1}$ where we may assume $B(A-1) \neq 0$.

We first prove that (a) implies (c). Since $\nabla(P(S-1)F^S) = P(S)F^{S+1}$,

$$\overline{P(S-1)F^S} \xmapsto{\nabla_A} \overline{P(S)F^{S+1}} = \overline{B(S)F^S}.$$

This shows that $\nabla_A(\overline{P(S-1)F^S})$ generates $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}$.

To show that (a) implies (b) suppose $\nabla_A(\overline{Q(S)F^S}) = 0$. This means $Q(S+1)F^{S+1} \in \sum(s_i - (a_i - 1)) \cdot \mathscr{D}_{X,\mathfrak{x}}[S]F^S$. Multiplying both sides by $B(S)$ gives $Q(S+1)B(S)F^{S+1} \in \sum(s_i - (a_i - 1)) \cdot \mathscr{D}_{X,\mathfrak{x}}[S]P(S)F^{S+1}$. So $Q(S)B(S-1)F^S \in \sum(s_i - a_i) \cdot \mathscr{D}_{X,\mathfrak{x}}[S]F^S$ and $\overline{Q(S)F^S}$ is zero in $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S}$. □

Remark 2.3.2. 1. In the classical setting where $F = (f)$ and $F^S = f^s$, (a), (b), and (c) of Proposition 2.3.1 are equivalent (see 6.3.15 in [16] for the equivalence of (a) and (c); the claims involving (b) follow by a similar diagram chase).

2. Suppose $A = (a, \ldots, a)$ and f is strongly Euler-homogeneous, Saito-holonomic, and tame. By Remark 2.2.20, there is a commutative square of $\mathscr{D}_{X,\mathfrak{x}}$- maps:

$$\begin{tikzcd}
\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \arrow[r, "\simeq"] \arrow[d, "\nabla_A"] & \frac{\mathscr{D}_{X,\mathfrak{x}}[s]f^s}{(s-a)\mathscr{D}_{X,\mathfrak{x}}[s]f^s} \arrow[d, "\nabla_a"] \\
\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \arrow[r, "\simeq"] & \frac{\mathscr{D}_{X,\mathfrak{x}}[s]f^s}{(s-(a-1))\mathscr{D}_{X,\mathfrak{x}}[s]f^s}.
\end{tikzcd}$$

If the conditions in Proposition 2.3.1 were equivalent, then the inclusion induced by the diagonal embedding $\mathrm{V}(B_{f,\mathfrak{x}}) \hookrightarrow \mathrm{V}(B_{F,\mathfrak{x}}) \bigcap \mathrm{V}(s_1 - s_2, \ldots, s_{r-1} - s_r)$, given in Proposition 2.2.18 would be surjective.

Example 2.3.3. Let $f = x(2x^2 + yz)$ and $F = (x, 2x^2 + yz)$. This is strongly Euler-homogeneous, Saito-holonomic (cf. Examples 2.2.1, 2.2.5), and tame ($n \leq 3$). Using Singular and Macaulay2 we compute $\mathrm{V}(B_{F,0}) = (s_1+1)(s_2+1)\prod_{k=3}^{r}(s_1 + 2s_2 + k)$ and $\mathrm{V}(B_{f,0}) = (s+1)^3(s+\frac{4}{3})(s+\frac{5}{3})$. In this case, the diagonal embedding $\mathrm{V}(B_{f,0}) \hookrightarrow \mathrm{V}(B_{F,0}) \bigcap \mathrm{V}(s_1 - s_2)$ of Proposition 2.2.18 is surjective and, see Remark 2.3.2, $\nabla_{-k+1,-k+1}$ is neither surjective nor injective for $k = 3, 4, 5$.

The rest of this section is devoted to proving that under the nice hypotheses of the previous section and in the language of Proposition 2.3.1, that (b) implies (c). Our proof makes use of a Koszul resolution over the central elements $S - A$.

Convention 2.3.1. A resolution is a (co)-complex with a unique (co)homology module at its end. An acyclic (co)-complex has no (co)homology. Given a (co)-complex ($C^\bullet$) $C_\bullet$ resolving A, the augmented (co)-complex ($C^\bullet \to A$) $C_\bullet \to A$ is acyclic.

Definition 2.3.2. *For a (not necessarily commutative) ring R and a sequence of central R-elements $a = a_1, \dots, a_k$ let $K^\bullet(a)$ be the Koszul co-complex induced by the elements a, cf. Section 6 in [21]. For a left R-module M, let $K^\bullet(a; M) = K^\bullet(a) \otimes M$ be the Koszul co-complex on M induced by a. We index $K^\bullet(a)$ so that the right most object is $K^0(a)$.*

The following lemma is immediate after considering $H^{-1}(K^\bullet(c_1, \dots, c_r; M))$:

Lemma 2.3.4. *Let R be a, possibly noncommutative, ring, M a left R-module, $m_i \in M$, and $c_1, \dots, c_r$ central elements of R. Assume $H^{-1}(K(c_1, \dots, c_r); M) = 0$. If $c_i m_i \in (c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_r)M$, then $m_i \in (c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_r)M$.*

Let $v_1, \dots, v_k$ be positive integers. If R is commutative and if $K^\bullet(a; M)$ is a resolution, we know $K^\bullet(a_1^{v_1}, \dots, a_k^{v_k}; M)$ is a resolution, cf. Exercise 6.16 in [21]. A routine induction argument (that we omit) using the the tensor product of Koszul co-complexes verifies that this is also true for general R and central a:

Proposition 2.3.5. *Let R be a, possibly non-commutative, ring, M a R-module, $c_1, \dots, c_r$ central elements of R, and $v_1, \dots, v_r \in \mathbb{Z}_+$. If $K^\bullet(c_1, \dots, c_r; M)$ is a resolution, then $K^\bullet(c_1^{v_1}, \dots, c_r^{v_r}; M)$ is a resolution.*

Now return to $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]F^S)$. Under the nice hypothesis of the previous section, $\mathrm{gr}_{(0,1,1)}(s_1), \dots, \mathrm{gr}_{(0,1,1)}(s_r)$ act like a regular sequence:

Proposition 2.3.6. *Let $f = f_1 \cdots f_r$ and let $F = (f_1, \dots, f_r)$. Suppose that for $\mathfrak{x} \in X$ the following hold:*

- *f has the strong Euler-homogeneity $E_{\mathfrak{x}}$ at $\mathfrak{x}$;*
- *$\widetilde{L_{F,\mathfrak{x}}} \subseteq \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ is Cohen–Macaulay of dimension $n + r$;*
- *$L_{f,\mathfrak{x}} + \mathrm{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}}) \cdot \mathrm{gr}_{(0,1)}(E_{\mathfrak{x}}) \subseteq \mathrm{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}})$ is Cohen–Macaulay of dimension n.*

Then $K^\bullet(S; \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])/\widetilde{L_{F,\mathfrak{x}}})$ *is co-complex of* $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$*-modules resolving* $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])/(\widetilde{L_{F,\mathfrak{x}}}, S) \simeq \mathrm{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}})/(L_{f,\mathfrak{x}} + \mathrm{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}}) \cdot \mathrm{gr}_{(0,1)}(E_{\mathfrak{x}}))$.

Proof. The last isomorphism is immediate from the definition of ψ_F and the construction of $\widetilde{L_{F,\mathfrak{x}}}$ and $L_{f,\mathfrak{x}}$, see Definition 2.2.8 and the preceding comments.

Multiplying $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ by s_k increases the degree of an element by one. So after doing the appropriate degree shifts, we may view $K^\bullet(S; \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]))$ as a sequence of graded modules with degree preserving maps. By Proposition 1.5.15 (c) of [22], exactness of such a sequence is a graded local property. The only $(0,1,1)$-graded maximal ideal $\mathfrak{m}^\star$ is generated by $\mathscr{O}_{X,\mathfrak{x}}$ and the irrelevant ideal. So localize $K^\bullet(S; \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])/\widetilde{L_{F,\mathfrak{x}}})$ at $\mathfrak{m}^\star$.

By Theorem 2.1.2 of [22], if both $(\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])/\widetilde{L_{F,\mathfrak{x}}})_{\mathfrak{m}^\star}$ and

$$(\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])/\widetilde{L_{F,\mathfrak{x}}} + \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])_{\mathfrak{m}^\star} \simeq (\mathrm{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}})/L_{f,\mathfrak{x}} + \mathrm{gr}_{(0,1)}(E_x))_{\mathfrak{m}^\star}$$

are Cohen–Macaulay and the difference in their dimensions is the length of the sequence S, then our localized Koszul co-complex is a resolution. Since the dimension of a graded-local ring equals the dimension after localization at the graded maximal ideal, cf. Corollary 13.7 of [20], we are done. □

For $a_1, \ldots, a_r \in \mathbb{C}$, label $S - A = s_1 - a_1, \ldots, s_r - a_r \in \mathscr{D}_{X,\mathfrak{x}}[S]$. Being central elements, $S - A$ yields the Koszul co-complex $K^\bullet(S - A; \mathscr{D}_{X,\mathfrak{x}}[S]F^S)$ of $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules. Its terminal cohomology module is $\mathscr{D}_{X,\mathfrak{x}}[S]F^S/(S - A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S$. We show that under our standard hypotheses on f, i.e. strongly Euler-homogeneous, Saito-holonomic, and tame, that $s_1 - a_1$, $\ldots$, $s_r - a_r$ behaves like a regular sequence.

Proposition 2.3.7. *Suppose* $f = f_1 \cdots f_r$ *is strongly Euler-homogeneous, Saito-holonomic, and tame and let* $F = (f_1, \ldots, f_r)$. *Then* $K^\bullet(S - A; \mathscr{D}_{X,\mathfrak{x}}[S]F^S)$ *resolves* $\mathscr{D}_{X,\mathfrak{x}}[S]F^S/(S - A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S$.

Proof. Under the total order filtration, $s_k - a_k$ has weight one. It is routine to define a filtration G, compatible with the total order filtration, on the augmented co-complex $K^\bullet(S - A; \mathscr{D}_{X,\mathfrak{x}}[S]F^S) \to \mathscr{D}_{X,\mathfrak{x}}[S]F^S/(S - A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S$ such $\mathrm{gr}_G(K^\bullet(S - A; \mathscr{D}_{X,\mathfrak{x}}[S]F^S) \to \mathscr{D}_{X,\mathfrak{x}}[S]F^S/(S - A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S)$ is isomorphic to

$K^\bullet(S;\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])/\widetilde{L_{F,\mathfrak{x}}}) \to \mathrm{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}})/(L_{f,\mathfrak{x}} + \mathrm{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}}) \cdot \mathrm{gr}_{(0,1)}(E_x))$. If this co-complex is acyclic, then a standard argument using the spectral sequence attached to a filtered co-complex proves that $K^\bullet(S-A;\mathscr{D}_{X,\mathfrak{x}}[S]F^S) \to \mathscr{D}_{X,\mathfrak{x}}[S]F^S/(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S$ is acyclic is well. The claim then follows by Theorem 2.2.2, Corollary 3.19 of [7], and Proposition 2.3.6. □

Finally we can prove the section's main theorem:

Theorem 2.3.2. *Let $f = f_1 \cdots f_r$ be strongly Euler-homogeneous, Saito-holonomic, and tame and let $F = (f_1, \ldots, f_r)$. If ∇_A is injective, then it is surjective.*

Proof. For this proof, and this proof alone, write $\widetilde{s_i} = s_i - (a_i - 1)$. Also, $\overline{(-)}$ denotes the image of $(-)$ in the appropriate quotient object.

The Plan: If there is some multivariate Bernstein–Sato polynomial $B(S)$ that does not vanish at $(a_1 - 1, \ldots, a_r - 1)$, then the claim follows by Proposition 2.3.1. So pick a multivariate Bernstein–Sato polynomial $B(S) = \sum A_k \widetilde{s_k}$, $A_k \in \mathbb{C}[S]$. The idea is to successively "remove" each s_k factor from each A_k. In doing so, we will produce a finite sequence of polynomials $B_0, B_i, \ldots$ satisfying the technical condition (2.3.1) introduced in Step 1, starting with our multivariate Bernstein–Sato polynomial, such that each polynomial uses fewer variables than its predecessor. The terminal polynomial will demonstrate that the cokernel of ∇_A vanishes.

The inductive construction of these polynomials is not hard but technical. Before doing it we prove three claims. The first is that a particular cohomology module of the Koszul co-complex of $\widetilde{s_1}, \ldots, \widetilde{s_r}$ on $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}$ vanishes. We use this to "remove" the $\widetilde{s_k}$ factors. The second and third claims are the technical details comprising the inductive algorithm used to produce these polynomials.

Claim 1: For all positive integers $v_1, \ldots, v_r$,

$$H^{-1}\left(K^\bullet\left(\widetilde{s_1}^{v_1}, \ldots, \widetilde{s_r}^{v_r}; \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}\right)\right) = 0.$$

Proof of Claim 1*:* The $\mathscr{D}_{X,\mathfrak{x}}$-map ∇_A is always injective and sends $F^S \mapsto F^{S+1}$. If ∇_A is also injective there is a short exact sequence of augmented co-complexes:

$$\begin{aligned}0 &\to (K^\bullet(S-A;\mathscr{D}_{X,\mathfrak{x}}[S]F^S) \to \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S}) \\ &\to (K^\bullet(S-(A-1);\mathscr{D}_{X,\mathfrak{x}}[S]F^S) \to \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}) \\ &\to (K^\bullet(S-(A-1);\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}) \to \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}) \to 0.\end{aligned}$$

The first map is induced by ∇ and by ∇_A on the augmented part; the second by quotient maps. By Proposition 2.3.7 and the canonical long exact sequence, the last (nonzero) augmented co-complex is acyclic. Claim 1 follows by Proposition 2.3.5.

Claim 2*:* Write $\overline{F^S}$ for the image of F^S in $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}$. Suppose there exists $P(S) \in \mathbb{C}[S]$, $1 \leq j < r$, positive integers $n_{j+1}, \ldots, n_r$, and an integer $m \geq \max\{n_{j+1}, \ldots, n_r\}$ such that

$$\left(\prod_{j+1\leq k\leq r} \widetilde{s_k}^{n_k}\right) P(S) \bullet \overline{F^S} \in (\widetilde{s_1}, \ldots, \widetilde{s_j}, \widetilde{s_{j+1}}^m, \ldots, \widetilde{s_r}^m)\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}.$$

Then for $m' = \min\{m-n_{j+1}, \ldots, m-n_r\}$ we have

$$P(S) \bullet \overline{F^S} \in (\widetilde{s_1}, \ldots, \widetilde{s_j}, \widetilde{s_{j+1}}^{m'}, \ldots, \widetilde{s_r}^{m'})\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}.$$

Proof of Claim 2*:* The idea is to use Claim 1 and Lemma 2.3.4 to "remove" each $\widetilde{s_k}^{n_k}$ factor one at a time. We first "remove" the $\widetilde{s_{j+1}}^{n_{j+1}}$ factor.

By hypothesis, there exists $Q_{j+1} \in \mathscr{D}_{X,\mathfrak{x}}[S]$ such that

$$\begin{aligned}&\left(\prod_{j+1\leq k\leq r} \widetilde{s_k}^{n_k}\right) P(S) \bullet \overline{F^S} - \widetilde{s_{j+1}}^m Q_{j+1} \bullet \overline{F^S} \\ &\qquad = \widetilde{s_{j+1}}^{n_{j+1}}\left(\left(\prod_{j+2\leq k\leq r} \widetilde{s_k}^{n_k}\right) P(S) \bullet \overline{F^S} - \widetilde{s_{j+1}}^{m-n_{j+1}} Q_{j+1} \bullet \overline{F^S}\right) \\ &\qquad \in (\widetilde{s_1}, \ldots, \widetilde{s_j}, \widetilde{s_{j+2}}^m, \ldots, \widetilde{s_r}^m)\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}.\end{aligned}$$

By Claim 1, $H^{-1}(K(\widetilde{s_1}, \dots, \widetilde{s_j}, \widetilde{s_{j+1}}^{n_{j+1}}, \widetilde{s_{j+2}}^{m}, \dots, \widetilde{s_r}^{m}; \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}))$ vanishes. So Lemma 2.3.4 implies

$$\left(\prod_{j+2 \leq k \leq r} \widetilde{s_k}^{n_k}\right) P(S) \bullet \overline{F^S} - \widetilde{s_{j+1}}^{m-n_{j+1}} Q_{j+1} \bullet \overline{F^S}$$
$$\in (\widetilde{s_1}, \dots, \widetilde{s_j}, \widetilde{s_{j+2}}^{m}, \dots, \widetilde{s_r}^{m}) \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}.$$

Rearrange to see

$$\left(\prod_{j+2 \leq k \leq r} \widetilde{s_k}^{n_k}\right) P(S) \bullet \overline{F^S} \in (\widetilde{s_1}, \dots, \widetilde{s_j}, \widetilde{s_{j+1}}^{m-n_{j+1}}, \widetilde{s_{j+2}}^{m}, \dots, \widetilde{s_r}^{m}) \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}.$$

Repeat this process on each remaining factor $\widetilde{s_k}^{n_k}$, $j+2 \leq k \leq r$ one at a time to conclude

$$P(S) \bullet \overline{F^S} \in (\widetilde{s_1}, \dots, \widetilde{s_j}, \widetilde{s_{j+1}}^{m-n_{j+1}}, \widetilde{s_{j+2}}^{m-n_{j+2}}, \dots, \widetilde{s_r}^{m-n_r}) \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}.$$

Claim 3*:* Suppose $B_j \in \mathbb{C}[s_{j+1}, \dots, s_r]$, where $j < r$, with $B_j \in \mathbb{C}[s_{j+1}, \dots, s_r] \cdot (\widetilde{s_{j+1}}, \dots, \widetilde{s_r})$ but $B_j \notin \mathbb{C}[\widetilde{s_{j+1}}, \dots, \widetilde{s_r}] \cdot (\widetilde{s_k})$ for all $j+1 \leq k \leq r$. Furthermore, assume that for $m \geq \max\{n_{j+1}, \dots, n_r\}$ we have

$$B_j \bullet F^S \in (\widetilde{s_1}, \dots, \widetilde{s_j}, \widetilde{s_{j+1}}^{m}, \dots, \widetilde{s_r}^{m}) \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}.$$

Then, relabeling the s_k if necessary, there exists $B_i \in \mathbb{C}[s_{i+1}, \dots s_r]$, where $j < i < r$, $B_t \notin \mathbb{C}[s_{i+1}, \dots, s_r] \cdot (\widetilde{s_k})$ for $i+1 \leq k \leq r$, so that for $m' = \min\{m - n_{j+1}, \dots, m - n_r\}$ we have

$$B_i \bullet F^S \in (\widetilde{s_1}, \dots, \widetilde{s_i}, \widetilde{s_{t+1}}^{m'}, \dots, \widetilde{s_r}^{m'}) \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}.$$

Proof of Claim 3*:* Note that the hypotheses imply $j < r-1$ so the promised choice of i is possible. Since $B_j \notin \mathbb{C}[s_{j+1}, \dots, s_r] \cdot (\widetilde{s_k})$ for all $j+1 \leq k \leq r$, there exists a largest $\emptyset \neq I = \{\widetilde{s_{i_1}}, \dots, \widetilde{s_{i_{|i|}}}\} \subsetneq \{j+1, \dots, r\}$ such that $B_j \notin \mathbb{C}[s_{j+1}, \dots, s_r] \cdot (\widetilde{s_{i_1}}, \dots, \widetilde{s_{i_{|I|}}})$. Relabel so that $I = \{j+1, \dots, i\}$. This means there exist positive integers n_k, polynomials $A_l \in \mathbb{C}[S]$, and a polynomial $B_i \in \mathbb{C}[s_{i+1}, \dots, s_r]$ such that

$$B_j = \left(\prod_{i+1 \leq k \leq r} \widetilde{s_k}^{n_k} \right) B_i + \sum_{1 \leq \ell \leq i} \widetilde{s}_\ell A_\ell.$$

We may make each n_k large enough so as to assume $B_i \notin \mathbb{C}[s_{i+1}, \dots, s_r] \cdot (\widetilde{s_k})$ for any $i + 1 \leq k \leq r$.

Therefore

$$\left(\prod_{i+1 \leq k \leq r} \widetilde{s_k}^{n_k} \right) B_i \bullet \overline{F^S} \in (\widetilde{s_1}, \dots, \widetilde{s}_i, \widetilde{s_{i+1}}^m, \dots, \widetilde{s_r}^m) \frac{\mathscr{D}_{X,\mathfrak{x}}[S] F^S}{\mathscr{D}_{X,\mathfrak{x}}[S] F^{S+1}}.$$

Then Claim 3 follows from Claim 2.

Proof of Theorem.

Step 1*:* We will inductively construct a sequence of polynomials B_{i_1}, $B_{i_2}, \dots,$ such that (after potentially relabelling the s_k) the following hold: $0 \leq i_t < r$ for each i_t; $i_t < i_{t+1}$; $B_{i_t} \in \mathbb{C}[s_{i_t+1}, \dots, s_r]$; for m_{i_t} arbitrarily large

$$B_{i_t} \bullet \overline{F^S} \in (\widetilde{s_1}, \dots, \widetilde{s_{i_t}}, \widetilde{s_{i_t+1}}^{m_{i_t}}, \dots, \widetilde{s_r}^{m_{i_t}}) \frac{\mathscr{D}_{X,\mathfrak{x}}[S] F^S}{\mathscr{D}_{X,\mathfrak{x}}[S] F^{S+1}}. \tag{2.3.1}$$

We terminate the induction once we produce a B_i such that, in addition to the above properties, $B_i \notin \mathbb{C}[s_{i+1}, \dots, s_r] \cdot (\widetilde{s_{i+1}}, \dots, \widetilde{s_r})$.

Base Case: Take a multivariate Bernstein–Sato polynomial $B(S) \in B_{F,\mathfrak{x}}$. If $B(S) \notin \mathbb{C}[s_1, \dots, s_r] \cdot (\widetilde{s_1}, \dots, \widetilde{s_r})$ then we are done: $B(S) = B_0$ works. (Recall $B(S) \bullet F^S \in \mathscr{D}_{X,\mathfrak{x}}[S] F^{S+1}$.) Otherwise find the largest $J \subsetneq [r]$ such that $B(S) \notin C[S] \cdot (\widetilde{s_{j_1}}, \dots, \widetilde{s_{j_{|J|}}})$. Re-label to assume $J = \{1, \dots, j\}, j < r$. (We allow $J = \emptyset$, in which case $j = 0$.) This means we can write $B(S)$ as

$$B(S) = \left(\prod_{j+1 \leq k \leq r} \widetilde{s_k}^{n_k} \right) B_j + \sum_{1 \leq t \leq j} \widetilde{s}_t A_t$$

where $B_j \in \mathbb{C}[s_{j+1}, \ldots, s_r]$ and each n_k a positive integer chosen large enough so that $B_j \notin \mathbb{C}[s_{j+1}, \ldots, s_r] \cdot (\widetilde{s_k})$, for $j+1 \leq k \leq r$, Because $B(S)$ is a multivariate Bernstein–Sato polynomial, $B(S) \bullet F^S \in \mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}$. Therefore,

$$\left(\prod_{j+1 \leq k \leq r} \widetilde{s_k}^{n_k}\right) B_j \bullet \overline{F^S} \in (\widetilde{s_1}, \ldots, \widetilde{s_j}) \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}. \tag{2.3.2}$$

Now (2.3.2) trivially implies that for all $m \geq 0$

$$\left(\prod_{j+1 \leq k \leq r} \widetilde{s_k}^{n_k}\right) B_j \bullet \overline{F^S} \in (\widetilde{s_1}, \ldots, \widetilde{s_j}, \widetilde{s_{j+1}}^m, \ldots, \widetilde{s_r}^m) \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}.$$

In particular, the above holds for m arbitrarily large. By Claim 2, there exists m_j arbitrarily large such that

$$B_j \bullet \overline{F^S} \in (\widetilde{s_1}, \ldots, \widetilde{s_j}, \widetilde{s_{j+1}}^{m_j}, \ldots, \widetilde{s_r}^{m_j}) \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}.$$

Then B_j is the first element in our sequence of polynomials.

Inductive Step: Suppose $B_j \in \mathbb{C}[s_{j+1}, \ldots, s_r]$ has already been defined. If the algorithm has not terminated, $j < r$ and $B_j \notin \mathbb{C}[s_{j+1}, \ldots, s_r] \cdot (\widetilde{s_k})$ for all $j+1 \leq k \leq r$. Then use Claim 3 to define B_i, where $j < i < r$. Note that if $j = r-1$ then $B_{r-1} \notin \mathbb{C}[s_r] \cdot (\widetilde{s_r})$ and so the algorithm terminates at B_{r-1}.

Step 2*:* Use the terminal polynomial $B_i \in \mathbb{C}[s_{i+1}, \ldots, s_r]$, $i < r$, produced by Step 1. This means $B_i \notin \mathbb{C}[s_{i+1}, \ldots, s_r] \cdot (\widetilde{s_{i+1}}, \ldots, \widetilde{s_r})$ and easily implies

$$B_i \bullet \overline{F^S} \in (\widetilde{s_1}, \ldots, \widetilde{s_r}) \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}.$$

On one hand, since B_i does not vanish at $(a_{i+1} - 1, \ldots, a_r - 1)$, $B_i\overline{F^S}$ and $\overline{F^S}$ generate the same submodule of $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(\widetilde{s_i}, \ldots, \widetilde{s_r})\mathscr{D}_{X,\mathfrak{x}}[S]F^S + \mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}$; on the other hand, $0 = B_i \bullet \overline{F^S} \in \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(\widetilde{s_i}, \ldots, \widetilde{s_r})\mathscr{D}_{X,\mathfrak{x}}[S]F^S + \mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}}$. Thus, $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(\widetilde{s_i}, \ldots, \widetilde{s_r})\mathscr{D}_{X,\mathfrak{x}}[S]F^S + \mathscr{D}_{X,\mathfrak{x}}[S]F^{S+1}} = 0$ (because it is generated by $\overline{F^S}$). That is, the cokernel of ∇_A vanishes. ∎

Using Theorem 2.3.2 we can show that the three conditions of Proposition 2.3.1 are equivalent in a very special and restricted case:

Proposition 2.3.8. *Suppose $f = f_1 \cdots f_r$ is a central, essential, indecomposable, and tame hyperplane arrangement, where each f_k is of degree d_k and the f_k are not necessarily reduced. Let $F = (f_1, \dots, f_r)$. If $A - 1 \in \{d_1 s_1 + \cdots + d_r s_r + n = 0\}$, then $A - 1 \in \mathrm{V}(B_{F,0})$ and ∇_A is neither surjective nor injective.*

Proof. An easy extension of the argument in Theorem 2.2.4 shows that both

$$\operatorname{ann}_{\mathscr{D}_{X,0}[S]} F^S + \mathscr{D}_{X,0}[S] \cdot f \subseteq \mathscr{D}_{X,0}[S] \cdot \mathfrak{m}_0 + \mathscr{D}_{X,0}[S] \cdot (-n - \sum d_k s_k). \tag{2.3.3}$$

and $A - 1 \in \mathrm{V}(B_{F,0})$. Now ∇_A is surjective precisely when

$$\mathscr{D}_{X,0}[S] = \operatorname{ann}_{\mathscr{D}_{X,0}[S]} F^S + \mathscr{D}_{X,0}[S] \cdot f + \sum \mathscr{D}_{X,0}[S] \cdot (s_k - (a_k - 1)).$$

By (2.3.3), if ∇_A is surjective,

$$\mathscr{D}_{X,0}[S] \subseteq \mathscr{D}_{X,0}[S] \cdot \mathfrak{m}_0 + \mathscr{D}_{X,0}[S] \cdot (-n - \sum d_k s_k) + \sum \mathscr{D}_{X,0}[S] \cdot (s_k - (a_k - 1)).$$

After evaluating each s_k at $a_k - 1$, we deduce $\mathscr{D}_{X,0} \subseteq \mathscr{D}_{X,0} \cdot \mathfrak{m}_0$. Therefore ∇_A is not surjective. By Theorem 2.3.2, ∇_A is not injective. ☐

2.4 Free Divisors, Lie–Rinehart Algebras, and ∇_A

In Definition 2.2.6 we defined tame divisors. A stronger condition is freeness:

Definition 2.4.1. *A divisor Y is* free *if it locally everywhere admits a defining equation f such that $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$ is a free $\mathscr{O}_{X,\mathfrak{x}}$-module.*

Freeness implies tameness because $\Omega_{X,\mathfrak{x}}(\log f)$ and $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$ are dual and if $\Omega_{X,\mathfrak{x}}(\log f)$ is free, then $\Omega^p_{X,\mathfrak{x}}(\log f) = \bigwedge^p \Omega_{X,\mathfrak{x}}(\log f)$ (see 1.7, 1.8 of [8]).

Throughout this section we upgrade our working hypotheses of strongly Euler-homogeneous, Saito-holonomic, and tame to reduced, strongly Euler-homogeneous, Saito-holonomic, and free. The goal is to investigate the surjectivity of the map ∇_A. Let's give a road map. First we compute Ext modules of $\mathscr{D}_{X,\mathfrak{x}}[S]F^S/(S - A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S$ using [9] and the rich theory of Lie–Rinehart algebras. Lifting a surjective ∇_A to these Ext-modules will produce an injective map. This injective map acts like ∇_{-A}. By Theorem 2.3.2, ∇_{-A} is surjective. Using duality again will show that ∇_A is injective.

2.4.1 Lie–Rinehart Algebras and the Spencer Co-Complex Sp.

Definition 2.4.2. *(Compare with [19]), [23] and the appendix of [3]) Fix a homomorphism of commutative rings $k \to A$. A* Lie–Rinehart algebra *L over (k, A) is a A-module L with* anchor map *$\rho : L \to \mathrm{Der}_k(A)$ that is A-linear, a k-Lie algebra map, and satisfies, for all λ, $\lambda' \in L$, $a \in A$,*

$$[\lambda, a\lambda'] = a[\lambda, \lambda'] + \rho(\lambda)(a)\lambda'.$$

We will usually drop ρ and replace $\rho(\lambda)(a)$ with $\lambda(a)$. A morphism $F : L \to L'$ of Lie–Rinehart algebras over (k, A) is a A-linear map that is a morphism of Lie-algebras satisfying $\lambda(a) = F(\lambda)(a)$.

Example 2.4.1.(a) $\mathrm{Der}_k(A)$ is a Lie–Rinehart algebra over (k, A) with the identity as the anchor map.

(b) Any A-submodule of $\mathrm{Der}_k(A)$ that is also a k-Lie algebra is a Lie–Rinehart algebra over (k, A), with anchor map induced by the inclusion into $\mathrm{Der}_k(A)$. In particular $\mathrm{Der}_{X,\mathfrak{x}}(-\log(f))$ is a Lie–Rinehart algebra over $(\mathbb{C}, \mathscr{O}_{X,\mathfrak{x}})$.

(c) If L is a Lie–Rinehart algebra over (k, A), then $L \oplus A$ is a Lie–Rinehart algebra over (k, A) with anchor map induced by the projection $L \oplus A \to L$, $(\lambda, a) \mapsto \lambda$. So $\mathrm{Der}_{X,\mathfrak{x}} \oplus \mathscr{O}^r_{X,\mathfrak{x}}$ and $\mathrm{Der}_{X,\mathfrak{x}}(-\log(f)) \oplus \mathscr{O}^r_{X,\mathfrak{x}}$ are Lie–Rinehart algebras over $(\mathbb{C}, \mathscr{O}_{X,\mathfrak{x}})$.

