SPECIALIZATION AND COMPLEXITY OF INTEGRAL CLOSURE OF IDEALS

by

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ABSTRACT

This dissertation is based on joint work with Lindsey Hill. There are two main parts, which are linked by the common theme of the integral closure of the Rees algebra.

In the first part of this dissertation, comprised of Chapter 3 and Chapter 4, we study the integral closure of the Rees algebra directly. In Chapter 3 we identify a bound for the multiplicity of the Rees algebra $R[It]$ of a homogeneous ideal I generated in the same degree, and combine this result with theorems of Ulrich and Vasconcelos in [34] to obtain upper bounds on the number of generators of the integral closure of the Rees algebra as a module over $R[It]$. We also find various other upper bounds for this number, and compare them in the case of a monomial ideal generated in the same degree. In Chapter 4, inspired by the large depth assumption on the integral closure of $R[It]$ in the results of Chapter 3, we obtain a lower bound for the depth of the associated graded ring and the Rees algebra of the integral closure filtration in terms of the dimension of the Cohen-Macaulay local ring R and the equimultiple ideal I. We finish the first part of this dissertation with a characterization of when the integral closure of $R[It]$ is Cohen-Macaulay for height 2 ideals.

In the second part of this dissertation, Chapter 5, we use the integral closure of the Rees algebra as a tool to discuss specialization of the integral closure of an ideal I. We prove that for ideals of height at least two in a large class of rings, the integral closure of I is compatible with specialization modulo general elements of I. This result is analogous to a result of Itoh and an extension by Hong and Ulrich which show that for ideals of height at least two in a large class of rings, the integral closure of I is compatible with specialization modulo generic elements of I. We then discuss specialization modulo a general element of the maximal ideal, rather than modulo a general element of the ideal I itself. In general it is not the case that the operations of integral closure and specialization modulo a general element of the maximal ideal are compatible, even under the assumptions of Theorem 5.2.4. We prove that the two operations are compatible for local excellent algebras over fields of characteristic zero whenever R/I is reduced with depth at least 2, and conclude with a class of ideals for which the two operations appear to be compatible based on computations in Macaulay2.

1. INTRODUCTION

A central theme of this dissertation, which is based on joint work with Lindsey Hill, is the integral closure of the Rees algebra. In Chapter 3 and Chapter 4 our main results provide information about the integral closure of the Rees algebra, while in Chapter 5 the integral closure of the Rees algebra is the tool we use to determine properties of the integral closure of an ideal.

Given an ideal I in a ring R, the integral closure of I is

$$\overline{I} = \{x \in R \,|\, x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \text{ for some } n \in \mathbb{N} \text{ and } a_i \in I^i\}.$$

That is, the integral closure of I consists of roots of monic polynomials with coefficients in the appropriate powers of I.

The integral closure of an ideal I of a ring R can alternatively be viewed as the degree one component of the integral closure of a particular ring called the Rees algebra. The Rees algebra can be viewed as a graded subring $R[It]$ of the polynomial ring $R[t]$, and is isomorphic as an R-module to

$$\oplus_{n=0}^{\infty} I^n.$$

Thus, the Rees algebra encodes information about all powers of the ideal I.

In the case where I is a monomial ideal of a polynomial ring, the integral closure of I has a geometric description – the exponent set of $\overline{I}$ is the set of lattice points of the convex hull of the exponent set of I itself (see [31, Section 1.4]). In general, the integral closure of an ideal is not easy to compute. As of writing this dissertation, the methods implemented in the computer algebra system Macaulay2 involve computing first the integral closure of $R[It]$, then reading off the degree-one component. Finding the integral closure of $R[It]$ actually provides the integral closure of every power of I, computing far more than the desired integral closure of I itself. Moreover, computing the integral closure of $R[It]$ is a computationally intensive process, possibly by computing the S_2-ification followed by the desingularization in codimension one (both of which are discussed in [35, Chapter 6]). Because of the work involved in finding the integral closure of an ideal in this way, it is helpful to understand how complex the integral closure of $R[It]$ may be to better bound the number of computations that must be performed.

In terms of geometric intuition, the Rees algebra is the algebraic object which corresponds to the geometric concept of blowing up along a subvariety to resolve singularities. More precisely, the projective scheme of the Rees algebra of an ideal I in a ring R is the blowing-up of the spectrum of R along the subscheme defined by I. Similarly, since a normal ring satisfies Serre's conditions R_1 and S_2, under slight assumptions the integral closure $\overline{S}$ of a ring S removes, in particular, the codimension one component of the singular locus of S. That is, both considering the Rees algebra of an ideal I and considering integral closure of a ring relate to resolution of singularities.

The main results of this dissertation are divided into three chapters: Chapter 3, which discusses the number of generators and embedding dimension of the integral closure of the Rees algebra; Chapter 4, which discusses the depth of the integral closure of the Rees algebra; and Chapter 5, which discusses conditions which ensure that taking the integral closure of an ideal commutes with taking a quotient modulo a sufficiently general element. More specifically, this dissertation is organized as follows.

Chapter 2 introduces terminology, notation, and properties of algebraic objects that will be used in more than one of Chapters 3 through 5. We will relegate preliminary material that is specific to a certain chapter to that individual chapter. In Section 2.1 we briefly introduce graded rings and modules, including the Hilbert function and multiplicity. Next, in Section 2.2 we define the Rees algebra, extended Rees algebra, associated graded ring, and fiber cone of an ideal. We briefly discuss some natural gradings on each ring, and provide information about their dimensions. In Section 2.3, we discuss integral closure. We first remind the reader of the integral closure of a ring, then define the integral closure of an ideal and discuss how it relates to the integral closure of the Rees algebra. We then provide some basic properties of integral closure and define the reduction of an ideal. Finally, in Section 2.4, we define and discuss various properties of rings, such as being analytically unramified or excellent, that are associated with integral closure.

In Chapter 3, we first discuss bounds by Ulrich and Vasconcelos on the number of generators of a finite birational extension of a ring as a module over the original ring, and on the embedding dimension of the extension ring. These bounds give us an idea of how complex the integral closure of the Rees algebra is, providing *a priori* bounds for computing the integral closure. The bounds given by Ulrich and Vasconcelos rely on the multiplicity of the ring. We then restrict to the integral closure of the Rees ring the case of homogeneous ideals in polynomial rings. In this case, we can explicitly compute bounds on the multiplicity. We do so in Theorem 3.2.4, then combine this bound with the results of Ulrich and Vasconcelos, obtaining the bounds discussed in Theorem 3.2.5 and Theorem 3.2.6. We conclude the chapter with a brief discussion of an alternate bound (Proposition 3.2.15) on the number of generators of the integral closure of the Rees algebra in the case of a monomial ideal generated in the same degree, and give a comparison of the two bounds.

In the case of non-monomial ideals, the results in Chapter 3 require that the integral closure of the Rees algebra have maximal or near-maximal depth. With this as our motivation, in Chapter 4 we discuss the depth of the integral closure of the Rees algebra, and of the associated graded ring related to the integral closure filtration $\mathcal{F} = \{\overline{I^n}\}$. As we journey to our main result of the chapter, Theorem 4.2.1, we first discuss previous results which link the depth of the Rees algebra and associated graded ring of an ideal, and which give lower bounds for the depth of the Rees algebra of an ideal and of a filtration.

In the final chapter of this dissertation, Chapter 5, we discuss when specialization by an element is compatible with integral closure of an ideal. We begin in Section 5.1, where we give a simple counterexample in which specialization by an element is not compatible with integral closure. We then define two "sufficiently random" types of elements – generic elements and general elements – and provide several results from the literature which guarantee that specialization by a generic element of an ideal I is compatible with integral closure of I. In the following section, Section 5.2, we prove our main result (Theorem 5.2.4), which shows that for ideals of height at least two in a large class of rings, the integral closure of the ideal is compatible with specialization modulo general elements of the ideal. Finally, in Section 5.3, we provide counterexamples to the related question of whether specialization by a general element of the maximal ideal of a local ring commutes with the integral closure of an ideal. We conclude by discussing cases where it seems that such specialization does in fact commute with integral closure of an ideal.

2. PRELIMINARIES

In this chapter, we discuss definitions and properties of algebraic objects that will be used throughout. Unless otherwise stated, we will assume all rings are commutative with 1.

2.1 Graded Rings and Modules

Definition 2.1.1. *A ring R is a* ***graded ring*** *if it can be decomposed into Abelian groups R_i as $R = \bigoplus_{i\in\mathbb{Z}} R_i$ such that for any integers i and j, $R_iR_j \subset R_{i+j}$. Here R_i is called the i^{th}* ***graded component*** *of R, and an element $f \in R_i$ is called* ***homogeneous of degree*** *i (or* ***a form of degree*** *i).*

Example 2.1.2. By giving the indeterminate t degree 1, the polynomial ring $k[t]$ over a field k is a graded ring with $R_0 = k$ and $R_i = kt^i$ for all $i \geq 0$. For any $a \in k$, an element at^2 is homogeneous of degree 2 under this grading.

More generally, for any ring R we may consider the polynomial ring $R[x_1, \ldots, x_n]$ in the variables $x_1, \ldots, x_n$. If we give each x_i degree 1, the 2^{nd} graded component of $R[x_1, \ldots, x_n]$ will consist of elements of the form $\sum_{1\leq i,j\leq n} a_{ij}x_ix_j$ where the coefficients a_{ij} are in R.

Definition 2.1.3. *A graded ring R is* ***nonnegatively graded*** *if $R = \bigoplus_{i\geq 0} R_i$. A graded ring R is called* ***standard graded*** *if $R = R_0[R_1]$; that is, if R is generated as an R_0-algebra by the degree-one component of R.*

Example 2.1.4. Both of the examples in Example 2.1.2 are standard graded. By assigning different degrees to the variables, we can change the grading so the rings are neither nonnegatively graded nor standard graded - for example, by giving t degree -1 in the polynomial ring $k[t]$.

Definition 2.1.5. *A* ***homogeneous ideal*** *is an ideal which can be generated by homogeneous elements (potentially of different degrees). An ideal is* ***generated in degree*** *n if it is a homogeneous ideal which has a generating set consisting entirely of elements which are homogeneous of degree n.*

Definition 2.1.6. *A ring R is called* ****local*** *if R has a unique maximal homogeneous ideal.*

Definition 2.1.7. *An R-module M is a **graded R-module** if it can be decomposed into Abelian groups M_i as $M = \bigoplus_{i\in\mathbb{Z}} M_i$ such that for any integers i and j, $R_iM_j \subset M_{i+j}$. M_i is called the i^{th} **graded component** of M, and an element $f \in M_i$ is called **homogeneous of degree** i.*

When discussing the graded components of a more complicated module M, we will often write $[M]_i$ instead of M_i in order to provide clarity.

We next recall the definitions of the Hilbert function, Hilbert series, Hilbert polynomial, and multiplicity of a graded module. For those interested in a more thorough discussion, we recommend Chapter 11 of Commutative Algebra by Atiyah and MacDonald [1] or Chapter 4 of Cohen-Macaulay Rings by Bruns and Herzog [2].

Definition 2.1.8. *Denote by $\lambda_R(M)$ the length of a module M over a ring R. (Length is frequently denoted instead by $\ell_R(M)$. However, we will reserve ℓ for the analytic spread of an ideal, defined in Definition 2.2.8.) Given a Noetherian graded ring R with R_0 Artinian, and a finitely generated graded R-module M, we define the **Hilbert function** of M, denoted $H_M : \mathbb{Z} \to \mathbb{N}_0$ by*

$$H_M(i) = \lambda_{R_0}(M_i)$$

*and the **Hilbert series** of M as the formal series*

$$h_M(t) = \sum_{i=-\infty}^{\infty} H_M(i)t^i.$$

Theorem 2.1.9. *[See [1, Chapter 11]] If in addition R is standard graded, then for integers $t \gg 0$, $h_M(t)$ is equal to a polynomial function $P_M(t)$. If $\dim M \geq 1$, then P_M is a polynomial in $\mathbb{Q}[x]$ with leading term $\dfrac{e_0}{(d-1)!}x^{d-1}$, where $d = \dim M$ and $e_0 \in \mathbb{Z}_{>0}$.*

Definition 2.1.10. *The polynomial P_M as in Theorem 2.1.9 is called the **Hilbert polynomial** of M.*

*The **multiplicity** of M is defined to be*

$$e(M) = \begin{cases} e_0 & \text{if } d \geq 1 \\ \lambda_R(M) & \text{if } d = 0 \text{ or } M = 0 \end{cases}$$

2.2 Blowup Algebras

In this section, we will define and give basic properties of four blowup algebras: the Rees algebra, the extended Rees algebra, the associated graded ring, and the fiber cone. These rings are called blowup algebras due to their connection to the notion of blowing-up from algebraic geometry. Here blowing-up has the connotation of zooming in on the point, and repeated blowups are used to resolve singularities. To be more precise about the relation between geometric blowups and blowup algebras, the projective scheme of the Rees algebra of an ideal I in a ring R is the blowing-up of the spectrum of R along the subscheme defined by I.

For the remainder of this work, we will use the algebraic meaning of the Rees algebra, as defined below.

Definition 2.2.1. *Given a ring R, an R-ideal I, and an indeterminate t, the* ***Rees algebra of*** *I, denoted in this work as $R[It]$ (and sometimes denoted as $\mathcal{R}(t)$ in other literature), is the R-subalgebra of the polynomial ring $R[t]$ generated by It. More precisely,*

$$R[It] = \bigoplus_{i=0}^{\infty} I^i t^i = R \oplus It \oplus I^2t^2 \oplus I^3t^3 \oplus \cdots \subset R[t].$$

As an R-module, the Rees algebra is isomorphic to

$$R[It] \cong \bigoplus_{i=0}^{\infty} I^i,$$

where I^i is defined to be R for $i \leq 0$. The Rees algebra provides information not only about the ring R, but also about all powers of the ideal I.

Notice that by giving t degree one, the Rees algebra inherits a natural standard grading from the polynomial ring $R[t]$, in which the degree i component is

$$[R[It]]_i = I^i t^i.$$

Definition 2.2.2. *The* ***extended Rees algebra*** *of an R-ideal I is the R-subalgebra of the Laurent polynomial ring $R[t,\, t^{-1}]$ (where t is an indeterminate of the ring R) generated by It and t^{-1}. That is,*

$$R[It,\, t^{-1}] = \bigoplus_{i=-\infty}^{\infty} I^i t^i = \cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus It \oplus I^2t^2 \oplus I^3t^3 \oplus \cdots \subset R[t, t^{-1}]$$

As an R-module, the extended Rees algebra is isomorphic to

$$R[It, t^{-1}] \cong \bigoplus_{i=-\infty}^{\infty} I^n,$$

where I^i is again defined to be R for $i \leq 0$. Like the Rees algebra, the extended Rees algebra provides information not only about the ring R, but also about all powers of the ideal I.

The extended Rees algebra also has a natural (but neither standard nor nonnegative) grading inherited from $R[t, t^{-1}]$ by giving t degree 1. Under this grading, the degree i component is again

$$\left[R[It, t^{-1}]\right]_i = I^i t^i.$$

Theorem 2.2.3 ([31, Theorem 5.1.4])**.** *If R is a Noetherian ring,*

$$\dim R[It,\, t^{-1}] = \dim R + 1.$$

If in addition I has positive height,

$$\dim R[It] = \dim R + 1.$$

Definition 2.2.4. *The **associated graded ring** of an R-ideal I, denoted $\operatorname{gr}_I(R)$, is the quotient of the extended Rees algebra of I modulo the ideal generated by the nonzerodivisor t^{-1}:*

$$\begin{aligned}\operatorname{gr}_I(R) &= R[It,\, t^{-1}]/t^{-1}R[It,\, t^{-1}]\\ &= R[It]/IR[It]\\ &\cong \bigoplus_{i\geq 0}(I^i/I^{i+1}),\end{aligned}$$

where we again define I^0 to be R.

Remark 2.2.5. The associated graded ring can be thought of as a graded ring by assigning to it the grading inherited from the Rees algebra; that is, $[\operatorname{gr}_I(R)]_i \cong I^i/I^{i+1}$ for $i \geq 0$. If R is Noetherian and $I \subset \operatorname{Rad}(R)$, the Jacobson radical of R, then the dimension of the associated graded ring is equal to the dimension of R (see, for instance, [31, Proposition 5.1.6]).

Definition 2.2.6. *If (R, m) is a Noetherian local ring, the **fiber cone** (or **special fiber ring**) of I, denoted $\mathcal{F}_I(R)$, is the tensor product of the Rees algebra with the residue field R/m:*

$$\mathcal{F}_I(R) = R[It] \otimes_R (R/m) = \frac{R[It]}{mR[It]} \cong \bigoplus_{i\geq 0} \frac{I^i}{mI^i}.$$

Remark 2.2.7. The fiber cone can also be written as a quotient of the associated graded ring or extended Rees algebra if $I \neq R$:

$$\mathcal{F}_I(R) = \frac{\operatorname{gr}_I(R)}{m\operatorname{gr}_I(R)} = \frac{R[It,\, t^{-1}]}{(m,\, t^{-1})R[It,\, t^{-1}]}.$$

It can be thought of as a graded ring by assigning to it the grading inherited from the Rees algebra; that is, $[\mathcal{F}_I(R)]_i \cong I^i/mI^i$ for $i \geq 0$.

The dimension of the fiber cone is an important invariant of an ideal, called the *analytic spread*. It is related to integral closure and reductions of ideals, as discussed in Remark 2.3.12.

Definition 2.2.8. *The **analytic spread** of an ideal I, denoted $\ell(I)$, is $\ell(I) = \dim \mathcal{F}_I(R)$.*

2.3 Integral Closure of Rings and Ideals

The integral closure of an ideal is closely related to the integral closure of the Rees algebra. As a reminder, the integral closure of a ring is a generalization of the algebraic closure of a field:

Definition 2.3.1. *For rings $R \subset S$, the* ***integral closure*** *of R in S is the collection of roots of monic polynomials with coefficients in R. That is,*

$$\overline{R}^S = \{x \in S \mid x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \text{ for some } n \in \mathbb{N} \text{ and } a_i \in R\}.$$

An element $x \in S$ satisfying such an equation is called ***integral over*** *R, and such an equation is called an* ***equation of integrality****.*

Recall that the integral closure of a ring R is an integrally closed subring of S; that is, $\overline{\overline{R}^S}^S = \overline{R}^S$. Additionally, integral closure preserves inclusion: if $R \subset T \subset S$, then $\overline{R}^S \subset \overline{T}^S$.

If $R \subset S$ is a homogeneous inclusion of graded rings, then $\overline{R}^S$ is also graded, and if S is non-negatively graded then so is $\overline{R}^S$. (These facts can be generalized to other gradings on rings, as seen in [31, Theorem 2.3.2].) The integral closures of the Rees algebra and extended Rees algebra are of particular interest, since the Rees algebras provide information about powers of ideals.

