COMPACTNESS, EXISTENCE AND PARTIAL REGULARITY IN HYDRODYNAMICS OF LIQUID CRYSTALS

by

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Dedicated to my parents Miancheng Du and Shaolong Du, and my brother Rongyu Du for their unconditional love and support.

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ABSTRACT

This thesis mainly focuses on the PDE theories that arise from the study of hydrodynamics of nematic liquid crystals.

In Chapter 1, we give a brief introduction of the Ericksen–Leslie director theory and Beris–Edwards Q-tensor theory to the PDE modeling of dynamic continuum description of nematic liquid crystals. In the isothermal case, we derive the simplified Ericksen–Leslie equations with general targets via the energy variation approach. Following this, we introduce a simplified, non-isothermal Ericksen–Leslie system and justify its thermodynamic consistency.

In Chapter 2, we study the weak compactness property of solutions to the Ginzburg–Landau approximation of the simplified Ericksen–Leslie system. In 2-D, we apply the Pohozaev type argument to show a kind of concentration cancellation occurs in the weak sequence of Ginzburg–Landau system. Furthermore, we establish the same compactness for non-isothermal equations with approximated director fields staying on the upper semi-sphere in 3-D. These compactness results imply the global existence of weak solutions to the limit equations as the small parameter tends to zero.

In Chapter 3, we establish the global existence of a suitable weak solution to the corotational Beris–Edwards system for both the Landau–De Gennes and Ball–Majumdar bulk potentials in 3-D, and then study its partial regularity by proving that the 1-D parabolic Hausdorff measure of the singular set is 0.

In Chapter 4, motivated by the study of un-corotational Beris–Edwards system, we construct a suitable weak solution to the full Ericksen–Leslie system with Ginzburg–Landau potential in 3-D, and we show it enjoys a (slightly weaker) partial regularity, which asserts that it is smooth away from a closed set of parabolic Hausdorff dimension at most $\frac{15}{7}$.

1. INTRODUCTION

1.1 Hydrodynamic Theory

The liquid crystals constitute states of matter intermediate between rigid crystalline solids and isotropic flowing fluids. They exhibit various optical patterns with suitable control of external electronic-magnetic field, and this property is the key to build the liquid crystal displays (LCDs). There is no need to emphasize the importance of materials of this kind since they appear in billions of smartphones, televisions and laptops. There are mainly three types of liquid crystal phases, *cholesterics, nematics* and *smectics* (see [1]). We will focus on the nematic phase, in which the molecules have orientational order but no preferred positional order. Based on different order parameters for the macroscopic description of molecules, there are mainly two PDE theories for hydrodynamics of nematic liquid crystals: Ericksen–Leslie director theory and Beris–Edwards Q-tensor theory.

1.1.1 Ericksen–Leslie director theory

In the nematic phase, the mean orientation of liquid crystal molecules can be represented by a unit vector field $\mathbf{d} : D(\subset \mathbb{R}^n) \times \mathbb{R}_+ \to \mathbb{S}^2$ $(n = 2, 3)$ called *director*. Based on this representation, the full Ericksen–Leslie system reads (cf. [2]–[4])

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, & \text{(Conservation of mass)} \\ \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = \rho F + \nabla \cdot \hat{\sigma}, & \text{(Balance of linear momentum)} \\ \nabla \cdot \mathbf{u} = 0, & \text{(Incompressibility)} \\ \rho_1(\partial_t \omega + \mathbf{u} \cdot \nabla \omega) = \rho_1 G + \hat{g} + \nabla \cdot \boldsymbol{\pi}, & \text{(Balance of angular momentum)} \end{cases} \tag{1.1.1}$$

where $\rho : D \times \mathbb{R}_+ \to \mathbb{R}_+$ denotes the fluid density, $\mathbf{u} : D \times \mathbb{R}_+ \to \mathbb{R}^3$ is the fluid velocity, ρ_1 is a inertial constant, $F : D \times \mathbb{R}_+ \to \mathbb{R}^3$ and $G : D \times \mathbb{R}_+ \to \mathbb{R}^3$ represent the external body forces. We introduce the following notations

$$\begin{aligned} A &= \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \quad && \Omega = \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^T), \\ \omega &= \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d}, && N = \omega - \Omega \mathbf{d} \end{aligned}$$

for the symmetric, anti-symmetric part of the velocity gradient, the material derivative and co-rotational derivative of $\mathbf{d}$. Meanwhile, $\hat{\sigma}, \boldsymbol{\pi}$ and $\hat{g}$ satisfy the following constitutive relations

$$\begin{aligned} \hat{\sigma} &= -P\mathrm{I}_3 - \rho\sigma^E + \sigma^L, \\ \boldsymbol{\pi} &= \beta \otimes \mathbf{d} + \rho \left(\frac{\partial W}{\partial \nabla \mathbf{d}} \right)^T, \\ \hat{g} &= \gamma \mathbf{d} - (\nabla \mathbf{d})\beta - \rho \frac{\partial W}{\partial \mathbf{d}} + g. \end{aligned}$$

