ON THE ABSTRACT STRUCTURE OF OPERATOR SYSTEMS AND APPLICATIONS TO QUANTUM INFORMATION THEORY

by

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In memory of Ed Effros whose work inspired so much of my own.

"For myself, I have but one requirement, that of success."

-Napoleon Bonaparte

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LIST OF SYMBOLS

H a Hilbert space

$\ell_2^n(H)$ the n-fold Hilbertian direct sum

$\mathcal{A}$ a (unital) C*-algebra

$[n] := \{1, \ldots, n\}$ $n \in \mathbb{N}$

$\mathbb{R}^+$ all positive real numbers

$\mathcal{C}$ matrix ordering on a $*$-vector space

$\mathcal{X}^*$ the Banach dual of the operator space $\mathcal{X}$.

$\mathcal{X}'$ the weak dual of a matrix ordered $*$-vector space $(\mathcal{X}, \mathcal{C})$

$\mathcal{X}^d$ the algebraic dual of a vector space $\mathcal{X}$

$\mathcal{X}_{\max}$ the 1-max structure on an Archimedean order unit space $\mathcal{X}$

$\mathcal{I}(\mathcal{X})$ the injective envelope of the operator system $\mathcal{X}$

$C_e^*(\mathcal{X})$ the C*-envelope of the operator system $\mathcal{X}$

$B(H)$ norm continuous linear operators on the Hilbert space H

$\langle \cdot, \cdot \rangle$ scalar pairing between $*$-vector spaces

$\langle\langle \cdot, \cdot \rangle\rangle$ matrix pairing between $*$-vector spaces

$\langle \cdot | \cdot \rangle_H$ inner product on the Hilbert space H

$(\mathcal{X}, \mathcal{C}, e)$ operator system

$\mathcal{C}(p)$ the matrix ordering induced by a positive contraction p

$\mathcal{C}(p_1, \ldots, p_N)$ matrix ordering induced by N positive contractions

J_N the $N \times N$ matrix with 1 in every entry

$\mathcal{C}_n^\infty$ the Archimedean closure of $\bigcup_{L \in \mathbb{N}} \mathcal{C}_n^L$

$\mathcal{C}^\infty$ the inductive limit of the nested increasing sequence of matrix orderings $\{\mathcal{C}^L\}_{L \in \mathbb{N}}$

$\pi_L : \mathcal{X} \to M_{2^{NL}}(\mathcal{X})$ the mapping $x \mapsto x \otimes J_{2^{NL}}$

$\mathcal{C}^\infty(p_1, \ldots, p_N)$ the inductive limit of the nested increasing sequence $\{\pi_L^{-1} \mathcal{C}(p_1, \ldots, p_N)^L\}_{L \in \mathbb{N}}$

$C(n, k)$ set of correlations with n-inputs and k-outputs

$(\mathcal{X}^x, \mathcal{C}^x, x)$ the operator system induced by the positive element x

$\alpha^x : \mathcal{X}^x \to [0, \infty)$ operator space norm induced by the positive element x

$(\mathcal{X}^x, \alpha^x)$ the operator space induced by the positive element x

$\otimes$ the algebraic tensor product

$\otimes_{\min}$ the minimal tensor product (either of operator spaces or operator systems)

$\otimes_{\wedge}$ the operator space projective tensor product

$\mathrm{CB}(\mathcal{X}, \mathcal{Y})$ the Banach space of completely bounded maps between operator spaces $\mathcal{X}$ and $\mathcal{Y}$

ABSTRACT

We introduce the notion of an abstract projection in an operator system and when a finite number of positive contractions in an operator system are all simultaneously abstract projections in that operator system. We extend this notion to Archimedean order unit spaces where we prove when a positive contraction is an abstract projection in some operator system, and furthermore when a finite number of positive contractions in an Archimedean order unit space are all simultaneously abstract projections in a single operator system. These methods are then used to provide new characterizations of both nonsignalling and quantum commuting correlations. In particular, we construct a universal Archimedean order unit space such that every quantum commuting correlation may be realized as the image of a unital linear positive map acting on the generators of that Archimedean order unit space. We also construct an Archimedean order unit space which is universal (in the same way) to nonsignalling correlations. We conclude with results concerning weak dual matrix ordered $*$-vector spaces and the operator systems they induce.

1. INTRODUCTION

With its origins dating back to the early 20^{th} century, operator algebras has had a profound impact in the world of mathematics with an abundance of applications to fields such as probability theory, ergodic theory and quantum information theory. In order to provide a rigorous mathematical foundation for quantum mechanics, John von Neumann and Francis Murray published a series of papers ([14], [15], [16]) where they provided the foundations for the field now known as von Neumann algebras. In later years the field was greatly expanded due to the development of the theory of C*-algebras. The name operator algebras comes from the (highly non-trivial) fact that both von Neumann algebras and C*-algebras may always be realized as a subalgebra of $B(H)$ for some Hilbert space H. This is to say that every von Neumann algebra or C*-algebra may be regarded as being contained in $B(H)$, the algebra of bounded linear operators on some Hilbert space H. As time progressed it was noticed by operator algebraists that if one were to ease the structure of a C*-algebra and consider more general objects, then one could deduce amazing results concerning the ambient algebras. It was using these methods that an incredible connection was made between the fields of operator algebras and quantum information theory.

Analogous with the development of quantum mechanics, it was realized that even in "classical" operator algebra theory, looking at the level one operator algebra provided insufficient information with regard to many natural constructions or questions one may have. This is to say that if $\mathcal{A}$ is the object in question, then one must not only look at $\mathcal{A}$ but rather at $M_n(\mathcal{A})$ for every natural number n. We point out if n is a natural number then $M_n(\mathcal{A})$ is also a C*-algebra with structure induced from the algebra $B(\ell_2^n(H))$, where $\ell_2^n(H)$ denotes the n-fold Hilbertian direct sum. An example of this is if one wishes to consider continuous maps on tensor products of C*-algebras. It is well known that given two linear maps between vector spaces then they induce a unique linear map between the algebraic tensor products of their domains and their ranges. This presents no problems since linear maps are the "morphisms" one considers when looking at vector spaces, but when we consider objects with nontrivial topological structure, such as C*-algebras, then this property becomes much more delicate. In general, given two continuous maps between C*-algebras it is not true that their induced linear map on the algebraic tensor product extends to a continuous map on the C*-algebra tensor product. By considering properties of matrices over $\mathcal{A}$ and induced maps on those spaces of matrices, one can circumvent such difficulties. Thus, it is precisely the matricial properties of $\mathcal{A}$ that one must consider to avoid problems.

Every C*-algebra contains a closed positive cone of elements. By our remarks above, if $\mathcal{A} \subset B(H)$ then the positive cone of $\mathcal{A}$, $\mathcal{A}^+$, is precisely the convex set of positive operators on H intersected with $\mathcal{A}$. There is also a closed positive cone in $M_n(\mathcal{A})$ for every natural number n, defined in a similar way. In particular, there is a noncommutative (matricial) order structure that one may consider on $\mathcal{A}$. The abstract analogue of such "noncommutative matrix ordered spaces" first appeared in the work of Choi and Effros in [4], where they were able to use such matricially ordered spaces to deduce various properties of C*-algebras and von Neumann algebras. These matricially ordered spaces came to be known as *operator systems*, but despite this work appearing in 1977, operator system theory was not a particularly dominate niche in operator algebras. Operator system theory reemerged in 2009 in [19], where Paulsen and Tomforde worked out the abstract theory for objects known as *Archimedean order unit spaces*. Such an object consists of a vector space with involution, a positive cone, and a "suitable" notion of a unit. Though this paper did not directly pertain to operator systems, it provided the foundation for later work in the theory. A couple of years later, in [12] and [11], both the tensor theory of operator systems and properties of quotients of operator systems were developed and the authors were able to provide operator system equivalences to the most famous open problem in operator algebras.

Beginning with a remark of Connes in [5], it was a conjecture made by Kirchberg in [13], which drove research in the theory for decades and into the present day. In particular, *Kichberg's conjecture* became the most famous problem in operator algebras. Given a discrete group G then one may always associate a maximal C*-algebra to it. This *full group C*-algebra* of G, denoted $C^*(G)$, is often a complicated object to understand. Kirchberg conjectured that when G is the free group on 2-generators, then there was only one possible C*-algebra tensor product associated to the algebraic tensor product of $C^*(F_2)$ with itself. Kirchberg went on to prove his conjecture was indeed equivalent to the remark made my Connes' 17 years earlier. In particular, Connes' remark had an affirmative answer if and only if Kircherg's conjecture was true. Using operator system theory, in [11], the authors were able to prove an operator system equivalence to Kirchberg's conjecture.

Once again consider a C*-algebra $\mathcal{A} \subset B(H)$. In a similar spirit to our remarks concerning matricial orderings on $\mathcal{A}$, there are also matricial norms on $\mathcal{A}$. Every C*-algebra has the relative operator norm induced by $B(H)$. If n is a natural number, then making the identification $M_n(B(H)) \simeq B(\ell_2^n(H))$, it is immediate that $M_n(\mathcal{A})$ has a relative norm induced by the algebra $B(\ell_2^n(H))$. Doing this for every natural number, we obtain a matricial norm structure on the C*-algebra. The abstract theory of matricial normed spaces first appeared in [20], and such objects came to be known as *operator spaces.* As remarked above, though the abstract ideas of operator systems appeared first, it was operator space theory that dominated the minds of many operator algebraists during the late '80s and throughout the '90s. Not only has operator space theory had many applications in operator algebras, but in recent years it has had incredible applications in quantum information theory.

Correlations are tuples of positive real numbers which model joint probability distributions. Sets of correlations have long been investigated by physicists, even going back to the work of Bell in [3]. Of particular interest is the set correlations determined by families of pairwise commuting projections on a Hilbert space. Various interesting sets of correlations are subsets of this set, and many questions have revolved around distinguishing such subsets. In [22], Tsirelson asked whether the set of correlations determined by suitable pairs of pairwise commuting projections on an arbitrary Hilbert space (*quantum commuting correlations*) could be approximated by similar correlations but where the projections are restricted to finite-dimensional Hilbert spaces (*quantum correlations*). Using fundamental techniques and ideas from operator space and operator system theory, in [9], Junge et al. made an incredible discovery where they proved *Tsirelson's problem* was indeed equivalent to Kirchberg's conjecture. Since then, there has been a surge of activity using operator space and operator system theory in answering various questions regarding such sets of correlations. According to a recent preprint, [8], it is not the case that quantum correlations approximate quantum commuting correlations for all input and output values.

One learns in their first course of functional analysis that projections on Hilbert spaces are the building blocks of normal operators defined on that Hilbert space. As we saw in the preceding paragraph, projections are also the building blocks of various sets of correlations. Despite many excellent results concerning these correlation sets, there is much we still do not understand about them. Following [1] and [2], in this manuscript we develop the notion of a projection in an operator system which mirrors the properties of a concrete projection in a C*-algebra. We have generalized this characterization to finite sets of positive contractions in an operator system and we have proven when such a finite number of positive contractions in an operator system are all simultaneously projections. In a similar spirit we have been able prove when a positive contraction in an Archimedean order unit space is a projection in some operator system, and furthermore when a finite number of positive contractions in an Archimedean order unit space are all simultaneously projections in a single operator system. Using our methods we have been able to provide a purely abstract characterization of the set of quantum commuting correlations. In particular, we establish a characterization of quantum commuting correlations viewed as actions of particular morphisms on classes of Archimedean order unit spaces. Thus, our work provides a new and abstract way one may view quantum commuting correlations.

A fundamental principle in the theory of Banach spaces is that of *local reflexivity.* In short, every Banach space $\mathcal{E}$ satisfies the property that if V is a finite-dimensional Banach space and $\mathcal{E}^{**}$ denotes the bidual of $\mathcal{E}$, then every contraction $u : V \to \mathcal{E}^{**}$ is the point-weak dual limit of a net of contractions $\{u_{\mathrm{i}}\}_{\mathrm{i}\in I}, u_{\mathrm{i}} : V \to \mathcal{E}$. The analogue of local reflexivity for operator spaces was introduced by Effros and Haagerup in [6]. It was quickly noticed that local reflexivity for operator spaces was much more delicate than its counterpart in Banach space theory. For one, it is not true in general that every operator space is locally reflexive. Furthermore, Effros and Haagerup showed that local reflexivity for operator spaces was connected to highly non-trivial properties concerning C*-algebras. Motivated by this, we provide the beginnings of a method to establish a notion of local reflexivity in the category of operator systems. In order to do this we introduce *weak dual matrix ordered $*$-vector spaces* and the canonical operator systems that they induce. Though operator systems are similar to operator spaces, one cannot simply transfer the characterization of local reflexivity to the category of operator systems. The obstacles arise precisely because of the lack of duality theory in operator systems. By definition, an operator system contains a "suitable" unit and this unit characterizes various properties of the operator system. In particular, every operator system has an *operator space norm* defined on it, and this norm is dependent on the unit of the system. Thus, if $\mathcal{X}$ is an operator system, then one may consider its Banach dual $\mathcal{X}^*$. The issue arises because the Banach dual may not have a unit. Thus, in general, the Banach dual of an operator system is not an operator system. As mentioned above, we partially circumvent this obstacle by considering the more relaxed structure of a *matrix ordered $*$-vector space*. We then consider dual pairs of such objects, so that one may consider the weak topology on the space and the weak-dual topology on its dual. In other words, we consider weak dual matrix ordered $*$-vector spaces. We then prove that every weak dual matrix ordered $*$-vector space induces a canonical collection of operator systems.

The manuscript is organized as follows: Section 2 discusses preliminary material for the manuscript. In Section 3 we develop the notion of a projection in an operator system and characterize when a finite number of positive contractions in an operator system are all simultaneously projections. In Section 4 we establish when a positive contraction in an Archimedean order unit space is a projection in some operator system and furthermore, when a finite number of positive contractions in an Archimedean order unit space are all simultaneously projections in a single operator system. In Section 5 we apply our methods to questions in quantum information theory. In particular, we construct a universal Archimedean order unit space such that every quantum commuting correlation must be induced by the action of a morphism on the generators of that Archimedean order unit space. In Section 6 we establish results concerning weak dual matrix ordered $*$-vector spaces and the operator systems they induce. In Section 7 we close with some remarks concerning ongoing work.

2. PRELIMINARIES

We begin with the theory of Archimedean order unit spaces. Consider a $*$-vector space $\mathcal{X}$, which is to say $\mathcal{X}$ is a vector space equipped with an involution $* : \mathcal{X} \to \mathcal{X}$. Given $x \in \mathcal{X}$, if $x = x^*$ then we will say x is *hermitian* and we will denote the real vector subspace of all hermitian elements of $\mathcal{X}$ by $\mathcal{X}_h$. A subset $C \subset \mathcal{X}_h$ is a *cone* if $\mathbb{R}^+ C \subset C$ and $C + C \subset C$. Here we let $\mathbb{R}^+$ denote all positive real numbers. The cone C induces a partial ordering on $\mathcal{X}_h$ by declaring $x \geq y$ if and only if $x - y \in C$. The pair $(\mathcal{X}, C)$ will be called an *ordered $*$-vector space*. If $C \cap -C = \{0\}$ then we will call C a *proper* cone and in this case the pair $(\mathcal{X}, C)$ will be called a *proper matrix ordered $*$-vector space*. An element $e \in \mathcal{C}$ be will called an *Archimedean order unit* if it satisfies the following two properties: (i) given any $x \in \mathcal{X}_h$ there exists $r > 0$ such that $re - x \in C$; (ii) $x \in C$ if and only if $\epsilon e + x \in C$ for all $\epsilon > 0$. Property (i) ensures that the positive cone majorizes the real hermitian subspace and Property (ii) is saying the positive cone C is order closed with respect to the element e. If an element $e \in \mathcal{X}_h$ satisfies Property (i) then we will call it an *order unit*.

Definition 2.0.1 ([19]). *If $(\mathcal{X}, C)$ is a proper ordered $*$-vector space and e is an Archimedean order unit, then the triple $(\mathcal{X}, C, e)$ will be called an* Archimedean order unit (AOU) space.

We will simply denote an AOU space as $\mathcal{X}$ when there is no confusion as to what the ordering or unit is for $\mathcal{X}$. If $(\mathcal{X}, C)$ is an ordered $*$-vector space then there is always canonical quotient one may take to form a proper ordered $*$-vector space. To this end, suppose $(\mathcal{X}, C)$ is an ordered $*$-vector space. Let $\mathcal{J} := \operatorname{span} C \cap -C$. Then one may consider the quotient $*$-vector space $\mathcal{X}/\mathcal{J}$ with proper ordering $C + \mathcal{J}$. In particular, if $e \in C$ is an Archimedean order unit for the ordered $*$-vector space, then $e + \mathcal{J}$ is an Archimedean order unit for the proper ordered $*$-vector space $(\mathcal{X}/\mathcal{J}, C + \mathcal{J})$ and therefore the triple $(\mathcal{X}/\mathcal{J}, C + \mathcal{J}, e + \mathcal{J})$ is an AOU space.

Let $u : (\mathcal{X}, C) \to (\mathcal{Y}, D)$ be a linear map between two ordered $*$-vector spaces. We call u *positive* if $u(C) \subset D$. If u is a linear isomorphism such that both u and u^{-1} are positive then we will call u an *order isomorphism*. If $u : \mathcal{X} \to \mathcal{Y}$ is an order isomorphism onto its range then we will sometimes say u is a *order embedding*. If $e^{\mathcal{X}}, e^{\mathcal{Y}}$ are two Archimedean order units for the spaces $(\mathcal{X}, C)$ and $(\mathcal{Y}, D)$, respectively, then we call u *unital* if $u(e^{\mathcal{X}}) = e^{\mathcal{Y}}$. A unital positive linear functional will be called a *state* of the AOU space $\mathcal{X}$.

If $\mathcal{X}$ and $\mathcal{Y}$ are two AOU spaces and if there exists a (unital) order isomorphism $u : \mathcal{X} \to \mathcal{Y}$ then we say the AOU spaces are *(unital) order isomorphic*. In this way we identify two AOU spaces. Fix an AOU space $(\mathcal{X}, C, e)$ and consider the map $\alpha^e : \mathcal{X}_h \to [0, \infty)$ defined by $\alpha^e(x) := \inf\{t \in \mathbb{R}^+ : te \pm x \in C\}$. It follows α^e is a seminorm on $\mathcal{X}_h$ and since e is Archimedean then it is necessarily a norm. In a natural way one may extend this norm to all of $\mathcal{X}$. An *order *-seminorm* on $\mathcal{X}$ is a seminorm $\left\| \cdot \right\| : \mathcal{X} \to [0, \infty)$ such that $\left\|x\right\| = \left\|x^*\right\|$ for all $x \in \mathcal{X}$ and $\left\| \cdot \right\||_{\mathcal{X}_h} = \alpha^e$. Of particular interest will be the *minimal order norm*, which we will denote by $\alpha^{\min} : \mathcal{X} \to [0, \infty)$ defined by $\alpha^{\min}(x) := \sup\{|\varphi(x)| : \varphi : \mathcal{X} \to \mathbb{C}$ is unital positive$\}$. In Section 4 we will be concerned with positive contractions in an AOU space $(\mathcal{X}, C, e)$. Such reference will always be made with regard to the order norm α^e.

The concrete analogue of AOU spaces dates back to the work of Kadison in [10]. AOU spaces are the abstract version of *function systems* which are self-adjoint, unital subspaces of the Banach space of continuous functions on some compact Hausdorff space. In particular, along with Kadison's work, it was shown in [19, Theorem 5.2] that given any AOU space $\mathcal{X}$ there exists a compact Hausdorff space V and a unital order embedding $\Phi : \mathcal{X} \to C(V)$ such that $\left\|\Phi(x)\right\|_\infty = \alpha^{\min}(x)$ for all $x \in \mathcal{X}$. Here we have let $\left\| \cdot \right\|_\infty : C(V) \to [0, \infty)$ denote the supremum norm on $C(V)$. In particular, given an AOU space $\mathcal{X}$, then $\mathcal{X}$ may be identified with the function system of complex-valued continuous affine functions on the state space of $\mathcal{X}$. In short, if we let $V := \{\varphi : \mathcal{X} \to \mathbb{C}\}$ denote the set of all unital positive maps on $\mathcal{X}$, then V is a weak-dual compact Hausdorff space and there exists a unital order isomorphism from $\mathcal{X}$ to $A(V)$, where $A(V) := \{F : V \to \mathbb{C} : F \text{ continuous affine}\}$.

We now review the theory of operator systems. Suppose $\mathcal{X}$ is a vector space with an involution $* : \mathcal{X} \to \mathcal{X}$. A *matrix ordering* on $\mathcal{X}$ is defined to be a collection $\mathcal{C} := \{\mathcal{C}_n\}_{n \in \mathbb{N}}$ of sets $\mathcal{C}_n \subset M_n(\mathcal{X})_h$ satisfying the following properties:

- $\mathcal{C}_n$ is a cone for every $n \in \mathbb{N}$. This is to say $\mathcal{C}_n + \mathcal{C}_n \subset \mathcal{C}_n$ and $\mathbb{R}^+\mathcal{C}_n \subset \mathcal{C}_n$ for every $n \in \mathbb{N}$.

- $a^*\mathcal{C}_n a \subset \mathcal{C}_m$ for every $a \in M_{n,m}$ and $n, m \in \mathbb{N}$. We will call this property *compatibility* of the family $\mathcal{C}$.

If $\mathcal{C}$ satisfies the additional property that $\mathcal{C}_n \cap -\mathcal{C}_n = \{0\}$ for every $n \in \mathbb{N}$, then we will call $\mathcal{C}$ a proper matrix ordering. The pair $(\mathcal{X}, \mathcal{C})$ will be called a *(proper) matrix ordered $*$-vector space* when $\mathcal{X}$ is a $*$-vector space and $\mathcal{C}$ is a (proper) matrix ordering. Given a natural number $n \in \mathbb{N}$ then we let $[n] := \{1, \ldots, n\}$.

We record the following well-known fact for convenience:

Lemma 2.0.1. *Let $\mathcal{X}$ be a $*$-vector space and suppose $\mathcal{C}$ is a matrix ordering such that $\mathcal{C}_1 \cap -\mathcal{C}_1 = \{0\}$. Then $\mathcal{C}$ is a proper matrix ordering.*

Proof. Assuming $\mathcal{C}_1$ is proper we claim $\mathcal{C}_n \cap -\mathcal{C}_n = \{0\}$ for every $n \in \mathbb{N}$. Suppose that $x \in \mathcal{C}_n \cap -\mathcal{C}_n$. Let $\{e_k\}_{k \in [n]} \subset \mathbb{C}^n$ denote the canonical basis vectors viewed as column vectors. It then follows for every $k \in [n]$ we have $x_{kk} = e_k^* x e_k \in \mathcal{C}_1 \cap -\mathcal{C}_1$. By assumption it must follow $x_{kk} = 0$. Let $k, l \in [n]$ such that $k \neq l$. Then $(e_k + e_l)^* x (e_k + e_l) = x_{lk} + x_{kl} = 2\operatorname{Re}(x_{lk}) \in \mathcal{C}_1 \cap -\mathcal{C}_1$ which implies $\operatorname{Re}(x_{lk}) = 0$. In a similar fashion $(e_k - \mathrm{i}e_l)^* x (e_k - \mathrm{i}e_l) = \mathrm{i}(x_{lk} - x_{kl}) = 2\mathrm{i}\operatorname{Im}(x_{lk}) \in \mathcal{C}_1 \cap -\mathcal{C}_1$ and therefore $\operatorname{Im}(x_{lk}) = 0$. Thus $x_{lk} = 0$, and it follows $x = 0$. This proves $\mathcal{C}_n$ is proper and thus $\mathcal{C}$ is a proper matrix ordering. $\square$

Throughout much of the manuscript we will be concerned with matrix ordered $*$-vector spaces that also contain a noncommutative order unit. To this end, consider a hermitian element $e \in \mathcal{X}_h$ which satisfies the following properties:

- Given $x \in M_n(\mathcal{X})_h$ there exists $r > 0$ such that $rI_n \otimes e - x \in \mathcal{C}_n$. This property ensures that there is a single element which majorizes the hermitian subspace of $\mathcal{X}$.

- An element $x \in M_n(\mathcal{X})$ is in $\mathcal{C}_n$ if and only if $\epsilon I_n \otimes e + x \in \mathcal{C}_n$ for every $\epsilon > 0$. This property ensures that each cone $\mathcal{C}_n$ is order closed with respect to the element e.

If an element $e \in \mathcal{X}_h$ satisfies Property (i) we will call it a *matrix order unit*. It is a well-known fact that if $(\mathcal{X}, \mathcal{C})$ is a proper matrix ordered $*$-vector space, then an element $e \in \mathcal{X}_h$ is an order unit if and only if it is a matrix order unit.

Proposition 2.0.1. *Given a matrix ordered $*$-vector space $\mathcal{X}$ then $e \in \mathcal{X}$ is an order unit if and only if it is a matrix order unit.*

Proof. Assume $e \in \mathcal{X}_h$ is an order unit and let $x \in M_n(\mathcal{X})_h$. We write $x = \sum_{\mathrm{i} \leq n} A_\mathrm{i} \otimes x_\mathrm{i} \in (M_n)_h \otimes \mathcal{X}_h$. For each i write $A_\mathrm{i} = P_\mathrm{i} - Q_\mathrm{i}$, where $P_\mathrm{i}, Q_\mathrm{i} \in M_n^+$. Choose $r > 0$ such that $re \pm x_\mathrm{i} \in \mathcal{X}^+$ for each i. We then see

$$r(\sum_{\mathrm{i} \leq n} P_\mathrm{i} + Q_\mathrm{i}) \otimes e - x = \sum_{\mathrm{i} \leq n} P_\mathrm{i} \otimes (re - x_\mathrm{i}) + \sum_{\mathrm{i} \leq n} Q_\mathrm{i} \otimes (re + x_\mathrm{i}) \in M_n(\mathcal{X})^+.$$

Simply choose $\tilde{r}$ such that $\tilde{r} I_n \geq r(\sum_{\mathrm{i} \leq n} P_\mathrm{i} + Q_\mathrm{i})$ which proves the claim. □

Definition 2.0.2 ([4])**.** *Let $(\mathcal{X}, \mathcal{C})$ be a proper matrix ordered $*$-vector space and let $e \in \mathcal{X}_h$ be an Archimedean matrix order unit. Then the triple $(\mathcal{X}, \mathcal{C}, e)$ is called an* operator system

Similar to the case for AOU spaces, when no confusion will arise we will simply denote an operator system as $\mathcal{X}$. Let X, Y be vector spaces and let $u : X \to Y$ be a linear map. Then u induces a linear map on the vector spaces of matrices over X, and Y. In particular, given $n \in \mathbb{N}$ we define the n^{th}*-amplification* of u as the map

$$u_n : M_n(X) \to M_n(Y), \ \sum_{\mathrm{ij}} e_\mathrm{i} e_\mathrm{j}^* \otimes x_{\mathrm{ij}} \mapsto \sum_{\mathrm{ij}} e_\mathrm{i} e_\mathrm{j}^* \otimes u(x_{\mathrm{ij}}).$$

If $(\mathcal{X}, \mathcal{C})$ and $(\mathcal{Y}, \mathcal{D})$ are matrix ordered $*$-vector spaces then we say a linear map $u : \mathcal{X} \to \mathcal{Y}$ is *completely positive* if $u_n(\mathcal{C}_n) \subset \mathcal{D}_n$ for every $n \in \mathbb{N}$. If $\mathcal{X}$ and $\mathcal{Y}$ are operator systems with Archimedean matrix order units $e^{\mathcal{X}}, e^{\mathcal{Y}}$, respectively, then u is *unital* if $u(e^{\mathcal{X}}) = e^{\mathcal{Y}}$. A unital completely positive map $u : \mathcal{X} \to M_n$ will sometimes be called a *matrix state* or a *matrix n-state*. In particular, this terminology will only be used when considering unital completely positive maps into a matrix algebra. If the map $u : \mathcal{X} \to \mathcal{Y}$ is a (unital) linear isomorphism and both u and u^{-1} are completely positive then we call u a *(unital) complete order isomorphism*. If u is a (unital) complete order isomorphism onto its range then we sometimes call u a *(unital) complete order embedding*. If $\mathcal{X}$ and $\mathcal{Y}$ are two operator systems and $u : \mathcal{X} \to \mathcal{Y}$ is a (unital) complete order isomorphism, then we say $\mathcal{X}$ and $\mathcal{Y}$ are *(unital) completely order isomorphic*.

It was realized in [4] that operator systems are precisely the abstract analogue of self-adjoint, unital subspaces of bounded operators on a Hilbert space. In particular, let H be a Hilbert space and consider the algebra of bounded linear operators on H, $B(H)$. Let $\mathcal{X} \subset B(H)$ be a subspace of $B(H)$ such that for all $x \in \mathcal{X}$ one has $x^* \in \mathcal{X}$, and furthermore suppose $\mathrm{Id}_H \in \mathcal{X}$. Consider the collection $\mathcal{C} := \{\mathcal{C}_n\}_{n\in\mathbb{N}}$ such that $\mathcal{C}_n := M_n(\mathcal{X}) \cap B(\ell_2^n(H))^+$ for each $n \in \mathbb{N}$. Here we have let $\ell_2^n(H)$ denote the n-fold Hilbertian direct sum. It is readily seen that $\mathcal{C}$ is a proper matrix ordering on $\mathcal{X}$ and furthermore, Id_H is an Archimedean matrix order unit for the proper matrix ordered $*$-vector space $(\mathcal{X}, \mathcal{C})$. Thus, $(\mathcal{X}, \mathcal{C}, \mathrm{Id}_H)$ is an operator system. Therefore we see that our characterization of operator systems as $*$-vector spaces with proper matrix orderings and an Archimedean matrix order unit is precisely the abstract characterization of unital self-adjoint subspaces of $B(H)$ where H is a Hilbert space.

Theorem 2.0.2 ([4, Theorem 4.4]). *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system. Then there exists a Hilbert space H, a unital self-adjoint subspace $\tilde{\mathcal{X}} \subset B(H)$ and a unital complete order isomorphism $\Phi : \mathcal{X} \to \tilde{\mathcal{X}}$. In particular, the operator system $\mathcal{X}$ may be realized as a unital self-adjoint subspace of $B(H)$ by making the identification $\mathcal{X} \simeq \Phi(\mathcal{X}) = \tilde{\mathcal{X}} \subset B(H)$.*

We saw earlier that every Archimedean order unit space may be identified with the continuous affine functions defined on its state space. A similar noncommutative analogue holds for operator systems. Let $V_n \subset M_n(\mathcal{X}^*)$ denote the set of all linear maps $x' : \mathcal{X} \to M_n$ such that x' is unital completely positive. Here we have let $\mathcal{X}^*$ denote the *Banach dual* of the operator system $\mathcal{X}$. Then $V := \{V_n\}_{n\in\mathbb{N}}$ is weak-dual compact matrix convex set. V is called the *matrix state space* of the operator system $\mathcal{X}$. V being *matrix convex* means that for each fixed $n \in \mathbb{N}$, that V_n is closed under sums of the form

$$\sum_{\mathrm{i}\leq n} \gamma_{\mathrm{i}}^* x'_{\mathrm{i}} \gamma_{\mathrm{i}},$$

where $x'_{\mathrm{i}} \in V_{n_{\mathrm{i}}}$ for each $\mathrm{i} \leq n, \gamma_{\mathrm{i}} \in M_{n_{\mathrm{i}},n}$ and $\sum_{\mathrm{i}\leq n} \gamma_{\mathrm{i}}^* \gamma_{\mathrm{i}} = I_n$. We call such a sum a *matrix convex combination*. A map $F := (F_n)_{n\in\mathbb{N}}, F_n : V_n \to M_n$ is *matrix affine* if for each $n \in \mathbb{N}$,

F_n preserves matrix convex combinations. We let $\mathcal{A}(V)$ denote all matrix affine functions $F := (F_n)_{n\in\mathbb{N}}$ such that $F_1 : V \to \mathbb{C}$ is continuous.