Definition 2.3.2. *Let I be an ideal of a ring R. The* ***integral closure*** *of I, denoted $\overline{I}$, consists of roots of monic polynomials whose coefficients are in the appropriate power of I; more specifically,*

$$\overline{I} = \{x \in R \mid x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \text{ for some } n \in \mathbb{N} \text{ and } a_i \in I^i\}.$$

Such an equation is called an ***equation of integrality*** *of x.*

Theorem 2.3.3 (Alternate Definition of $\overline{I}$)**.** *[31, Proposition 5.2.1] Let I be an ideal of a ring R. The* ***integral closure*** *of I can equivalently be defined as the degree-one component of the integral closure of the (extended) Rees algebra of I:*

$$\overline{I}t = \left[\overline{R[It]}^{R[t]}\right]_1 = \left[\overline{R[It, t^{-1}]}^{R[t,t^{-1}]}\right]_1$$

Moreover,

$$\overline{R[It]}^{R[t]} = R \oplus \overline{I}t^1 \oplus \overline{I^2}t^2 \oplus \cdots = \bigoplus_{i\geq 0} \overline{I^i}t^i$$

and

$$\overline{R[It,\, t^{-1}]}^{R[t,\, t^{-1}]} = \cdots \oplus Rt^{-1} \oplus R \oplus \overline{I}t^1 \oplus \overline{I^2}t^2 \oplus \cdots = \bigoplus_{i\in\mathbb{Z}} \overline{I^i}t^i$$

If R is integrally closed in its total ring of quotients ($\overline{R}^{\mathrm{Quot}(R)} = R$) then $\overline{R[It]}^{R[t]} = \overline{R[It]}^{\mathrm{Quot}(R[t])}$ and $\overline{R[It, t^{-1}]}^{R[t,t^{-1}]} = \overline{R[It, t^{-1}]}^{\mathrm{Quot}(R[t,t^{-1}])}$ (see, for instance, [31, Proposition 5.2.4]). From this point on, unless otherwise stated we will use the notation $\overline{R[It]}$ to refer to the integral closure of $R[It]$ in $R[t]$, and the notation $\overline{R[It, t^{-1}]}$ to refer to the integral closure of $R[It, t^{-1}]$ in $R[t, t^{-1}]$.

Definition 2.3.4. *An ideal is* ***integrally closed*** *if $\overline{I} = I$.*

An element x is said to be ***integral over*** *I if $x \in \overline{I}$. An ideal K is said to be* ***integral over*** *I if every element of K is integral over I; that is, if $K \subset \overline{I}$.*

An ideal is ***normal*** *if $\overline{R[It]} = R[It]$; that is, if every power of I is integrally closed.*

Remark 2.3.5 (Properties of $\overline{I}$). There are many important properties of integral closure that follow quickly from either Theorem 2.3.3 or Definition 2.3.2.

(a) $\overline{I}$ is an ideal.

(b) $\overline{I}$ is integrally closed.

(c) $I \subset \overline{I}$.

(d) $\overline{I} \subset \sqrt{I}$. In particular, radical and prime ideals are integrally closed.

(e) If $I \subset J$ are ideals, then $\overline{I} \subset \overline{J}$.

(f) **Persistence:** If $\varphi : R \to S$ is a ring homomorphism and I is an R-ideal, then $\varphi(\overline{I}) \subset \overline{\varphi(I)S}$.

(g) **Contraction:** If $\varphi : R \to S$ is a ring homomorphism and I is an S-ideal, then $\varphi^{-1}(\overline{I})$ is integrally closed.

(h) If $I \subset J$ and $J \subset K$ are both integral extensions, then $I \subset K$ is also an integral extension.

Proof. (a) By Theorem 2.3.3, we can see that $\overline{I}$ is an ideal since R is the degree 0 component of the ring $\overline{R[It]}$ and $\overline{I}t$ is the degree-one component of the same ring.

(b) By Definition 2.3.2, we can see that $I \subset \overline{I}$ since every $x \in I$ is the root of an equation of integrality $x - x = 0$.

(c) By Theorem 2.3.3, we can see that $\overline{I} = \overline{\overline{I}}$ since the corresponding property holds for rings; that is, since $\overline{R[It]}$ is integrally closed in $R[t]$.

(d) By Definition 2.3.2, we can see that $\overline{I} \subset \sqrt{I}$ since an equation of integrality shows $x^n \in I$.

(e) By Definition 2.3.2, we can see that integral closure of an ideal preserves inclusion since an equation of integrality over I is also an equation of integrality over a larger ideal J. Alternatively, we can see this by Theorem 2.3.3, since integral closure of rings preserves inclusion.

(f) By Definition 2.3.2, we can see that persistence of integral closure holds since the image of an equation of integrality $f(x) = 0$ of x over I becomes an equation of integrality of $\varphi(x)$ over $\varphi(I)S$.

(g) Let $x \in \overline{\varphi^{-1}(\overline{I})}$. Then x has an equation of integrality f over $\varphi^{-1}(\overline{I})$. So $\varphi(f)$ is an equation of integrality of $\varphi(x)$ over $\overline{I}$, hence $\varphi(x) \in \overline{\overline{I}} = \overline{I}$ and $x \in \varphi^{-1}(\overline{I})$.

(h) This follows from the transitivity of integral closure of rings.

■

Remark 2.3.6. Remark 2.3.5 shows that the integral closure of an ideal satisfies the properties of a *closure operation*: idempotence (b), extension (c), and order-preservation (e). For more on closure operations, see [3].

For more properties of integral closure, the interested reader should refer to [31].

Remark 2.3.7. Integral closure of an ideal behaves well with respect to localization and quotients by minimal primes. Thus, we can often reduce to the case where R is a local domain to check that an element belongs to the integral closure.

(a) ([31, Remark 1.3.2]) Integral closure commutes with localization; in other words, $W^{-1}(\overline{I}) = \overline{W^{-1}I}$ for any multiplicatively closed set W of R. Therefore, $r \in \overline{I}$ if and only if $rR_p \in \overline{I_p}$ for all $p \in \text{m-Spec}(R)$.

(b) ([31, Proposition 1.1.5]) $r \in \overline{I}$ if and only if $r + p \in \overline{I(R/p)}$ for every $p \in \operatorname{Min}(R)$.

(c) ([31, Proposition 1.1.5]) $r \in \overline{I}$ if and only if $r(R_{\text{red}}) \subset \overline{IR_{\text{red}}}$

When proving things about the integral closure of an ideal, it is frequently helpful to consider instead a *reduction* of the ideal, a possibly smaller ideal with the same integral closure.

Definition 2.3.8. *An ideal J is a **reduction** of I if $J \subset I$ and there is a non-negative integer n such that $I^{n+1} = JI^n$. The smallest such n is called the **reduction number of** I **with respect to** J, and is denoted $r_J(I)$.*

*J is a **minimal reduction** of I if it is minimal with respect to inclusion, and I is **basic** if I is its own minimal reduction. The (absolute) **reduction number** of I is denoted $r(I)$ and defined to be*

$$r(I) = \min\{r_J(I) \mid J \textit{ is a minimal reduction of } I\}.$$

Remark 2.3.9. Notice that in the above definition, JI^n is always contained in I^{n+1} since $J \subset I$; if J is a reduction of I then the reverse containment also holds.

Reductions and integral closure are closely related.

Proposition 2.3.10 ([31, Corollary 1.2.5]). *Let $J \subset I$ be ideals, with I finitely generated. The following are equivalent :*

(a) J is a reduction of I

(b) I is integral over J

If R is Noetherian, these are also equivalent to

(c) $R[It]$ is a finite $R[Jt]$-module

Remark 2.3.11. If J is a reduction of a finitely generated ideal I, then $\overline{I} = \overline{J}$ because $\overline{J} \subset \overline{I} \subset \overline{\overline{J}} = \overline{J}$. Conversely, if $\overline{I} = \overline{J}$, then J is a reduction of I. Thus, minimal reductions of I are the smallest ideals contained in I with the same integral closure.

Remark 2.3.12 ([31, Theorem 8.3.5 and Proposition 8.3.7]). If R is a Noetherian local ring, then minimal reductions of an ideal I always exist. If R has infinite residue field, then a reduction is a minimal reduction if and only if it is generated by $\ell(I)$ elements, and an ideal generated by $\ell(I)$ sufficiently general elements of I is a minimal reduction. For a definition of general elements, see Definition 5.1.3.

We now define two properties of ideals which are related to integral closure of ideals:

Definition 2.3.13. *An ideal I is called **unmixed** if $\operatorname{ht} I = \operatorname{ht} p$ for every associated prime $p \in \operatorname{Ass}(R/I)$.*

Definition 2.3.14. *Let (R, m) be a Noetherian local ring of dimension d. A **system of parameters** of R is a set of elements $a_1, \ldots, a_d$ such that $(a_1, \ldots, a_d)$ is an m-primary ideal.*

*An ideal I is called a **parameter ideal** if it is generated by a system of parameters.*

Definition 2.3.15. *An ideal I is called an **equimultiple ideal** if $\ell(I) = \operatorname{ht} I$.*

Remark 2.3.16. In a Noetherian local ring (R, m), for any proper ideal I we have that $\operatorname{ht} I \leq \ell(I) \leq \dim R$ (see, for instance, [31, Corollary 8.3.9]). Thus an m-primary ideal in such a ring is always equimultiple.

2.4 Types of Rings

We frequently need to assume a ring is "nice enough" in order to prove properties related to integral closure. For reference, we will define such assumptions on a ring in this section, and discuss consequences of these properties.

Definition 2.4.1. *An M**-regular sequence** of an R-module M is a sequence of elements $a_1, \ldots, a_n$ in R such that for $1 \leq i \leq n$, a_i is a nonzerodivisor on $M/(a_1, \ldots, a_{i-1})M$ and $M/(a_1, \ldots, a_n)M \neq 0$. We will use the term regular sequence to refer to an R-regular sequence.*

Definition 2.4.2. *Let R be a Noetherian ring, let M be a finite R-module, and let I be an R-ideal.*

*(a) If $IM \neq M$, we define the $I-$**depth** of M (or the **grade** of I on M), denoted $\operatorname{depth}_I(M)$, to be the maximal length of an M-regular sequence contained in I.*

*(b) If $I \neq R$, the **grade of** I, denoted $\operatorname{grade}(I)$, is the maximal length of an R-regular sequence contained in I – that is, $\operatorname{grade}(I) = \operatorname{grade}(I, R)$.*

*(c) If R is a local ring with maximal ideal m and $M \neq 0$, we define the **depth** of M, denoted $\operatorname{depth}(M)$, to be the maximal length of an M-regular sequence – that is, $\operatorname{depth}(M) = \operatorname{depth}_m(M)$.*

Proposition 2.4.3 ([23, Theorem 16.7])**.** *Let R be a Noetherian ring, let I be an R-ideal, and let M be a finite R-module with $IM \neq M$. We can equivalently define depth and grade in terms of non-vanishing of Ext modules:*

(a) $\operatorname{depth}_I(M) = \inf\{i \in \mathbb{Z}_{\geq 0} \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}$ and

(b) $\operatorname{grade}(M) = \inf\{i \in \mathbb{Z}_{\geq 0} \mid \operatorname{Ext}^i_R(M, R) \neq 0\}$

Definition 2.4.4. *A ring R **satisfies Serre's condition** S_n if*

$$\operatorname{depth}(R_p) \geq \min\{\dim R_p, n\}.$$

Equivalently, a ring satisfies S_n if R_p is Cohen-Macaulay for every prime p of height at most n, and has depth at least n otherwise.

*A ring R **satisfies Serre's condition** R_n if R_p is a regular local ring for all primes $p \in \operatorname{Spec}(R)$ with $\dim R_p \leq n$.*

Definition 2.4.5. *A ring R is **reduced** if either of the following equivalent conditions are satisfied:*

(a) $\sqrt{0} = 0$

(b) *R has no nonzero nilpotent elements*

If R is Noetherian, these conditions are equivalent to:

(c) *R satisfies Serre's conditions R_0 and S_1*

Definition 2.4.6. *A domain R is **normal** if $\overline{R}^{\operatorname{Quot}(R)} = R$, where $\operatorname{Quot}(R)$ denotes the total ring of quotients of R.*

*A ring R (which is not necessarily a domain) is **normal** if R_m is a normal domain for all $m \in \text{m-Spec}((R))$.*

Remark 2.4.7. For a Noetherian ring R, the following conditions are equivalent:

(a) R is a normal ring.

(b) R satisfies Serre's conditions R_1 and S_2.

(c) R is reduced and integrally closed in $\operatorname{Quot}(R)$.

The property of being *analytically unramified* is closely related to the integral closure of the powers of an ideal.

Definition 2.4.8. *A local ring (R, m) is **analytically unramified** if its completion $\widehat{R}$ is reduced.*

*A ring is **locally analytically unramified** if the rings R_p are analytically unramified for all $p \in \operatorname{Spec}(R)$.*

Remark 2.4.9. In [29], Rees proved two conditions are equivalent to being analytically unramified.

If (R, m) is an analytically unramified Noetherian local ring, and I is an R-ideal, then $\overline{R[It]}$ is a finite $R[It]$ module. This property of analytically unramified rings is beneficial when considering how complex the integral closure of the Rees algebra may be. It follows directly from Rees' theorem in [29] that a Noetherian local ring is analytically unramified if and only if for every ideal I there is an integer k such that $\overline{I^{n+k}} \subset I^n$ for all $n \geq 0$.

The second equivalence proved by Rees is that a Noetherian local domain (R, m) is analytically unramified if and only if for every finitely generated R-algebra S with $R \subset S \subset \operatorname{Quot}(R)$, $\overline{S}^{\operatorname{Quot}(R)}$ is finitely generated as a module over S.

For more information about analytically unramified rings, we refer the reader to Chapter 9 of Swanson and Huneke's Integral Closure of Ideals, Rings, and Modules [31].

Given an ideal I, it is always true that $\operatorname{ht} I + \dim R/I \leq \dim R$. It is frequently beneficial to be able to have an equality instead: $\operatorname{ht} I + \dim R/I = \dim R$. This is not always the case, but is true given certain reasonable conditions on the ring.

Definition 2.4.10. *A ring R is **catenary** if for any two primes $p \subset q$, every chain of prime ideals $p = p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n = q$ which cannot be extended to a larger chain of prime ideals has the same length.*

*A ring R is **universally catenary** if it is Noetherian and every finitely generated R-algebra is catenary.*

Theorem 2.4.11 (Dimension Formula [31, Theorem B.3.2])**.** *A universally catenary Noetherian ring R satisfies the dimension formula. That is, for any finitely generated extension S of R which is a domain, and for any $Q \in \operatorname{Spec}(S)$ and $P = Q \cap R$,*

$$\dim S_Q = \dim R_P + \operatorname{trdeg}_R S - \operatorname{trdeg}_{\kappa(P)} \kappa(Q).$$

Remark 2.4.12. If R is a catenary local domain, then for every ideal I, $\operatorname{ht} I + \dim R/I = \dim R$.

Definition 2.4.13. *A ring R is **equidimensional** if for all $p \in \operatorname{Min}(R)$, $\dim R/p = \dim R$.*

Remark 2.4.14. Notice that Definition 2.4.13 does not require that all maximal ideals $m \in$ m-Spec(R) have the same height. We will abide by the convention that rings which satisfy this property will be called equicodimensional, and that rings which are both equidimensional and equicodimensional are called biequidimensional.

Definition 2.4.15. *A Noetherian local ring ring R is* ***formally equidimensional*** (*sometimes also called* ***quasi-unmixed***) *if the completion $\widehat{R}$ is equidimensional.*

A Noetherian ring R is ***locally formally equidimensional*** *if R_p is formally equidimensional for every prime p of R.*

Remark 2.4.16. To determine whether a ring is locally formally equidimensional, it is enough to check for maximal ideals by [31, Theorem B.5.2].

Rees proved that in a formally equidimensional ring, the integral closure of an ideal is the unique largest ideal with the same multiplicity:

Theorem 2.4.17 (Rees [28, Theorem 3.2])**.** *If (R, m) is a formally equidimensional (Noetherian local) ring and if I is an m-primary ideal, then $\overline{I}$ is the unique largest ideal containing I with $e(I) = e(\overline{I})$, where $e(I)$ denotes the multiplicity of I.*

Theorem 2.4.18 (Ratliff [25, Theorem 3.6])**.** *A Noetherian domain is locally formally equidimensional if and only if it is universally catenary.*

Lemma 2.4.19. *Any ideal I in a Noetherian local ring (R, m) which is both equidimensional and catenary satisfies*

$$\dim R/I + \operatorname{ht} I = \dim R \tag{2.1}$$

Proof. Notice that it is enough to prove that Equation (2.1) is true for every prime $p \in$ Spec(R):

It is always true that $\dim R/I + \operatorname{ht} I \leq \dim R$. If we have equality for every prime p, then for any ideal I we may choose a prime $p \in V(I)$ with $\operatorname{ht} I = \operatorname{ht} p$. Notice that $\dim R/I \geq \dim R/p$, thus equality for p implies equality for I.

Now, assume $p \in \operatorname{Spec}(R)$. Since R is equidimensional, every minimal prime q has the same dimension, $\dim R/q = \dim R$. Choose a minimal prime $q \in \operatorname{Min}(R)$ contained in p.

Since R is also catenary, we can extend the chain of primes $q \subset p \subset m$ to a saturated chain; the length must be $\dim R/q$, which is equal to $\dim R$. ■

In general, catenary rings are not equidimensional. If a catenary ring is also Noetherian local and satisfies Serre's condition S_2, then the ring is equidimensional.

Lemma 2.4.20. *Every Noetherian local ring which satisfies Serre's condition S_2 and is catenary is equidimensional.*

Before we prove Lemma 2.4.20, we will state the algebraic version of Hartshorne's Connectedness Lemma, which we will use in the proof.

Lemma 2.4.21 (Hartshorne's Connectedness Lemma)**.** *[10, Proposition 2.1] Let R be a Noetherian local ring, and let I and J be non-nilpotent ideals (that is, $I \not\subset \sqrt{0}$ and $J \not\subset \sqrt{0}$) such that their product IJ is nilpotent. Then* $\operatorname{grade}(I+J) \leq 1$. (*Equivalently,* $\operatorname{Spec}(R) \setminus V(K)$ *is connected for an ideal K with grade at least 2.*)

Proof of Lemma 2.4.20. Let R be a Noetherian local ring which is both S_2 and catenary. Suppose toward contradiction that R is not equidimensional. We may write $\operatorname{Min}(R) = \{p_1, \ldots, p_s, p_{s+1}, \ldots, p_n\}$, and without loss of generality we may assume that the first s primes, $p_1, \ldots, p_s$ are exactly the minimal primes for which $\dim R/p_i = \dim R$. Let $I = p_1 \cap \cdots \cap p_s$ and let $J = p_{s+1} \cap \cdots \cap p_n$.

Notice I and J are both non-nilpotent: Suppose J is nilpotent. Then $J \subset \sqrt{0} = I \cap J$, hence $p_{s+1} \cdot \cdots \cdot p_n \subset p_{s+1} \cap \cdots \cap p_n = J \subset p_1$, hence for some $i \neq 1$ some $p_i \subset p_1$. This is a contradiction since $p_i \neq p_1$ and p_1 is a minimal prime. Similarly, I is not nilpotent. However, $IJ \subset I \cap J = \sqrt{0}$ is nilpotent.