Definition 2.4.3. *Let L be a Lie–Rinehart algebra over (k, A) with $k \to A$. Suppose R is a ring (not necessarily a Lie–Rinehart algebra) and $A \to R$ a ring homomorphism that makes R central over k, i.e. images of elements of k are central elements in R. Then a k-linear map $g : L \to R$ is* admissible *if:*

(a) $g(a\lambda) = ag(\lambda)$, *for* $a \in A$, $\lambda \in L$ *(*g *is a morphism of* A*-modules);*

(b) $g([\lambda, \lambda']) = [g(\lambda), g(\lambda')]$, *for* λ, $\lambda' \in L$ *(*g *is a morphism of Lie-algebras);*

(c) $g(\lambda)a - ag(\lambda) = \lambda(a)1_R$ *for* $\lambda \in L$, $a \in A$.

The following theorem will be our definition of the *universal algebra* $U(L)$:

Theorem 2.4.1. (cf. [23]) *For any Lie–Rinehart algebra* L *over* (k, A) *there exists a ring* $U(L)$, *a ring homomorphism* $A \to U(L)$ *making* $U(L)$ *central over* k, *and an admissible map* $\theta : L \to U(L)$ *that is universal in the following sense: for any ring* R *with a ring homomorphism* $A \to R$ *making* R *central over* k, *and any admissible map* $g : L \to R$, *there is a unique ring homomorphism* $h : U(L) \to R$ *such that* $h \circ \theta = g$. *The natural map* $\theta : L \to U(L)$ *induces a filtration on* $U(L)$ *given by the powers of images of* θ.

We omit the proof of the following proposition. It uses the (not provided) explicit construction of $U(L)$ and standard universal object arguments.

Proposition 2.4.2. *Given a Lie–Rinehart algebra* L *over* (k, A), *consider the direct sum* $L \oplus A$. *This is a Lie–Rinehart algebra over* (k, A) *with anchor map induced by projection:* $L \oplus k \twoheadrightarrow L \to \mathrm{Der}_k(A)$. *Then* $U(L \oplus A) \simeq U(L)[s]$. *Moroever, the natural filtration on* $U(L \oplus A)$ *corresponds to a "total order filtration" on* $U(L)[S]$, *i.e. a filtration where* s *has weight one.*

Example 2.4.3.(a) The universal Lie–Rinehart algebra of $\mathrm{Der}_{X,\mathfrak{x}}$ over $(\mathbb{C}, \mathscr{O}_{X,\mathfrak{x}})$ is $\mathscr{D}_{X,\mathfrak{x}}$. The natural filtration is the order filtration.

(b) By repeated application of Proposition 2.4.2, the universal Lie–Rinehart algebra of $\mathrm{Der}_{X,\mathfrak{x}} \oplus \mathscr{O}^r_{X,\mathfrak{x}}$ over $(\mathbb{C}, \mathscr{O}_{X,\mathfrak{x}})$ is $\mathscr{D}_{X,\mathfrak{x}}[s_1, \ldots, s_r]$. The natural filtration is the total order filtration $F_{(0,1,1)}$.

(c) For $F = (f_1, \ldots, f_r)$ a decomposition of $f = f_1 \cdots f_r$, the annihilating derivations $\theta_{F,\mathfrak{x}}$ constitute a Lie–Rinehart algebra over $(\mathbb{C}, \mathscr{O}_{X,\mathfrak{x}})$. The $\mathscr{O}_{X,\mathfrak{x}}$-map ψ_F : $\mathrm{Der}_{X,\mathfrak{x}}(-\log f) \to \theta_{F,\mathfrak{x}}$ is an isomorphism of Lie–Rinehart algebras over $(\mathbb{C}, \mathscr{O}_{X,\mathfrak{x}})$. So there is a containment of Lie–Rinehart algebras over $(\mathbb{C}, \mathscr{O}_{X,\mathfrak{x}})$: $\theta_{F,\mathfrak{x}} \subseteq \mathrm{Der}_{X,\mathfrak{x}} \oplus \mathscr{O}^r_{X,\mathfrak{x}}$.

(d) The universal algebra of the Lie–Rinehart algebra $\mathrm{Der}_{X,\mathfrak{r}}[S]$ over $(\mathbb{C}[S], \mathscr{O}_{X,\mathfrak{r}}[S])$ is $\mathscr{D}_{X,\mathfrak{r}}[S]$. Note that s_k is contained in the 0^{th} filtered part and the filtration is induced by the order filtration.

We care about the formalism of Lie–Rinehart algebras because we want to construct complexes of the universal algebras. Given two Lie–Rinehart algebras $L \subseteq L'$, the following gives a complex of $U(L')$-modules.

Definition 2.4.4. *(Compare with 1.1.8 of [19]) Let L and L' be Lie–Rinehart algebras over (k, A). The* Cartan–Eilenberg–Chevalley–Rinehart–Spencer co-complex *associated to $L \subseteq L'$ and the left $U(L)$-module E is the co-complex $Sp_{L,L'}(E)$. Here*

$$Sp^{-r}(L, L') := U(L') \otimes_A \bigwedge^r L \otimes_A E$$

and the $U(L')$-linear differential

$$d^{-r} : Sp^{-r}_{L,L'}(E) \to Sp^{-(r-1)}_{L,L'}(E)$$

is given by

$$\begin{aligned} d^{-r}(P \otimes \lambda_1 \wedge \cdots \wedge \lambda_r \otimes e) = &\sum_{i=1}^{r}(-1)^{i-1} P\lambda_i \otimes \widehat{\lambda_i} \otimes e - \sum_{i=1}^{r}(-1)^{i-1} P \otimes \widehat{\lambda_i} \otimes \lambda_i e \\ &+ \sum_{1 \leq i < j \leq r} (-1)^{i+j} P \otimes [\lambda_i, \lambda_j] \wedge \widehat{\lambda_{i,j}} \otimes e. \end{aligned} \tag{2.4.1}$$

(Here $\widehat{\lambda_{i,j}}$ is the wedge of the of all the λ's except λ_i and λ_j.) There is a natural augmentation map

$$U(L') \otimes_A E \to U(L') \otimes_{U(L)} E.$$

When $E = A$, write $Sp_{L,L'}(A)$ as $Sp_{L,L'}$.

In general, the cohomology of $\mathrm{Sp}_{L,L'}(E)$ is mysterious. In principal, it can be computed using the spectral sequence associated to the filtration of $U(L')$ promised by Theorem 2.4.1. In the classical case of a Lie algebra, the Poincaré-Birkhoff-Witt theorem says that the natural associated graded ring of universal algebra of g is canonically isomorphic (as algebras) to the symmetric algebra of g. Rinehart proved, cf. [23], the analogous result for L: the natural associated graded ring of $U(L)$ is isomorphic to $\mathrm{Sym}_A(L)$. A spectral sequence argument gives the following:

Proposition 2.4.4. (Proposition 1.5.3 in [19]) *Suppose $L \subseteq L'$ are Lie–Rinehart algebras over (k, A) and E a left $U(L)$-module free over A. Moreover, suppose L, L' are free A-modules of finite rank such that a basis of L forms a regular sequence in the symmetric algebra $\mathrm{Sym}_A(L')$. Then $\mathrm{Sp}_{L,L'}(E)$ is a finite free $U(L')$-resolution of $U(L') \otimes_{U(L)} E$.*

We may use Proposition 2.4.4 to resolve $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$, provided f is nice enough:

Proposition 2.4.5. *Suppose $f = f_1 \cdots f_r$ is reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let $F = (f_1, \ldots, f_r)$. Then $\mathrm{Sp}_{\theta_{F,\mathfrak{x}}, \mathrm{Der}_{X,\mathfrak{x}} \oplus \mathscr{O}_{X,\mathfrak{x}}^r}$ is a free $\mathscr{D}_{X,\mathfrak{x}}[S]$-resolution of $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$.*

Proof. We argue as in Section 1.6 of [24]. First, note that by Proposition 6.3 of [25] and Corollary 1.9 of [26], that for reduced free divisors being Saito-holonomic is equivalent to being Koszul free, where Koszul free means there is a basis $\delta_1, \ldots, \delta_n$ of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$ that $\mathrm{gr}_{(0,1)}(\delta_1) \ldots \mathrm{gr}_{(0,1)}(\delta_n)$ is a regular sequence in $\mathrm{gr}_{(0,1)}(\mathscr{D}_{X,\mathfrak{x}})$. Let $\delta_1, \ldots, \delta_n$ be such a basis. Then $s_1, \ldots, s_n, \psi_{F,\mathfrak{x}}(\delta_1), \ldots, \psi_{F,\mathfrak{x}}(\delta_n)$ is a regular sequence in $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$. As these elements are all $(0,1,1)$-homogeneous, we may rearrange them and conclude $\psi_{F,\mathfrak{x}}(\delta_1), \ldots, \psi_{F,\mathfrak{x}}(\delta_n)$ is a regular sequence in $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \simeq \mathrm{Sym}_{\mathscr{O}_{X,\mathfrak{x}}}(\mathrm{Der}_{X,\mathfrak{x}} \oplus \mathscr{O}_{X,\mathfrak{x}}^r)$. Now Proposition 2.4.4 implies that $\mathrm{Sp}_{\theta_{F,\mathfrak{x}}, \mathrm{Der}_{X,\mathfrak{x}} \oplus \mathscr{O}_{X,\mathfrak{x}}^r}$ is a free $\mathscr{D}_{X,\mathfrak{x}}[S]$-resolution and inspecting the terminal map of this co-complex shows it resolves $\mathscr{D}_{X,\mathfrak{x}}[S]/\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_{F,\mathfrak{x}}$, which, by Theorem 2.2.3, is isomorphic to $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$. ◻

When f is strongly Euler-homogeneous, Saito-holonomic, and tame we showed in Proposition 2.3.7 that there is a Koszul co-complex resolution of $\mathscr{D}_{X,\mathfrak{x}}[S]F^S/(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S$. Using $\mathrm{Sp}_{\theta_{F,\mathfrak{x}}, \mathrm{Der}_{X,\mathfrak{x}} \oplus \mathscr{O}_{X,\mathfrak{x}}^r}$ we construct a free $\mathscr{D}_{X,\mathfrak{x}}[S]$-resolution of $\mathscr{D}_{X,\mathfrak{x}}[S]F^S/(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S$.

Proposition 2.4.6. *Suppose $f = f_1 \cdots f_r$ is reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let $F = (f_1, \dots, f_r)$. Then there is a finite, free resolution of $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules*

$$\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} Sp_{\theta_{F,\mathfrak{x}}, \mathrm{Der}_{X,\mathfrak{x}} \oplus \mathscr{O}^r_{X,\mathfrak{x}}} \to \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]}.$$

Proof. By Proposition 2.4.5, it is enough to prove that, for $k \geq 1$,

$$\mathrm{Tor}^k_{\mathscr{D}_{X,\mathfrak{x}}[S]}(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]}, \mathscr{D}_{X,\mathfrak{x}}[S]F^S) = 0.$$

As $K^\bullet(S-A; \mathscr{D}_{X,\mathfrak{x}}[S])$ resolves $\mathscr{D}_{X,\mathfrak{x}}[S]/(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]$, use Proposition 2.3.7. □

2.4.2 Dual of $\mathscr{D}_{X,\mathfrak{x}}[S]F^S/(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S$.

Now that we have resolutions, we can proceed to our first goal: to compute the $\mathscr{D}_{X,\mathfrak{x}}$-dual of $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S}$.

Definition 2.4.5. *(Compare with Appendix A of [9]) Consider a Lie–Rinehart algebra L over (k, A) that is A-projective of constant rank n. There is an equivalence of categories from right $U(L)$-modules Q to the left $U(L)$-modules given by $Q^{left} = \mathrm{Hom}_A(w_L, Q)$ where w_L is the dualizing module of L, namely, $w_L = \mathrm{Hom}_A(\bigwedge^n L, A)$. Regard $\mathscr{D}_{X,\mathfrak{x}}$ as the universal algebra of the Lie–Rinehart algebra $\mathrm{Der}_{X,\mathfrak{x}}$ over $(\mathbb{C}, \mathscr{O}_{X,\mathfrak{x}})$ and $\mathscr{D}_{X,\mathfrak{x}}[S]$ as the universal algebra of the Lie–Rinehart algebra $\mathrm{Der}_{X,\mathfrak{x}}[S]$ over $(\mathbb{C}[S], \mathscr{O}_{X,\mathfrak{x}}[S])$. In the appropriate derived category of left modules, where N is a left $U(L)$-module, let:*

$$\mathbb{D}(N) := (RHom_{\mathscr{D}_{X,\mathfrak{x}}}(N, \mathscr{D}_{X,\mathfrak{x}})^{left};$$

$$\mathbb{D}_S(N) := (RHom_{\mathscr{D}_{X,\mathfrak{x}}[S]}(N, \mathscr{D}_{X,\mathfrak{x}}[S])^{left}.$$

The following demystifies how $(-)^{\text{left}}$ works for the above universal algebras. Its proof is entirely similar to the classical case of $(-)^{\text{left}}$ for $\mathscr{D}_{X,\mathfrak{x}}$-modules.

Lemma 2.4.7. *Take a $\ell \times m$ matrix M with entries in $\mathscr{D}_{X,\mathfrak{x}}[S]$ so that multiplication on the left gives a map of right $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules $\mathscr{D}_{X,\mathfrak{x}}[S]^m \to \mathscr{D}_{X,\mathfrak{x}}[S]^\ell$. Here an element $\mathscr{D}_{X,\mathfrak{x}}[S]^m$ is* *a column vector. For some fixed coordinate system, define the map* $\tau : \mathscr{D}_{X,\mathfrak{x}}[S] \to \mathscr{D}_{X,\mathfrak{x}}[S]$, $\tau(x^{\boldsymbol{u}}\partial^{\boldsymbol{v}}s^{\boldsymbol{w}}) = (-\partial)^{\boldsymbol{v}}x^{\boldsymbol{u}}s^{\boldsymbol{w}}$. *Extend* τ *to* $\mathscr{D}_{X,\mathfrak{x}}[S]^m$ *in an obvious way and to* M *by applying* τ *to each entry. Then there is a commutative diagram of left* $\mathscr{D}_{X,\mathfrak{x}}[S]$*-modules, where elements in the bottom row are row vectors and* $(-)^T$ *denotes the transpose:*

$$\begin{tikzcd} (\mathscr{D}_{X,\mathfrak{x}}[S]^m)^{\text{left}} \arrow[r, "M^{\text{left}}"] \arrow[d, "\simeq"] & (\mathscr{D}_{X,\mathfrak{x}}[S]^\ell)^{\text{left}} \arrow[d, "\simeq"] \\ \mathscr{D}_{X,\mathfrak{x}}[S]^m \arrow[r, "\cdot\tau(M)^T"] & \mathscr{D}_{X,\mathfrak{x}}[S]^\ell. \end{tikzcd}$$

Given a right $\mathscr{D}_{X,\mathfrak{x}}$*-linear map* $M : \mathscr{D}_{X,\mathfrak{x}}^m \to \mathscr{D}_{X,\mathfrak{x}}^\ell$*, there is an entirely similar commutative diagram of left-*$\mathscr{D}_{X,\mathfrak{x}}$ *modules (where* τ *has the obvious definition).*

The first step in computing $\mathbb{D}(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A\mathscr{D}_{X,\mathfrak{x}}[S])})$ is finding a resolution–this is Proposition 2.4.6. The second is the following technical lemma:

Lemma 2.4.8. *Suppose* $f = f_1 \cdots f_r$ *is reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let* $F = (f_1, \ldots, f_r)$*. As complexes of free* $\mathscr{D}_{X,\mathfrak{x}}$*-modules,*

$$\mathbb{D}\Big(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} Sp_{\theta_{F,\mathfrak{x}},\mathrm{Der}_{X,\mathfrak{x}} \oplus \mathscr{O}_{X,\mathfrak{x}}^r}\Big)$$
$$\simeq \frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathbb{D}_S\left(Sp_{\theta_{F,\mathfrak{x}},\mathrm{Der}_{X,\mathfrak{x}} \oplus \mathscr{O}_{X,\mathfrak{x}}^r}\right)$$

Proof. For brevity, abbreviate $\mathrm{Sp}_{\theta_{F,\mathfrak{x}},\mathrm{Der}_{X,\mathfrak{x}} \oplus \mathscr{O}_{X,\mathfrak{x}}^r}$ to $\mathrm{Sp}^\bullet$. Write the differential as $d^{-k} : \mathrm{Sp}^{-k} \to \mathrm{Sp}^{-(k-1)}$.

We will first compute the objects and maps of $\mathbb{D}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathrm{Sp}\right)$. Since Sp is a co-complex of finite, free $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules, $\mathrm{Sp}^{-k} \simeq \mathscr{D}_{X,\mathfrak{x}}[S]^{\binom{n}{k}}$. Therefore, as $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules,

$$\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathrm{Sp}^{-k} \simeq \frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]}^{\binom{n}{k}}. \tag{2.4.2}$$

On the LHS of (2.4.2), we have the differential $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} d^{-k}$. Think of d^{-k} as a matrix. On the RHS of (2.4.2) the differential is $\operatorname{eval}_A(d^{-k})$: the matrix d^{-k} except each s_i is replaced with a_i. As right $\mathscr{D}_{X,\mathfrak{x}}$-modules,

$$\begin{aligned}\operatorname{Hom}_{\mathscr{D}_{X,\mathfrak{x}}}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathrm{Sp}^{-k}, \mathscr{D}_{X,\mathfrak{x}}\right) &\simeq \operatorname{Hom}_{\mathscr{D}_{X,\mathfrak{x}}}\left(\mathscr{D}_{X,\mathfrak{x}}^{\binom{n}{k}}, \mathscr{D}_{X,\mathfrak{x}}\right) \\ &\simeq \mathscr{D}_{X,\mathfrak{x}}^{\binom{n}{k}}.\end{aligned}$$

Making the above identification, $\operatorname{Hom}_{\mathscr{D}_{X,\mathfrak{x}}}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathrm{Sp}, \mathscr{D}_{X,\mathfrak{x}}\right)$ has a differential given by multiplication on the left by $(\operatorname{eval}_A(d^{-k}))^T$–the transpose of $\operatorname{eval}_A(d^{-k})$. To make the Hom complex a complex of left modules we apply the equivalence of categories $(-)^{\text{left}}$. By Lemma 2.4.7 we get a complex of left $\mathscr{D}_{X,\mathfrak{x}}$ modules isomorphic to the following, with differential given by matrix multiplication on the right

$$A_\bullet := \ldots \to \mathscr{D}_{X,\mathfrak{x}}^{\binom{n}{k-1}} \xrightarrow{\cdot\tau((\operatorname{eval}_A(d^{-k}))^T)^T} \mathscr{D}_{X,\mathfrak{x}}^{\binom{n}{k}} \to \ldots.$$

Now we compute the objects and maps of $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathbb{D}_S(\mathrm{Sp})$. As right $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules, $\operatorname{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}\left(\mathrm{Sp}^{-k}, \mathscr{D}_{X,\mathfrak{x}}[S]\right) \simeq \mathscr{D}_{X,\mathfrak{x}}[S]^{\binom{n}{k}}$. The induced differential is multiplication on the left by $(d^{-k})^T$. By Lemma 2.4.7, we can identify the complex obtained by applying $(-)^{\text{left}}$ with a complex whose terms are $\mathscr{D}_{X,\mathfrak{x}}[S]^{\binom{n}{k}}$ and whose differentials are $\tau((d^{-k})^T)^T$. As left $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules (and so as left $\mathscr{D}_{X,\mathfrak{x}}$-modules),

$$\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathscr{D}_{X,\mathfrak{x}}[S]^{\binom{n}{k}} \simeq \frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]}^{\binom{n}{k}}. \tag{2.4.3}$$

The RHS of (2.4.3) is isomorphic as a left $\mathscr{D}_{X,\mathfrak{x}}$-module to $\mathscr{D}_{X,\mathfrak{x}}^{\binom{n}{k}}$. With this identification, the differentials of the complex $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathbb{D}_S(\mathrm{Sp})$ are given by $\operatorname{eval}_A(\tau((d^{-k})^T)^T)$. Thus the complex of left $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathbb{D}_S(\mathrm{Sp})$ is isomorphic as a complex of left $\mathscr{D}_{X,\mathfrak{x}}$-modules to

$$B_\bullet := \ldots \to \mathscr{D}_{X,\mathfrak{x}}^{\binom{n}{k-1}} \xrightarrow{\cdot\operatorname{eval}_A(\tau((d^{-k})^T)^T)} \mathscr{D}_{X,\mathfrak{x}}^{\binom{n}{k}} \to \ldots.$$

We will be done once we show that $A_\bullet$ and $B_\bullet$ are isomorphic complexes of $\mathscr{D}_{X,\mathfrak{x}}$-modules. Because $\tau((\mathrm{eval}_A(d^{-k}))^T)^T = \tau(\mathrm{eval}_A(d^{-k})) = \mathrm{eval}_A(\tau(d^{-k})) = \mathrm{eval}_A(\tau((d^{-k})^T)^T)$, $A_\bullet$ and $B_\bullet$ have the same differentials. $\square$

So we have reduced our problem to, in light of Proposition 2.4.5, computing $\mathbb{D}_S(\mathscr{D}_{X,\mathfrak{x}}[S]F^S)$. In Corollary 3.6 of [9] Narváez–Macarro does this for $\mathscr{D}_{X,\mathfrak{x}}[s]f^s$ with similar working hypotheses as ours and Maisonobe shows in [11] that this result generalizes to $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$ as well. In our language, cf. the proof of Proposition 2.4.5, this result is as follows:

Proposition 2.4.9. (Proposition 6 in [11]) *Let $f = f_1 \cdots f_r$ be reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let $F = (f_1, \dots, f_r)$. Then, in the category of left derived $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules, there is a canonical isomorphism*

$$\mathbb{D}_S(\mathscr{D}_{X,\mathfrak{x}}[S]F^S) \simeq \mathscr{D}_{X,\mathfrak{x}}[S]F^{-S-1}[n].$$

Theorem 2.4.2. *Suppose $f = f_1 \cdots f_r$ is reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let $F = (f_1, \dots, f_r)$. Then in the derived category of $\mathscr{D}_{X,\mathfrak{x}}$-modules there is a $\mathscr{D}_{X,\mathfrak{x}}$-isomorphism χ_A given by*

$$\begin{aligned} \chi_A : \mathbb{D}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S}\right) &\xrightarrow{\simeq} \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^{-S-1}}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^{-S-1}}[n] \\ &\xrightarrow{\simeq} \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(-A-1)\mathscr{D}_{X,\mathfrak{x}}[S]F^S}[n]. \end{aligned} \tag{2.4.4}$$

Proof. The $\mathscr{D}_{X,\mathfrak{x}}$-linear involution on $\mathscr{D}_{X,\mathfrak{x}}[S]$ defined by sending each $s_k \mapsto -s_k - 1$ induces a $\mathscr{D}_{X,\mathfrak{x}}$-linear map $\mathscr{D}_{X,\mathfrak{x}}[S]F^{-S-1} \simeq \mathscr{D}_{X,\mathfrak{x}}[S]F^S$. This gives the second isomorphism of (2.4.4). Considerations using this map and Proposition 2.4.5 show that $\mathrm{Tor}^k_{\mathscr{D}_{X,\mathfrak{x}}[S]}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S}, \mathscr{D}_{X,\mathfrak{x}}[S]F^{-S-1}\right)$ vanishes for $k \geq 1$. Proposition 2.4.9 then implies the acylicity of the augmented co-complex

$$\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathbb{D}_S\left(\mathrm{Sp}_{\theta_{F,\mathfrak{x}}, \mathrm{Der}_{X,\mathfrak{x}} \oplus \mathscr{O}^r_{X,\mathfrak{x}}}\right) \to \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^{-S-1}}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^{-S-1}}.$$

This, Proposition 2.4.6, and Lemma 2.4.8 give the first isomorphism of (2.4.4). $\square$

Remark 2.4.10. When f is reduced, strongly Euler-homogeneous, Saito-holonomic, and free, this immediately implies $\mathscr{D}_{X,\mathfrak{x}}[S]F^S/(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S$ is a holonomic $\mathscr{D}_{X,\mathfrak{x}}$-module. Without freeness, computing Ext is currently intractable.

2.4.3 Free Divisors and ∇_A.

Recall from Definition 2.3.1 the $\mathscr{D}_{X,\mathfrak{x}}$-linear map

$$\nabla_A : \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \to \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}$$

induced by $s_k \mapsto s_k + 1$, for each k. If f is reduced, strongly Euler-homogeneous, Saito-holonomic, and free, by Proposition 2.4.2 the complexes $\mathbb{D}(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S})$ and $\mathbb{D}(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S})$ can be identified with modules (i.e. Ext vanishes in all but one place). ∇_A lifts to a map between the resolutions of $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S}$ and $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}$ and to the Hom of those resolutions. Therefore ∇_A induces a map (thinking of these as modules)

$$\mathbb{D}(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S}) \longrightarrow \mathbb{D}(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}).$$

Name this map $\mathbb{D}(\nabla_A)$.

Theorem 2.4.3. *Suppose $f = f_1 \cdots f_r$ is reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let $F = (f_1, \ldots, f_r)$. Let χ_A be the $\mathscr{D}_{X,\mathfrak{x}}$-isomorphism of Theorem 2.4.2. Then there is a commutative diagram*

$$\begin{tikzcd}
\mathbb{D}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]}\right) \arrow[r, "\simeq", "\chi_A"'] & \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(-A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \\
\mathbb{D}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]}\right) \arrow[u, "\mathbb{D}(\nabla_A)"] \arrow[r, "\simeq", "\chi_{A-1}"'] & \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(-A))\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \arrow[u, "\nabla_{-A}"']
\end{tikzcd}.$$

Proof. First consider the $\mathscr{D}_{X,\mathfrak{x}}$-linear map $\nabla : \mathscr{D}_{X,\mathfrak{x}}[S]F^S \to \mathscr{D}_{X,\mathfrak{x}}[S]F^S$ given by sending $s_i \to s_{i+1}$ for all i. By Propositon 2.4.5, the co-complex of free $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules

$\mathrm{Sp}_{\theta_{F,\mathfrak{x}},\mathrm{Der}_{X,\mathfrak{x}}\oplus\mathscr{O}^r_{X,\mathfrak{x}}}$ resolves $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$. For readability, in this proof we will write this co-complex as Sp. Regarding this as a co-complex of $\mathscr{D}_{X,\mathfrak{x}}$-modules, we may lift ∇ to a chain map. A straightforward computation using (2.4.1) and the definition of $\psi_{F,\mathfrak{x}}$ shows that one such lift is given by

$$\begin{tikzcd}
\mathrm{Sp}^{-k} \arrow[r, "\simeq"] \arrow[d, dashed] & \mathscr{D}_{X,\mathfrak{x}}[S]^{\binom{n}{k}} \arrow[d, "\sigma_{-k}"] \\
\mathrm{Sp}^{-k} \arrow[r, "\simeq"] & \mathscr{D}_{X,\mathfrak{x}}[S]^{\binom{n}{k}},
\end{tikzcd}$$

where the dashed line is the lift of ∇ at the $-k$ slot and σ_{-k} multiplies each component of the direct sum by f on the right and sends each s_i to s_{i+1} in every component.

We may use the finite, free $\mathscr{D}_{X,\mathfrak{x}}$-resolution of $\mathscr{D}_{X,\mathfrak{x}}[S]F^S/(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S$ by $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathrm{Sp}$ to lift ∇_A to a chain map, cf. Proposition 2.4.6. One such lift is given by

$$\begin{tikzcd}
\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathrm{Sp}^{-k} \arrow[r, "\simeq"] \arrow[d, dashed, "\ell_{-k}(\nabla_A)"] & \frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]}^{\binom{n}{k}} \arrow[d, "\sigma^A_{-k}"] \\
\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathrm{Sp}^{-k} \arrow[r, "\simeq"] & \frac{\mathscr{D}_{X,\mathfrak{x}}[S](S-(A-1))}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]}^{\binom{n}{k}},
\end{tikzcd} \tag{2.4.5}$$

where the $\ell_{-k}(\nabla_A)$ is the lift of ∇_A the $-k$ slot and σ^A_{-k} is induced by σ_{-k}. That is, σ^A_{-k} is given by multiplying each component of the direct sum by f on the right.

Apply $\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}}(-,\mathscr{D}_{X,\mathfrak{x}})^{\mathrm{left}}$ to the chain map given by the $l_{-k}(\nabla_A)$. Then (2.4.5) implies that at the $-n$ slot we have

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathrm{Sp}^{-n}, \mathscr{D}_{X,\mathfrak{x}}\right)^{\mathrm{left}} & \frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \arrow[l, "\simeq"'] \\
\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]} \otimes_{\mathscr{D}_{X,\mathfrak{x}}[S]} \mathrm{Sp}^{-n}, \mathscr{D}_{X,\mathfrak{x}}\right)^{\mathrm{left}} \arrow[u, dashed, "\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}}(\ell_{-n}(\nabla_A),\mathscr{D}_{X,\mathfrak{x}})^{\mathrm{left}}"] & \frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-(A-1)\mathscr{D}_{X,\mathfrak{x}}[S]}, \arrow[l, "\simeq"'] \arrow[u, "\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}}(\sigma^A_{-n},\mathscr{D}_{X,\mathfrak{x}})^{\mathrm{left}}"']
\end{tikzcd} \tag{2.4.6}$$

where $\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}}(\sigma^A_{-n},\mathscr{D}_{X,\mathfrak{x}})^{\mathrm{left}}$ is simply multiplication by f on the right. Since $\mathbb{D}(\mathscr{D}_{X,\mathfrak{x}}[S]F^S/(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S)$ has nonzero homology only at the $-n$ slot, we may identify this complex with that homology module and the map $\mathbb{D}(\nabla_A)$ is induced by

$\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}}(\sigma^A_{-n}, \mathscr{D}_{X,\mathfrak{x}})^{\mathrm{left}}$, i.e. by multiplication by f on the right. So (2.4.6) and Theorem 2.4.2 give the following commutative diagram

$$\begin{tikzcd}
\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]} \arrow[r, two heads] & \mathbb{D}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S}\right) \arrow[r, "\simeq", "\chi_A"'] & \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(-A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \\
\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]} \arrow[u, "\cdot f"] \arrow[r, two heads] & \mathbb{D}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}\right) \arrow[u, "\mathbb{D}(\nabla_A)"] \arrow[r, "\simeq", "\chi_{A-1}"'] & \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(-A))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}. \arrow[u, dashed]
\end{tikzcd}$$

A straightforward diagram chase shows that the dashed map is ∇_{-A}. $\square$

Theorem 2.4.4. *Suppose $f = f_1 \cdots f_r$ is reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let $F = (f_1, \ldots, f_r)$. Then ∇_A is injective if and only if it is surjective.*

Proof. By Theorem 2.3.2, we may assume ∇_A is surjective. So we have a short exact sequence of holonomic left $\mathscr{D}_{X,\mathfrak{x}}$-modules:

$$0 \to N \to \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \xrightarrow{\nabla_A} \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \to 0.$$

Using the long exact sequence of Ext and basic properties of holonomic modules, one checks ∇_A is surjective if and only if $\mathbb{D}(\nabla_A)$ is injective. Similarly, ∇_A is injective if and only if $\mathbb{D}(\nabla_A)$ is surjective. We are done by the following:

$$\begin{aligned}
\nabla_A \text{ is surjective} &\iff \mathbb{D}(\nabla_A) \text{ is injective} && \text{[Duality]} \\
&\iff \nabla_{-A} \text{ is injective} && \text{[Theorem 2.4.3]} \\
&\implies \nabla_{-A} \text{ is surjective} && \text{[Theorem 2.3.2]} \\
&\iff \mathbb{D}(\nabla_A) \text{ is surjective} && \text{[Theorem 2.4.3]} \\
&\iff \nabla_A \text{ is injective} && \text{[Duality]}.
\end{aligned}$$

$\square$

2.5 Free Divisors and the Cohomology Support Loci

In this short section, we assume $f_1, \ldots, f_r$ are mutually distinct and irreducible germs at $\mathfrak{x} \in X$ that vanish at $\mathfrak{x}$. Let $f = f_1 \cdots f_r$. Take a small open ball $B_{\mathfrak{x}}$ about $\mathfrak{x}$ and let $U_{\mathfrak{x}} = B_{\mathfrak{x}} \setminus \operatorname{Var}(f)$. Define $U_{\mathfrak{y}}$ for $\mathfrak{y} \in \operatorname{Var}(f)$ and $\mathfrak{y}$ near $\mathfrak{x}$ similarly.