Proposition 2.0.2 ([23, Proposition 3.5])**.** *Given any operator system $\mathcal{X}$ then there exists a unital complete order isomorphism $\Phi : \mathcal{X} \to \mathcal{A}(V)$, where V denotes the matrix state space of the operator system $\mathcal{X}$.*

Consider a matrix ordered $*$-vector space $(\mathcal{X}, \mathcal{C})$. Then by letting $\mathcal{J} := \operatorname{span} \mathcal{C}_1 \cap -\mathcal{C}_1$ we may consider the quotient space $\mathcal{X}/\mathcal{J}$ along with the collection $\mathcal{C}+\mathcal{J}$. It readily follows that the pair $(\mathcal{X}/\mathcal{J}, \mathcal{C}+\mathcal{J})$ is a proper matrix ordered $*$-vector space. In particular if $(\mathcal{X}, \mathcal{C})$ is a matrix ordered $*$-vector space and $e \in \mathcal{X}_h$ is an Archimedean matrix order unit then it readily follows $(\mathcal{X}/\mathcal{J}, \mathcal{C}+\mathcal{J}, e+\mathcal{J})$ is an operator system. See Proposition 3.1.2.

An operator system $\mathcal{Y}$ is called *injective* if given any inclusion of operator systems $\mathcal{X} \subset \tilde{\mathcal{X}}$ then every unital completely positive map $u : \mathcal{X} \to \mathcal{Y}$ extends to a unital completely positive map $\tilde{u} : \tilde{\mathcal{X}} \to \mathcal{Y}$ such that $\tilde{u}|_{\mathcal{X}} = u$.

Fix an operator system $\mathcal{X}$. It was shown in [7] that there exists an injective operator system $\mathcal{I}(\mathcal{X})$ and a unital complete order embedding $\mathrm{j} : \mathcal{X} \to \mathcal{I}(\mathcal{X})$ such that if $\mathcal{E}$ is another injective operator system satisfying $\mathcal{X} \subset \mathcal{E} \subset \mathcal{I}(\mathcal{X})$ then $\mathcal{E}$ is unital completely order isomorphic to $\mathcal{I}(\mathcal{X})$. This is to say $\mathcal{I}(\mathcal{X})$ is the minimal injective object containing $\mathcal{X}$ as an operator subsystem and furthermore it is unique up to complete order isomorphism. We call $\mathcal{I}(\mathcal{X})$ the *injective envelope* of the operator system $\mathcal{X}$. The *C*-envelope*, $C_e^*(\mathcal{X})$, of the operator system $\mathcal{X}$ is the C*-algebra generated by $\mathrm{j}(\mathcal{X})$. In particular, the C*-envelope satisfies the following universal property: given any pair $(\mathcal{A}, \iota)$ where $\mathcal{A}$ is a unital C*-algebra and $\iota : \mathcal{X} \to \mathcal{A}$ is a unital complete order embedding such that $\mathcal{A} = C^*(\iota(\mathcal{X}))$, then there exists a unique $*$-epimorphism $\sigma : \mathcal{A} \to C_e^*(\mathcal{X})$ satisfying $\sigma\iota = \mathrm{j}$. Thus, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{A} \arrow[dr, dashed, "\sigma"] & \\ \mathcal{X} \arrow[u, "\iota"] \arrow[r, "\mathrm{j}"'] & C_e^*(\mathcal{X}) \end{tikzcd}$$

Remark 2.0.3. For the convenience of the reader we discuss the steps in showing the existence of an injective envelope of an operator system. Let $\mathcal{X} \subset B(H)$ be an operator system. Following the standard terminology, a linear map $\varphi : B(H) \to B(H)$ will be called an $\mathcal{X}$*-map* if it is completely positive and $\varphi|_{\mathcal{X}} = \mathrm{Id}_{\mathcal{X}}$. Given an $\mathcal{X}$-map $\varphi : B(H) \to B(H)$ then the map $p_\varphi : B(H) \to [0, \infty)$ defined by $p_\varphi(x) := \left\|\varphi(x)\right\|$, will be called an $\mathcal{X}$*-seminorm*. If $\varphi : B(H) \to B(H)$ is an $\mathcal{X}$-map such that $\varphi\varphi = \varphi$ then we call φ an $\mathcal{X}$*-projection*. In the construction of the injective envelope, one first shows that there exists a minimal $\mathcal{X}$-seminorm. This is done by cosidering a net of $\mathcal{X}$-maps $\{\varphi_{\mathrm{i}}\}_{\mathrm{i}\in I}$ such that the induced $\mathcal{X}$-seminorms $\{p_{\varphi_{\mathrm{i}}}\}_{\mathrm{i}\in I}$ form a descending chain. One proves such chain has a lower bound at which point one invokes Zorn's Lemma. The next step is proving that if $\varphi : B(H) \to B(H)$ is an $\mathcal{X}$-map such that $p_\varphi : B(H) \to [0, \infty)$ is a minimal $\mathcal{X}$-seminorm then φ is necessarily a minimal $\mathcal{X}$-projection and $\varphi(B(H))$ is an injective envelope. One then proves that the injective envelope is unique up to isomorphism (in the category).

Consider an AOU space $(\mathcal{X}, C, e)$. We will call a matrix ordering $\mathcal{C}$ an *operator system structure* on $\mathcal{X}$ if $\mathcal{C}_1 = C$ and the triple $(\mathcal{X}, \mathcal{C}, e)$ is an operator system. It was proven in [18] that one may construct a unique minimal, and unique maximal operator system structure on an AOU space. The maximal operator system structure will play a pivotal role in the early sections of the manuscript. Consider the collection $D^{\max} := \{D_n^{\max}\}_{n\in\mathbb{N}}$ where for each $n \in \mathbb{N}$ we have

$$D_n^{\max} := \{a^* x a : x = \bigoplus_{\mathrm{i}\in[m]} x_{\mathrm{i}}, x_{\mathrm{i}} \in C, a \in M_{m,n}, m \in \mathbb{N}\}.$$

Furthermore define the collection $C^{\max} := \{C_n^{\max}\}_{n\in\mathbb{N}}$ where for each $n \in \mathbb{N}$ we have

$$C_n^{\max} := \{x \in M_n(\mathcal{X}) : \epsilon I_n \otimes e + x \in D_n^{\max},\ \forall \epsilon > 0\}.$$

It is readily seen that the triple $(\mathcal{X}, C^{\max}, e)$ is an operator system which we call the *1-max (or maximal) operator system* and we call $C^{\max}$ the *1-max (or maximal) operator system structure on* $\mathcal{X}$. The 1-max operator system structure satisfies the property that if $\mathcal{C}$ is any other operator system structure on $\mathcal{X}$ then it follows $C^{\max} \subset \mathcal{C}$, which is to say for every $n \in \mathbb{N}$ we have $C_n^{\max} \subset \mathcal{C}_n$. When no confusion will arise we will denote the triple $(\mathcal{X}, C^{\max}, e)$ by $\mathcal{X}_{\max}$.

3. PROJECTIONS IN OPERATOR SYSTEMS

3.1 Single Projections in Operator Systems

Consider a concrete operator system $\mathcal{X} \subset B(H)$. Recall that the matrix ordering $\mathcal{C}$ of $\mathcal{X}$ is defined for each $n \in \mathbb{N}$ by

$$\mathcal{C}_n := M_n(\mathcal{X}) \cap M_n(B(H)))^+.$$

Suppose that $p \in \mathcal{C}_1$ is an element of $\mathcal{X}$ that acts as a projection on the Hilbert space H. Such a p induces a collection of subsets $\tilde{\mathcal{C}}(p) := \{\tilde{\mathcal{C}}(p)_n\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$ we have

$$\tilde{\mathcal{C}}(p)_n := \{x \in M_n(\mathcal{X})_h : (I_n \otimes p)x(I_n \otimes p) \in B(\ell_2^n(H))^+\}.$$

Note that $\mathcal{C} \subset \tilde{\mathcal{C}}(p)$.

Proposition 3.1.1. *Let $\mathcal{X} \subset B(H)$ be an operator system and let $p \in \mathcal{C}_1$ be a projection on the Hilbert space H. Then $\{\tilde{\mathcal{C}}(p)_n\}_{n\in\mathbb{N}}$ is a matrix ordering on $\mathcal{X}$ and if $p \leq e$ then p is an Archimedean matrix order unit for the matrix ordered $*$-vector space $(\mathcal{X}, \tilde{\mathcal{C}}(p))$.*

Proof. We begin by showing that $\tilde{\mathcal{C}}(p)$ is a matrix ordering on $\mathcal{X}$. Note that the compression by such an element p is a linear completely positive map. In particular, if $\lambda \in \mathbb{R}^+$ and $x \in \tilde{\mathcal{C}}(p)_n$ then $(I_n \otimes p)(\lambda x)(I_n \otimes p) = \lambda(I_n \otimes p)x(I_n \otimes p) \in \lambda B(\ell_2^n(H))^+ \subset B(\ell_2^n(H))^+$. Similarly linearity and complete positivity of the compression by the projection p readily implies that $\tilde{\mathcal{C}}(p)_n + \tilde{\mathcal{C}}(p)_n \subset \tilde{\mathcal{C}}(p)_n$. Finally if $a \in M_{n,m}$ then

$$\begin{aligned}(I_m \otimes p)(a^*xa)(I_m \otimes p) &= \left[p\left(\sum_{k,l=1}^{n} \overline{a}_{k\mathrm{i}} x_{kl} a_{l\mathrm{j}}\right)p\right]_{\mathrm{i,j}=1}^{m} = \left[\sum_{k,l=1}^{n} \overline{a}_{k\mathrm{i}} p x_{kl} p a_{l\mathrm{j}}\right]_{\mathrm{i,j}=1}^{m} \\ &= a^*(I_n \otimes p)x(I_n \otimes p)a \in a^*B(\ell_2^n(H))^+a.\end{aligned}$$

Since $a^*B(\ell_2^n(H))^+a \subset B(\ell_2^m(H))^+$ this proves that $\tilde{\mathcal{C}}(p)$ is a matrix ordering on $\mathcal{X}$.

We now show $p \in \mathcal{C}_1$ is an Archimedean matrix order unit for $(\mathcal{X}, \tilde{\mathcal{C}}(p))$. We first show that p is a matrix order unit. By Proposition 2.0.1 it suffices to show that p is an order unit. First note that since e is an Archimedean matrix order unit for $(\mathcal{X}, \mathcal{C})$ then for $x \in \mathcal{X}_h$ there exists $r > 0$ such that $re - x \in \mathcal{C}_1$ and thus $rp - pxp \in B(H)^+$. In particular, we see that p is an order unit, and thus matrix order unit, for operators of the form $pxp, x \in \mathcal{X}_h$. Furthermore suppose $x \in M_n(\mathcal{X})_h$ and that for all $\epsilon > 0$ we assume that $\epsilon(I_n \otimes p) + (I_n \otimes p)x(I_n \otimes p) \in B(\ell_2^n(H))^+$. Since $p \leq e$ it follows

$$\epsilon(I_n \otimes e) + (I_n \otimes p)x(I_n \otimes p) \geq \epsilon(I_n \otimes p) + (I_n \otimes p)x(I_n \otimes p) \in B(\ell_2^n(H))^+.$$

Since e is an Archimedean matrix order unit for $B(\ell_2^n(H))$ it follows that $(I_n \otimes p)x(I_n \otimes p) \in B(\ell_2^n(H))^+$ and in particular $x \in \tilde{\mathcal{C}}(p)_n$. Thus, p is an Archimedean matrix order unit for operators of the form pxp where $x \in \mathcal{X}_h$.

Thus, given $x \in \mathcal{X}_h$ let $r > 0$ such that $rp - pxp \in B(H)^+$. If we consider $rp - x$ then $p(rp - x)p = rp - pxp \in B(H)^+$ which implies that $rp - x \in \tilde{\mathcal{C}}(p)_1$. This proves that p is a matrix order unit for $(\mathcal{X}, \tilde{\mathcal{C}}(p))$. Finally suppose that $x \in M_n(\mathcal{X})$ such that for all $\epsilon > 0$ we have $\epsilon(I_n \otimes p) + x \in \tilde{\mathcal{C}}(p)_n$. This implies $\epsilon(I_n \otimes p) + (I_n \otimes p)x(I_n \otimes p) \in B(\ell_2^n(H))^+$ for all $\epsilon > 0$. By our earlier remarks this necessarily implies that $(I_n \otimes p)x(I_n \otimes p) \in B(\ell_2^n(H))^+$ which implies that $x \in \tilde{\mathcal{C}}(p)_n$. This proves that p is an Archimedean matrix order unit for the matrix ordered $*$-vector space $(\mathcal{X}, \tilde{\mathcal{C}}(p))$. $\square$

We now consider a quotient $*$-vector space of the matrix ordered $*$-vector space $(\mathcal{X}, \tilde{\mathcal{C}}(p))$. Define $\mathcal{J} := \operatorname{span} \tilde{\mathcal{C}}(p)_1 \cap -\tilde{\mathcal{C}}(p)_1$. It is readily checked that $\mathcal{J}$ is a $*$-closed subspace of $\mathcal{X}_h$.

Lemma 3.1.1. *Let $\mathcal{X} \subset B(H)$ be an operator system and let $p \in \mathcal{C}_1$ be a projection on H. For $n \in \mathbb{N}$ define $\mathcal{J}_n :=$ span$\tilde{\mathcal{C}}(p)_n \cap -\tilde{\mathcal{C}}(p)_n$ then $M_n(\mathcal{J}) = \mathcal{J}_n$ for all $n \in \mathbb{N}$.*

Proof. Let $x \in M_n(\mathcal{J})$ which we write as $x = \sum_{\mathrm{ij}} e_{\mathrm{i}} e_{\mathrm{j}}^* \otimes x_{\mathrm{ij}}$ where $x_{\mathrm{ij}} \in \mathcal{J}$. Thus for each $\mathrm{i}, \mathrm{j} \in [n]$ we write $x_{\mathrm{ij}} = \sum_k \lambda_{\mathrm{ij}k} x_{\mathrm{ij}k}$ where $x_{\mathrm{ij}k} \in \tilde{\mathcal{C}}(p)_1 \cap -\tilde{\mathcal{C}}(p)_1$. It readily follows that

$$(I_n \otimes p)x(I_n \otimes p) = (I_n \otimes p)\left(\sum_{\mathrm{ij}k} \lambda_{\mathrm{ij}k} e_{\mathrm{i}} e_{\mathrm{j}}^* \otimes x_{\mathrm{ij}k}\right)(I_n \otimes p) = \sum_{\mathrm{ij}k} \lambda_{\mathrm{ij}k} e_{\mathrm{i}} e_{\mathrm{j}}^* \otimes p x_{\mathrm{ij}k} p.$$

It then follows $(I_n \otimes p)x(I_n \otimes p) = 0$ since $\pm p x_{\mathrm{ij}k} p \in B(H)^+$ for all $\mathrm{i}, \mathrm{j}, k$. Thus $x \in \tilde{\mathcal{C}}(p)_1 \cap -\tilde{\mathcal{C}}(p)_1$.

Conversely suppose $x \in \mathcal{J}_n$ which we write as $x = \sum_k \lambda_k x_k$ where $x_k \in \tilde{\mathcal{C}}(p)_n \cap -\tilde{\mathcal{C}}(p)_n$. For each k we write $x_k = \sum_{\text{ij}} e_{\text{i}} e_{\text{j}}^* \otimes x_{k\text{ij}} \in M_n(\mathcal{X})$. Since $(I_n \otimes p) x_k (I_n \otimes p) = 0$ for each k implies that $p x_{k\text{ij}} p = 0$ for all i, j. Thus $x_{k\text{ij}} \in \tilde{\mathcal{C}}(p)_1 \cap -\tilde{\mathcal{C}}(p)_1$ which implies $x = \sum_{\text{ij}k} e_{\text{i}} e_{\text{j}}^* \otimes \lambda_k x_{k\text{ij}} \in M_n(\mathcal{J})$. This finishes the proof. $\square$

In this setting we thus consider the quotient $*$-vector space $\mathcal{X}/\mathcal{J}$ where $*$ denotes the relative involution on cosets. Finally if $\tilde{\mathcal{C}}(p)$ denotes the matrix ordering induced by the projection p we consider the collection $\tilde{\mathcal{C}}(p) + \mathcal{J} := \{\tilde{\mathcal{C}}(p)_n + M_n(\mathcal{J})\}_{n \in \mathbb{N}}$.

Theorem 3.1.2. *Let $\mathcal{X} \subset B(H)$ be an operator system and let $p \in \mathcal{C}_1$ be a projection on the Hilbert space H. Furthermore let $\mathcal{J} := span\, \tilde{\mathcal{C}}(p)_1 \cap -\tilde{\mathcal{C}}(p)_1$. Then $(\mathcal{X}/\mathcal{J}, \tilde{\mathcal{C}}(p) + \mathcal{J}, p + \mathcal{J})$ is an operator system.*

Proof. We need only check that $\tilde{\mathcal{C}}(p) + \mathcal{J}$ is a proper matrix ordering and that $p + \mathcal{J}$ is an Archimedean matrix order unit. Note that since $\tilde{\mathcal{C}}(p)$ is a matrix ordering on $\mathcal{X}$ and $\mathcal{J}$ is a $*$-closed subspace it readily follows that $\mathbb{R}^+ \tilde{\mathcal{C}}(p) + \mathcal{J} \subset \tilde{\mathcal{C}}(p) + \mathcal{J}$ and $(\tilde{\mathcal{C}}(p) + \mathcal{J}) + (\tilde{\mathcal{C}}(p) + \mathcal{J}) \subset \tilde{\mathcal{C}}(p) + \mathcal{J}$. Furthermore, compatibility of $\tilde{\mathcal{C}}(p)$ implies if $a \in M_{n,k}$ then $a^* \tilde{\mathcal{C}}(p)_n a + \mathcal{J}_k \subset \tilde{\mathcal{C}}(p)_k + \mathcal{J}_k$ and therefore $\tilde{\mathcal{C}}(p) + \mathcal{J}$ is a matrix ordering on the quotient $*$-vector space $\mathcal{X}/\mathcal{J}$. If $\pm x + M_n(\mathcal{J}) \in \tilde{\mathcal{C}}(p)_n + M_n(\mathcal{J})$ then $\pm x \in \tilde{\mathcal{C}}(p)_n$ which implies $x \in \mathcal{J}_n$. By Lemma 3.1.1 it follows since $M_n(\mathcal{J}) = \mathcal{J}_n$ we have $x + M_n(\mathcal{J}) = 0 + M_n(\mathcal{J})$ and therefore the matrix ordering $\tilde{\mathcal{C}}(p) + \mathcal{J}$ is proper.

Given $x + \mathcal{J} \in (\mathcal{X}/\mathcal{J})_h$ it follows $x \in \mathcal{X}_h$ and therefore there exists $r > 0$ such that $rp - x \in \tilde{\mathcal{C}}(p)_1$. In particular we have $(rp + \mathcal{J}) - (x + \mathcal{J}) \in \tilde{\mathcal{C}}(p)_1 + \mathcal{J}$. This proves $p + \mathcal{J}$ is an order unit and thus is a matrix order unit. Finally if $x + M_n(\mathcal{J}) \in M_n(\mathcal{X}/\mathcal{J})$ such that for $\epsilon > 0$ it follows $(\epsilon(I_n \otimes p) + x) + M_n(\mathcal{J}) \in \tilde{\mathcal{C}}(p)_n + M_n(\mathcal{J})$ then necessarily we have $\epsilon(I_n \otimes p) + x \in \tilde{\mathcal{C}}(p)_n$ for all $\epsilon > 0$. Since $\tilde{\mathcal{C}}(p)$ is Archimedean closed with respect to p it follows $x \in \tilde{\mathcal{C}}(p)_n$ and consequently $x + M_n(\mathcal{J}) \in \tilde{\mathcal{C}}(p)_n + M_n(\mathcal{J})$. Therefore $(\mathcal{X}/\mathcal{J}, \tilde{\mathcal{C}}(p) + \mathcal{J}, p + \mathcal{J})$ is an operator system. $\square$

Thus we have shown that any positive element of $\mathcal{X}$ which acts as a projection the the Hilbert space H in turn induces a natural operator system. We thus arrive at the following definition:

Definition 3.1.1. *Let $\mathcal{X} \subset B(H)$ be an operator system and let $p \in \mathcal{C}_1$ be a positive element of $\mathcal{X}$ which acts as a projection on H and such that $p \leq e$. If $\tilde{\mathcal{C}}(p)$ denotes the matrix ordering induced by p and if we let $\mathcal{J} := span\, \tilde{\mathcal{C}}(p)_1 \cap -\tilde{\mathcal{C}}(p)_1$ then we call the triple $(\mathcal{X}/\mathcal{J}, \tilde{\mathcal{C}}(p) + \mathcal{J}, p + \mathcal{J})$ the* concrete compression operator system *relative to p. We will denote the concrete compression operator system relative to p by $p\mathcal{X}p$, regarded as linear operators on pH.*

Throughout the rest of the manuscript we will be primarily concerned with operator systems induced by the direct sum of projections. In particular, we wish to formulate the abstract analogue of the concrete compression operator system induced by the direct sum of a projection p and its orthogonal $p^\perp$. Our results imply that if $\mathcal{X} \subset B(H)$ is an operator system with $p \in \mathcal{C}_1$ then since $p \leq e$ it follows that $p^\perp := e - p \in \mathcal{C}_1$ and acts as a projection on H. We thus define the matrix ordering $\mathcal{C}(p)$ on $M_2(\mathcal{X})$ to be defined for each $n \in \mathbb{N}$ as

$$\mathcal{C}(p)_n := \{x \in M_2(\mathcal{X})_h : (I_n \otimes (p \oplus p^\perp))x(I_n \otimes (p \oplus p^\perp)) \in B(\ell_2^{2n}(H))^+\}.$$

This yields the following corollary:

Corollary 3.1.3. *Let $\mathcal{X} \subset B(H)$ be an operator system and let $p \in \mathcal{C}_1$ such that p acts as a projection on H. Let $\mathcal{C}(p)$ be the collection of subsets $\{\mathcal{C}(p)_n\}_{n\in\mathbb{N}}$ where $\mathcal{C}(p)_n := \{x \in M_{2n}(\mathcal{X})_h : (I_n \otimes (p \oplus p^\perp))x(I_n \otimes (p \oplus p^\perp)) \in B(\ell_2^{2n}(H))^+\}$. Define $\mathcal{J} := span\, \mathcal{C}(p)_1 \cap -\mathcal{C}(p)_1$. Then the triple $(M_2(\mathcal{X})/\mathcal{J}, \mathcal{C}(p) + \mathcal{J}, p \oplus p^\perp + \mathcal{J})$ is an operator system. In particular, $\mathcal{C}(p) + \mathcal{J}$ is a proper matrix ordering on $M_2(\mathcal{X})/\mathcal{J}$ and $p \oplus p^\perp + \mathcal{J}$ is an Archimedean matrix order unit.*

In order to formulate an abstract analogue of our previously defined concrete compression operator systems, we begin by showing that given any matrix ordered $*$-vector space $(\mathcal{X}, \mathcal{C})$ and an element $e \in \mathcal{C}_1$ that acts as an Archimedeam matrix order unit, then there is always a canonical operator system obtained by taking a particular quotient.

Lemma 3.1.4. *Let $\mathcal{X}$ be a $*$-vector space and let $\mathcal{C}$ be a matrix ordering on $\mathcal{X}$. For each $n \in \mathbb{N}$ we define the $*$-subspace $\mathcal{J}_n := span\, \mathcal{C}_n \cap -\mathcal{C}_n$, where we will let $\mathcal{J} := \mathcal{J}_1$. Then $M_n(\mathcal{J}) = \mathcal{J}_n$ for all $n \in \mathbb{N}$.*

Proof. First suppose that $x \in \mathcal{J}_n$. Then we write $x = \sum_k \lambda_k x_k$, where $x_k \in \mathcal{C}_n \cap -\mathcal{C}_n$. For each k we write $\lambda_k = a_k + \mathrm{i} b_k$ which yields $x = \sum_k \lambda_k x_k = \sum_k a_k x_k + \mathrm{i} \sum_k b_k x_k$. We see that if we let $y_1 := \sum_k a_k x_k$ and $y_2 := \sum_k b_k x_k$ then both $y_1, y_2 \in \mathcal{C}_n \cap -\mathcal{C}_n$. In particular this implies that if $x \in \mathcal{J}_n$ then we may write $x = y_1 + \mathrm{i} y_2$ such that $y_{\mathrm{i}} \in \mathcal{C}_n \cap -\mathcal{C}_n$.

We will prove that given any $x \in \mathcal{J}_1$ that $e_k e_l^* \otimes x \in \mathcal{J}_n$. This will prove that $M_n(\mathcal{J}) \subset \mathcal{J}_n$. By our remarks above we write $x = y_1 + \mathrm{i} y_2$ where $y_1, y_2 \in \mathcal{C}_1 \cap -\mathcal{C}_1$. Since $\mathcal{C}$ is a matrix ordering necessarily implies that for $k, l \in [n]$ we have $e_k e_k^* \otimes \pm y_1$, and $e_k e_k^* \otimes \pm y_2$ are in $\mathcal{C}_n$ which necessarily implies $\pm e_k e_k^* \otimes x \in \mathcal{J}_n$. Similarly $(e_k + e_l)(e_k + e_l)^* \otimes \pm x \in \mathcal{J}_n$ and $(e_k + \mathrm{i} e_l)(e_k + \mathrm{i} e_l)^* \otimes \pm x \in \mathcal{J}_n$. It thus follows

$$((e_k + e_l)(e_k + e_l)^* - e_k e_k^* - e_l e_l^*) \otimes x = (e_k e_l^* + e_l e_k^*) \otimes x \in \mathcal{J}_n. \tag{3.1}$$

Similarly it follows

$$\mathrm{i}((e_k + \mathrm{i} e_l)(e_k + \mathrm{i} e_l)^* - e_k e_k^* - e_l e_l^*) \otimes x = (e_k e_l^* - e_l e_k^*) \otimes x \in \mathcal{J}_n. \tag{3.2}$$

Summing these equations together implies that $e_k e_l^* \otimes x \in \mathcal{J}_n$ which proves the first inclusion.

Consider now $x \in \mathcal{J}_n$ which by our remarks above we write as $x = y_1 + \mathrm{i} y_2$ where $y_s \in \mathcal{C}_n \cap -\mathcal{C}_n$. We will show that if $y_1 = \sum_{kl} e_k e_l^* \otimes y_{1kl}$ that $y_{1kl} \in \mathcal{J}$ for each k, l. The proof for y_2 will be the same. These together will prove that $x \in M_n(\mathcal{J})$. By the assumption that $\mathcal{C}$ is a matrix ordering necessarily implies that given $k, l \in [n]$ that $e_k^* y_1 e_k = y_{1kk}, e_l^* y_1 e_l = y_{1ll}$ are contained in $\mathcal{J}$. Similarly, $(e_k + e_l)^* y_1 (e_k + e_l) = y_{1kk} + y_{1kl} + y_{1lk} + y_{1ll} \in \mathcal{J}$ and $(e_k + \mathrm{i} e_l)^* y_1 (e_k + \mathrm{i} e_l) = y_{1kk} + \mathrm{i} y_{1kl} - \mathrm{i} y_{1lk} + y_{1ll} \in \mathcal{J}$. Thus we see that $y_{1kl} + y_{1lk} \in \mathcal{J}$ and $y_{1kl} - y_{1lk} \in \mathcal{J}$ which implies $y_{1kl} \in \mathcal{J}$ as desired. $\square$

Similar to our results above we thus are able to prove that every matrix ordered $*$-vector space e with Archimedean matrix order unit e induces a canonical operator system.

Proposition 3.1.2. *Let $(\mathcal{X}, \mathcal{C})$ be a matrix ordered $*$-vector space and suppose that e is an Archimedean matrix order unit. Let $\mathcal{J} := \mathrm{span}\, \mathcal{C}_1 \cap -\mathcal{C}_1$. Then the triple $(\mathcal{X}/\mathcal{J}, \mathcal{C} + \mathcal{J}, e + \mathcal{J})$ is an operator system.*

Proof. The proof that $\mathcal{C} + \mathcal{J}$ is a matrix ordering is similar to the method of proof used in Theorem 3.1.2 . Suppose that $\pm x + M_n(\mathcal{J}) \in \mathcal{C}_n + M_n(\mathcal{J})$. This implies $\pm x \in \mathcal{C}_n$ and thus $x \in \mathcal{J}_n$ which by Lemma 3.1.4 implies $x \in M_n(\mathcal{J})$. In particular, $x + M_n(\mathcal{J}) = 0 + M_n(\mathcal{J})$. This proves that $\mathcal{C} + \mathcal{J}$ is a proper matrix ordering on $\mathcal{X}/\mathcal{J}$. The proof that $e + \mathcal{J}$ is an Archimedean matrix order unit is a direct consequence of the fact that e is an Archimedean matrix order unit for the pair $(\mathcal{X}, \mathcal{C})$. $\square$

Throughout the manuscript we will be considering matrix ordered $*$-vector spaces which may not be a priori proper. Thus, time and again we will rely on the techniques of the above results to consider the canonical induced operator systems. We first relate the order structure as defined in the concrete compression operator systems to the initial ordering of the operator system.

Note that if $\mathcal{X} \subset B(H)$ is an operator system and $p \in \mathcal{C}_1$ is a positive contraction which acts as a projection on H then given $x \in \mathcal{C}_1$, since conjugation by p is a positive map (on $B(H)$) it follows that $pxp \in B(H)^+$. In the next result we determine an equivalent condition for the converse.