Here $P : D \times \mathbb{R}_+ \to \mathbb{R}$ is the pressure. $\sigma^E : D \times \mathbb{R}_+ \to \mathbb{R}^{3\times 3}$ and $\sigma^L : D \times \mathbb{R}_+ \to \mathbb{R}^{3\times 3}$ are the Ericksen stress tensor and Leslie stress tensor, respectively. We have that

$$\begin{aligned} \sigma^E &= \frac{\partial W(\mathbf{d}, \nabla \mathbf{d})}{\partial \nabla \mathbf{d}} \odot \nabla \mathbf{d}, \\ \sigma^L &= \mu_1 (\mathbf{d}^T A \mathbf{d})(\mathbf{d} \otimes \mathbf{d}) + \mu_2 N \otimes \mathbf{d} + \mu_3 N \otimes \mathbf{d} + \mu_4 A + \mu_5 A\mathbf{d} \otimes \mathbf{d} + \mu_6 \mathbf{d} \otimes A\mathbf{d}, \end{aligned}$$

where $W = W(\mathbf{d}, \nabla \mathbf{d})$ is the Oseen–Frank energy density of the director field:

$$\begin{aligned} W(\mathbf{d}, \nabla \mathbf{d}) := {} & \frac{k_1}{2}(\nabla \cdot \mathbf{d})^2 + \frac{k_2}{2}|\mathbf{d} \times (\nabla \times \mathbf{d})|^2 + \frac{k_3}{2}|\mathbf{d} \cdot (\nabla \times \mathbf{d})|^2 \\ & + \frac{1}{2}(k_2 + k_4)[\operatorname{tr}(\nabla \mathbf{d})^2 - (\nabla \cdot \mathbf{d})^2]. \end{aligned} \tag{1.1.2}$$

$\beta : D \times \mathbb{R}_+ \to \mathbb{R}^3$ and $\gamma : D \times \mathbb{R}_+ \to \mathbb{R}$ are Lagrangian multipliers due to the constraint that $|\mathbf{d}| = 1$.

$$g = \lambda_1 N + \lambda_2 A\mathbf{d}$$

represents the kinematic transport effect on the director field. $\mu_{\rm i}, k_{\rm j}, \lambda_k$ are given material constants. For simplicity, we make the following assumptions:

(One constant approximation)	$k_1 = k_2 = k_3 = 1, k_4 = 0,$
(Homogeneous fluid)	$\rho \equiv 1,$
(Small inertial effect)	$\rho_1 \equiv 0,$
(Absence of external forces)	$F = G \equiv 0.$

As a consequence, (1.1.1) can be written as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = -\nabla\cdot(\nabla\mathbf{d}\odot\nabla\mathbf{d}) + \nabla\cdot\sigma^L, \\ \nabla\cdot\mathbf{u} = 0, \\ \partial_t\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d} - \Omega\mathbf{d} + \dfrac{\lambda_2}{\lambda_1}A\mathbf{d} = \dfrac{1}{-\lambda_1}(\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}) + \dfrac{\lambda_2}{\lambda_1}(\mathbf{d}^T A\mathbf{d})\mathbf{d}. \end{cases} \tag{1.1.3}$$

1.1.2 Beris–Edwards Q-tensor theory

Due to the head-to-tail symmetry of the molecules, the sign of the director $\mathbf{d}$ has no physical meaning. Thus it is better to use the matrices $(\mathbf{d}\otimes\mathbf{d})_{\rm ij} := \mathbf{d}_{\rm i}\mathbf{d}_{\rm j}$ which takes the same values for $\pm\mathbf{d}$. After normalizing the trace of the matrices, we can use the so-called Q-tensor as the order parameter for the liquid crystals. The Beris–Edwards Q-tensor system modeling the hydrodynamic flow of liquid crystal materials was proposed by Beris and Edwards in the 1980s [5]:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} - \nabla\cdot(\nabla Q\odot\nabla Q) + \nabla\cdot(\tau+\sigma), \\ \nabla\cdot\mathbf{u} = 0, \\ \partial_t Q + \mathbf{u}\cdot\nabla Q - S(Q,\nabla\mathbf{u}) = \Gamma H + 3\Gamma(Q:H)Q, \end{cases} \tag{1.1.4}$$

where $\mathbf{u} : D\times\mathbb{R}_+ \to \mathbb{R}^3$ represents the velocity field of the flow, $Q : D\times\mathbb{R}_+ \to \mathcal{S}_0^{(3)}$, the set of traceless, symmetric 3×3 matrices, is a matrix field that represents the statistical macroscopic molecular orientation of the nematic liquid crystal material, and $P : \mathbb{T}^3 \times \mathbb{R}_+ \to \mathbb{R}$ represents the pressure function.