Definition 2.5.1. *(*Compare with Section 1, [17]) Let $M(U)$ denote the rank one local systems on U. Define the *cohomology support loci of f near* $\mathfrak{x}$, denoted as $V(U_{\mathfrak{x}}, B_{\mathfrak{x}})$, by:

$$V(U_{\mathfrak{x}}, B_{\mathfrak{x}}) := \bigcup_{\mathfrak{y} \in D \text{ near } \mathfrak{x}} \operatorname{res}_{\mathfrak{y}}^{-1}(\{L \in M(U_{\mathfrak{y}}) \mid H^{\bullet}(U_{\mathfrak{y}}, L) \neq 0\}),$$

where $\operatorname{res}_{\mathfrak{y}} : M(U_{\mathfrak{x}}) \to M(U_{\mathfrak{y}})$ is given by restriction. This agrees with the notion of "uniform cohomology support locus" given in [6], cf. Remark 2.8 [17] and [27].

Convention 2.5.1. For $A \in \mathbb{C}^r$ and $k \in \mathbb{Z}$, let $A - k$ denote $(a_1 - k, \ldots, a_r - k)$.

Let j be the embedding of $U_{\mathfrak{x}} \hookrightarrow B_{\mathfrak{x}}$. For $L \in M(U_{\mathfrak{x}})$, $Rj_{\star}(L[n])$ is a perverse sheaf (hence of finite length). In Theorem 1.5 of [17], the authors prove that

$$V(U_{\mathfrak{x}}, B_{\mathfrak{x}}) = \{L \in M(U_{\mathfrak{x}}) \mid Rj_{\star}(L[n]) \text{ is not a simple perverse sheaf on } B_{\mathfrak{x}}\}. \tag{2.5.1}$$

Using this Budur proves in Theorem 1.5 of [6], cf. Remark 4.2 of [17], that

$$\operatorname{Exp}(\mathrm{V}(B_{F,\mathfrak{x}})) \supseteq V(U_{\mathfrak{x}}, B_{\mathfrak{x}}). \tag{2.5.2}$$

Here $M(U_{\mathfrak{x}})$ are identified with representations $\{\pi_1(U_{\mathfrak{x}}) \to \mathbb{C}^{\star}\} \subseteq \mathbb{C}^{\star^r}$.

While we cannot prove the converse containment to (2.5.2), we can prove a weaker statement about simplicity of modules:

Theorem 2.5.2. *Suppose $f = f_1 \cdots f_r$ and $F = (f_1, \ldots, f_r)$, where the f_k are mutually distinct and irreducible hypersurface germs at $\mathfrak{x}$ vanishing at $\mathfrak{x}$. Suppose f is reduced, strongly Euler-homogeneous, Saito-holonomic, and free. If $A \in \mathbb{C}^r$ such that the rank one local system $L_{\operatorname{Exp}(A)} \notin V(U_{\mathfrak{x}}, B_{\mathfrak{x}})$, then, for all $k \in \mathbb{Z}$, the map ∇_{A+k} is an isomorphism and $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A+k))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}$ is a simple $\mathscr{D}_{X,\mathfrak{x}}$-module.*

Proof. For all $A \in \mathbb{C}^r$ there is a cyclic $\mathscr{D}_{X,\mathfrak{x}}$-module $\mathscr{D}_{X,\mathfrak{x}}F^A$ defined similarly to $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$. Moreover, there is a commutative diagram of $\mathscr{D}_{X,\mathfrak{x}}$-modules and maps

$$\begin{tikzcd} \frac{\mathscr{D}_X[S]F^S}{(S-(A+k))\mathscr{D}_X[S]F^S} \arrow[r, two heads, "p_{A+k}"] \arrow[d, "\nabla_{A+k}"] & \mathscr{D}_{X,\mathfrak{x}}F^{A+k} \arrow[d, hook] \\ \frac{\mathscr{D}_X[S]F^S}{(S-(A+k-1))\mathscr{D}_X[S]F^S} \arrow[r, two heads, "p_{A+k-1}"] & \mathscr{D}_{X,\mathfrak{x}}F^{A+k-1}. \end{tikzcd} \tag{2.5.3}$$

By Theorem 5.2 in [6], $\mathscr{D}_{X,\mathfrak{x}}F^A$ is regular, holonomic and

$$DR(\mathscr{D}_{X,\mathfrak{x}}F^{A-k}) = Rj_\star L_{\mathrm{Exp}(A)}[n], \text{ for } k \in \mathbb{N},\ k \gg 0;$$
$$DR(\mathscr{D}_{X,\mathfrak{x}}F^{A+k}) = IC(L_{\mathrm{Exp}(A)}[n]), \text{ for } k \in \mathbb{N},\ k \gg 0.$$

Here DR is the de Rham functor and $L_{\mathrm{Exp}(A)}$ is the local system given by a representation $\pi_1(U_{\mathfrak{x}}) \to \mathbb{C}^{\star^r}$. Because of (2.5.1), our hypotheses on $L_{\mathrm{Exp}(A)}$ imply $\mathscr{D}_{X,\mathfrak{x}}F^{A+k}$ is simple for all $k \in \mathbb{Z}$. So to prove the theorem, it suffices to show that the $\mathscr{D}_{X,\mathfrak{x}}$-maps p_{A+k} and ∇_{A+k} of (2.5.3) are isomorphisms for all $k \in \mathbb{Z}$.

By Proposition 3.2 and 3.3 of [28] there exists an integer $t \in \mathbb{Z}$ such that $p_{A+t-1-j}$ is an isomorphism for all $j \in \mathbb{Z}_{\geq 0}$. By the commutativity of (2.5.3), ∇_{A+t} is surjective. By Theorem 2.4.4, ∇_{A+t} is an isomorphism. Thus p_{A+t} is as well. Repeat this procedure to finish the proof. □

2.6 Initial Ideals

Suppose the commutative Noetherian ring R is a domain containing a field $\mathbb{K}$. Consider the polynomial ring over many variables $R[X]$, graded by the total degree of a non-negative integral vector $\mathbf{u}$. Let I be an ideal contained in $(X) \cdot R[X]$. We closely follow the treatment of Bruns and Conca in [29] to obtain our main result, Proposition 2.6.6, which establishes a relationship between the initial ideal $\mathrm{in}_{\mathbf{u}}$ of I with respect to the $\mathbf{u}$-grading and I itself. This is a weaker analogue to Proposition 3.1 of loc. cit. and is integral to the strategy of Section 2.

Remark 2.6.1.(a) The *monomials* of $R[X]$ are the elements $x^{\mathbf{v}} = \prod x_i^{v_i}$ for $\mathbf{v}$ a non-negative integral vector.

(b) Just as in the case $R = \mathbb{K}$ we can declare a *monomial ordering* $>$ on $R[X]$. This ordering is Artinian, with least element $1 \in R$.

(c) Every $f \in R[X]$ has a unique expression in monomials: $f = \sum r_i m_i$, $r_i \in R$, m_i a monomial, $m_i > m_{i+1}$, for some total ordering $>$ of the monomials.

(d) Let the *initial term* of f be $\mathrm{in}_>(f) := r_1 m_1$, where we appeal to the unique expression of f above. For V a R-submodule of $R[X]$ let $\mathrm{in}_>(V)$ be the R-submodule generated by all the $\mathrm{in}_>(f)$ elements for $f \in V$.

(e) Given a nonnegative integral vector $\mathbf{u} = (u_1, \dots u_n)$ there is a canonical grading on $R[X]$ given by $\mathbf{u}(x_i) = u_i$. Every monomial $\prod x_i^{v_i}$ is $\mathbf{u}$-homogeneous of degree $\sum v_i u_i$ and every element $f \in R[X]$ has a unique decomposition into $\mathbf{u}$-homogeneous pieces. The *degree* $\mathbf{u}(f)$ is the largest degree of a monomial of f; the *initial term* $\mathrm{in}_{\mathbf{u}}(f)$ is the sum of the monomials of f of largest degree.

Definition 2.6.1. *Let $f \in R[X]$, $f = \sum r_i m_i$ its monomial expression, $\boldsymbol{u}$ a non-negative integral vector defining a grading on $R[X]$. We introduce a new variable t by letting $T =$ R[X][t]. Define the* homogenization *of f with respect to $\boldsymbol{u}$ to be*

$$\mathrm{hom}_{\boldsymbol{u}}(f) := \sum r_i m_i t^{\boldsymbol{u}(f) - \boldsymbol{u}(m_i)} \in T.$$

For a R-submodule V of $R[X]$ let

$$\mathrm{hom}_{\boldsymbol{u}}(V) := \textit{ the } R[t]\textit{-submodule generated by } \{\mathrm{hom}_{\boldsymbol{u}}(f) \mid f \in V\}.$$

Remark 2.6.2.(a) If I is an ideal of $R[X]$, $\mathrm{hom}_{\mathbf{u}}(I)$ is an ideal of T.

(b) Let $\mathbf{u}'$ be the non-negative integral vector $(\mathbf{u}, 1)$ and extend the grading on $R[X]$ to T by declaring t to have degree 1. Then $\mathrm{hom}_{\mathbf{u}}(f)$ is a $\mathbf{u}'$-homogeneous of degree $\mathbf{u}(f)$.

Proposition 2.6.3. *Suppose R is a Noetherian domain containing the field $\mathbb{K}$ and let I be an ideal of $R[X]$. Then*

$$\frac{T}{(\hom_{\boldsymbol{u}}(I), t)} \simeq \frac{R[X]}{\mathrm{in}_{\boldsymbol{u}}(I)} \quad \text{and} \quad \frac{T}{(\hom_{\boldsymbol{u}}(I), t-1)} \simeq \frac{R[X]}{I}.$$

Proof. Argue as in Proposition 2.4 of [29]. □

As in the classical case, t is regular on $T/\hom_{\mathbf{u}}(I)$ (cf. Proposition 2.3 (d) in [29]). The argument is similar so we only outline the basic steps.

Definition 2.6.2. *Let τ be a monomial ordering on $R[X]$, $\boldsymbol{u}$ a non-negative integral vector. Let n, m be monomials in $R[X]$. Define a monomoial ordering $\tau\boldsymbol{u}$ on $R[X]$:*

$$[m >_{\tau\boldsymbol{u}} n] \iff [\boldsymbol{u}(m) > \boldsymbol{u}(n)] \quad \text{or} \quad [\boldsymbol{u}(m) = \boldsymbol{u}(n),\, m >_\tau n].$$

Define a monomial ordering $\tau\boldsymbol{u}'$ on T:

$$\begin{aligned}[mt^i >_{\tau\boldsymbol{u}'} nt^j] \iff & [\boldsymbol{u}'(mt^i) > \boldsymbol{u}'(nt^j)] \text{ or,} \\ & [\boldsymbol{u}'(mt^i) = \boldsymbol{u}'(nt^j) \text{ and } i < j] \text{ or,} \\ & [\boldsymbol{u}'(mt^i) = \boldsymbol{u}'(nt^j) \text{ and } i < j \text{ and } m >_\tau n].\end{aligned}$$

Lemma 2.6.4. (Compare with 2.3(c) in [29]) *Suppose R is a Noetherian domain containing the field $\mathbb{K}$. For V a R-submodule of $R[X]$,*

$$\mathrm{in}_{\tau\boldsymbol{u}}(V)R[t] = \mathrm{in}_{\tau\boldsymbol{u}'}(\hom_{\boldsymbol{u}}(V)).$$

Proof. Argue similarly to Proposition 2.3 (c) in [29]. □

Proposition 2.6.5. (Compare with 2.3(d) in [29]) *Suppose R is a Noetherian domain containing the field $\mathbb{K}$. Let $I \subseteq X \cdot R[X]$ be an ideal of $R[X]$ and $\boldsymbol{u}$ a nonnegative integral vector. Then $T/\hom_{\boldsymbol{u}}(I)$ is a torsion-free $\mathbb{K}[t]$ module.*

Proof. We give a sketch. Suppose $h \in T$, $s(t) \in \mathbb{K}[t]$ such that $s(t)h \in \hom_{\mathbf{u}}(I)$. We must show that $h \in \hom_{\mathbf{u}}(I)$.

Because $\tau\mathbf{u}'$ is a monomial order, $\operatorname{in}_{\tau\mathbf{u}'}(s(t)h) = s_k t^k \operatorname{in}_{\tau\mathbf{u}'}(h)$, for $s_k \in K$. By hypothesis and Lemma 2.6.4, $s_k t^k \operatorname{in}_{\tau\mathbf{u}'}(h) \in \operatorname{in}_{\tau\mathbf{u}}(I)R[t]$. By comparing monomials and using the fact we can "divide" an equation by t if both sides are multiples of t, careful bookkeeping yields the following: there exists $g \in \hom_{\mathbf{u}}(I)$ such that $h - g < h$ and $s(t)(h-g) \in \hom_{\mathbf{u}}(I)$. Repeat the process to continually peel off initial terms and conclude either $h \in \hom_{\mathbf{u}}(I)$ or there exists $0 \neq r \in R \cap \operatorname{in}_{\tau\mathbf{u}}(I)$. Because $I \subseteq X \cdot R[X]$, we have $\operatorname{in}_{\tau\mathbf{u}}(I) \subseteq X \cdot R[X]$. Therefore no such r exists and the claim is proved. □

The following is the section's main proposition:

Proposition 2.6.6. (Compare with 3.1 in [29]) *Suppose R is a Noetherian domain containing the field $\mathbb{K}$. Let $I \subseteq X \cdot R[X]$ be an ideal of $R[X]$ and $\boldsymbol{u}$ a non-negative integral vector. Then the following hold:*

(a) If $R[X]/\operatorname{in}_{\boldsymbol{u}}(I)$ is Cohen–Macaulay, then $R[X]/I$ is Cohen–Macaulay;

(b) $\dim(R[X]/\operatorname{in}_{\boldsymbol{u}}(I)) \geq \dim(R[X]/I)$.

Proof. (a). We follow the argument of Proposition 3.1 in [29]: first, we show that Cohen–Macaulayness percolates from $T/(\hom_{\mathbf{u}}(I), t)$ to $T/\hom_{\mathbf{u}}(I)$; second, that it descends from $T/\hom_{\mathbf{u}}(I)$ to $T/(\hom_{\mathbf{u}}(I), t-1)$.

First, the percolation. Since $\mathbf{u}'(t) = 1$, any maximal $\mathbf{u}'$-graded ideal $\mathfrak{m}^\star$ of $T/\hom_{\mathbf{u}}(I)$ contains t. Consider the commutative diagram

$$\begin{array}{ccc} T/\hom_{\mathbf{u}}(I) & \longrightarrow & T_m/(\hom_{\mathbf{u}}(I))_{\mathfrak{m}^\star} \\ \downarrow & & \downarrow \\ T/(\hom_{\mathbf{u}}(I), t) & \longrightarrow & T_{\mathfrak{m}^\star}/(\hom_{\mathbf{u}}(I), t)_{\mathfrak{m}^\star}, \end{array}$$

with horizontal maps localization at $\mathfrak{m}^\star$, vertical maps quotients by t.

It suffices to show that $T/\hom_{\mathbf{u}}(I)$ is Cohen–Macaulay after localizing at a maximal $\mathbf{u}'$-graded ideal $\mathfrak{m}^\star$ (cf. Exercise 2.1.27 [22]). Since $t \in \mathfrak{m}^\star$, by assumption $T_{\mathfrak{m}^\star}/(\hom_{\mathbf{u}}(I), t)_{\mathfrak{m}^\star}$ is Cohen–Macaulay. And since t is a non-zero divisor on $T_{\mathfrak{m}^\star}/\hom_{\mathbf{u}}(I)_{\mathfrak{m}^\star}$ by Proposition 2.6.5, we see $T_{\mathfrak{m}^\star}/\hom_{\mathbf{u}}(I)_{\mathfrak{m}^\star}$ is Cohen–Macaulay (cf. Theorem 2.1.3 in [22]).

It remains to show that Cohen–Macaulayness descends from $T/\hom_{\mathbf{u}}(I)$ to $T/(\hom_{\mathbf{u}}(I), t-1)$. By the universal property of localization we have:

$$\begin{tikzcd} T/\hom_{\mathbf{u}}(I) \arrow[r] \arrow[d] & T/(\hom_{\mathbf{u}}(I), t-1) \\ (T/\hom_{\mathbf{u}}(I))[t^{-1}]. \arrow[ur, "\gamma"] & \end{tikzcd} \tag{2.6.1}$$

It is well known (cf. Proposition 1.5.18 in [22]) that

$$(T/\hom_{\mathbf{u}}(I))[t^{-1}] \simeq ((T/\hom_{\mathbf{u}}(I))[t^{-1}])_0[y, y^{-1}].$$

So γ of (2.6.1) induces, where $-_0$ denotes the degree 0 elements, the ring maps:

$$T/(\hom_{\mathbf{u}}, t-1) \simeq \frac{(T/\hom_{\mathbf{u}}(I))[t^{-1}]}{(t-1)(T/\hom_{\mathbf{u}}(I))[t^{-1}])} \simeq ((T/\hom_{\mathbf{u}}(I))[t^{-1}])_0.$$

We have

$$(T/\hom_{\mathbf{u}}(I))[t^{-1}] \simeq (T/(\hom_{\mathbf{u}}(I), t-1))[y, y^{-1}]. \tag{2.6.2}$$

Therefore, since Cohen–Macualayness is preserved under localization at a non-zero divisor, all we need to show is that if $B[y, y^{-1}]$ is a Laurent polynomial ring that is Cohen–Macaulay then B is an Cohen–Macaulay. To see this take a $\mathfrak{m} \in \operatorname{mSpec}(B)$ and look at $(\mathfrak{m}, y-1) \in \operatorname{Spec}(B[y])$ and the corresponding prime ideal in $B[y, y^{-1}]$.

Now we move onto (b). The descent part of part (a) gives us the plan:

$$\begin{aligned} \dim(T/(\hom_{\mathbf{u}}(I), t-1)) &= \dim((T/(\hom_{\mathbf{u}}(I), t-1))[y, y^{-1}]) - 1 \\ &= \dim((T/\hom_{\mathbf{u}}(I))[t^{-1}]) - 1 \\ &\leq \dim(T/\hom_{\mathbf{u}}(I)) - 1 \\ &= \dim(T/(\hom_{\mathbf{u}}(I), t)). \end{aligned}$$

The second equality follows by (2.6.2). The inequality is not an equality because localization may lower dimension. For the last equality use the fact dimension of a graded ring can be computed by looking only at the height of the graded maximal ideals (Corollary 13.7 [20]). In $T/\hom_{\mathbf{u}}(I)$, t is contained in all graded maximal ideals; since it is a non-zero divisor, its associated primes are not minimal. $\square$

Remark 2.6.7.(a) This proposition generalizes the common geometric picture for $R = \mathbb{K}$. In this setting the map $\mathbb{K}[t] \to T/\hom_{\mathbf{u}}(I)$ gives a flat family whose generic fiber is R/I and whose special fiber is $R/\operatorname{in}_{\mathbf{u}}(I)$. In our generality, it is easy to extend Proposition 2.6.5 and show that $\mathbb{K}[t] \hookrightarrow T/\hom_{\mathbf{u}}(I)$ is a flat ring map whose special fiber is $R[X]/\operatorname{in}_{\mathbf{u}}(I)$ and whose generic fiber is $R[X]/I$.

(b) In Section 2 we study ideals $I \subseteq (Y, S) \cdot \mathscr{O}_{X,\mathfrak{x}}[Y, S]$ where $\mathscr{O}_X$ is an analytic structure sheaf and the $\mathbf{u}$-grading assigns 1 to the y-terms and 0 to the s-terms. Proposition 2.6.6 applies with $R = \mathscr{O}_{X,\mathfrak{x}}$.

3. COMBINATORIALLY DETERMINED ZEROES OF BERNSTEIN–SATO IDEALS FOR TAME AND FREE ARRANGEMENTS[1]

3.1 Introduction

Consider a central, not necessarily reduced, hyperplane arrangement cut out by $f \in \mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n]$. Given a factorization $f = f_1 \cdots f_r$, not necessarily into linear terms, and letting $F = (f_1, \ldots, f_r)$, there is a free $\mathbb{C}[X][\frac{1}{f}][s_1, \ldots, s_r]$-module generated by the symbol $F^S = f_1^{s_1} \cdots f_r^{s_r}$. This module has an $\mathrm{A}_n(\mathbb{C})[S] = \mathrm{A}_n(\mathbb{C})[s_1, \ldots, s_r]$-module structure, where $\mathrm{A}_n(\mathbb{C})[S]$ is a polynomial ring extension over the Weyl algebra, given by the formal rules of calculus. We will denote the $\mathrm{A}_n(\mathbb{C})[S]$-module generated by F^S as $\mathrm{A}_n(\mathbb{C})[S]F^S$. For f' and $g \in \mathbb{C}[X]$ dividing f we study the polynomials $B(S) \in \mathbb{C}[S] = \mathbb{C}[s_1, \ldots, s_r]$ satisfying the functional equation

$$B(S)f'F^S \in \mathrm{A}_n(\mathbb{C})[S]gf'F^S. \tag{3.1.1}$$

The ideal populated by said polynomials is the *Bernstein–Sato ideal* $B_{f'F}^g$. When $f' = 1$ and $g = f$ this defines the multivariate Bernstein–Sato ideal in the sense of Budur [6] and we simply write B_F; if we further restrict to the trivial factorization $F = (f)$ then we obtain the classical functional equation whose corresponding ideal, which we denote by B_f, has as its monic generator the *Bernstein–Sato polynomial*.

The roots of the Bernstein–Sato polynomial encode various data about the singular locus of f. Malgrange and Kashiwara, cf. [3], [4], famously proved that exponentiating the local version of the Bernstein–Sato polynomial's roots recovers the eigenvalues of the algebraic monodromy action on nearby Milnor fibers. In [6], Budur conjectured the analogous claim for the multivariate Bernstein–Sato ideal B_F associated to a factorization of f into irreducibles: exponentiating the ideal's zero locus recovers the cohomology support locus of the complement of $\mathrm{Var}(f)$. A proof of this (for germs f that need not be arrangements) has recently been announced by Budur, Veer, Wu, and Zhou, cf. [10]. Beyond these mon-

[1]↑A version of this chapter has been published in the Journal of Singularities as [30].

odromy results, zeroes of Bernstein–Sato polynomials are related to many other invariants: multiplier ideals, log canonical thresholds, F-pure thresholds, etc.

However, even in the case of arrangements, formulae for Bernstein–Sato ideals, polynomials, or their zero loci are very rare. Walther has found a formula for the Bernstein–Sato polynomial for generic arrangements in [31], Maisonobe has shown the Bernstein–Sato ideal B_F for a generic arrangement factored into linear forms is principal and found the corresponding formula for a generator, cf. [32], and Saito has shown that the roots of the Bernstein–Sato polynomial of a reduced and central arrangement f lie in $(-2 + \frac{1}{\deg(f)}, 0) \cap \mathbb{Q}$, cf. [33]. On the other hand, Walther has shown that, in general, the roots of the Bernstein–Sato polynomial are not combinatorially determined, that is, they cannot be computed from the arrangement's intersection lattice, cf. [7] and Example 3.4.22. The multivariate Bernstein–Sato ideal B_F is not even guaranteed to be principal, cf. [15] for a counter-example in the local case. To our knowledge, there are no systematic studies of the more general type of Bernstein–Sato ideal $B_{f'F}^{g}$ though it does play a role in [31].

Our starting point is the program of Maisonobe in [11] wherein he proves the Bernstein–Sato ideal of a central, reduced, and free (in the sense of Saito [8]) arrangement equipped with its factorization into linear forms is principal and gives a combinatorial formula for its generator. While the approach is similar, we encounter many technical difficulties because our results are significantly more general: we consider the more general functional equation (3.1.1) and we often relax the assumptions of f being factored into linear forms, being free, and being reduced.

In Section 2, we consider a larger class of analytic germs $f \in \mathscr{O}_X$ than just central, reduced, and free arrangements and we consider any factorization $f = f_1 \cdots f_r$. In Chapter 2, we proved that $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} F^S$ is generated by derivations, that is, by differential operators of order at most one under a natural filtration, under the hypotheses of tameness (a sliding condition on projective dimension), strongly Euler-homogeneous (a hypothesis that a particular logarithmic derivation exists locally everywhere), and Saito-holonomicity (a finiteness condition on the logarithmic stratification). We use similar techniques to generalize these results from Chapter 2 in Theorem 3.2.21:

Theorem 3.1.1. *Suppose $f = f_1 \cdots f_r$ is tame, strongly Euler-homogeneous, and Saito-holonomic, $f' \in \mathscr{O}_{X,\mathfrak{x}}[\frac{1}{f}]$ is compatible with f, and $F = (f_1, \ldots, f_r)$. Then the $\mathscr{D}_{X,\mathfrak{x}}[S]$-annihilator of $f'F^S$ is generated by derivations.*

In Section 3, we replace the hypothesis of tame with free and prove a version of the symmetry of $B^g_{f'F}$ that was first identified by Narváez-Macarro in [9] in the case of Bernstein–Sato polynomials and generalized to B_F by Maisonobe in [11]. This follows from computing the $\mathscr{D}_{X,\mathfrak{x}}[S]$-dual of $\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S$. Without freeness, computing these $\mathscr{D}_{X,\mathfrak{x}}[S]$-duals is currently intractible. While we are certain one could use Narváez-Macarro's Lie-Rinehart strategy, we instead opt for Maisonobe's approach, which itself relies on a computation of the trace of an adjoint action first proved by Castro–Jiménez and Ucha in Theorem 4.1.4 of [34]; we give a different proof of this in Section 5. With $\mathbb{D}$ denoting the $\mathscr{D}_{X,\mathfrak{x}}[S]$-dual $\mathrm{RHom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(-, \mathscr{D}_{X,\mathfrak{x}}[S])^{\mathrm{left}}$, in Theorem 3.3.9 we prove:

Theorem 3.1.2. *Suppose $f = f_1 \cdots f_r \in \mathscr{O}_X$ is free, strongly Euler-homogeneous, and Saito-holonomic and $f_{\mathrm{red}} \in \mathscr{O}_{X,\mathfrak{x}}$ is a Euler-homogeneous reduced defining equation for f at $\mathfrak{x}$. Let $F = (f_1, \ldots, f_r)$, let $f' \in \mathscr{O}_{X,\mathfrak{x}}$ be compatible with f, and let $g \in \mathscr{O}_{X,\mathfrak{x}}$ such that $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot g$. Then*

$$\mathbb{D}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S] \cdot gf'F^S}\right) \simeq \frac{\mathscr{D}_{X,\mathfrak{x}}[S](gf'f_{\mathrm{red}})^{-1}F^{-S}}{\mathscr{D}_{X,\mathfrak{x}}[S](f'f_{\mathrm{red}})^{-1}F^{-S}}[n+1].$$

The main application is Theorem 3.3.16 which identifies technical conditions on f', g, and F such that $B^g_{f'F}$ is invariant under a non-trivial involution of $\mathbb{C}[S]$.

In Section 4 we return to hyperplane arrangements and first show that the nice structure of $\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}S}\, f'F^S$ from Theorem 3.2.21 allows us to adapt Maisonobe's arguments to estimate $B^g_{f'F}$ for any factorization. In particular we complement Walther's result that the roots of Bernstein–Sato polynomial are not combinatorial for even tame arrangements, cf. [7]. Namely, we prove in Theorem 3.4.21 the roots lying in $[-1, 0)$ are combinatorial:

Theorem 3.1.3. *Let f be a central, not necessarily reduced, tame hyperplane arrangement. Suppose f' divides f; let $g = \frac{f}{f'}$. Then the roots $\mathrm{V}(B^g_{f'f})$ lying in $[-1, 0)$ are combinatorially determined:*

$$\mathrm{V}(B^g_{f'f}) \cap [-1, 0) = \bigcup_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \; \bigcup_{j_X = r(X) + d'_X}^{d_X} \frac{-j_X}{d_X}.$$

Setting $f' = 1$ gives all the roots of the Bernstein–Sato polynomial of f lying in $[-1, 0)$.

If we assume further that f is free, then we can then use the symmetry property of Theorem 3.3.16 to more accurately estimate $\mathrm{V}(B_{f'F}^{g})$, where $\mathrm{V}(-)$ always refers to the zero locus of the ideal in question. In this setting there is a computation for the multivariate Bernstein–Sato ideal of a reduced, free f that has been factored into linear forms due to Maisonobe [11], but no results about other factorizations, non-reduced f, or even the Bernstein–Sato polynomial. We fill in much of this gap. With $P_{f'F,X}^{g} \in \mathbb{C}[S]$ the explicit linear polynomial from Definition 3.4.10, we obtain the following, which in particular shows that the roots of the Bernstein–Sato polynomial for any power of a reduced, central, and free arrangement are combinatorially determined:

Theorem 3.1.4. *Suppose $f = f_1 \cdots f_r$ is a central, not necessarily reduced, free hyperplane arrangement, $F = (f_1, \cdots, f_r)$, f' divides f, and $g = \frac{f}{f'}$. If (f', F) is an unmixed pair up to units and if $\deg(f') \leq 4$, then $\mathrm{V}(B_{f'F}^{g})$ is a hypersurface and*

$$\mathrm{V}(B_{f'F}^{g}) = \mathrm{V}\left(\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X = 0}^{d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X} \left(P_{f'F,X}^{g} + j_X \right) \right). \tag{3.1.2}$$

If L is a factorization of $f = l_1 \cdots l_d$ into irreducibles and $\deg(f') \leq 4$, then

$$B_{f'L}^{g} = \prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X = 0}^{d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X} \left(P_{f'L,X}^{g} + j_X \right)$$

and so $B_{f'L}^{g}$ principal. If $f' = 1$ and f is reduced, then for any F

$$\mathrm{V}(B_F) = \mathrm{V}\left(\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X = 0}^{d_{X,\mathrm{red}} + d_X - 2r(X)} \left(P_{F,X}^{g} + j_X \right) \right). \tag{3.1.3}$$

In particular, if f is reduced or is a power of a central, reduced, and free hyperplane arrangement, then the roots of the Bernstein–Sato polynomial of f are given by (3.1.3).

In Remark 3.4.28 we discuss how to use new results to get a combinatorial formula for the roots of the Bernstein–Sato polynomial corresponding to any central, free f, that is, to f that may not be a power of a reduced arrangement. In the case of line arrangements, we are also able to compute $\mathrm{V}(B_{f'F}^{g})$ for any suitable choice of f', g, and F without the technical condition of unmixed up to units, cf. Theorem 3.4.25 and Definition 3.3.14.