Lemma 3.1.5. *Let $\mathcal{X} \subset B(H)$ be an operator system and let $p \in \mathcal{C}_1$ be a positive contraction that acts as a projection on the Hilbert space H. Then for any $x \in \mathcal{X}_h$ it follows $pxp \in B(H)^+$ if and only if for all $\epsilon > 0$ there exists $t > 0$ such that $x + \epsilon p + tp^\perp \in \mathcal{C}_1$.*

Proof. The if direction is immediate by our remarks preceding the lemma along with the assumption that e is an Archimedean matrix order unit for $B(H)$. For the converse, suppose that we write $H = pH \oplus p^\perp H$. It then follows an operator $u \in B(H)$ is positive if and only if both $pup, p^\perp up^\perp$ are both positive and for all $\xi = \xi_1 + \xi_2 \in pH \oplus p^\perp H$ we have

$$\left|\langle pup^\perp \xi_2 | \xi_1 \rangle\right|^2 \leq \langle pup\xi_1 | \xi_1 \rangle \langle p^\perp up^\perp \xi_2 | \xi_2 \rangle.$$

Let $\epsilon > 0$ and choose $t > \|x\|$ such that $\epsilon(t - \|x\|) > \|x\|^2$. Let $u := x + \epsilon p + tp^\perp$. We then have $pup^\perp = pxp^\perp, p^\perp up = p^\perp xp$. Furthermore, $pup = pxp + \epsilon p \in B(H)^+$ since

$pxp \in B(H)^+$. It follows $p^\perp u p^\perp = p^\perp x p^\perp + tp^\perp \geq tp^\perp - \|x\| p^\perp = (t - \|x\|)p^\perp \in B(H)^+$. It then follows

$$\left|\langle pup^\perp \xi_2 | \xi_1 \rangle\right|^2 \leq \|x\|^2 \|\xi_1\|^2 \|\xi_2\|^2 < \epsilon(t - \|x\|)\|\xi_1\|^2 \|\xi_2\|^2.$$

It is then immediate

$$\epsilon\|\xi_1\|^2 = \langle \epsilon p \xi_1 | \xi_1 \rangle \leq \langle pxp\xi_1 | \xi_1 \rangle + \langle \epsilon p \xi_1 | \xi_1 \rangle = \langle pup\xi_1 | \xi_1 \rangle,$$

and

$$(t - \|x\|)\|\xi_2\|^2 = \langle (t - \|x\|)p^\perp \xi_2 | \xi_2 \rangle \leq \langle p^\perp u p^\perp \xi_2 | \xi_2 \rangle.$$

We then obtain

$$\left|\langle pup^\perp \xi_2 | \xi_1 \rangle\right|^2 \leq \langle pup\xi_1 | \xi_1 \rangle \langle p^\perp u p^\perp \xi_2 | \xi_2 \rangle. \tag{3.3}$$

Thus, $u = x + \epsilon p + tp^\perp \in \mathcal{X} \cap B(H)^+$ which is precisely the cone $\mathcal{C}_1$. □

Definition 3.1.2. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $p \in \mathcal{X}$ be a positive contraction. Define the collection $\mathcal{C}(p) := \{\mathcal{C}(p)_n\}_{n\in\mathbb{N}}$ with base space $M_2(\mathcal{X})$ for each $n \in \mathbb{N}$ by*

$$\mathcal{C}(p)_n := \{x \in M_{2n}(\mathcal{X})_h : \forall \epsilon > 0\ \exists t > 0 \text{ such that} \tag{3.4}$$

$$x + \epsilon I_n \otimes (p \oplus p^\perp) + tI_n \otimes (p^\perp \oplus p) \in \mathcal{C}_{2n}\}. \tag{3.5}$$

Theorem 3.1.6. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $p \in \mathcal{C}_1$ be a positive contraction. Then $\mathcal{C}(p)$ is a matrix ordering on $M_2(\mathcal{X})$. Furthermore if $\mathcal{J} := span\ \mathcal{C}(p)_1 \cap -\mathcal{C}(p)_1$ then the triple $(M_2(\mathcal{X})/\mathcal{J}, \mathcal{C}(p) + \mathcal{J}, (p \oplus p^\perp) + \mathcal{J})$ is an operator system.*

Proof. We begin by showing that $\mathcal{C}(p)$ is a matrix ordering on $M_2(\mathcal{X})$. Let $y \in \mathcal{C}(p)_n$ and let $\lambda > 0$. Then if $\epsilon > 0$ choose $t > 0$ such that $y + \frac{\epsilon}{\lambda}(I_n \otimes (p \oplus p^\perp)) + t(I_n \otimes (p^\perp \oplus p)) \in \mathcal{C}_{2n}$. It then follows $\lambda y + \epsilon(I_n \otimes (p \oplus p^\perp)) + \lambda t(I_n \otimes (p^\perp \oplus p)) \in \lambda \mathcal{C}_{2n} \subset \mathcal{C}_{2n}$. Thus, $\mathbb{R}^+\mathcal{C}(p) \subset \mathcal{C}(p)$. Consider now $y_1, y_2 \in \mathcal{C}(p)_n$ and let $\epsilon > 0$. Then there exists $t_i > 0$ such that $y_i + \frac{\epsilon}{2}(I_n \otimes (p \oplus p^\perp)) + t_i(I_n \otimes (p^\perp \oplus p)) \in \mathcal{C}_{2n}$. Let $t := \max\{t_1, t_2\}$. This implies $(y_1 + y_2) + \epsilon(I_n \otimes (p \oplus p^\perp)) + t(I_n \otimes (p^\perp \oplus p)) \in$

$\mathcal{C}_{2n}$. Therefore $\mathcal{C}(p) + \mathcal{C}(p) \subset \mathcal{C}(p)$. It remains to show compatibility of the collection $\mathcal{C}(p)$. Let $y \in \mathcal{C}(p)_n$ and suppose $a \in M_{n,k}$. We claim $(a \otimes I_2)^* y (a \otimes I_2) \in \mathcal{C}(p)_k$. Let $\epsilon > 0$ and let $t > 0$ such that $y + \frac{\epsilon}{\|a\|^2}(I_n \otimes (p \oplus p^\perp)) + t(I_n \otimes (p^\perp \oplus p)) \in \mathcal{C}_{2n}$. Then it necessarily follows

$$(a \otimes I_2)^* y (a \otimes I_2) + \frac{\epsilon}{\|a\|^2}(a \otimes I_2)^*(I_n \otimes (p \oplus p^\perp))(a \otimes I_2) \\ + t(a \otimes I_2)^*(I_n \otimes (p^\perp \oplus p))(a \otimes I_2)$$

is an element of $\mathcal{C}_{2k}$. Since we have $(a \otimes I_2)^*(I_n \otimes (p \oplus p^\perp))(a \otimes I_2) \leq \|a\|^2(I_k \otimes (p \oplus p^\perp))$ and $(a \otimes I_2)^*(I_n \otimes (p^\perp \oplus p))(a \otimes I_2) \leq \|a\|^2(I_k \otimes (p^\perp \oplus p))$ then it follows

$$(a \otimes I_2)^* y (a \otimes I_2) + \epsilon(I_k \otimes (p \oplus p^\perp)) + t\|a\|^2(I_k \otimes (p^\perp \oplus p)) \in \mathcal{C}_{2k}.$$

Consequently $(a \otimes I_2)^* y (a \otimes I_2) \in \mathcal{C}(p)_k$ which proves $\mathcal{C}(p)$ is a matrix ordering on $M_2(\mathcal{X})$.

It is then an immediate consequence of our previous results that $\mathcal{C}(p) + \mathcal{J}$ is a proper matrix ordering on the quotient $*$-vector space $M_2(\mathcal{X})/\mathcal{J}$.

It remains to show $(p \oplus p^\perp) + \mathcal{J}$ is an Archimedean matrix order unit for $M_2(\mathcal{X})/\mathcal{J}$. By Proposition 2.0.1 it suffices to show $(p \oplus p^\perp) + \mathcal{J}$ is an order unit. To this end let $x \in M_2(\mathcal{X})/\mathcal{J}$ and choose $r > 0$ such that $r(I_2 \otimes e) - x \in \mathcal{C}_2$. It then follows if $\epsilon > 0$ and we let $t = r$ we have

$$(r(p \oplus p^\perp) - x) + \epsilon(p \oplus p^\perp) + r(p^\perp \oplus p) = (r(I_2 \otimes e) - x) + \epsilon(p \oplus p^\perp) \in \mathcal{C}_2.$$

Thus, $(r(p \oplus p^\perp) + \mathcal{J}) - (x + \mathcal{J}) \in \mathcal{C}(p)_1 + \mathcal{J}$ which proves $(p \oplus p^\perp) + \mathcal{J}$ is an order unit and therefore a matrix order unit.

Consider $x \in M_n(M_2(\mathcal{X})/\mathcal{J})$ such that $\delta(I_n \otimes (p \oplus p^\perp)) + x \in \mathcal{C}(p)_n$ for all $\delta > 0$. Furthermore let $\epsilon > 0$. It then follows that there exists $t > 0$ such that $(\frac{\epsilon}{2}(I_n \otimes (p \oplus p^\perp)) + x) + \frac{\epsilon}{2}(I_n \otimes (p \oplus p^\perp)) + t(I_n \otimes (p^\perp \oplus p)) \in \mathcal{C}_{2n}$. Thus, we see $x + \epsilon(I_n \otimes (p \oplus p^\perp)) + t(I_n \otimes (p^\perp \oplus p)) \in \mathcal{C}_{2n}$ which implies $x \in \mathcal{C}(p)_n$ and consequently $x + M_n(\mathcal{J}) \in \mathcal{C}(p)_n + M_n(\mathcal{J})$ which proves $(p \oplus p^\perp) + \mathcal{J}$ is an Archimedean matrix order unit. Thus, $(M_2(\mathcal{X})/\mathcal{J}, \mathcal{C}(p) + \mathcal{J}, (p \oplus p^\perp) + \mathcal{J})$ is an operator system. $\square$

Definition 3.1.3. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $p \in \mathcal{C}_1$ be a positive contraction. If $\mathcal{C}(p)$ denotes the matrix ordering induced by the positive contraction p and if $\mathcal{J} := span\, \mathcal{C}(p)_1 \cap -\mathcal{C}(p)_1$ then we call the triple $(M_2(\mathcal{X})/\mathcal{J}, \mathcal{C}(p) + \mathcal{J}, (p \oplus p^{\perp}) + \mathcal{J})$ the* abstract compression operator system *relative to the positive contraction p. We will denote the abstract compression operator system relative to p by $M_2(\mathcal{X})/\mathcal{J}$ when no confusion will arise.*

Corollary 3.1.7. *Let $\mathcal{X} \subset B(H)$ be an operator system and let $p \in \mathcal{C}_1$ be a positive contraction which acts as a projection on H. Then if $\mathcal{J} := \mathcal{C}(p)_1 \cap -\mathcal{C}(p)_1$ then $(p \oplus p^{\perp})\mathcal{X}(p \oplus p^{\perp})$ is completely order isomorphic to the abstract compression operator system $M_2(\mathcal{X})/\mathcal{J}$.*

Proof. This is an immediate consequence of Lemma 3.1.5 and Theorem 3.1.6. In particular, one needs only show that given $x \in M_{2n}(\mathcal{X})_h$ then $(I_n \otimes (p \oplus p^{\perp})x(I_n \otimes (p^{\perp} \oplus p)) \in B(\ell_2^{2n}(H))^+$ if and only if for every $\epsilon > 0$ there exists $t > 0$ such that

$$x + \epsilon I_n \otimes (p \oplus p^{\perp}) + tI_n \otimes (p^{\perp} \oplus p) \in \mathcal{C}_{2n}.$$

Since $I_n \otimes (p \oplus p^{\perp})$ is a projection on $\ell_2^{2n}(H)$ then by Lemma 3.1.5 it follows $(I_n \otimes (p \oplus p^{\perp})x(I_n \otimes (p^{\perp} \oplus p)) \in B(\ell_2^{2n}(H))^+$ if and only if for every $\epsilon > 0$ there exists $t > 0$ such that

$$x + \epsilon I_n \otimes (p \oplus p^{\perp}) + tI_n \otimes (p^{\perp} \oplus p) \in \mathcal{C}_{2n}.$$

In particular, since the matrix orderings coincide then this shows the identity $(p \oplus p^{\perp})\mathcal{X}(p \oplus p^{\perp}) \to M_2(\mathcal{X})/\mathcal{J}$ is a unital complete order isomorphism. $\square$

We point out that our results generalize to arbitrary direct sums of finite length. In particular if $\{p_1, \ldots, p_N\}$ is a set of positive contractions on an operator system $(\mathcal{X}, \mathcal{C}, e)$ then if we let $P := \oplus_{\mathrm{i}} p_{\mathrm{i}}$ and $P^{\perp} := \oplus p_{\mathrm{i}}^{\perp}$ then we may consider $M_N(\mathcal{X})/\mathcal{J}$, the abstract compression operator system relative to P, where $\mathcal{J} := \mathrm{span}\, \mathcal{C}(P)_1 \cap -\mathcal{C}(P)_1$.

Throughout the manuscript, given $n \in \mathbb{N}$ then we let $J_n \in M_n$ denote the $n \times n$ matrix with 1 in every entry.

Theorem 3.1.8. *Let $\mathcal{X} \subset B(H)$ an operator system and let $p \in \mathcal{C}_1$ be a positive contraction which acts as a projection on H. Then given $x \in M_n(\mathcal{X})$ it follows $x \otimes J_2 \in \mathcal{C}(p)_n$ if and only if $x \in \mathcal{C}_n$.*

Proof. First consider $x, y, z \in \mathcal{X}$ such that $x, z \in \mathcal{X}_h$. We begin by proving that $\begin{pmatrix} x & y \\ y^* & z \end{pmatrix} \in \mathcal{C}(p)_1$ if and only if $pxp + pyp^\perp + p^\perp y^* p + p^\perp z p^\perp \in B(H)^+$. If $\begin{pmatrix} x & y \\ y^* & z \end{pmatrix} \in \mathcal{C}(p)_1$ then by Lemma 3.1.5 we have

$$(p \oplus p^\perp) \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} (p \oplus p^\perp) \in B(\ell_2^2(H))^+. \tag{3.6}$$

Conjugating by the matrix $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ yields the desired conclusion. Conversely if $pxp + pyp^\perp + p^\perp y^* p + p^\perp z p^\perp \in B(H)^+$ then once again by Lemma 3.1.5 we need only show $(p \oplus p^\perp) \begin{pmatrix} x & y \\ y^* & z \end{pmatrix} (p \oplus p^\perp) =: S \in B(\ell_2^2(H))^+$. Write $H = pH \oplus p^\perp H$ and let $\xi, \eta \in H$. If we denote $\xi' := p\xi$ and $\eta' := p^\perp \eta$ then it follows

$$\langle S(\xi \oplus \eta) | (\xi \oplus \eta) \rangle = \langle (pxp + pyp^\perp + p^\perp y^* p + p^\perp z p^\perp)(\xi' + \eta') | (\xi' + \eta') \rangle \in \mathbb{R}^+,$$

and thus $\begin{pmatrix} x & y \\ y^* & z \end{pmatrix} \in \mathcal{C}(p)_1$.

We go on to prove our claim that $x \otimes J_2 \in \mathcal{C}(p)_n$ if and only if $x \in \mathcal{C}_n$. Let $x \in \mathcal{C}_n$. Then $x \otimes J_2 \in \mathcal{C}_{2n}$ and therefore if $\epsilon > 0$ and we let $t = \epsilon$ then it follows

$$x \otimes J_2 + \epsilon I_n \otimes (p \oplus p^\perp) + \epsilon I_n \otimes (p^\perp \oplus p) \in \mathcal{C}_{2n}.$$

This proves $x \otimes J_2 \in \mathcal{C}(p)_n$. Conversely, suppose $x \in M_n(\mathcal{X})_h$ such that $x \otimes J_2 \in \mathcal{C}(p)_n$. Let $\epsilon > 0$ be arbitrary. Then there exists $t > 0$ such that

$$x \otimes J_2 + \epsilon I_n \otimes (p \oplus p^{\perp}) + t I_n \otimes (p^{\perp} \oplus p) \in \mathcal{C}_{2n}.$$

Applying a canonical shuffle we rewrite the above sum as

$$\begin{pmatrix} x & x \\ x & x \end{pmatrix} + \epsilon \begin{pmatrix} I_n \otimes p & 0 \\ 0 & I_n \otimes p^{\perp} \end{pmatrix} + t \begin{pmatrix} I_n \otimes p^{\perp} & 0 \\ 0 & I_n \otimes p \end{pmatrix} \in \mathcal{C}_{2n}.$$

This implies $\begin{pmatrix} x & x \\ x & x \end{pmatrix} \in \mathcal{C}(I_n \otimes p)_1$ and by the first part of our proof this implies

$$(I_n \otimes p)x(I_n \otimes p) + (I_n \otimes p)x(I_n \otimes p^{\perp}) + (I_n \otimes p^{\perp})x(I_n \otimes p) + (I_n \otimes p^{\perp})x(I_n \otimes p^{\perp})$$

is an element of $B(\ell_2^n(H))^+$. Since $p + p^{\perp} = \mathrm{Id}_H$ then we have $x \in \mathcal{C}_n$. This concludes the proof. $\square$

Motivated by the above theorem we arrive at the following definition:

Definition 3.1.4. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $p \in \mathcal{C}_1$ be a nonzero positive contraction. If $\mathcal{C}(p)$ denotes the matrix ordering induced by p and if $\mathcal{J} := span\, \mathcal{C}(p)_1 \cap -\mathcal{C}(p)_1$ then we will say p is an* abstract projection *if the map*

$$\boldsymbol{\pi}_p : \mathcal{X} \to M_2(\mathcal{X})/\mathcal{J}, x \mapsto x \otimes J_2 + \mathcal{J},$$

is a complete order embedding. If $p = 0$ then we call p the zero abstract projection.

It follows if $(\mathcal{X}, \mathcal{C}, e)$ is an operator system then e is an abstract projection. We consider the map $\boldsymbol{\pi}_e : \mathcal{X} \to M_2(\mathcal{X})/\mathcal{J}$ where $\mathcal{J} := \mathrm{span}\, \mathcal{C}(e)_1 \cap -\mathcal{C}(e)_1$. First note that if $x \in \mathcal{C}_n$ then

$x \otimes J_2 \in \mathcal{C}_{2n}$. One may check this by taking a concrete realization of $\mathcal{X}$ and showing the property holds. In particular, if $\epsilon > 0$ is arbitrary and we let $t = \epsilon$ then we have

$$x \otimes J_2 + \epsilon I_n \otimes (e \oplus 0) + \epsilon I_n \otimes (0 \oplus e) \in \mathcal{C}_{2n}.$$

Thus $x \otimes J_2 \in \mathcal{C}(e)_n$. If $x \in M_n(\mathcal{X})$ such that $(\boldsymbol{\pi}_e)_n(x) \in \mathcal{C}(e)_n + M_n(\mathcal{J})$ then this implies for all $\epsilon > 0$ there exists $t > 0$ such that

$$x \otimes J_2 + \epsilon I_n \otimes (e \oplus 0) + tI_n \otimes (0 \oplus e) \in \mathcal{C}_{2n}.$$

By first applying a canonical shuffle $M_n \otimes M_2 \to M_2 \otimes M_n$ and then compressing the above expression by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (here we are compressing to the 1×1 block) implies $x + \epsilon I_n \otimes e \in \mathcal{C}_n$. Here we have used the assumption that $\mathcal{C}$ is a matrix ordering and is therefore compatible. Since $\epsilon > 0$ is arbitrary and the matrix ordering $\mathcal{C}$ is Archimedean closed with respect to e necessarily implies $x \in \mathcal{C}_n$. This proves $\boldsymbol{\pi}_e$ is a complete order embedding. Thus we see in an abstract operator system $(\mathcal{X}, \mathcal{C}, e)$ that the "identity" e is indeed an abstract projection. This coincides with the immediate fact that if H is a Hilbert space then the identity operator $\mathrm{Id}_H : H \to H$ is a projection. In a similar fashion to our remarks regarding the abstract projection, e, one shows that the zero element 0 is also an abstract projection in $(\mathcal{X}, \mathcal{C}, e)$. Similarly this coincides with the immediate fact that if we once again consider $B(H)$, that 0 is trivially a projection on H.

Consider the operator system $\mathcal{X}$ and let $p \in \mathcal{C}_1$ be an abstract projection. We claim $p^\perp := e - p$ is also an abstract projection. By assumption it follows $x \in \mathcal{C}_n$ if and only if for all $\epsilon > 0$ there exists $t > 0$ such that

$$x \otimes J_2 + \epsilon I_n \otimes (p \oplus p^\perp) + tI_n \otimes (p^\perp \oplus p) \in \mathcal{C}_{2n}.$$

If we conjugate by the unitary $I_n \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then necessarily we have

$$x \otimes J_2 + \epsilon I_n \otimes (p^\perp \oplus p) + tI_n \otimes (p \oplus p^\perp) \in \mathcal{C}_{2n}.$$

In particular, $x \in \mathcal{C}_{2n}$ if and only if $(\boldsymbol{\pi}_{p^\perp})_n(x) \in \mathcal{C}(p^\perp)_n$; i.e., $\boldsymbol{\pi}_{p^\perp}$ is a complete order embedding. A similar argument shows if $p^\perp$ is an abstract projection then p is also an abstract projection. Thus, as in the case for bounded operators on a Hilbert space, a positive contraction p is an abstract projection if and only if $p^\perp$ is an abstract projection.

Remark 3.1.9. Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system. If we consider the abstract projection e then it follows for all $n \in \mathbb{N}$, if $\tilde{\mathcal{C}}(e)$ denotes the matrix ordering on $\mathcal{X}$ induced by e then

$$\tilde{\mathcal{C}}(e)_n := \{x \in M_n(\mathcal{X})_h : \forall \epsilon > 0,\ x + \epsilon I_n \otimes e \in \mathcal{C}_n\}$$

which by the fact that $\mathcal{C}$ is Archimedean closed with respect to e yields $\tilde{\mathcal{C}}(e)_n = \mathcal{C}_n$. Thus, $\mathcal{J} := \operatorname{span} \tilde{\mathcal{C}}(e)_1 \cap -\tilde{\mathcal{C}}(e)_1 = \{0\}$. Thus, $\mathcal{X}/\mathcal{J} \simeq \mathcal{X}$. Therefore we see that e behaves as the identity operator projection would in the concrete setting of $B(H)$.

If we consider now $p = 0$ it follows that $\tilde{\mathcal{C}}(0)$ is the matrix ordering defined for each $n \in \mathbb{N}$ by

$$\tilde{\mathcal{C}}(0)_n := \{x \in M_n(\mathcal{X})_h : \exists t > 0 \text{ such that } x + tI_n \otimes e \in \mathcal{C}_n\}.$$

Since e is a matrix order unit it follows $\tilde{\mathcal{C}}(0)_1 = \mathcal{X}_h$, i.e., for each $n \in \mathbb{N}$ it follows $\tilde{\mathcal{C}}(0)_n = M_n(\mathcal{X})_h$. This implies if $\mathcal{J} := \operatorname{span} \tilde{\mathcal{C}}(0)_1 \cap -\tilde{\mathcal{C}}(0)_1$ then $\mathcal{X}/\mathcal{J} = \mathcal{X}/\mathcal{X}_h = \{0\}$ which is a direct result of the fact that $\mathcal{X} = \mathcal{X}_h \oplus \mathrm{i}\mathcal{X}_h$. Thus we see that the zero abstract projection behaves as the identically zero operator on a Hilbert space H.

Remark 3.1.10. Given an operator system $\mathcal{X}$ then our conditions that a positive contraction $p \in \mathcal{X}$ be an abstract projection imply a very important property regarding the norm. It must be the case that if p is a nonzero abstract projection in $(\mathcal{X}, \mathcal{C}, e)$ then $\alpha(p) = 1$ where $\alpha : \mathcal{X} \to [0, \infty)$ denotes the order norm induced by the Archimedean order unit e. Suppose

$0 < \alpha(p) < 1$. Let $r := \alpha(p)$. Then it follows $re - p \in \mathcal{C}_1$. Recalling that $e = p + p^\perp$ we see $re - p = rp + rp^\perp - p = (r-1)p + rp^\perp$. If $\epsilon > 0$ is arbitrary then it follows $(r-1)p + \epsilon p + rp^\perp \in \mathcal{C}_1$ and therefore we see $(r-1)p \in \tilde{\mathcal{C}}(p)_1$ where we will let $\tilde{\mathcal{C}}(p)$ denote the matrix ordering on $\mathcal{X}$ defined for each $n \in \mathbb{N}$ as

$$\tilde{\mathcal{C}}(p)_n := \{x \in M_n(\mathcal{X})_h : \forall \epsilon > 0 \ \exists t > 0 \text{ such that } x + \epsilon p + tp^\perp \in \mathcal{C}_n\}.$$

As our results in the beginning of Section 3.1 imply, it follows if $\mathcal{J} := \operatorname{span} \tilde{\mathcal{C}}(p)_1 \cap -\tilde{\mathcal{C}}(p)_1$ then the triple $(\mathcal{X}/\mathcal{J}, \tilde{\mathcal{C}}(p) + \mathcal{J}, p + \mathcal{J})$ is an operator system. Since $p \in \mathcal{C}_1$ it follows $p \in \tilde{\mathcal{C}}(p)_1$. Furthermore since $(r-1)p = -(1-r)p \in \tilde{\mathcal{C}}(p)_1$ then $-p \in \tilde{\mathcal{C}}(p)_1$. Thus, we see that $p \in \mathcal{J}$ and consequently the abstract compression operator system $(\mathcal{X}/\mathcal{J}, \tilde{\mathcal{C}}(p) + \mathcal{J}, p + \mathcal{J})$ is trivial. In particular, if we assumed that p was an abstract projection with $\alpha(p) < 1$ then $\pi_p(p) = 0 + \mathcal{J}$ and thus π_p would not be injective, contradicting the assumption that it was a complete order embedding, and thus a linear isomorphism.

Corollary 3.1.11. *Let $\mathcal{X} \subset B(H)$ be an operator system and let $p \in \mathcal{X}$ be a positive contraction that acts as a projection on H. Then p is an abstract projection in $\mathcal{X}$.*

Proof. Immediate from Lemma 3.1.5, Corollary 3.1.7, and Theorem 3.1.8. □

Thus far we have proven that our notion of an abstract projection satisfies many properties as that of a concrete projection on some Hilbert space. In particular, Corollary 3.1.11 implies that every concrete projection is an abstract projection. We now go on to prove that every abstract projection is a concrete projection on some Hilbert space H.

Proposition 3.1.3. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $p \in \mathcal{X}$ be an abstract projection. Let $\mathcal{C}(p)$ be the induced matrix ordering on $M_2(\mathcal{X})$ induced by p and let $\mathcal{J} :=$ span$\mathcal{C}(p)_1 \cap -\mathcal{C}(p)_1$. Then given any matrix state $\varphi : M_2(\mathcal{X})/\mathcal{J} \to M_n$ there exists another matrix state $\varphi' : M_2(\mathcal{X})/\mathcal{J} \to M_{n'}$ such that $\varphi'((p \oplus 0) + \mathcal{J})$ and $\varphi'((0 \oplus p^\perp) + \mathcal{J})$ are both projections in $M_{n'}$ and if $x, y, z \in M_n(\mathcal{X})$ then*

$$\varphi'_n\left(\begin{pmatrix} x & y \\ y^* & z \end{pmatrix} + M_n(\mathcal{J})\right) \in M_{nn'}^+$$

if and only if

$$\varphi_{2n}\left(\begin{pmatrix} x & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y^* & 0 & 0 & z \end{pmatrix} + M_{2n}(\mathcal{J})\right) \in M_{n^2}^+.$$

Proof. For notational convenience we will let $\hat{x} \in M_n(M_2(\mathcal{X})/\mathcal{J})$ denote the coset $x+M_n(\mathcal{J})$ where $x \in M_{2n}(\mathcal{X})$. Since $\varphi(\widehat{p \oplus 0}) = I_n - \varphi(\widehat{0 \oplus p^\perp})$ it follows $\left[\varphi(\widehat{p \oplus 0}), \varphi(\widehat{0 \oplus p^\perp})\right] = 0$. Here we have let $[\cdot,\cdot]$ denote the commutator defined by $[a,b] := ab - ba, a, b \in M_n$. Choose an orthonormal basis of ℓ_2^n such that both $\varphi(\widehat{p \oplus 0})$ and $\varphi(\widehat{0 \oplus p^\perp})$ are diagonal. We write $P := \varphi(\widehat{p \oplus 0})$ and $P^\perp := \varphi(\widehat{0 \oplus p^\perp})$ which by reordering our basis we may write as

$$P = \begin{pmatrix} I_k & & & & \\ & x_{k+1} & & & \\ & & \ddots & & \\ & & & x_{k'} & \\ & & & & 0_{n-k'} \end{pmatrix} \in M_n, \text{and} \quad P^\perp = \begin{pmatrix} 0_k & & & & \\ & y_{k+1} & & & \\ & & \ddots & & \\ & & & y_{k'} & \\ & & & & I_{n-k'} \end{pmatrix} \in M_n$$

such that $0 \le k \le k' \le n$ and $x_{\rm i} + y_{\rm i} = 1$ with $x_{\rm i}, y_{\rm i} \in (0,1)$ for each i. We consider the following operators

$$A := \begin{pmatrix} I_k & & & & \\ & x_{k+1}^{-1/2} & & & 0_{k',n-k'} \\ & & \ddots & & \\ & & & x_{k'}^{-1/2} & \end{pmatrix} \in M_{k',n},$$

and

$$B := \begin{pmatrix} & y_{k+1}^{-1/2} & & & \\ 0_{n-k,k} & & \ddots & & \\ & & & y_{k'}^{-1/2} & \\ & & & & I_{n-k'} \end{pmatrix} \in M_{n-k,n}.$$

It then follows that $APA^* = I_{k'}$ and $BP^{\perp}B^* = I_{n-k}$. Let $m := k' + n - k$ and define the map $\varphi' : M_2(\mathcal{X})/\mathcal{J} \to M_m$ by

$$\widehat{\begin{pmatrix} x & y \\ w & z \end{pmatrix}} \mapsto (A \oplus B) \begin{pmatrix} \varphi\widehat{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}} & \varphi\widehat{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}} \\ \varphi\widehat{\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}} & \varphi\widehat{\begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}} \end{pmatrix} (A^* \oplus B^*).$$

Then $\varphi' : M_2(\mathcal{X})/\mathcal{J} \to M_m$ is unital completely positive and $\varphi'(\widehat{p \oplus 0}) = I_{k'} \oplus 0_{n-k}$ and $\varphi'(\widehat{0 \oplus p^{\perp}}) = 0_{k'} \oplus I_{n-k}$. In particular we see $\varphi'(\widehat{p \oplus 0}) + \varphi'(\widehat{0 \oplus p^{\perp}}) = I_m$. In order to verify the final claim it suffices to show if $\begin{pmatrix} x & y \\ y^* & z \end{pmatrix} \in M_2(\mathcal{X})_h$ then the nonzero entries of

$$\varphi\left(\widehat{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}}\right), \varphi\left(\widehat{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}}\right), \varphi\left(\widehat{\begin{pmatrix} 0 & 0 \\ y^* & 0 \end{pmatrix}}\right), \varphi\left(\widehat{\begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}}\right),$$

are all supported in the proper corners. This is to say that the nonzero entries of $\varphi\left(\widehat{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}}\right)$ are contained in its upper left $k' \times k'$ corner, the nonzero entries of $\varphi\left(\widehat{\begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}}\right)$ are contained in its lower right $(n-k) \times (n-k)$ corner, and the nonzero entries of $\varphi\left(\widehat{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}}\right)$ are contained in its upper right $k' \times (n-k)$ corner. This in turn proves the claim since $\varphi' : M_2(\mathcal{X})/\mathcal{J} \to M_m$ is simply the compression of

$$\begin{pmatrix} \varphi\widehat{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}} & \varphi\widehat{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}} \\ \varphi\widehat{\begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}} & \varphi\widehat{\begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}} \end{pmatrix}$$

to its support, and thus a $m \times m$ submatrix, followed by conjugation by the invertible matrix

$$\begin{pmatrix} I_k & & & & & & & \\ & x_{k+1}^{-\frac{1}{2}} & & & & & & \\ & & \ddots & & & & & \\ & & & x_{k'}^{-\frac{1}{2}} & & & & \\ & & & & y_{k+1}^{-\frac{1}{2}} & & & \\ & & & & & \ddots & & \\ & & & & & & y_{k'}^{-\frac{1}{2}} & \\ & & & & & & & I_{n-k'} \end{pmatrix}.$$

We begin by proving the claim for $\varphi\left(\widehat{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}}\right)$. Thus, for $x \in \mathcal{X}_h$ we may assume that $(p \oplus p^{\perp}) \pm \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{C}(p)_1$ since $\widehat{p \oplus p^{\perp}}$ is an Archimedean order unit for the abstract compression operator system $M_2(\mathcal{X})/\mathcal{J}$. Since $\mathcal{C}(p)$ is a matrix ordering this necessarily implies for all $\epsilon > 0$ there exists $t > 0$ such that $p \pm x + \epsilon p + tp^{\perp} \in \mathcal{C}_1$ and consequently $\begin{pmatrix} p \pm x & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{C}(p)_1$. Positivity of φ implies that $\varphi\left(\widehat{\begin{pmatrix} p \pm x & 0 \\ 0 & 0 \end{pmatrix}}\right) = P \pm \varphi\left(\widehat{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}}\right) \in M_n^+$ which necessarily implies that the nonzero entries of $\varphi\left(\widehat{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}}\right)$ sit in the upper left $k' \times k'$

block. A similar argument holds in showing that the matrix $\varphi\widehat{\begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix}}$ has support contained in its lower right $(n-k)\times(n-k)$ block. It remains to prove the claim for $\varphi\left(\widehat{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}}\right)$.