Thus by Hartshorne's Connectedness Lemma, Lemma 2.4.21, we know $\operatorname{grade}(I+J) \leq 1$. Since R is S_2, this implies $\operatorname{ht}(I+J) \leq 1$. Hence there is a prime q of height 1 containing $I+J$. Notice that R/I is equidimensional of dimension $d = \dim R$ since its minimal primes $p_1/I, \ldots, p_s/I$ correspond to the minimal primes of R of maximal dimension. The ring R/I is also a catenary ring because it is a factor ring of a catenary ring. Therefore $\dim R/q = \dim R/I - \operatorname{ht} q/I = d-1$ (where $\operatorname{ht} q/I = 1$ because I is an intersection of minimal primes). Now, since the height of q in R is 1 and $J \subset q$, for some j between $s+1$ and n we must have that p_j is properly contained in q. Recall that by definition p_j does not have maximal dimension. However, since $p_j \subsetneq q$ we must have that $\dim R/p_j \geq d$, a contradiction.

■

In the remainder of this section, we lead up to the definition of excellent rings and provide a few properties of excellent rings. In practice, almost all rings in number theory and in algebraic geometry are excellent, including fields, complete Noetherian local rings, and any finitely generated algebra over or localization of an excellent ring.

Before defining an excellent ring, we define two conditions an excellent ring must satisfy: that of being a G-ring and a J-2 ring.

Definition 2.4.22. *A ring R is called a **G-ring** if it is Noetherian and for every prime p of R, and for every $q \in \operatorname{Spec}(R_p)$, $\widehat{R_p} \otimes_{R_p} K$ is a regular ring for any finite field extension K of the residue field $k(q) = R_q/qR_q$.*

Definition 2.4.23. *A ring R is **J-2** if for any finitely generated R-algebra S, the set of primes at which S_p is regular is an open subset of* $\operatorname{Spec}(S)$.

Definition 2.4.24. *A ring R is **excellent** if it is a (Noetherian) G-ring that is J-2 and universally catenary.* (*If a ring satisfies all these properties except being universally catenary, it is called quasi-excellent.*)

Remark 2.4.25. If R is semilocal, one does not need to require J-2 in the definition of excellent (see [22, 259]).

Definition 2.4.26. *A Noetherian ring R is a **Nagata ring** if for every $p \in \operatorname{Spec}(R)$, $\overline{R/p}^L$ is a finite R/p-module for any finite field extension L of* $\operatorname{Quot}(R/p)$.

Remark 2.4.27. An excellent ring is a Nagata ring (see [22, Theorem 78]).

Remark 2.4.28. Let R be a reduced excellent ring. Then $\overline{R}^{\operatorname{Quot}(R)}$ is finitely generated over R as a module.

Proof. Let $S = \times(R/p_i)$, where the product ranges over all $p_i \in \operatorname{Min}(R)$. Notice that $\operatorname{Quot}(R) = \operatorname{Quot}(S)$ and $R \subset S$ is a finite extension, thus $\overline{R} = \overline{S}$, where the integral closures are taken in $\mathrm{Quot}(R) = \mathrm{Quot}(S)$. Notice that $\overline{S} = \times\overline{R/p_i}$, where the integral closure of R/p_i is taken in $\mathrm{Quot}(R/p_i)$. Because R is Nagata, $\overline{R/p_i}$ is a finite R/p_i-module for each p_i, and hence $R \subset S \subset \overline{S} = \overline{R}$ are all finite extensions. Thus $R \subset \overline{R}$ is a finite extension. ■

Example 2.4.29. A few examples of excellent rings are:

(a) fields

(b) Dedekind domains of characteristic zero

(c) complete local rings

(d) finitely generated algebras over excellent rings

(e) homomorphic images of excellent rings

(f) localizations of excellent rings

For more information on excellent rings, we direct the interested reader to Chapter 13 of Matsumura's Commutative Algebra [22].

3. COMPLEXITY OF INTEGRAL CLOSURE

In this chapter, we discuss bounds on the number of generators and embedding dimension of $\overline{R[It]}^{R[t]}$, which we denote as $\overline{R[It]}$. Before we do so, we discuss a few preliminaries which are specific to this chapter.

3.1 Complexity Preliminaries

Definition 3.1.1. *Let M be a finitely generated graded module over a positively graded *local ring R with R_0 local. We will denote the minimal number of generators of M as an R-module by $\mu_R(M)$.*

Remark 3.1.2. Notice Definition 3.1.1 encompasses the case where M is a module over a local ring R by letting $R = R_0$.

Definition 3.1.3. *Let B be a reduced, finitely generated k-algebra over a field k. That is, B is an affine k-algebra and can be presented non-uniquely as $B \cong k[x_1, \ldots, x_n]/I$, where I is an $k[x_1, \ldots, x_n]$-ideal. The **embedding dimension** of B, denoted* $\operatorname{embdim}(B)$*, is*

$$\operatorname{embdim}(B) = \min\{n \mid k[x_1, \ldots, x_n]/I \textit{ is a presentation of } B\}.$$

Definition 3.1.4. *A ring extension $A \subset B$ is called **finite** if B can be generated by finitely many elements as an A-module, and is called **birational** if* $\operatorname{Quot}(A) = \operatorname{Quot}(B)$.

Remark 3.1.5. Let I be an R-ideal of positive grade, and consider the Rees algebra $A = R[It]$ and its integral closure $B = \overline{R[It]}$. Notice $A \subset B \subset R[t]$ and that every nonzerodivisor remains a nonzerodivisor under these inclusions, hence

$$\operatorname{Quot}(A) \subset \operatorname{Quot}(B) \subset \operatorname{Quot}(R[t]). \tag{3.1}$$

Since I has positive grade, there exists $x \in I$ such that x is a nonzerodivisor, and hence $t = x^{-1}xt \in \operatorname{Quot}(A)$. Thus $\operatorname{Quot}(A) = \operatorname{Quot}(R[t])$, hence the inequalities in Equation (3.1) are equalities, thus $A \subset B$ is a birational extension.

Also, recall that if R is an analytically unramified Noetherian local ring, then $R[It] \subset \overline{R[It]}$ is a module-finite extension by a theorem of Rees (see Remark 2.4.9).

Definition 3.1.6. *A ring homomorphism $\varphi : R \to S$ is said to be **flat** if S is a flat R-module with the module structure given by φ - that is, if tensoring with S is left-exact. Equivalently, we say that S is a flat R-algebra.*

*Similarly, φ is said to be **faithfully flat** if S is a faithfully flat R-module with the module structure given by φ. That is, φ is faithfully flat if a sequence of R-linear maps $\mathfrak{L}$ is exact if and only if $\mathfrak{L} \otimes_R S$ is exact. Equivalently, we say that S is a faithfully flat R-algebra.*

In the case where R is a polynomial ring, we can often say more about the complexity of the Rees algebra since R is particularly nice.

Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k with homogeneous maximal ideal $m = (x_1, \ldots, x_d)$. Often, polynomial rings are endowed with the grading obtained by giving $\deg x_i = 1$. Under this grading, R is a standard graded ring over the field k (recall the definition of a standard graded ring from Definition 2.1.3).

Let I be an R-ideal generated by homogeneous elements $f_1, \ldots, f_s$ with $\deg f_i = n$ for all $1 \leq i \leq s$. We say that I is **generated in the same degree** n (or **generated in degree** n). In this case a natural grading on the Rees algebra $R[It]$ is the bi-grading obtained by giving each x_i degree $(1, 0)$ and giving t degree $(-n, 1)$. Under this grading, $R[It]$ is a standard bigraded ring, but is not a standard graded k-algebra.

Since much is known about standard graded rings, it can be helpful to consider $R[It]$ as a standard graded ring over k. We may naturally consider $R[It]$ to be a standard graded k-algebra by considering instead the total degree given by the sum of the bigraded degrees. Then $R[It] = k[x_1, \ldots, x_d, f_1t, \ldots, f_st]$, where $\deg x_i = 1$ for $1 \leq i \leq d$ and $\deg f_it = \deg f_i + \deg t = 1$ for $1 \leq i \leq s$. In this way, we may obtain additional information about the complexity of the Rees algebra and its integral closure.

In Lemma 3.2.3, we find the multiplicity of a Rees algebra graded in this way (recall the definition of multiplicity from Definition 2.1.8). This result is similar to a result of Verma in [36], which he obtains through the notion of mixed multiplicities.

3.2 Number of Generators of the Integral Closure of the Rees Algebra

In the case where A is a reduced, equidimensional, finitely generated algebra over a field k, and when B is a finite and birational ring extension of A, Ulrich and Vasconcelos in [34] give bounds for the number of generators of B as an A-module, and for the embedding dimension of B over k. These results rely on the depth of B being fairly large. (We discuss the depth of the specific instance where $B = \overline{R[It]}$ in Section 4.2.) In the case where B is Cohen-Macaulay, one obtains the smallest upper bounds:

Theorem 3.2.1 (Ulrich-Vasconcelos [34, Theorem 2.1])**.** *Let A be a reduced and equidimensional affine algebra of dimension d over a field k. For $T = k[x_1, \dots, x_d]$ a Noether normalization of A and $K = \mathrm{Quot}(T)$, let $\mathrm{rank}_T(A)$ be the dimension of the K-vector space $A \otimes_T K$. Let $e = \inf\{\mathrm{rank}_T(A) \,|\, T$ is a Noether normalization of $A\}$. If $A \subset B$ is a finite and birational ring extension, and B is Cohen-Macaulay, then*

$$\mu_A(B) \leq e$$

and

$$\mathrm{embdim}(B) \leq e + d - 1.$$

In the case where A is standard graded and B instead has large depth ($\mathrm{depth}_A(B) \geq \dim A - 1$), Ulrich and Vasconcelos were able to find larger bounds for the number of generators and embedding dimension of B:

Theorem 3.2.2 (Ulrich-Vasconcelos [34, Theorem 3.2])**.** *Let k be a field of characteristic zero and let A be a reduced and equidimensional standard graded k–algebra of dimension d and multiplicity e. If $A \subset B$ is a finite and birational extension of graded rings with $\mathrm{depth}_A(B) \geq d - 1$, then*

$$\mu_A(B) \leq (e-1)^2 + 1$$

and

$$\mathrm{embdim}(B) \leq (e-1)^2 + d + 1.$$

In the case where R is a polynomial ring, $A = R[It]$, $B = \overline{R[It]}$, and I is a homogeneous R-ideal generated in the same degree, we can consider A to be a standard-graded algebra over a field and determine a bound for the multiplicity of A:

Lemma 3.2.3. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k with homogeneous maximal ideal $m = (x_1, \ldots, x_d)$, and give $R[m^n t]$ a standard grading by setting $\deg x_i = 1$ and $\deg t = 1-n$. Then the multiplicity of $R[m^n t]$ under this grading is $e(R[m^n t]) = \dfrac{n^d - 1}{n-1}$.*

Proof. The Rees algebra of m^n is $R[m^n t] = k[m, m^n t]$. Since m^n is generated by elements of exponent degree n, we can consider $R[m^n t]$ to be a standard graded algebra over k as before, by giving each x_i degree 1 and giving t degree $1-n$. Under this grading, the degree s component of the Rees algebra is obtained from s-fold products of the variables and generators of $m^n t$. Moreover, since $R[m^n t]$ is generated over k by its degree 1 component, the degree s component of the Rees algebra is the direct sum of the products of $R_{s-k} \cdot (R_n t)^k$.

$$\begin{aligned} R[m^n t]_s &= R_s \oplus R_{s-1} \cdot (R_n t) \oplus \cdots \oplus R_{s-k} \cdot (R_n t)^k \oplus \cdots \oplus (R_n t)^s \\ &= R_s \oplus R_{s+n-1} t \oplus \cdots \oplus R_{s+(n-1)k} t^k \oplus \cdots \oplus R_{ns} t^s \end{aligned}$$

As a k-vector space, this is isomorphic to

$$[m^s]_s \oplus \left[m^{s+n-1}\right]_{s+n-1} \oplus \cdots \oplus \left[m^{s+(n-1)j}\right]_{s+(n-1)j} \oplus \cdots \oplus [m^{ns}]_{ns}$$

or equivalently

$$\bigoplus_{j=0}^{s} \left[m^{s+(n-1)j}\right]_{s+(n-1)j}.$$

Thus, the Hilbert function $H(s)$ of $R[m^n t]$ under this grading is given by the sum of the number of generators of $m^{s+(n-1)j}$ for $j = 0, \ldots, s$.

Notice that the number of monomials of degree n in d variables is equal to the number of d-tuples of non-negative integers whose sum is n. This number is given by a combinatorial argument often referred to as the stars and bars method (see, for example, [4]). (As an aside, to the best of our knowledge this method is not associated with the flag of the same name, but is rather named for the illustrative approach of assigning indistinguishable objects (stars) and dividers (bars) into $n+d-1$ blank spaces.) Using this method, we know that the number of monomials of degree n in d variables is given by $\binom{n+d-1}{d-1}$. For simplicity, we will use the notation $H(s)$ to represent $H_{R[m^n t]}(s)$. Then for all s,

$$H(s) = \sum_{j=0}^{s} \binom{s+(n-1)j+d-1}{d-1}.$$

Recall that because m^n has positive height, $\dim R[m^n t] = d+1$. Thus, to determine the multiplicity we wish to find the coefficient of s^d, the top degree of s. In the next several steps, we rewrite $H(s)$ to better determine this coefficient.

$$\begin{aligned} H(s) &= \sum_{j=0}^{s} \frac{(s+(n-1)j+d-1)(s+(n-1)j+d-2)\cdots(s+(n-1)j+2)(s+(n-1)j+1)}{(d-1)!} \\ &= \sum_{j=0}^{s} \frac{(s+(n-1)j)^{d-1}}{(d-1)!} + \text{ lower degree terms in } s. \end{aligned}$$

Dropping the lower degree terms, it is enough for us to consider the leading coefficient of s in the simpler expression

$$\begin{aligned} \sum_{j=0}^{s} \frac{(s+(n-1)j)^{d-1}}{(d-1)!} &= \sum_{j=0}^{s} \frac{1}{(d-1)!} \sum_{i=0}^{d-1} \binom{d-1}{i} s^i ((n-1)j)^{d-i-1} \text{ by a binomial expansion} \\ &= \sum_{j=0}^{s} \sum_{i=0}^{d-1} \frac{1}{(d-i-1)!\, i!} s^i (n-1)^{d-i-1} j^{d-i-1} \\ &= \sum_{i=0}^{d-1} \frac{1}{(d-i-1)!\, i!} s^i (n-1)^{d-i-1} \sum_{j=0}^{s} j^{d-i-1} \\ &\quad \text{since no other terms depend on } j. \end{aligned}$$

Now, the leading term of $\sum_{j=0}^{s} j^{d-i-1}$ is $\dfrac{1}{d-i} \cdot s^{d-i}$ (see, for instance, [21]). Thus, to determine the coefficient of the largest power of s, it is enough to consider the expression

$$\sum_{i=0}^{d-1} \frac{1}{(d-i-1)!\,i!}\, s^i (n-1)^{d-i-1} \left(\frac{1}{d-i} \cdot s^{d-i}\right) = \sum_{i=0}^{d-1} \frac{1}{(d-i)!\,i!}\, s^{i+d-i}(n-1)^{d-i-1}$$
$$= \sum_{i=0}^{d-1} \frac{(n-1)^{d-i-1}}{(d-i)!\,i!}\, s^d.$$

So the coefficient of s^d in $H(s)$ is $\sum_{i=0}^{d-1} \dfrac{(n-1)^{d-i-1}}{(d-i)!\,i!}$, and the multiplicity of $R[It]$ is thus $d! \cdot \sum_{i=0}^{d-1} \dfrac{(n-1)^{d-i-1}}{(d-i)!\,i!}$. It remains to show this is equal to $\dfrac{n^d-1}{n-1}$. This comes from slight adjustments to rewrite the sum in the form of a binomial expansion. First, we adjust the exponent of $n-1$:

$$\begin{aligned} e(R[It]) &= \sum_{i=0}^{d-1} \frac{d!}{(d-i)!\,i!}(n-1)^{d-i-1} \\ &= \sum_{i=0}^{d-1} \binom{d}{i}(n-1)^{d-i-1} \\ &= \frac{1}{n-1} \cdot \sum_{i=0}^{d-1} \binom{d}{i}(n-1)^{d-i}. \end{aligned}$$

Next, we adjust the bounds of the sum by adding and subtracting $\binom{d}{d}(n-1)^0 = 1$:

$$e(R[It]) = \frac{1}{n-1} \cdot \left(-1 + \sum_{i=0}^{d} \binom{d}{i}(n-1)^{d-i}\right).$$

Finally, we write $(n-1)^{d-i}$ as $(n-1)^{d-i} \cdot 1^i$ and use the binomial expansion to obtain

$$\begin{aligned} e(R[It]) &= \frac{1}{n-1} \cdot \left(-1 + ((n-1)+1)^d\right) \\ &= \frac{n^d-1}{n-1}. \end{aligned}$$

Thus $e(R[m^n t]) = \dfrac{n^d-1}{n-1}$ (or equivalently, $1 + n + n^2 + \cdots + n^{d-1}$). ■

Theorem 3.2.4. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k, and let I be a non-zero proper homogeneous ideal generated in the same degree n, and consider again the standard grading on $R[It]$ given by setting $\deg x_i = 1$ and $\deg t = 1 - n$. Then $e(R[It]) \leq \dfrac{n^d - 1}{n - 1}$.*

Proof. With this grading, $R[It]$ is a graded subring of $R[m^n t]$, where $R[m^n t]$ is graded as in Lemma 3.2.3. Since both rings have the same dimension, but the graded components of $R[It]$ grow more slowly than the graded components of $R[m^n t]$, we see that the multiplicity of $R[It]$ is at most the multiplicity of $R[m^n t]$. ∎

We can combine Theorem 3.2.4 with the two results of Ulrich and Vasconcelos to obtain bounds on the number of generators of $\overline{R[It]}$ as an $R[It]$-module and on the embedding dimension of $\overline{R[It]}$, for I a homogeneous ideal generated in the same degree.

Theorem 3.2.5. *Let k be a field, let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over k, and let I be a non-zero, proper homogeneous ideal generated in the same degree n. Let $A = R[It]$ and let $B = \overline{R[It]}$. If B is Cohen-Macaulay, then*

$$\mu_A(B) \leq \frac{n^d - 1}{n - 1}$$

and

$$\operatorname{embdim}(B) \leq \frac{n^d - 1}{n - 1} + d - 1.$$

Theorem 3.2.6. *Let k be a field of characteristic zero, let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over k, and let I be a non-zero, proper homogeneous ideal generated in the same degree n. Let $A = R[It]$ and let $B = \overline{R[It]}$. If $\operatorname{depth}(B) \geq d$ then*

$$\mu_A(B) \leq \left(\frac{n^d - 1}{n - 1} - 1\right)^2 + 1$$

and

$$\operatorname{embdim}(B) \leq \left(\frac{n^d - 1}{n - 1} - 1\right)^2 + d + 1.$$

Remark 3.2.7. Before we prove Theorem 3.2.5 and Theorem 3.2.6, we note that the Rees algebra of an ideal of positive height in an equidimensional ring is equidimensional:

Let $\mathfrak{p}$ be any minimal prime of $R[It]$. By the discussion in [31, Section 5.1], we know that $\mathfrak{p} = pR[t] \cap R[It]$ for some $p \in \operatorname{Min}(R)$. Then $R[It]/\mathfrak{p}$ is the Rees algebra of $(I+p)/p \subset R/p$, which has dimension equal to $\dim R/p + 1$ by Theorem 2.2.3. Since R is equidimensional, the dimension of any minimal prime of $R[It]$ is $\dim R + 1$.