Unfortunately our methods are not appropriate for determining the multiplicity of roots of the Bernstein–Sato polynomial so we cannot conclude this polynomial is combinatorial for free arrangements. These multiplicities are mysterious, although in [33] Saito proves various results about them in the general (i.e. in the non-free) setting. Notably he shows that -1 has multiplicity equal to the arrangement's rank.

In Section 5 we make use of our results involving the more general functional equation (3.1.1) to study the smallest arrangement $\mathrm{V}(f')$ that when added to the arrangement $\mathrm{V}(g)$ makes $\mathrm{V}(f'g)$ free, i.e. the smallest arrangement f' that *frees* g. For arbitrary divisors g, it is unknown whether or not such a divisor f' exists. There are some positive results, but the methodologies are very particular to the type of divisors considered. For example, Mond and Schulze identified certain classes of germs that are freed by a adjoint divisors–these germs are related to discriminants of versal deformations, cf. [35]. Other cases of freeing divisors are considered in [36] and [37]. However, Yoshinaga [38] has communicated to us a way, based on the combinatorics of g, to find an arrangement f' that frees an arrangement g. In Theorem 3.5.4 we prove the degree of f' is related to roots of the Bernstein–Sato polynomial of g.

Theorem 3.1.5. *Suppose that g is a central, reduced, tame hyperplane arrangement of rank n, v an integer such that $1 < v \leq n-1$, and $\deg(g)$ is co-prime to v. If $\frac{-2\deg(g)+v}{\deg(g)}$ is a root of the Bernstein–Sato polynomial of g and if f' is a central arrangement that frees g, then $\deg(f') \geq n - v$.*

In Section 6 we prove a conjecture of Budur's in the case of central, reduced, and free hyperplane arrangments. The recently announced paper [10] gives a general proof using entirely different methods.

3.2 Bernstein–Sato Ideals and the $\mathscr{D}_{X,\mathfrak{x}}[S]$-module $\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S$

In this section we introduce some of our working hypotheses on $f \in \mathscr{O}_X$. These are needed to utilize results from Chapter 2 and [7] which will be needed throughout the chapter. We generalize Theorem 2.29 of Chapter 2 and discuss how Bernstein–Sato varieties attached to different factorizations of f relate to each other.

3.2.1 Hypotheses on f

Let X be a smooth analytic space or $\mathbb{C}$-scheme of dimension n and $\mathscr{O}_X$ be the analytic structure sheaf. Pick $f \in \mathscr{O}_X$ to be regular with divisor $Y = \operatorname{Div}(f)$ and ideal sheaf $\mathscr{I}_Y$. In general, we make no reducedness assumption on Y.

Definition 3.2.1. Let $\operatorname{Der}_X(-\log Y)$ be the $\mathscr{O}_X$-sheaf of *logarithmic derivations on* Y, that is, the sheaf generated locally by the vector fields δ such that $\delta \bullet \mathscr{I}_Y \subseteq \mathscr{I}_Y$. If $Y = \operatorname{Div}(f)$ then we also label $\operatorname{Der}_X(-\log f) = \operatorname{Der}_X(-\log Y)$. Define the *derivations that kill* f to be

$$\operatorname{Der}_X(-\log_0 f) = \{\delta \in \operatorname{Der}_X(-\log f) \mid \delta \bullet f = 0\}.$$

Remark 3.2.2.(a) It is easily checked that $\operatorname{Der}_X(-\log Y)$ depends on $\mathscr{I}_Y$ and not the choice of generators of $\mathscr{I}_Y$.

(b) By Lemma 3.4 of [18], $\operatorname{Der}_{X,\mathfrak{x}}(-\log fg) = \operatorname{Der}_{X,\mathfrak{x}}(-\log f) \cap \operatorname{Der}_{X,\mathfrak{x}}(-\log g)$. This is not always true when restricting to derivations that kill f.

(c) $\operatorname{Der}_{X,\mathfrak{x}}(-\log f)$ is closed under taking commutators.

At points we will be interested in when $\operatorname{Der}_X(-\log Y)$ has a particularly nice structure.

Definition 3.2.3. The divisor $Y = \operatorname{Div}(f)$ is *free* when $\operatorname{Der}_X(-\log Y)$ is locally everywhere a free $\mathscr{O}_X$-module. Similarly $f \in \mathscr{O}_{X,\mathfrak{x}}$ is free when $\operatorname{Der}_{X,\mathfrak{x}}(-\log f)$ is a free $\mathscr{O}_{X,\mathfrak{x}}$-module.

In [8], Saito introduced the logarithmic differential forms which are, in some sense, a dual notion to logarithmic derivations.

Definition 3.2.4. Let Ω^k_X be the sheaf of differential k-forms on X and $d : \Omega^k_X \to \Omega^{k+1}_X$ the standard differential. Define the sheaf of *logarithmic k-forms along f* by

$$\Omega^k_X(\log f) = \{w \in \frac{1}{f}\Omega^k_X \mid df \wedge w \in \Omega^{k+1}_X\}.$$

An element $f \in \mathscr{O}_X$ is *tame* if the projective dimension of the logarithmic k-forms along f is at most k in each stalk. A divisor Y is tame if it locally everywhere admits tame defining equations.

Remark 3.2.5.(a) The logarithmic 1-forms are dual to the logarithmic differentials: $\mathrm{Hom}_{\mathscr{O}_{X,\mathfrak{x}}}(\mathrm{Der}_{X,\mathfrak{x}}(-\log f), \mathscr{O}_{X,\mathfrak{x}}) \simeq \Omega^1_X(\log f)$. When f is free, $\Omega^k_X(\log f) \simeq \bigwedge^k \Omega^1_X(\log f)$, cf. 1.6 and page 270 of [8].

(b) If $\dim(X) = n \leq 3$ then any divisor Y is automatically tame. This follows from the reflexivity of logarithmic k-forms, cf. [8].

The logarithmic derivations can also be used to stratify X:

Definition 3.2.6. (Compare to 3.3 and 3.8 of [8]) There is a relation on X induced by the logarithmic derivations along Y. Two points $\mathfrak{x}$ and $\mathfrak{y}$ are equivalent if there exists an open U containing them and a $\delta \in \mathrm{Der}_U(-\log Y \cap U)$ such that: (i) δ vanishes nowhere on U; (ii) an integral curve of δ passes through $\mathfrak{x}$ and $\mathfrak{y}$. The transitive closure of this relation stratifies X into equivalence classes whose irreducible components are the *logarithmic strata*. These strata constitute the *logarithmic stratification*.

We say Y is *Saito-holonomic* when the logarithmic stratification is locally finite.

Example 3.2.7. By 3.14 of [8] hyperplane arrangements are Saito-holonomic.

Finally, we define some homogeneity conditions on $f \in \mathscr{O}_X$.

Definition 3.2.8. We say $f \in \mathscr{O}_{X,\mathfrak{x}}$ is *Euler-homogeneous* when there exists $\delta \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f)$ such that $\delta \bullet f = f$. If δ may be picked to vanish at $\mathfrak{x}$, then f is *strongly Euler-homogeneous.*

The element $f \in \mathscr{O}_X$ is (strongly) Euler-homogeneous if it is so at each point. The divisor Y is (strongly) Euler-homogeneous if it locally everywhere admits a defining equation that is (strongly) Euler-homogeneous.

Remark 3.2.9. If $f \in \mathscr{O}_{X,\mathfrak{x}}$ and $u \in \mathscr{O}_{X,\mathfrak{x}}$ is a unit, then f is strongly Euler-homogeneous if and only if uf is, cf. Remark 2.8 of [7].

Example 3.2.10. Hyperplane arrangements are strongly Euler-homogeneous.

Our working hypotheses on f will often be "tame, strongly Euler-homogeneous, and Saito-holonomic" or "free, strongly Euler-homogeneous, and Saito-holonomic." In light of Examples 3.2.7 and 3.2.10, if f cuts out a hyperplane arrangement only tameness or freeness need be assumed.

3.2.2 The $\mathscr{D}_{X,\mathfrak{x}}[S]$-Annihilator of $f' F^S$

Let $\mathscr{D}_X$ be the sheaf of $\mathbb{C}$-linear differential operators with coefficients in $\mathscr{O}_X$ and $\mathscr{D}_X[S]$ be the polynomial ring extension induced by adding r central variables $S = s_1, \ldots, s_r$.

Definition 3.2.11. Consider the free $\mathscr{O}_X[S][\frac{1}{f}]$-module generated by the symbol $F^S = f_1^{s_1} \cdots f_r^{s_r}$. This is endowed with a $\mathscr{D}_X[S]$-action by specifying the action of a $\mathbb{C}$-linear derivation δ on $\mathscr{O}_X$. For any $g \in \mathscr{O}_X[\frac{1}{f}]$, declare

$$\delta \bullet (s_i g F^S) = s_i(\delta \bullet g) F^S + s_i g (\sum_k \frac{\delta \bullet f_k}{f_k} s_k) F^S.$$

Let $\mathscr{D}_X[S]F^S$ be the $\mathscr{D}_X[S]$-module generated by F^S. For $g \in \mathscr{O}_X[\frac{1}{f}]$, let $\mathscr{D}_X[S]gF^S$ be the $\mathscr{D}_X[S]$-module generated by gF^S.

Remark 3.2.12. When executing the above construction with only one s, we use the notation $\mathscr{D}_X[s]f^s$. This is the classical, univariate situation.

In Proposition 2.7 of Chapter 2 we showed both that there is a canonical way to associate elements of $\operatorname{Der}_X(-\log f)$ to elements of $\operatorname{ann}_{\mathscr{D}_X[S]} F^S$ and that when f is tame, strongly

Euler-homogeneous, and Saito-holonomic, $\operatorname{ann}_{\mathscr{D}_{X,x}[S]} F^S$ is generated by said elements. In this subsection we prove the analogous claims for $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S$, provided f' is chosen such that $f^N f' \in \mathscr{O}_{X,\mathfrak{x}}$ and $f^M \in \mathscr{O}_{X,\mathfrak{x}} \cdot f^N f'$ for suitable choices of $N, M \geq 0$. First, we show how to associate elements of $\operatorname{Der}_{X,x}(-\log f)$ to $\operatorname{ann}_{\mathscr{D}_{X,x}[S]} f'F^S$ in an entirely similar way as in the prequel; second, we show that these elements generate $\operatorname{ann}_{\mathscr{D}_{X,x}[S]} f'F^S$ when f is tame, strongly Euler-homogeneous, and Saito-holonomic.

Definition 3.2.13. The *total order filtration* $F_{(0,1,1)}$ on $\mathscr{D}_{X,\mathfrak{x}}[S]$ assigns, in local coordinates, every ∂_{x_k} weight one, every s_k weight one, and every element of $\mathscr{O}_X$ weight zero. We will denote the elements of weight at most l by $F^l_{(0,1,1)}$ or $F^l_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$.

Definition 3.2.14. Write $f \in \mathscr{O}_{X,\mathfrak{x}}$ as $f = ul_1^{p_1} \cdots l_q^{p_q}$ where the l_t are pairwise distinct irreducibles, $p_t \in \mathbb{Z}_+$, and u is a unit in $\mathscr{O}_{X,\mathfrak{x}}$. We say $f' \in \mathscr{O}_{X,\mathfrak{x}}[\frac{1}{f}]$ is *compatible* with f if there exists a unit $u' \in \mathscr{O}_{X,\mathfrak{x}}$ and integers $v_t \in \mathbb{Z}$ such that

$$f' = ul_1^{v_1} \cdots l_q^{v_q}.$$

In this case, v_t is the *multiplicity of* l_t.

By Remark 3.2.2, if $f = ul_1^{p_1} \cdots l_q^{v_q}$ a factorization of f into irreducibles at $\mathfrak{x}$, u a unit, then if $\delta \in \operatorname{Der}_{X,\mathfrak{x}}(-\log f)$, $\frac{\delta \bullet l_t}{l_t} \in \mathscr{O}_{X,\mathfrak{x}}$. So for f' compatible with f,

$$\delta \bullet f'F^S = (\delta \bullet f')F^S + f'(\sum_k \frac{\delta \bullet f_k}{f_k} s_k)F^S = (\frac{\delta \bullet f'}{f'} + \sum_k \frac{\delta \bullet f_k}{f_k} s_k) f'F^S,$$

where $(\frac{\delta \bullet f'}{f'} + \sum_k \frac{\delta \bullet f_k}{f_k} s_k) \in \mathscr{O}_{X,\mathfrak{x}}[S]$. Indeed, $\frac{\delta \bullet f'}{f'} = \sum v_t \frac{\delta \bullet l_t}{l_t} \in \mathscr{O}_{X,\mathfrak{x}}$ and similarly $\frac{\delta \bullet f_k}{f_k} \in \mathscr{O}_X$.

Definition 3.2.15. Suppose f' is compatible with f. If $f = f_1 \cdots f_r$ and $F = (f_1, \ldots, f_r)$, then there is a map of $\mathscr{O}_{X,x}$-modules

$$\psi_{f'F,\mathfrak{x}} : \operatorname{Der}_{X,x}(-\log f) \to \operatorname{ann}_{\mathscr{D}_{X,x}[S]} f'F^S \cap F^1_{(0,1,1)}$$

given by

$$\psi_{f'F,\mathfrak{x}}(\delta) = \delta - \sum_k \frac{\delta \bullet f_k}{f_k} s_k - \frac{\delta \bullet f'}{f'}.$$

The $\mathscr{O}_{X,\mathfrak{x}}$-module of *annihilating derivations* along $f'F$ is defined as

$$\theta_{f'F,\mathfrak{x}} = \psi_{f'F,\mathfrak{x}}(\mathrm{Der}_{X,\mathfrak{x}}(-\log f))$$

and $\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S$ is *generated by derivations* when

$$\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S = \mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_{f'F,\mathfrak{x}}.$$

When $f' = 1$ we write $\psi_{F,\mathfrak{x}}$ and $\theta_{F,\mathfrak{x}}$.

Arguing as in Proposition 2.7 of Chapter 2 we see that:

Proposition 3.2.16. (Compare to Proposition 2.7 of Chapter 2) *Suppose f' is compatible with f. If $f = f_1 \cdots f_r$ and $F = (f_1, \ldots, f_r)$, then $\psi_{f'F,\mathfrak{x}}$ is an isomorphism.*

Proof. Suppose $\delta - \sum_k b_k s_k - b \in \mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F \cap F^1_{(0,1,1)}$ where $b_k, b \in \mathscr{O}_{X,\mathfrak{x}}$. Since $f'F^S$ generates a free $\mathscr{O}_{X,\mathfrak{x}}[S][\frac{1}{f}]$-module we deduce

$$(\sum_k \frac{\delta \bullet f_k}{f_k} s_k - b_k s_k) + (\frac{\delta \bullet f'}{f'} - b) = 0$$

and hence

$$\delta \in \bigcap_k \mathrm{Der}_{X,\mathfrak{x}}(-\log f_k) = \mathrm{Der}_{X,\mathfrak{x}}(-\log f).$$

So the map $\delta - \sum_k b_k s_k - b \mapsto \delta$ sends $\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F \cap F^1_{(0,1,1)}$ to $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$. Its inverse is $\psi_{f'F,\mathfrak{x}}$. $\square$

Remark 3.2.17. By definition, $\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S$ is closed under taking commutators; hence $\theta_{f'F,\mathfrak{x}}$ is as well. As $\psi_{f'F,\mathfrak{x}}$ is an isomorphism, a basic computation shows $\psi_{f'F,\mathfrak{x}}$ respects taking commutators.

In Chapter 2 we generalized an approach of Walther's in [7]: we looked at the associated graded object of $\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} F^S$ under the total order filtration $F_{(0,1,1)}$. As $\psi_{F,\mathfrak{x}}(\mathrm{Der}_{X,\mathfrak{x}}(-\log f)) \subseteq \mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} F^S$ the following definition is natural:

Definition 3.2.18. Suppose f is strongly Euler-homogeneous. The *generalized Liouville ideal* $\widetilde{L_{F,\mathfrak{x}}} \subseteq \operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ is generated by the symbols of elements in $\psi_F(\operatorname{Der}_{X,\mathfrak{x}}(-\log f))$ under the total order filtration. That is,

$$\widetilde{L_{F,\mathfrak{x}}} = \operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot \operatorname{gr}_{(0,1,1)}(\psi_{F,\mathfrak{x}}(\operatorname{Der}_{X,\mathfrak{x}}(-\log f))).$$

Remark 3.2.19.(a) The strongly Euler-homogeneous assumption in the above definition ensures that algebraic properties of $\widetilde{L_{F,x}}$ do not depend on choice of defining equations for each f_k at x. See Remark 2.15 of Chapter 2 for details.

(b) By Corollary 2.28 of Chapter 2, if $f \in \mathscr{O}_X$ is tame, strongly Euler-homogeneous, and Saito-holonomic then $\widetilde{L_{F,\mathfrak{x}}} = \operatorname{gr}_{(0,1,1)}(\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} F^S)$.

(c) For $\delta \in \operatorname{Der}_{X,\mathfrak{x}}(-\log f)$, note that

$$\begin{aligned} \operatorname{gr}_{(0,1,1)}(\psi_{f'F,\mathfrak{x}}(\delta)) &= \operatorname{gr}_{(0,1,1)}(\delta - \sum_k \frac{\delta \bullet f_k}{f_k} s_k - \frac{\delta \bullet f'}{f'}) \\ &= \operatorname{gr}_{(0,1,1)}(\delta - \sum_k \frac{\delta \bullet f_k}{f_k} s_k) \\ &= \operatorname{gr}_{(0,1,1)}(\psi_{F,\mathfrak{x}}(\delta)). \end{aligned}$$

Since $\widetilde{L_{F,\mathfrak{x}}} \subseteq \operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ has, by definition, generators $\{\operatorname{gr}_{(0,1,1)}(\psi_{F,\mathfrak{x}}(\delta)) \mid \delta \in \operatorname{Der}_{X,\mathfrak{x}}(-\log f)\}$, we deduce

$$\begin{aligned} \widetilde{L_{F,\mathfrak{x}}} &= \operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot \{\operatorname{gr}_{(0,1,1)}(\psi_{f'F,\mathfrak{x}}(\delta)) \mid \delta \in \operatorname{Der}_{X,\mathfrak{x}}(-\log f)\} \\ &= \operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot \operatorname{gr}_{(0,1,1)}(\theta_{f'F,\mathfrak{x}}) \\ &\subseteq \operatorname{gr}_{(0,1,1)}(\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S). \end{aligned}$$

By the preceding remark, $\widetilde{L_{F,\mathfrak{x}}}$ approximates $\operatorname{gr}_{(0,1,1)}(\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S)$. Arguing as in Corollary 2.28 of Chapter 2 we prove the approximation is in fact an equality:

Theorem 3.2.20. *Suppose $f = f_1 \cdots f_r$ is tame, strongly Euler-homogeneous and Saito-holonomic. Let $F = (f_1, \ldots, f_r)$ and suppose $f' \in \mathscr{O}_{X,\mathfrak{x}}[\frac{1}{f}]$ is compatible with f. Then*

$$\mathrm{gr}_{(0,1,1)}(\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S) = \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot \mathrm{gr}_{(0,1,1)}(\theta_{f'F,\mathfrak{x}}).$$

Proof. For the first part of this proof we mimic Proposition 2.25 of Chapter 2. In Definition 2.24 of loc. cit. we introduced a $\mathscr{O}_{X,\mathfrak{x}}$-linear ring homomorphism $\phi_{F,\mathfrak{x}} : \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \to R(\mathrm{Jac}(f_1), \ldots, \mathrm{Jac}(f_r))$ where $R(\mathrm{Jac}(f_1), \ldots, \mathrm{Jac}(f_r))$ is the multi-Rees algebra associated to the r Jacobian ideals $\mathrm{Jac}(f_1), \ldots, \mathrm{Jac}(f_r)$. Using local coordinates ∂_{x_i} and identifying $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ with $\mathscr{O}_{X,\mathfrak{x}}[Y][S]$ via $\mathrm{gr}_{(0,1,1)}(\partial_{x_i}) = y_i$, the map $\phi_{F,\mathfrak{x}}$ is given by

$$y_i \mapsto \sum_k \frac{f}{f_k}(\partial_{x_i} \bullet f_k) s_k \text{ and } s_k \mapsto f s_k.$$

Proposition 2.26 of loc. cit. shows $\ker(\phi_{F,\mathfrak{x}})$ is a prime ideal of dimension $n + r$.

Select $P \in \mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F$ of weight l under the total order filtration $F_{(0,1,1)}$. For any Q of weight l, $f^l Q \bullet f'F^S \in \mathscr{O}_{X,\mathfrak{x}}[S]F^S$. Now, for $g \in \mathscr{O}_{X,\mathfrak{x}}[S][\frac{1}{f}]$, write $\partial_{x_i} \bullet gf'F^S = (\partial_{x_i} \bullet g + g\frac{\partial_{x_i} \bullet f'}{f'} + g\sum_k \frac{\partial_{x_i} \bullet f_k}{f_k} s_k) f'F^S$. Thus, if applying a partial derivative to $gf'F^S$ causes the s-degree (under the natural filtration) of the $\mathscr{O}_{X,\mathfrak{x}}[S]$-coefficient of $f'F^S$ to increase, the terms of higher s-degree are precisely $g\sum_k \frac{\partial_{x_i} \bullet f_k}{f_k}$. A straightforward computation then shows that the S-lead term of $f^l Q \bullet f'F^S$ is exactly $\phi_{F,\mathfrak{x}}(\mathrm{gr}_{(0,1,1)}(Q)) f'F^S \in \mathscr{O}_{X,\mathfrak{x}}[S] f'F^S$. Since $f'F^S$ generates a free $\mathscr{O}_{X,\mathfrak{x}}[S][\frac{1}{f}]$-module and since $P \bullet f'F^S = 0$, we conclude $\mathrm{gr}_{(0,1,1)}(P) \in \ker(\phi_{F,\mathfrak{x}})$.

By Remark 3.2.19 we deduce:

$$\begin{aligned} \widetilde{L_{F,\mathfrak{x}}} \subseteq \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot \mathrm{gr}_{(0,1,1)}(\theta_{f'F,\mathfrak{x}}) &\subseteq \mathrm{gr}_{(0,1,1)}(\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S) \\ &\subseteq \ker(\phi_{F,\mathfrak{x}}). \end{aligned} \tag{3.2.1}$$

Since f is tame, strongly Euler-homogeneous, and Saito-holonomic, by Theorem 2.23 of loc. cit., $\widetilde{L_{F,\mathfrak{x}}}$ is a prime ideal of dimension $n + r$. So the outer ideals of (3.2.1) are prime ideals of dimension $n + r$ and the containments are equalities. □

Theorem 3.2.21. *Suppose $f = f_1 \cdots f_r$ is tame, strongly Euler-homogeneous, and Saito-holonomic, $f' \in \mathscr{O}_{X,\mathfrak{x}}[\frac{1}{f}]$ is compatible with f, and $F = (f_1, \dots, f_r)$. Then the $\mathscr{D}_{X,\mathfrak{x}}[S]$-annihilator of $f'F^S$ is generated by derivations.*

Proof. By Theorem 3.2.20, for $P \in \operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S$, we can find $L \in \mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_{f'F,\mathfrak{x}}$ such that P and L have the same initial term with respect to the total order filtration. Since $P - L$ annihilates $f'F^S$ and, by construction, has a smaller weight than P, we can argue inductively as in Theorem 2.29 of Chapter 2 now using Theorem 3.2.20 instead of Corollary 2.28 of Chapter 2. The induction argument therein will also terminate in this setting since $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S \cap \mathscr{O}_{X,\mathfrak{x}} = 0$. □

The following corollary will let us study the Weyl algebra version of the annihilator of $f'F^S$ when f' and f are global algebraic.

Corollary 3.2.22. *If X is the analytic space of a smooth $\mathbb{C}$-scheme, then the statement of Theorem 3.2.21 holds in the algebraic category.*

Proof. See Corollary 2.30 of Chapter 2. □

We will also be interested in the $\mathscr{D}_{X,\mathfrak{x}}[S]$-module generated by the symbol $F^{-S} = f_1^{-s_1} \cdots f_r^{-s_r}$ which is defined in the same way as $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$. Most of our previous definitions apply to F^{-S} as well, in particular, if f' is compatible with f let $\psi_{f'F,\mathfrak{x}}^{-S}$ and $\theta_{F,\mathfrak{x}}^{-S}$ be as before, except with the signs of the s_k switched.

Theorem 3.2.23. *Suppose $f = f_1 \cdots f_r$ is tame, strongly Euler-homogeneous, and Saito-holonomic, f' is compatible with f, and $F = (f_1, \cdots, f_r)$. Then the $\mathscr{D}_{X,\mathfrak{x}}[S]$-annihilator of $f'F^{-S}$ is generated by derivations in that*

$$\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^{-S} = \mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_{f'F,\mathfrak{x}}^{-S}.$$

If X is the analytic space of a smooth $\mathbb{C}$-scheme, then this holds in the algebraic category as well.

Proof. It is sufficient to prove the generated by derivations statement. For this argue as in Theorem 3.2.21 except replace $\widetilde{L_{F,\mathfrak{x}}}$ and $\phi_{F,\mathfrak{x}}$ with their images under the $\operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ automorphism induced by $s_k \mapsto -s_k$. □

3.2.3 Bernstein–Sato Ideals

Recall the univariate *functional equation*, with $b(s) \in \mathbb{C}[s]$, $P(s) \in \mathscr{D}_{X,\mathfrak{x}}[s]$:

$$b(s)f^s = P(s)f^{s+1}.$$

The polynomials $b(s)$ generate the *Bernstein–Sato ideal* $B_{f,\mathfrak{x}}$ of f. The monic generator of this ideal is the *Bernstein–Sato polynomial*; the reduced locus of its variety is $\mathrm{V}(B_{f,\mathfrak{x}})$. We will be interested in multivariate generalizations of this functional equation.

Definition 3.2.24. Let $f', g_1, \ldots, g_u \in \mathscr{O}_{X,\mathfrak{x}}$ and I the ideal generated by the $g_1, \ldots g_u$. Consider the functional equation

$$B(S)f'F^S = \sum_t P_t g_t f' F^S \in \mathscr{D}_{X,\mathfrak{x}}[S] \cdot I f' F^S$$

where $f = f_1 \cdots f_r$, $F = (f_1, \ldots, f_r)$, $P_t \in \mathscr{D}_{X,\mathfrak{x}}[S]$, and $B(S) \in \mathbb{C}[S]$. The polynomials $B(S)$ satisfying this functional equation constitute the *Bernstein–Sato ideal* $B^I_{f'F,\mathfrak{x}}$. Note that

$$B^I_{f'F,\mathfrak{x}} = \mathbb{C}[S] \cap (\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot I).$$

When $I = (f)$ we will write $B^I_{f'F,\mathfrak{x}} = B_{f'F,\mathfrak{x}}$ and when $I = (g)$ we will write $B^g_{f'F,\mathfrak{x}}$. When in the univariate case, i.e. $r = 1$, we will write $B_{f'F,\mathfrak{x}} = B_{f'f,x}$ and $B^g_{f'F,\mathfrak{x}} = B^g_{f'f,\mathfrak{x}}$. When in the global algebraic case we define similar objects using $\mathrm{A}_n(\mathbb{C})\,[S]$ instead of $\mathscr{D}_{X,\mathfrak{x}}[S]$–in this case we drop the $(-)_\mathfrak{x}$ subscript. Finally by $\mathrm{V}(-)$ we always mean the reduced locus of the appropriate variety.

We will want to compare the Bernstein–Sato ideals corresponding to different factorizations.

Definition 3.2.25. Let $f = f_1 \cdots f_r$ and $F = (f_1, \ldots, f_r)$. Write $[r]$ as the disjoint union of the intervals I_t where $1 \leq t \leq m$ and consider the *coarser* factorization $H = (h_1, \ldots, h_m)$ where $f = h_1 \cdots h_m$ and $h_t = \prod_{i \in I_t} f_i$. Define S_H to be the ideal of $\mathbb{C}[S]$ generated by $s_i - s_j$ for all $i, j \in I_t$ and for all t.

Proposition 3.2.26. *Let $f = f_1 \cdots f_r$ be tame, strongly Euler-homogeneous, and Saito-holonomic. Let $F = (f_1, \ldots, f_r)$, let $I \subseteq \mathscr{O}_{X,\mathfrak{x}}$, and let H be a coarser factorization. If $f' \in \mathscr{O}_{X,\mathfrak{x}}$ such that $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot f'$, then the image of $B^I_{f'F,\mathfrak{x}}$ modulo S_H lies in $B^I_{f'H,\mathfrak{x}}$.*

Proof. As f' is compatible with f, $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S$ and $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'H^S$ are both generated by derivations. Since $\operatorname{Der}_{X,\mathfrak{x}}(-\log f) \subseteq \operatorname{Der}_{X,\mathfrak{x}}(-\log f')$, we can easily get a result similar to Proposition 2.33 of Chapter 2 and, from that, a result similar to Proposition 2.32 of loc. cit. The argument is essentially the same as the proof of Proposition 3.5.3 of this chapter. $\square$

Example 3.2.27. For $f = xy^2(x+y)^2$ and $F = (xy, y(x+y), x+y)$,

$$B_F = (s_1 + 1) \prod_{j=0}^{1} (s_1 + s_2 + 1 + j)(s_2 + s_3 + 1 + j)\left(\prod_{m=0}^{4} (2s_1 + 2s_2 + s_3 + 2 + m)\right).$$

While Proposition 3.2.26 can estimate B_f, it estimates multiplicities poorly. Indeed, going modulo $(s_1 - s_2, s_1 - s_3, s_2 - s_3)$ we find

$$(s+1)^3(2s+1)^2 \prod_{m=0}^{4} (5s + 2 + m) \in B_f = \mathbb{C}[s] \cdot (s+1)(2s+1) \prod_{m=0}^{4} (5s + 2 + m).$$

3.3 $\mathscr{D}_{X,\mathfrak{x}}[S]$-Dual of $\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S$

In [9], Narváez-Macarro computed the $\mathscr{D}_{X,\mathfrak{x}}[s]$-dual of $\mathscr{D}_{X,\mathfrak{x}}[s]f^s$ when f is reduced, free, and quasi-homogeneous; in [11] Maisonbe generalized this approach to compute the $\mathscr{D}_{X,\mathfrak{x}}[S]$-dual of $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$ where f is as in [9], $f = f_1 \cdots f_r$, and $F = (f_1, \ldots, f_r)$. In this section we will use Maisonobe's approach to compute the $\mathscr{D}_{X,x}[S]$-dual of $\mathscr{D}_{X,x}[S]f'F^S$ where $f \in \mathscr{O}_X$ is free, strongly Euler-homogeneous, Saito-holonomic, not necessarily reduced but admitting a reduced Euler-homogeneous defining equation f_{red} at $\mathfrak{x}$, $f' \in \mathscr{O}_{X,\mathfrak{x}}$ is compatible with f, and $F = (f_1, \ldots, f_r)$ corresponds to any factorization, not necessarily into irreducibles, of

$f = f_1 \cdots f_r$. The strategy hinges on a formula for the trace of the adjoint first proved by Castro–Jiménez and Ucha in Theorem 4.1.4 of [34]. We supply a different proof in Proposition 3.6.12.

In the second subsection, we note that this duality computation lets us argue as in Maisonobe's Proposition 20 of [39] and prove that the radical of $B_{f'F,\mathfrak{x}}$ is principal. In the third subsection, we show that $B^g_{f'F,\mathfrak{x}}$ is fixed under a non-trivial involution when f', F, and g satisfy a technical condition, cf. Definition 3.3.14.