Since $\widehat{p \oplus p^{\perp}}$ is an order unit we may assume after rescaling that $\begin{pmatrix} p & y \\ y^* & p^{\perp} \end{pmatrix} \in \mathcal{C}(p)_1$. Since the mapping

$$\begin{pmatrix} x & y \\ w & z \end{pmatrix} \mapsto \begin{pmatrix} x & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ w & 0 & 0 & z \end{pmatrix}$$

is completely positive then

$$\begin{pmatrix} p & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y^* & 0 & 0 & p^{\perp} \end{pmatrix} \in \mathcal{C}(p)_2.$$

Consequently it follows

$$\varphi_2\left(\widehat{\begin{pmatrix} p & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y^* & 0 & 0 & p^{\perp} \end{pmatrix}}\right) = \begin{pmatrix} P & \varphi\left(\widehat{\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}}\right) \\ \varphi\left(\widehat{\begin{pmatrix} 0 & 0 \\ y^* & 0 \end{pmatrix}}\right) & P^{\perp} \end{pmatrix} \in M_{2n}^{+}.$$

This proves the claim and therefore finishes the proof. $\square$

Lemma 3.1.12. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $p \in \mathcal{X}$ be an abstract projection. Let $\mathcal{C}(p)$ be the matrix ordering on $M_2(\mathcal{X})$ induced by p and let $\mathcal{J} := span\mathcal{C}(p)_1 \cap -\mathcal{C}(p)_1$. Then the map $\pi_p : \mathcal{X} \to M_2(\mathcal{X})/\mathcal{J}$ is unital.*

Proof. It suffices to prove that

$$\pm\begin{pmatrix} 0 & p \\ p & p \end{pmatrix}, \pm\begin{pmatrix} p^\perp & p^\perp \\ p^\perp & 0 \end{pmatrix} \in \mathcal{C}(p)_1.$$

It will then follow that $\boldsymbol{\pi}_p(e) = e \otimes J_2 + \mathcal{J} = (p \oplus p^\perp) + \mathcal{J}$ which by Theorem 3.1.6 proves the claim. We prove the claim for $\pm\begin{pmatrix} 0 & p \\ p & p \end{pmatrix}$ and the argument for $\pm\begin{pmatrix} p^\perp & p^\perp \\ p^\perp & 0 \end{pmatrix}$ is similar. Let $\epsilon > 0$ and let $t = \frac{1}{\epsilon}$. It then follows

$$\begin{pmatrix} 0 & p \\ p & p \end{pmatrix} + \epsilon(p \oplus p^\perp) + \frac{1}{\epsilon}(p^\perp \oplus p) = \begin{pmatrix} \epsilon & 1 \\ 1 & 1 + \frac{1}{\epsilon} \end{pmatrix} \otimes p + \begin{pmatrix} \frac{1}{\epsilon} & 0 \\ 0 & \epsilon \end{pmatrix} \otimes p^\perp \in \mathcal{C}_2.$$

Similarly, if $\epsilon > 0$ is arbitary and we choose $t = (1 + \frac{1}{\epsilon})$ then

$$\begin{pmatrix} 0 & -p \\ -p & -p \end{pmatrix} + \epsilon(p \oplus p^\perp) + (1 + \frac{1}{\epsilon})(p^\perp \oplus p) = \begin{pmatrix} \epsilon & -1 \\ -1 & \frac{1}{\epsilon} \end{pmatrix} \otimes p + \begin{pmatrix} 1 + \frac{1}{\epsilon} & 0 \\ 0 & \epsilon \end{pmatrix} \otimes p^\perp \in \mathcal{C}_2.$$

This, along with our initial remarks, concludes the proof. $\square$

Theorem 3.1.13. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $p \in \mathcal{X}$ be an abstract projection. Then there exists a Hilbert space H and a unital complete order embedding $\boldsymbol{\pi} : \mathcal{X} \to B(H)$ such that $\boldsymbol{\pi}(p)$ is a projection on H.*

Proof. For each $n \in \mathbb{N}$ we consider the set

$$A_n := \{\psi : M_2(\mathcal{X})/\mathcal{J} \to M_n, \text{ such that } \psi \text{ is unital completely positive}\},$$

i.e., we are letting A_n denote the convex set of matrix states of level n on the operator system $M_2(\mathcal{X})/\mathcal{J}$. Consider the map $\varphi := \bigoplus_{n\in\mathbb{N}} \bigoplus_{\psi \in A_n} \psi : M_2(\mathcal{X})/\mathcal{J} \to \bigoplus_{n\in\mathbb{N}} \bigoplus_{\psi\in A_n} M_n^\psi, M_n^\psi = M_n$ for each $\psi \in A_n$, which is necessarily a unital complete order isomorphism onto the range. Using Proposition 3.1.3 we replace each $\psi : M_2(\mathcal{X})/\mathcal{J} \to M_n$ with the unital completely positive map $\psi' : M_2(\mathcal{X})/\mathcal{J} \to M_m$ and we then obtain a unital completely positive map $\varphi' : M_2(\mathcal{X})/\mathcal{J} \to \bigoplus_{m\in\mathbb{N}} \bigoplus_{\psi'\in A_m} M_m^{\psi'} \subset B(H)$ where $H := \bigoplus_{m\in\mathbb{N}} \bigoplus_{\psi'\in A_m} \ell_2^{m_{\psi'}}$, and such that

$\varphi'(\widehat{p \oplus 0})$ is a projection on H. We claim that φ' is a complete order embedding. It will suffice to show that for $x, z \in M_n(\mathcal{X})_h$, and $y \in M_n(\mathcal{X})$, if $\widehat{\begin{pmatrix} x & y \\ y^* & z \end{pmatrix}} \notin \mathcal{C}(p)_n$ then there exists a natural number m and $\psi \in A_m$ such that $\psi_n\left(\widehat{\begin{pmatrix} x & y \\ y^* & z \end{pmatrix}}\right) \notin M_{mn}^+$. If $\widehat{\begin{pmatrix} x & y \\ y^* & z \end{pmatrix}} \notin \mathcal{C}(p)_n$ then it follows

$$\widehat{\begin{pmatrix} x & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y^* & 0 & 0 & z \end{pmatrix}} \notin \mathcal{C}(p)_{2n}.$$

In particular there exists a matrix state $\psi : M_2(\mathcal{X})/\mathcal{J} \to M_m$ such that

$$\psi_{2n}\left(\widehat{\begin{pmatrix} x & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y^* & 0 & 0 & z \end{pmatrix}}\right) \notin M_{2mn}^+.$$

Let $\psi' : M_2(\mathcal{X})/\mathcal{J} \to M_{m'}$ be the induced unital completely positive map whose existence comes from Proposition 3.1.3. This implies that $\psi'_n\left(\widehat{\begin{pmatrix} x & y \\ y^* & z \end{pmatrix}}\right) \notin M_{m'n}^+$ and since $\psi' \in A_{m'}$ this proves our claim. Thus, $\varphi' : M_2(\mathcal{X})/\mathcal{J} \to B(H)$ is a unital complete order embedding. Let $\pi_p : \mathcal{X} \to M_2(\mathcal{X})/\mathcal{J}$ be the unital complete order embedding into the abstract compression operator system. The desired map is then $\pi := \varphi'\pi_p : \mathcal{X} \to B(H)$. $\square$

Theorem 3.1.14. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $p \in \mathcal{X}$ be an abstract projection. Then p is a projection in $C_e^*(\mathcal{X})$.*

Proof. Using Theorem 3.1.13 we let H be a Hilbert space, and $\pi : \mathcal{X} \to B(H)$ a unital complete order embedding such that $\pi(p)$ is a projection in $B(H)$. Let $\mathcal{A} := C^*(\pi(\mathcal{X}))$, i.e., $\mathcal{A}$ is the unital C*-algebra generated by the image of $\mathcal{X}$ under π. Using the universal property of $C_e^*(\mathcal{X})$ there exists a unique $*$-epimorphism $\sigma : \mathcal{A} \to C_e^*(\mathcal{X})$ such that $\sigma\boldsymbol{\pi} = \mathrm{j}$ where $\mathrm{j} : \mathcal{X} \to C_e^*(\mathcal{X})$ denotes the canonical embedding into the C*-envelope. We verify that $\mathrm{j}(p)$ is a self-adjoint idempotent. Since j is positive then it is necessarily self-adjoint which is to say that given any $x \in \mathcal{X}$ that $\mathrm{j}(x^*) = \mathrm{j}(x)^*$. Consequently, since $p \in \mathcal{X}_h$ we have that $\mathrm{j}(p) = \mathrm{j}(p)^*$. For our final claim we see

$$\mathrm{j}(p)^2 = (\sigma\boldsymbol{\pi}(p))(\sigma\boldsymbol{\pi}(p)) = \sigma(\boldsymbol{\pi}(p)^2) = \sigma(\boldsymbol{\pi}(p)) = \mathrm{j}(p).$$

This finishes the proof. □

Combining Definition 3.1.4, Theorem 3.1.13, and Theorem 3.1.14 we thus arrive at the following theorem which we use to completely characterize abstract projections in an operator system.

Theorem 3.1.15. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $p \in \mathcal{X}$ be a positive contraction. Then the following are equivalent:*

1. *p is an abstract projection.*
2. *There exists a Hilbert space H and a unital complete order embedding $\boldsymbol{\pi} : \mathcal{X} \to B(H)$ such that $\boldsymbol{\pi}(p)$ is a projection.*
3. *p is a projection in $C_e^*(\mathcal{X})$.*

3.2 Multiple Projections in Operator Systems

Expanding on the methods of Section 3.1 we now develop our methods in order to detect when a finite number of positive contractions in an operator system are all simultaneously abstract projections. In particular, our goal is to avoid using the C*-envelope as we had done in Theorem 3.1.15. The reason for doing this is we wish to determine when a finite number of positive contractions in an AOU space are all simultaneously abstract projections relative to the same operator system structure. Our new methods are necessary since when dealing with AOU spaces one cannot simply appeal to the C*-envelope since no such universal object is guaranteed to exist in the category of AOU spaces with morphisms being positive maps.

Definition 3.2.1. *Let* $(\mathcal{X}, \mathcal{C}, e)$ *be an operator system and let* $\{p_1, \ldots, p_N\} \subset \mathcal{X}$ *be a finite set of positive contractions in* $\mathcal{X}$*. For each* $\mathrm{i} \in [N]$ *we define*

$$P_{\mathrm{i}}^N := I_{2^{\mathrm{i}-1}} \otimes (p_{\mathrm{i}} \oplus p_{\mathrm{i}}^{\perp}) \otimes J_{2^{N-\mathrm{i}}},$$

and

$$Q_{\mathrm{i}}^N := I_{2^{\mathrm{i}-1}} \otimes (p_{\mathrm{i}}^{\perp} \oplus p_{\mathrm{i}}) \otimes J_{2^{N-\mathrm{i}}}.$$

Similarly, for each $\mathrm{i} \in [N]$ *we define*

$$\widehat{P}_{\mathrm{i}}^N := J_{2^{\mathrm{i}-1}} \otimes (p_{\mathrm{i}} \oplus p_{\mathrm{i}}^{\perp}) \otimes J_{2^{N-\mathrm{i}}}$$

and

$$\widehat{Q}_{\mathrm{i}}^N := J_{2^{\mathrm{i}-1}} \otimes (p_{\mathrm{i}}^{\perp} \oplus p_{\mathrm{i}}) \otimes J_{2^{N-\mathrm{i}}}.$$

For each $\mathrm{i} \in [N]$ *it follows* $P_{\mathrm{i}}^N, Q_{\mathrm{i}}^N, \widehat{P}_{\mathrm{i}}^N, \widehat{Q}_{\mathrm{i}}^N \in M_{2^N}(\mathcal{X})$*. We define the collection* $\mathcal{C}(p_1, \ldots, p_N)$ *for each* $n \in \mathbb{N}$ *by* $x \in \mathcal{C}(p_1, \ldots, p_N)_n$ *if and only if* $x \in M_{n2^N}(\mathcal{X})_h$ *and for all* $\epsilon_1, \ldots, \epsilon_N > 0$ *there exists* $t_1, \ldots, t_N > 0$ *such that*

$$x + \sum_{\mathrm{i}} \epsilon_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^N + \sum_{\mathrm{i}} t_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^N \in \mathcal{C}_{n2^N}, \tag{3.7}$$

with the additional property that if we replace $0 < \epsilon_{\mathrm{i}}' < \epsilon_{\mathrm{i}}$ *then we may choose* $t_{\mathrm{i}}' > t_{\mathrm{i}}$ *such that Equation* (3.7) *is still satisfied. We define the collection* $\widehat{\mathcal{C}}(p_1, \ldots, p_N)$ *analogously with* $\widehat{P}_{\mathrm{i}}^N$ *and* $\widehat{Q}_{\mathrm{i}}^N$ *in place of* P_{i}^N *and* Q_{i}^N *in Equation* (3.7)*.*

Note that given any $n \in \mathbb{N}$ then both $\mathcal{C}(p_1, \ldots, p_N)_n$ and $\widehat{\mathcal{C}}(p_1, \ldots, p_N)_n$ are nonempty since if one chooses $x \in \mathcal{C}_{n2^N}$ then letting $t_{\mathrm{i}} = \epsilon_{\mathrm{i}}$ for each $\mathrm{i} \in [N]$ as in Definition 3.2.1, then Equation (3.7) is satisfied.

Lemma 3.2.1. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $\{p_1, \ldots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions. Let $\mathcal{C}(p_1, \ldots, p_N)$ and $\widehat{\mathcal{C}}(p_1, \ldots, p_N)$ be as in Definition 3.2.1. Then $\widehat{\mathcal{C}}(p_1, \ldots, p_N) \subset \mathcal{C}(p_1, \ldots, p_N)$.*

Proof. First note that $J_2 \leq 2I_2$. In particular if we fix $n \in \mathbb{N}$ then $I_n \otimes \widehat{P}_{\mathrm{i}}^N \leq 2^{\mathrm{i}-1} I_n \otimes P_{\mathrm{i}}^N$. Consider $x \in \widehat{\mathcal{C}}(p_1, \ldots, p_N)_n$. Thus, for all $\epsilon_1, \ldots, \epsilon_N > 0$ there exists $t_1, \ldots, t_N > 0$ such that

$$x + \sum_{\mathrm{i}} \frac{\epsilon_{\mathrm{i}}}{2^{\mathrm{i}-1}} I_n \otimes \widehat{P}_{\mathrm{i}}^N + \sum_{\mathrm{i}} t_{\mathrm{i}} I_n \otimes \widehat{Q}_{\mathrm{i}}^N \in \mathcal{C}_{n2^N}.$$

By our beginning remarks it then follows

$$x + \sum_{\mathrm{i}} \epsilon_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^N + \sum_{\mathrm{i}} 2^{\mathrm{i}-1} t_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^N \in \mathcal{C}_{n2^N}.$$

This proves that $x \in \mathcal{C}(p_1, \ldots, p_N)_n$. □

Proposition 3.2.1. *Let $\mathcal{X} \subset B(H)$ be an operator system and let $\{p_1, \ldots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions such that each contraction acts as a projection on H. Then the following are equivalent:*

1. $x \in \mathcal{C}_n$.
2. $x \otimes J_{2^N} \in \mathcal{C}(p_1, \ldots, p_N)_n$.
3. $x \otimes J_{2^N} \in \widehat{\mathcal{C}}(p_1, \ldots, p_N)_n$.

Proof. Since $\mathcal{C} \otimes J_{2^N} \subset \widehat{\mathcal{C}}(p_1, \ldots, p_N) \subset \mathcal{C}(p_1, \ldots, p_N)$ we readily see that (1) implies (3) implies (2). We will prove by induction on N that if $x \otimes J_{2^N} \in \mathcal{C}(p_1, \ldots, p_N)_n$ then $x \in \mathcal{C}_n$.

The base case is given by Theorem 3.1.8. Thus, suppose that our claim holds for $N-1$ contractions. Consider $x \in M_n(\mathcal{X})$ such that $x \otimes J_{2^N} \in \mathcal{C}(p_1, \ldots, p_N)_n$. Thus given arbitrary $\epsilon_1, \ldots, \epsilon_N > 0$ there exists $t_1, \ldots, t_N > 0$ such that

$$x \otimes J_{2^N} + \sum_{\mathrm{i}} \epsilon_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^N + \sum_{\mathrm{i}} t_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^N \in \mathcal{C}_{n2^N}$$

which we rewrite as

$$x \otimes J_{2^N} + \sum_{\mathrm{i}=1}^{N-1} \epsilon_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^N + \epsilon_N I_{n2^{N-1}} \otimes (p_N \oplus p_N^{\perp}) + \sum_{\mathrm{i}=1}^{N-1} t_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^N + t_N I_{n2^{N-1}} \otimes (p_N^{\perp} \oplus p_N).$$

Fix $\mathrm{i} \in [N-1]$. Notice $I_n \otimes P_{\mathrm{i}}^N = I_n \otimes P_{\mathrm{i}}^{N-1} \otimes J_2$ and similarly $I_n \otimes Q_{\mathrm{i}}^N = I_n \otimes Q_{\mathrm{i}}^{N-1} \otimes J_2$. Rewriting the first term of the above equation as $x \otimes J_{2^{N-1}} \otimes J_2$ we rewrite the above equation as

$$\left[x \otimes J_{2^{N-1}} + \sum_{\mathrm{i}=1}^{N-1} \epsilon_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^{N-1} + \sum_{\mathrm{i}=1}^{N-1} t_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^{N-1}\right] \otimes J_2 + \epsilon_N I_{n2^{N-1}} \otimes (p_N \oplus p_N^{\perp})$$

$$+ t_N I_{n2^{N-1}} \otimes (p_N^{\perp} \oplus p_N) \in \mathcal{C}_{n2^N}.$$

Since $I_{n2^{N-1}} \otimes p_N$ is a projection on the Hilbert space $\ell_2^{n2^{N-1}}(H)$ then by Theorem 3.1.8 we conclude

$$x \otimes J_{2^{N-1}} + \sum_{\mathrm{i}=1}^{N-1} \epsilon_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^{N-1} + \sum_{\mathrm{i}=1}^{N-1} t_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^{N-1} \in \mathcal{C}_{n2^{N-1}}.$$

By the inductive hypothesis we conclude $x \in \mathcal{C}_n$ and thus (1) holds. This finishes the proof. □

Lemma 3.2.2. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $\{p_1, \ldots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions. Let $\mathcal{C}(p_1, \ldots, p_N)$ denote the collection of sets as defined in Definition 3.2.1. Then $(M_{2^N}(\mathcal{X}), \mathcal{C}(p_1, \ldots, p_N))$ is a matrix ordered $*$-vector space.*

Proof. The proof is similar to that of Theorem 3.1.6. We show $\mathcal{C}(p_1, \ldots, p_N)$ is closed under direct sums and conjugation by matrices of the form $a \otimes I_{2^N}$. Let $x \in \mathcal{C}(p_1, \ldots, p_N)_n$ and $y \in \mathcal{C}(p_1, \ldots, p_N)_m$. Then if $\epsilon_1, \ldots, \epsilon_N, \delta_1, \ldots, \delta_N > 0$ there exists $s_1, \ldots, s_N, t_1, \ldots, t_N > 0$ such that

$$x + \sum_{\mathrm{i}} \epsilon_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^N + \sum_{\mathrm{i}} s_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^N \in \mathcal{C}_{n2^N},$$

and

$$y + \sum_{\mathrm{i}} \delta_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^N + \sum_{\mathrm{i}} t_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^N \in \mathcal{C}_{n2^N}.$$

Set $r_i := \max\{s_i, t_i\}$ then it follows

$$x \oplus y + \sum_i \epsilon I_{n+m} \otimes P_i^N + \sum_i r_i I_{n+m} \otimes Q_i^N \in \mathcal{C}_{(n+m)2^N}$$

and therefore $x \oplus y \in \mathcal{C}(p_1, \dots, p_N)_{n+m}$.

Suppose $x \in \mathcal{C}(p_1, \dots, p_N)_n$ and let $a \in M_{n,k}$. We claim $(a \otimes I_{2^N})^* x (a \otimes I_{2^N}) \in \mathcal{C}(p_1, \dots, p_N)_k$. Let $\epsilon_1, \dots, \epsilon_N > 0$ and let $t_1, \dots, t_N > 0$ such that

$$x + \sum_i \frac{\epsilon_i}{\|a\|^2} I_n \otimes P_i^N + \sum_i t_i I_n \otimes Q_i^N \in \mathcal{C}_{n2^N}.$$

Conjugating by $a \otimes I_{2^N}$ we have

$$(a \otimes I_{2^N})^* x (a \otimes I_{2^N}) + \sum_i \frac{\epsilon_i}{\|a\|^2} a^* a \otimes P_i^N + \sum_i t_i a^* a \otimes Q_i^N \in \mathcal{C}_{k2^N},$$

from which it follows

$$(a \otimes I_{2^N})^* x (a \otimes I_{2^N}) + \sum_i \epsilon_i I_k \otimes P_i^N + \sum_i \|a^* a\| t_i I_k \otimes Q_i^N \in \mathcal{C}_{k2^N}.$$

Thus $\mathcal{C}(p_1, \dots, p_N)$ is compatible and this finishes the proof. □

Lemma 3.2.3. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $\{p_1, \dots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions. Then $I_{2^N} \otimes e$ is an Archimedean matrix order unit for the matrix ordered $*$-vector space $(M_{2^N}(\mathcal{X}), \mathcal{C}(p_1, \dots, p_N))$.*

Proof. We begin by showing $I_{2^N} \otimes e$ is a matrix order unit which by using Proposition 2.0.1 implies we need only show it is an order unit. Let $x \in M_{2^N}(\mathcal{X})_h$. Since e is a matrix order unit implies there exists $r > 0$ such that $rI_{2^N} \otimes e - x \in \mathcal{C}_{2^N} \subset \mathcal{C}(p_1, \dots, p_N)_1$. This implies $I_{2^N} \otimes e$ is indeed an order unit. Assume $x \in M_{n2^N}(\mathcal{X})$ such that for all $\epsilon > 0$ it follows $\epsilon I_{n2^N} \otimes e + x \in \mathcal{C}(p_1, \dots, p_N)_n$. Thus, if $\epsilon_1, \dots, \epsilon_N > 0$ there exists $t_1, \dots, t_N > 0$ such that

$$x + \frac{\epsilon_N}{2} I_{n2^N} \otimes e + \sum_{i=1}^{N-1} \epsilon_i I_n \otimes P_i^N + \frac{\epsilon_N}{2} I_n \otimes P_N^N + \sum_{i=1}^{N} t_i I_n \otimes Q_i^N \in \mathcal{C}_{n2^N}.$$

Since

$$I_{n2^N} \otimes e = I_n \otimes I_{2^{N-1}} \otimes (e \oplus e)$$

we see

$$I_n \otimes P_N^N + I_n \otimes Q_N^N = I_n \otimes I_{2^{N-1}} \otimes (p_N \oplus p_N^{\perp}) + I_n \otimes I_{2^{N-1}} \otimes (p_N^{\perp} \oplus p_N)$$

and consequently,

$$I_{n2^N} \otimes e = I_n \otimes P_N^N + I_n \otimes Q_N^N.$$

It then follows

$$x + \sum_{i=1}^{N} \epsilon_i I_n \otimes P_i^N + \sum_{i=1}^{N-1} t_i I_n \otimes Q_i^N + (t_N + \frac{\epsilon_N}{2}) I_n \otimes Q_N^N \in \mathcal{C}_{n2^N}.$$

Thus, $x \in \mathcal{C}(p_1, \ldots, p_N)_n$ which implies $I_{2^N} \otimes e$ is an Archimedean matrix order unit. □

Proposition 3.2.2. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $\{p_1, \ldots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions. Let $\mathcal{C}(p_1, \ldots, p_N)$ denote the matrix ordering on $M_{2^N}(\mathcal{X})$ as defined in Definition 3.2.1. If $\mathcal{J} := span\,\mathcal{C}(p_1, \ldots, p_N)_1 \cap -\mathcal{C}(p_1, \ldots, p_N)_1$ then the triple*

$$(M_{2^N}(\mathcal{X})/\mathcal{J}, \mathcal{C}(p_1, \ldots, p_N) + \mathcal{J}, I_{2^N} \otimes e + \mathcal{J})$$

is an operator system.

Proof. The proof is immediate by Lemma 3.2.2 and Lemma 3.2.3. □

Proposition 3.2.3. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $\{p_1, \ldots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions. Let $\mathcal{C}(p_1, \ldots, p_N)$ be the matrix ordering as defined in Definition 3.2.1 and let $\mathcal{J} := span\,\mathcal{C}(p_1, \ldots, p_N)_1 \cap -\mathcal{C}(p_1, \ldots, p_N)_1$. Then the mapping*

$x \mapsto x \otimes J_{2^N} + \mathcal{J}$ from the operator system $(\mathcal{X}, \mathcal{C}, e)$ to the quotient operator system $(M_{2^N}(\mathcal{X})/\mathcal{J}, \mathcal{C}(p_1, \ldots, p_N) + \mathcal{J}, I_{2^N} \otimes e + \mathcal{J})$ is unital. In particular,

$$e \otimes J_{2^N} + \mathcal{J} = I_2 \otimes e \otimes J_{2^{N-1}} + \mathcal{J} = I_{2^2} \otimes e \otimes J_{2^{N-2}} + \mathcal{J} = \cdots = I_{2^N} \otimes e + \mathcal{J}.$$

Proof. Fix $\mathrm{i} \in [N]$ and consider $I_{2^{\mathrm{i}-1}} \otimes e \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J}$. Since $e = p_\mathrm{i} + p_\mathrm{i}^\perp$ we may rewrite $I_{2^{\mathrm{i}-1}} \otimes e \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J}$ as

$$I_{2^{\mathrm{i}-1}} \otimes e \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J} = (I_{2^{\mathrm{i}-1}} \otimes p_\mathrm{i} \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J}) + (I_{2^{\mathrm{i}-1}} \otimes p_\mathrm{i}^\perp \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J}).$$

We claim

$$I_{2^{\mathrm{i}-1}} \otimes p_\mathrm{i} \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J} = I_{2^{\mathrm{i}-1}} \otimes (p_\mathrm{i} \oplus 0) \otimes J_{2^{N-\mathrm{i}}} + \mathcal{J} \tag{3.8}$$

and

$$I_{2^{\mathrm{i}-1}} \otimes p_\mathrm{i}^\perp \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J} = I_{2^{\mathrm{i}-1}} \otimes (0 \oplus p_\mathrm{i}^\perp) \otimes J_{2^{N-\mathrm{i}}} + \mathcal{J}. \tag{3.9}$$

Our methods are similar to those of Lemma 3.1.12. We first consider $I_{2^{\mathrm{i}-1}} \otimes p_\mathrm{i} \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J}$ which we rewrite as

$$I_{2^{\mathrm{i}-1}} \otimes \begin{pmatrix} p_\mathrm{i} & p_\mathrm{i} \\ p_\mathrm{i} & p_\mathrm{i} \end{pmatrix} \otimes J_{2^{N-\mathrm{i}}} + \mathcal{J},$$

of which Equation (3.8) will follow if we prove that

$$\pm I_{2^{\mathrm{i}-1}} \otimes \begin{pmatrix} 0 & p_\mathrm{i} \\ p_\mathrm{i} & p_\mathrm{i} \end{pmatrix} \otimes J_{2^{N-\mathrm{i}}} \in \mathcal{C}(p_1, \ldots, p_N)_1.$$

Let $\epsilon_1, \ldots, \epsilon_N > 0$. For each $\mathrm{j} \neq \mathrm{i}$ set $t_\mathrm{j} = \epsilon_\mathrm{j}$ and set $t_\mathrm{i} = 1 + \frac{1}{\epsilon_\mathrm{i}}$. It follows

$$\begin{pmatrix} 0 & p_\mathrm{i} \\ p_\mathrm{i} & p_\mathrm{i} \end{pmatrix} + \epsilon_\mathrm{i} \begin{pmatrix} p_\mathrm{i} & 0 \\ 0 & p_\mathrm{i}^\perp \end{pmatrix} + t_\mathrm{i} \begin{pmatrix} p_\mathrm{i}^\perp & 0 \\ 0 & p_\mathrm{i} \end{pmatrix} = \begin{pmatrix} \epsilon_\mathrm{i} & 1 \\ 1 & 2 + \frac{1}{\epsilon_\mathrm{i}} \end{pmatrix} \otimes p_\mathrm{i} + \begin{pmatrix} 1 + \frac{1}{\epsilon_\mathrm{i}} & 0 \\ 0 & \epsilon_\mathrm{i} \end{pmatrix} \otimes p_\mathrm{i}^\perp \in \mathcal{C}_2.$$

Furthermore,

$$-\begin{pmatrix} 0 & p_{\mathrm{i}} \\ p_{\mathrm{i}} & p_{\mathrm{i}} \end{pmatrix} + \epsilon_{\mathrm{i}} \begin{pmatrix} p_{\mathrm{i}} & 0 \\ 0 & p_{\mathrm{i}}^{\perp} \end{pmatrix} + t_{\mathrm{i}} \begin{pmatrix} p_{\mathrm{i}}^{\perp} & 0 \\ 0 & p_{\mathrm{i}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathrm{i}} & -1 \\ -1 & \frac{1}{\epsilon_{\mathrm{i}}} \end{pmatrix} \otimes p_{\mathrm{i}} + \begin{pmatrix} 1+\frac{1}{\epsilon_{\mathrm{i}}} & 0 \\ 0 & \epsilon_{\mathrm{i}} \end{pmatrix} \otimes p_{\mathrm{i}}^{\perp} \in \mathcal{C}_2$$

This implies

$$\pm I_{2^{\mathrm{i}-1}} \otimes \begin{pmatrix} 0 & p_{\mathrm{i}} \\ p_{\mathrm{i}} & p_{\mathrm{i}} \end{pmatrix} \otimes J_{2^{N-\mathrm{i}}} + \epsilon_{\mathrm{i}} P_{\mathrm{i}}^{N} + t_{\mathrm{i}} Q_{\mathrm{i}}^{N} \in \mathcal{C}_{2^N}.$$

Consequently, it follows

$$\pm I_{2^{\mathrm{i}-1}} \otimes \begin{pmatrix} 0 & p_{\mathrm{i}} \\ p_{\mathrm{i}} & p_{\mathrm{i}} \end{pmatrix} \otimes J_{2^{N-\mathrm{i}}} + \sum_{\mathrm{j}=1}^{N} \epsilon_{\mathrm{j}} P_{\mathrm{j}}^{N} + \sum_{\mathrm{j}=1}^{N} t_{\mathrm{j}} Q_{\mathrm{j}}^{N} \in \mathcal{C}_{2^N},$$

which implies

$$\pm I_{2^{\mathrm{i}-1}} \otimes \begin{pmatrix} 0 & p_{\mathrm{i}} \\ p_{\mathrm{i}} & p_{\mathrm{i}} \end{pmatrix} \otimes J_{2^{N-\mathrm{i}}} \in \mathcal{C}(p_1, \dots, p_N)_1.$$

This proves Equation (3.8) and Equation (3.9) is proven in a similar manner. Notice when we add the right side of Equation (3.8) with the right side of Equation (3.9) we obtain

$$(I_{2^{\mathrm{i}-1}} \otimes (p_{\mathrm{i}} \oplus 0) \otimes J_{2^{N-\mathrm{i}}} + \mathcal{J}) + (I_{2^{\mathrm{i}-1}} \otimes (0 \oplus p_{\mathrm{i}}^{\perp}) \otimes J_{2^{N-\mathrm{i}}} + \mathcal{J}) = P_{\mathrm{i}}^{N} + \mathcal{J}.$$

When we add the left side of Equation (3.8) with the left side of Equation (3.9) we obtain

$$(I_{2^{\mathrm{i}-1}} \otimes p_{\mathrm{i}} \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J}) + (I_{2^{\mathrm{i}-1}} \otimes p_{\mathrm{i}}^{\perp} \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J}) = I_{2^{\mathrm{i}-1}} \otimes e \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J}.$$

Combining these yields

$$I_{2^{\mathrm{i}-1}} \otimes e \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J} = P_{\mathrm{i}}^{N} + \mathcal{J}, \quad \text{for each } \mathrm{i} \in [N].$$

Once again fix $\mathrm{i} \in [N]$ and suppose $\epsilon_1, \ldots, \epsilon_N > 0$. For $\mathrm{j} \neq \mathrm{i}$ let $t_\mathrm{j} = \epsilon_\mathrm{j}$ and set $t_\mathrm{i} = 1$. It follows

$$\pm Q_\mathrm{i}^N + \sum_\mathrm{j} \epsilon_\mathrm{j} P_\mathrm{j}^N + \sum_\mathrm{j} t_\mathrm{j} Q_\mathrm{j}^N \in \mathcal{C}_{2^N},$$

and consequently $\pm Q_\mathrm{i}^N \in \mathcal{C}(p_1, \ldots, p_N)_1$. This implies

$$P_\mathrm{i}^N + \mathcal{J} = (P_\mathrm{i}^N + Q_\mathrm{i}^N) + \mathcal{J} = I_{2^\mathrm{i}} \otimes e \otimes J_{2^{N-\mathrm{i}}} + \mathcal{J}, \quad \text{for each } \mathrm{i} \in [N].$$

It thus follows

$$e \otimes J_{2^N} + \mathcal{J} = I_2 \otimes e \otimes J_{2^{N-1}} + \mathcal{J} = I_{2^2} \otimes e \otimes J_{2^{N-2}} + \mathcal{J} = \cdots = I_{2^N} \otimes e + \mathcal{J}$$

and in particular, the mapping $x \mapsto x \otimes J_{2^N} + \mathcal{J}$ is unital. $\square$

Proposition 3.2.4. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $p \in \mathcal{X}$ be a positive contraction. Let $\mathcal{C}(p)$ denote the matrix ordering, induced by p, on $M_2(\mathcal{X})$ and let $\mathcal{J} := \operatorname{span} \mathcal{C}(p)_1 \cap -\mathcal{C}(p)_1$. If $\pi_p : \mathcal{X} \to M_2(\mathcal{X})/\mathcal{J}$ denotes the mapping taking $x \mapsto x \otimes J_2 + \mathcal{J}$ then π_p is completely positive.*

Proof. Fix $n \in \mathbb{N}$ and consider $x \in \mathcal{C}_n$. Then $x \otimes J_2 \in \mathcal{C}_{2n}$ and consequently if $\epsilon > 0$ is arbitrary then by letting $t = \epsilon$ it follows

$$x \otimes J_2 + \epsilon I_n \otimes (p \oplus p^\perp) + \epsilon I_n \otimes (p^\perp \oplus p) \in \mathcal{C}_{2n}.$$

This implies that $(\pi_p)_n(x) = x \otimes J_2 + M_n(\mathcal{J}) \in \mathcal{C}(p)_n + M_n(\mathcal{J})$ which proves the claim. $\square$

We now prove when a finite set of positive contractions in an operator system are all simultaneously abstract projections.