Proof of Theorem 3.2.5 and Theorem 3.2.6. Notice in this case, $A \subset R[t]$ is reduced (in fact, it is a domain). Additionally, it is equidimensional by Remark 3.2.7 since it is the Rees algebra of an equidimensional ring. As discussed previously, we may give t degree $1-n$ to give A a standard grading. Since R is a domain, $\operatorname{grade} I > 0$, and hence $\dim A = d+1$. Finally, by Remark 3.1.5 the extension $A \subset B$ is finite and birational since R is an excellent reduced ring, hence analytically unramified. We found an upper bound for the multiplicity of $R[It]$ as a standard graded algebra in Theorem 3.2.4, which we can substitute into Theorem 3.2.2 and Theorem 3.2.1. ∎

We can extend the previous result about the number of generators of $\overline{R[It]}$ to homogeneous ideals that are not generated in the same degree by considering a related homogeneous ideal, which we will call J, which is generated in the same degree in the polynomial ring $R[a]$.

Lemma 3.2.8. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k and let I be a homogeneous R-ideal with minimal homogeneous generators $v_1, \ldots, v_s$. Let $\deg v_1 = m_1, \ldots, \deg v_s = m_s$ with $m_1 \le m_2 \le \ldots \le m_s$. Let $S = R[a]$, where a is an indeterminate over R, and let $J = (a^{m_s - m_1}v_1, a^{m_s-m_2}v_2, \ldots, v_s)$. Notice all generators of J have degree m_s, so J is generated in the same degree. Then $\mu_{R[It]}(\overline{R[It]}) \le \mu_{S[Jt]}(\overline{S[Jt]})$.*

Proof. Because localization preserves generating sets, we may localize $S[Jt]$ at a to see that

$$\mu_{S[Jt]_a}(\overline{S[Jt]}_a) \le \mu_{S[Jt]}(\overline{S[Jt]}).$$

Thus, it is enough to show that

$$\mu_{R[It]}(\overline{R[It]}) \leq \mu_{S[Jt]_a}(\overline{S[Jt]}_a). \tag{3.2}$$

Now,

$$\begin{aligned}
S[Jt]_a &= R[a][(v_1 a^{m_s - m_1}, v_2 a^{m_s - m_2}, \ldots, v_{s-1} a^{m_s - m_{s-1}}, v_s)t]_a \\
&= R[a][(v_1, \ldots, v_s)t]_a \text{ since } a \text{ is a unit} \\
&= (R[a])_a[(v_1, \ldots, v_s)(R[a])_a t]
\end{aligned}$$

So we can rewrite $S[Jt]_a$ as $R[a]_a[IR[a]_a t]$ or as $R[It] \otimes_R R[a]_a$, where $R[a]_a = R \otimes_k k[a]_a$. Similarly, since localization and adjoining variables both commute with integral closure, $\overline{S[Jt]}_a = \overline{(S[Jt])_a} = \overline{R[a]_a[IR[a]_a t]} = \overline{R[It]} \otimes_R R[a]_a$, so the right hand side of Equation (3.2) can be rewritten as $\mu_{R[IRt] \otimes_R R[a]_a}(\overline{R[It]} \otimes_R R[a]_a)$. It remains to show that

$$\mu_{R[It]}(\overline{R[It]}) \leq \mu_{R[IRt] \otimes_R R[a]_a}(\overline{R[It]} \otimes_R R[a]_a).$$

This is in fact an equality because $R \to R[a]_a$ is a flat extension of *local rings which maps the maximal homogeneous ideal to the maximal homogeneous ideal, thus $- \otimes_R R[a]_a$ preserves minimal presentations of modules. ■

Theorem 3.2.9 (Bound 1)**.** *Let R be a polynomial ring in d variables over a field k of characteristic 0. Let I be a nonzero proper monomial ideal with N being the maximal degree of some minimal monomial generating set. Then* $\mu_{R[It]}(\overline{R[It]}) \leq \left(\dfrac{N^{d+1} - 1}{N - 1} - 1\right)^2 + 1.$

Proof. Combine Lemma 3.2.8 with Theorem 3.2.5. Notice in this case, $\overline{S[Jt]}$ is normal because R is normal, and is a toric ring (which we will not define) because it is generated over k by monomials, thus by a theorem of Hochster in [11, Theorem 1], $\overline{S[Jt]}$ is Cohen-Macaulay. Observe that $\dim S = d + 1$. ■

In the case where the ideal I is a monomial ideal, we obtain an alternate bound for the minimal number of generators of $\overline{R[It]}$ by combining two results:

Theorem 3.2.10 (Singla [30, Theorem 5.1])**.** *Let $I \subset R = k[x_1, \dots, x_d]$ be a monomial ideal, and let J be its minimal monomial reduction ideal. Let ℓ be the analytic spread of I, as defined in Definition 2.2.8. Then $\overline{I^m} = J\overline{I^{m-1}}$ for all $m \geq \ell$.*

Theorem 3.2.10 has immediate application for the integral closure of the Rees algebra:

Corollary 3.2.11. *Let $I \subset R = k[x_1, \dots, x_d]$ be a monomial ideal, and let ℓ be the analytic spread of I. Then as a module $\overline{R[It]}$ is generated over $R[It]$ by elements of t-degree at most $\ell - 1$.*

Proof. As in Theorem 3.2.10, let $R = k[x_1, \dots, x_d]$, and let I be a monomial ideal with analytic spread ℓ. Define the typical grading on the Rees algebra $R[It]$ by setting $\deg t = 1$; that is, $[R[It]]_i = I^i t^i$. Since $J \subset I$, Theorem 3.2.10 means that for all $n \geq 0$,

$$\overline{I^{\ell+n}} \subset I\overline{I^{\ell+n-1}} \subset I \cdot I\overline{I^{\ell+n-2}} = I^2\overline{I^{\ell+n-2}} \subset \cdots \subset I^{n+1}\overline{I^{\ell-1}}.$$

Translating this to $\overline{R[It]}$ gives information about the generating degree of $\overline{R[It]}$ as a module over $R[It]$:

$$\left[\overline{R[It]}\right]_{\ell+n} \subset \overline{I^{\ell+n}}t^{\ell+n} \subset I^{n+1}\overline{I^{\ell-1}}t^{\ell+n} = [R[It]]_{n+1} \cdot \left[\overline{R[It]}\right]_{\ell-1}.$$

Thus, as a module $\overline{R[It]}$ is generated by elements of t-degree at most $\ell - 1$. ■

Moreover, for monomial ideals there is an upper bound for the degrees of generators of $\overline{I}$ in terms of the generating degrees of I:

Theorem 3.2.12 ([31, Proposition 1.4.9])**.** *Let $I \subset R = k[x_1, \dots, x_d]$ be a monomial ideal. Let N be the upper bound on the degrees of minimal monomial generators of I. Then the generators of the integral closure of I have degree at most $N + d - 1$.*

Remark 3.2.13. In the setting of Theorem 3.2.12, a bound for the degrees of generators of $\overline{I^k}$ is $kN + d - 1$ since an upper bound on the degrees of minimal monomial generators of I^k is kN.

Proposition 3.2.14 (Bound 2)**.** *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k, let I be a monomial R-ideal with minimal monomial generators $v_1, \ldots, v_s$, and let ℓ be the analytic spread of I. Without loss of generality, we may assume that* $\deg v_1 \leq \deg v_2 \leq \cdots \leq \deg v_s$*. Let $m = \deg v_1$ and let $N = \deg v_s$. Then*

$$\mu_{R[It]}(\overline{R[It]}) \leq (\ell - 1)\left[\binom{(\ell-1)N + 2d - 1}{d} - \binom{m + d - 1}{d}\right] + 1.$$

Proof. By Theorem 3.2.10, we know that $\overline{R[It]}$ is generated as a module over $R[It]$ in t-degree at most $\ell - 1$. Thus the number of generators of $\overline{R[It]}$ as a module over $R[It]$ can be bounded by the number of monomials needed to generate $\overline{R[It]}$ in t-degrees 0 through $\ell - 1$.

A rough bound on this number is the sum of the number of monomials needed to generate $\overline{I^k}$ for $1 \leq k \leq \ell - 1$, plus 1 to generate the degree one component. That is,

$$\mu_{R[It]}(\overline{R[It]}) \leq \left[\sum_{k=1}^{\ell-1} \mu(\overline{I^k})\right] + 1.$$

Notice that the generators of the integral closure of I have degree at least m by a degree argument:

If $x \in \overline{I}$ is homogeneous, then x satisfies an equation of integrality

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

where $a_i \in I^i$ are homogeneous. Since $\deg a_i \geq mi$, it follows that $\deg x \geq m$.

More generally, the generators of the integral closure of I^k have degree at least km by the same degree argument.

Thus the degrees of generators of $\overline{I^k}$ are between km and $kN + d - 1$, where the upper bound is given by Remark 3.2.13. Since we know the number of monomials whose degrees are between km and $kN + d - 1$, this gives a bound on $\mu(\overline{I^k})$. Thus

$$
\begin{aligned}
\mu_{R[It]}(\overline{R[It]}) &\leq \left[\sum_{k=0}^{\ell-1} (\text{number of monomials which generate } \overline{I^k}/I^k)\right] + 1 \\
&\leq \left[\sum_{k=1}^{\ell-1} (\text{number of monomials of degree } km \text{ through } kN+d-1)\right] + 1 \\
&= \left[\sum_{k=1}^{\ell-1} (\text{number of monomials of degree at most } kN+d-1)\right. \\
&\qquad \left. -(\text{number of monomials of degree at most } km-1)\right] + 1
\end{aligned}
$$

Recall that the number of monomials of degree exactly n in d variables is $\binom{n+d-1}{d-1}$ (see the proof of Lemma 3.2.3). Notice that the number of monomials of degree at most n in $k[x_1, \ldots, x_d]$ is equal to the number of monomials of degree exactly n in $k[x_1, \ldots, x_{d+1}]$, with the correspondence coming from mapping x_{d+1} to 1. Thus the number of monomials of degree at most n in d variables is $\binom{n+d}{d}$. So we can rewrite the above bound as

$$
\begin{aligned}
\mu_{R[It]}(\overline{R[It]}) &\leq \left[\sum_{k=1}^{\ell-1} \binom{(kN+d-1)+d}{d} - \binom{(km-1)+d}{d}\right] + 1 \\
&= \left[\sum_{k=1}^{\ell-1} \binom{kN+2d-1}{d} - \binom{km+d-1}{d}\right] + 1
\end{aligned}
$$

To obtain a closed form for the bound, we note that for all k, $1 \leq k \leq \ell - 1$. Thus for all k in question,

$$
\binom{kN+2d-1}{d} - \binom{km+d-1}{d} \leq \binom{(\ell-1)N+2d-1}{d} - \binom{m+d-1}{d}
$$

and hence

$$
\mu_{R[It]}(\overline{R[It]}) \leq (\ell-1)\left[\binom{(\ell-1)N+2d-1}{d} - \binom{m+d-1}{d}\right] + 1
$$

■

If I is a monomial ideal generated in the same degree n, we can say more. Notice that in this case, $\overline{I}$ is also generated by monomials of degree n. We can see this geometrically by considering the Newton polyhedron (see [31, Section 1.4] for more information). Moreover, the minimal generators of I are minimal generators of $\overline{I}$ because $I \subset \overline{I}$ are both generated in the same degree, and $\mu(I^k) \geq \mu(I)$ because I^k is generated in degree nk, thus the k^{th} power of every minimal generator of I is a minimal generator of I^k. Using these facts, we can modify the bound in Proposition 3.2.14:

Proposition 3.2.15 (Bound 2′)**.** *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k, let I be a monomial R-ideal with minimal monomial generators $v_1, \ldots, v_s$, all of degree n, and let ℓ be the analytic spread of I. Then*

$$\mu_{R[It]}(\overline{R[It]}) \leq (\ell - 1)\left[\binom{(\ell-1)n + d - 1}{d-1} - \mu(I)\right] + 1.$$

Proof. By Theorem 3.2.10, we know that $\overline{R[It]}$ is generated as a module over $R[It]$ in t-degree at most $\ell - 1$. Thus the number of generators of $\overline{R[It]}$ as a module over $R[It]$ can be bounded by the number of monomials needed to generate $\overline{R[It]}$ in t-degrees 0 through $\ell - 1$.

A rough bound on this number is the sum of the number of monomials needed to generate $\overline{I^k}/I^k$ for $1 \leq k \leq \ell - 1$, plus 1 to generate the degree zero component. Because I is a monomial ideal generated in the same degree, $\mu(\overline{I^k}/I^k) \leq \mu(\overline{I^k}) - \mu(I^k)$. Since generators of the $\overline{I^k}$ have degree nk, we can write

$$\begin{aligned}
\mu_{R[It]}(\overline{R[It]}) &\leq \left[\sum_{k=0}^{\ell-1} \text{ (number of monomials which generate } \overline{I^k}/I^k)\right] + 1 \\
&\leq \left[\sum_{k=1}^{\ell-1} \mu(\overline{I^k}) - \mu(I^k)\right] + 1 \\
&\leq \left[\sum_{k=1}^{\ell-1} \text{ (number of monomials of degree } kn) \ - \mu(I^k)\right] + 1
\end{aligned}$$

As in the proof of Lemma 3.2.3, the number of monomials of degree exactly n in d variables is $\binom{n+d-1}{d-1}$. So we can rewrite the above bound as

$$\mu_{R[It]}(\overline{R[It]}) \leq \left[\sum_{k=1}^{m-1} \binom{kn+d-1}{d-1} - \mu(I^k)\right] + 1$$

To obtain a closed form for the bound, we note that for all k, $1 \leq k \leq \ell - 1$ and $\mu(I^k) \geq \mu(I)$. Thus for all k in question,

$$\binom{(kn+d-1}{d-1} - \mu(I^k) \leq \binom{(\ell-1)n+d-1}{d-1} - \mu(I)$$

and hence

$$\mu_{R[It]}(\overline{R[It]}) \leq (\ell-1)\left[\binom{(\ell-1)n+d-1}{d-1} - \mu(I)\right] + 1$$

■

Example 3.2.16. In the case where I is generated in the same degree n, Theorem 3.2.9 gives that $\mu_{R[It]}(\overline{R[It]}) \leq \left(\frac{n^d-1}{n-1} - 1\right)^2 + 1$ while Proposition 3.2.15 gives the inequality $\mu_{R[It]}(\overline{R[It]}) \leq (\ell-1)\left[\binom{(\ell-1)n+d-1}{d-1} - \mu(I)\right] + 1$. For ideals generated in small degree in a polynomial ring R of large dimension, Theorem 3.2.9 may give a better bound. For example, if $n = 2$, $d = 7$, and $\ell = 7$, then Theorem 3.2.9 gives a bound of 15,877 while Proposition 3.2.15 gives a bound on the order of 100,000 (with the exact value depending on $\mu(I)$).

On the other hand, if the generating degree is large and the number of variables is small, then the bound given by Proposition 3.2.15 is likely better. For example, if $n = 6$, $d = 3$, and $\ell = 3$, then Theorem 3.2.9 gives a bound of 1,765 while Proposition 3.2.15 gives a bound of at most 183 (with the exact value again depending on $\mu(I)$).

It is also important to note that for many examples, these bounds are not particularly close to the actual number of generators of $\overline{R[It]}$ as an $R[It]$ module. For example, if I is a normal ideal then the actual number of generators of $\overline{R[It]}$ as a module over $R[It]$ is 1 because the two rings are equal. As an explicit example, $I = (xyz^4, y^3z^3, y^4z^2, y^5z) \subset \mathbb{Q}[x, y, z]$ is a normal ideal generated in degree 6 in a ring of dimension 3 whose analytic spread is 3, thus the bounds given above are not close to the actual number of generators. This leads to the question of whether there is a better bound for certain classes of ideals, perhaps by decreasing the estimate for the number of generators of $\overline{I^k}/I^k$ in the proof of Proposition 3.2.14.

4. DEPTH OF THE INTEGRAL CLOSURE OF THE REES ALGEBRA

4.1 Depth Preliminaries

Recall that the bounds on complexity given by Ulrich and Vasconcelos in [34] required $\overline{R[It]}$ to have large depth. With this as our motivation, in this section we discuss some previous results and provide our own result on the depth of $\overline{R[It]}$.

Before discussing the depth of the integral closure of the Rees algebra, we recall several results about depth and discuss a few results about the depth of the Rees algebra itself.

We will frequently make use of the Depth Lemma to calculate the depth of modules. This lemma can be proved using the long exact sequence of Ext modules induced by an exact sequence, and the definition of depth in terms of non-vanishing of Ext modules.

Lemma 4.1.1 (Depth Lemma [2, Proposition 1.2.9])**.** *Let R be a Noetherian ring, let I be an R-ideal, and let M, M', and M'' be finite R-modules such that the sequence*

$$0 \longrightarrow M \longrightarrow M \longrightarrow M \longrightarrow 0$$

is exact. Then

(a) $\operatorname{depth}_I(M) \geq \min\{\operatorname{depth}_I(M'), \operatorname{depth}_I(M'')\}$

(b) $\operatorname{depth}_I(M') \geq \min\{\operatorname{depth}_I(M), \operatorname{depth}_I(M'')+1\}$

(c) $\operatorname{depth}_I(M'') \geq \min\{\operatorname{depth}_I(M), \operatorname{depth}_I(M')-1\}$

Definition 4.1.2. *We will denote the set of primes $p \in \operatorname{Spec}(R)$ containing an ideal I as $\mathbf{V}(I)$.*

The depth of the Rees algebra $R[It]$ is closely related to the depth of the associated graded ring $\operatorname{gr}_I(R)$, as well as to the reduction number of I. Recall that if R is Noetherian and $I \subset \operatorname{Rad}(R)$ then $\dim \operatorname{gr}_I(R) = \dim R$. There are many results about when $R[It]$ is Cohen-Macaulay. We briefly mention a few of these results, leaving out many others (for instance, results in [8], [33], [32], [7]).

Recall that by $r(I)$ we mean the reduction number of an ideal I, as defined in Definition 2.3.8. Goto and Shimoda prove that Cohen-Macaulayness of the Rees algebra is related to Cohen-Macaulayness of the associated graded ring for an m-primary ideal I in a Cohen-Macaulay local ring.

Theorem 4.1.3 (Goto-Shimoda [9, Remark 3.10]). *Let (R, m) be a Cohen-Macaulay local ring of positive dimension with infinite residue field R/m, and let I be an m-primary ideal. Then $R[It]$ is Cohen-Macaulay if and only if the two following conditions are satisfied:*

(a) $\operatorname{gr}_I(R)$ *is Cohen-Macaulay*

(b) $r(I) \leq \dim R - 1$.