Convention 3.3.1. A resolution is a (co)-complex with a unique (co)-homology module at its end. An acyclic (co)-complex has no non-trivial (co)-homology. Given a (co)-complex ($C^\bullet$) $C_\bullet$ resolving A, the augmented (co)-complex ($C^\bullet \to A$) $C_\bullet \to A$ is acyclic.

3.3.1 Computing the Dual

Our argument begins at essentially the same place as Narváez-Macarro's and Maisonobe's: the Spencer co-complex.

Definition 3.3.2. Let $f = f_1 \cdots f_r \in \mathscr{O}_{X,\mathfrak{x}}$ be free, let $F = (f_1, \ldots, f_r)$, and let $f' \in \mathscr{O}_{X,\mathfrak{x}}$ be compatible with f. Consider $g_1, \ldots, g_u \in \mathscr{O}_{X,\mathfrak{x}}$ such that $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot g_j$ for all $1 \leq j \leq u$, and let $I \subseteq \mathscr{O}_{X,\mathfrak{x}}$ be the ideal generated by $g_1, \ldots, g_u$. We will define $\mathrm{Sp}^I_{\theta_{f'F,\mathfrak{x}}}$, the *extended Spencer co-complex* associated to f' and I. When $I = (g)$, write $\mathrm{Sp}^g_{f'F}$. This will be a mild generalization of the normal Spencer complex, cf. A.18 of [9].

Let E be the free submodule of $\mathscr{O}^u_{X,\mathfrak{x}}$ prescribed by the basis $e_1, \ldots, e_u$ where $e_j = (0, \ldots, g_j, \ldots, 0)$. We define an anti-commutative map

$$\sigma : (\theta_{f'F,\mathfrak{x}} \oplus E) \times (\theta_{f'F,\mathfrak{x}} \oplus E) \to \theta_{f'F,\mathfrak{x}} \oplus E$$

that is essentially the commutator on $F^1_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$. The map is determined by its anti-commutativity and the following assignments:

$$\sigma(\lambda_i,\lambda_j)=\begin{cases}[\lambda_i,\lambda_j], & \lambda_i,\lambda_j\in\theta_{f'F,\mathfrak{x}},\\ 0, & \lambda_i,\lambda_j\in E,\\ \frac{\delta\bullet(bg_j)}{g_j}e_j, & \lambda_i=\psi_{f'F,\mathfrak{x}}(\delta_i)\text{ for }\delta_i\in\operatorname{Der}_{X,\mathfrak{x}}(-\log f),\ \lambda_j=be_j.\end{cases}$$

Abbreviate $\operatorname{Sp}^I_{\theta_{f'F,\mathfrak{x}}}$ as $\operatorname{Sp}^\bullet$. Then the objects of our complex are

$$\operatorname{Sp}^{-m}=\mathscr{D}_{X,\mathfrak{x}}[S]\otimes_{\mathscr{O}_{X,\mathfrak{x}}}\bigwedge^m(\theta_{f'F,\mathfrak{x}}\oplus E)$$

and the differentials $d^{-m}:\operatorname{Sp}^{-m}\mapsto\operatorname{Sp}^{-m+1}$ are given by

$$\begin{aligned}d^{-m}(P\otimes\lambda_1\wedge\cdots\wedge\lambda_m)=&\sum_{i=1}^{r}(-1)^{i-1}P\lambda_i\otimes\widehat{\lambda_i}\\&+\sum_{1\le i<j\le m}(-1)^{i+j}P\otimes\sigma(\lambda_i,\lambda_j)\wedge\widehat{\lambda_{i,j}}.\end{aligned}$$

Here $\widehat{\lambda_i}$ is the wedge, in increasing order, of all the $\lambda_1,\ldots,\lambda_r$ except for λ_i; $\widehat{\lambda_{i,j}}$ is the same except now excluding both λ_i and λ_j. To be clear, we interpret Pe_j as $Pg_j\in\mathscr{D}_{X,\mathfrak{x}}[S]$; in particular, $d^{-1}(P\otimes e_j)=Pg_j$. There is a natural augmentation map

$$\operatorname{Sp}^0=\mathscr{D}_{X,\mathfrak{x}}[S]\mapsto\frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{\mathscr{D}_{X,\mathfrak{x}}[S]\cdot\theta_{f'F,\mathfrak{x}}+\mathscr{D}_{X,\mathfrak{x}}[S]\cdot I}.$$

Remark 3.3.3.(a) Since $\operatorname{Der}_{X,\mathfrak{x}}(-\log f)$ is closed under taking commutators, so is $\theta_{f'F,\mathfrak{x}}$, see also Example 4.7 of Chapter 2. And as g_j divides f for all $1\le j\le u$, we know $\operatorname{Der}_{X,\mathfrak{x}}(-\log f)\subseteq\operatorname{Der}_{X,\mathfrak{x}}(-\log g_j)$ for all j. Thus σ, and consequently the differentials, are well-defined.

(b) That the extended Spencer co-complex is in fact a co-complex is a straightforward computation mirroring the case of the standard Spencer co-complex.

(c) We have assumed f is free so that $\operatorname{Sp}^I_{\theta_{f'F,\mathfrak{x}}}$ will be a finite, free co-complex of $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules. We may fix a basis of $\theta_{f'F,\mathfrak{x}}$, extend it to a basis of $\theta_{f'F,\mathfrak{x}}\oplus E$ using the prescribed basis of E, and then compute differentials. Label this basis $\lambda_1, \dots, \lambda_{n+u}$. Let $\sigma(\lambda_i, \lambda_j) = \sum_{k=1}^{n+u} c_k^{i,j} \lambda_k$ be the unique expression of $\sigma(\lambda_i, \lambda_j)$. Then

$$\begin{aligned} d^{-m}(\lambda_1 \wedge \cdots \wedge \lambda_m) &= \sum_{i=1}^{m} (-1)^{i-1} \lambda_i \otimes \widehat{\lambda_i} \\ &+ \sum_{1 \leq i < j \leq m} (-1)^{i+j} c_i^{i,j} \otimes (-1)^{i-1} \widehat{\lambda_j} + (-1)^{i+j} c_j^{i,j} \otimes (-1)^j \widehat{\lambda_i} \\ &= \sum_{i=1}^{m} \left((-1)^{i-1} \lambda_i + \sum_{j<i} (-1)^{i-1} c_j^{j,i} + \sum_{i<j} (-1)^i c_j^{i,j} \right) \otimes \widehat{\lambda_i}. \end{aligned}$$

We can naturally encode this as matrix multiplication on the right.

The following calculation relies on Castro–Jiménez and Ucha's formula for adjoints appearing in Theorem 4.1.4 of [34]; cf. Proposition 3.6.12 for our proof. See also Lemma 1 and Proposition 6 of [11]. Before stating the Proposition, let us recall the side-changing functor for $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules. We use the notation of Appendix A of [9].

Definition 3.3.4. (Compare to Appendix A of [9]) We will define the equivalence of categories between right $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules and left $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules. First, regard $\mathrm{Der}_{X,\mathfrak{x}}[S]$ as a free $\mathscr{O}_{X,\mathfrak{x}}[S]$-module of rank n. Then the dualizing module $\omega_{\mathrm{Der}_{X,\mathfrak{x}}[S]}$ of $\mathrm{Der}_{X,\mathfrak{x}}[S]$ is defined as

$$\omega_{\mathrm{Der}_{X,\mathfrak{x}}[S]} = \mathrm{Hom}_{\mathscr{O}_{X,\mathfrak{x}}[S]} \left(\bigwedge^{n} \mathrm{Der}_{X,\mathfrak{x}}[S], \mathscr{O}_{X,\mathfrak{x}}[S] \right).$$

This naturally carries a right $\mathscr{D}_{X,\mathfrak{x}}[S]$-module structure by A.20 of [9]. The aforementioned equivalence of categories is given by associated to every right $\mathscr{D}_{X,\mathfrak{x}}[S]$-module Q the left $\mathscr{D}_{X,\mathfrak{x}}[S]$-module Q^{left} defined by

$$Q^{\mathrm{left}} = \mathrm{Hom}_{\mathscr{O}_{X,\mathfrak{x}}[S]} \left(\omega_{\mathrm{Der}_{X,\mathfrak{x}}[S]}, Q \right).$$

That Q^{left} is a left $\mathscr{D}_{X,\mathfrak{x}}[S]$-module follows from A.2 of [9]; that this gives an equivalence of categories follows from the discussion before A.25 of loc. cit.

Remark 3.3.5. Despite the s-terms, this side-changing functor is defined entirely similarly to the side-changing functor for $\mathscr{D}_{X,\mathfrak{x}}$-modules. So just as in the $\mathscr{D}_{X,\mathfrak{x}}[S]$-module case, if we fix coordinates (x, ∂_x) we can describe the transition from right to left $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules in elementary terms. Define $\tau : \mathscr{D}_{X,\mathfrak{x}}[S] \to \mathscr{D}_{X,\mathfrak{x}}[S]$ by $\tau(x^{\alpha}\partial_x^{\beta}s^{\gamma}) = (-\partial_x^{\beta})x^{\alpha}s^{\gamma}$ where α, β, and γ are multi-indices. Then $(-)^{\text{left}}$ sends the cyclic right $\mathscr{D}_{X,\mathfrak{x}}[S]$-module $\mathscr{D}_{X,\mathfrak{x}}[S]/J$ is to the left $\mathscr{D}_{X,\mathfrak{x}}[S]$-module $\mathscr{D}_{X,\mathfrak{x}}[S]/\tau(J)$. See 1.2 of [40] for details in a similar case.

Proposition 3.3.6. *Let $f = f_1 \cdots f_r \in \mathscr{O}_{X,\mathfrak{x}}$ be free, $F = (f_1, \dots, f_r)$, $f_{\text{red}} \in \mathscr{O}_{X,\mathfrak{x}}$ a Euler-homogeneous reduced defining equation for f at $\mathfrak{x}$, and $I \subseteq \mathscr{O}_{X,\mathfrak{x}}$ the ideal generated by $g_1, \dots, g_u$ with $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot g_v$ for each g_v. Write $g = g_1 \cdots g_u$. Then we can compute the terminal homology module of $\operatorname{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(Sp^{I}_{\theta_{f'F,x}}, \mathscr{D}_{X,\mathfrak{x}}[S])^{\text{left}}$:*

$$H_{-n-u}\left(\operatorname{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(Sp^{I}_{\theta_{f'F,x}}, \mathscr{D}_{X,\mathfrak{x}}[S])^{\text{left}}\right) \simeq \frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta^{-S}_{(f'gf_{\text{red}})^{-1}F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot I}.$$

Proof. We will show that the image of $\operatorname{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(d^{-n-u}, \mathscr{D}_{X,\mathfrak{x}}[S])^{\text{left}}$ is $\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta^{-S}_{(f'gf_{\text{red}})^{-1}F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot I$. It suffices to do this in local coordinates $x_1, \dots, x_n$. Select a basis $\delta_1, \dots, \delta_n$ of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f)$, label $\lambda_i = \psi_{f'F,\mathfrak{x}}(\delta_i)$ and label $\lambda_{n+j} = e_j = (0, \dots, g_j, \dots, 0)$ for $1 \leq j \leq u$, cf. Definition 3.3.2. Then $\lambda_1, \dots, \lambda_{n+u}$ is a basis of $\psi_{f'F,\mathfrak{x}} \oplus E$. Consequently, we may uniquely write $\sigma(\lambda_i, \lambda_j) = \sum_{k=1}^{n+u} c_k^{i,j}\lambda_k$ with $c_k^{i,j} \in \mathscr{O}_{X,\mathfrak{x}}$.

Let us compute the $c_k^{i,j}$ terms in cases. First assume $i, j \leq n$. Then $\sigma(\lambda_i, \lambda_j) = [\psi_{f'F,\mathfrak{x}}(\delta_i), \psi_{f'F,\mathfrak{x}}(\delta_j)] = [\delta_i, \delta_j]$, where the last equality follows since $\psi_{f'F,\mathfrak{x}}$ respects taking commutators, cf. Remark 3.2.17. Thus $c_1^{i,j}, \dots, c_n^{i,j}$ satisfy $[\delta_i, \delta_j] = \sum_{k=1}^{n} c_k^{i,j}\delta_k$; moreover, if $k \geq n+1$, then $c_k^{i,j} = 0$. Second, assume $i \leq n$ and $j \leq u$. By definition $\sigma(\lambda_i, \lambda_{n+j}) = \frac{\delta \bullet g_j}{g_j}\lambda_{n+j}$ and so $c_{n+j}^{i,n+j} = \frac{\delta_i \bullet g_j}{g_j}$ and $c_k^{i,n+j} = 0$ for $k \neq n+j$. Similarly for $j \leq n$ and $i \leq u$, $c_{n+j}^{n+j,i} = -\frac{\partial_i \bullet g_j}{g_j}$ and $c_k^{n+j,i} = 0$ for all $k \neq n+j$. Finally, assume $i, j \leq u$. Then $\sigma(\lambda_{n+i}, \lambda_{n+j}) = 0$ and $c_k^{n+i,n+j} = 0$ for all k.

Using Remark 3.3.3, d^{-n-u} is given, where $i \leq n$ and $v \leq u$, by multiplying on the right by the matrix

$$\left[\cdots \quad (-1)^{i-1}(\psi_{f'F,\mathfrak{x}}(\delta_i) - \sum_{j=1}^{n} c_j^{i,j} - \sum_{v=1}^{u} \tfrac{\delta \bullet g_v}{g_v}) \quad \cdots \quad (-1)^{n+v-1}g_v \quad \cdots\right]. \tag{3.3.1}$$

The dual map is given by transposing (3.3.1) and applying τ, the standard right-to-left map (cf. Remark 3.3.5), to each each entry where τ is inert on $\mathscr{O}_{X,\mathfrak{x}}[S]$ and sends $h\partial_{x_i}$ to $-\partial_{x_i}h$,

$h \in \mathscr{O}_{X,\mathfrak{x}}[S]$. Write $\delta_i = \sum_e h_{e,i}\partial_{x_e}$ and observe that $\tau(\delta_i) = -\delta_i - \sum_e \partial_{x_e} \bullet h_{e,i}$. Therefore $\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(d^{-n-u}, \mathscr{D}_{X,\mathfrak{x}}[S])^{\mathrm{left}}$ is given by right multiplication by

$$\begin{bmatrix} \vdots \\ (-1)^{i-1}(-\delta_i - \sum_{k=1}^{r} \frac{\delta_i \bullet f_k}{f_k} s_k - \frac{\delta_i \bullet f'}{f'} - \sum_{e=1}^{n} \partial_{x_e} \bullet h_{e,i} - \sum_{j=1}^{n} c_j^{i,j} - \sum_{v=1}^{u} \frac{\delta \bullet g_v}{g_v}) \\ \vdots \\ (-1)^{n+v-1} g_v \\ \vdots \end{bmatrix} \tag{3.3.2}$$

Assume $n \geq 2$. We could have chosen $\delta_1, \dots, \delta_n$ to be a preferred basis of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}}) = \mathrm{Der}_{X,\mathfrak{x}}(-\log f)$, cf. Definition 3.6.11, making $\delta_1, \dots, \delta_{n-1} \in \mathrm{Der}_{X,\mathfrak{x}}(-\log_0 f)$ and δ_n a Euler-homogeneity for f_{red}. By the trace-adjoint formula of Proposition 3.6.12:

$$\sum_j c_j^{i,j} = -\sum_e \partial_{x_e} \bullet h_{e,i} \text{ for } i \neq n; \; \sum_j c_j^{n,j} = -\sum_e \partial_{x_e} \bullet h_{e,n} + 1 \text{ for } i = n.$$

Recall $g = g_1 \cdots g_u$. Since $\delta_i \bullet f_{\mathrm{red}} = 0$ for $i \leq n-1$ and since δ_n is Euler-homogeneous on f_{red}, (3.3.2) simplifies to

$$\begin{bmatrix} \vdots \\ (-1)^{i}(\psi^{-S}_{(f' g f_{\mathrm{red}})^{-1}F,\mathfrak{x}})(\delta_i) \\ \vdots \\ (-1)^{n}(\psi^{-S}_{(f' g f_{\mathrm{red}})^{-1}F,\mathfrak{x}}(\delta_n) \\ \vdots \\ (-1)^{n+v-1} g_v \\ \vdots \end{bmatrix}.$$

Thus the image of $\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(d^{-n-u}, \mathscr{D}_{X,\mathfrak{x}}[S])^{\mathrm{left}}$ is $\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta^{-S}_{(f' g f_{\mathrm{red}})^{-1}F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot I$, proving the proposition for $n \geq 2$.

As for $n = 1$, we can assume $f_{\mathrm{red}} = x$ and $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}})$ is freely generated by its Euler-homogeneity. Simplifying (3.3.2) is then an easy calculation. $\square$

We endow $\mathrm{Sp}^{I}_{f'F,\mathfrak{x}}$ with a chain co-complex filtration that is based on a construction of Gros and Narváez-Macarro, cf. page 85 of [41].

Proposition 3.3.7. *Let $f = f_1 \cdots f_r$ be free, $F = (f_1, \ldots, f_r)$, and let f' and I be as in Definition 3.3.2. Abbreviate $Sp^{I}_{\theta_{f'F,\mathfrak{x}}}$ to $Sp^{\bullet}$. Define a filtration $G^{\bullet}$ on $Sp^{\bullet}$ by*

$$G^p\, Sp^{-m} = \bigoplus_j \left(F^{p-m+j}_{(0,1,1)} \mathscr{D}_{X,\mathfrak{x}}[S] \otimes_{\mathscr{O}_{X,\mathfrak{x}}} \bigwedge^{m-j} \theta_{f'F,\mathfrak{x}} \wedge \bigwedge^{j} E \right).$$

If $\delta_1, \ldots, \delta_n$ is a basis of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$, then $\mathrm{gr}_G(Sp^{\bullet})$ is isomorphic to the following Koszul co-complex on $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$:

$$K^{\bullet}(\mathrm{gr}_{(0,1,1)}(\psi_{F,\mathfrak{x}}(\delta_1)), \ldots, \mathrm{gr}_{(0,1,1)}(\psi_{F,\mathfrak{x}}(\delta_n)), g_1, \ldots, g_u; \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])). \tag{3.3.3}$$

Moreover, $G^{\bullet}$ naturally gives a filtration on $Hom_{\mathscr{D}_{X,\mathfrak{x}}[S]}(Sp^{\bullet}, \mathscr{D}_{X,\mathfrak{x}}[S])^{left}$ whose associated graded complex is isomorphic to

$$K^{\bullet}(\mathrm{gr}_{(0,1,1)}(-\psi_{F,\mathfrak{x}}(\delta_1)), \ldots, \mathrm{gr}_{(0,1,1)}(-\psi_{F,\mathfrak{x}}(\delta_n)), g_1, \ldots, g_u; \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])). \tag{3.3.4}$$

Proof. That $G^{\bullet}$ is a chain filtration and that the associated graded co-complex is isomorphic to the Koszul complex (3.3.3) follows from the definitions. As for the dual statement, it is enough to note that τ, the standard right-to-left map (cf. Lemma 4.13 of Chapter 2), preserves weight 0 entries (under the total order filtration) and sends weight 1 entries $\delta + p(S)$ to $-\delta + p(S) +$ error terms, where δ is a derivation and both $p(S)$ and the error terms lie in $\mathscr{O}_{X,\mathfrak{x}}[S]$. □

We now add hypotheses to the settings of Propositions 3.3.6 and 3.3.7. First, we assume $I = \mathscr{O}_{X,x} \cdot g$ is principal; second, we assume f is not only free but also strongly Euler-homogeneous and Saito-holonomic. This will let us use results from Chapter 2. The filtration $G^{\bullet}$ will demonstrate that $\mathrm{Sp}^{g}_{f'F}$ and its dual are resolutions.

Definition 3.3.8. For M a left $\mathscr{D}_{X,\mathfrak{x}}[S]$-module, denote the $\mathscr{D}_{X,\mathfrak{x}}[S]$-dual of M by

$$\mathbb{D}(M) = \mathrm{RHom}_{\mathscr{D}_{X,x}[S]}(M, \mathscr{D}_{X,\mathfrak{x}}[S])^{\mathrm{left}}.$$

Theorem 3.3.9. *Suppose $f = f_1 \cdots f_r \in \mathscr{O}_X$ is free, strongly Euler-homogeneous, and Saito-holonomic and $f_{\text{red}} \in \mathscr{O}_{X,\mathfrak{x}}$ is a Euler-homogeneous reduced defining equation for f at $\mathfrak{x}$. Let $F = (f_1, \dots, f_r)$, let $f' \in \mathscr{O}_{X,\mathfrak{x}}$ be compatible with f, and let $g \in \mathscr{O}_{X,\mathfrak{x}}$ such that $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot g$. Then*

$$\mathbb{D}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S] \cdot gf'F^S}\right) \simeq \frac{\mathscr{D}_{X,\mathfrak{x}}[S](gf'f_{\text{red}})^{-1}F^{-S}}{\mathscr{D}_{X,\mathfrak{x}}[S](f'f_{\text{red}})^{-1}F^{-S}}[n+1].$$

Proof. We first show that (3.3.3) and (3.3.4) are both resolutions; in fact, showing (3.3.3) is a resolution proves (3.3.4) is as well. Let $\delta_1, \dots, \delta_n$ be a basis of $\text{Der}_{X,\mathfrak{x}}(-\log f)$. Since $\text{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ is graded local and $\text{gr}_{(0,1,1)}(\psi_{F,\mathfrak{x}}(\delta_i))$ and f all live in the graded maximal ideal, it is sufficient to prove that the Koszul co-complex (3.3.3) is a resolution after localization at the graded maximal ideal. By Theorem 2.23 of Chapter 2, $\widetilde{L_{F,\mathfrak{x}}}$ is Cohen–Macaulay and prime of dimension $n + r$. Therefore $\widetilde{L_{F,\mathfrak{x}}} + \text{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot f$ has dimension $n + r - 1$. Moreover, this ideal's dimension does not change after localization at the graded maximal ideal. Theorem 2.1.2 of [22] then implies (3.3.3) is a resolution after said localization, finishing this part of the proof.

Since (3.3.3) is a resolution, a standard spectral sequence argument associated to the filtered co-complex of $\text{Sp}^g_{f'F,\mathfrak{x}}$ implies $\text{Sp}^g_{f'F,\mathfrak{x}}$ is a resolution. By Theorem 3.2.21 and the definition of the augmentation map it resolves $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$. Similar reasoning verifies that $\text{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(\text{Sp}^g_{f'F,\mathfrak{x}}, \mathscr{D}_{X,\mathfrak{x}}[S])^{\text{left}}$ is a resolution. Because f_{red} is Euler homogeneous, the claim follows by Proposition 3.3.6 and Theorem 3.2.23. $\square$

Remark 3.3.10. We are skeptical that (3.3.3) is a resolution for any non-principal, non-pathological I. Possible candidates are linear free divisors f with many factors, even though the non-pathological examples in $n \leq 4$ fail, cf. [42].

3.3.2 Principality of Bernstein–Sato Ideals

Here we discuss the principality of the radical of $B^g_{f'F,\mathfrak{x}}$. The argument is essentially the same as Proposition 20 of [39], but we do not have to appeal to tame pure extensions because of our hypotheses on f.

We will need some homological definitions for modules over non-commutative rings, cf. Appendix IV of [16] for a detailed treatment. We say a $\mathscr{D}_{X,\mathfrak{x}}[S]$-module M has *grade* j if $\mathrm{Ext}^k_{\mathscr{D}_{X,\mathfrak{x}}[S]}(M, \mathscr{D}_{X,\mathfrak{x}}[S])$ vanishes for all $k < j$ and is nonzero for $k = j$. We say M is *pure* of grade j if every nonzero submodule of M has grade j. We also need the following filtration on $\mathscr{D}_{X,\mathfrak{x}}[S]$:

Definition 3.3.11. Define the *order filtration* $F_{(0,1,0)}$ on $\mathscr{D}_X[S]$ by designating, in local coordinates, every ∂_{x_k} weight one and every element of $\mathscr{O}_X[S]$ weight zero. Let $\mathrm{gr}_{(0,1,0)}(\mathscr{D}_X[S])$ denote the associated graded object and note that locally $\mathrm{gr}_{(0,1,0)}(\mathscr{D}_X[S]) \simeq \mathscr{O}_X[Y][S]$, with $\mathrm{gr}_{(0,1,0)}(\partial_{x_k}) = y_k$. For a coherent $\mathscr{D}_X[S]$-module M and any good filtration Γ on M relative to $F_{(0,1,0)}$, the *characteristic ideal* $J^{\mathrm{rel}}(M) \subseteq \mathrm{gr}_{(0,1,1)}(\mathscr{D}_X[S])$ is defined as

$$J^{\mathrm{rel}}(M) = \sqrt{\mathrm{ann}_{\mathrm{gr}_{(0,1,0)}(\mathscr{D}_X[S])}\, \mathrm{gr}_\Gamma(M)}$$

and is independent of the choice of good filtration.

Proposition 3.3.12. (Compare to Proposition 20 of [39]) *Suppose $f = f_1 \ldots f_r \in \mathscr{O}_X$ is free, strongly Euler-homogeneous, and Saito-holonomic such that the reduced divisor of f is Euler-homogeneous. Let $F = (f_1, \ldots, f_r)$ and select $f' \in \mathscr{O}_X$ and $g \in \mathscr{O}_X$ such that f lies in both $\mathscr{O}_X \cdot f'$ and $\mathscr{O}_X \cdot g$. Then for all $\mathfrak{x}$, $\sqrt{B^g_{f'F,\mathfrak{x}}}$ is principal.*

Proof. Since f' is a section generating a holonomic $\mathscr{D}_X$-module, by Proposition 13 of [39] there is a conical Lagrangian variety $\Lambda \subseteq T^\star X$ so that $\mathrm{V}(J^{\mathrm{rel}}(\mathscr{D}_X[S]f'F^S)) = \Lambda \times C^r$. So $\mathrm{V}(J^{\mathrm{rel}}(\frac{\mathscr{D}_X[S]f'F^S}{\mathscr{D}_X[S]gf'F^S})) \subseteq \Lambda \times \mathbb{C}^r$, that is, in the language of Maisonobe, $\frac{\mathscr{D}_X[S]f'F^S}{\mathscr{D}_X[S]gf'F^S}$ is *majoré par une Lagrangian.* By Proposition 8 of [39], there exist conical Lagrangians $T^\star_{X_\alpha}X$ and algebraic varieties $S_\alpha \subseteq \mathbb{C}^r$ such that

$$\mathrm{V}\left(J^{\mathrm{rel}}(\frac{\mathscr{D}_X[S]f'F^S}{\mathscr{D}_X[S]gf'F^S})\right) = \cup_\alpha T^\star_{X_\alpha}X \times S_\alpha. \tag{3.3.5}$$

By Proposition 9 of [39], $\mathrm{V}(B^g_{f'F,\mathfrak{x}}) = \cup_{\mathfrak{x}\in X_\alpha} S_\alpha$.

Now to show the radical of $B^g_{f'F,\mathfrak{x}}$ is principal, it suffices to show S_α is of dimension $r-1$ for each α such that $\mathfrak{x} \in X_\alpha$; that is, by the description of $T^\star_{X_\alpha}X$, it suffices to show

$J^{\mathrm{rel}}(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S})$ is equidimensional of dimension $n+r-1$. By Theorem 3.3.9, $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$ has grade $n+1$. Using Theorem 3.3.9 again and the characterization of pure modules in terms of double Ext modules, cf. Proposition IV.2.6 of [16], we deduce $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$ is a pure $\mathscr{D}_{X,\mathfrak{x}}[S]$-module of grade $n+1$. By Theorem IV.5.2 of [16], $J^{\mathrm{rel}}(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S})$ is equidimensional and every minimal prime of the characteristic ideal has codimension $n+1$, completing the proof. $\square$

The next proposition lays out a criterion for $B_{f'F,\mathfrak{x}}^{g}$ to be principal. The argument is that of the last paragraph of Theorem 2 of [11].

Proposition 3.3.13. (Compare to Theorem 2 of [11]) *Let f, F, f', and g be as in Proposition 3.3.12 and suppose that $\sqrt{B_{f'F,\mathfrak{x}}^{g}} = \mathbb{C}[S]\cdot b(S)$, i.e. it is principal. Suppose that $(B_{f'F,\mathfrak{x}} : \sqrt{B_{f'F,\mathfrak{x}}})$ contains a polynomial $a(S)$ such that $\mathrm{V}(\mathbb{C}[S]\cdot b(S)) \cap \mathrm{V}(\mathbb{C}[S]\cdot a(S))$ has irreducible components of dimension at most $r-2$. Then $B_{f'F,\mathfrak{x}}^{g}$ equals its radical and is principal.*

Proof. It suffices to show $b(S)\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$ is zero. If it is nonzero, it is a submodule of the pure module $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$ of grade $n+1$ and so is itself pure of the same grade. Reasoning as in Proposition 3.3.12, cf. Proposition 9 of [39] in particular, all the minimal primes of $\mathbb{C}[S]$-annihilator of $b(S)\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$ have dimension $r-1$. But the variety of this annihilator is contained inside $\mathrm{V}(\mathbb{C}[S]\cdot b(S)) \cap \mathrm{V}(\mathbb{C}[S]\cdot a(S))$ which is of dimension $r-2$ by hypothesis. As this is impossible, $b(S)\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$ must be zero. $\square$

3.3.3 Symmetry of Some Bernstein–Sato Varieties

As Theorem 3.3.9 generalizes Corollary 3.6 of [9] and Proposition 6 of [11], one would hope $B_{f'F,\mathfrak{x}}^{g}$ has a symmetry generalizing Theorem 4.1 of [9] and Proposition 8 of [11]. However, without reducedness and with the addition of f', symmetry seems to depend on the factorization of f.

Definition 3.3.14. Suppose f has a factorization into irreducibles $l_1^{v_1} \cdots l_q^{v_q}$ at $\mathfrak{x}$ where the l_t are distinct and $v_t \in \mathbb{Z}_+$. Let $f = f_1 \cdots f_r$ be some other factorization of f and let $F = (f_1, \ldots, f_r)$. We say the factorization $f = f_1 \cdots f_r$ is *unmixed* if the following hold:

(i) for each k, there exists $d_k \in \mathbb{Z}_+$ and $J_k \subseteq [q]$ such that $f_k = \prod_{j \in J_k} l_j^{d_k}$;

(ii) if $i, j \in J_k$, then $v_i = v_j$.

F is *unmixed* when it corresponds to an unmixed factorization; F is *unmixed up to units* if there exists units $u_1, \ldots, u_r$ such that $uF = (u_1 f_1, \ldots, u_r f_r)$ is unmixed. Given an unmixed factorization, let the *repeated multiplicity* of F be $\{m_k\}_k$ where, for any $j \in J_k$ (and thus all), m_k is the multiplicity of l_j with respect to f.

For $f' \in \mathscr{O}_{X,\mathfrak{x}}$ compatible with f, we say (f', F) is an *unmixed pair* if:

(i)' F is unmixed;

(ii)' $f' = \prod_k \prod_{j \in J_k} l_j^{d'_k}$ for $d'_k \in \mathbb{Z}$.

The pair (f', F) is an *unmixed pair up to units* if F is unmixed up to units and f' satisfies (ii)' after possibly multiplying by a unit. For (f', F) an unmixed pair up to units, the *pairs of repeated powers* of (f', F) are $\{(d'_k, d_k)\}_k$.