Theorem 3.2.4. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $\{p_1, \ldots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions. Then the following are equivalent:*

1. *Each p_i is an abstract projection in $\mathcal{X}$.*

2. *For each* $\mathrm{i} \in [N]$ *the map* $\pi_{p_\mathrm{i}} : \mathcal{X} \to M_2(\mathcal{X})/\mathcal{J}_\mathrm{i}, x \mapsto x \otimes J_2 + \mathcal{J}_\mathrm{i}$*, where* $\mathcal{J}_\mathrm{i} := span\,\mathcal{C}(p_\mathrm{i})_1 \cap -\mathcal{C}(p_\mathrm{i})_1$ *is a complete order embedding.*

3. *The map* $x \mapsto x \otimes J_{2^N} + \mathcal{J}$*, where* $\mathcal{J} := span\,\mathcal{C}(p_1, \ldots, p_N)_1 \cap -\mathcal{C}(p_1, \ldots, p_N)_1$*, is a complete order embedding from* $(\mathcal{X}, \mathcal{C}, e)$ *to*

$$(M_{2^N}(\mathcal{X})/\mathcal{J}, \mathcal{C}(p_1, \ldots, p_N) + \mathcal{J}, I_{2^N} \otimes e + \mathcal{J}).$$

Proof. The equivalence of (1) and (2) is Theorem 3.1.15. Furthermore it is immediate from Proposition 3.2.1 that (1) implies (3). We thus prove (3) implies (2).

Assume (3) holds and fix $\mathrm{i} \in [N]$. We claim $\pi_{p_\mathrm{i}} : \mathcal{X} \to M_2(\mathcal{X})/\mathcal{J}_\mathrm{i}$ is a complete order embedding. By Proposition 3.2.4 we need only show $\pi_{p_\mathrm{i}}^{-1} : \pi_{p_\mathrm{i}}(\mathcal{X}) \to \mathcal{X}$ is completely positive. Let $x \in M_n(\mathcal{X})$ such that $(\pi_{p_\mathrm{i}})_n(x) \in \mathcal{C}(p_\mathrm{i})_n + M_n(J_\mathrm{i})$. Thus for every $\epsilon > 0$ there exists $t_\mathrm{i} > 0$ such that

$$x \otimes J_2 + \epsilon I_n \otimes (p_\mathrm{i} \oplus p^\perp) + t I_n \otimes (p_\mathrm{i}^\perp \oplus p_\mathrm{i}) \in \mathcal{C}_{2n}.$$

By tensoring on the left by the positive matrix $J_{2^{N-\mathrm{i}}}$ and by tensoring on the right by the positive matrix $J_{2^{\mathrm{i}-1}}$ we obtain

$$J_{2^{N-\mathrm{i}}} \otimes (x \otimes J_2) \otimes J_{2^{\mathrm{i}-1}} + \epsilon_\mathrm{i} J_{2^{N-\mathrm{i}}} \otimes I_n \otimes (p_\mathrm{i} \oplus p_\mathrm{i}^\perp) \otimes J_{2^{\mathrm{i}-1}} + t_\mathrm{i} J_{2^{N-\mathrm{i}}} \otimes I_n \otimes (p_\mathrm{i}^\perp \oplus p_\mathrm{i}) \otimes J_{2^{\mathrm{i}-1}},$$

which is then an element of $\mathcal{C}_{n2^N}$. By applying the canonical shuffle $M_{2^{N-\mathrm{i}}} \otimes M_n \otimes M_2 \otimes M_{2^{\mathrm{i}-1}} \to M_n \otimes M_{2^{N-\mathrm{i}}} \otimes M_2 \otimes M_{2^{\mathrm{i}-1}}$, followed by the shuffle $M_n \otimes M_{2^{N-\mathrm{i}}} \otimes M_2 \otimes M_{2^{\mathrm{i}-1}} \to M_n \otimes M_{2^{\mathrm{i}-1}} \otimes M_2 \otimes M_{2^{N-\mathrm{i}}}$, it follows

$$x \otimes J_{2^N} + \epsilon_\mathrm{i} I_n \otimes \widehat{P}_\mathrm{i}^N + t_\mathrm{i} \widehat{Q}_\mathrm{i}^N \in \mathcal{C}_{n2^N},$$

where $\widehat{P}_{\mathrm{i}}^N$ and $\widehat{Q}_{\mathrm{i}}^N$ are as in Definition 3.2.1. Let $\epsilon_{\mathrm{j}} > 0$ be arbitrary for $\mathrm{j} \neq \mathrm{i}$ and set $t_{\mathrm{j}} = \epsilon_{\mathrm{j}}$ for $\mathrm{j} \neq \mathrm{i}$. Then

$$x \otimes J_{2^N} + \sum_{\mathrm{j}} \epsilon_{\mathrm{j}} I_n \otimes \widehat{P}_{\mathrm{j}}^N + \sum_{\mathrm{j}} t_{\mathrm{j}} I_n \otimes \widehat{Q}_{\mathrm{j}}^N \in \mathcal{C}_{n2^N}.$$

Consequently, $x \otimes J_{2^N} \in \widehat{\mathcal{C}}(p_1, \ldots, p_N)_n \subset \mathcal{C}(p_1, \ldots, p_N)_n$ which by our assumption implies $x \in \mathcal{C}_n$. This proves (2) and concludes the proof. $\square$

We have thus far shown that our notion of an abstract projection in an operator system has similar properties to projections in C*-algebras. In Theorem 3.1.15 we characterized when a positive contraction is necessarily an abstract projection. Furthermore, given a finite number of positive contractions in an operator system, in Theorem 3.2.4 we proved when all the positive contractions are simultaneously abstract projections. In the next chapter we will extend our methods to Archimedean order unit spaces.

4. PROJECTIONS IN ARCHIMEDEAN ORDER UNIT SPACES

Building off of our results in Section 3 our goal of this section will be to characterize when finite sets of positive contractions in an AOU space are simultaneously abstract projections relative to the same operator system structure.

4.1 Single Projections in AOU spaces

Given an operator system $(\mathcal{X}, \mathcal{C}, e)$ and a positive contraction $p \in \mathcal{X}$ then we once again let $\mathcal{C}(p)$ denote the matrix ordering on $M_2(\mathcal{X})$ defined for each $n \in \mathbb{N}$ by

$$\mathcal{C}(p)_n := \{x \in M_{2n}(\mathcal{X})_h : \forall \epsilon > 0 \ \exists t > 0 \text{ such that } x + \epsilon I_n(p \oplus p^\perp) + tI_n \otimes (p^\perp \oplus p) \in \mathcal{C}_{2n}\}.$$

Furthermore, we will let $\mathcal{J} := \operatorname{span} \mathcal{C}(p)_1 \cap -\mathcal{C}(p)_1$.

We consider the operator system $(M_2(\mathcal{X})/\mathcal{J}, \mathcal{C}(p) + \mathcal{J}, I_2 \otimes e + \mathcal{J})$.

Remark 4.1.1. In Subsection 3.1 we considered abstract compression operator systems where the Archimedean matrix order unit was given by $p \oplus p^\perp + \mathcal{J}$ where $p \in \mathcal{X}$ was our initial contraction. Similar to our proof in Lemma 3.1.12 it follows that $p \oplus p^\perp + \mathcal{J} = I_2 \otimes e + \mathcal{J}$. Let $\pi_p : \mathcal{X} \to M_2(\mathcal{X})/\mathcal{J}$ denote the map $x \mapsto x \otimes J_2 + \mathcal{J}$. It was proven in Lemma 3.1.12 that $e \otimes J_2 + \mathcal{J} = (p \oplus p^\perp) + \mathcal{J}$. Note that $I_2 \otimes e + \mathcal{J} = (p \oplus p^\perp) + \mathcal{J} + (p^\perp \oplus p) + \mathcal{J}$. We claim that $\pm(0 \oplus p), \pm(p^\perp \oplus 0) \in \mathcal{C}(p)_1$. This is proven following the methods of Lemma 3.1.12. Another way to see this is by utilizing Proposition 3.2.3. In the proof of Proposition 3.2.3 we saw if $\{p_1, \ldots, p_N\} \subset \mathcal{X}$ is a finite set of positive contractions then for each $\mathrm{i} \in [N]$ one has

$$I_{2^{\mathrm{i}-1}} \otimes e \otimes J_{2^{N-\mathrm{i}+1}} + \mathcal{J} = P_{\mathrm{i}}^N + \mathcal{J} = I_{2^{\mathrm{i}}} \otimes e \otimes J_{2^{N-\mathrm{i}}} + \mathcal{J}.$$

Let $N = 1$. Then we have $e \otimes J_2 + \mathcal{J} = P_1^1 + \mathcal{J} = I_2 \otimes e + \mathcal{J}$.

Lemma 4.1.2. *Let $(\mathcal{X}, C, e)$ be an AOU space and let $\mathcal{C}$ be an operator system structure on $\mathcal{X}$. Furthermore, let $p \in \mathcal{X}$ be a positive contraction and let $\mathcal{C}(p)$ denote the matrix* *ordering induced by* p*. Set* $\mathcal{J} := \text{span } \mathcal{C}(p)_1 \cap -\mathcal{C}(p)_1$*. For each* $n \in \mathbb{N}$ *consider the set* $(\pi_p)_n^{-1}(\mathcal{C}(p)_n + M_n(\mathcal{J}))$*, where*

$$\pi_p : (\mathcal{X}, \mathcal{C}, e) \to (M_2(\mathcal{X})/\mathcal{J}, \mathcal{C}(p) + \mathcal{J}, I_2 \otimes e + \mathcal{J})$$

denotes the mapping $\pi_p(x) = x \otimes J_2 + \mathcal{J}$*. Then the collection* $\pi_p^{-1}(\mathcal{C}(p)+\mathcal{J}) := \{(\pi_p)_n^{-1}(\mathcal{C}(p)_n + M_n(\mathcal{J}))\}_{n\in\mathbb{N}}$ *is a matrix ordering on* $\mathcal{X}$ *and* $\mathcal{C}(p)_n = \pi_p^{-1}(\mathcal{C}(p) + \mathcal{J})(p)_n$ *for each* $n \in \mathbb{N}$*.*

Proof. For notational convenience, throughout this proof we denote $\pi_p^{-1}(\mathcal{C}(p) + \mathcal{J})$ by $\mathcal{D}$. Thus, we claim that $\mathcal{C}(p) = \mathcal{D}(p)$. Since $\mathcal{C}(p) + \mathcal{J}$ is a matrix ordering then it follows immediately that $\mathcal{D}$ is a matrix ordering. Using Proposition 3.2.4 then given any $x \in \mathcal{C}_n$ it necessarily follows that $(\pi_p)_n(x) \in \mathcal{C}(p)_n + M_n(\mathcal{J})$, i.e., $x \in \mathcal{D}_n$. Thus, $\mathcal{C} \subset \mathcal{D}$.

Suppose $x \in \mathcal{C}(p)_n$ and let $\epsilon > 0$ be arbitrary. Then there exists $t > 0$ such that

$$x + \epsilon I_n \otimes (p \oplus p^\perp) + tI_n \otimes (p^\perp \oplus p) \in \mathcal{C}_{2n},$$

and since $\mathcal{C}_{2n} \subset \mathcal{D}_{2n}$ it follows that $x \in \mathcal{D}(p)_n$ and therefore $\mathcal{C}(p) \subset \mathcal{D}(p)$.

Conversely, suppose that $x \in \mathcal{D}(p)_n$ which implies that for all $\epsilon > 0$ there exists $t > 0$ such that

$$x + \epsilon I_n \otimes (p \oplus p^\perp) + tI_n \otimes (p^\perp \oplus p) \in \mathcal{D}_{2n}$$

and thus

$$(\pi_p)_{2n}(x + \epsilon I_n \otimes (p \oplus p^\perp) + tI_n \otimes (p^\perp \oplus p)) \in \mathcal{C}(p)_{2n} + M_{2n}(\mathcal{J}).$$

In particular given any $\delta > 0$ there exists $r > 0$ such that

$$\Big[x + \epsilon I_n \otimes (p \oplus p^\perp) + tI_n \otimes (p^\perp \oplus p)\Big] \otimes J_2 + \delta I_{2n} \otimes (p \oplus p^\perp) + rI_{2n} \otimes (p^\perp \oplus p) \in \mathcal{C}_{4n}.$$

Thus, given any $\epsilon > 0$ there exists $t, r > 0$ such that

$$\left[x + \frac{\epsilon}{2} I_n \otimes (p \oplus p^\perp) + t I_n \otimes (p^\perp \oplus p)\right] \otimes J_2 + \frac{\epsilon}{2} I_{2n} \otimes (p \oplus p^\perp) + r I_{2n} \otimes (p^\perp \oplus p) \in \mathcal{C}_{4n}$$

After applying the canonical shuffle $M_{2n} \otimes M_2 \to M_2 \otimes M_{2n}$ the above expression becomes

$$J_2 \otimes \left[x + \frac{\epsilon}{2} I_n \otimes (p \oplus p^\perp) + t I_n \otimes (p^\perp \oplus p)\right] + \frac{\epsilon}{2}(p \oplus p^\perp) \otimes I_{2n} + r(p^\perp \oplus p) \otimes I_{2n}.$$

Let $A \in M_{4n}$ denote the $4n \times 4n$ permutation matrix which exchanges the i^{th} and $(\mathrm{i} + 2n)^{\mathrm{th}}$ columns for $\mathrm{i} = 2, 4, \ldots, 2n$. Conjugating the above expression by A then compressing to the 1×1 corner implies that for all $\epsilon > 0$ there exists $t, r > 0$ such that

$$x + \epsilon I_n \otimes (p \oplus p^\perp) + (t + r) I_n \otimes (p^\perp \oplus p) \in \mathcal{C}_{2n}$$

and consequently $x \in \mathcal{C}(p)_n$ which finishes the proof. □

Using the above lemma we are now able to characterize when a positive contraction in an AOU space is an abstract projection relative to some operator system structure.

Theorem 4.1.3. *Let $(\mathcal{X}, C, e)$ be an AOU space and let $p \in \mathcal{X}$ be a positive contraction. Then the following are equivalent:*

1. *There exists an operator system structure $\mathcal{C}$, on $\mathcal{X}$, such that p is an abstract projection in $(\mathcal{X}, \mathcal{C}, e)$.*
2. *The map $\pi_p : (\mathcal{X}, C) \to (M_2(\mathcal{X})/\mathcal{J}, C^{max}(p)_1 + \mathcal{J}), x \mapsto x \otimes J_2 + \mathcal{J}$ is an order embedding.*

Proof. We begin by showing (1) implies (2). Suppose $\mathcal{C}$ is an operator system structure on $\mathcal{X}$ such that $p \in \mathcal{X}$ is an abstract projection. This is to say that $(\mathcal{X}, \mathcal{C}, e)$ is an operator system and $\mathcal{C}_1 = C$. If $x \in C = C_1^{\max}$ then by Proposition 3.2.4 we know $\pi_p(x) \in C^{\max}(p)_1 + \mathcal{J}$. Conversely, if $\pi_p(x) \in C^{\max}(p)_1 + \mathcal{J}$, then for all $\epsilon > 0$ there exists $t > 0$ such that

$$x \otimes J_2 + \epsilon(p \oplus p^\perp) + t(p^\perp \oplus p) \in C_2^{\max}.$$

By properties of the maximal operator system structure we have $C^{\max} \subset \mathcal{C}$. Consequently,

$$x \otimes J_2 + \epsilon(p \oplus p^{\perp}) + t(p^{\perp} \oplus p) \in \mathcal{C}_2.$$

Thus, $\pi_p(x) \in \mathcal{C}(p)_1 + \mathcal{J}$, which by our assumption implies $x \in \mathcal{C}_1 = C$. This proves (2).

We now show (2) implies (1). To this end suppose that

$$\pi_p : (\mathcal{X}, C, e) \to (M_2(\mathcal{X})/\mathcal{J}, C^{\max}(p)_1 + \mathcal{J})$$

is an order embedding. We consider the matrix ordering $\pi_p^{-1}(C^{\max}(p) + \mathcal{J})$ on $\mathcal{X}$ which, for notational convenience, we will denote as $\mathcal{D}$. By assumption we have $\mathcal{D}_1 = C$. We claim $\pi_p : (\mathcal{X}, \mathcal{D}, e) \to (M_2(\mathcal{X})/\mathcal{J}, \mathcal{D}(p) + \mathcal{J}, I_2 \otimes e + \mathcal{J})$ is a complete order embedding. By Lemma 4.1.2 we have $\mathcal{D}(p) = C^{\max}(p)$. Thus, if $x \in M_n(\mathcal{X})$ such that $(\pi_p)_n(x) \in \mathcal{D}(p)_n + M_n(\mathcal{J})$ then $(\pi_p)_n(x) \in C^{\max}(p)_n + M_n(\mathcal{J})$. By construction it must follow $x \in \mathcal{D}_n$. $\square$

We end this section with some remarks regarding our methods above. Using Theorem 4.1.3 we are able to detect when a positive contraction in an AOU space is an abstract projection relative to some operator system structure on the $*$-vector space. But this theorem has strict limitations. In particular, if p_1 and p_2 are two positive contractions in an AOU space $(\mathcal{X}, C, e)$ then by Theorem 4.1.3 there exist operator system structures $\mathcal{K}$ and $\mathcal{L}$, on $\mathcal{X}$, such that p_1 is an abstract projection in $(\mathcal{X}, \mathcal{K}, e)$ and p_2 is an abstract projection in $(\mathcal{X}, \mathcal{L}, e)$. The issue arises in the fact that it is not necessary that $\mathcal{K} = \mathcal{L}$. We wish to develop a method to determine when a finite set of positive contractions in an AOU space are all simultaneously abstract projections in the same operator system.

4.2 Multiple Projections in AOU spaces

Expanding on methods from Subsection 3.2, the goal of this section is to prove when a finite set of positive contractions in an AOU space are simultaneously abstract projections relative to an operator system structure on the AOU space.

Definition 4.2.1. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $\{p_1, \dots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions in $\mathcal{X}$. For each $\mathrm{i} \in [N]$ let $P_{\mathrm{i}}^N, Q_{\mathrm{i}}^N \in M_{2^N}(\mathcal{X})$ be as in Definition 3.2.1. Fix $L \in \mathbb{N}$ and define the operators on $M_{2^{NL}}(\mathcal{X})$,*

$$P_{\mathrm{ij}}^{NL} := I_{2^{N(\mathrm{j}-1)}} \otimes P_{\mathrm{i}}^N \otimes J_{2^{N(L-\mathrm{j})}},$$
$$Q_{\mathrm{ij}}^{NL} := I_{2^{N(\mathrm{j}-1)}} \otimes Q_{\mathrm{i}}^N \otimes J_{2^{N(L-\mathrm{j})}},$$

where $\mathrm{j} \in [L]$. For each $L \in \mathbb{N}$ we define the collection $\mathcal{C}(p_1, \dots, p_N)^L := \{\mathcal{C}(p_1, \dots, p_N)_n^L\}_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$, we have $x \in \mathcal{C}(p_1, \dots, p_N)_n^L$ if and only if given any $N \times L$ matrix $(\epsilon_{\mathrm{ij}})_{\mathrm{ij}}$ of strictly positive real numbers there exists an $N \times L$ matrix $(t_{\mathrm{ij}})_{\mathrm{ij}}$ of strictly positive real numbers such that

$$x + \sum_{\mathrm{ij}} \epsilon_{\mathrm{ij}} I_n \otimes P_{\mathrm{ij}}^{NL} + \sum_{\mathrm{ij}} t_{\mathrm{ij}} I_n \otimes Q_{\mathrm{ij}}^{NL} \in \mathcal{C}_{2^{NL}}, \tag{4.1}$$

with the property that if one replaces ϵ_{ij} with $0 < \epsilon'_{\mathrm{ij}} < \epsilon_{\mathrm{ij}}$ then we may choose $t'_{\mathrm{ij}} > t_{\mathrm{ij}}$ such that Equation (4.1) still holds.

Another way to view the collection $\mathcal{C}(p_1, \dots, p_N)^L$ is that it is the collection of cones as defined in Definition 3.2.1 when the positive contractions $p_1, \dots, p_N$ are repeated sequentially L times. This leads to an immediate proposition.

Proposition 4.2.1. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $\{p_1, \dots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions. Fix $L \in \mathbb{N}$, then the following hold:*

1. *Let $\mathcal{J}^L := span\, \mathcal{C}(p_1, \dots, p_N)_1^L \cap -\mathcal{C}(p_1, \dots, p_N)_1^L$. Then the triple*

   $$(M_{2^{NL}}(\mathcal{X})/\mathcal{J}^L, \mathcal{C}(p_1, \dots, p_N)^L + \mathcal{J}, I_{2^{NL}} \otimes e + \mathcal{J}^L)$$

   is an operator system and the mapping $x \mapsto x \otimes J_{2^{NL}} + \mathcal{J}$ is unital.

2. *The positive contractions $p_1, \dots, p_N$ are all abstract projection in $\mathcal{X}$ if and only if the mapping $x \mapsto x \otimes J_{2^{NL}} + \mathcal{J}$ from $(\mathcal{X}, \mathcal{C}, e)$ to*

   $$(M_{2^{NL}}(\mathcal{X})/\mathcal{J}^L, \mathcal{C}(p_1, \dots, p_N)^L + \mathcal{J}, I_{2^{NL}} \otimes e + \mathcal{J}^L)$$

is a complete order embedding.

Proof. Statement (1) is immediate from Proposition 3.2.2 and Proposition 3.2.3. Statement (2) is immediate from Theorem 3.2.4. $\square$

Definition 4.2.2. *Let $\mathcal{X}$ be a $*$-vector space and fix a nonzero hermitian element $e \in \mathcal{X}_h$. Suppose for every $L \in \mathbb{N}$ there exists a matrix ordering $\mathcal{C}^L$ such that $(\mathcal{X}, \mathcal{C}^L, e)$ is an operator system. We say the collection $\{\mathcal{C}^L\}_{L\in\mathbb{N}}$ of matrix orderings is* nested increasing *if for all $n \in \mathbb{N}$ it follows $\mathcal{C}_n^L \subset \mathcal{C}_n^{L+1}$ for each $L \in \mathbb{N}$. Given a nested increasing sequence $\{\mathcal{C}^L\}_{L\in\mathbb{N}}$ we define the* inductive limit *$\mathcal{C}^\infty$ to be the collection $\mathcal{C}^\infty := \{\mathcal{C}_n^\infty\}_{n\in\mathbb{N}}$ defined for each $n \in \mathbb{N}$ by $x \in \mathcal{C}_n^\infty$ if and only if for all $\epsilon > 0$ there exists $L \in \mathbb{N}$ such that $x + I_n \otimes e \in \mathcal{C}_n^L$. In particular, $\mathcal{C}_n^\infty$ is the Archimedean closure of $\bigcup_{L\in\mathbb{N}} C_n^L$ with respect to e.*

Lemma 4.2.1. *Let $\mathcal{X}$ be a $*$-vector space and consider a nonzero hermitian element $e \in \mathcal{X}_h$. Suppose for every $L \in \mathbb{N}$ there exists a matrix ordering $\mathcal{C}^L$ such that $(\mathcal{X}, \mathcal{C}^L, e)$ is an operator system and suppose the collection $\{\mathcal{C}^L\}_{L\in\mathbb{N}}$ forms a nested increasing sequence. Then $(\mathcal{X}, \mathcal{C}^\infty)$ is a matrix ordered $*$-vector space and e is an Archimedean matrix order unit.*

Proof. We prove that $\mathcal{C}^\infty$ is a matrix ordering by showing that it is closed under direct sums and is compatible. First consider the collection $\mathcal{C}' := \{\mathcal{C}'_n\}_{n\in\mathbb{N}}$ where $\mathcal{C}'_n := \bigcup_{L\in\mathbb{N}} \mathcal{C}_n^L$. If $x \in \mathcal{C}'_n$ and $\lambda > 0$ then since $x \in \mathcal{C}_n^L$ for some L implies $\lambda x \in \lambda \mathcal{C}_n^L \subset \mathcal{C}_n^L$ and thus $\mathbb{R}^+\mathcal{C}' \subset \mathcal{C}'$. If $x \in \mathcal{C}'_n$ and $y \in \mathcal{C}'_m$ then there exists $L_x, L_y \in \mathbb{N}$ such that $x \in \mathcal{C}_n^{L_x}$ and $y \in \mathcal{C}_m^{L_y}$. Let $L := \max\{L_x, L_y\}$. Since the collection $\{\mathcal{C}^L\}_{L\in\mathbb{N}}$ is nested increasing implies $x \in \mathcal{C}_n^L$ and $y \in \mathcal{C}_m^L$ and consequently $x \oplus y \in \mathcal{C}_{n+m}^L$ which implies $x \oplus y \in \mathcal{C}'_{n+m}$. The compatibility of $\mathcal{C}'$ is immediate since each $\mathcal{C}^L$ is compatible. Since for each $n \in \mathbb{N}$, $\mathcal{C}_n^\infty$ is the Archimedean closure of C'_n, then this proves $(\mathcal{X}, \mathcal{C}^\infty)$ is a matrix ordered $*$-vector space. Let $x \in M_n(\mathcal{X})_h$. Then for each $L \in \mathbb{N}$ there exists $r > 0$ such that $rI_n \otimes e - x \in \mathcal{C}_n^L$. In particular, $rI_n \otimes e - x \in \mathcal{C}'_n$. Thus, e is a matrix order unit for the pair $(\mathcal{X}, \mathcal{C}^\infty)$. By definition of $\mathcal{C}^\infty$ we have that e is necessarily an Archimedean matrix order unit and this finishes the proof. $\square$

Proposition 4.2.2. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $\{p_1, \dots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions. For each $L \in \mathbb{N}$ define the mapping $\boldsymbol{\pi}_L : \mathcal{X} \to M_{2^{NL}}(\mathcal{X})$ defined* *by $\pi_L(x) := x \otimes J_{2^{NL}}$. Then the collection $\{\pi_L^{-1}\mathcal{C}(p_1, \ldots, p_N)^L\}_{L\in\mathbb{N}}$ is a nested increasing sequence of matrix orderings on $\mathcal{X}$.*

Proof. For every $L \in \mathbb{N}$ we have $\pi_L^{-1}\mathcal{C}(p_1, \ldots, p_N)^L := \{(\pi_L)_n^{-1}(\mathcal{C}(p_1, \ldots, p_N)_n^L\}_{n\in\mathbb{N}}$ is a matrix ordering on $\mathcal{X}$. Indeed, this follows by Lemma 3.2.2 and methods similar to Lemma 4.1.2. We therefore show that the collection $\{\pi_L^{-1}\mathcal{C}(p_1, \ldots, p_N)^L\}_{L\in\mathbb{N}}$ forms a nested increasing sequence. To this end, fix $n, L \in \mathbb{N}$ and suppose $x \in M_n(\mathcal{X}) \cap (\pi_L)_n^{-1}(\mathcal{C}(p_1, \ldots, p_N)_n^L)$ and therefore $x \otimes J_{2^{NL}} \in \mathcal{C}(p_1, \ldots, p_N)_n^L$. We claim $x \otimes J_{2^{N(L+1)}} \in \mathcal{C}(p_1, \ldots, p_N)_n^{L+1}$ which will finish the proof.

By our assumptions it follows that given an $N \times L$ matrix $(\epsilon_{\mathrm{ij}})_{\mathrm{i}\in[N],\mathrm{j}\in[L]}$, of strictly positive real numbers, there exists an $N \times L$ matrix $(t_{\mathrm{ij}})_{\mathrm{i}\in[N],\mathrm{j}\in[L]}$ of strictly positive real numbers such that

$$x \otimes J_{2^{NL}} + \sum_{\mathrm{ij}} \epsilon_{\mathrm{ij}} I_n \otimes P_{\mathrm{ij}}^{NL} + \sum_{\mathrm{ij}} t_{\mathrm{ij}} I_n \otimes Q_{\mathrm{ij}}^{NL} \in \mathcal{C}_{n2^{NL}}.$$

We tensor this expression on the right by the positive matrix J_{2^N} which yields

$$x \otimes J_{2^{N(L+1)}} + \sum_{\mathrm{ij}} \epsilon_{\mathrm{ij}} I_n \otimes P_{\mathrm{ij}}^{NL} \otimes J_{2^N} + \sum_{\mathrm{ij}} t_{\mathrm{ij}} I_n \otimes Q_{\mathrm{ij}}^{NL} \otimes J_{2^N} \in \mathcal{C}_{n2^{N(L+1)}}.$$

For each $\mathrm{i} \in [N]$ let $\epsilon_{\mathrm{i},(L+1)} > 0$ be arbitrary and set $t_{\mathrm{i},(L+1)} = \epsilon_{\mathrm{i},(L+1)}$. Notice for each $\mathrm{i} \in [N]$ and $\mathrm{j} \in [L]$ we have

$$P_{\mathrm{ij}}^{NL} \otimes J_{2^N} = I_{2^{N(\mathrm{j}-1)}} \otimes P_{\mathrm{i}}^N \otimes J_{2^{N(L-\mathrm{j})}} \otimes J_{2^N} = I_{2^{N(\mathrm{j}-1)}} \otimes P_{\mathrm{i}}^N \otimes J_{2^{N((L+1)-\mathrm{j})}} = P_{\mathrm{ij}}^{N(L+1)},$$

and a similar calculation holds for $Q_{\mathrm{ij}}^{NL} \otimes J_{2^N}$. We therefore arrive at the following expression

$$x \otimes J_{2^{N(L+1)}} + \sum_{\mathrm{i}=1}^{N} \sum_{\mathrm{j}=1}^{L+1} \epsilon_{\mathrm{ij}} I_n \otimes P_{\mathrm{ij}}^{N(L+1)} + \sum_{\mathrm{i}=1}^{N} \sum_{\mathrm{j}=1}^{L+1} t_{\mathrm{ij}} I_n \otimes Q_{\mathrm{ij}}^{N(L+1)} \in \mathcal{C}_{n2^{N(L+1)}}.$$

This proves $x \otimes J_{2^{N(L+1)}} \in \mathcal{C}(p_1, \ldots, p_N)_n^{(L+1)}$ which proves that the collection

$$\{\pi_L^{-1}\mathcal{C}(p_1, \ldots, p_N)^L\}_{L\in\mathbb{N}}$$

is nested increasing. $\square$

Definition 4.2.3. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $\{p_1, \ldots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions in $\mathcal{X}$. We define the matrix ordering $\mathcal{C}(p_1, \ldots, p_N)^\infty$ on $\mathcal{X}$ to be the inductive limit of the collection $\{\pi_L^{-1}\mathcal{C}(p_1, \ldots, p_N)^L\}_{L \in \mathbb{N}}$ where $\pi_L : \mathcal{X} \to M_{2^{NL}}(\mathcal{X})$ is the mapping $x \mapsto x \otimes J_{2^{NL}}$.*

Proposition 4.2.3. *Let $(\mathcal{X}, \mathcal{C}, e)$ be an operator system and let $\{p_1, \ldots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions. Suppose the matrix ordering $\mathcal{C}(p_1, \ldots, p_N)^\infty$ satisfies the condition that $\mathcal{C}(p_1, \ldots, p_N)_1^\infty \cap -\mathcal{C}(p_1, \ldots, p_N)_1^\infty = \{0\}$. Then the triple $(\mathcal{X}, \mathcal{C}(p_1, \ldots, p_N)^\infty, e)$ is an operator system and p_i is an abstract projection in $(\mathcal{X}, \mathcal{C}(p_1, \ldots, p_N)^\infty, e)$ for each $\mathrm{i} \in [N]$.*

Proof. We have by Proposition 4.2.2 that for each $L \in \mathbb{N}$ it follows $\pi_L^{-1}(\mathcal{C}(p_1, \ldots, p_N)^L)$ is a matrix ordering on $\mathcal{X}$ and furthermore $\{\pi_L^{-1}(\mathcal{C}(p_1, \ldots, p_N))\}_{L \in \mathbb{N}}$ is a nested increasing sequence. Along with Lemma 4.2.1 we necessarily have $(\mathcal{X}, \mathcal{C}(p_1, \ldots, p_N)^\infty)$ is a matrix ordered $*$-vector space and e is an Archimedean matrix order unit. Since $\mathcal{C}(p_1, \ldots, p_N)_1^\infty$ is proper then by applying Lemma 2.0.1 we have $(\mathcal{X}, \mathcal{C}(p_1, \ldots, p_N)^\infty, e)$ is an operator system.