Notice that when these conditions are satisfied, we get that

$$\operatorname{depth}(R[It]) = \operatorname{depth}(\operatorname{gr}_I(R)) + 1. \tag{4.1}$$

In the case where $\operatorname{gr}_I(R)$ is *not* Cohen-Macaulay, that is, when $\operatorname{depth}(\operatorname{gr}_I(R)) \lneq \dim \operatorname{gr}_I(R)$, Huckaba and Marley showed that for an m-primary ideal I in a local Cohen-Macaulay ring (R, m) with positive dimension, Equation (4.1) still holds:

Theorem 4.1.4 (Huckaba-Marley [13, Theorem 2.1]). *Let (R, m) be a Cohen-Macaulay local ring and let I be an m-primary ideal. If $\operatorname{gr}_I(R)$ is not Cohen-Macaulay, then*

$$\operatorname{depth}(R[It]) = \operatorname{depth}(\operatorname{gr}_I(R)) + 1.$$

Soon after, Huckaba and Marley were able to remove the assumption that I is m-primary:

Theorem 4.1.5 (Huckaba-Marley [14, Corollary 3.12]). *Let (R, m) be a Cohen-Macaulay local ring and let I be an R-ideal. If $\operatorname{gr}_I(R)$ is not Cohen-Macaulay, then*

$$\operatorname{depth}(R[It]) = \operatorname{depth}(\operatorname{gr}_I(R)) + 1.$$

In the same paper, Huckaba and Marley show Equation (4.1) holds when R is not Cohen-Macaulay (with no assumption on the ideal) if $\operatorname{depth}(\operatorname{gr}_I(R)) \lneq \operatorname{depth}(R)$:

Theorem 4.1.6 (Huckaba-Marley [14, Theorem 3.10])**.** *Let* (R, m) *be a Noetherian local ring and let* I *be an* R*-ideal. If* $\operatorname{depth}(\operatorname{gr}_I(R)) < \operatorname{depth}(R)$*, then*

$$\operatorname{depth}(R[It]) = \operatorname{depth}(\operatorname{gr}_I(R)) + 1.$$

Later, Johnston and Katz generalized the result of Goto and Shimoda:

Theorem 4.1.7 (Johnston-Katz [18, Theorem 2.3])**.** *Let* R *be a Cohen-Macaulay local ring with infinite residue field and let* $I \subset R$ *be an ideal of positive height. Then* $R[It]$ *is Cohen-Macaulay if and only if* $\operatorname{gr}_I(R)$ *is Cohen-Macaulay and* $r(I_p) \leq \ell(I_p)$ *for all prime ideals* $p \in V(I)$ *such that* $\ell(I_p) = \dim R_p$*.*

We finish this section with a few lemmas.

Definition 4.1.8. *An ideal* I *is called a* ***complete intersection*** *if* I *can be generated by a (possibly empty) regular sequence.*

Remark 4.1.9. In general, $\operatorname{grade}(I) \leq \operatorname{ht} I \leq \mu(I)$ (see [23, 132]). Notice that I is a complete intersection if and only if $\operatorname{grade}(I) = \operatorname{ht} I = \mu(I)$.

Lemma 4.1.10. *Let* R *be a Noetherian ring satisfying Serre's condition* S_k*. Then* $\operatorname{ht} I = \operatorname{grade}(I)$ *for every ideal* I *of height at most* k*.*

Proof. Since the grade of I can be computed locally (see [2, Proposition 1.2.10]), we can write

$$\begin{aligned}
\operatorname{grade}(I) &= \min\{\operatorname{depth}(R_p) \,|\, p \supset I\} \\
&= \min\{\{\operatorname{depth}(R_p) \,|\, p \supset I,\, \dim R_p \leq k\} \cup \{\operatorname{depth}(R_p) \,|\, p \supset I,\, \dim R_p > k\}\} \\
&= \min\{\{\dim R_p \,|\, p \supset I,\, \dim R_p \leq k\} \cup \{k\}\} \\
&\quad \text{(by } S_k\text{, } R_p \text{ is Cohen-Macaulay if } \dim R_p \leq k\text{, and otherwise } \operatorname{depth}(R_p) \geq k) \\
&= \operatorname{ht} I
\end{aligned}$$

■

Remark 4.1.11. Let R be a Noetherian local ring with infinite residue field which satisfies Serre's condition S_k, and let I be an equimultiple ideal of height at most k. Then any minimal reduction of I is a complete intersection.

Proof. Let J be a minimal reduction of I. We show $\operatorname{grade}(J) = \mu(J)$, and hence J is a complete intersection.

Now,

$$\begin{aligned}
\mu(J) &= \ell(I) \text{ since } k \text{ is infinite, see Remark 2.3.12} \\
&= \operatorname{ht} I \text{ since } I \text{ is equimultiple} \\
&= \operatorname{ht} J \text{ since } J \subset I \text{ and } JI^n = I^{n+1} \text{ for some } n \text{ implies } \sqrt{I} = \sqrt{J} \\
&= \operatorname{grade}(J) \text{ since } R \text{ satisfies Serre's condition } S_k \text{ (see Lemma 4.1.10)}
\end{aligned}$$

■

We now define filtrations and discuss generalizations of several previously mentioned results on the Cohen-Macaulayness and depth of the Rees algebra to Rees algebras of filtrations.

Definition 4.1.12. *Let R be a ring and let I be a proper R-ideal. A non-increasing sequence of ideals $\mathcal{F} = \{I_i \,|\, i \in \mathbb{N}_0\}$ with $I_iI_j \subset I_{i+j}$ is called a **multiplicative filtration**. If $II_i \subset I_{i+1}$, then the filtration is called an I**-filtration**. In the remainder of this chapter, unless otherwise noted, we will use $\mathcal{F}$ to denote a multiplicative I-filtration $\{I_i\}$.*

Definition 4.1.13. *The **Rees algebra of** $\mathcal{F}$, which we will denote as $\mathcal{R}(\mathcal{F})$, is the graded $R[It]$-algebra*

$$\mathcal{R}(\mathcal{F}) = \bigoplus_{i=0}^{\infty} I_i t^i.$$

It is typically graded by $[\mathcal{R}(\mathcal{F})]_i = I_i t^i$.

Definition 4.1.14. *The **associated graded ring of** $\mathcal{F}$, which we will denote as $G(\mathcal{F})$, is the graded $R[It]$-algebra*

$$G(\mathcal{F}) = \bigoplus_{i=0}^{\infty} I_i/I_{i+1}.$$

It is typically graded by $[G(\mathcal{F})]_i = I_i/I_{i+1}$.

Definition 4.1.15. *An ideal $J \subset I$ is a* ***reduction*** *of $\mathcal{F}$ if $I_{n+1} = JI_n$ for all $n >> 0$. The* ***reduction number of $\mathcal{F}$ with respect to*** *J, denoted $r_J(\mathcal{F})$, is the smallest non-negative integer r such that $I_{n+1} = JI_n$ for all $n \geq r$.*

We are interested in the integral closure filtration $\mathcal{F} = \{\overline{I^n}\}$ of an R-ideal I. Notice that this is, in fact, a multiplicative I-filtration since $\overline{I^i}\,\overline{I^j} \subset \overline{I^{i+j}}$ and $I\overline{I^i} \subset \overline{I}\,\overline{I^i} \subset \overline{I^{i+1}}$ for all i and j. When $\mathcal{F}$ is the integral closure filtration, $\mathcal{R}(\mathcal{F}) = \oplus_{i=0}^{\infty}\overline{I^i}t^i$, which is simply $\overline{R[It]}$, the integral closure of the Rees algebra of I in $R[t]$. Also notice that for any reduction J of I we have that $R[Jt] \subset R[It] \subset \mathcal{R}(\mathcal{F}) = \overline{R[It]}$ are all integral extensions. Hence $R[Jt] \subset \mathcal{R}(\mathcal{F})$ is an integral extension of graded rings.

Many of the results above relating the depth of the Rees algebra to the depth of the associated graded ring can be generalized to the Rees algebra and associated graded ring of filtrations. In particular, Nishida provides a bound for the depth of the associated graded ring of a filtration. This result generalizes a previous result of Ghezzi in [6, Theorem 3.2.10] for the filtration $\{I^n\}$, which itself was a generalization of several of the previously discussed results. In the case where $\mathcal{F}$ is the integral closure filtration, Nishida's result gives the following:

Theorem 4.1.16 (Nishida [24, Theorem 1.1])**.** *Let R be a Cohen-Macaulay local ring. Let $\mathcal{F} = \{\overline{I^n}\}$ be the integral closure filtration of an ideal I. Let $\ell = \ell(I)$ be the analytic spread of I, and assume there exist elements $a_1, \ldots, a_\ell$ in $\overline{I}$ such that $J = (a_1, \ldots, a_\ell)$ is a reduction of $\mathcal{F}$ with reduction number $r = r_J(\mathcal{F})$. For $1 \leq i \leq \ell$, let $J_i = (a_1, \ldots, a_i)$. Assume the following conditions are satisfied:*

(a) *For all $p \in V(I)$ with $\operatorname{ht} p = i < \ell$, $(J_i)_p$ is a reduction of $\mathcal{F}_p$ with $r_{J_i}(\mathcal{F}_p) \leq \max\{0, r - \ell + i\}$*

(b) *For $0 \leq i < \ell - r$, $\operatorname{depth}(R/(J_i : \overline{I}))_p \geq \operatorname{ht} p - i$ for all $p \in \operatorname{Spec}(R)$*

(c) *For $0 < i < \ell - r$, $a_i \notin q$ for any $q \in \operatorname{Ass}(R/J_{i-1}) \setminus V(I)$*

(d) *If* $0 \leq i < \ell$ *and* $n \leq i - \ell + r$*, then* $\operatorname{depth}_{(R/\overline{I^n})_p}(\geq) \min\{\operatorname{ht} p - i, \ell - i\}$ *for all* $p \in V(I)$.

Then the depth of $G(\mathscr{F})$ *is at least*

$$\min\{\{d\} \cup \{\operatorname{depth}(R/\overline{I^n}) + i \,|\, 0 \leq i \leq \ell \text{ and } n \leq r - \ell + i\}\}.$$

In our main result of this chapter, Theorem 4.2.1, we give alternate conditions for which we can find a lower bound on the depth of the integral closure of the Rees algebra and the associated graded ring of the integral closure filtration.

4.2 Depth of the Integral Closure of the Rees Algebra

For the remainder of this chapter, we discuss our own results regarding the depth of the integral closure of the Rees algebra, culminating in a particularly nice characterization of when $\overline{R[It]}$ is Cohen-Macaulay for height 2 ideals.

The following lemma gives a formula for the depth of the integral closure of the Rees algebra. A major assumption is that the reduction number of the integral closure filtration is at most one. When this is the case, then by definition for any $n \geq 1$

$$\overline{I^n} = J\overline{I^{n-1}} = J^2\overline{I^{n-2}} = \cdots = J^{n-1}\overline{I}$$

and thus $\mathcal{R}(\mathcal{F}) = R[\overline{I}t]$ and we can use known results about the Rees algebra of an ideal. In Theorem 4.2.2 we discuss conditions that imply that the reduction number $r_J(\mathcal{F})$ is at most 1.

Theorem 4.2.1. *Let* (R, m) *be a Cohen-Macaulay local ring of dimension* d*, and let* I *be an equimultiple ideal of height* g *with* $\operatorname{depth}(R/\overline{I}) = k$*. Let* $\mathcal{F} = \{\overline{I^n}\}$ *be the integral closure filtration of* I *and suppose* $r_J(\mathcal{F}) \leq 1$ *for some minimal reduction* J *of* I*. Additionally, assume either that* J *is a complete intersection or that* R/m *is infinite. Then*

$$\operatorname{depth}(G(\mathcal{F})) \geq k + g$$

and

$$\operatorname{depth}(\mathcal{R}(\mathcal{F})) \geq \min\{d, \;\; k+g+1\}.$$

In particular, if $R/\overline{I}$ is Cohen-Macaulay then $G(\mathcal{F})$ is Cohen-Macaulay.

Moreover, if $g \geq 2$ then we obtain the equalities

$$\operatorname{depth}(G(\mathcal{F})) = k+g$$

and

$$\operatorname{depth}(\mathcal{R}(\mathcal{F})) = k+g+1,$$

thus $\mathcal{R}(\mathcal{F})$ is Cohen-Macaulay if and only if $R/\overline{I}$ is Cohen-Macaulay.

Proof. Let J be a minimal reduction of I such that $r_J(\mathcal{F}) \leq 1$. By assumption, J is a complete intersection or R/m is infinite. In either case, J is a complete intersection. Notice this implies that $\mathcal{F} = \{\overline{I}^n\}$ (since $\overline{I}^n \subset \overline{I^n} \subset J^{n-1}\overline{I} \subset \overline{I}^n$ implies equality throughout). Thus $\mathcal{R}(\mathcal{F}) = R[\overline{I}t]$ and $G(\mathcal{F}) = G(\overline{I})$.

If $\operatorname{ht} I = 0$, then $\overline{I^n} = \sqrt{0}$ for all $n \geq 1$ and $\sqrt{0} = \overline{I^2} \subset 0 \cdot \overline{I} = 0$. Thus $\mathcal{R}(\mathcal{F}) = G(\mathcal{F}) = R$. So we may assume $\operatorname{ht} I > 0$.

Notice since J is generated by a regular sequence, hence by a quasi-regular sequence, $G(J) \cong (R/J)[T_1, \dots T_g]$. Moreover, since $J^k \otimes_R (R/\overline{I}) \cong (J^k/J^{k+1}) \otimes_R (R/\overline{I})$, the cokernel of the natural inclusion map $\overline{I}R[Jt] \hookrightarrow R[Jt]$ is $R[Jt] \otimes_R (R/\overline{I}) \cong G(J) \otimes_R (R/\overline{I})$.

We apply the Depth Lemma to the short exact sequence

$$0 \longrightarrow \overline{I}R[Jt] \longrightarrow R[Jt] \longrightarrow (R/\overline{I})[T_1, \dots, T_g] \longrightarrow 0$$

Because J is generated by a regular sequence, R/J is Cohen-Macaulay and hence so is $G(J) \cong (R/J)[T_1, \dots T_g]$. Hence by the work of Johnston and Katz (see Theorem 4.1.7), $R[Jt]$ is Cohen-Macaulay. Thus $\operatorname{depth}(R[Jt]) = \dim R[Jt] = d+1$ since $\operatorname{grade}(J) > 0$. We also have that $\operatorname{depth}((R/\overline{I})[T_1, \dots, T_g]) = k+g$. Notice $k+g = \operatorname{depth}(R/\overline{I}) + \operatorname{ht} I \leq \dim R/\overline{I} + \operatorname{ht} \overline{I} \leq d$. Thus by applying the Depth Lemma, we see $\operatorname{depth}(\overline{I}R[Jt]) = k+g+1$.

Next we apply the Depth Lemma to the short exact sequence

$$0 \longrightarrow \overline{I}tR[\overline{I}t] \longrightarrow R[\overline{I}t] \longrightarrow R \longrightarrow 0 \tag{4.2}$$

By the above, we have $\operatorname{depth}(\overline{I}tR[\overline{I}t]) = k + g + 1$. Since R is Cohen-Macaulay, $\operatorname{depth}(R) = d$. Thus by the Depth Lemma $\operatorname{depth}(R[\overline{I}t]) \geq \min\{d,\, k+g+1\}$. This proves the second statement of the proposition.

Next, to find the depth of $G(\mathcal{F})$ we apply the Depth Lemma to the short exact sequence

$$0 \longrightarrow \overline{I}R[\overline{I}t] \longrightarrow R[\overline{I}t] \longrightarrow G(\overline{I}) \longrightarrow 0$$

and obtain $\operatorname{depth}(G(\overline{I})) \geq \min\{d,\, k+g\} = k+g$.

Moreover, if $g \geq 2$, then by Johnston and Katz's theorem (see Theorem 4.1.7) we have that $G(\overline{I})$ is Cohen-Macaulay if and only if $R[\overline{I}t]$ is Cohen-Macaulay. If $k+g = d$ then $G(\overline{I})$ and $R[\overline{I}t]$ are Cohen-Macaulay, necessarily of depths $k+g$ and $k+g+1$, by the above.

If $k+g < d$ then by the above $\operatorname{depth}(R[\overline{I}t]) = \min\{d,\, k+g+1\} < d+1$. So $R[\overline{I}t]$ is not Cohen-Macaulay. Hence by Johnston and Katz's theorem (see Theorem 4.1.7) $G(\overline{I})$ is not Cohen-Macaulay, and thus by the theorem of Huckaba and Marley (see Theorem 4.1.6), $\operatorname{depth}(G(\overline{I})) = \operatorname{depth}(R[\overline{I}t]) - 1 = k+g$. ■

The previous result relies on the reduction number of $\mathcal{F}$ being small. In the following theorem, we discuss conditions that ensure this.

Theorem 4.2.2. *Let (R, m, k) be a Noetherian local ring that is universally catenary and satisfies Serre's conditions S_3 and R_2, and let I be an equimultiple ideal of height 2. Then the integral closure filtration $\mathcal{F}$ of I has reduction number at most 1 over any minimal reduction of I.*

Proof. Let J be a minimal reduction of I, and let $R(X) = R[X]_{mR[X]}$. Then $JR(X)$ is a reduction of $IR(X)$. If K is a minimal reduction of $JR(X)$ and hence of $IR(X)$ and we prove that the reduction number of the integral closure filtration $\{\overline{I^nR(X)}\}$ with respect to K is at most 1, then the reduction number with respect to $JR(X)$ is at most 1. Thus

$JR(X)\overline{I^n}R(X) = JR(X)\overline{I^nR(X)} = \overline{I^{n+1}R(X)} = \overline{I^{n+1}}R(X)$ for all n and so $r_J(\mathcal{F}) \leq 1$. Letting $R = R(X)$ and $J = K$, we may assume the residue field R/m is infinite.

Then since R satisfies S_2 and has an infinite residue field, J is generated by a regular sequence of length two by Remark 4.1.11. We will prove that $J^n\overline{I} = \overline{I^{n+1}}$ for all $n \geq 1$, which we will use to show that $r_J(\mathcal{F}) \leq 1$. Since it is always true that

$$J^n\overline{I} \subset \overline{I^n}\,\overline{I} \subset \overline{I^{n+1}}$$

for all n, to check that $J^n\overline{I} = \overline{I^{n+1}}$ it suffices to check equality locally at associated primes of $J^n\overline{I}$.

We claim that the ideal $\overline{I}J^n$ is unmixed of height 2. From the short exact sequence

$$0 \longrightarrow J^n/J^n\overline{I} \longrightarrow R/J^n\overline{I} \longrightarrow R/J^n \longrightarrow 0$$

it follows that

$$\mathrm{Ass}(R/J^n\overline{I}) \subset \mathrm{Ass}(R/J^n) \cup \mathrm{Ass}(J^n/J^n\overline{I}).$$

We show that $\mathrm{Ass}(R/J^n)$ and $\mathrm{Ass}(J^n/\overline{I}J^n)$ consist only of height 2 prime ideals.

From the short exact sequence

$$0 \longrightarrow J^{n-1}/J^n \longrightarrow R/J^n \longrightarrow R/J^{n-1} \longrightarrow 0$$

we obtain

$$\mathrm{Ass}(R/J^n) \subset \mathrm{Ass}(R/J^{n-1}) \cup \mathrm{Ass}(J^{n-1}/J^n).$$

Proceeding inductively, we see that

$$\mathrm{Ass}(R/J^n) \subset \bigcup_{k=1}^{n} \mathrm{Ass}(J^{k-1}/J^k).$$

Since $J^{k-1}/J^k = [G(J)]_{k-1}$ and $G(J)$ is a polynomial ring over R/J (because J is generated by a regular sequence), we see that J^{k-1}/J^k is a free (R/J)-module. Therefore,

$\operatorname{Ass}(J^{k-1}/J^k) \subset \operatorname{Ass}(R/J)$ for $1 \leq k \leq n$. Hence $\operatorname{Ass}(R/J^n) \subset \operatorname{Ass}(R/J)$. Since J is a complete intersection of height 2 and R satisfies Serre's condition S_3, $\operatorname{Ass}(R/J)$ consists of prime ideals of height 2. Therefore, $\operatorname{Ass}(R/J^n)$ consists only of primes of height 2; that is, J^n is unmixed of height 2.