Lemma 3.3.15. *Write $f = l_1^{v_1} \cdots l_q^{v_q}$ where the l_i are distinct and irreducible; $f_k = \prod_{j \in J_k} l_j^{d_k}$; $f_{\text{red}} = l_1 \cdots l_q$. Assume that f' and g are compatible with f, $F = (f_1, \ldots, f_r)$ a factorization of f, (f', F) and (g, F) are unmixed pairs with pairs of repeated powers $\{(d'_k, d_k)\}_k$ and $\{(d''_k, d_k)\}_k$, and $\{m_k\}_k$ the repeated multiplicities of F. If $\varphi : \mathbb{C}[S] \to \mathbb{C}[S]$ is the automorphism of $\mathbb{C}$-algebras induced by*

$$\varphi(s_k) = -s_k - \frac{1}{m_k} - \frac{2d'_k}{d_k} - \frac{d''_k}{d_k},$$

then for $\delta \in \operatorname{Der}_{X,\mathfrak{x}}(-\log f)$, and after extending φ to $\mathscr{D}_{X,\mathfrak{x}}[S]$,

$$\varphi(\psi^{-S}_{(f' g f_{\text{red}})^{-1} F, \mathfrak{x}}(\delta)) = \psi^{S}_{f' F, \mathfrak{x}}(\delta).$$

Proof. This is a straightforward computation once we observe that v_j is the sum of all the d_k such that l_j divides f_k. $\square$

Theorem 3.3.16. *Suppose $f = f_1 \cdots f_r \in \mathscr{O}_X$ is free, strongly Euler-homogeneous, and Saito-holonomic, and while f is not necessarily reduced, suppose that it admits a strongly Euler-homogeneous reduced defining equation at $\mathfrak{x}$. Let $F = (f_1, \cdots, f_r)$ and select $g \in \mathscr{O}_{X,\mathfrak{x}}$ such that $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot g$. Assume that f' and g are compatible with f, (f', F) and (g, F) are unmixed pairs up to units with pairs of repeated powers $\{(d'_k, d_k)\}_k$ and $\{(d''_k, d_k)\}_k$, and $\{m_k\}_k$ are the repeated multiplicities of F. If $\varphi : \mathbb{C}[S] \to \mathbb{C}[S]$ is the automorphism of $\mathbb{C}$-algebras induced by*

$$\varphi(s_k) = -s_k - \frac{1}{m_k} - \frac{2d'_k}{d_k} - \frac{d''_k}{d_k},$$

then

$$B(S) \in B^g_{f'F,\mathfrak{x}} \iff \varphi(B(S)) \in B^g_{f'F,\mathfrak{x}}.$$

Proof. We first reduce to the case that (f', F) and (g, F) are unmixed pairs. It follows from the functional equation that if u is a unit in $\mathscr{O}_{X,\mathfrak{x}}$, then $B^g_{f'F,\mathfrak{x}} = B^g_{uf'F,\mathfrak{x}}$ and $B^g_{f'F,\mathfrak{x}} = B^{ug}_{f'F,\mathfrak{x}}$. To finish the reduction, we must also verify that if $F' = (u_1 f_1, \ldots, u_r f_r)$ for units $u_1, \ldots, u_r$ in $\mathscr{O}_{X,\mathfrak{x}}$, then $B^g_{f'F,\mathfrak{x}} = B^g_{f'F',\mathfrak{x}}$. This follows by arguing as in Lemma 10 (i) of [15] wherein the claim is proved for $f' = 1$ and $g = f$.

By the $\mathbb{C}[S]$-linearity of $\mathbb{D}$, cf. Remark 3.2 of [9], and by Theorem 3.3.9,

$$B(S) \in \operatorname{ann}_{\mathbb{C}[S]} \frac{\mathscr{D}_{X,\mathfrak{x}}[S] f' F^S}{\mathscr{D}_{X,\mathfrak{x}}[S] \cdot g f' F^S} \implies B(S) \in \operatorname{ann}_{\mathbb{C}[S]} \frac{\mathscr{D}_{X,\mathfrak{x}}[S](g f' f_{\text{red}})^{-1} F^{-S}}{\mathscr{D}_{X,\mathfrak{x}}[S] \cdot (f' f_{\text{red}}) F^{-S}}$$

where we may assume f_{red} is as in Lemma 3.3.15, cf. Remark 3.2.9. In other words,

$$\begin{aligned} B(S) &\in \mathbb{C}[S] \cap (\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_{f'F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot g) \\ &\implies B(S) \in \mathbb{C}[S] \cap (\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta^{-S}_{(f' g f_{\text{red}})^{-1} F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot g). \end{aligned}$$

By Lemma 3.3.15, φ induces a $\mathscr{D}_{X,\mathfrak{x}}$-automorphism that sends $\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta^{-S}_{(f' g f_{red})^{-1} F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot g$ to $\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_{f'F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot g$. Therefore $\varphi(B^I_{f'F,\mathfrak{x}}) \subseteq B^I_{f'F,\mathfrak{x}}$. The reverse containment follows from the fact φ is an involution. $\square$

Remark 3.3.17. Suppose f, f', and F are as in Theorem 3.3.16, and I is the ideal generated by $g_1, \ldots, g_u$ such that $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot g_j$. If $\mathrm{Sp}^{g}_{f'F,\mathfrak{x}}$ and its $\mathscr{D}_{X,\mathfrak{x}}[S]$-dual are both resolutions, then φ fixes $B^{I}_{f'F,\mathfrak{x}}$. Note that φ depends only on the product of the g_j.

Let us catalogue some of the most useful versions of the theorem:

Corollary 3.3.18. *Suppose $f = f_1 \cdots f_r \in \mathscr{O}_X$ is free, strongly Euler-homogeneous, and Saito-holonomic, and while f is not necessarily reduced, suppose that it admits a strongly Euler-homogeneous reduced defining equation at $\mathfrak{x}$. Let $F = (f_1, \ldots, f_r)$ and φ be as in Theorem 3.3.16.*

(a) *Suppose that $F = (l_1, \ldots, l_1, \ldots, l_q)$ with each l_t appearing v_t times, and f' and g any elements of $\mathscr{O}_{X,\mathfrak{x}}$ dividing f. Then $\varphi(B^{g}_{f'F,\mathfrak{x}}) = B^{g}_{f'F,\mathfrak{x}}$.*

(b) *Suppose f is reduced, F corresponds to any factorization, $f' = \prod_{k' \in K'} f'_k$, $g = \prod_{k \in K} f_k$, for $K', K \subseteq [r]$. Then $\varphi(B^{g}_{f'F,\mathfrak{x}}) = B^{g}_{f'F,\mathfrak{x}}$.*

(c) *Suppose f' divides $f = f_1 \cdots f_r$, $F = (f_1, \ldots, f_r)$ and $g = \frac{f}{f'}$. If (f', F) is an unmixed pair up to units, then $\varphi(B^{g}_{f'F,\mathfrak{x}}) = B^{g}_{f'F,\mathfrak{x}}$*

(d) *Suppose $f = f^{k}_{\mathrm{red}}$ and $F = (f^{k}_{\mathrm{red}})$. Then $\varphi(s) = -s - 1 - \frac{1}{k}$ and $\varphi(B_{f^k,\mathfrak{x}}) = B_{f^k,\mathfrak{x}}$.*

Proof. All that must be checked is that the appropriate things are unmixed pairs up to units. For example, in (a) and (b), F is unmixed up to units because it is a factorization into irreducibles, possibly with repetition, and because f is reduced, respectively. In both cases, d_k, d'_k, and d''_k are all 1. □

The symmetry property for the Bernstein–Sato polynomial of a reduced divisor forces all its roots to lie inside $(-2, 0)$, cf. [9]. We have the following generalization for powers of reduced divisor:

Corollary 3.3.19. *Suppose f is reduced, free, strongly Euler-homogeneous, and Saito-holonomic. Then $\mathrm{V}(B_{f^k}) \subseteq (-1 - \frac{1}{k}, 0)$. If $b_{f^k,\min}$ is the smallest root of the Bernstein–Sato polynomial of f^k, then $b_{f^k,\min} \to -1$ as $k \to \infty$.*

Proof. Since freeness, strongly Euler-homogeneous, and Saito-holonomicity pass from f_{red} to f^k we may use Corollary 3.3.18 to improve the well known containment $\mathrm{V}(B_{f^k,\mathfrak{x}}) \subseteq (-\infty, 0)$ to $\mathrm{V}(B_{f^k,\mathfrak{x}}) \subseteq (-1 - \frac{1}{k}, 0)$. The rest follows since $-1 \in \mathrm{V}(B_{f^k,\mathfrak{x}})$. $\square$

3.4 Bernstein–Sato Varieties for Tame and Free Arrangements

In this section we study the global Bernstein–Sato ideals $B^g_{f'F}$ where f is a central, not necessarily reduced, tame hyperplane arrangement, f' divides f, $g = \frac{f}{f'}$, and F corresponds to the factorization $f = f_1 \cdots f_r$, which need not be into linear forms. We always assume $\mathscr{O}_{X,\mathfrak{x}} \cdot f' \neq \mathscr{O}_{X,\mathfrak{x}} \cdot f$. We revisit the arguments of Maisonobe in [11] giving full details for our versions of Lemma 2 and Proposition 9 in the first subsection and Proposition 10 in the second. We generalize the strategy of Lemma 2 and Proposition 9 to compute a principal ideal containing $B^g_{f'F}$ for tame hyperplane arrangements and any F; we generalize Proposition 10 to find an element of $B^g_{f'F}$ when f is not necessarily reduced, not necessarily tame, and F is the total factorization of f into linear forms. As Maisonobe does in Theorem 2 of loc. cit., in the third subsection we use the symmetry of $B^g_{f'F}$ when f is free and (f', F) is an unmixed pair up to units to provide rather precise estimates of $\mathrm{V}(B^g_{f'F})$. In certain situations, these estimates compute $\mathrm{V}(B^g_{f'F})$.

Definition 3.4.1. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a central, not necessarily reduced, hyperplane arrangement of degree d whose factorization into homogeneous linear forms is $f = l_1 \cdots l_d$. Associated to f is the intersection lattice $L(A)$, partially ordered by reverse inclusion and with smallest element $\mathbb{C}^n$. We call any $X \in L(A)$ an *edge* of $L(A)$. The *rank* of X is the length of a maximal chain in $L(A)$ with smallest element $\mathbb{C}^n$ and largest element X. We denote the rank of X by $r(X)$; for example, $r(\mathrm{V}(l_i)) = 1$. Given an edge $X \in L(A)$ we define $J(X)$ to be the subset of $[d]$ identifying the hyperplanes that contain X, that is:

$$X = \bigcap_{j \in J(X)} \mathrm{V}(l_j).$$

Note that because f is not necessarily reduced $J(X)$ may contain indices i and j such that $\mathrm{V}(l_i) = \mathrm{V}(l_j)$. Given an edge X, there is the subarrangement A_X which has the defining equation

$$f_X = \prod_{j \in J(X)} l_j.$$

The degree of f_X is denoted d_X. So $d_X = |J(X)|$. The edge X is *decomposable* if there is a change of coordinates $y_1 \sqcup y_2$, y_1 and y_2 disjoint, such that $f_X = pq$ where p and q are hyperplane arrangements using variables only from y_1 and y_2 respectively. Otherwise X is *indecomposable*.

Consider a potentially different factorization $f = f_1 \cdots f_r$ where each f_k is of degree d_k. Since each f_k is a product of some of the l_m, let $S_k \subseteq [d]$ identify the linear forms comprising f_k, that is,

$$f_k = \prod_{m \in S_k} l_m.$$

The factorization $f = f_1 \cdots f_r$ induces a factorization of f_X. Define $S_{X,k} \subseteq [d]$ by

$$S_{X,k} = J_X \cap S_k.$$

Then f_X inherits the factorization $f_X = f_{X,1} \cdots f_{X,r}$ where

$$f_{X,k} = \prod_{j \in S_{X,k}} l_j.$$

We say $f_{X,k}$ has degree $d_{X,k}$. We also write $F_X = (f_{X,1}, \ldots, f_{X,r})$.

Any hyperplane arrangement has a reduced equation f_{red} of degree d_{red}. We define $f_{X,\mathrm{red}}$, $d_{X,\mathrm{red}}$, $f_{X,k,\mathrm{red}}$, and $d_{X,k,\mathrm{red}}$ similarly.

If f' of degree d' divides f, then all the previous constructions apply to f'. Define $f'_{\mathrm{red}}, d'_{\mathrm{red}}, f'_X, d'_X, f'_{X,\mathrm{red}}, d'_{X,\mathrm{red}}, f'_{X,k}, d'_{X,k}, f'_{X,k,\mathrm{red}}, d'_{X,k,\mathrm{red}}$ in the natural ways.

We will be working with the Weyl algebra $\mathrm{A}_n(\mathbb{C}) = \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ where the *global Bernstein–Sato ideal* $B^g_{f'F}$ is defined similarly to $B^g_{f'F,\mathfrak{x}}$ except using $\mathrm{A}_n(\mathbb{C})\,[S]$ operators. Write $B^g_{f'f}$ when $F = (f)$ corresponds to the trivial factorization $f = f$. We use the notation $\theta_{f'F}$ and $\psi_{f'F}$ for the algebraic, global versions of $\theta_{f'F,\mathfrak{x}}$ and $\psi_{f'F,\mathfrak{x}}$.

By Corollary 3.2.22 and Examples 3.2.7 and 3.2.10, if f is tame and f' divides f, then $\text{ann}_{\mathrm{A}_n(\mathbb{C})\,[S]}\, f'F^S$ is generated by derivations. Moreover, f_{red} is strongly Euler-homogeneous itself. Finally, since f is central, the $\mathbb{C}^\star$-action on $\mathrm{V}(f)$ can be used to show $B^g_{f'F} = B^g_{f'F,0}$. Therefore we can apply the results of the previous sections.

Finally, recall that for any central hyperplane arrangement $f \in \mathbb{C}[x_1, \ldots, x_n]$ of degree d, the *Euler derivation* $E = x_1\partial_1 + \cdots x_n\partial_n$ satisfies $E \bullet f = df$. Thus $\frac{1}{d}E$ is a strong Euler-homogeneity for f at the origin.

3.4.1 An Ideal Containing $B^g_{f'F}$

We compute a principal ideal containing $B^g_{f'F}$ where f is a central, indecomposable, and tame hyperplane arrangement, f' divides f, $g = \frac{f}{f'}$, and F corresponds to any factorization. The argument tracks Lemma 2 and Proposition 9 of [11] but we have replaced freeness with tameness, reduced with non-reduced, added f', and we will use any factorization F instead of the factorization into linear forms. Though the approach is similar to Maisonobe's, we provide detail for the sake of the reader.

Definition 3.4.2. The *right normal form* of $P \in \mathrm{A}_n(\mathbb{C})\,[S]$ is the unique expression

$$P = \sum_{\mathbf{u}} \partial^{\mathbf{u}} P_{\mathbf{u}}$$

where $P_{\mathbf{u}} \in \mathbb{C}[X][S]$. The *right constant term* of P is $P_{\mathbf{0}}$. Note that for $P, Q \in \mathrm{A}_n(\mathbb{C})\,[S]$, the right constant term of $P + Q$ is $P_{\mathbf{0}} + Q_{\mathbf{0}}$.

Convention 3.4.3. Let $\mathbb{C}[X]_t$ be the subspace of homogeneous polynomials in $\mathbb{C}[X]$ of degree t and let $\mathbb{C}[X]_{\geq t}$ be the ideal of $\mathbb{C}[X]$ generated by the homogeneous polynomials of degree at least t. Denote by $\mathbb{C}[X]_t[S]$ and $\mathbb{C}[X]_{\geq t}[S]$ the $\mathbb{C}[S]$-modules generated by $\mathbb{C}[X]_t$ and $\mathbb{C}[X]_{\geq t}$ respectively.

Lemma 3.4.4. *Consider a derivation $\delta = \sum_i a_i \partial_{x_i}$ and a polynomial $c \in \mathbb{C}[X][S]$. If $P \in A_n(\mathbb{C})[S]$ has right constant term $P_{\mathbf{0}}$, then $P \cdot (\delta - c)$ has right constant term*

$$-(\sum_i \partial_{x_i} \bullet a_i)P_{\mathbf{0}} - \delta \bullet (P_{\mathbf{0}}) - cP_{\mathbf{0}}.$$

Proof. Consider the right normal form $\sum \partial^{\mathbf{u}} P_{\mathbf{u}}$ of P. Then

$$\begin{aligned} P \cdot (\delta - c) &= \sum_{\mathbf{u}} \partial^{\mathbf{u}} (\delta P_{\mathbf{u}} - \delta \bullet P_{\mathbf{u}} - P_{\mathbf{u}} c) \\ &= \sum_{\mathbf{u}} \partial^{\mathbf{u}} ((\sum_i \partial_i a_i - \sum_i \partial_i \bullet a_i) P_{\mathbf{u}} - \delta \bullet P_{\mathbf{u}} - P_{\mathbf{u}} c) \\ &= \sum_{\mathbf{u}} \partial^{\mathbf{u}} \sum_i \partial_i a_i P_{\mathbf{u}} + \sum_{\mathbf{u}} \partial^{\mathbf{u}} ((-\sum_i \partial_i \bullet a_i) P_{\mathbf{u}} - \delta \bullet (P_{\mathbf{u}}) - cP_{\mathbf{u}}). \end{aligned}$$

Because $\sum_{\mathbf{u}} \partial^{\mathbf{u}} \sum_i \partial_i a_i P_{\mathbf{u}}$ has constant term 0, the lemma follows. ☐

Lemma 3.4.5. *Suppose $\delta \in \operatorname{Der}_X(-\log f)$ can be written as $\sum_{i=1}^n a_i \partial_i$ where each a_i is a homogeneous polynomial of degree t in $\mathbb{C}[X]$. Let $f = f_1 \cdots f_r$ where each f_k is homogeneous, $F = (f_1, \dots, f_r)$, and f' is a homogeneous polynomial dividing f. If $P \in A_n(\mathbb{C})[S]$, then the right constant term of $P \cdot \psi_{f'F}(\delta)$ lies in $\mathbb{C}[X]_{\geq t-1}[S]$.*

Proof. Recall $\psi_{f'F}(\delta) = \delta - \sum \frac{\delta \bullet f_k}{f_k} s_k - \frac{\delta \bullet f'}{f'}$. By the choice of δ, $-\sum_{k=1}^r \frac{\delta \bullet f_k}{f_k} s_k - \frac{\delta \bullet f'}{f'} \in \mathbb{C}[X]_{t-1}[S]$. By Lemma 3.4.4, the right constant term of $P \cdot \psi_F(\delta)$ is

$$(-\sum_i \partial_i \bullet a_i) P_0 - \delta \bullet P_0 - (\sum_k \frac{\delta \bullet f_k}{f_k} s_k) P_0 - \frac{\delta \bullet f'}{f'} P_{\mathbf{0}}.$$

Let m be the smallest nonnegative integer such that $P_0 \in \mathbb{C}[X]_{\geq m}[S]$. Because $\partial_i \bullet a_i \in \mathbb{C}[X]_{t-1}$ and $\delta \bullet P_0 \in \mathbb{C}[X]_{\geq t+m-1}[S]$ the claim follows. ☐

There is a natural $\mathbb{C}[X]$-isomorphism between $\operatorname{Der}_X(-\log_0 f)$ and the first syzygies of the Jacobian ideal $J(f)$, i.e. the ideal of $\mathbb{C}[X]$ generated by the partials of f. If f is homogeneous, so is $J(f)$ and so is its first syzygy module.

Definition 3.4.6. For f homogeneous, define $\operatorname{mdr}(f)$ to be

$$\operatorname{mdr}(f) = \min\{t \mid \text{ there exists a homogeneous syzygy of } J(f) \text{ of degree } t\}.$$

Remark 3.4.7.(a) It known that a central hyperplane arrangement of f of rank ≥ 2 is indecomposable if and only if $\operatorname{mdr}(f) \geq 2$. For one direction use the first part of Theorem 5.13 of [7]; for the other, use the two disjoint Euler derivations induced by the coordinate change.

(b) Identify $\operatorname{Der}_X(-\log_0 f)$ and first syzygies of $J(f)$ to conclude that we may pick a generating set $\delta_1, \ldots, \delta_m$ of $\operatorname{Der}_X(-\log_0 f)$ such that $\delta_j = \sum_{i=1}^r a_{j,i}\partial_i$ and each $a_{j,i} \in \mathbb{C}[X]$ is homogeneous of degree at least $\operatorname{mdr}(f)$.

We can now prove our version of Lemma 2 from [11]. The argument is similar but we defer applying any symmetry of $B_{f'F}^g$ until later.

Theorem 3.4.8. (Compare to Lemma 2 in [11]) *Let f be a central, not necessarily reduced, indecomposable and tame hyperplane arrangement of rank $n \geq 2$ and let $F = (f_1, \ldots, f_r)$ correspond to any factorization $f = f_1 \cdots f_r$. If f' divides f and $g = \frac{f}{f'}$, then*

$$B_{f'F}^g \subseteq \mathbb{C}[S] \cdot \prod_{j=0}^{\operatorname{mdr}(f)+d-d'-3} \left(\sum_k d_k s_k + n + d' + j \right).$$

Proof. To begin, we choose two polynomials. First fix $0 \neq B(S) \in B_{f'F}^g$. By definition of $B_{f'F,\mathfrak{x}}^g$, the polynomial $B(S)$ lies in $\operatorname{ann}_{A_n(\mathbb{C})[S]} f'F + A_n(\mathbb{C})[S] \cdot g$. Second, pick a nonzero homogeneous polynomial $v \in \mathbb{C}[X]$ such that (i) $\deg(v) \leq \operatorname{mdr}(f) - 2$ and (ii) there exists a point $\alpha \in V(g) \setminus V(v)$. By Remark 3.4.7 such a choice of v is possible. Note that $vB(S) \in \operatorname{ann}_{A_n(\mathbb{C})[S]} f'F + A_n(\mathbb{C})[S] \cdot g$.

Let $\delta_1, \ldots, \delta_m$ generate $\operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f)$ where $\delta_j = \sum_j a_{j,i}\partial_i$; let E by the Euler derivation. By Remark 3.4.7, we may assume $\{a_{j,i}\}_i$ are all homogeneous polynomials of the same degree where that degree is at least $\operatorname{mdr}(f)$. Corollary 3.2.22 implies there exist $L, P, Q_2, \ldots, Q_m \in A_n(\mathbb{C})[S]$ such that

$$vB(S) = Lg + P\psi_{f'F}(E) + \sum_{j=2}^m Q_j \psi_{f'F}(\delta_j). \tag{3.4.1}$$

Express both sides of (3.4.1) in their right normal form. First consider the right hand side of (3.4.1). By Lemma 3.4.5, the right constant term of $Q_j\psi_{f'F}(\delta_j)$ is in $\mathbb{C}[X]_{\geq \operatorname{mdr}(f)-1}[S]$.

Write the right constant term $L_{\mathbf{0}}$ of L as $L_{\mathbf{0}} = \sum_t L_{\mathbf{0}}^t$ where $L_{\mathbf{0}}^t \in \mathbb{C}[X]_t[S]$; similarly, write the right constant term $P_{\mathbf{0}}$ of P as $P_{\mathbf{0}} = \sum_t P_{\mathbf{0}}^t$ where $P_{\mathbf{0}}^t \in \mathbb{C}[X]_t[S]$. The right constant term of Lg is $L_{\mathbf{0}}g$. By Lemma 3.4.4, the right constant term of $P\psi_{f'F}(E)$ is

$$\begin{aligned}\sum_t -nP_{\mathbf{0}}^t - E \bullet P_{\mathbf{0}}^t - \left(\sum_k \frac{E \bullet f_k}{f_k} s_k\right) P_{\mathbf{0}}^t - \frac{E \bullet f'}{f'} P_{\mathbf{0}}^t \\ = \sum_t \left(-n - t - \sum_k d_k s_k - d'\right) P_{\mathbf{0}}^t.\end{aligned}$$

On the other hand, the right constant term of $vB(S)$ is $vB(S)$ itself. Note that $vB(S) \in \mathbb{C}[X]_{\deg(v)}[S]$ and, by the choice of v, $\deg(v) < \operatorname{mdr}(f) - 1$. So when we write the right constant term of both sides of (3.4.1), the left hand side is $vB(S)$ and the right hand side can be written using only terms in $\mathbb{C}[X]_{\deg(v)}[S]$. We deduce

$$vB(S) = L_{\mathbf{0}}^{\deg(v)} g + \left(-n - \deg(v) - d' - \sum_k d_k s_k\right) P_{\mathbf{0}}^{\deg(v)}. \tag{3.4.2}$$

The equation (3.4.2) occurs in $\mathbb{C}[X]_{\deg(v)}[S]$ and so the equality is still true when regarding all the elements as belonging to $\mathbb{C}[X][S]$. By the choice of v, there exists $\alpha \in \mathrm{V}(g) \setminus \mathrm{V}(v)$. The polynomial $P_{\mathbf{0}}^{\deg(v)}$ cannot vanish at α, lest $B(S) = 0$. By evaluating (3.4.2) at α we see

$$B(S) \in \mathbb{C}[S] \cdot \left(-n - \deg(v) - d' - \sum_k d_k s_k\right). \tag{3.4.3}$$

As $\deg(v)$ is flexible,

$$B_{f'F,\mathfrak{x}}^g \subseteq \mathbb{C}[S] \cdot \prod_{j=0}^{\operatorname{mdr}(f)-2} \left(\sum_k d_k s_k + n + d' + j\right). \tag{3.4.4}$$

Now suppose $(f) \subseteq (f'') \subseteq (f')$ and let $g'' = \frac{f}{f''}$. Since f is a hyperplane arrangement we can choose f'' to be of any degree between d' and $d-1$. Because $B_{f'F}^g \subseteq B_{f'',F}^{g''}$, the containment (3.4.4) can be improved to

$$B_{f'F}^g \subseteq \mathbb{C}[S] \cdot \prod_{j=0}^{\operatorname{mdr}(f)+d-d'-3} \left(\sum_k d_k s_k + n + d' + j\right).$$

$\square$

Remark 3.4.9.(a) It is easy to see, see Corollary 6 in [15] for the B_F statement, that

$$B^g_{f'F} = \bigcap_{\mathfrak{x}\in\mathbb{C}^n} B^g_{f'F,\mathfrak{x}}.$$

(b) Recall the notation of Definition 3.4.1. Given an edge $X \in L(A)$, there exists a $\mathfrak{x} \in X$ such that $\mathfrak{x} \notin \mathrm{V}(l_m)$ for all $m \notin J(X)$. By definition,

$$F_X = (f_{X,1}, \ldots, f_{X,r}) = (\prod_{j\in S_{x,1}} l_j, \ldots, \prod_{j\in S_{X,r}} l_j).$$

We may write F as

$$F = (\prod_{m\in S_1\setminus S_{X,1}} l_m \prod_{j\in S_{X,1}} l_j, \ldots, \prod_{m\in S_r\setminus S_{X,r}} l_m \prod_{j\in S_{X,r}} l_j).$$

So at $\mathfrak{x}$, the decompositions F and F_X differ by multiplying each component by a unit at $\mathfrak{x}$. Arguing as in Lemma 10 of [15] (see also the first paragraph of the proof of Theorem 3.3.16), we deduce

$$B^g_{f'F,\mathfrak{x}} = B^{g_X}_{f'_X F_X,\mathfrak{x}}.$$

Since $\mathfrak{x}$ and 0 both lie in the maximal edge of f_X, $B^{g_X}_{f'_X F_X,0} = B^{g_X}_{f'_X F_X,\mathfrak{x}}$. The centrality of f_X, and the consequent $\mathbb{C}^\star$-action on $\mathrm{V}(f_X)$, implies

$$B^{g_X}_{f'_X F_X,0} = B^{g_X}_{f'_X F_X}.$$

(c) Putting (a) and (b) together yields

$$B^g_{f'F} = \bigcap_{X\in L(A)} B^{g_X}_{f'_X F_X}.$$

The following definition will help simplify notation.

Definition 3.4.10. Let $f = f_1 \cdots f_r$ be any factorization of a central hyperplane arrangement and $F = (f_1, \ldots, f_r)$. Suppose f' divides f; $g = \frac{f}{f'}$. For any indecomposable edge X define the polynomial

$$P^g_{f'F,X} = \sum_k d_{X,k} s_k + r(X) + d'_X \in \mathbb{C}[S].$$

Remark 3.4.9 and Theorem 3.4.8 prove our version of Proposition 9 in [11]:

Theorem 3.4.11. (Compare to Proposition 9 of [11]) *Suppose f is a central, tame, not necessarily reduced, hyperplane arrangement of rank n and let $F = (f_1, \ldots, f_r)$ correspond to any factorization $f = f_1 \cdots f_r$. Let f' divide f and $g = \frac{f}{f'}$. For indecomposable edges X of rank ≥ 2 define*

$$p_{f'F,X}(S) = \prod_{j_X=0}^{mdr(f_X)+d_X-d'_X-3} \left(P^g_{f'F,X} + j_X\right).$$

For indecomposable edges X of rank one define

$$p_{f'F,X}(S) = \prod_{j_X=0}^{d_X-d'_X-1} \left(P^g_{f'F,X} + j_X\right).$$

Then

$$B^g_{f'F} \subseteq \mathbb{C}[S] \cdot \operatorname{lcm}\left\{p_{f'F,X}(S) \mid X \in L(A),\ X \text{ indecomposable}\right\}.$$

Proof. By Remark 3.4.9,

$$B^g_{f'F} = \left(\bigcap_{\substack{X \in L(A) \\ r(X) \geq 2}} B^{g_X}_{f'_X F_X} \right) \bigcap \left(\bigcap_{\substack{X \in L(A) \\ r(X) = 1}} B^{g_X}_{f'_X F_X} \right)$$

If X is an edge of rank ≥ 2, then Theorem 3.4.8 combined with Definition 3.4.10 says

$$B^{g_X}_{f'_X F_X} \subseteq \mathbb{C}[S] \cdot \prod_{j_X=0}^{\mathrm{mdr}(f_X)+d_X-d'_x-3} (P^g_{f'F,X} + j_X).$$

Therefore, once we prove that for rank one edges X

$$B^{g_X}_{f'_X F_X} \subseteq \mathbb{C}[S] \cdot \prod_{j_X=0}^{d_X - d'_X - 1} \left(P^g_{f'F,X} + j_X \right),$$

then the claim will follow.

For the rank one edges, argue as in Theorem 3.4.8. Since the rank is one, we can get an equation resembling (3.4.1) without any $\psi_{f'F}(\delta)$ terms and with $v = 1$. Now looking at the right constant terms, since $B(S) \in \mathbb{C}[S]$ and $L_{\mathbf{0}}g$ is not, we deduce (3.4.3) holds with $\deg(v) = 0$. The other factors of $p_{f'F}$ are found using the containment $B^g_{f'F} \subseteq B^{g''}_{f''F}$, as in the final paragraph of Theorem 3.4.8. $\square$

3.4.2 An Element of $B^g_{f'F}$

Here we drop the assumption of tameness and compute an element of $B^g_{f'F}$ for $f = f_1 \cdots f_r$ any factorization of a central, not necessarily reduced, hyperplane arrangement f and where f' and g are as before. The bulk of the argument tracks Proposition 10 of [11], however we have removed the reducedness hypothesis. Again, we provide detail for the reader's sake.

We begin with some basic facts about differential operators. First, consider a product of functions fg with factorizations $f = f_1 \dots f_r$ and $g = g_1 \dots, g_u$. Let $F = (f_1, \dots, f_r)$ and $G = (g_1, \dots, g_u)$ and $FG = (f_1, \dots, f_r, g_1, \dots, g_u)$.