It remains to show that each p_i is an abstract projection in $(\mathcal{X}, \mathcal{C}(p_1, \ldots, p_N)^\infty, e)$. This is to say that the map taking $x \mapsto x \otimes J_{2^N} + \mathcal{J}$ is a complete order embedding from $(\mathcal{X}, \mathcal{C}(p_1, \ldots, p_N)^\infty, e)$ to

$$(M_{2^N}(\mathcal{X})/\mathcal{J}, \mathcal{C}(p_1, \ldots, p_N)^\infty(p_1, \ldots, p_N) + \mathcal{J}, I_{2^N} \otimes e + \mathcal{J}),$$

where $\mathcal{J} := \operatorname{span} \mathcal{C}(p_1, \ldots, p_N)^\infty(p_1, \ldots, p_N)_1 \cap -\mathcal{C}(p_1, \ldots, p_N)^\infty(p_1, \ldots, p_N)_1$. Therefore let $x \in M_n(\mathcal{X})$ such that $x \otimes J_{2^N} + \mathcal{J} \in \mathcal{C}(p_1, \ldots, p_N)^\infty(p_1, \ldots, p_N)_n + \mathcal{J}$. Thus, $x \otimes J_{2^N} \in \mathcal{C}(p_1, \ldots, p_N)^\infty(p_1, \ldots, p_N)_n$ which implies given arbitrary $\epsilon_1, \ldots, \epsilon_N > 0$ there exists $t_1, \ldots, t_N > 0$ such that

$$x \otimes J_{2^N} + \sum_\mathrm{i} \epsilon_\mathrm{i} I_n \otimes P_\mathrm{i}^N + \sum_\mathrm{i} t_\mathrm{i} I_n \otimes Q_\mathrm{i}^N \in \mathcal{C}(p_1, \ldots, p_N)_{n2^N}^\infty.$$

This implies given any $\epsilon > 0$ there exists $L \in \mathbb{N}$ such that

$$x \otimes J_{2^N} + \sum_{\mathrm{i}} \epsilon_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^N + \sum_{\mathrm{i}} t_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^N + \epsilon I_{n2^N} \otimes e \in (\boldsymbol{\pi}_L)_{n2^N}^{-1}(C(p_1, \ldots, p_N)_{n2^N}^L),$$

which implies

$$\left(x \otimes J_{2^N} + \sum_{\mathrm{i}} \epsilon_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^N + \sum_{\mathrm{i}} t_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^N + \epsilon I_{n2^N} \otimes e\right) \otimes J_{2^{NL}} \in C(p_1, \ldots, p_N)_{n2^N}^L.$$

In particular we have

$$\begin{aligned}&\left(x \otimes J_{2^N} + \sum_{\mathrm{i}} \epsilon_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^N + \sum_{\mathrm{i}} t_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^N + \epsilon I_{n2^N} \otimes e\right) \otimes J_{2^{NL}} + M_{n2^N}(\mathcal{J}^L) \\ &\in C(p_1, \ldots, p_N)_{n2^N}^L + M_{n2^N}(\mathcal{J}^L).\end{aligned}$$

where $\mathcal{J}^L := \operatorname{span} \mathcal{C}(p_1, \ldots, p_N)_1^L \cap -\mathcal{C}(p_1, \ldots, p_N)_1^L$. By Proposition 3.2.3 we have

$$I_{2^N} \otimes e \otimes J_{2^{NL}} + M_{n2^N}(\mathcal{J}^L) = e \otimes J_{2^{N(L+1)}} + M_{n2^N}(\mathcal{J}^L),$$

and therefore

$$I_{n2^N} \otimes e \otimes J_{2^{NL}} + M_{n2^N}(\mathcal{J}^L) = I_n \otimes e \otimes J_{2^{N(L+1)}} + M_{n2^N}(\mathcal{J}^L).$$

Thus our expression above becomes

$$\begin{aligned}&\left(x \otimes J_{2^N} + \sum_{\mathrm{i}} \epsilon_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^N + \sum_{\mathrm{i}} t_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^N + \epsilon I_n \otimes e \otimes J_{2^N}\right) \otimes J_{2^{NL}} + M_{n2^N}(\mathcal{J}^L) \\ &\in C(p_1, \ldots, p_N)_{n2^N}^L + M_{n2^N}(\mathcal{J}^L),\end{aligned}$$

and in particular

$$\left(x \otimes J_{2^N} + \sum_{\mathrm{i}} \epsilon_{\mathrm{i}} I_n \otimes P_{\mathrm{i}}^N + \sum_{\mathrm{i}} t_{\mathrm{i}} I_n \otimes Q_{\mathrm{i}}^N + \epsilon I_n \otimes e \otimes J_{2^N}\right) \otimes J_{2^{NL}} \in C(p_1, \ldots, p_N)_{n2^N}^L.$$

By definition of $C(p_1,\dots,p_N)^L_{n2^N}$ we have for any $N\times L$ matrix $(\delta_{kl})_{k\in[N],l\in[L]}$ of strictly positive real numbers there exists an $N\times L$ matrix $(r_{kl})_{k\in[N],l\in[L]}$ of strictly positive real numbers such that

$$\left(x\otimes J_{2^N}+\sum_{\mathrm{i}}\epsilon_{\mathrm{i}}I_n\otimes P^N_{\mathrm{i}}+\sum_{\mathrm{i}}t_{\mathrm{i}}I_n\otimes Q^N_{\mathrm{i}}+\epsilon I_n\otimes e\otimes J_{2^N}\right)\otimes J_{2^{NL}}$$
$$+\sum_{kl}\delta_{kl}I_{n2^N}\otimes P^{NL}_{kl}+\sum_{kl}r_{kl}I_{n2^N}\otimes Q^{NL}_{kl}\in\mathcal{C}_{n2^{N(L+1)}}.$$

Note

$$I_{n2^N}\otimes P^{NL}_{kl}=I_n\otimes I_{2^N}\otimes I_{2^{N(l-1)}}\otimes P^N_k\otimes J_{2^{N(L-l)}}=I_n\otimes I_{2^{N((l+1)-1)}}\otimes P^N_k\otimes J_{2^{N(L+1-(l+1))}}$$
$$=I_n\otimes P^{N(L+1)}_{k(l+1)}.$$

Furthermore, we see $I_n\otimes P^{N(L+1)}_{k1}=I_n\otimes P^N_{\mathrm{i}}\otimes J_{2^{NL}}$. Similar properties hold for $I_n\otimes Q^N_{\mathrm{i}}\otimes J_{2^{NL}}$ and $I_{n2^N}\otimes Q^{NL}_{kl}$.

For each $\mathrm{i}\in[N]$ set $\delta_{\mathrm{i}(L+1)}=\epsilon_{\mathrm{i}}$ and similarly set $r_{\mathrm{i}(L+1)}=t_{\mathrm{i}}$. It then follows

$$(x+\epsilon I_n\otimes e)\otimes J_{2^{N(L+1)}}+\sum_{\mathrm{i}=1}^{N}\sum_{l=1}^{L+1}\delta_{\mathrm{i}l}I_n\otimes P^{N(L+1)}_{\mathrm{i}l}+\sum_{\mathrm{i}=1}^{N}\sum_{l=1}^{L+1}r_{\mathrm{i}l}I_n\otimes Q^{N(L+1)}_{\mathrm{i}l}\in\mathcal{C}_{n2^{N(L+1)}}.$$

This proves that $(x+\epsilon I_n\otimes e)\otimes J_{2^{N(L+1)}}\in\mathcal{C}(p_1,\dots,p_N)^{(L+1)}_n$, and thus $x+\epsilon I_n\otimes e\in(\boldsymbol{\pi}_{L+1})^{-1}_n(\mathcal{C}(p_1,\dots,p_N)^{(L+1)}_n)$. Since $\epsilon>0$ was arbitrary we conclude $x\in\mathcal{C}(p_1,\dots,p_N)^{\infty}_n$. Thus, the mapping $x\mapsto x\otimes J_{2^N}+\mathcal{J}$ from $(\mathcal{X},\mathcal{C},e)$ to

$$(M_{2^N}(\mathcal{X})/\mathcal{J},\mathcal{C}(p_1,\dots,p_N)^{\infty}(p_1,\dots,p_N)+\mathcal{J},I_{2^N}\otimes e+\mathcal{J})$$

is a complete order embedding. By Theorem 3.2.4 this proves our claim. □

We are now able to prove when a finite number of positive contractions in an AOU space are all simultaneously abstract projections relative to a single operator system structure.

Theorem 4.2.2. *Let $(\mathcal{X},C,e)$ be an AOU space and let $\{p_1,\dots,p_N\}\subset\mathcal{X}$ be a finite set of positive contractions in $\mathcal{X}$. Then the following are equivalent:*

1. *There exists a Hilbert space H and a unital order embedding $\pi : \mathcal{X} \to B(H)$ such that $\pi(p_i)$ is a projection on H for each $i \in [N]$.*

2. *There exists an operator system structure $\mathcal{C}$, on $\mathcal{X}$, such that each p_i is an abstract projection in $(\mathcal{X}, \mathcal{C}, e)$.*

3. $C = C^{max}(p_1, \ldots, p_N)_1^\infty$.

Proof. (1) and (2) are equivalent by Theorem 3.2.4. Thus, we need only show that (2) and (3) are equivalent.

To this end, suppose that there exists an operator system structure $\mathcal{C}$ such that each $p_i \in \mathcal{X}$ is an abstract projection in $(\mathcal{X}, \mathcal{C}, e)$. We claim $C = C^{\max}(p_1, \ldots, p_N)_1^\infty$. If $x \in C$ then since $C = C_1^{\max}$ and $C_1^{\max} \subset C^{\max}(p_1, \ldots, p_N)_1^L$ for each $L \in \mathbb{N}$ implies that $C \subset C^{\max}(p_1, \ldots, p_N)_1^\infty$. Conversely, suppose $x \in C^{\max}(p_1, \ldots, p_N)_1^\infty$. Thus, for $\epsilon > 0$ arbitrary there exists $L \in \mathbb{N}$ such that $x + \epsilon e \otimes J_{2^{NL}} \in C^{\max}(p_1, \ldots, p_N)_1^L$. Since $C^{\max} \subset \mathcal{C}$ implies for every $\epsilon > 0$ there exists $L \in \mathbb{N}$ such that $x + \epsilon e \otimes J_{2^{NL}} \in \mathcal{C}(p_1, \ldots, p_N)_1^L$. By Proposition 4.2.1 we have that $x + \epsilon e \in \mathcal{C}_1 = C$ and since C is Archimedean closed with respect to e readily implies $x \in C$. This proves (3).

Suppose that $C = C^{\max}(p_1, \ldots, p_N)_1^\infty$. This implies the cone $C^{\max}(p_1, \ldots, p_N)_1^\infty$ is proper and thus by Proposition 4.2.3 we have each p_i is an abstract projection in the operator system $(\mathcal{X}, C^{\max}(p_1, \ldots, p_N)^\infty, e)$. By assumption we have $C^{\max}(p_1, \ldots, p_N)^\infty$ is the desired operator system structure. $\square$

Corollary 4.2.3. *Let $(\mathcal{X}, C, e)$ be an AOU space and let $\{p_1, \ldots, p_N\} \subset \mathcal{X}$ be a finite set of positive contractions in $\mathcal{X}$. If the cone $C^{max}(p_1, \ldots, p_N)_1^\infty$ is proper then there exists a Hilbert space H a unital order embedding $\pi : (\mathcal{X}, C^{max}(p_1, \ldots, p_N)_1^\infty, e) \to B(H)$ such that $\pi(p_i)$ is a projection on H for each $i \in [N]$.*

Proof. Since the cone $C^{\max}(p_1, \ldots, p_N)_1^\infty$ is proper then by Proposition 4.2.3 the triple $(\mathcal{X}, C^{\max}(p_1, \ldots, p_N)^\infty, e)$ is an operator system and for each $i \in [N]$, p_i is an abstract projection in $(\mathcal{X}, C^{\max}(p_1, \ldots, p_N)^\infty, e)$. If we consider the AOU space $(\mathcal{X}, C^{\max}(p_1, \ldots, p_N)_1^\infty, e)$ then by (1) and (2) from Theorem 4.2.2 we know there exists a Hilbert space H and a unital order embedding $\boldsymbol{\pi} : (\mathcal{X}, C^{\max}(p_1, \ldots, p_N)_1^\infty, e) \to B(H)$ such that $\boldsymbol{\pi}(p_{\mathrm{i}})$ is an abstract projection for each $\mathrm{i} \in [N]$. $\square$

5. APPLICATIONS TO QUANTUM INFORMATION THEORY

Using our notions of abstract projections in operator systems and AOU spaces, we now present new characterizations of the set of quantum commuting correlations. Fix two natural numbers $n, k \in \mathbb{N}$. We define a *correlation* to be a tuple $p := \{p(ab|xy) : a, b \in [k], x, y \in [n]\}$ such that $p(ab|xy) \in \mathbb{R}^+$ for all $a, b \in [k], x, y \in [n]$ and such that $\sum_{ab} p(ab|xy) = 1$ for each $x, y \in [n]$. We refer to n as the number of *inputs* and k as the number of *outputs*. The set of all correlations with n-inputs and k-outputs will be denoted by $C(n, k)$. Given a correlation $p \in C(n, k)$ we consider the operators

$$p_A(a|x) := \sum_b p(ab|xy), \text{ and } \quad p_B(b|y) := \sum_a p(ab|xy),$$

where $x, y \in [n]$, and $a, b \in [k]$. The correlation p is *nonsignalling* if both p_A and p_B are well-defined which is to say $p_A(a|x)$ is independent of the input $y \in [n]$ and $p_B(b|y)$ is independent of the input $x \in [n]$. We denote the set of all nonsignalling correlations with n-inputs and k-outputs by $C_{\text{ns}}(n, k)$. It readily follows that $C_{\text{ns}}(n, k)$ is a convex subset of $\mathbb{R}^{n^2k^2}$. Of particular interest to operator algebraists has been the study of correlation sets which are subsets of the nonsignalling correlations. In particular, all sets of correlations that we consider will be subsets of the nonsignalling correlations with the appropriate input and output sizes.

Let H be a Hilbert space. A *projection-valued measure* is a set of projections

$$\{E_1, \dots, E_N\} \subset B(H)$$

such that $\sum_{\text{i}} E_{\text{i}} = I_H$ where $\text{i} \in [N]$ and $\text{Id}_H : H \to H$ denotes the identity operator on H. If $p \in C_{\text{ns}}(n, k)$ is a nonsignalling correlation then we will say p is *quantum commuting* if there exists a Hilbert space H, a unit vector $\eta \in H$ and pairwise commuting projection-valued measures $\{E_{xa}\}_{a=1}^k, \{F_{yb}\}_{b=1}^k$ for each $x, y \in [n]$ such that $p(ab|xy) = \langle \eta | E_{xa} F_{yb} \eta \rangle$ for each $x, y \in [n]$ and $a, b \in [k]$. The set of all quantum commuting correlations with n-inputs and k-outputs will be denoted $C_{\text{qc}}(n, k)$. If we require the Hilbert space above to be finite-dimensional then we will say the correlation is a *quantum* correlation. The set of all quantum correlations with n-inputs and k-outputs will be denoted $C_{\mathrm{q}}(n,k)$. The closure $\overline{C_{\mathrm{q}}(n,k)}$ will be called the *quantum approximate* correlations with n-inputs and k-outputs and will be denoted $C_{\mathrm{qa}}(n,k)$. It follows

$$C_{\mathrm{q}}(n,k) \subset C_{\mathrm{qa}}(n,k) \subset C_{\mathrm{qc}}(n,k) \subset C_{\mathrm{ns}}(n,k). \tag{5.1}$$

Question 5.0.1 ([22]). *Does it follow $C_{qa}(n,k) = C_{qc}(n,k)$ for all $n,k \in \mathbb{N}$?*

It was proven in [21], that for certain input values n and output values k, the set of quantum correlations $C_{\mathrm{q}}(n,k)$ is not closed. Until recently, it was unknown if for particular values of n and k if the set of quantum approximate correlations was a proper subset of the quantum commuting correlations. In particular, according to the authors in [8], the set of quantum approximate correlations is a proper subset of the set of quantum commuting correlations for very large n and k.

The goal of the rest of this section is to present new and purely abstract characterizations of the set of quantum commuting correlations. We begin with a well-known fact (see e.g. [1, Proposition 6.2]).

Proposition 5.0.1. *Fix $n,k \in \mathbb{N}$, and let $p := \{p(ab|xy) : a,b \in [k], x,y \in [n]\} \in C(n,k)$. Then the following are equivalent:*

1. *$p \in C_{qc}(n,k)$ (resp. $p \in C_q(n,k)$)*

2. *There exists a (resp. finite-dimensional) C*-algebra $\mathcal{A}$, projection-valued measures $\{E_{xa}\}_{a=1}^k, \{F_{yb}\}_{b=1}^k \subset \mathcal{A}$ for each $x,y \in [n]$, such that*

$$E_{xa}F_{yb} = F_{yb}E_{xa}$$

for each a,b,x,y and a state $\varphi : \mathcal{A} \to \mathbb{C}$ such that $\varphi(E_{xa}F_{yb}) = p(ab|xy)$ for each a,b,x,y.

3. *There exists a (resp. finite-dimensional) Hilbert space H, an operator system $\mathcal{X} \subset B(H)$, projection-valued measures $\{E_{xa}\}_{a=1}^k, \{F_{yb}\}_{b=1}^k$ for each $x,y \in [n]$,*

such that $E_{xa}F_{yb} \in \mathcal{X}$ *for each* a, b, x, y, $E_{xa}F_{yb} = F_{yb}E_{xa}$ *for each* a, b, x, y, *and a state* $\varphi : \mathcal{X} \to \mathbb{C}$ *such that* $\varphi(E_{xa}F_{yb}) = p(ab|xy)$ *for each* a, b, x, y.

Definition 5.0.1. *Fix* $n, k \in \mathbb{N}$, *and let* $(\mathcal{X}, \mathcal{C}, e)$ *be an operator system. We say* $\mathcal{X}$ *is* nonsignalling *if* $\mathcal{X} = span\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$ *where each* $Q(ab|xy) \in \mathcal{X}^+$, *and such that for each* $x, y \in [n]$ *we have* $\sum_{ab} Q(ab|xy) = e$. *Furthermore for each* $x, y \in [n]$ *the operators*

$$E(a|x) := \sum_b Q(ab|xy), \text{ and } \quad F(b|y) := \sum_a Q(ab|xy)$$

are well-defined. We will call the set $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$ *the set of generators of* $\mathcal{X}$. *We say* $\mathcal{X}$ *is a* quantum commuting *operator system if each generator* $Q(ab|xy)$ *is an abstract projection in* $\mathcal{X}$.

Theorem 5.0.2. *Fix* $n, k \in \mathbb{N}$ *and let* $p \in C(n, k)$. *Then* p *is a nonsignalling correlation if and only if there exists a nonsignalling operator system* $\mathcal{X}$ *with generators* $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$ *and a unital completely positive map* $\varphi : \mathcal{X} \to \mathbb{C}$ *such that* $\varphi(Q(ab|xy)) = p(ab|xy)$ *for all* $x, y \in [n]$ *and* $a, b \in [k]$. *Similarly,* p *is a quantum commuting correlation if and only if there exists a quantum commuting operator system* $\mathcal{X}$ *with generators* $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$ *and a unital completely positive map* $\varphi : \mathcal{X} \to \mathbb{C}$ *such that* $\varphi(Q(ab|xy)) = p(ab|xy)$ *for all* $x, y \in [n]$ *and* $a, b \in [k]$.

Proof. We begin with the nonsignalling case. Let $\mathcal{X}$ be a nonsignalling operator system and let $\varphi : \mathcal{X} \to \mathbb{C}$ be a state. For each $x, y \in [n], a, b \in [k]$ set

$$p(ab|xy) := \varphi(Q(ab|xy))$$

and let $p := \{p(ab|xy) : a, b \in [k], x, y \in [n]\}$. It readily follows that $p(ab|xy) \in \mathbb{R}^+$ for each x, y, a, b and furthermore $1 = \varphi(e) = \sum_{ab} \varphi(Q(ab|xy)) = \sum_{ab} p(ab|xy)$. Thus, $p \in C(n, k)$. By similar reasoning it follows the marginal operators p_A and p_B are well-defined. This proves $p \in C_{\text{ns}}(n, k)$. Conversely, let $p \in C_{\text{ns}}(n, k)$. Let $\mathcal{X} := \text{span}\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$ where for each a, b, x, y we have $Q(ab|xy) := p(ab|xy)$. Thus, if we let $H := \mathbb{C}$ then $\mathcal{X} \subset B(H)$

is an operator system such that for each $x, y \in [n]$ the sum $\sum_{ab} Q(ab|xy) = e$, where the Archimedean matrix order unit e is 1. And similarly, both marginal operators $E(a|x)$ and $F(b|y)$ are well-defined for each $a, b \in [k]$ and $x, y \in [n]$. Define the map $\varphi : \mathcal{X} \to \mathbb{C}$ by $\varphi(\lambda) = \lambda$. Then φ is a state and satisfies $\varphi(Q(ab|xy)) = p(ab|xy)$ for all $a, b \in [k]$ and $x, y \in [n]$.

We now prove our claim for quantum commuting correlations. If $p \in C_{\mathrm{qc}}(n, k)$ then by Proposition 5.0.1 there exists a Hilbert space H and projection-valued measures $\{E_{xa}\}_{a=1}^{k}$ and $\{F_{yb}\}_{b=1}^{k}$ for each $x, y \in [n]$ such that $E_{xa}F_{yb} = F_{yb}E_{xa}$ for each a, b, x, y. For each $a, b \in [k]$ and $x, y \in [n]$ let $Q(ab|xy) := E_{xa}F_{yb}$. If we let $\mathcal{X} = \mathrm{span}\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$, then $\mathcal{X}$ is an operator subsystem of $B(H)$ with generators $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$ and it is readily verified that $\mathcal{X}$ is a quantum commuting operator system that satisfies the claim by using the state φ as provided in Proposition 5.0.1.

Conversely, let $\mathcal{X}$ be a quantum commuting operator system with the set of generators $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$. By Theorem 3.1.15 it follows each abstract projection $Q(ab|xy)$ is a projection in $C_e^*(\mathcal{X})$. For notational convenience we will write $\mathcal{D} := C_e^*(\mathcal{X})$. Let H be a Hilbert space such that $\mathcal{D}$ is represented faithfully on H. It follows $Q(ab|xy)Q(a'b'|xy) = 0$ if either $a \neq a'$ or $b \neq b'$ since by assumption $\sum_{ab} Q(ab|xy) = e$. Note, here e denotes the Archimedean matrix order unit of $\mathcal{X}$, which by our assumption on $\mathcal{D}$ is also equal to Id_H. Furthermore, if we consider the marginal operators defined for each $a, b \in [k], x, y \in [n]$,

$$E(a|x) = \sum_{b} Q(ab|xy), \quad F(b|y) = \sum_{a} Q(ab|xy),$$

then it is readily checked that both $\{E(a|x)\}_{a \in [k]}$ and $\{F(b|y)\}_{b \in [k]}$ are projection-valued measures on H. It also follows

$$E(a|x)F(b|y) = \sum_{a'b'} Q(ab'|xy)Q(a'b|xy) = Q(ab|xy).$$

Both of these observations follow from the properties and orthogonality of the projections $Q(ab|xy)$. Let $\varphi : \mathcal{X} \to \mathbb{C}$ be a state on the operator system $\mathcal{X}$ and let $\widetilde{\varphi} : \mathcal{D} \to \mathbb{C}$ be a (unital) completely positive extension of it obtained from the Arveson-Wittstock extension theorem. Let $p := \{p(ab|xy) : a, b \in [k], x, y \in [n]\}$ be the tuple defined by $p(ab|xy) := \widetilde{\varphi}(E(a|x)F(b|y)) = \widetilde{\varphi}(Q(ab|xy)) = \varphi(Q(ab|xy))$. By Proposition 5.0.1 it then follows $p \in C_{\text{qc}}(n, k)$. This finishes the proof. □

The goal of the rest of this section is to construct an AOU space which is universal with respect to quantum commuting correlations. This is to say that a correlation $p \in C(n, k)$ is quantum commuting if and only if there exists a unital positive map on said AOU space such that the correlation is determined by the action of the unital positive map on the generators of the AOU space. We will first construct such an object which is universal with respect to nonsignalling correlations, then we will apply methods from Section 3.2 and Section 4.2 to obtain such a universal object with respect to quantum commuting correlations.

Definition 5.0.2. *A* nonsignalling *vector space $\mathcal{X}$ is a vector space $\mathcal{X}$ such that $\mathcal{X} = \mathrm{span}\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$, where the $Q(ab|xy)$ are called the* generators *of $\mathcal{X}$ and they satisfy the following properties:*

1. *For each $x, y \in [n]$ it follows $\sum_{ab} Q(ab|xy) = e$, for some fixed nonzero e.*
2. *The vectors $E(a|x) := \sum_b Q(ab|xy)$ and $F(b|y) := \sum_a Q(ab|xy)$ are well-defined for all $x, y \in [n]$ and $a, b \in [k]$.*

The element e as in (1) will be called the unit *of the vector space $\mathcal{X}$. If $\mathcal{X}$ is a nonsignalling vector space then we will let $n(\mathcal{X})$ and $k(\mathcal{X})$ denote the number of inputs and outputs, respectively. Thus, $\mathcal{X} = \mathrm{span}\{Q(ab|xy) : a, b \in k(\mathcal{X}), x, y \in n(\mathcal{X})\}$.*

Example 5.0.3. Let $n, k \in \mathbb{N}$. Let D_k denote the set of diagonal $k \times k$ matrices. Let E_a denote the diagonal matrix with 1 for its a^{th} diagonal entry and zeroes elsewhere. Let $\mathcal{X} \subset D_k^{\otimes 2n}$ denote the vector space spanned by the operators $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$ defined by

$$Q(ab|xy) := I_k^{\otimes x-1} \otimes E_a \otimes I_k^{\otimes n-x} \otimes I_k^{\otimes y-1} \otimes E_b \otimes I_k^{\otimes n-y}$$

where I_k denotes the $k \times k$ identity matrix and $I_k^{\otimes n}$ denotes the n-fold tensor product of I_k with itself (understanding $I_k^{\otimes 0} = 1$). It readily follows that $\mathcal{X}$ is a nonsignalling vector space. The marginal vectors are respectively defined as

$$E(a|x) = I_k^{\otimes x-1} \otimes E_a \otimes I_k^{\otimes n-x} \otimes I_k^{\otimes n} \quad \text{and} \quad F(b|y) = I_k^{\otimes n} \otimes I_k^{\otimes y-1} \otimes E_b \otimes I_k^{\otimes n-y},$$

for each $x, y \in [n], a, b \in [k]$. To see this fix $a \in [k], x \in [n]$ and note

$$E(a|x) = \sum_b I_k^{\otimes x-1} \otimes E_a \otimes I_k^{\otimes n-x} \otimes I_k^{\otimes y-1} \otimes E_b \otimes I_k^{\otimes n-y} = I_k^{\otimes x-1} \otimes E_a \otimes I_k^{\otimes n-x} \otimes I_k^{\otimes n}.$$

A similar calculation shows $F(b|y)$ is as claimed. It is clear that each marginal vector is well-defined and furthermore we see

$$\sum_{ab} Q(ab|xy) = \sum_{ab} I_k^{\otimes x-1} \otimes E_a \otimes I_k^{\otimes n-x} \otimes I_k^{\otimes y-1} \otimes E_b \otimes I_k^{\otimes n-y} = I_k^{\otimes 2n}.$$

Thus, $\mathcal{X}$ is a nonsignalling vector space. We now show that $\dim(\mathcal{X}) = (n(k-1)+1)^2$. First note

$$\begin{aligned}
E(a|x)F(b|y) &= (I_k^{\otimes x-1} \otimes E_a \otimes I_k^{\otimes n-x} \otimes I_k^{\otimes n})(I_k^{\otimes n} \otimes I_k^{\otimes y-1} \otimes E_b \otimes I_k^{\otimes n-y}) \\
&= ((I_k^{\otimes x-1} \otimes E_a \otimes I_k^{\otimes n-x}) \otimes I_k^{\otimes n})(I_k^{\otimes n} \otimes (I_k^{\otimes y-1} \otimes E_b \otimes I_k^{\otimes n-y})) \\
&= I_k^{\otimes x-1} \otimes E_a \otimes I_k^{\otimes n-x} \otimes I_k^{\otimes y-1} \otimes E_b \otimes I_k^{\otimes n-y} \\
&= Q(ab|xy).
\end{aligned}$$

Here we are taking the product in $D_k^{\otimes 2n}$. If we let $\mathcal{X}_A := \operatorname{span}\{E(a|x) : a \in [k], x \in [n]\}$ and $\mathcal{X}_B := \operatorname{span}\{F(b|y) : b \in [k], y \in [n]\}$ then $\mathcal{X} = \mathcal{X}_A\mathcal{X}_B$. We see $\mathcal{X}_A$ is spanned by the set

$$S = \{E(a|x) : a \in [k-1], x \in [n]\} \cup \{I_k^{\otimes 2n}\},$$

which is immediate from the fact that $E(k|x) = I_k^{\otimes 2n} - (\sum_{a=1}^{k-1} E(a|x))$. It follows S is linearly independent and hence a basis for $\mathcal{X}_A$. Thus $\dim(\mathcal{X}_A) = n(k-1)+1$. In a similar fashion it follows $\dim(\mathcal{X}_B) = n(k-1)+1$. We conclude

$$\dim(\mathcal{X}) = \dim(\mathcal{X}_A)\dim(\mathcal{X}_B) = (n(k-1)+1)^2.$$

Proposition 5.0.2. *For each $n, k \in \mathbb{N}$, there exists a nonsignalling vector space $\mathcal{X}_{ns}$ with $n(\mathcal{X}_{ns}) = n$, $k(\mathcal{X}_{ns}) = k$, and generators $\{Q_{ns}(a,b|x,y) : a,b \in [n], x,y \in [k]\}$ satisfying the following universal property: if $\mathcal{Y}$ is another nonsignalling vector space with $n(\mathcal{Y}_{ns}) = n$, $k(\mathcal{Y}_{ns}) = k$, and generators $\{Q(a,b|x,y) : a,b \in [k], x,y \in [n]\}$, then there exists a linear map $\varphi : \mathcal{X}_{ns} \to \mathcal{Y}$ satisfying $\varphi(Q_{ns}(a,b|x,y)) = Q(a,b|x,y)$. Moreover* $\dim(\mathcal{X}_{ns}) = (n(k-1)+1)^2$.