Next, notice that

$$\begin{aligned} J^n/J^n\overline{I} &\cong J^n \otimes_R R/\overline{I} \\ &\cong (J^n \otimes_R R/J) \otimes_{R/J} R/\overline{I} \\ &\cong J^n/J^{n+1} \otimes_{R/J} R/\overline{I} \end{aligned}$$

By the above argument, J^n/J^{n+1} is a free R/J-module, hence

$$\begin{aligned} J^n/J^{n+1} \otimes_{R/J} R/\overline{I} &\cong (\oplus R/J) \otimes_{R/J} R/\overline{I} \\ &\cong \oplus R/\overline{I}. \end{aligned}$$

Since $J^n/J^n\overline{I}$ is a free $R/\overline{I}$-module, $\operatorname{Ass}(J^n/\overline{I}J^n) \subset \operatorname{Ass}(R/\overline{I})$. Since J is a reduction of I, $\overline{I} = \overline{J}$.

Notice that since R is universally catenary and satisfies S_2, R is equidimensional by Lemma 2.4.20. Thus R is formally equidimensional by a theorem of Ratliff in [26]. Then applying another theorem of Ratliff ([27, Theorem 2.12]) to the parameter ideal J (recall the definition of a parameter ideal from Definition 2.3.14), we see that every associated prime of $\overline{J} = \overline{I}$ has height 2. Therefore, $J^n\overline{I}$ is unmixed of height 2. This proves the claim.

Now, let p be an associated prime of $J^n\overline{I}$. Then localizing at p, a prime of height 2, we may assume that R is a regular local ring of dimension 2 because R satisfies Serre's condition R_2. Then

$$\overline{J^{n+1}} \subset J^n$$

by work of Lipman and Sathaye ([19, Theorem 1]). By work of Itoh ([16, Theorem 1]),

$$J^n \cap \overline{J^{n+1}} = J^n\overline{J} \text{ for all } n \geq 1.$$

Since J is a reduction of I, it follows that J^n is a reduction of I^n for any n, and hence $\overline{I^n} = \overline{J^n}$ for any n. Therefore we may rewrite the two results above as

$$\overline{I^{n+1}} \subset J^n$$

and

$$J^n \cap \overline{I^{n+1}} = J^n\overline{I}.$$

Combining these two results,

$$\overline{I^{n+1}} \subset J^n\overline{I}.$$

This confirms that $\overline{I^{n+1}} = J^n\overline{I}$ for all $n \geq 1$. Then for $n \geq 1$,

$$\overline{I^{n+1}} = J^n\overline{I} = J(J^{n-1}\overline{I} \subset J\overline{I^n}.$$

Hence $r_J(\mathcal{F}) \leq 1$.

∎

We conclude this chapter with an application of Theorem 4.2.1 and Theorem 4.2.2, which provides a nice characterization of when the integral closure of the Rees algebra is Cohen-Macaulay.

Theorem 4.2.3. *Let (R, m) be a Cohen-Macaulay local ring that satisfies Serre's condition R_2, and let I be an equimultiple ideal of height 2 with* $\operatorname{depth}(R/\overline{I}) = k$. *Then* $\operatorname{depth}(\overline{R[It]}) = k + 3$. *In particular, $R/\overline{I}$ is Cohen-Macaulay if and only if $\overline{R[It]}$ is Cohen-Macaulay.*

Proof. If R/m is not infinite, we may again reduce to the case where it is infinite as in the proof of Theorem 4.2.2, by considering $R(X)$. Let $\mathcal{F} = \{\overline{I^n}\}$ be the integral closure filtration of I, and let J be a minimal reduction of I. Then by Theorem 4.2.2, $r_J(\mathcal{F}) \leq 1$. Hence by Theorem 4.2.1, $\operatorname{depth}(\overline{R[It]}) = \operatorname{depth}(\mathcal{R}(\mathcal{F})) = k + 3$.

Since R is local, equidimensional and catenary, $\dim(R/\overline{I}) = \dim R - 2$ by Lemma 2.4.19. Therefore, $\overline{R[It]}$ is Cohen-Macaulay if and only if $k = \dim R - 2$, which is true if and only if $R/\overline{I}$ is Cohen-Macaulay. ■

Remark 4.2.4. In Theorem 4.2.3, unlike in Theorem 4.2.1, we do not need to assume that either J is a complete intersection or k is infinite since we do not need to assume that $JR(X)$ is a minimal reduction of $\mathcal{F}R(X)$.

5. SPECIALIZATION OF INTEGRAL CLOSURE

5.1 Specialization Preliminaries

In this chapter we prove that for ideals of height at least two in a large class of rings, the integral closure of the ideal is compatible with specialization modulo general elements of the ideal.

By specialization, we mean taking the quotient of a ring R by an element of R. Saying that integral closure is compatible with specialization for a certain class of elements means that for those elements,

$$\overline{I}\,(R/(a)) = \overline{I\,(R/(a))}.$$

If a is in I, this is equivalent to saying that $\overline{I}/(a) = \overline{I/(a)}$. Notice that by Remark 2.3.5 (in particular, by persistence, Item (f)), it is always true that $\overline{I}\,(R/(a)) \subset \overline{I\,(R/(a))}$. It is not always the case that the reverse containment holds, even for m-primary monomial ideals in a polynomial ring, as seen in the following example.

Example 5.1.1. Let $R = k[x, y]$ be a polynomial ring in two variables over a field k, and let I be the integrally closed ideal $I = (x^2, xy, y^2)$. After specialization by the element x^2, we see that x is integral over $I/(x^2)$ since it satisfies the equation of integral dependence $x^2 = 0$ in $R/(x^2)$. Since $x \notin I/(x^2)$, we see that $I/(x^2)$ is not integrally closed, even though I was integrally closed, so $\overline{I}/(x^2) \subsetneq \overline{I/(x^2)}$.

Our results, and other results from the literature, show that in the case where a is "sufficiently random" (which we define more precisely in Definition 5.1.2 and Definition 5.1.3), then for a large class of ideals in a large class of rings, it *is* true that integral closure is compatible with specialization.

Definition 5.1.2. *Let R be a ring, let $I = (a_1, \ldots, a_n)$ be an ideal generated by n elements, and let $S = R[X_1, \ldots, X_n]$ be a polynomial ring over R in n variables. If (R, m) is a local ring, we may alternatively let $S = R[X_1, \ldots, X_n]_{mR[X_1,\ldots,X_n]}$ be the localization of a polynomial ring. An element $a = \sum_{i=1}^{n} X_i a_i$ is called a **generic element** of I.*

We now define a general element of an ideal I in an algebra over an infinite field.

Definition 5.1.3. *Let k be an infinite field, let R be a k-algebra, and let $I = (a_1, \dots, a_n)$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n$, let $a_\alpha = \sum_{i=1}^{n} \alpha_i a_i$. Then a property P holds for a **general element** of I if there is a dense open subset U in affine n-space k^n such that P holds for a_α whenever $\alpha \in U$.*

Remark 5.1.4. We can extend the definition of a general element to an ideal $I = (a_1, \dots, a_n)$ of a local ring (R, m) which has infinite residue field k. For $\alpha = (\alpha_1, \dots, \alpha_n) \in R^n$, we say a property P holds for a **general element** of I if there is a dense open subset U of affine n-space k^n such that the property P holds for $a_\alpha = \sum_{i=1}^{n} \alpha_i a_i$ whenever the image $\overline{\alpha}$ of $\alpha = (\alpha_1, \dots, \alpha_n) \in R^n$ in k^n belongs to U.

Remark 5.1.5. Since the intersection of finitely many dense open subsets of k^n, where k is an infinite field, is again a dense open subset of k^n, it follows that if finitely many properties $P_1, \dots, P_s$ hold for a general element of I, then the intersection of all the properties, $P_1 \wedge \cdots \wedge P_s$, also holds for a general element of I.

In [17] Itoh proved specialization by a generic element of I is compatible with the integral closure of I when I is a parameter ideal in an analytically unramified local Cohen-Macaulay ring of dimension at least 2 (recall the definition of a parameter ideal from Definition 2.3.14):

Theorem 5.1.6 (Itoh [17, Theorem 1])**.** *Let (R, m) be an analytically unramified Cohen-Macaulay local ring of dimension $d \geq 2$ with R/m infinite. Let $I = (a_1, \dots, a_d)$ be a parameter ideal of R. Let $S = R[X_1, \dots, X_d]_{mR[X_1,\dots,X_d]}$ be the localization of a polynomial ring in the variables $X_1, \dots, X_d$, and let $x = \sum_{i=1}^{d} X_i a_i$ be a generic element of I in S. Then $\overline{IS/(x)S} = \overline{I}(S/(x)S)$.*

Hong and Ulrich extend Itoh's result to a larger class of ideals in a larger class of rings in [12]:

Theorem 5.1.7 (Hong-Ulrich [12, Theorem 2.1])**.** *Let R be a Noetherian, locally equidimensional, universally catenary ring such that $R_{\text{red}} = R/\sqrt{0}$ is locally analytically unramified. Let $I = (a_1, \dots, a_n)$ be an R-ideal of height at least 2. Let $S = R[X_1, \dots, X_n]$ be a polynomial ring in the variables $X_1, \dots, X_n$, and let $x = \sum_{i=1}^{n} X_i a_i$ be a generic element of I in S. Then $\overline{IS/(x)S} = \overline{IS}/(x)S$.*

Remark 5.1.8. Notice that Itoh proves specialization commutes with integral closure for parameter ideals in an analytically unramified, Cohen-Macaulay local ring. Such a ring R satisfies the conditions of Hong and Ulrich's theorem: it is a Noetherian, locally equidimensional, universally catenary ring such that R_{red} is locally analytically unramified. Since R has dimension at least 2, any parameter ideal I will also have height at least 2. Finally, if $\overline{I/(x)} = \overline{I}/(x)$ where x is a generic element of I in a polynomial ring over R, then specialization also commutes with integral closure of I after localization, as Hong and Ulrich state in [12, Corollary 2.3]. So the result of Hong and Ulrich is truly an extension of Itoh's result.

These results have many applications. In [16], Itoh uses that specialization by generic elements commutes with integral closure to prove that for an ideal I generated by a regular sequence $x_1, \ldots, x_r$ in a Noetherian local ring, $I^n \cap \overline{I^{n+1}} = I^n \overline{I}$ (Huneke independently proved this result in [15]); Huneke's proof of this theorem partially inspired the creation of tight closure. Hong and Ulrich use Theorem 5.1.7 to show the integral closure of a module M is compatible with specialization by a generic element of the module through the use of a construction known as the generic Bourbaki ideal.

We can also use specialization of integral closure to obtain a quick proof of a result of Ma, Quy, and Smirnov:

Corollary 5.1.9 (Ma-Quy-Smirnov [20, Corollary 12])**.** *Let (R, m) be a Noetherian local formally equidimensional ring. Then for every m-primary integrally closed ideal I, we have $e(I) \geq \lambda(R/I)$.*

Proof. Let $\widehat{R}$ represent the completion of R with respect to m. Since $e(I) = e(I\widehat{R})$, since $\lambda(R/I) = \lambda(\widehat{R}/I\widehat{R})$, and since $I\widehat{R}$ is still integrally closed (see, for instance, [31, Lemma 9.1.1]), we may pass to $\widehat{R}$ to assume that R is not only Noetherian local and locally equidimensional, but also complete. Since $\widehat{R}$ is a Noetherian complete local ring, it is in addition universally catenary, and $\widehat{R}$ is analytically unramified, so the assumptions of [12, Corollary 2.3] are satisfied. Thus, if $\operatorname{ht} I \geq 2$, we may specialize by a generic element x of I to decrease the height by 1. Notice this preserves the assumptions of being Noetherian local, equidimensional (since we may assume x is part of a system of parameters), and excellent.

Case 1: Assume that $\mathrm{ht}(I) = \dim R = 0$. Then since $\dim \mathrm{gr}_I(R) = \dim R = 0$, $e(I) = e(\mathrm{gr}_I(R)) = \lambda(\mathrm{gr}_I(R))$. Since R/I embeds into $\mathrm{gr}_I(R)$, $\lambda(R/I) \leq \lambda(\mathrm{gr}_I(R))$ and hence $\lambda(R/I) \leq e(I)$.

Case 2: Assume that $\mathrm{ht}(I) = \dim R = 1$. Since I is integrally closed, $\sqrt{0} \subset I$, and therefore $R_{red}/IR_{red} \cong R/I$. Therefore, $\lambda(R_{red}/IR_{red}) = \lambda(R/I)$. Since $e(IR_{red}) \leq e(I)$, it suffices to prove that $\lambda(R/I) \leq e(I)$ when R is reduced. Since $\dim R = 1$ and R is reduced, R is Cohen-Macaulay. Let $a \in S = R[X_1, \ldots, X_n]_{mR[X_1,\ldots,X_n]}$ be a generic element of I. Then (a) is a parameter ideal in a Cohen-Macaulay ring, and hence $\lambda(S/(a)) = e((a))$. Since $S/(a)$ surjects onto S/IS, $\lambda(R/I) = \lambda(S/IS) \leq \lambda(S/(a))$. Since (a) is a reduction of IS, $e((a)) = e(IS) = e(I)$. Therefore, $\lambda(R/I) \leq e(I)$.

If $\mathrm{ht}\, I = \dim R \geq 2$, we can use specialization to reduce to one of the previous cases. Let x be a generic element of I. Then $IS/(x)$ is integrally closed by [12, Corollary 2.3] , with height $\mathrm{ht}\, I - 1$ and $\lambda(S/(x)/IS/(x)) = \lambda(S/IS) = \lambda(R/I)$ (see, for instance, [31, Lemma 8.4.2]). By [31, Lemma 11.1.9], $e(I) = e(IS) = e(IS/(x))$ because we may assume x is a superficial element of IS. Thus, $\lambda(S/(x)/IS/(x)) \leq e(IS/(x))$ implies $\lambda(R/I) \leq e(I)$. ■

Remark. If R contains a field of characteristic 0, we can alternatively use our main result of this chapter, Theorem 5.2.4, to obtain this result.

Specialization by a generic element involves extending the original ring R, either to a polynomial ring over R or a localization thereof. This is often sufficient for proofs using induction on the height of the ideal when considering properties preserved under faithfully flat extensions. However, there are cases where we may not want to change the ring. The core of the ideal, which is the intersection of all reductions of an ideal, may not be preserved under ring change. Similarly, properties of the residue field, such as being perfect or algebraically closed, may not pass to extensions of R. To preserve the original ring we need to instead consider specialization by general elements of the ideal, which come from the original ring. In our main result, Theorem 5.2.4, we show that for many k-algebras, where k is a field of characteristic zero, specialization by a general element of the ideal is compatible with integral closure if the height of the ideal is at least 2.

An important breakthrough for specialization by general elements came in Flenner's paper [5]. In his paper, Flenner proves there is a local version of Bertini's second classical theorem, positively answering a question posed by Grothendieck about whether Serre's conditions pass to specializations of rings. We recall Flenner's theorems in Theorem 5.1.10 and Corollary 5.1.11. Recall that many properties, such as a ring being normal or reduced, are equivalent to satisfying certain of Serre's conditions. This result is vital to the proof of our main theorem, as it allows us to pass Serre's conditions to specializations of $\overline{R[It]}$.

Theorem 5.1.10 (Flenner [5, Corollary 4.7])**.** *Let (S, m) be a local excellent k-algebra over the field k of characteristic 0, let $I = (x_1, \ldots, x_n) \subset m$, and let $U = \operatorname{Spec}(S) \setminus V(I)$, where $V(I) = \{p \in \operatorname{Spec}(R) \mid I \subset p\}$. Assume that for primes $p \in U$, the ring S_p satisfies Serre's conditions S_r and/or R_s. For general $\alpha \in k^n$, let $x_\alpha = \sum_{i=1}^{n} \alpha_i x_i$ be a general element of I in S. Then for $p \in U \cap V(x_\alpha)$ the ring $(S/x_\alpha S)_p$ also satisfies the conditions S_r and/or R_s.*

In particular, normality is preserved by specialization by a general element in the following way:

Corollary 5.1.11 (Flenner [5, Corollary 4.8])**.** *Let (S, m) be a local excellent k-algebra over the field k of characteristic 0 and let $I = (x_1, \ldots, x_n) \subset m$. Let $\operatorname{Nor}(S) = \{p \in \operatorname{Spec}(S) \mid S_p$ is normal$\}$, and let $D(I) = \{p \in \operatorname{Spec}(R) \mid I \not\subset p\}$. For general $\alpha \in k^n$, let $x_\alpha = \sum_{i=1}^{n} \alpha_i x_i$, as in Theorem 5.1.10. Then*

$$\operatorname{Nor}(S) \cap V(x_\alpha) \cap D(I) \subset \operatorname{Nor}(S/x_\alpha S).$$

Next, we note a lemma of Hong and Ulrich which they use in [12] to prove their result of specialization of integral closure by generic elements. We will use a slight modification of this statement in our proof of specialization by a general element. We note that the statement of this lemma includes the Rees valuations of I. For this dissertation, it is enough to know that Rees valuations exist and are unique, therefore the number e in the below lemma is well defined; more information about Rees valuations can be found in [31, Chapter 10]. The statement of item (c) below is a consequence of [12, Lemma 1.1(c)], although it is not the stated result.

Lemma 5.1.12 (Hong-Ulrich [12, Lemma 1.1])**.** *Let R be a Noetherian, equidimensional, universally catenary local ring of dimension d such that $R_{\mathrm{red}} = R/\sqrt{0}$ is analytically unramified. Let $I = (a_1, \ldots, a_n)$ be a proper R-ideal with $\operatorname{ht} I > 0$ and write $\mathcal{A} = R[It, t^{-1}]$ for the extended Rees ring of I. Let $v_1, \ldots, v_r$ be the Rees valuations of I, and let e be the least common multiple of the values $v_1(I), \ldots, v_r(I)$ of I. Let u be a variable with $u^e = t$ and $\deg u = 1/e$. Write $\mathcal{S} = \mathcal{A}\,[u^{-1}]$ and let $\overline{\mathcal{S}}$ be the integral closure of $\mathcal{S}$ in $R[u, u^{-1}]$.*

(a) Let $\overline{\mathcal{S}_{red}}$ denote the integral closure of $\mathcal{S}_{red}$ in $R_{\mathrm{red}}[u, u^{-1}]$. Then the R-algebra $\overline{\mathcal{S}_{red}}$ is finitely generated and graded by $(1/e)\mathbb{Z}$, has a unique maximal homogeneous ideal, which is a maximal ideal, and is equidimensional of dimension $d+1$.

(b) One has the equality $\overline{\mathcal{S}}/u^{-1}\overline{\mathcal{S}} = \overline{\mathcal{S}_{red}}/u^{-1}\overline{\mathcal{S}_{red}}$. This R-algebra is finitely generated and graded by $(1/e)\mathbb{Z}_{\geq 0}$, has a unique maximal homogeneous ideal, is equidimensional of dimension d, and is reduced.

(c) $\operatorname{grade}(It(\overline{\mathcal{A}}/t^{-1}\overline{\mathcal{A}})) \geq 1$.