Definition 3.4.12. Let $P \in \mathrm{A}_n(\mathbb{C})\,[S]$ and consider $\mathrm{A}_n(\mathbb{C})\,[S](FG)^S$. Relabel the s_k so that we may write $\mathrm{A}_n(\mathbb{C})\,[S,T]F^S G^T = \mathrm{A}_n(\mathbb{C})\,[S]f_1^{s_1} \cdots f_r^{s_r} g_1^{t_1} \cdots g_u^{t_u}$ and consider P as in $\mathrm{A}_n(\mathbb{C})\,[S,T]$. As there is an $\mathrm{A}_n(\mathbb{C})\,[S]$-action on F^S there is a naturally defined $\mathrm{A}_n(\mathbb{C})\,[S,T]$ action. Denote by $P \bullet F^S$ the result of letting P act on F^S.

Lemma 3.4.13. *Let $P \in A_n(\mathbb{C})[S]$ of total order k, i.e. $P \in F^k_{(0,1,1)} A_n(\mathbb{C})[S]$. Then*

$$PF^S G^T - (P \bullet F^S) G^T \in A_n(\mathbb{C})[S,T] F^S G^{T-k}.$$

Proof. It is sufficient to prove the following:

Claim: If $h \in \mathbb{C}[X][S][T]$, there exists $Q_{\mathbf{u}}$ of total order at most $|\mathbf{u}|$ such that

$$\partial^{\mathbf{u}} h F^S G^T - h(\partial^{\mathbf{u}} \bullet F^S) G^T = Q_{\mathbf{u}} F^S G^{T-|\mathbf{u}|}.$$

We prove this by induction on $|\mathbf{u}|$. The base case is straightforward. For the inductive step, observe:

$$\begin{aligned} \partial_1 \partial^{\mathbf{u}} h F^S G^T &= \partial_1 [h(\partial^{\mathbf{u}} \bullet F) G^T + Q_{\mathbf{u}} F^S G^{T-|\mathbf{u}|}] \\ &= (\partial_1 \bullet h)(\partial^{\mathbf{u}} \bullet F^S) G^T + h(\partial_1 \partial^{\mathbf{u}} \bullet F^S) G^T \\ &+ h(\partial^{\mathbf{u}} \bullet F)(g \sum_k t_k \frac{\partial_1 \bullet g_k}{g_k}) G^{T-1} + \partial_1 Q_{\mathbf{u}} F^S G^T. \end{aligned} \tag{3.4.5}$$

Since $\partial_1 \bullet h \in \mathbb{C}[X][S][T]$ the induction hypothesis implies

$$(\partial_1 \bullet h)(\partial^{\mathbf{u}} \bullet F^S) G^T \in F_{(0,1,1)}^{|\mathbf{u}|} \operatorname{A}_n(\mathbb{C}) [S][T] F^S G^{T-|\mathbf{u}|}.$$

Similarly, since $h(g \sum_k t_k \frac{\partial_1 \bullet g_k}{g_k}) \in \mathbb{C}[S][T]$, by induction

$$h(\partial^{\mathbf{u}} \bullet F^S)(g \sum_k t_k \frac{\partial_1 \bullet g_k}{g_k}) G^{T-1} \in F_{(0,1,1)}^{|\mathbf{u}|} \operatorname{A}_n(\mathbb{C}) [S][T] F^S G^{T-|\mathbf{u}|-1}.$$

Rearranging (3.4.5) proves the claim and hence the lemma. ☐

We also need the following elementary lemma.

Lemma 3.4.14. *Let $E = x_1 \partial_1 + \cdots + x_n \partial_n$ be the Euler derivation. Then*

$$\prod_{j=0}^{t} (E + n + j) = \sum_{\substack{u_1, \ldots, u_n \\ u_1 + \cdots + u_n = t+1}} \binom{t+1}{u_1, \cdots, u_n} \partial^{\boldsymbol{u}} x^{\boldsymbol{u}}.$$

Proof. This also succumbs to induction on t after utilizing Pascal's formula for multinomial coefficients. ☐

Definition 3.4.15. Consider a central, essential, not necessarily reduced, hyperplane arrangement of rank n defined by $f = l_1 \cdots l_d$, where the l_k are homogeneous linear forms.

Write $L = (l_1, \ldots, l_d)$. For an edge $X \in L(A)$ and with $J(X)$ as in Definition 3.4.1, define the ideal $\Gamma_L \subseteq \mathbb{C}[x_1, \ldots, x_n]$ by

$$\Gamma_L = \sum_{\substack{X \in L(A) \\ r(X) = n-1}} \mathbb{C}[x_1, \ldots, x_n] \cdot \prod_{k \notin J(X)} l_k.$$

Lemma 3.4.16. *Consider a central, essential, not necessarily reduced, hyperplane arrangement of rank n defined by $f = l_1 \cdots l_d$, where the l_k are homogeneous linear forms. Let $L = (l_1, \ldots, l_d)$ and denote the ideal of $\mathbb{C}[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$ by $\mathfrak{m}$. Then there exists an integer k such that $\mathfrak{m}^k \subseteq \Gamma_L$.*

Proof. It suffices to show Γ_L is $\mathfrak{m}$-primary since $\mathfrak{m}$ is maximal and $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian. So we need only show $\mathrm{V}(\Gamma_L) = \{0\}$. Suppose $0 \neq p \in \mathrm{V}(\Gamma_L)$. Since $\mathrm{V}(\Gamma_L)$ is the intersection of unions of central hyperplanes, we deduce $\mathrm{V}(\Gamma_L)$ contains a codimension $n-1$ line. We may find a largest edge X containing said line; if X is not of codimension $n-1$ enlarge X further to a codimension $n-1$ edge. So for all $k \notin J(X)$, $\mathrm{V}(l_k)$ will not contain this line and hence will not contain p. But $p \in \mathrm{V}(\Gamma_F) \subseteq \mathrm{V}(\prod_{k \notin J(X)} l_k) = \cup_{k \notin J(X)} \mathrm{V}(l_k)$, contradicting $p \in \mathrm{V}(\Gamma_L)$. $\square$

Remark 3.4.17. We need essentiality in the above lemma lest the maximal edge of $L(A)$ have rank $n-1$ forcing $\Gamma_F = 1$. Without this condition, the X selected in the above proof could be the maximal edge of $L(A)$.

Recall the notation of Definition 3.4.1. We proceed to the subsection's main idea, which is a generalization of Proposition 10 of [11] and is proved similarly.

Theorem 3.4.18. (Compare to Proposition 10 of [11]) *Consider a central, not necessarily reduced, hyperplane arrangement $f = l_1 \cdots l_d$ where the l_k are linear terms and let $L = (l_1, \ldots, l_d)$. Suppose that f' divides f; let $g = \frac{f}{f'}$. Then there is a positive integer N such that*

$$\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j=0}^{N} \left(P^g_{f'L,X} + j \right) \in B^g_{f'F}.$$

Proof. We prove this by induction on the rank of $L(A)$ and first deal with the inductive step. So we may assume the rank is n and f is essential. If f is decomposable into $f_1 f_2$, then f' (resp. g) inherts a decomposition $f'_1 f'_2$ (resp. $g_1 g_2$). If F_1 (resp. F_2) is the associated factorization of f_1 (resp. f_2) into linear forms and if $b_1 \in B^{g_1}_{f'_1 F_1}$ and $b_2 \in B^{g_2}_{f'_2 F_2}$, then $b_1 b_2 \in B^{g}_{f' F}$. In this case the induction hypothesis applies to $B^{g_1}_{f'_1 F_1}$ and $B^{g_2}_{f'_2 F_2}$. So we may assume f is indecomposable.

Let $\mathfrak{m}$ be the ideal in $\mathbb{C}[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$. On the one hand, Lemma 3.4.14 implies that for all positive integers t

$$\prod_{j=0}^{t}(s_1 + \cdots + s_d + n + d' + j) f' L^S = \prod_{j=0}^{t}(E + n + j) f' L^S \in \mathrm{A}_n(\mathbb{C}) \cdot \mathfrak{m}^{t+1} f' L^S.$$

By Lemma 3.4.16, for any positive integer m there exists an integer N large enough so that

$$\prod_{j=0}^{N}(s_1 + \cdots + s_d + n + d' + j) f' L^S \in \sum_{\substack{X \in L(A) \\ r(X) = n-1}} \mathrm{A}_n(\mathbb{C})\,[S](\prod_{k \notin J(X)} l_k)^m f'_X L^S. \tag{3.4.6}$$

Note we have folded some of the factors of f' into $(\prod_{k \notin J(X)} l_k)^m$.

By induction, for each such edge X of rank less than n, there exists a differential operator P_X of total order k_X and a polynomial $b_X \in \mathbb{C}[S]$ such that $P_X \prod_{i \in J(X)} l_i^{s_i + 1} = b_X f'_X \prod_{i \in J(X)} l_i^{s_i}$. Fix m large enough so that $m > \max\{k_X \mid X \in L(A),\ X \text{ codimension } n-1\}$. Consequently, choose N large enough so that (3.4.6) holds for this fixed m. Lemma 3.4.13 implies

$$\begin{aligned} b_X(\prod_{k \notin J(X)} l_k)^m f'_X L^S &= (b_X f'_X \prod_{i \in J(X)} l_i^{s_i})(\prod_{k \notin J(X)} l_k^{s_k + m}) \\ &\in \mathrm{A}_n(\mathbb{C})\,[S](\prod_{i \in J(X)} l_i^{s_i+1})(\prod_{k \notin J(X)} l_k^{s_k + m - k_X}) \\ &\subseteq \mathrm{A}_n(\mathbb{C})\,[S] L^{S+1}. \end{aligned} \tag{3.4.7}$$

Combining (3.4.6) and (3.4.7) we deduce

$$\prod_{j=0}^{N}(s_1 + \cdots + s_d + n + d' + j)(\prod_{\substack{X \in L(A) \\ r(X) = n-1}} b_X) f' L^S \in \mathrm{A}_n(\mathbb{C})\,[S] L^{S+1}. \tag{3.4.8}$$

The result follows by the inductive description of each b_X and the definition of $P^g_{f'L,X}$. Note we may have to replace either the N chosen in (3.4.8) or the N coming from the inductive hypothesis with a larger integer so that the final polynomial is in the promised form. There is no harm in this as it can only only add linear factors to the polynomial appearing in (3.4.8) and does not change the containment.

All that remains is the base case, but this is obvious by a direct computation using Lemma 3.4.14. $\square$

This theorem only gives an element of $B^g_{f'L}$ when L is a factorization into linear forms. If f is tame we can find an element no matter the factorization.

Corollary 3.4.19. *Let $f = f_1 \cdots f_r$ be a central, not necessarily reduced, tame hyperplane arrangement where the f_k are not necessarily linear forms. Let $F = (f_1, \ldots, f_r)$. Suppose f' divides f; let $g = \frac{f}{f'}$. If L corresponds to the factorization of f into linear terms, then there exists a positive integer N such that*

$$\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j=0}^{N} \left(P^g_{f'L,X} + j\right) \text{ modulo } S_F \in B^g_{f'F},$$

where S_F is as in Definition 3.2.25.

Proof. Use Proposition 3.2.26. $\square$

Just as in the last part of Theorem 2 of [11], 3.4.18 also implies $B^g_{f'L}$ is principal. (Here we very much need L to correspond to a factorization into linear forms.)

Corollary 3.4.20. *Consider the central, not necessarily reduced, free hyperplane arrangement $f = l_1 \cdots l_d$, where the l_k are linear forms, and let $L = (l_1, \cdots, l_d)$. Suppose f' divides f; let $0 \neq g$ divide $\frac{f}{f'}$. Then $B^g_{f'L}$ equals its radical and is principal.*

Proof. Let $P(S)$ be the polynomial of Theorem 3.4.18. If g divides $\frac{f}{f'}$, then by said theorem $P(S) \in B^g_{f'L}$. The claim then follows by Proposition 3.3.12 and Proposition 3.3.13 since $P(S)$ cuts out a reduced hyperplane arrangement. $\square$

3.4.3 Computations and Estimates

We now have combinatorial determined ideal subsets and supsets of $B^g_{f'F}$. In general, $\mathrm{V}(B_f)$ is not combinatorially determined. However, if f is tame, then $\mathrm{V}(B_f) \cap [-1, 0]$ is combinatorial.

Theorem 3.4.21. *Let f be a central, not necessarily reduced, tame hyperplane arrangement. Suppose f' divides f; let $g = \frac{f}{f'}$. Then the roots $\mathrm{V}(B^g_{f'f})$ lying in $[-1, 0)$ are combinatorially determined:*

$$\mathrm{V}(B^g_{f'f}) \cap [-1, 0) = \bigcup_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \bigcup_{j_X = r(X) + d'_X}^{d_X} \frac{-j_X}{d_X}.$$

Setting $f' = 1$ gives the roots of the Bernstein–Sato polynomial of f lying in $[-1, 0)$.

Proof. We find a subset and supset of $B^g_{f'F}$ using Corollary 3.4.19 and Theorem 3.4.11 respectively. Their varieties will be equal after intersecting with $[-1, 0)$ once we verify the following inequalities for indecomposable edges X: $r(X) + \mathrm{mdr}(f) + d_X - 3 \geq d_X$ if $r(X) \geq 2$; $1 + d_X - 1 \geq d_X$ if $r(X) = 1$. The second is trivial. The first is as well: since X is indecomposable $\mathrm{mdr}(f) \geq 2$. $\square$

Example 3.4.22. In [7], Walther showed the Bernstein–Sato polynomial of an arrangement is not combinatorially determined. He gives the following two arrangements that have the same intersection lattice, but the former has $\frac{-18+2}{9}$ as a root and the latter does not:

$$f = xyz(x + 3z)(x + y + z)(x + 2y + 3z)(2x + y + z)(2x + 3y + z)(2x + 3y + 4z);$$
$$g = xyz(x + 5z)(x + y + z)(x + 3y + 5z)(2x + y + z)(2x + 3y + z)(2x + 3y + 4z).$$

Because these arrangements are rank 3 they are automatically tame, cf. Remark 3.2.5. The above theorem says the roots of the b-polynomials agree inside $[-1, 0)$. In Remark 4.14.(iv) of [33], Saito shows that their roots agree except for $\frac{-18+2}{9}$.

For the rest of the subsection we restrict to free hyperplane arrangments. In [11], Maisonobe used the symmetry of B_L, when L corresponded to a factorization of a reduced

f into linear terms, to make his estimates of B_L so precise they actually computed B_L, cf. Theorem 2 in loc. cit. We use the symmetry of $B^g_{f'F}$ given by φ of Theorem 3.3.16 similarly, but our situation is more technical because of the addition of f', the lack of reducedness, and our focus on different factorizations F.

Lemma 3.4.23. *Let $f = f_1 \cdots f_r$ be an unmixed factorization of a central hyperplane arrangement and let $F = (f_1, \ldots, f_r)$. Suppose f' divides f; $g = \frac{f}{f'}$. If (f', F) is an unmixed pair and φ the $\mathbb{C}[S]$-automorphism prescribed in Theorem 3.3.16, then*

$$\varphi(P^g_{f'F,X}) = -(P^g_{f'F,X} + d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X).$$

Proof. First notation. Factor $f = l_1^{v_1} \cdots l_q^{v_q}$, where the l_t pairwise distinct irreducibles. Let $\{m_k\}$ be the repeated multiplicities of F; $\{d'_k, d_k\}_k$ and $\{d''_k, d_k\}_k$ the repeated powers of the unmixed pairs (f', F) and (g, F). Because $f'g = f$, the formulation of φ in Theorem 3.3.16 can be simplified:

$$\begin{aligned}
\varphi(\sum_k d_{X,k} s_k) &= -\sum_k d_{X,k}(s_k + \frac{1}{m_k} + \frac{2d'_k}{d_k} + \frac{d''_k}{d_k}) \\
&= -\sum_k d_{X,k}(s_k + \frac{1}{m_k} + \frac{d'_k}{d_k} + 1) \\
&= -\sum_k d_{X,k}(s_k + \frac{1}{m_k}) - \sum_k d_{X,k,\mathrm{red}} d'_k - d_X \\
&= -\sum_k d_{X,k}(s_k + \frac{1}{m_k}) - d'_X - d_X.
\end{aligned}$$

After rearranging, we will be done once we show that $\sum_k \frac{d_{X,k}}{m_k} = d_{X,red}$.

Fix $k \in [r]$. Observe:

$$\prod_{\substack{t \in [q] \\ v_t = m_k}} l_t^{m_k} = \prod_{\substack{i \in [r] \\ m_i = m_k}} f_i = \prod_{\substack{i \in [r] \\ m_i = m_k}} \prod_{\substack{t \in [q] \\ f_i \in (l_t)}} l_t^{d_i}. \tag{3.4.9}$$

Equality will still hold in (3.4.9) if we further restrict t to the integers such that l_t divides f_X. The degrees of the resulting polynomials are equal:

$$\begin{aligned} m_k \left|\{l_t \mid v_t = m_k; f_X \in (l_t)\}\right| &= \sum_{\substack{i\in[r] \\ m_i=m_k}} d_i \left|\{l_t \mid f_i, f_X \in (l_t)\}\right| \\ &= \sum_{\substack{i\in[r] \\ m_i=m_k}} d_i \; d_{X,i,\mathrm{red}} \\ &= \sum_{\substack{i\in[r] \\ m_i=m_k}} d_{X,i}. \end{aligned} \tag{3.4.10}$$

Therefore

$$\begin{aligned} \sum_k \frac{d_{X,k}}{m_k} = \sum_{p\in\{m_k\}} \sum_{\substack{i\in[r] \\ m_i=p}} \frac{d_{X,k}}{p} &= \sum_{p\in\{m_k\}} \left|\{l_t \mid v_t = p; f_X \in (l_t)\}\right| \\ &= \sum_{p\in\{v_t\}} \left|\{l_t \mid v_t = p; f_X \in (l_t)\}\right| \\ &= d_{X,\mathrm{red}}. \end{aligned} \tag{3.4.11}$$

□

First we use Theorem 3.4.18 and the symmetry of $B^g_{f'L}$ to find an element of $B^g_{f'L}$ that more accurately approximates the Bernstein–Sato ideal.

Proposition 3.4.24. *Consider the central, not necessarily reduced, free hyperplane arrangement $f = l_1 \cdots l_d$, where the l_k are linear forms, and let $L = (l_1, \ldots, l_d)$. Suppose f' divides f; let $g = \frac{f}{f'}$. Then*

$$\prod_{\substack{X\in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{d_{x,\mathrm{red}}+d_X-2r(X)-d'_X} \left(P^g_{f'L,X} + j_X\right) \in B^g_{f'L}. \tag{3.4.12}$$

Proof. By Theorem 3.4.18 there exists a positive integer N such that

$$\prod_{\substack{X\in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{N} \left(P^g_{f'L,X} + j_X\right) \in B^g_{f'L}. \tag{3.4.13}$$

Since (f', L) are an unmixed pair up to units by virtue of L being a factorization into linear forms, by Theorem 3.3.16/Corollary 3.3.18 and Lemma 3.4.23

$$\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{N} \left(P_{f'L,X}^{g} + d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X - j_X \right) \in B_{f'L}^{g}. \tag{3.4.14}$$

By Corollary 3.4.20, $B_{f'L}^{g}$ is principal. Comparing the irreducible factors of the elements given in (3.4.13) and (3.4.14) proves the claim. $\square$

When the rank of f is at most 2, and so f is automatically free, we can compute $\mathrm{V}(B_{f'F}^{g})$ for any factorization F of f and we can compute $B_{f'L}^{g}$ for L a factorization into linear terms.

Theorem 3.4.25. *Suppose that f is a central, not necessarily reduced, hyperplane arrangement of rank at most 2 and let $F = (f_1, \dots, f_r)$ correspond to any factorization $f = f_1 \cdots f_r$. Let f' divide f and $g = \frac{f}{f'}$. Then*

$$\mathrm{V}(B_{f'F}^{g}) = \mathrm{V}\left(\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X} \left(P_{f'F,X}^{g} + j_X \right) \right). \tag{3.4.15}$$

If L is a factorization of $f = l_1 \cdots l_d$ into irreducibles, then

$$B_{f'L}^{g} = \prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X} \left(P_{f'L,X}^{g} + j_X \right). \tag{3.4.16}$$

Proof. If f is indecomposable, then by Saito's criterion for freeness, cf. page 270 of [8], $\mathrm{mdr}(f) = d_{\mathrm{red}} - 1$. So in this case Theorem 3.4.11 implies

$$B_{f'F}^{g} \subseteq \sqrt{\mathbb{C}[S] \cdot \prod_{j_0=0}^{d_{\mathrm{red}} + d - d' - 4} \left(P_{f'F,0}^{g} + j_0 \right) \prod_{\substack{X \in L(A) \\ r(X)=1}} \prod_{j_X=0}^{d_X - d'_X - 1} \left(P_{f'F,X}^{g} + j_X \right)}. \tag{3.4.17}$$

Proposition 3.4.24 and Proposition 3.2.26 together imply

$$\sqrt{\mathbb{C}[S]\cdot \prod_{\substack{X\in L(A)\\ X \text{ indecomposable}}} \prod_{j_X=0}^{d_{x,\text{red}}+d_X-2r(X)-d'_X} \left(P^g_{f'F,X}+j_X\right)} \subseteq \sqrt{B^g_{f'F}}, \tag{3.4.18}$$

where we have included radicals because the image of a polynomial modulo S_F may have multiplicands with large multiplicities, cf. Example 3.2.27. Combining (3.4.17) and (3.4.18) and simplifying $d_{x,\text{red}}+d_X-2r(X)-d'_X$ for rank 2 and rank 1 edges proves (3.4.15).

Because L is a factorization into irreducibles, even if f is not reduced the polynomial on the right hand side of (3.4.16) is reduced. Therefore (3.4.15) and Corollary 3.4.20 implies (3.4.16). The case of f decomposable follows by similar reasoning. □

If f is of rank greater than 2, $\operatorname{mdr}(f)$ can be small and so the estimate in Theorem 3.4.11 will not be precise enough for our purposes. In this case, we impose symmetry on $B^g_{f'F}$ to obtain the following estimates:

Theorem 3.4.26. *Suppose that $f=f_1\cdots f_r$ is a central, not necessarily reduced, free hyperplane arrangement, $F=(f_1,\cdots,f_r)$, f' divides f, and $g=\frac{f}{f'}$. Then*

$$\sqrt{\mathbb{C}[S]\cdot \prod_{\substack{X\in L(A)\\ X \textit{ indecomposable}}} \prod_{j_X=0}^{d_{X,\text{red}}+d_X-2r(X)-d'_X} \left(P^g_{f'F,X}+j_X\right)} \subseteq \sqrt{B^g_{f'F}}. \tag{3.4.19}$$

If we assume (f',F) is an unmixed pair up to units, then

$$B^g_{f'F} \subseteq \sqrt{\mathbb{C}[S]\cdot \prod_{\substack{X\in L(A)\\ X \textit{ indecomposable}}} \prod_{j_X\in\Xi_X} \left(P^g_{f'F,X}+j_X\right)}, \tag{3.4.20}$$

where, for each indecomposable edge X, Ξ_X is the, possibly empty, set of nonnegative integers defined by

$$\begin{cases} [0, d_{X,\text{red}}+d_X-2r(X)-d'_X] & r(X)\le 2 \\ [0, d_X-d'_X-1]\cup[d_{X,\text{red}}-2r(X)+1, d_{X,\text{red}}+d_X-2r(X)-d'_X] & r(X)\ge 3. \end{cases}$$

Proof. The inclusion (3.4.19) is proved in exactly the same way as (3.4.18), so we need to only prove (3.4.20). Arguing as in the beginning of Theorem 3.3.16, we may assume (f', F) is an unmixed pair. Theorem 3.4.11 implies

$$B_{f'F}^{g} \subseteq \sqrt{\mathbb{C}[S] \cdot \prod_{\substack{X \in L(A) \\ X \text{ indecomposable} \\ r(X) \geq 3}} \prod_{j_X = 0}^{d_X - d'_X - 1} \left(P_{f'F,X}^{g} + j_X\right)}. \tag{3.4.21}$$

The symmetry of $B_{f'F,X}^{g}$, cf. Theorem 3.3.16/Corollary 3.3.18, Lemma 3.4.23, and (3.4.21) imply

$$\begin{aligned} B_{f'F}^{g} &\subseteq \sqrt{\mathbb{C}[S] \cdot \prod_{\substack{X \in L(A) \\ X \text{ indecomposable} \\ r(X) \geq 3}} \prod_{j_X = 0}^{d_X - d'_X - 1} P_{f'F,X}^{g} + d_{X,\text{red}} + d_X - 2r(X) - d'_X - j_X} \\ &= \sqrt{\mathbb{C}[S] \cdot \prod_{\substack{X \in L(A) \\ X \text{ indecomposable} \\ r(X) \geq 3}} \prod_{j_X = d_{x,\text{red}} - 2r(X) + 1}^{d_{x,\text{red}} + d_X - 2r(X) - d'_X} \left(P_{f'F,X}^{g} + j_X\right)}. \end{aligned} \tag{3.4.22}$$

At the edges of rank two or one we have an ideal containment similar to (3.4.17). Combining this, (3.4.21), and (3.4.22) and using the fact that $\mathbb{C}[S]$ is a UFD proves (3.4.20). $\square$

If d' is small enough, the previous result does not just estimate–it computes.

Corollary 3.4.27. (Compare to Theorem 2 of [11]) *Suppose $f = f_1 \cdots f_r$ is a central, not necessarily reduced, free hyperplane arrangement, $F = (f_1, \cdots, f_r)$, f' divides f, and $g = \frac{f}{f'}$. If (f', F) is an unmixed pair up to units and if $d' \leq 4$, then*

$$\mathrm{V}(B_{f'F}^{g}) = \mathrm{V}\left(\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X = 0}^{d_{X,\text{red}} + d_X - 2r(X) - d'_X} \left(P_{f'F,X}^{g} + j_X\right) \right). \tag{3.4.23}$$

If L is a factorization of $f = l_1 \cdots l_d$ into irreducibles and $d' \leq 4$, then

$$B^g_{f'L} = \prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{d_{X,\mathrm{red}}+d_X-2r(X)-d'_X} \left(P^g_{f'L,X} + j_X\right). \tag{3.4.24}$$

If $f' = 1$ and f is reduced, then for any F

$$\mathrm{V}(B_F) = \mathrm{V}\left(\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{d_{X,\mathrm{red}}+d_X-2r(X)} \left(P^g_{F,X} + j_X\right)\right). \tag{3.4.25}$$

In particular, if f is reduced or is a power of a central, reduced, and free hyperplane arrangement, then the roots of the Bernstein–Sato polynomial of f are given by (3.4.25).

Proof. Because of Theorem 3.4.26, proving (3.4.23) amounts to showing that $\Xi_X = [0, d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X]$ for each X of rank at least 3. This occurs if $d'_X \leq 2(r(X)-1)$. So (3.4.23) is true. Since (f', L) is always an unmixed pair up to units, Corollary 3.4.20 proves (3.4.24). Equation (3.4.25) follows from (3.4.23) and the fact $(1, F)$ is always an unmixed pair up to units when f is reduced, cf. Corollary 3.3.18. For the final claim, it suffices to note that $(1, F)$ for $F = (f)$ is an unmixed pair up to units provided f is reduced or f is a power of a central, reduced hyperplane arrangement. $\square$

Remark 3.4.28.(a) Let us outline how to strengthen the final claim of Corollary 3.4.27 to Bernstein–Sato polynomials for all non-reduced, free f. In the recently announced paper [10], Budur, Veer, Wu, and Zhou consider local, analytic f that satisfy a vanishing Ext criterion. Namely, that $\mathrm{Ext}^k_{\mathscr{D}_{X,\mathfrak{x}}[S]}(\mathscr{D}_{X,\mathfrak{x}}[S]F^S, \mathscr{D}_{X,\mathfrak{x}}[S])$ vanishes for all but one value of k. (We let F corresponds to any factorization of f.) In Proposition 3.4.3 they characterize elements of $\mathrm{V}(B_{F,\mathfrak{x}})$ in terms of the non-vanishing of a certain tensor product. It is easy to show that this is equivalent to the non-surjectivity of the $\mathscr{D}_{X,\mathfrak{x}}$-map ∇_A. This is the map $\mathscr{D}_{X,\mathfrak{x}}[S]F^S/(s_1 - a_1, \ldots, s_r - a_r) \cdot \mathscr{D}_{X,\mathfrak{x}}[S]F^S \to \mathscr{D}_{X,\mathfrak{x}}[S]F^S/(s_1 - (a_1 - 1), \ldots, s_r - (a_r - 1)) \cdot \mathscr{D}_{X,\mathfrak{x}}[S]F^S$ induced by sending each s_k to s_{k+1}. Here A corresponds to $(a_1, \ldots, a_r) \in \mathbb{C}^r$. See Section 3 of Chapter 2, Proposition 2 of [6], or Section 6 in this chapter for more details on ∇_A. If f corresponds to a free, possibly non-reduced, arrangement, it follows from Theorem 3.3.9 that the vanishing Ext condition of [10] holds. Moreover, using the commutative diagram in Remark 3.3 of Chapter 2, the non-surjectivity of the map ∇_A is equivalent to the non-surjectivity of the classical map ∇_a. (This is the same as ∇_A for $r = 1$.) The non-surjectivity of ∇_a is known to characterize the roots of the Bernstein–Sato polynomial of an arbitrary f. So when L corresponds to a factorization of our possibly non-reduced arrangement f into irreducibles, we can use the above procedure to show that intersecting $\mathrm{V}(B_L)$ with the diagonal gives $\mathrm{V}(B_f)$, again, see Remark 3.3 of Chapter 2. Using the formula for $\mathrm{V}(B_L)$ in (3.4.24), we then obtain the expected formula (3.4.25) for $\mathrm{V}(B_f)$ without requiring the reduced hypothesis.

(b) The above strategy for computing $\mathrm{V}(B_f)$ for f a central, reduced, free hyperplane arrangement can also be executed without appeal to [10] thanks to Proposition 3.7.1.

(c) In light of Proposition 3.4.3 of [10], the assumption of "unmixed pair up to units" does not seem to be necessary. Rather, it seems there should be a version of this result for $f'F^S$ so that computing $B^g_{f'L}$ would be sufficient for computing $\mathrm{V}(B^g_{f'F})$.

3.5 Freeing Hyperplane Arrangements

In this short section we consider the problem of embedding a central hyperplane arrangement g inside a central, free hyperplane arrangement. Equivalently, given such a g we consider central hyperplane arrangements f such that fg is free. (Note that we have somewhat switched notation for reasons that will become clear in Proposition 3.5.3.)

Definition 3.5.1. We say the central arrangement f *frees* the central arrangement g if fg is free.

For g an arbitrary divisor, it is unknown if such an f exists. In [35], Mond and Schulze find some general instances of the freeing divisor f; see also [37], [43], [36]. Returning to arrangements g, both Abe and Wakefield identify some situations in [44] and [45] respectively where f is a hyperplane and fg is free. For g a central hyperplane arrangement, Masahiko Yoshinaga [38] has communicated to us an algorithm, depending only on the intersection lattice of g, that always produces such an f. Accordingly, we make the following definition, noting nothing is lost by assuming reducedness.

Definition 3.5.2. For g a central, reduced hyperplane arrangement, define

$$\mu_g = \min\{\deg(f) \mid f \text{ is a central arrangement that frees } g\}.$$

We will highlight a connection between small roots of the Bernstein–Sato polynomial of a tame g and lower bounds for μ_g. First some notation.