Proof. Let $\mathcal{Y}$ be an arbitrary nonsignalling vector space as in the above statement. Let $X := \ell_2^{n^2k^2}$ and denote the set of the canonical basis elements by $\{Q'(ab|xy) : a,b \in [k], x,y \in [n]\}$. Define a linear map $\varphi : X \to \mathcal{Y}$ by $\varphi(Q'(ab|xy)) = Q(ab|xy)$. Define the linear subspace J to be the span of the following vectors:

$$\begin{aligned} F(x,y|x',y') &:= \sum_{a,b} Q'(a,b|x,y) - \sum_{a,b} Q'(a,b|x',y'), \\ G(a|x,z,w) &:= \sum_{c} Q'(a,c|x,z) - \sum_{c} Q'(a,c|x,w), \quad \text{and} \\ H(b|y,z,w) &:= \sum_{d} Q'(d,b|z,y) - \sum_{d} Q'(d,b|w,y). \end{aligned}$$

We then consider the quotient vector space $\mathcal{X}_{\mathrm{ns}} := X/J$, where for each $a,b \in [k], x,y \in [n]$ we denote $Q_{\mathrm{ns}}(ab|xy) := Q'(ab|xy) + J$. It follows $J \subset \ker\varphi$ and consequently φ descends to a linear map $\widetilde{\varphi} : \mathcal{X}_{\mathrm{ns}} \to \mathcal{Y}$ such that $Q_{\mathrm{ns}}(ab|xy) \mapsto Q(ab|xy)$. If $e_{\mathrm{ns}} := \sum_{ab} Q_{\mathrm{ns}}(ab|xy)$ then it is readily verified that $\mathcal{X}_{\mathrm{ns}}$ is a nonsignalling vector space and satisfies our initial claim.

It remains to check the dimension of $\mathcal{X}_{\mathrm{ns}}$. Given any $a,b \in [k]$ and $x,y \in [n]$ let $E_{\mathrm{ns}}(a|x)$ and $F_{\mathrm{ns}}(b|y)$ denote the respective marginal vectors in $\mathcal{X}_{\mathrm{ns}}$. Consider the set

$$S := \{e_{\mathrm{ns}}, Q_{\mathrm{ns}}(a,b|x,y), E_{\mathrm{ns}}(a|x), F_{\mathrm{ns}}(b|y) : a,b \in [k-1], x,y \in [n]\}.$$

We see $\dim S = 1 + n^2(k-1)^2 + 2n(k-1) = (n(k-1)+1)^2$. We claim that S is a basis for $\mathcal{X}_{\text{ns}}$. In order to show that S is a spanning set for $\mathcal{X}_{\text{ns}}$ we show that S contains the vectors $Q_{\text{ns}}(k,b|x,y), Q_{\text{ns}}(a,k|x,y)$, and $Q_{\text{ns}}(k,k|x,y)$ for each $a,b \in [k-1]$ and $x,y \in [n]$. Fix $b \in [k]$ and $x,y \in [n]$. Then

$$F_{\text{ns}}(b|y) - \sum_{a=1}^{k-1} Q_{\text{ns}}(a,b|x,y) = \sum_{a=1}^{k} Q_{\text{ns}}(a,b|x,y) - \sum_{a=1}^{k-1} Q_{\text{ns}}(a,b|x,y) = Q_{\text{ns}}(k,b|x,y).$$

Thus $Q_{\text{ns}}(k,b|x,y) \in \operatorname{span} S$. A similar observation shows that $Q_{\text{ns}}(a,k|x,y) \in \operatorname{span} S$ for each $a \in [k]$ and $x,y \in [n]$. Now fix $x,y \in [n]$. Then

$$e_{\text{ns}} - \sum_{a,b=1}^{k-1} Q_{\text{ns}}(a,b|x,y) - \sum_{b=1}^{k-1} Q_{\text{ns}}(k,b|x,y) - \sum_{a=1}^{k-1} Q_{\text{ns}}(a,k|x,y) = Q_{\text{ns}}(k,k|x,y).$$

Thus $Q_{\text{ns}}(k,k|x,y) \in \operatorname{span} B$. We conclude that $\dim(\mathcal{X}_{\text{ns}}) \leq |S| = (n(k-1)+1)^2$. Suppose we consider the nonsignalling vector space from Example 5.0.3, which we denote by $\mathcal{X}$. By the universal property of $\mathcal{X}_{\text{ns}}$ there exists a linear surjection $\varphi : \mathcal{X}_{\text{ns}} \to \mathcal{X}$. In particular, $\dim \mathcal{X}_{\text{ns}} \geq \dim \mathcal{X} = (n(k-1)+1)^2$. This proves $\dim \mathcal{X}_{\text{ns}} = |S|$ and consequently S is a basis for $\mathcal{X}_{\text{ns}}$. $\square$

We thus know that $\mathcal{X}_{\text{ns}}$ is isomorphic to the nonsignalling vector space from Example 5.0.3.

Definition 5.0.3. *Fix $n,k \in \mathbb{N}$ and let $\mathcal{X}_{ns}$ denote the universal nonsignalling vector space with $n(\mathcal{X}_{ns}) = n$ and $k(\mathcal{X}_{ns}) = k$. Define the following sets:*

$$(\mathcal{X}_{ns})_h := \{\sum t(ab|xy) Q_{ns}(ab|xy) : t(ab|xy) \in \mathbb{R},\ \textit{for each } a,b \in [k], x,y \in [n]\},$$
$$D_{ns} := \{\sum t(ab|xy) Q_{ns}(ab|xy) : t(ab|xy) \in \mathbb{R}^+,\ \textit{for each } a,b \in [k], x,y \in [n]\}.$$

Given $z = \sum s(ab|xy) Q_{ns}(ab|xy)$ then define the map $* : \mathcal{X}_{ns} \to \mathcal{X}_{ns}$ by

$$z^* := \sum \overline{s(ab|xy)} Q_{ns}(ab|xy).$$

Proposition 5.0.3. *Fix $n, k \in \mathbb{N}$ and let $\mathcal{X}_{ns}$ denote the universal nonsignalling vector space with $n(\mathcal{X}_{ns}) = n$ and $k(\mathcal{X}_{ns}) = k$. Let $(\mathcal{X}_{ns})_h$ and D_{ns} be as in Definition 5.0.3. Then the triple $(\mathcal{X}_{ns}, D_{ns}, e_{ns})$ is an AOU space.*

Proof. First note that the map $* : \mathcal{X}_{\text{ns}} \to \mathcal{X}_{\text{ns}}$ is a well-defined involution being the involution on the nonsignalling vector space from Example 5.0.3. Furthermore, $z \in (\mathcal{X}_{\text{ns}})_h$ if and only if $z = z^*$. We also see $D_{\text{ns}} \subset (\mathcal{X}_{\text{ns}})_h$ and in particular, $D_{\text{ns}} \cap -D_{\text{ns}} = \{0\}$. It remains to show that e_{ns} is an interior point of D_{ns}. The set of extreme points of D_{ns} is exactly $\{tQ_{\text{ns}}(ab|xy) : t \in [0, \infty)\}$. We then see that e_{ns} is in the interior of D_{ns} since we may write e_{ns} as a convex combination of such extreme points, e.g., $e_{\text{ns}} = \frac{1}{n^2k^2} \sum_{abxy} k^2 Q_{\text{ns}}(ab|xy)$. This finishes the proof. $\square$

Let $\mathcal{X}_{\text{ns}}$ be the universal nonsignalling vector space with n inputs and k outputs and consider the AOU space $(\mathcal{X}_{\text{ns}}, D_{\text{ns}}, e_{\text{ns}})$. We consider the maximal operator system structure $\mathcal{D}_{\text{ns}}$ on $\mathcal{X}_{\text{ns}}$ which implies $(\mathcal{D}_{\text{ns}})_1 = D_{\text{ns}}$. By Proposition 5.0.3 it then follows that $(\mathcal{X}_{\text{ns}}, \mathcal{D}_{\text{ns}}, e_{\text{ns}})$ is an operator system.

Proposition 5.0.4. *Fix $n, k \in \mathbb{N}$ and let $\mathcal{X}_{ns}$ denote the universal nonsignalling vector space with $n(\mathcal{X}_{ns}) = n$ and $k(\mathcal{X}_{ns}) = k$. Then the operator system $(\mathcal{X}_{ns}, \mathcal{D}_{ns}, e_{ns})$ satisfies the following property: given any nonsignalling operator system $\mathcal{Y}$ such that $n(\mathcal{Y}) = n$ and $k(\mathcal{Y}) = k$, then if $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$ is the set of generators of $\mathcal{Y}$, it follows, the map $\pi : \mathcal{X}_{ns} \to \mathcal{Y}, Q_{ns}(ab|xy) \mapsto Q(ab|xy)$ is unital completely positive.*

Proof. Let $\mathcal{C}$ denote the proper matrix ordering on $\mathcal{Y}$. The map π is well-defined and linear by Proposition 5.0.2. Since $\mathcal{Y}$ is a nonsignalling operator system then π is unital. If $\sum s(ab|xy)Q_{\text{ns}}(ab|xy) \in D_{\text{ns}}$ then $\pi(\sum s(ab|xy)Q_{\text{ns}}(ab|xy)) = \sum s(ab|xy)Q(ab|xy) \in \mathcal{C}_1$ since $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\} \subset \mathcal{C}_1$ and each $s(ab|xy) \in \mathbb{R}^+$. In particular, we see the map $\pi : (\mathcal{X}_{\text{ns}}, D_{\text{ns}}, e_{ns}) \to \mathcal{Y}$ is unital positive. Since $\mathcal{D}_{\text{ns}}$ denotes the maximal operator system structure then it follows $\pi : (\mathcal{X}_{\text{ns}}, \mathcal{D}_{\text{ns}}, e_{\text{ns}}) \to \mathcal{Y}$ is unital completely positive. $\square$

Theorem 5.0.4. *Fix $n, k \in \mathbb{N}$ and let $\mathcal{X}_{ns}$ denote the universal nonsignalling vector space with $n(\mathcal{X}_{ns}) = n$ and $k(\mathcal{X}_{ns}) = k$. Let $p \in C(n, k)$. Then $p \in C_{ns}(n, k)$ if and only if there* *exists a unital positive map* $\varphi : (\mathcal{X}_{ns}, D_{ns}, e_{ns}) \to \mathbb{C}$ *such that* $p(ab|xy) = \varphi(Q_{ns}(ab|xy))$ *for all* $a, b \in [k]$ *and* $x, y \in [n]$.

Proof. First suppose $p \in C_{\mathrm{ns}}(n,k)$. Then there exists a nonsignalling operator system $(\mathcal{Y}, \mathcal{C}, e)$ with generators $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$, and a state $\varphi : \mathcal{Y} \to \mathbb{C}$ such that $p(ab|xy) = \varphi(Q(ab|xy))$ for all $a, b \in [k]$ and $x, y \in [n]$. By Proposition 5.0.4 the map $\boldsymbol{\pi} : (\mathcal{X}_{\mathrm{ns}}, D_{\mathrm{ns}}, e_{\mathrm{ns}}) \to \mathcal{Y}, Q_{\mathrm{ns}}(ab|xy) \mapsto Q(ab|xy)$ is unital positive and consequently the composition $\varphi\boldsymbol{\pi} : (\mathcal{X}_{\mathrm{ns}}, D_{\mathrm{ns}}, e_{\mathrm{ns}}) \to \mathbb{C}$ is a state with $\varphi\boldsymbol{\pi}(Q_{\mathrm{ns}}(ab|xy)) = p(ab|xy)$.

Conversely, consider the triple $(\mathcal{X}_{\mathrm{ns}}, D_{\mathrm{ns}}, e_{\mathrm{ns}})$ and let $\varphi : (\mathcal{X}_{\mathrm{ns}}, D_{\mathrm{ns}}, e_{\mathrm{ns}}) \to \mathbb{C}$ be a state. It necessarily follows $\varphi : (\mathcal{X}_{\mathrm{ns}}, \mathcal{D}_{\mathrm{ns}}, e_{\mathrm{ns}}) \to \mathbb{C}$ is also a state and by Theorem 5.0.2 it follows if $p := \{p(ab|xy) : a, b \in [k], x, y \in [n]\}$, where $p(ab|xy) := \varphi(Q_{\mathrm{ns}}(ab|xy))$ then $p \in C_{\mathrm{ns}}(n,k)$. This finishes the proof. □

We now wish to prove a result analogous to Theorem 5.0.4 which we do by employing our results from Section 4.2.

Definition 5.0.4. *Fix* $n, k \in \mathbb{N}$ *and let* $\mathcal{X}_{ns}$ *denote the universal nonsignalling vector space with* $n(\mathcal{X}_{ns}) = n$ *and* $k(\mathcal{X}_{ns}) = k$*. Consider the operator system* $(\mathcal{X}_{ns}, \mathcal{D}_{ns}, e_{ns})$ *and let* $\{p_1, \dots, p_N\}$ *be some enumeration of the set of generators* $\{Q_{ns}(ab|xy) : a, b \in [k], x, y \in [n]\}, N = n^2k^2$*. We denote by* $\mathcal{D}_{qc} := \mathcal{D}_{ns}(p_1, \dots, p_n)^{\infty}$ *the inductive limit of the matrix orderings* $\{\boldsymbol{\pi}_L^{-1}(\mathcal{D}_{ns}(p_1, \dots, p_N)^L)\}_{L\in\mathbb{N}}$*, and let* $D_{qc} := \mathcal{D}_{ns}(p_1, \dots, p_N)_1^{\infty}$

It will be shown that the matrix ordering $\mathcal{D}_{\mathrm{qc}}$, and thus D_{qc}, is independent of the choice of enumeration of the generators $\{p_1, \dots, p_N\}$. In the next result let

$$P_{\mathrm{i}}^N(x) := I_{2^{\mathrm{i}-1}} \otimes (x \oplus x^{\perp}) \otimes J_{2^{N-\mathrm{i}}} \quad \text{and} \quad P_{\mathrm{i,j}}^{N,L}(x) := I_{2^{N(\mathrm{j}-1)}} \otimes P_{\mathrm{i}}^N(x) \otimes J_{2^{N(L-\mathrm{j})}}$$

for any positive contraction x, where $x^{\perp} := e - x$, $\mathrm{i} \in [N]$, and $\mathrm{j} \in [L]$.

Proposition 5.0.5. *Fix* $n, k \in \mathbb{N}$ *and let* H *be a Hilbert space. For each* $x, y \in [n]$ *let* $\{E_{xa}\}_{a\in[k]}$ *and* $\{F_{yb}\}_{b\in[k]}$ *be projection-valued measures such that* $E_{xa}F_{yb} = F_{yb}E_{xa}$ *for all* $a, b \in [k]$ *and* $x, y \in [n]$*. Let* $\mathcal{Y} := span\{E_{xa}F_{yb} : a, b \in [k], x, y \in [n]\}$*. Let* $\boldsymbol{\pi} : (\mathcal{X}_{ns}, \mathcal{D}_{qc}, e_{ns}) \to \mathcal{Y}$ *be defined by* $\boldsymbol{\pi}(Q_{ns}(ab|xy)) = E_{xa}F_{yb}$*. Then* $\boldsymbol{\pi}$ *unital completely positive.*

Proof. For notational convenience we will denote each generator $E_{xa}F_{yb}$ as $Q(ab|xy)$. Thus, $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$ will denote the set of generators of the operator system $\mathcal{Y}$. Note for each $x, y \in [n]$ we have

$$\sum_{ab} Q(ab|xy) = \sum_{ab} E_{xa}F_{yb} = \mathrm{Id}_H,$$

which follows by the assumption that $\{E_{xa}\}_{a\in[k]}$ and $\{F_{yb}\}_{b\in[k]}$ are projection-valued measures. Furthermore if $x \in [n]$ and $a \in [k]$ are fixed we see

$$E(a|x) = \sum_b Q(ab|xy) = E_{xa}\sum_b F_{yb} = E_{xa}\,\mathrm{Id}_H = E_{xa}.$$

A similar observation holds for $F(b|y)$ for each $y \in [n]$ and $b \in [k]$. Thus, the marginal operators are well-defined. Finally, since for each $a, b \in [k]$ and $x, y \in [n]$ we have $E_{xa}F_{yb} = F_{yb}E_{xa}$ it follows that each positive operator $Q(ab|xy)$ is an abstract projection. This proves that $\mathcal{Y}$ is a quantum commuting operator system. By Proposition 5.0.4 it follows the map $\pi : (\mathcal{X}_{\mathrm{ns}}, \mathcal{D}_{\mathrm{ns}}, e_{\mathrm{ns}}) \to \mathcal{Y}, Q_{\mathrm{ns}}(ab|xy) \mapsto Q(ab|xy)$, is unital completely positive. We thus claim $\pi : (\mathcal{X}_{\mathrm{ns}}, \mathcal{D}_{\mathrm{qc}}, e_{\mathrm{ns}}) \to \mathcal{Y}$ is completely positive.

Recall, $N := n^2k^2$. For each $L \in \mathbb{N}$ let $P^{NL}_{\mathrm{ij}}(p_{\mathrm{i}}) := P^{NL}_{\mathrm{ij}}$ and let $Q^{NL}_{\mathrm{ij}}(p_{\mathrm{i}}) := Q^{NL}_{\mathrm{ij}}$, where $\{p_1, \ldots, p_N\}$ is some enumeration of the positive contractions $\{Q_{\mathrm{ns}}(ab|xy) : a, b \in [k], x, y \in [n]\}$. In a similar fashion, if $\{\widehat{p}_1, \ldots, \widehat{p}_N\}$ is some enumeration of the generators $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\} \subset B(H)$ then we let $\widehat{P}^{NL}_{\mathrm{ij}}(\widehat{p}_{\mathrm{i}}) := \widehat{P}^{NL}_{\mathrm{ij}}$ and let $\widehat{Q}^{NL}_{\mathrm{ij}}(\widehat{p}_{\mathrm{i}}) := \widehat{Q}^{NL}_{\mathrm{ij}}$. Let $x \in (\mathcal{D}_{\mathrm{qc}})_n$. Thus for all $\epsilon > 0$ there exists $L \in \mathbb{N}$ such that $(x + \epsilon I_n \otimes e_{\mathrm{ns}}) \otimes J_{2^{NL}} \in \mathcal{D}_{\mathrm{ns}}(p_1, \ldots, p_N)^L_n$. Therefore given an arbitrary $N \times L$ matrix $(\epsilon_{\mathrm{ij}})_{\mathrm{i}\in[N],\mathrm{j}\in[L]}$ of strictly positive real numbers there exists another $N \times L$ matrix $(t_{\mathrm{ij}})_{\mathrm{i}\in[N],\mathrm{j}\in[L]}$ such that

$$(x + \epsilon I_n \otimes e_{\mathrm{ns}}) \otimes J_{2^{NL}} + \sum_{\mathrm{ij}} \epsilon_{\mathrm{ij}} I_n \otimes P^{NL}_{\mathrm{ij}} + \sum_{\mathrm{ij}} t_{\mathrm{ij}} I_n \otimes Q^{NL}_{\mathrm{ij}} \in (\mathcal{D}_{\mathrm{ns}})_{n2^{NL}}.$$

Since the map π is completely positive with respect to the matrix ordering $\mathcal{D}_{\text{ns}}$ we take the $n2^{NL\text{th}}$-amplification of the above expression which yields

$$\begin{aligned}
&\pi_{n2^{NL}}\left((x+\epsilon I_n\otimes e_{\text{ns}})\otimes J_{2^{NL}}+\sum_{\text{ij}}\epsilon_{\text{ij}}I_n\otimes P_{\text{ij}}^{NL}+\sum_{\text{ij}}t_{\text{ij}}I_n\otimes Q_{\text{ij}}^{NL}\right)\\
&=(\pi_n(x)+\epsilon I_n\otimes \text{Id}_H)\otimes J_{2^{NL}}+\sum_{\text{ij}}\epsilon_{\text{ij}}I_n\otimes \widehat{P}_{\text{ij}}^{NL}+\sum_{\text{ij}}t_{\text{ij}}I_n\otimes \widehat{Q}_{\text{ij}}^{NL}\in B(\ell_2^{n2^{NL}}(H))^+.
\end{aligned}$$

Since the generators $Q(ab|xy)$ in $B(H)$ are abstract projections in $\mathcal{Y}$ it follows by Proposition 3.2.1

$$(\pi_n(x)+\epsilon I_n\otimes I_H)\in B(\ell_2^n(H))^+.$$

Since the cone $B(\ell_2^n(H))^+$ is Archimedean closed with respect to Id_H it follows $\pi_n(x)\in B(\ell_2^n(H))^+$. This implies π is unital completely positive with respect to the operator system $(\mathcal{X}_{\text{ns}},\mathcal{D}_{\text{qc}},e_{\text{ns}})$ which finishes the proof. □

Corollary 5.0.5. *Fix $n,k\in\mathbb{N}$ and let $\mathcal{X}_{ns}$ denote the universal nonsignalling vector space with $n(\mathcal{X}_{ns})=n$ and $k(\mathcal{X}_{ns})=k$. Let $\{Q_{ns}(ab|xy):a,b\in[k],x,y\in[n]\}$ be the set of generators of $\mathcal{X}_{ns}$ and let $\{p_1,\dots,p_N\}$ be some enumeration of the generators where $N=n^2k^2$. Let $\mathcal{D}_{qc}:=\mathcal{D}_{ns}(p_1,\dots,p_N)^\infty$ and let $D_{qc}:=\mathcal{D}_{ns}(p_1,\dots,p_N)_1^\infty$. Then $D_{qc}\cap -D_{qc}=\{0\}$ and thus $(\mathcal{X}_{ns},D_{qc},e_{ns})$ is an AOU space and $(\mathcal{X}_{ns},\mathcal{D}_{qc},e_{ns})$ is a quantum commuting operator system. Furthermore, the matrix ordering $\mathcal{D}_{qc}$ is independent of the enumeration $\{p_1,\dots,p_N\}$ of the generators $\{Q_{ns}(ab|xy):a,b\in[k],x,y\in[n]\}$. Thus, if $\{p_1',\dots,p_N'\}$ is another such enumeration of the nonsignalling generators and $\mathcal{D}_{qc}'$ is the corresponding inductive limit $\mathcal{D}_{qc}':=\mathcal{D}_{ns}(p_1',\dots,p_N')^\infty$, then the identity map*

$$Id:(\mathcal{X}_{ns},\mathcal{D}_{qc}',e_{ns})\to(\mathcal{X}_{ns},\mathcal{D}_{qc},e_{ns})$$

is a (unital) complete order isomorphism.

Proof. Let $\mathcal{Y} \subset B(\ell_2^{k^{2n}})$ be the nonsignalling vector space from Example 5.0.3. We then recall

$$Q(ab|xy) := I_k^{\otimes x-1} \otimes E_a \otimes I_k^{\otimes n-x} \otimes I_k^{\otimes y-1} \otimes E_b \otimes I_k^{\otimes n-y}.$$

Furthermore we saw the marginal vectors were

$$E(a|x) = I_k^{\otimes x-1} \otimes E_a \otimes I_k^{\otimes n-x} \otimes I_k^{\otimes n} \quad \text{and} \quad F(b|y) = I_k^{\otimes n} \otimes I_k^{\otimes y-1} \otimes E_b \otimes I_k^{\otimes n-y}.$$

It was established that $\mathcal{Y}$ was a nonsignalling vector space and in fact isomorphic to $\mathcal{X}_{ns}$. We claim $\mathcal{Y}$ is a quantum commuting operator system with abstract projections $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$. Note the involution, matrix ordering, and Archimedean matrix order unit are all inherited from $B(\ell_2^{k^{2n}})$. Furthermore we see

$$\begin{aligned} Q(ab|xy)^2 &= (I_k^{\otimes x-1} \otimes E_a \otimes I_k^{\otimes n-x} \otimes I_k^{\otimes y-1} \otimes E_b \otimes I_k^{\otimes n-y})^2 \\ &= I_k^{\otimes x-1} \otimes E_a \otimes I_k^{\otimes n-x} \otimes I_k^{\otimes y-1} \otimes E_b \otimes I_k^{\otimes n-y} \\ &= Q(ab|xy). \end{aligned}$$

Each generator $Q(ab|xy)$ is hermitian and thus the set $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$ consists of projections in $B(\ell_2^{k^{2n}})$. By Proposition 5.0.5 we know the map $\boldsymbol{\pi} : (\mathcal{X}_{\mathrm{ns}}, \mathcal{D}_{\mathrm{qc}}, e_{\mathrm{ns}}) \to \mathcal{Y}, Q_{\mathrm{ns}}(ab|xy) \mapsto Q(ab|xy)$, is unital completely positive. This necessarily implies $(\mathcal{D}_{\mathrm{qc}})_1 \cap -(\mathcal{D}_{\mathrm{qc}})_1 \subset \ker \boldsymbol{\pi}$ since if $\pm x \in (\mathcal{D}_{\mathrm{qc}})_1$ then $\pm \boldsymbol{\pi}(x) \in \mathcal{Y}^+$ which implies $\boldsymbol{\pi}(x) = 0$. By assumption we know $\dim \mathcal{X}_{\mathrm{ns}} = \dim \mathcal{Y}$ and therefore $\boldsymbol{\pi}$ must be injective and consequently $(\mathcal{D}_{\mathrm{qc}})_1 \cap -(\mathcal{D}_{\mathrm{qc}})_1 = \{0\}$. By applying Lemma 2.0.1 we know $\mathcal{D}_{\mathrm{qc}}$ is a proper matrix ordering which implies the triple $(\mathcal{X}_{\mathrm{ns}}, \mathcal{D}_{\mathrm{qc}}, e_{\mathrm{ns}})$ is an operator system and consequently the triple $(\mathcal{X}_{\mathrm{ns}}, D_{\mathrm{qc}}, e_{\mathrm{ns}})$ is an AOU space. The claim that $(\mathcal{X}_{\mathrm{ns}}, \mathcal{D}_{\mathrm{qc}}, e_{\mathrm{ns}})$ is a quantum commuting operator system is a direct result of Proposition 4.2.3. Indeed since $(\mathcal{D}_{\mathrm{qc}})_1 = \mathcal{D}_{\mathrm{ns}}(p_1, \ldots, p_N)_1^\infty$ then by applying Proposition 4.2.3 it follows each $p_{\mathrm{i}}, \mathrm{i} \in [N]$, is an abstract projection.

To finish the proof we consider two enumerations, $\{p_1, \ldots, p_N\}$ and $\{p'_1, \ldots, p'_N\}$, of the set of generators $\{Q_{\mathrm{ns}}(ab|xy) : a, b \in [k], x, y \in [n]\} \subset \mathcal{X}_{\mathrm{ns}}$. Consider the two quantum commuting operator systems $(\mathcal{X}_{\text{ns}}, \mathcal{D}_{\text{qc}}, e_{\text{ns}})$ and $(\mathcal{X}_{\text{ns}}, \mathcal{D}'_{\text{qc}}, e_{\text{ns}})$, which correspond to the respective enumerations. By Proposition 5.0.5 it necessarily follows both

$$\text{Id} : (\mathcal{X}_{\text{ns}}, \mathcal{D}'_{\text{qc}}, e_{\text{ns}}) \to (\mathcal{X}_{\text{ns}}, \mathcal{D}_{\text{qc}}, e_{\text{ns}})$$

and

$$\text{Id} : (\mathcal{X}_{\text{ns}}, \mathcal{D}_{\text{qc}}, e_{\text{ns}}) \to (\mathcal{X}_{\text{ns}}, \mathcal{D}'_{\text{qc}}, e_{\text{ns}})$$

are completely positive. In particular, $\text{Id} : (\mathcal{X}_{\text{ns}}, \mathcal{D}'_{\text{qc}}, e_{\text{ns}}) \to (\mathcal{X}_{\text{ns}}, \mathcal{D}_{\text{qc}}, e_{\text{ns}})$ is a (unital) complete order isomorphism and consequently $\mathcal{D}_{\text{qc}} = \mathcal{D}'_{\text{qc}}$. This finishes the proof. $\square$

We are now able to characterize when a correlation is necessarily quantum commuting.

Theorem 5.0.6. *Fix $n, k \in \mathbb{N}$ and let $\mathcal{X}_{ns}$ denote the universal nonsignalling vector space with $n(\mathcal{X}_{ns}) = n$ and $k(\mathcal{X}_{ns}) = k$. Let $\{Q_{ns}(ab|xy) : a, b \in [k], x, y \in [n]\}$ be the set of generators of $\mathcal{X}_{ns}$ and let $\{p_1, \dots, p_N\}$ be some enumeration of the generators where $N = n^2k^2$. Then a correlation $p \in C(n,k)$ is quantum commuting if and only if there exists a unital positive map $\varphi : (\mathcal{X}_{ns}, D_{qc}, e_{ns}) \to \mathbb{C}$ such that $p(ab|xy) = \varphi(Q_{ns}(ab|xy))$ for each $a, b \in [k]$ and $x, y \in [n]$.*

Proof. Assume $p \in C_{\text{qc}}(n,k)$. Then by Theorem 5.0.2 there exists a quantum commuting operator system $(\mathcal{Y}, \mathcal{C}, e)$ with set of generators $\{Q(ab|xy) : a, b \in [k], x, y \in [n]\}$ and a unital positive map $\varphi : \mathcal{Y} \to \mathbb{C}$ such that $p(ab|xy) = \varphi(Q(ab|xy))$ for all $a, b \in [k], x, y \in [n]$. If we consider the universal nonsignalling vector space $\mathcal{X}_{\text{ns}}$ with the set of generators $\{Q_{\text{ns}}(ab|xy) : a, b \in [k], x, y \in [n]\}$ then by Corollary 5.0.5 we know $(\mathcal{X}_{\text{ns}}, \mathcal{D}_{\text{qc}}, e_{\text{ns}})$ is a quantum commuting operator system and consequently, $(\mathcal{X}_{\text{ns}}, D_{\text{qc}}, e_{\text{ns}})$ is an AOU space with the property that if $\boldsymbol{\pi} : (\mathcal{X}_{\text{ns}}, D_{\text{qc}}, e_{\text{ns}}) \to \mathcal{Y}$ denotes the map $Q_{\text{ns}}(ab|xy) \mapsto Q(ab|xy)$, then $\boldsymbol{\pi}$ is unital positive. Thus, if we consider $\varphi\boldsymbol{\pi} : (\mathcal{X}_{\text{ns}}, D_{\text{qc}}, e_{\text{ns}}) \to \mathbb{C}$ then $\varphi\boldsymbol{\pi}$ is a state on $(\mathcal{X}_{\text{ns}}, D_{\text{qc}}, e_{\text{ns}})$ and $\varphi\boldsymbol{\pi}(Q_{\text{ns}}(ab|xy)) = p(ab|xy)$ for each $a, b \in [k]$ and $x, y \in [n]$.

Conversely, consider the AOU space $(\mathcal{X}_{\text{ns}}, D_{\text{qc}}, e_{\text{ns}})$ and let $\varphi : (\mathcal{X}_{\text{ns}}, D_{\text{qc}}, e_{\text{ns}}) \to \mathbb{C}$ be a state. This implies $\varphi : (\mathcal{X}_{\text{ns}}, \mathcal{D}_{\text{qc}}, e_{\text{ns}}) \to \mathbb{C}$ is unital positive and it therefore follows

$\varphi : (\mathcal{X}_{\rm ns}, \mathcal{D}_{\rm qc}, e_{\rm ns}) \to \mathbb{C}$ completely positive. Consequently, if we consider the correlation $p \in C(n,k)$ defined by $p(ab|xy) := \varphi(Q_{\rm ns}(ab|xy))$ for all $a, b \in [k]$ and $x, y \in [n]$ then by Theorem 5.0.2 we must have $p \in C_{\rm qc}(n,k)$. □

Remark 5.0.7. We conclude with some remarks on generalizations of the results that we have presented in Section 5. In particular, such generalizations follow from the results thus presented. The notions of nonsignalling vector spaces and nonsignalling operator systems are readily generalized to the multipartite situation, where one considers correlations of the form $p(a_1a_2\dots a_n|x_1x_2\dots x_n)$. Such multipartite correlations describe the scenario where n spacially distinct parties each perform a measurement on their respective quantum system. One then considers quantum commuting correlations arising from n mutually commuting C*-algebras in a common Hilbert space. To describe such multipartite correlations using the methods presented thus far, we redefine nonsignalling operator systems to be generated by operators $\{Q(a_1\dots a_n|x_1\dots x_n)\}$ satisfying

$$\sum_{a_1\dots a_n} Q(a_1\dots a_n|x_1\dots x_n) = e$$

and such that the marginal operators

$$E_{\rm i}(a_{\rm i}|x_{\rm i}) = \sum_{a_{\rm j}, {\rm j}\neq {\rm i}} Q(a_1\dots a_n|x_1\dots x_n),$$

are well-defined. If one requires the generators to be abstract projections then this yields a quantum commuting operator system. The constructions of the universal nonsignalling and quantum commuting operator systems proceed in the same manner as the bipartite case. Our work also readily generalizes to the setting of matricial correlation sets, as described in [17]. In particular, it is readily seen that the matrix affine dual of the matricial nonsignalling and quantum commuting correlations are precisely the operator systems $(\mathcal{X}_{ns}, \mathcal{D}_{ns}, e_{ns})$ and $(\mathcal{X}_{ns}, \mathcal{D}_{qc}, e_{ns})$, respectively, using work in [23].