Remark 5.1.13. Using the notation of Lemma 5.1.12, let x be a general element of I. Notice xt is a general element of It. Since it is a general condition to avoid finitely many primes, by item (c) above we may assume that xt is regular on $\overline{\mathcal{A}}/t^{-1}\overline{\mathcal{A}}$. equivalently, we may assume that t^{-1}, xt is a regular sequence on $\overline{\mathcal{A}}$.

A final ingredient in the proof of our main theorem in this chapter, Theorem 5.2.4, is the vanishing of local cohomology of the integral closure of the extended Rees ring of I. We first recall the definition and a few relevant properties of local cohomology, and refer the reader to [2, Section 3.5] for a more detailed discussion in the case $I = m$.

Definition 5.1.14. *Let R be a ring and let I be an R-ideal. For an R-module M, set*

$$\begin{aligned}\Gamma_I(M) &= \{x \in M \mid I^n x = 0 \textit{ for some } n \geq 0\}\\ &= \bigcup_{n\geq 0} (0 :_M I^n)\\ &\cong \varinjlim_n \operatorname{Hom}_R(R/I^n, M).\end{aligned}$$

*The functor $\Gamma_I(-)$ is called the **section functor** with respect to I.*

*The i^{th} **local cohomology functor** with respect to I, denoted $H^i_I(-)$, is the i^{th} right derived functor of $\Gamma_I(-)$, and is naturally isomorphic to*

$$H^i_I(-) \cong \varinjlim \operatorname{Ext}^i_R(R/I^n, -).$$

Remark 5.1.15. (Properties of Local Cohomology)

(a) Because $H^i_I(-)$ is a right derived functor, a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

induces a natural long exact sequence of local cohomology

$$0 \longrightarrow \Gamma_I(M') \longrightarrow \Gamma_I(M) \longrightarrow \Gamma_I(M'') \longrightarrow H^1_I(M') \longrightarrow \cdots$$

(b) If I and J are finitely generated ideals with $\sqrt{I} = \sqrt{J}$, then $H^i_I(M) = H^i_J(M)$ for all i.

(c) The I-depth of a module is the smallest i such that local cohomology does not vanish: If R is a Noetherian ring and M is a finite R-module, then

$$\operatorname{depth}_I(M) = \min\{i \mid H^i_I(M) \neq 0\}.$$

(d) Local cohomology can be computed via the Čech complex: Let $I = (x_1, \ldots, x_n)$, and let $\underline{x} = x_1, \ldots, x_n$. The Čech complex of M with respect to $\underline{x}$, denoted $C^\bullet(\underline{x}; M)$, is defined as

$$C^\bullet(\underline{x}, M) = C^\bullet(x_1) \otimes_R \cdots \otimes_R C^\bullet(x_n) \otimes_R M$$

where

$$C^\bullet(x) : 0 \longrightarrow R \longrightarrow R_{x_i} \longrightarrow 0$$

is the cochain complex concentrated in cohomological degrees 0 and 1. Here R_{x_i} denotes the localization of R at the element x_i. If R is Noetherian, then $H^i_I(M) \cong H^i(C^\bullet(\underline{x}; M))$, the i^{th} cohomology of the Čech complex of M.

Hong and Ulrich use the following theorem in their proof of compatibility of integral closure with specialization by generic elements. We are able to use the same result in our proof that integral closure is compatible with specialization by general, rather than generic, elements.

Theorem 5.1.16 (Hong-Ulrich [12, Theorem 1.2])**.** *Let R be a Noetherian, locally equidimensional, universally catenary ring such that R_{red} is locally analytically unramified. Let I be a proper R-ideal with $\operatorname{ht} I > 0$, $\mathcal{A} = R[It, t^{-1}]$ the extended Rees ring of I, and $\overline{\mathcal{A}}$ the integral closure of $\mathcal{A}$ in $R[t, t^{-1}]$. Let J be an $\mathcal{A}$-ideal of height at least 3 generated by t^{-1} and homogeneous elements of positive degree. Then $[H^2_J(\overline{\mathcal{A}})]_n = 0$ for all $n \leq 0$, where $[\ \ ]_n$ denotes the degree n component.*

Lemma 5.1.17. *Let R be a Noetherian ring, M an R-module, and $J = (a_1, \ldots, a_n)$ an R-ideal. If $H^0_J(M) = M$, then $H^i_J(M) = 0$ for $i \geq 1$.*

Proof. Consider the Čech complex

$$C^\bullet(\underline{a}, M) : 0 \longrightarrow M \xrightarrow{\varphi} \oplus_{i=1}^n M_{a_i} \longrightarrow \oplus_{0 \leq i \leq j \leq n} M_{a_i a_j} \longrightarrow \cdots$$

and recall that $H^i_J(M) \cong H^i(C^\bullet(\underline{a}, M))$ for all i. In particular, $M = H^0_J(M) = \ker \varphi$, thus $\varphi = 0$ and hence $M_{a_i} = 0$ for all i. Thus any further localization of M is also zero. Therefore the Čech complex has the form

$$C^\bullet(\underline{a}, M) : 0 \longrightarrow M \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

So $H^i_J(M) = 0$ for all $i \geq 1$. ■

5.2 Specialization by General Elements

In this section we prove in Theorem 5.2.4 that integral closure of an ideal is compatible with specialization by a general element of the ideal. First, we state and prove a few technical lemmas necessary to reduce to the case where R is a normal reduced ring. Lemma 5.2.1 allows us to restrict our focus to reduced rings, Lemma 5.2.2 allows us to restrict our focus further to normal rings, and Lemma 5.2.3 shows that the height of an ideal is preserved under these reductions.

Lemma 5.2.1. *Let R be an algebra over a field k with $|k| = \infty$, let $R_{\text{red}} := R/\sqrt{0}$, and let J be an R-ideal. Let x be an element of R, which may or may not be an element of J. If specialization by x is compatible with integral closure for the image of J in R_{red}, then specialization and integral closure are also compatible for the ideal J. Stated symbolically, if $\overline{JR_{\text{red}}}\,(R_{\text{red}}/(x)R_{\text{red}}) = \overline{J\,(R_{\text{red}}/(x)R_{\text{red}})}$, then $\overline{J}\,(R/(x)) = \overline{J\,(R/(x))}$.*

Proof. By persistence of integral closure (see Remark 2.3.5) applied to the natural map $R \to R/(x)$, it is always true that $\overline{J}\,(R/(x)) \subset \overline{J\,(R/(x))}$. It remains to show that the reverse containment holds if $\overline{JR_{\text{red}}}\,(R_{\text{red}}/(x)R_{\text{red}}) = \overline{J\,(R_{\text{red}}/(x)R_{\text{red}})}$.

Let φ denote the natural map from $R/(x)$ to $R_{\text{red}}/(x)R_{\text{red}}$ given by $a + (x) \mapsto a + (x) + (\sqrt{0})$.

By assumption $\overline{JR_{\text{red}}}\,(R_{\text{red}}/(x)R_{\text{red}}) = \overline{J\,(R_{\text{red}}/(x)R_{\text{red}})}$, and hence we have equality of the preimages:

$$\varphi^{-1}\left(\overline{JR_{\text{red}}}\,(R_{\text{red}}/(x)R_{\text{red}})\right) = \varphi^{-1}\left(\overline{J\,(R_{\text{red}}/(x)R_{\text{red}})}\right).$$

Since integral closure modulo the nilradical lifts, $\overline{JR_{\text{red}}} = \overline{J}R_{\text{red}}$ (see Remark 2.3.7). Thus

$$\begin{aligned}\varphi^{-1}\left(\overline{JR_{\text{red}}}\,(R_{\text{red}}/(x)R_{\text{red}})\right) &= \varphi^{-1}(\overline{J}\,(R_{\text{red}}/(x)R_{\text{red}})) \\ &= \{a + (x) \mid a + (x) + \sqrt{0} \in \overline{J} + (x) + \sqrt{0})\}.\end{aligned}$$

Notice $\sqrt{0} \subset \overline{J}$ for any ideal J because a nilpotent element satisfies an equation of integrality $x^n = 0$. Thus we can rewrite the above as

$$\begin{aligned}\varphi^{-1}\left(\overline{JR_{\text{red}}}\left(R_{\text{red}}/(x)R_{\text{red}}\right)\right) &= \{a + (x) \mid a + (x) \in \overline{J} + (x))\} \\ &= \overline{J}\left(R/(x)\right).\end{aligned}$$

Hence to show that $\overline{J\left(R/(x)\right)} \subset \overline{J}\left(R/(x)\right)$ it remains to show that $\overline{J\left(R/(x)\right)} \subset \varphi^{-1}(\overline{J\left(R_{\text{red}}/(x)R_{\text{red}}\right)})$.

It is always true that

$$\overline{J\left(R/(x)\right)} \subset \varphi^{-1}\left(\varphi\left(\overline{J\left(R/(x)\right)}\right)\right). \tag{5.1}$$

Moreover, we can again apply persistence to the natural map $R/(x) \to R_{\text{red}}/(x)R_{\text{red}}$ given by $a + (x) \mapsto a + (x) + \sqrt{0}$ to see that

$$\varphi\left(\overline{J\left(R/(x)\right)}\right) \subset \overline{J\left(R_{\text{red}}/(x)R_{\text{red}}\right)}. \tag{5.2}$$

Taking preimages of Equation (5.2) and combining it with Equation (5.1), we conclude that $\overline{J\left(R/(x)\right)} = \overline{J}\left(R/(x)\right)$, as desired. ■

Lemma 5.2.2. *Let R be an algebra over a field k with $|k| = \infty$, let $m \in \text{m-Spec}(\overline{R})$, let I be an R-ideal, and let x be an element of I. If specialization by x is compatible with integral closure of the image of I in $\overline{R}_m$ for every maximal ideal m, then specialization and integral closure are also compatible for the ideal I. Stated symbolically, if $\overline{I\overline{R}_m}\left(\overline{R}_m/(x)\overline{R}_m\right) = \overline{I\left(\overline{R}_m/(x)\overline{R}_m\right)}$ for all $m \in \text{m-Spec}(\overline{R})$, then $\overline{I}\left(R/(x)\right) = \overline{I\left(R/(x)\right)}$.*

Proof. As a first step, we show that under these assumptions

$$\overline{I\overline{R}}\left(\overline{R}/(x)\overline{R}\right) = \overline{I\left(\overline{R}/(x)\overline{R}\right)} \tag{5.3}$$

Since both are ideals of $\overline{R}/(x)\overline{R}$, it suffices to check that the two are equal locally at maximal ideals $m \in \text{m-Spec}(\overline{R}/(x)\overline{R})$. Identifying $\text{m-Spec}(\overline{R}/(x)\overline{R})$ with the maximal ideals of $\overline{R}$ which contain x, we have that

$$\begin{aligned}
\left(\overline{I\left(\overline{R}/(x)\overline{R}\right)}\right)_m &= \overline{I\left(\overline{R}_m/(x)\overline{R}_m\right)} \\
&\quad \text{since integral closure commutes with localization by Remark 2.3.7} \\
&= \overline{I\overline{R}_m}\left(\overline{R}_m/(x)\overline{R}_m\right) \text{ by assumption} \\
&= \left(\overline{I\overline{R}}\left(\overline{R}/(x)\overline{R}\right)\right)_m \text{ by Remark 2.3.7}
\end{aligned}$$

This proves the desired equality of Equation (5.3).

We use the above to show that $\overline{I}\,(R/(x)) = \overline{I\,(R/(x))}$.

By persistence of integral closure (Remark 2.3.5) applied to the natural map $R \to R/(x)$, it is always true that $\overline{I}\,(R/(x)) \subset \overline{I\,(R/(x))}$.

It remains to show that the reverse containment holds if Equation (5.3) holds. Let φ denote the natural map from $R/(x)$ to $\overline{R}/(x)\overline{R}$, which is induced by the composition of the natural maps $R \hookrightarrow \overline{R} \twoheadrightarrow \overline{R}/(x)\overline{R}$. Then by Equation (5.3),

$$\varphi^{-1}\left(\overline{I\overline{R}}\left(\overline{R}/(x)\overline{R}\right)\right) = \varphi^{-1}\left(\overline{I\left(\overline{R}/(x)\overline{R}\right)}\right).$$

Since $\overline{R}$ is an integral extension of R, $\overline{I\overline{R}} \cap R = \overline{I}$ (see [31, Proposition 1.6.1]). Thus

$$\begin{aligned}
\varphi^{-1}\left(\overline{I\overline{R}}\left(\overline{R}/(x)\overline{R}\right)\right) &= \{a + (x)R \,|\, a \in R \text{ and } a + (x)\overline{R} \in \overline{I\overline{R}} + (x)\overline{R}\} \\
&= \{a + (x)R \,|\, a \in R \cap \overline{I\overline{R}}\} \\
&= \overline{I}\,(R/(x))
\end{aligned}$$

where the second equality holds because $x \in \overline{I}$.

Therefore, it suffices to show that $\overline{I\,(R/(x))} \subset \varphi^{-1}\left(\overline{I\left(\overline{R}/(x)\overline{R}\right)}\right)$.

As in the proof of Lemma 5.2.1, by persistence and taking preimages we see that

$$\varphi^{-1}\left(\varphi\left(\overline{I\,(R/(x))}\right)\right) \subset \varphi^{-1}\left(\overline{I\left(\overline{R}/(x)\overline{R}\right)}\right).$$

Since it is always true that

$$\overline{I\left(R/(x)\right)} \subset \varphi^{-1}\left(\varphi\left(\overline{I\left(R/(x)\right)}\right)\right),$$

we conclude that $\overline{I\left(R/(x)\right)} \subset \varphi^{-1}\left(\overline{I\left(\overline{R}/(x)\overline{R}\right)}\right)$. Hence $\overline{I}\left(R/(x)\right) = \overline{I\left(R/(x)\right)}$.

■

Lemma 5.2.3. *Let (R, m) be a local equidimensional excellent ring. Let I be an R-ideal. Then* $\operatorname{ht} I\overline{R_{\text{red}}} = \operatorname{ht} I$*.*

Proof. Notice that the properties of being local, equidimensional, and excellent pass from R to R_{red}. Moreover, $\operatorname{ht} I = \operatorname{ht} IR_{\text{red}}$. Thus by replacing R with R_{red}, we may assume R is a reduced local equidimensional excellent ring. It remains to show that $\operatorname{ht} I\overline{R} = \operatorname{ht} I$.

We first note that $\overline{R}$ is finitely generated over R by Remark 2.4.28 since it is reduced and excellent. Since R is excellent, it is by definition (Definition 2.4.24) universally catenary. Therefore, $\overline{R}$ is catenary.

We next show that $\overline{R}$ is locally equidimensional of the same dimension at every maximal ideal. Notice $R \subset \overline{R}$ is a birational extension – that is, $\operatorname{Quot}(R) = \operatorname{Quot}(\overline{R})$ – since $R \subset \overline{R} \subset \operatorname{Quot}(R)$ and it can be shown that every nonzerodivisor of $\overline{R}$ is a unit of $\operatorname{Quot}(R)$. Since $\operatorname{Quot}(R)$ is the ring of fractions of R with respect to the complement of the union of the associated primes of R, and hence all minimal primes are preserved, there is a one-to-one correspondence between $\operatorname{Min}(R)$ and $\operatorname{Min}(\operatorname{Quot}(R))$. Thus there are one-to-one correspondences

$$\operatorname{Min}(R) \xleftrightarrow{\text{1-1}} \operatorname{Min}(\operatorname{Quot}(R)) = \operatorname{Min}(\operatorname{Quot}(\overline{R})) \xleftrightarrow{\text{1-1}} \operatorname{Min}(\overline{R})$$

By this one-to-one correspondence, we see that every minimal prime of $\overline{R}$ contracts to a minimal prime of R.

Now let $n \in \text{m-Spec}(\overline{R})$. Let $q \in \operatorname{Min}(\overline{R})$ be contained in n. Notice $n \cap R$ must be equal to m, the unique maximal ideal of R, since the contraction of a maximal ideal in any ring integral over R must be a maximal ideal of R. As we showed above, $p = q \cap R$ must be a minimal prime of R. Since R is equidimensional and local, $\dim\left(R/p\right)_m = \dim R$. Notice that the dimension formula (Theorem 2.4.11) applies because $R/p \subset \overline{R}/q$ is an extension of domains, R/p is a universally catenary Noetherian ring, and $\overline{R}/q$ is a finitely generated algebra over R/p. By the dimension formula,

$$\dim\left(\overline{R}/q\right)_n = \dim\left(R/p\right)_m + \operatorname{trdeg}_{R/p}\overline{R}/q - \operatorname{trdeg}_{\kappa(m/p)}\kappa(n/q)$$

The above transcendence degrees are both zero since they are transcendence degrees of integral extensions, hence $\dim\left(\overline{R}/q\right)_n = \dim\left(R/p\right)_m = \dim R = \dim\overline{R}$. Thus $\overline{R}$ is locally equidimensional of the same dimension at every maximal ideal n.

Moreover, since $\overline{R}$ is catenary and locally equidimensional of the same dimension at every maximal ideal, one sees that for any prime $p \in \operatorname{Spec}(\overline{R})$, $\dim\overline{R}/p + \operatorname{ht} p = \dim\overline{R}$, and hence the same holds for any ideal of $\overline{R}$. Thus

$$\begin{aligned}\operatorname{ht}\overline{I} &= \dim R - \dim R/\overline{I}\\ &= \dim\overline{R} - \dim R/\overline{I}\\ &= \left(\dim\overline{R}/\overline{I\overline{R}} + \operatorname{ht}\overline{I\overline{R}}\right) - \dim R/\overline{I}\end{aligned}$$

By [31, Proposition 1.6.1], $\overline{I\overline{R}} \cap R = \overline{R}$, and thus the ring extension $R/\overline{I} = R/(\overline{I\overline{R}} \cap R) \to \overline{R}/(\overline{I\overline{R}})$ coming from the integral extension $R \to \overline{R}$ is also an integral extension. So $\dim\overline{R}/\overline{I\overline{R}} = \dim R/\overline{I}$, and thus by the above computation $\operatorname{ht}\overline{I} = \operatorname{ht}\overline{I\overline{R}}$.

Finally, an ideal and its integral closure have the same height since they have the same radical (see [31, Lemma 8.1.10] and recall that I is a reduction of $\overline{I}$).

We conclude that $\operatorname{ht} I = \operatorname{ht}\overline{I} = \operatorname{ht}\overline{I\overline{R}} = \operatorname{ht} I\overline{R}$. ■

We are now ready to prove our main theorem.

Theorem 5.2.4. *Let (R, m) be a local equidimensional excellent k-algebra, where k is a field of characteristic 0. Let $I = (a_1, ..., a_n)$ be an R-ideal such that $\operatorname{ht} I \geq 2$. Then for a general element x of I, as defined in Definition 5.1.3, we have that*

$$\overline{I}/(x) = \overline{I/(x)}.$$

Proof. Notice that by persistence (Remark 2.3.5) applied to the natural map $R \to R/(x)$, it is always true that $\overline{I}/(x) \subset \overline{I/(x)}$. This proof shows the reverse containment.