Consider a reduced hyperplane arrangement $l_1 \cdots l_d$ and write it as a product fg. Let $F = (f_1, \dots, f_r)$ and $G = (g_1, \dots, g_u)$ correspond to the factorizations $f = f_1 \cdots f_r$ and $g = g_1 \cdots g_u$ into linear terms and let FG correspond to the factorization $l_1 \cdots l_d = f_1 \cdots f_r \cdot g_1 \cdots g_u$. When considering the $\mathrm{A}_n(\mathbb{C})\,[S]$-module generated $(FG)^S$, we will re-label so this is an $\mathrm{A}_n(\mathbb{C})\,[S,T]$-module generated by $f_1^{s_1} \cdot f_r^{s_r} g_1^{t_1} \cdot g_u^{t_u}$. Finally, let $S+1$ denote the $\mathbb{C}[S]$ ideal generated by $s_1+1, \dots, s_r+1$ and let $\Delta_{S+1} : \mathbb{C}^u \to \mathbb{C}^{r+u} = \mathbb{C}^d$ be the embedding given by $(a_1, \dots, a_u) \mapsto (-1, \dots, -1, a_1, \dots, a_u)$.

We need the following result:

Proposition 3.5.3. *Let f, g, F, G be as in the preceding paragraph. Suppose fg is tame. Then*

$$\Delta_{S+1}(\mathrm{V}(B_G)) \subseteq \mathrm{V}(B_{fFG}^{g}) \cap \{s_1 = -1, \dots, s_r = -1\} \subseteq \mathbb{C}^{u+r}.$$

Proof. Define $I = \mathrm{A}_n(\mathbb{C})\,[S,T] \cdot \mathrm{ann}_{\mathrm{A}_n(\mathbb{C})\,[T]}\, G^T + \mathrm{A}_n(\mathbb{C})\,[S,T] \cdot g + \mathrm{A}_n(\mathbb{C})\,[S,T] \cdot (S+1)$. If $P \in I \cap \mathbb{C}[S,T]$, then

$$P \text{ modulo } \mathrm{A}_n(\mathbb{C})\,[S,T] \cdot (S+1) \in \mathbb{C}[T] \cap (\mathrm{A}_n(\mathbb{C})\,[T] \cdot \mathrm{ann}_{\mathrm{A}_n(\mathbb{C})\,[T]}\, G^T + \mathrm{A}_n(\mathbb{C})\,[T] \cdot g).$$

So

$$I \cap \mathbb{C}[S,T] \subseteq \mathbb{C}[S,T] \cdot B_G + \mathbb{C}[S,T] \cdot (S+1).$$

As the reverse equality is obvious,

$$I \cap \mathbb{C}[S,T] = \mathbb{C}[S,T] \cdot B_G + \mathbb{C}[S,T] \cdot (S+1).$$

For δ a logarithmic derivation of fg,

$$\psi_{fFG}(\delta) = \delta - \sum_k s_k \frac{\delta \bullet f_k}{f_k} - \sum_m t_m \frac{\delta \bullet g_m}{g_m} - \frac{\delta \bullet f}{f}.$$

Under the map $\mathrm{A}_n(\mathbb{C})\,[S,T] \mapsto \mathrm{A}_n(\mathbb{C})\,[S,T]/\,\mathrm{A}_n(\mathbb{C})\,[S,T] \cdot (S+1)$,

$$\psi_{fFG}(\delta) \mapsto \delta - \sum_m t_m \frac{\delta \bullet g_m}{g_m} = \psi_G(\delta) \in \operatorname{ann}_{\mathrm{A}_n(\mathbb{C})\,[T]} G^T.$$

Therefore

$$I \supseteq \mathrm{A}_n(\mathbb{C})\,[S,T] \cdot \theta_{fFG} + \mathrm{A}_n(\mathbb{C})\,[S,T] \cdot g + \mathrm{A}_n(\mathbb{C})\,[S,T] \cdot (S+1).$$

Intersecting with $\mathbb{C}[S,T]$ and using Corollary 3.2.22, we deduce

$$\mathbb{C}[S,T] \cdot B_G + \mathbb{C}[S,T] \cdot (S+1) \supseteq B^g_{f'FG} + \mathbb{C}[S,T] \cdot (S+1).$$

Taking varieties finishes the proof. □

By Theorem 1 of [33], $\mathrm{V}(B_g) \subseteq (\frac{-2d+1}{d}, 0)$, g any central arrangement; by the formula (3.4.25) for $\mathrm{V}(B_g)$, the presence of roots $\frac{-2d+v}{d}$, $1 < v \leq n-1$ suggests g is not free. While this is not true because $\frac{-2d+v}{d}$ might not be written in lowest terms, the following outlines how such roots can measure the distance g is from being free.

Theorem 3.5.4. *Suppose that g is a central, reduced, tame hyperplane arrangement of rank n, v an integer such that $1 < v \leq n-1$, and $\deg(g)$ is co-prime to v. If $\frac{-2\deg(g)+v}{\deg(g)}$ is a root of the Bernstein–Sato polynomial of g, then $\mu_g \geq n - v$.*

Proof. Suppose f is a reduced, central hyperplane arrangement such that fg is free. We use the notation of the preceeding proposition and paragraphs. It suffices to prove $\deg(f) \geq n-v$.

By Proposition 3.2.26 (or Proposition 2.32 of Chapter 2) if $\frac{-2\deg(g)+v}{\deg(g)}$ is a root of the Bernstein–Sato polynomial of g then $(\frac{-2\deg(g)+v}{\deg(g)}, \ldots, \frac{-2\deg(g)+v}{\deg(g)}) \in \mathrm{V}(B_G)$, where G corresponds to the factorization of g into linear terms. By Proposition 3.5.3,

$$\Delta_{S+1}(\frac{-2\deg(g)+v}{\deg(g)}, \ldots, \frac{-2\deg(g)+v}{\deg(g)}) \in \mathrm{V}(B_{fFG}^{g}) \cap \mathrm{V}(\mathbb{C}[S][T] \cdot (S+1)).$$

By Theorem 3.4.26, there exists an indecomposable edge X associated to the intersection lattice of fg, and an integer j_X satisfying $0 \leq j_X \leq 2\deg(g_X) + 2\deg(f_X) - 2r(X) - \deg(f_X)$ such that $\Delta_{S+1}(\frac{-2\deg(g)+v}{\deg(g)}, \ldots, \frac{-2\deg(g)+v}{\deg(g)})$ lies in the intersection of $\mathrm{V}(\mathbb{C}[S][T] \cdot (S+1))$ and

$$\{\sum_k \deg(f_{X,k})s_k + \sum_m \deg(g_{X,m})t_m + r(X) + \deg(f_X) + j_X = 0\}.$$

That is,

$$-\deg(f_X) + \deg(g_X)(\frac{-2\deg(g)+v}{\deg(g)}) + r(X) + \deg(f_X) + j_X = 0. \tag{3.5.1}$$

Since v is co-prime to $\deg(g)$, $\frac{\deg(g_X)v}{\deg(g)}$ can only be an integer if $\deg(g_X) = \deg(g)$. This implies $X = 0$ and $r(X) = n$. Rearranging (3.5.1) and using the upper bound on j_X we see

$$\deg(f_X) \geq r(X) - 2\deg(g_X) + \deg(g_X)\frac{2\deg(g) - v}{\deg(g)}. \tag{3.5.2}$$

Because $\deg(g_X) = \deg(g)$ and $X = 0$, (3.5.2) simplifies to

$$\deg(f) \geq n - v.$$

□

This method of argument is more versatile than the theorem suggests. In practice, information about the intersection lattice lets us drop the co-prime condition.

Example 3.5.5. Let $g = xyzw(x+y+z)(y-z+w)$. This example is studied in [46], Example 5.7, and [47], Example 5.8. In the latter, Saito verifies that $\frac{-2*6+2}{6}$ is a root of the Bernstein–Sato polynomial. Since proj dim $\Omega^1(\log g) = 1$ and $n = 4$, g is tame. Suppose

f is a central, reduced hyperplane arrangement such that fg is free. Argue as in Theorem 3.5.4 until arriving at (3.5.1). If there is an indecomposable edge $X \neq 0$ associated to the intersection lattice of fg such that (3.5.1) holds, then $\deg(g_X)$ must equal 3 so that $\frac{2\deg(g_X)}{6}$ is an integer. Then g_X corresponds to the intersection of three hyperplanes of g; all such edges have rank 3 (as edges of $\mathrm{V}(g)$). So X has rank at least 3 as an edge of the intersection lattice of fg. Equation (3.5.2) becomes $\deg(f_X) \geq 3 - 2*3 + 3*\frac{10}{6} = 2$. On the other hand, if (3.5.1) is satisfied at $X = 0$, then argument of Theorem 3.5.4 applies and $\deg(f) \geq 2$. Hence $\mu_g \geq 2$.

3.6 Trace of Adjoints

Let f be free and a defining equation for a divisor Y at $\mathfrak{x}$ and $f = l_1^{d_1} \cdots l_r^{d_r}$ its unique factorization into irreducibles, up to multiplication by a unit. So any reduced defining equation f_{red} for Y at $\mathfrak{x}$ is, up to multiplication by a unit, $f_{\text{red}} = l_1 \cdots l_d$. In this section we find formulae involving the commutators of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f)$, which by Remark 3.2.2, equals $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$. These formulae are crucial to the proof of Proposition 3.3.6 and the precise description of the dual of $\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S$. Consequently, the formulae are one of the main reasons certain Bernstein–Sato ideals have the symmetry property we used throughout the chapter. These results were first proved by Castro–Jiménez and Ucha in Theorem 4.1.4 of [34]; here we include a different proof.

Definition 3.6.1. Let f_{red} be free and $\delta_1, \ldots, \delta_n$ a basis of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$. Define a matrix $\operatorname{Ad}_{\delta i}$ whose (j,k) entry is $c_k^{i,j}$, where $c_k^{i,j} \in \mathscr{O}_{X,\mathfrak{x}}$ are determined by

$$\operatorname{ad}_{\delta_i}(\delta_j) = [\delta_i, \delta_j] = \sum_k c_k^{i,j} \delta_k.$$

Remark 3.6.2. Note $\operatorname{Ad}_{\delta_i}$ does not determine the map $\operatorname{ad}_{\delta_i} : \operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}}) \to \operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ since said map is not $\mathscr{O}_{X,\mathfrak{x}}$-linear. Moreover, $\operatorname{Ad}_{\delta_i}$ depends on a choice of basis of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$.

We will eventually find, given a coordinate system, a particular basis $\delta_1, \ldots, \delta_n$ of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ so that $\operatorname{tr}\operatorname{Ad}_{\delta_i}$, the trace of $\operatorname{Ad}_{\delta_i}$, admits a nice formula. We collect some elementary facts about the interactions between $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}})$ and $\Omega^{\bullet}(\log f_{\mathrm{red}})$. Recall by Saito, cf. 1.6 of [8], the following: the inner product between $\mathrm{Der}_{X,\mathfrak{x}}(\log f_{\mathrm{red}})$ and $\Omega^1(\log f)$ shows $\Omega^1(\log f_{\mathrm{red}})$ is the $\mathscr{O}_{X,\mathfrak{x}}$-dual of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}})$; $\Omega^{\bullet}(\log f_{\mathrm{red}})$ is closed under taking inner products with logarithmic vector fields; $\Omega^{\bullet}(\log f_{\mathrm{red}})$ is closed under taking Lie derivatives along logarithmic vector fields of f_{red}; if f_{red} is free then $\Omega^k(\log f_{\mathrm{red}}) = \bigwedge^k \Omega^1(\log f_{\mathrm{red}})$.

Definition 3.6.3. For $w \in \Omega^k(\log f_{\mathrm{red}})$ and $\delta \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}})$ let $\iota_{\delta}(w) \in \Omega^{k-1}(\log f_{\mathrm{red}})$ denote the *inner product* of w and δ. Since f_{red} is free, the induced map $\Omega^1(\log f_{\mathrm{red}}) \times \mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}}) \to \mathscr{O}_{X,\mathfrak{x}}$ is a perfect pairing. Given a basis $\delta_1, \ldots, \delta_n$ of $\mathrm{Der}_{X,\mathfrak{x}}(\log f_{\mathrm{red}})$ we may select a dual basis $\delta_1^{\star}, \ldots, \delta_n^{\star}$ of $\Omega^1(\log f_{\mathrm{red}})$ such that

$$\iota_{\delta_i}(\delta_i^{\star}) = 1 \text{ and } \iota_{\delta_i}(\delta_j^{\star}) = 0 \text{ for } i \neq j.$$

Definition 3.6.4. For $w \in \Omega^k(\log f_{\mathrm{red}})$ and $\delta \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}})$ let $L_{\delta_i}(w) \in \Omega^k(\log f_{\mathrm{red}})$ denote the *Lie derivative* of w along δ_i. Let $\delta_1, \ldots, \delta_n$ and $\delta_1^{\star}, \ldots, \delta_n^{\star}$ be as in Definition 3.6.3. Then there exists a unique choice of $b_k^{i,j} \in \mathscr{O}_{X,\mathfrak{x}}$ such that

$$L_{\delta_i}(\delta_j^{\star}) = \sum_k b_k^{i,j} \delta_k^{\star}.$$

Define the matrix Lie_{δ_i} to have (j,k) entry $b_k^{i,j}$.

Remark 3.6.5. Just like Ad_{δ_i}, the matrix Lie_{δ_i} does not determine the map $L_{\delta_i} : \Omega^1(\log f_{\mathrm{red}}) \to \Omega^1(\log f_{\mathrm{red}})$; moreover, Lie_{δ_i} depends on the choice of basis $\delta_1, \ldots, \delta_n$ of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$ which in turn determines the basis $\delta_1^{\star}, \ldots, \delta_n^{\star}$ of $\Omega^1(\log f)$.

We need the following elementary lemma. It is well known for vector fields and differential forms and can easily be shown to hold in the logarithmic case by writing a logarithmic differential form as $\frac{1}{f_{\mathrm{red}}} w$ where w is a differential form.

Lemma 3.6.6. *Let $X, Y \in \mathrm{Der}_{X,\mathfrak{x}}(\log f_{\mathrm{red}})$. Then as maps from $\Omega^k(\log f_{\mathrm{red}}) \to \Omega^{k-1}(\log f_{\mathrm{red}})$, we have*

$$\iota_{[X,Y]} = [L_X, \iota_Y].$$

Proposition 3.6.7. *If f_{red} is free and $\delta_1, \ldots, \delta_n$ is a basis for $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}})$, then*

$$\mathrm{Ad}_{\delta_i} = -\,\mathrm{Lie}_{\delta_i}^T.$$

Proof. On one hand,

$$\iota_{\mathrm{ad}_{\delta_i}(\delta_j)}(\delta_t^\star) = \iota_{\sum_k c_k^{i,j}\delta_k}(\delta_t^\star) = c_t^{i,j}.$$

On the other hand,

$$[L_{\delta_i}, \iota_{\delta_j}](\delta_t^\star) = -\iota_{\delta_j}(L_{\delta_i}(\delta_t^\star)) = -\iota_{\delta_j}(\sum_k b_k^{i,t}\delta_k^\star) = -b_j^{i,t},$$

as the Lie derivative of a vector field on a constant is zero. Now use Lemma 3.6.6. ☐

Since f_{red} is free, $\Omega^n(\log f_{\mathrm{red}})$ is a free, cyclic $\mathscr{O}_{X,\mathfrak{x}}$-module generated by $\delta_1^\star \wedge \cdots \wedge \delta_n^\star$. Moreover:

Proposition 3.6.8. *Let f_{red} be free and $\delta_1, \ldots, \delta_n$ be a basis for $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}})$. Then*

$$L_{\delta_i}(\delta_1^\star \wedge \cdots \wedge \delta_n^\star) = -\,\mathrm{tr}\,\mathrm{Ad}_{\delta_i}(\delta_1^\star \wedge \cdots \wedge \delta_n^\star).$$

Proof. By basic facts of Lie derivatives:

$$\begin{aligned} L_{\delta_i}(\delta_1^\star \wedge \cdots \wedge \delta_n^\star) &= \sum_j \delta_1^\star \wedge \cdots \wedge \delta_{j-1}^\star \wedge L_{\delta_i}(\delta_j^\star) \wedge \delta_{j+1}^\star \wedge \cdots \wedge \delta_n^\star \\ &= \sum_j \delta_1^\star \wedge \cdots \wedge \delta_{j-1}^\star \wedge (\sum_k b_k^{i,j}\delta_k^\star) \wedge \delta_{j+1}^\star \wedge \cdots \wedge \delta_n^\star \\ &= (\sum_k b_k^{i,k})(\delta_1^\star \wedge \cdots \wedge \delta_n^\star). \end{aligned}$$

The result follows by Proposition 3.6.7. ☐

We will also need the following standard definition and proposition from differential geometry.

Definition 3.6.9. Consider local coordinates $x_1, \dots, x_n$. Let δ be a vector field. Then $\operatorname{div}(\delta)$ is the *divergence* of δ with respect to the n-form $dx_1 \wedge \cdots \wedge dx_n$ and is defined by:

$$L_\delta(dx_1 \wedge \cdots \wedge dx_n) = \operatorname{div}(\delta)(dx_1 \wedge \cdots \wedge dx_n).$$

Proposition 3.6.10. *In local coordinates $x_1, \cdots, x_n$, write the vector field δ as $\delta = \sum_k h_k \frac{\partial}{\partial_{x_k}}$, where $h_k \in \mathscr{O}_{X,\mathfrak{x}}$. Then $\operatorname{div}(\delta)$ with respect to $dx_1 \wedge \cdots \wedge dx_n$ satisfies the formula*

$$\operatorname{div}(\delta) = \sum_k \frac{\partial}{\partial_{x_k}} \bullet h_k.$$

Proof. Write $dx = dx_1 \wedge \cdots \wedge dx_n$. By Cartan's formula, $L_\delta(dx) = d(\iota_\delta(dx))$. Using the skew-symmetric properties of the inner product we deduce:

$$\begin{aligned} d(\iota_\delta(dx)) &= d(\sum_k (-1)^{k-1}(dx_1 \wedge \cdots \wedge \iota_\delta(dx_k) \wedge \cdots \wedge dx_n)) \\ &= d(\sum_k (-1)^{k-1} h_k (dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n)) \\ &= (\sum_k \frac{\partial}{\partial_{x_k}} \bullet h_k) dx. \end{aligned}$$

□

Consider a basis $\delta_1, \dots, \delta_n$ of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}})$. Then for any choice of coordinates $x_1, \dots, x_n$, there exists a corresponding unit $u \in \mathscr{O}_{X,\mathfrak{x}}$ such that $\delta_1^\star \wedge \cdots \wedge \delta_n^\star = \frac{u}{f_{\mathrm{red}}} dx_1 \wedge \cdots \wedge dx_n$. See the proof of the first theorem on page 270 of [8] for justification. Clearly $u\delta_1, \dots, \delta_n$ is still a basis of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}})$ and since $\frac{1}{u}\delta_1^\star = (u\delta_1)^\star$, the logarithmic forms $(u\delta_1)^\star, \delta_2^\star, \dots, \delta_n^\star$ constitute a dual basis of $\Omega^1(\log f_{\mathrm{red}})$ satisfying:

$$(u\delta_1)^\star \wedge \delta_2^\star \wedge \cdots \wedge \delta_n^\star = \frac{1}{f_{\mathrm{red}}} dx_1 \wedge \cdots \wedge dx_n.$$

This shows, as long as $n \geq 2$, that one can always find a basis of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}})$ satisfying the conditions of the following definition:

Definition 3.6.11. Let f_{red} have Euler homogeneity E at $\mathfrak{x}$. Having fixed a coordinate system $x_1, \dots, x_n$, consider a basis $\delta_1, \dots, \delta_n$ of $\text{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ such that $\delta_n = E$ and $\delta_1, \dots, \delta_{n-1}$ is a basis of $\text{Der}_{X,\mathfrak{x}}(-\log_0 f_{\text{red}})$. Such a basis is a *preferred basis* of $\text{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ if, in addition,

$$\delta_1^\star \wedge \cdots \wedge \delta_n^\star = \frac{1}{f_{\text{red}}} dx_1 \wedge \cdots \wedge dx_n.$$

We are finally ready to state the main formula of this section.

Proposition 3.6.12. *Let f_{red} be free with Euler homogeneity E. Given a coordinate system $x_1, \dots, x_n$, let $\delta_1, \dots, \delta_n$ be a preferred basis of* $\text{Der}_{X,x}(-\log f_{\text{red}})$. *Write $\delta_i = \sum_k h_{k,i} \frac{\partial}{\partial_{x_k}}$. Then*

(i) $\text{tr}\,\text{Ad}_{\delta_i} = -\sum_k \frac{\partial}{\partial_{z_k}} \bullet h_{k,i}$ *for $i \neq n$;*

(ii) $\text{tr}\,\text{Ad}_{\delta_n} = -\sum_k \frac{\partial}{\partial_{z_k}} \bullet h_{k,n} + 1$.

Proof. Write $dx = dx_1 \wedge \cdots \wedge dx_n$. Because $\delta_1, \cdots, \delta_n$ is a preferred basis of $\text{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ and by standard properties of the Lie derivative

$$\begin{aligned} L_{\delta_i}(\delta_1^\star \wedge \cdots \wedge \delta_n^\star) &= L_{\delta_i}(\frac{1}{f_{\text{red}}} dx) = L_{\delta_i}(\frac{1}{f_{\text{red}}})dx + \frac{1}{f_{\text{red}}} L_{\delta_i}(dx) \\ &= L_{\delta_i}(\frac{1}{f_{\text{red}}})dx + (\frac{1}{f_{\text{red}}} \sum_k \frac{\partial}{\partial_{x_k}} \bullet h_{k,i})dx. \end{aligned} \tag{3.6.1}$$

Note that the last equality of (3.6.1) follows by Proposition 3.6.10. When $i \neq n$, $L_{\delta_i}(\frac{1}{f_{\text{red}}}) = 0$; when $i = n$, $L_{\delta_n}(\frac{1}{f_{\text{red}}}) = -\frac{1}{f_{\text{red}}}$. The result follows by the definition of a preferred basis together with Proposition 3.6.8. $\square$

3.7 Budur's Conjecture for Central, Reduced, Free Arrangements

In [6], Budur conjectured that exponentiating $\text{V}(B_{F,\mathfrak{x}})$ (here $F = (f_1, \dots, f_r)$ is collection of polynomials) gives the support of the Sabbah specialization functor, generalizing the fact that exponentiating the roots of the Bernstein–Sato polynomial gives the support of the nearby cycle functor, cf. Conjecture 2 of loc. cit. In the same paper he reduced this conjecture to proving, in language we will shortly define, that if $A-1 \in \mathrm{V}(B_{F,\mathfrak{x}})$ then a certain $\mathscr{D}_{X,\mathfrak{x}}$-linear map ∇_A is not surjective, cf. Proposition 2 of loc. cit. For $f = f_1 \cdots f_r$ a central, reduced, and free hyperplane arrangement and $F = (f_1, \ldots, f_r)$ an arbitrary factorization of f we provide a proof here. Theorem 3.5.3 of the recently announced paper [10] gives a general proof of the conjecture by proving the claim about ∇_A for general points in the codimension one components of $\mathrm{V}(B_{F,\mathfrak{x}})$. Our method relies on the computation of $\mathrm{V}(B_{F,0})$ given in Corollary 3.4.27 and the behavior of ∇_A under duality, cf. Section 4 of Chapter 2.

First, let us clarify our terminology. (See also Section 3 of Chapter 2 for more details). For $a_1, \ldots, a_r \in \mathbb{C}$, denote by $S - A$ the sequence $s_1 - a_1, \ldots, s_r - a_r$. Similarly, let A and $A-1$ denote the tuple $a_1, \ldots, a_r$ and $a_1 - 1, \ldots, a_r - 1$ respectively. There is an injective $\mathscr{D}_{X,\mathfrak{x}}$-linear map $\nabla : \mathscr{D}_{X,\mathfrak{x}}[S]F^S \to \mathscr{D}_{X,\mathfrak{x}}[S]F^S$ given by sending every s_k to $s_k + 1$ and identifying F^{S+1} with fF^S. This induces the $\mathscr{D}_{X,\mathfrak{x}}$-linear map

$$\nabla_A : \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \to \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S-(A-1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}.$$

By Proposition 2 of [6], to prove Budur's conjecture in our setting, it suffices to prove the following:

Proposition 3.7.1. *Let $f = f_1 \cdots f_r$ be a central, reduced, and free hyperplane arrangement where the f_k are not necessarily linear forms. Let $F = (f_1, \ldots, f_r)$. If $A - 1 \in \mathrm{V}(B_{F,0})$, then*

$$\nabla_A : \frac{\mathscr{D}_{\mathbb{C}^n,0}[S]F^S}{(S-A)\mathscr{D}_{\mathbb{C}^n,0}[S]F^S} \to \frac{\mathscr{D}_{\mathbb{C}^n,0}[S]F^S}{(S-(A-1))\mathscr{D}_{\mathbb{C}^n,0}[S]F^S}$$

is not surjective.

Proof. Since the f_k are globally defined we may consider the global version of ∇_A. Since f is central, there is a natural $\mathbb{C}^\star$-action on $\mathrm{V}(f)$; moreover, ∇_A is equivariant with respect to this action. Therefore ∇_A is surjective at 0 if and only if it is surjective at all $\mathfrak{x} \in \mathrm{V}(f)$. So it suffices to prove ∇_A is not surjective for

$$A - 1 \in \bigcup_{j=0}^{2d-2n} \{\left(\sum d_k s_k\right) + n + j = 0\},$$

when f is indecomposable of rank n and degree d, cf. Corollary 3.4.27 and Remark 3.4.9.

Since f is reduced, $\mathrm{V}(B_{F,0})$ is invariant under the map φ on $\mathbb{C}[S]$ induced by $s_k \mapsto -s_k - 2$, cf. Theorem 3.3.16 or Proposition 8 of [11]. This map sends $\{(\sum d_k s_k) + n + j = 0\}$ to $\{(\sum d_k s_k) + n + (2d - 2n - j) = 0\}$. Theorem 4.18 and Theorem 4.19 of Chapter 2 prove that the invariance of φ forces ∇_A to be surjective if and only if ∇_{-A} is surjective. So if we show ∇_A is not surjective for all $A - 1 \in \{(\sum d_k s_k) + n + j = 0\}$ then we will have also shown ∇_{-A} is not surjective for all $-A - 1 \in \{(\sum d_k s_k) + 2d - n - j = 0\}$. Thus it suffices to prove ∇_A is not surjective for

$$A - 1 \in \bigcup_{j=0}^{d-n} \{\left(\sum d_k s_k\right) + n + j = 0\}.$$

Let f' divide f, where the degree d' of f' is less than d. Just as ∇_A is induced by the $\mathscr{D}_{\mathbb{C}^n,0}$-injection $\nabla : \mathscr{D}_{\mathbb{C},0}[S]F^S \to \mathscr{D}_{\mathbb{C}^n,0}F^S$ sending each s_k to $s_k + 1$, there is an induced $\mathscr{D}_{\mathbb{C}^n,0}$-map

$$\nabla_A^{f'} : \frac{\mathscr{D}_{\mathbb{C}^n,0}[S]F^S}{(S - A)\mathscr{D}_{\mathbb{C}^n,0}[S]F^S} \to \frac{\mathscr{D}_{\mathbb{C}^n,0}[S]f'F^S}{(S - (A-1))\mathscr{D}_{\mathbb{C}^n,0}[S]f'F^S}.$$

Moreover, the non-injectivity of $\nabla_A^{f'}$ implies the non-injectivity of ∇_A. Arguing as in Section 3 of Chapter 2, we can prove a version of Theorem 3.11 of loc. cit. for $\nabla_A^{f'}$: if $\nabla_A^{f'}$ is injective, then it is surjective. By Theorem 4.19 of loc. cit., it thus suffices to prove $\nabla_A^{f'}$ is not surjective for

$$A - 1 \in \{\left(\sum d_k s_k\right) + n + d' = 0\}.$$

Now we are in the situation of Theorem 3.4.8, where instead of looking for $vB(S) \in \mathrm{ann}_{\mathscr{D}_{\mathbb{C}^n,0}[S]} f'F^S + \mathscr{D}_{\mathbb{C}^n,0}[S] \cdot g$, where $g = \frac{f}{f'}$, we are considering the following possibility:

$$1 \in \mathrm{ann}_{\mathscr{D}_{\mathbb{C}^n,0}[S]} f'F^S + \mathscr{D}_{\mathbb{C}^n,0}[S] \cdot g + (S - (A-1))\mathscr{D}_{\mathbb{C}^n,0}[S]. \tag{3.7.1}$$

Suppose, towards contradiction, (3.7.1) holds, i.e. $\nabla_A^{f'}$ is surjective. We argue as in Theorem 3.4.8, except letting $B(S)$ and v be 1, and obtain an equation resembling (3.4.1) except with additional terms on the right hand side from $(S - (A-1))\mathscr{D}_{\mathbb{C}^n,0}[S]$. Look at the right constant terms of this version of (3.4.1), evaluate each s_k at $a_k - 1$, and regard every summand as a power series. This gives an equality of elements in $\mathscr{O}_{X,0}$; denote by $\mathfrak{m}_0$ the maximal ideal of $\mathscr{O}_{X,0}$. By the argument of Theorem 3.4.8, the only piece of the right hand side outside of $\mathfrak{m}_0$ can come from $L_{\mathbf{0}}g$ as the relevant pieces from $P\psi_{f'F,0}(E)$ and the $(S-(A-1)\mathscr{D}_{\mathbb{C}^n,0}[S]$ terms vanished after sending each s_k to $a_k - 1$ and there are no such pieces from the $Q_j\psi_{f'F}(\delta_j)$ terms by Lemma 3.4.5. Certainly $g \in \mathfrak{m}_0$. Thus the entire right hand side lies in $\mathfrak{m}_0$. Since $1 \notin \mathfrak{m}_0$, our assumption that (3.7.1) holds is actually impossible, and the claim is proved. $\square$

Remark 3.7.2.(a) One can argue similarly for non-reduced f if we assume F is unmixed up to units and we check Theorem 4.18 and Theorem 4.19 of Chapter 2 for F unmixed up to units. In particular, this applies when F is a factorization into linear terms. We leave this to the reader.

(b) In this case, we obtain the expected formula (3.4.25) for the roots of Bernstein–Sato polynomial of an appropriate f by Remark 3.7.2.(a) and the strategy outlined in Remark 3.4.28.(a). This approach does not rely on [10].

(c) The primary purpose of Theorem 3.5.3 of [10] is to analyze $\text{Exp}(\text{V}(B_{F,0}))$. When f is simply a central, reduced hyperplane arrangement and L is a factorization of f into linear forms, $\text{Exp}(\text{V}(B_{L,0})$ can be explicitly computed by Theorem 3.4.18 (or Maisonobe's Proposition 10 of [11]) and Corollary 2 of [6]. In this case, Budur's conjecture holds without appeal to [10]. Similar approaches work for non-reduced f and different factorizations F of f, cf. Corollary 3.4.19 and also Remark 6.10 of [6].

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VITA

Daniel Bath sprouted from the earth in Maryland and hated mathematics throughout his infancy. After moving to Pennsylvania to attend Swarthmore College he sort of matured but at least learned his hate was misguided–in fact he really liked math. He migrated west to Indiana so as to pursue a doctorate in mathematics at Purdue University. Within a vita he would never say anything poor about Indiana or Purdue, but he certainly enjoyed working with his advisor, his mathematical peers, living with his partner, and petting his dog. At the end of 2021 he will, yet again, uproot himself, travelling to Belgium to work as a postdoctoral scholar at KU Leuven. He is excited about this mathematically and geographically.