6. WEAK DUAL MATRIX ORDERED *-VECTOR SPACES AND RELATIVE ARCHIMEDEANIZATIONS

Consider a proper matrix ordered $*$-vector space $(\mathcal{X}, \mathcal{C})$. Given $x \in \mathcal{C}_1$ we define the following collection $\mathcal{C}^x$ where for each $n \in \mathbb{N}$

$$\mathcal{C}_n^x := \{y \in M_n(\mathcal{X}) : \forall \epsilon > 0, \quad \epsilon I_n \otimes x + y \in \mathcal{C}_n\}.$$

Thus, $\mathcal{C}^x$ is nothing but the Archimedean closure of the proper matrix ordering $\mathcal{C}$ with respect to $x \in \mathcal{C}_1$.

Proposition 6.0.1. *Given a proper matrix ordered $*$-vector space $(\mathcal{X}, \mathcal{C})$ then $\mathcal{C}^x$ is a matrix ordering on X.*

Proof. Let $y \in \mathcal{C}_n^x$. Then if $\lambda > 0$ and $\epsilon > 0$ is arbitrary it follows $\epsilon I_n \otimes x + \lambda y = \lambda(\frac{\epsilon}{\lambda} I_n \otimes x + y) \in \lambda \mathcal{C}_n \subset \mathcal{C}_n$. Thus, $\mathbb{R}^+ \mathcal{C}^x \subset \mathcal{C}^x$. If $y_1, y_2 \in \mathcal{C}_n^x$ and $\epsilon > 0$ is arbitrary it follows

$$\epsilon I_n \otimes x + (y_1 + y_2) = (\frac{\epsilon}{2} I_n \otimes x + y_1) + (\frac{\epsilon}{2} I_n \otimes x + y_2) \in \mathcal{C}_n + \mathcal{C}_n \subset \mathcal{C}_n.$$

Thus $\mathcal{C}^x + \mathcal{C}^x \subset \mathcal{C}^x$. We now check that $\mathcal{C}^x$ is compatible. Let $y \in \mathcal{C}_n^x$ and let $a \in M_{n,k}$. Since $a^* a \leq \|a\|^2 I_k$ it follows

$$\epsilon I_k \otimes x + a^* y a \geq \frac{\epsilon}{\|a\|^2} a^* a \otimes x + a^* y a = a^* (\frac{\epsilon}{\|a\|^2} I_n \otimes x + y) a \in a^* \mathcal{C}_n a \subset \mathcal{C}_k.$$

Thus $a^* \mathcal{C}_n^x a \subset \mathcal{C}_k^x$ and therefore $\mathcal{C}^x$ is a matrix ordering on $\mathcal{X}$. □

We recall some notions regarding duality for matrix ordered $*$-vector spaces. Given two vector spaces X, X_1 then we say that X and X_1 are *in duality* if there exists a bilinear map

$$\langle \cdot, \cdot \rangle : X \times X_1 \to \mathbb{C}$$

such that for $x \in X$ then $x = 0$ if and only $\langle x, x_1 \rangle = 0$ for all $x_1 \in X_1$, and similarly for $x_1 \in X_1$ then $x_1 = 0$ if and only if $\langle x, x_1 \rangle = 0$ for all $x \in X$. If two spaces are in duality then each induces a weak topology on the other. We will denote the weak topology on X as $w(X, X_1)$ and similarly the weak topology on X_1 as $w(X_1, X)$. Thus, when X and X_1 are in duality we will write $\langle X, X_1 \rangle$ and say it is a *dual pair* of vector spaces. Given two dual pairs $\langle X, X_1 \rangle$ and $\langle Y, Y_1 \rangle$ then we let $B_w(X, Y)$ denote the weak-to-weak continuous maps: i.e., the $w(X, X_1)$-to-$w(Y, Y_1)$ continuous maps. We will write $X' := B_w(X, \mathbb{C})$ and call it the *dual (or weak dual)* of X. Thus, if $\langle X, X_1 \rangle$ is a dual pair of vector spaces then we may identify X_1 with X' and X with X''. Furthermore, if $\langle X, X' \rangle$ is a dual pair of vector spaces then $\langle M_n(X), M_n(X') \rangle$ is a dual pair of vector spaces under the pairing

$$\langle \cdot, \cdot \rangle : M_n(X) \times M_n(X') \to \mathbb{C}, \quad \text{defined by} \quad \langle x, x' \rangle := \sum_{\mathrm{ij}} \langle x_{\mathrm{ij}}, x'_{\mathrm{ij}} \rangle. \tag{6.1}$$

We will also make use of the *matrix pairing* defined in the following way: if $\langle X, X' \rangle$ is a dual pair of vector spaces then $\langle M_m(X), M_n(X') \rangle$ is a dual pair with pairing

$$\langle\langle \cdot, \cdot \rangle\rangle : M_n(X) \times M_m(X') \to M_{mn}, \quad \text{defined by} \quad \langle\langle x, x' \rangle\rangle := [\langle x_{\mathrm{ij}}, x'_{kl} \rangle]. \tag{6.2}$$

The weak topology induced by Equation (6.1) coincides with that induced by Equation (6.2)

Let $(\mathcal{X}, \mathcal{C})$ be a proper matrix ordered $*$-vector space with vector space dual $\mathcal{X}'$. We say that $\mathcal{X}'$ is a $*$-vector dual of $\mathcal{X}$ if $\mathcal{X}'$ is a self-adjoint subspace of $\mathcal{X}^d := L(\mathcal{X}, \mathbb{C})$, the algebraic dual of $\mathcal{X}$. We may then define a *dual matrix ordering* on $\mathcal{X}'$ in the following way: let $n \in \mathbb{N}$ and define

$$\mathcal{C}'_n := \{x' \in M_n(\mathcal{X}') : \text{ the map } x' : \mathcal{X} \to M_n, \text{ is completely positive}\}$$

where $x' : \mathcal{X} \to M_n$ is defined as $x'(x) := \sum_{\mathrm{ij}} e_{\mathrm{i}} e_{\mathrm{j}}^* \otimes x'_{\mathrm{ij}}(x)$. Then the collection $\mathcal{C}'$ denotes a matrix ordering on $\mathcal{X}'$ and thus $(\mathcal{X}', \mathcal{C}')$ is a proper matrix ordered $*$-vector space and when $\mathcal{X}'$ is given this matrix ordering we will call $\mathcal{X}'$ the *matrix ordered dual* of $\mathcal{X}$. It follows by the Bipolar theorem that $\mathcal{X}$ is the matrix ordered dual of $\mathcal{X}'$ if and only if $\overline{\mathcal{C}}_n^w = \mathcal{C}_n$ for all $n \in \mathbb{N}$ where here we are taking the weak closure of $\mathcal{C}_n$.

Definition 6.0.1. *A matrix ordered $*$-vector space $(\mathcal{X}, \mathcal{C})$ will be called a* weak dual matrix ordered $*$-vector space *if there exists another matrix ordered $*$-vector space $(\mathcal{X}_1, \mathcal{D})$ such that $\langle \mathcal{X}, \mathcal{X}_1 \rangle$ is a dual pair of matrix ordered $*$-vector spaces. In particular, $\overline{\mathcal{C}}_n^{w(\mathcal{X},\mathcal{X}_1)} = \mathcal{C}_n$ for all $n \in \mathbb{N}$. If $\langle \mathcal{X}, \mathcal{X}_1 \rangle$ is a dual pair of matrix ordered $*$-vector spaces when $\mathcal{X}_1 = \mathcal{X}^d$ then we will say $\mathcal{X}$ has the* natural weak dual structure.

It necessarily follows that every operator system has the natural weak dual structure.

Whenever $(\mathcal{X}, \mathcal{C})$ is a weak dual matrix ordered $*$-vector space then we will let $\mathcal{X}'$ denote its weak dual matrix ordered $*$-vector space.

Lemma 6.0.1. *Let $(\mathcal{X}, \mathcal{C})$ be a proper weak dual matrix ordered $*$-vector space and let $x \in \mathcal{C}_1$. Then the matrix ordering $\mathcal{C}^x$ is proper.*

Proof. By Lemma 2.0.1 it suffices to show that $\mathcal{C}_1^x$ is proper. To this end, let $\pm y \in \mathcal{C}_1^x$. By Proposition 6.0.1 it follows $\pm ry \in \mathcal{C}_1^x$ for all $r > 0$. This implies for all $\epsilon > 0$ we have $\epsilon x \pm ry \in \mathcal{C}_1$ and consequently if $\epsilon = 1$ we obtain $\frac{1}{r}x \pm y \in \mathcal{C}_1$. We consider the dual pair $\langle \mathcal{X}, \mathcal{X}' \rangle$ of matrix ordered $*$-vector spaces. Let $x' \in \mathcal{C}_1'$ be arbitrary. It follows $\langle \frac{1}{r}x \pm y, x' \rangle \in \mathbb{R}^+$. Since $\mathbb{R}^+$ is closed it follows $\pm\langle y, x' \rangle = \lim_{r\to\infty} \frac{1}{r}\langle x, x' \rangle \pm \langle y, x' \rangle \in \mathbb{R}^+$. By [4, Lemma 4.3] the map

$$\Lambda : \mathcal{X} \to B_w(\mathcal{X}', \mathbb{C}), \quad \text{defined by} \quad \Lambda(y)(x') := \langle y, x' \rangle,$$

is a (complete) order isomorphism. Thus, $\Lambda(\pm y) \in B_w(\mathcal{X}', \mathbb{C})^+$ since $x' \in \mathcal{C}_1'$ was arbitrary and $\langle \pm y, x' \rangle = \Lambda(\pm y)x'$. Since Λ is an order isomorphism it follows $\pm y \in \mathcal{C}_1$ which implies $y = 0$ since $\mathcal{C}$ is proper. This completes the proof. $\square$

Remark 6.0.2. We point out in the proof of Lemma 6.0.1 above, in showing $\mathcal{C}^x$ is proper, one could directly appeal to the assumption that $\mathcal{X}$ is a weak dual space and thus $\overline{\mathcal{C}}^{w(\mathcal{X},\mathcal{X}')} = \mathcal{C}$. In particular, the cone $\mathcal{C}_1$ is $w(\mathcal{X}, \mathcal{X}')$-closed. We introduce the map $\Lambda : \mathcal{X} \to B_w(\mathcal{X}', \mathbb{C})$ since it plays a vital role throughout the section.

Let $(\mathcal{X}, \mathcal{C})$ be a weak dual matrix ordered $*$-vector space with $x \in \mathcal{C}_1$. Consider $\mathcal{X}^x := \mathcal{X}_h^x \oplus \imath \mathcal{X}_h^x$ where

$$\mathcal{X}_h^x := \{y \in \mathcal{X}_h : rx \pm y \in \mathcal{C}_1 \text{ for some } r > 0\}. \tag{6.3}$$

As in Proposition 6.0.1 we define the collection $\mathcal{C}^x$ where for each $n \in \mathbb{N}$

$$\mathcal{C}_n^x := \{y \in M_n(\mathcal{X}^x)_h : \forall \epsilon > 0,\ \epsilon I_n \otimes x + y \in \mathcal{C}_n\}. \tag{6.4}$$

Lemma 6.0.3. *Given a proper matrix ordered $*$-vector space $(\mathcal{X}, \mathcal{C})$ then $\mathcal{X}_h^x$ is a real vector space.*

Proof. Suppose $y \in \mathcal{X}_h^x$ and let $\lambda \in \mathbb{R}$. Let $r > 0$ such that $rx \pm y \in \mathcal{C}_1$.

- If $\lambda > 0$ then $\lambda rx \pm \lambda y = \lambda(rx \pm y) \in \lambda \mathcal{C}_1 \subset \mathcal{C}_1$. Thus λr is the desired positive real number.
- If $\lambda < 0$ then $-\lambda rx \pm \lambda y = -\lambda(rx \pm -y) = -\lambda(rx \pm y) \in -\lambda \mathcal{C}_1 \subset \mathcal{C}_1$. Thus, $-\lambda r$ is the desired positive real number.

Thus, $\mathbb{R}\mathcal{X}_h^x \subset \mathcal{X}_h^x$. Let $y_1, y_2 \in \mathcal{X}_h^x$ and let $r_i > 0$ such that $r_i x \pm y_i \in \mathcal{C}_1$. If $r := \max\{r_1, r_2\}$ then we see

$$2rx \pm (y_1 + y_2) = (rx \pm y_1) + (rx \pm y_2) \in \mathcal{C}_1 + \mathcal{C}_1 \subset \mathcal{C}_1.$$

Clearly $0 \in \mathcal{X}_h^x$ since $rx \in \mathcal{C}_1$ for all $r > 0$. This finishes the proof. □

Definition 6.0.2. *Let $(\mathcal{X}, \mathcal{C})$ be a proper matrix ordered $*$-vector space and let $x \in \mathcal{C}_1$. Then we call $\mathcal{X}_h^x$ the* majorization subspace with respect to x. *If $y \in \mathcal{X}_h^x$ and $r > 0$ such that $rx \pm y \in \mathcal{C}_1$ then we call r a* majorization constant of y.

Proposition 6.0.2. *Given a weak dual matrix ordered $*$-vector space $(\mathcal{X}, \mathcal{C})$ and $x \in \mathcal{C}_1$ then $\mathcal{C}^x$ is a proper matrix ordering on $\mathcal{X}^x$ and $(\mathcal{X}^x, \mathcal{C}^x, x)$ is an operator system.*

Proof. We begin by showing that $\mathcal{C}^x$ is a proper matrix ordering. Let $y \in \mathcal{C}_n^x$ and let $\lambda \in \mathbb{R}^+$. Then if $\epsilon > 0$ it follows $\epsilon I_n \otimes x + \lambda x = \lambda(\frac{\epsilon}{\lambda} I_n \otimes x + y) \in \lambda \mathcal{C}_1 \subset \mathcal{C}_1$. Thus $\mathbb{R}^+ \mathcal{C}^x \subset \mathcal{C}^x$. Let $y_i \in \mathcal{C}_n^x$. Then for $\epsilon > 0$ it follows $\epsilon I_n \otimes x + (y_1 + y_2) = (\frac{\epsilon}{2} I_n \otimes x + y_1) + (\frac{\epsilon}{2} I_n \otimes x + y_2) \in \mathcal{C}_n + \mathcal{C}_n \subset \mathcal{C}_n$. Thus $y_1 + y_2 \in \mathcal{C}_n^x$. If $y \in \mathcal{C}_n^x$ and $a \in M_{n,k}$ then since $a^* a \leq \|a\|^2 I_k$ and $a^* a \otimes x = a^*(I_n \otimes x)a$ we have for $\epsilon > 0$ that

$$\epsilon I_k \otimes x + a^* y a \geq_{\mathcal{C}} \frac{\epsilon}{\|a\|^2} a^* a \otimes x + a^* y a = a^* (\frac{\epsilon}{\|a\|^2} I_n \otimes x + y) a \in a^* \mathcal{C}_n a \subset \mathcal{C}_k.$$

Thus $a^* y a \in \mathcal{C}_k^x$ which proves that $\mathcal{C}^x$ is a matrix ordering. The ordering $\mathcal{C}^x$ will be proper if we can show that $\mathcal{C}_1^x$ is proper by once again invoking 2.0.1. If $\pm y \in \mathcal{C}_1^x$ then $\pm y \in \mathcal{X}_h^x$ by Lemma 6.0.3. At this point the argument proceeds just as in Lemma 6.0.1. Thus $\mathcal{C}^x$ is proper and consequently $(\mathcal{X}^x, \mathcal{C}^x)$ is a proper matrix ordered $*$-vector space.

It remains to show x is an Archimedean matrix order unit. If $y \in \mathcal{X}_h^x$ then by definition of the majorization subspace relative to x it follows there exists $r > 0$ such that $rx \pm y \in \mathcal{C}_1$. Note that if $\epsilon > 0$ then since $\epsilon x \in \mathcal{C}_1$ then $\epsilon x + (rx \pm y) = (\epsilon + r)x \pm y \in \mathcal{C}_1$. This holds for all $\epsilon > 0$ and therefore $rx \pm y \in \mathcal{C}_1^x$ which proves x is an order unit and therefore x is a matrix order unit by Proposition 2.0.1. We remark it is immediate that $rx \pm y \in \mathcal{X}_h^x$ since $\pm y \in \mathcal{X}_h^x$ and $rx \in \mathcal{X}_h^x$. Given $y \in M_n(\mathcal{X}^x)$ suppose that for all $\epsilon > 0$ it follows $\epsilon I_n \otimes x + y \in \mathcal{C}_n^x$. This implies that for all $\widetilde{\epsilon} > 0$ we have $(\widetilde{\epsilon} + \epsilon) I_n \otimes x + y \in \mathcal{C}_n$. Let $\widetilde{\epsilon} = \epsilon$ which yields $2\epsilon I_n \otimes x + y \in \mathcal{C}_n$ and thus $\epsilon I_n \otimes x + \frac{1}{2} y \in \mathcal{C}_n$. Since this holds for all $\epsilon > 0$ it follows $\frac{1}{2} y \in \mathcal{C}_n^x$ and consequently $y \in \mathcal{C}_n^x$. Therefore x is an Archimedean matrix order unit. This finishes the proof. □

Thus, as a result of Proposition 6.0.2 if $(\mathcal{X}, \mathcal{C})$ is a proper weak dual matrix ordered $*$-vector space it follows that every element of $\mathcal{C}_1$ induces an operator system, and consequently, an operator space. If $x \in \mathcal{C}_1$ and $\alpha^x : \mathcal{X}^x \to [0, \infty)$ denotes the operator space norm induced by x then we may consider $(\mathcal{X}^x, \mathcal{C}^x, \alpha^x, x)$.

Definition 6.0.3. *Let $(\mathcal{X}, \mathcal{C})$ be a proper weak dual matrix ordered $*$-vector space and let $x \in \mathcal{C}_1$. Then the triple $(\mathcal{X}^x, \mathcal{C}^x, x)$ will be called the* operator system relative to x. *The pair*

$(\mathcal{X}^x, \alpha^x)$ *will be called the* operator space relative to x. *The collection* $\{(\mathcal{X}^x, \mathcal{C}^x, \alpha^x, x)\}_{x \in \mathcal{C}_1}$ *will be called the* canonical collection of operator systems induced by the pair $(\mathcal{X}, \mathcal{C})$.

Remark 6.0.4. Let $(\mathcal{X}, \mathcal{C})$ be a proper weak dual matrix ordered $*$-vector space. Then $\mathcal{C}_1$ *generates* $\mathcal{X}_h$ if $\mathcal{X}_h = \mathcal{C}_1 - \mathcal{C}_1$. In particular, $\mathcal{C}_1$ generates $\mathcal{X}_h$ if and only if given any $y \in \mathcal{X}_h$ there exists $x \in \mathcal{C}_1$ such that $x \geq_{\mathcal{C}} y$. If $\mathcal{C}_1$ generates $\mathcal{X}_h$ then if $y \in \mathcal{X}_h$ we write $y = x_1 - x_2$ for $x_{\mathrm{i}} \in \mathcal{C}_1$. Consequently, $x_1 - y \in \mathcal{C}_1$. Conversely, if there exists $x \in \mathcal{C}_1$ such that $x - y \in \mathcal{C}_1$ then $y = x - (x - y)$.

Definition 6.0.4. *Given a proper weak dual matrix ordered $*$-vector space $(\mathcal{X}, \mathcal{C})$ then if $\mathcal{X}_h = \mathcal{C}_1 - \mathcal{C}_1$ we will say $\mathcal{C}_1$ has* non-void radial kernel.

Proposition 6.0.3. *Let $(\mathcal{X}, \mathcal{C})$ be a proper weak dual matrix ordered $*$-vector space such that $\mathcal{C}_1$ has non-void radial kernel. Then for all $y \in \mathcal{X}_h$ there exists $x \in \mathcal{C}_1$ such that $y \in \mathcal{X}_h^x$. In particular, given $y_1, y_2 \in \mathcal{X}_h$ then there exists $x \in \mathcal{C}_1$ such that $y_1, y_2 \in \mathcal{X}_h^x$.*

Proof. Let $y \in \mathcal{X}_h$ which we write as $y = y_1 - y_2$ for $y_{\mathrm{i}} \in \mathcal{C}_1$. It follows that there exists $x_{\mathrm{i}} \in \mathcal{C}_1$ such that $y_{\mathrm{i}} \in \mathcal{X}_h^{x_{\mathrm{i}}}$. Note here that we may choose $x_{\mathrm{i}} := y_{\mathrm{i}}$, since $y_{\mathrm{i}} \in \mathcal{C}_1^{y_{\mathrm{i}}}$. Therefore let $r_{\mathrm{i}} > 0$ be the respective majorization constants for y_{i}. Furthermore let $r = \max\{r_1, r_2\}$ and let $x := x_1 + x_2$. We claim $y \in \mathcal{X}_h^x$. Notice

$$2rx \pm y = 2rx \pm (y_1 - y_2) = (rx \pm y_1) + (rx \pm -y_2).$$

Since $r \geq r_1$ it follows $(rx \pm y_1) \geq_{\mathcal{C}} (r_1 x \pm y_1) \in \mathcal{C}_1$. Similarly, since $r \geq r_2$ we have $rx + y_2, rx - y_2 \in \mathcal{C}_1$. Thus, $2rx \pm y = 2rx \pm (y_1 - y_2) = (rx \pm y_1) + (rx \pm -y_2) \in \mathcal{C}_1 + \mathcal{C}_1 \subset \mathcal{C}_1$. This proves that $y \in \mathcal{X}_h^x$.

Consider now $y_1, y_2 \in \mathcal{X}_h$. By the first part of the proof we have $y_{\mathrm{i}} \in \mathcal{X}_h^{x_{\mathrm{i}}}$ for some $x_{\mathrm{i}} \in \mathcal{C}_1$. Once again we let $r := \max\{r_1, r_2\}$, where each $r_{\mathrm{i}} > 0$ is the respective majorization constant of y_{i}, and set $x := x_1 + x_2$. Then $x \in \mathcal{C}_1$ and we see

$$rx \pm y_1 = (rx_1 \pm y_1) + rx_2 \in \mathcal{C}_1$$

$$rx \pm y_2 = rx_1 + (rx_2 \pm y_2) \in \mathcal{C}_1.$$

This proves $y_1, y_2 \in \mathcal{X}_h^x$ which finishes the proof. $\square$

Definition 6.0.5. *Let* $(\mathcal{X}, \mathcal{C})$ *be a proper weak dual matrix ordered* $*$*-vector space and let* $y \in \mathcal{X}_h$*. Then the* sufficiency number *relative to* y *is the cardinality of the set* J *such that for all* $\mathrm{j} \in J$ *one has* $y \in \mathcal{C}_1^{x_{\mathrm{j}}}$*. Note that* J *may be empty, and in this case we say* $y \in \mathcal{X}_h$ *has a sufficiency number of 0.*

Remark 6.0.5. Consider a dual pair $\langle \mathcal{X}, \mathcal{X}' \rangle$ of matrix ordered $*$-vector spaces. By identifying $\mathcal{X} \simeq \Lambda(\mathcal{X})$ we may realize $\mathcal{X}$ as maps on the weak dual $\mathcal{X}'$. Thus, given $x \in \mathcal{X}$ and $x' \in \mathcal{X}'$, if we are identifying x with $\Lambda(x)$ then we will write the action as $\langle x', x \rangle$. In particular, we will always write the element being acted on in the left-hand side of the pairing $\langle \cdot, \cdot \rangle$. Since $\mathcal{X} = (\mathcal{X}')'$ this should not introduce any ambiguity.

Lemma 6.0.6. *Let* $(\mathcal{X}, \mathcal{C})$ *be a proper weak dual matrix ordered* $*$*-vector space such that* $\mathcal{C}_1$ *has non-void radial kernel and let* $y \in \mathcal{X}_h$*. If the sufficiency number of* y *is at least 1 then* $y \in \mathcal{C}_1$*.*

Proof. Let J be the set such that $|J|$ is the sufficiency number of y. By assumption $|J| \geq 1$. Thus, $y \in \bigcap_{\mathrm{j} \in J} \mathcal{C}_1^{x_{\mathrm{j}}}$, $x_{\mathrm{j}} \in \mathcal{C}_1$, and therefore for all $\epsilon > 0$ we have $\epsilon x_{\mathrm{j}} + y \in \mathcal{C}_1$. Let $1 \leq N < \infty$. We claim $y \in \mathcal{C}_1$ which occurs if and only if $\Lambda(y) \in B_w(\mathcal{X}', \mathbb{C})^+$. To this end let $x' \in \mathcal{C}_1'$ be arbitrary and let $\epsilon > 0$. Choose $\epsilon_{\mathrm{j}} > 0$ such that $\sum_{\mathrm{j}=1}^N \epsilon_{\mathrm{j}} \langle x', x_{\mathrm{j}} \rangle \leq \epsilon 1$. By assumption we have $\sum_{\mathrm{j}=1}^N \epsilon_{\mathrm{j}} x_{\mathrm{j}} + y \in \mathcal{C}_1$ and therefore

$$0 \leq \langle x', \sum_{\mathrm{j}=1}^N \epsilon_{\mathrm{j}} x_{\mathrm{j}} + y \rangle = \langle x', \sum_{\mathrm{j}=1}^N \epsilon_{\mathrm{j}} x_{\mathrm{j}} \rangle + \langle x', y \rangle \leq \epsilon 1 + \langle x', y \rangle.$$

Since $\epsilon > 0$ is arbitrary we have $\langle x', y \rangle \geq 0$. Therefore since $x' \in \mathcal{C}_1'$ is also arbitrary we have $\Lambda(y) \in B_w(\mathcal{X}', \mathbb{C})^+$ and this proves $y \in \mathcal{C}_1$. $\square$

Conjecture 6.0.7. Given a proper weak dual matrix ordered $*$-vector space $(\mathcal{X}, \mathcal{C})$ such that $\mathcal{C}_1$ has non-void radial kernel then there exists a complete order embedding from $\mathcal{X}$ to $\bigoplus_{x \in \mathcal{C}_1} \mathcal{X}^x$.

Remark 6.0.8. Let $(\mathcal{X}, \mathcal{C})$ be a proper weak dual matrix ordered $*$-vector space and let

$$\{(\mathcal{X}^x, \mathcal{C}^x, \alpha^x, x)\}_{x \in \mathcal{C}_1}$$

denote the canonical collection of operator systems induced by $(\mathcal{X}, \mathcal{C})$. For each $x \in \mathcal{C}_1$ let H^x be a Hilbert space such that $\Phi^x : \mathcal{X}^x \to B(H^x)$ is a unital complete order isomorphism. In particular identify

$$\bigoplus_{x \in \mathcal{C}_1} \mathcal{X}^x \simeq \bigoplus_{x \in \mathcal{C}_1} \Phi^x(\mathcal{X}^x) \subset B(H),$$

where $H := \ell_2(\{H^x : x \in \mathcal{C}_1\})$ is the Hilbertian direct sum. An affirmative answer to Conjecture 6.0.7 will imply that such a pair $(\mathcal{X}, \mathcal{C})$ may be realized concretely inside $B(H)$ for some H. Though recent progress has been made, Conjecture 6.0.7 is still not proven.

7. CLOSING REMARKS

7.1 Tsirelson's Problem

We conclude this manuscript with some final remarks. The motivation for the development of the methods in Section 3 and Section 4 was to understand the structure of correlation sets better. In particular, we saw in Section 5 that our methods provided new characterizations of the sets of nonsignalling, and more importantly, the quantum commuting correlations. Though, this is only half of the story. We recall that Tsirelson asked if for all $n, k \in \mathbb{N}$ it follows $C_{\text{qa}}(n, k) = C_{\text{qc}}(n, k)$. As already mentioned, according to the recent preprint [8], the equality does not hold, but we still do not have an good grasp of these objects. Using a hybrid from the theory of Archimedean order unit spaces and operator system theory, we are working to complete the puzzle by providing a characterization, similar to those of Section 5, for quantum correlations. Thus, with such a characterization we will have a much better understanding of quantum correlations, and consequently a better understanding of Tsirelson's problem.

7.2 Local Reflexivity

As explained in the Introduction, local reflexivity becomes an extremely delicate issue when leaving the realm of Banach spaces. Let $\lambda > 0$. An operator space $\mathcal{X}$ is called λ-*locally reflexive* if for every finite-dimensional operator space V and complete contraction $u : V \to \mathcal{X}^{**}$ then there exists a net $\{u_{\text{i}}\}_{\text{i}\in I}, u_{\text{i}} : V \to \mathcal{X}$, of maps such that for each $\text{i} \in I$, $\left\|u_{\text{i}}\right\|_{\text{cb}} \leq \lambda$ and $u_{\text{i}} \to u$ in the point $w(\mathcal{X}^{**}, \mathcal{X}^*)$-topology. For the sake of simplicity we will assume $\lambda = 1$. Translating such an approximation property to the realm of operator systems does not present a problem, but suppose we consider an equivalent formulation of local reflexivity which may also be taken as a definition: an operator space $\mathcal{X}$ is 1-locally reflexive if for every finite-dimensional operator space V it follows

$$V^* \otimes_\wedge \mathcal{X}^* \simeq (V \otimes_{\min} \mathcal{X})^*. \tag{7.1}$$

In Equation 7.1 the identification is a *completely isometric linear isomorphism (complete isometry)*, and we have let $\otimes_\wedge$ denote the operator space *projective tensor product*, and $\otimes_{\min}$ denote the *minimal operator space tensor product.* In proving such an equivalence one takes advantage of the well-behaved duality theory for operator spaces. In particular, since

$$(V^* \otimes_\wedge \mathcal{X}^*)^* \simeq \mathrm{CB}(V^*, \mathcal{X}^{**}) \simeq V \otimes_{\min} \mathcal{X}^{**},$$

then if we assume $\mathcal{X}$ is 1-locally reflexive, one must prove

$$V \otimes_{\min} \mathcal{X}^{**} \simeq (V \otimes_{\min} \mathcal{X})^{**},$$

as operator spaces. Note $\mathrm{CB}(V,^*, \mathcal{X}^{**})$ denotes the Banach space of completely bounded maps between the operator spaces V^* and $\mathcal{X}^{**}$. The converse is an immediate result of Goldstine's theorem. It is precisely this tensor characterization of local reflexivity that presents problems when one moves to the realm of operator systems. If one only considers finite dimensional operator systems then duality theory works well, but for arbitrary operator systems it does not. Ongoing work uses methods presented in Section 6 to circumvent such duality difficulties and it is our hope that our current work will shed light onto the proper interpretation of local reflexivity for operator systems.

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VITA

Roy Araiza was born in San Jose, California where he obtained his Bachelor of Arts in Pure Mathematics from San Jose State University in the Silicon Valley. Beginning Fall 2015 Roy then began his graduate studies in mathematics at Purdue University where he worked in operator space theory and operator system theory. During his time at Purdue, Roy wrote three manuscripts in operator algebras. He has given 19 invited lectures and many more contributed lectures. Roy has been the organizer for the Junior Operator Algebras Seminar and co-organizer for the Operator Algebras Seminar, both at Purdue. In 2019, along with Thomas Sinclair, Roy organized the conference QLA Meets QIT at Purdue University. Roy was a visiting researcher at the Institute for Pure and Applied Mathematics at the University of California, Los Angeles, during Spring 2018. An active member of the Math Alliance, Roy has served on various panels for undergraduates and has worked to introduce operator algebras to early-career graduate students. Beginning Fall 2021 Roy will be joining the Department of Mathematics at the University of Illinois at Urbana-Champaign as a J.L. Doob Research Assistant Professor.