We may reduce to the case where R is in addition a local normal ring: By Lemma 5.2.1, we may replace R by R_{red} to assume R is a reduced local equidimensional excellent k-algebra. Since R is a reduced excellent local ring, $\overline{R}$ is finitely generatd over R. This shows that $\overline{R}_m$ is excellent and that $\overline{R}$ is semilocal. Hence x is still a general element of $I\overline{R}_m$ for each of the finitely many maximal ideals m of $\overline{R}$. By Lemma 5.2.2 we may replace R with $\overline{R}_m$ for any $m \in$ m-Spec$(\overline{R})$ to assume in addition that R is a local normal ring (hence R is also a domain). Also, by Lemma 5.2.3 we may assume that I still has height at least 2 after these reductions. For the remainder of the proof, we will replace R with $\overline{(R_{\text{red}})}_m$ to assume that R is a local, normal, equidimensional excellent k-algebra, where k is a field of characteristic 0, and that $I = (a_1, \ldots, a_n)$ has height at least 2.

To simplify notation, let $\mathcal{A} = R[It, t^{-1}]$ denote the extended Rees algebra of I, and let $\overline{\mathcal{A}} = \overline{R[It, t^{-1}]}^{R[t,t^{-1}]}$ denote the integral closure of $\mathcal{A}$ in $R[t, t^{-1}]$. Notice that because R is excellent, $\overline{\mathcal{A}}$ is finitely generated over $\mathcal{A}$ by Remark 2.4.28 and is thus Noetherian. Similarly, let $\mathcal{B} = (R/(x))\left[(I/(x))\, t, t^{-1}\right]$ denote the extended Rees algebras of the $R/(x)$-ideal $I/(x)$, and let $\overline{\mathcal{B}}$ denote the integral closure of $\mathcal{B}$ in $(R/(x))\,[t, t^{-1}]$. Define J to be the $\mathcal{A}$-ideal $(It, t^{-1})\,\mathcal{A}$.

We note that $\mathcal{A}$ is catenary and locally equidimensional at the unique maximal homogeneous ideal $\mathfrak{m} = (m, It, t^{-1})\,\mathcal{A}$: Since R is excellent it is by definition universally catenary (see Definition 2.4.24 and Definition 2.4.10). Since $\mathcal{A}$ is a finitely generated algebra over the universally catenary ring R, $\mathcal{A}$ is catenary. Since R is equidimensional, the localization of $\mathcal{A}$ at the maximal homogeneous ideal $\mathfrak{m}$ is equidimensional of dimension $\dim R + 1$ by the proof of [31, Theorem 5.1.4(3)] and the fact that minimal primes of $\mathcal{A}$ come from minimal primes of R, as discussed on [31, p. 99].

We use these properties of $\mathcal{A}$ to show that $\operatorname{ht} J$ is at least 3, which is needed to apply Theorem 5.1.16. Notice J is homogeneous, and thus all minimal primes of J are homogeneous. So $\operatorname{ht} J = \operatorname{ht} J_{\mathfrak{m}}$ since $\mathfrak{m}$ is the unique maximal homogeneous ideal. Now, $\mathcal{A}_{\mathfrak{m}}$ is a local catenary equidimensional ring, hence

$$\begin{aligned} \operatorname{ht} J_{\mathfrak{m}} &= \dim \mathcal{A}_{\mathfrak{m}} - \dim(\mathcal{A}/J)_{\mathfrak{m}} \\ &\geq \dim \mathcal{A}_{\mathfrak{m}} - \dim \mathcal{A}/J \end{aligned}$$

Notice $\mathcal{A}/J \cong R/I$, and recall $\dim \mathcal{A}_{\mathfrak{m}} = \dim R + 1$ since $\operatorname{ht} I > 0$. Thus we can rewrite

$$\begin{aligned} \operatorname{ht} J_{\mathfrak{m}} &\geq \dim \mathcal{A} - \dim \mathcal{A}/J \\ &= \dim R + 1 - \dim R/I \\ &= \dim R + 1 - (\dim R - \operatorname{ht} I) \\ &= \operatorname{ht} I + 1 \\ &\geq 3. \end{aligned}$$

Thus

$$\operatorname{ht} J \geq 3 \tag{5.4}$$

The natural map

$$R[t, t^{-1}] \twoheadrightarrow (R/(x))[t, t^{-1}]$$

$$a \longmapsto a + (x)$$

induces a natural map

$$\mathcal{A} = R[It, t^{-1}] \twoheadrightarrow \mathcal{B} = (R/(x))[(I/(x))t, t^{-1}]$$

which in turn induces

$$\overline{\mathcal{A}} \xrightarrow{\psi} \overline{\mathcal{B}}$$

Because $xt \in \ker \psi$, we obtain a natural map φ

$$\overline{\mathcal{A}}/(xt)\overline{\mathcal{A}} \xrightarrow{\varphi} \overline{\mathcal{B}}.$$

We ultimately wish to show $\frac{\overline{I/(x)}}{\overline{I}/(x)} = 0$. Notice $\frac{\overline{I/(x)}}{\overline{I}/(x)} = [\operatorname{coker}\varphi]_1$, the degree 1 component of the cokernel.

As a first step towards showing the cokernel is zero in degree 1, we show that φ_p is an isomorphism for $p \in \operatorname{Spec}(\overline{\mathcal{A}}) \setminus V(J\overline{\mathcal{A}})$. Such a prime p must either avoid t^{-1} or not contain all of It. We consider the two cases separately.

Case 1: Suppose $t^{-1} \notin p$. Notice $\overline{\mathcal{A}}_{t^{-1}} \cong R[t,t^{-1}]$ and similarly $\overline{\mathcal{B}}_{t^{-1}} \cong (R/(x))[t,t^{-1}]$, so

$$\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_{t^{-1}} \cong \frac{\overline{\mathcal{A}}_{t^{-1}}}{xt\overline{\mathcal{A}}_{t^{-1}}} \cong \frac{R[t,t^{-1}]}{xtR[t,t^{-1}]} = R[t,t^{-1}]/xR[tt^{-1}] \cong \frac{R}{(x)}[t,t^{-1}] \cong \overline{\mathcal{B}}_{t^{-1}}.$$

Therefore any further localization is an isomorphism. So φ_p is an isomorphism if $t^{-1} \notin p$.

Case 2: Now let p be any prime $p \in \operatorname{Spec}(\overline{\mathcal{A}}) \setminus V(It\overline{\mathcal{A}})$. We will use that φ_q is an isomorphism if $t^{-1} \notin q$ to show that φ_p is also an isomorphism.

We first show φ_p is injective. Notice that $(\overline{A}/xt\overline{A})_p$ is $(xt\overline{\mathcal{A}})_p$ is contained in $(xtR[t,t^{-1}] \cap \overline{\mathcal{A}})_p$. We can see that $(xtR[t,t^{-1}] \cap \overline{\mathcal{A}})_p/(xt\overline{\mathcal{A}}_p) = \ker\varphi_p$ by considering the natural maps above. Thus, to show that φ_p is injective it is enough to show that $(xt\overline{\mathcal{A}})_p = (xtR[t,t^{-1}] \cap \overline{\mathcal{A}})_p$ locally at associated primes of $xt\overline{\mathcal{A}}_p$.

Notice $\overline{\mathcal{A}}_p$ is normal since R is normal, and as the localization of a finitely generated algebra over an excellent ring, it is also excellent. Since xt is a general element of It, by Flenner's Bertini Theorem (Corollary 5.1.11) $(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})_p$ is normal and hence a domain. Since $xt\overline{\mathcal{A}}_p$ isa prime ideal, it suffices to show the desired equality locally at $q = xt\overline{\mathcal{A}}_p$. Notice $\operatorname{ht} q\overline{\mathcal{A}}_p \leq 1$. Since t^{-1}, xt is an $\overline{\mathcal{A}}$-regular sequence by Remark 5.1.13, we may assume $t^{-1} \notin q$. Thus the desired equality holds locally at q by the previous case. Thus φ_p is injective for all $p \in \operatorname{Spec}(\overline{\mathcal{A}}) \setminus V(It\overline{\mathcal{A}})$.

We now show that for the same primes $p \in \operatorname{Spec}(\overline{\mathcal{A}}) \setminus V(It\overline{\mathcal{A}})$, φ_p is also a surjection. Since $\mathcal{A}$ surjects onto $\mathcal{B}$ we have that the ring extension $\operatorname{Im}(\mathcal{A})_{p\cap\mathcal{A}} = \mathcal{B}_{p\cap\mathcal{A}} \subset \overline{\mathcal{B}}_{p\cap\mathcal{A}}$ is an integral extension. Because $\mathcal{A} \subset \overline{\mathcal{A}}$, $\operatorname{Im}(\overline{\mathcal{A}})_{p\cap\mathcal{A}} \subset \overline{\mathcal{B}}_{p\cap\mathcal{A}}$ is also an integral extension. Thus $\operatorname{Im}(\overline{\mathcal{A}})_p = \operatorname{Im}\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p \subset \overline{\mathcal{B}}_p$ is also an integral extension.

Notice $\operatorname{Im}(\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p) \subset \overline{\mathcal{B}}_p$, t^{-1} is a non-zerodivisor on both rings, and after making t^{-1} invertible both rings are equal. Hence they have the same total ring of quotients.

Recall that $\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p$ is normal and φ_p is injective. Hence $\operatorname{Im}(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})_p$ is also normal. It follows that φ_p must be a surjection. Therefore φ_p is an isomorphism, proving our claim.

To simplify notation, let K be the kernel of φ and C be the cokernel of φ, and recall that it suffices to show that $C_1 = 0$. In order to do so, we will identify C with a submodule of $H^2_J(\overline{\mathcal{A}})$. Because φ_p is an isomorphism for all $p \notin V(J\overline{\mathcal{A}})$ we have that $K_p = 0$ at such primes. Thus $H^0_J(K) = K$, which implies that $H^i_J(K) = 0$ for all $i > 0$ by Lemma 5.1.17. Similarly $H^0_J(C) = C$. Because $t^{-1} \in J$ is a regular element on $\overline{\mathcal{B}}$, we also have that $H^0_J(\overline{\mathcal{B}}) = \{x \in \overline{\mathcal{B}} \mid xJ^n = 0 \text{ for some } n \geq 0\} = 0$.

From the long exact sequence of local cohomology induced by the exact sequence

$$0 \longrightarrow K \longrightarrow \overline{\mathcal{A}}/xt\overline{\mathcal{A}} \xrightarrow{\varphi} \operatorname{Im}(\varphi) \longrightarrow 0$$

we obtain $H^i_J(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}) \cong H^i_J(\operatorname{Im}(\varphi))$ for all $i \geq 1$ since $H^i_J(K)$ vanishes for $i \geq 1$.

From the long exact sequence of local cohomology induced by the exact sequence

$$0 \longrightarrow \operatorname{Im}(\varphi) \longrightarrow \overline{\mathcal{B}} \longrightarrow C \longrightarrow 0$$

we obtain the exact sequence

$$0 = H^0_J(\overline{\mathcal{B}}) \longrightarrow H^0_J(C) \longrightarrow H^1_J(\operatorname{Im}(\varphi)) \cong H^1_J(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}).$$

Therefore $C = H^0_J(C) \hookrightarrow H^1_J(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})$.

By Remark 5.1.13 we have $\operatorname{depth}_J(\overline{\mathcal{A}}) \geq 2$. Thus $H^1_J(\overline{\mathcal{A}}) = 0$. Hence from the short exact sequence

$$0 \longrightarrow xt\overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}/xt\overline{\mathcal{A}} \longrightarrow 0$$

we obtain the exact sequence

$$0 = H^1_J(\overline{\mathcal{A}}) \longrightarrow H^1_J(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}) \longrightarrow H^2_J(xt\overline{\mathcal{A}}).$$

Therefore $C \hookrightarrow H^2_J(xt\overline{\mathcal{A}})$.

Since x is a general element of I, we may assume x is a nonzerodivisor on R, and hence that xt is a nonzerodivisor on $\overline{\mathcal{A}} \subset R[t,t^{-1}]$. Thus we have an isomorphism of graded modules $xt\overline{\mathcal{A}} \cong \overline{\mathcal{A}}(-1)$, and hence $C \hookrightarrow H^2_J(\overline{\mathcal{A}}(-1))$. So $[C]_n \hookrightarrow [H^2_J(\mathcal{A}(-1))]_n \cong [H^2_J(\overline{\mathcal{A}})]_{n-1}$. By Theorem 5.1.16, we have $[H^2_J(\overline{\mathcal{A}})]_{n-1} = 0$ for all $n \leq 1$. In particular, $[C]_1 = 0$, that is, $\overline{I/(x)} = \overline{I}/(x)$. ■

If we additionally assume that R is normal, we can extend our main theorem to sufficiently large powers of I.

Proposition 5.2.5. *Let (R, m) be a local normal excellent k-algebra, where k is a field of characteristic 0. Let $I = (a_1, ..., a_n)$ be an R-ideal such that $\operatorname{ht} I \geq 2$, and let x be a general element of I. Then $(\overline{I^s} + (x))/(x) = \overline{(I/(x))^s}$ for s sufficiently large.*

Proof. As in Theorem 5.2.4, let $\mathcal{A}$ and $\mathcal{B}$ denote the extended Rees algebras of I and $I/(x)$, respectively, and let J denote the $\mathcal{A}$-ideal (It, t^{-1}). Consider again the natural map $\varphi : \overline{\mathcal{A}}/xt\overline{\mathcal{A}} \to \overline{\mathcal{B}}$, and denote the cokernel of φ by C. By the same proof as in Theorem 5.2.4, we can show that $H^0_J(C) = C$. Because $C_s = \frac{\overline{(I^s+(x))/(x)}}{(\overline{I^s}+(x))/(x)}$, it suffices to show $C_s = 0$ for all $s \gg 0$, where C_s denotes the degree s component of C.

Next we show that we may assume $R/(x)$ is a reduced excellent ring. Since R is excellent, $R/(x)$ is excellent. Since R is normal and hence reduced, by Flenner's Bertini theorem (Theorem 5.1.10) $R/(x)$ satisfies Serre's condition R_0 locally at primes which do not contain I. Since I has height at least 2 and Serre's condition R_0 can be checked locally at primes of height 0, $R/(x)$ satisfies R_0. Moreover, since R satisfies Serre's condition S_2 and x is a nonzerodivisor, $R/(x)$ satisfies Serre's condition S_1. Therefore $R/(x)$ is reduced. Since $(R/(x))[t, t^{-1}]$ is reduced and contains $\overline{\mathcal{B}}$, we see that $\overline{\mathcal{B}}$ is also a reduced ring. Furthermore, $\mathcal{B}$ is finitely generated as an $R/(x)$-algebra and hence is excellent. Therefore $\mathcal{B}$ is a finite $\mathcal{B}$-module by Remark 2.4.28. Since $\mathcal{B}$ is a Noetherian ring, $\overline{\mathcal{B}}^{\mathrm{Quot}(\mathcal{B})}$ is a Noetherian $\mathcal{B}$-module and therefore, $\overline{\mathcal{B}} \subset \overline{\mathcal{B}}^{\mathrm{Quot}(\mathcal{B})}$ is finitely generated as a $\mathcal{B}$-module. Since $\mathcal{A} \twoheadrightarrow \mathcal{B}$, $\overline{\mathcal{B}}$ is also finitely generated as a $\mathcal{A}$-module. Since $\overline{\mathcal{B}}$ is finitely generated as an $\mathcal{A}$-module, so is its homomorphic image C.

We show that since C is finitely generated as an $\mathcal{A}$-module, $C_s = 0$ for s sufficiently large. Let $z_1, \ldots, z_r$ be a set of generators of C. Since $H^0_J(C) = C$, by definition for $1 \leq i \leq r$ there exists k_i such that $J^{k_i}z_i = 0$ for $1 \leq i \leq r$. Therefore $C_s = 0$ for $s > \max\{k_i + \deg(z_i) \mid 1 \leq i \leq r\}$. Therefore, $\overline{(I^s} + (x))/(x) = \overline{(I/(x))^s}$ for $s > \max\{k_i + \deg(z_i) \mid 1 \leq i \leq r\}$. ■

Remark 5.2.6. In Proposition 5.2.5 we add the assumption that R is normal. This is because we are unable to reduce to the normal case as in Theorem 5.2.4 since the proof of Lemma 5.2.2 does not work when x is not an element of the ideal we are taking the integral closure of.

5.3 Specialization by General Elements of the Maximal Ideal

Even in the case of a monomial ideal in a polynomial ring, it is not necessarily true that specialization by a general element of the maximal ideal is compatible with integral closure. The issue with specialization by a general element x of the maximal ideal is similar to the issue we encountered in Proposition 5.2.5: x is not necessarily in the ideal we are taking the integral closure of. There are many counterexamples where integral closure does not commute with specialization by a general element of the maximal ideal, which can be computed quickly with the aid of Macaulay2.

Example 5.3.1. The following examples show that integral closure may not be compatible with specialization by a general element of the maximal ideal of the ring even if the ideal and ring satisfy all the assumptions of Theorem 5.2.4.

(a) Integral closure is not compatible with specialization by a general element of the maximal ideal if $\dim R/I \leq 1$. Consider an ideal I which is reduced and generated by general quadrics in $R = \mathbb{Q}[x, y, z]$. Modulo a general element a of the maximal ideal (x, y, z), an integrally closed ideal must be a power of the maximal ideal, and thus must have at least 3 generators. Thus $I(R/(a))$ is not integrally closed.

(b) We can also find counterexamples when $\dim R/I = 2$. Using Macaulay2, one can compute that $I = (x^2, yw, xz^2) \subset R = \mathbb{Q}[x, y, z, w]$ is an integrally closed ideal of height 2 and reduction number zero. However, $I(R/(a))$ is not integrally closed for a general element $a = \alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 w$ of $m = (x, y, z, w)$ because $w^2(\alpha_3 z + \alpha_4 w) \in \overline{I(R/(a))} \setminus I(R/(a))$.

However, if I is an ideal such that R/I is reduced and of depth at least 2, then the two operations are compatible:

Proposition 5.3.2. *Let (R, m) be a local excellent algebra over a field k of characteristic 0, and let $m = (x_1, \ldots, x_n)$. Let I be an ideal such that R/I is reduced with* $\operatorname{depth}(R/I) \geq 2$. *For general $\alpha \in k^n$, let $x_\alpha = \sum_{i=1}^n \alpha_i x_i$ be a general element of the maximal ideal. Then $(I, a)/(a) \subset R/(a)$ is integrally closed.*

Proof. By Theorem 5.1.10, the ring $R/(I, a)$ is reduced locally on the punctured spectrum (that is, locally at primes $p \neq m$). Since $\operatorname{depth}(R/I) \geq 2$ and a is a general element of m, a is regular on R. Thus $\operatorname{depth}(R/(I, a)) \geq 1$. It follows that $R/(I, a)$ is reduced. Hence the containments

$$(I, a)/(a) \subset \overline{(I, a)/(a)} \subset \sqrt{(I, a)/(a)}$$

are all equalities, i.e. $(I, a)/(a)$ is integrally closed. ■

This raises the question of whether specialization and integral closure are compatible for an ideal which is reduced for which R/I has dimension at least 2, and depth equal to 1. Notice that reduced monomial ideals are exactly the squarefree monomial ideals. Interestingly, it appears from computation that specialization by a general element of the maximal ideal does commute with integral closure for squarefree monomial ideals. We would be interested in knowing whether there are any counterexamples to this, or whether it is can be proved that for squarefree monomial ideals, specialization by a general element of the maximal ideal commutes with integral closure.

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