$$\begin{aligned} S(\nabla \mathbf{u}, Q) =& (\xi D + \Omega)\left(Q + \frac{1}{3}\mathrm{I}_3\right) + \left(Q + \frac{1}{3}\mathrm{I}_3\right)(\xi D - \Omega) \\ & - 2\xi \left(Q + \frac{1}{3}\mathrm{I}_3\right)\mathrm{tr}(Q\nabla \mathbf{u}), \end{aligned} \tag{1.1.5}$$

where

$$D = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right) \text{ and } \Omega = \frac{1}{2}\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^\top\right).$$

Denote

$$H = -\frac{\delta E(Q)}{\delta Q} = \Delta Q - f(Q)$$

where

$$E(Q) = \int_{\mathbb{T}^3}\left(\frac{1}{2}|\nabla Q|^2 + F(Q)\right)dx,$$

Here $F(Q)$ denotes the bulk energy density for the tensor field. It could be either the Landau–De Gennes polynomial potential or Ball–Majumdar entropy potential. We will give the detailed description of these two potentials in Chapter 3. ν, Γ, ξ are material coefficients reflecting viscosity, relaxation time and co-rotational effect. The symmetric part of the stress tensor reads

$$\begin{aligned} \tau =& \tau(Q, H) = -\xi\left(Q + \frac{1}{3}\mathrm{I}_3\right)H - \xi H\left(Q + \frac{1}{3}\mathrm{I}_3\right) \\ & + 2\xi Q : H\left(Q + \frac{1}{3}\mathrm{I}_3\right), \end{aligned}$$

and σ is the anti-symmetric part:

$$\sigma = \sigma(Q, H) = QH - HQ.$$

The system (1.1.4) can be derived from the bracket formalism with the so-called *master equation* (See [5, Chapter 11, pp. 546-549]). However, in the recent mathematical literatures the dissipative term $3\Gamma(Q : H)Q$ is neglected:

$$
\begin{cases}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nu \Delta \mathbf{u} - \nabla \cdot (\nabla Q \odot \nabla Q) + \nabla \cdot (\tau + \sigma), \\
\nabla \cdot \mathbf{u} = 0, \\
\partial_t Q + \mathbf{u} \cdot \nabla Q - S(Q, \nabla \mathbf{u}) = \Gamma H.
\end{cases}
\tag{1.1.6}
$$

1.2 Energetic Variational Approach (EnVarA)

The Energetic Variational Approach (EnVarA) provides a general framework for the derivation of PDE models for the hydrodynamics of dissipative complex fluids. Roughly speaking, it establishes the balance between the conservative force and the dissipative force, and these forces could be obtained via the Least Action Principle (LAP) and the Maximum Dissipation Principle (MDP) from the energy dissipation law. These underlying physical principles are motivated by work of Onsager [6] and Rayleigh [7], and we refer the interested readers to [8] for more details. In recent years, the EnVarA has been intensively employed by Liu and his collaborators on modeling of various kind of multiscale and multiphysics complex fluid system with dynamical boundary/interface effect [9]–[12], as well as designing numerical schemes in simulations [13]–[15]. In this section, we aim to present the application of EnVarA on the simplified Ericksen–Leslie system with general targets following the derivation as in [4], [16].

The EnVarA starts with the following energy dissipation law:

$$
\frac{dE^{\text{total}}}{dt} = -\Delta, \tag{1.2.1}
$$

where $E^{\text{total}} = E^{\text{kinematic}} + E^{\text{Helmholtz}}$ denotes the total energy (Hamiltonian) of the whole system and Δ is the rate of the energy dissipation. In terms of the first variation, the LAP and MDP read

$$
\begin{aligned}
&(\mathbf{LAP}) \quad \frac{\delta \int_0^T E^{\text{kinematic}} dt}{\delta x} = f_{\text{inertial}} = f_{\text{conservative}} = \frac{\int_0^T E^{\text{Helmholtz}} dt}{\delta x},\\
&(\mathbf{MDP}) \quad \frac{\delta \Delta}{\delta \mathbf{u}} = 2 f_{\text{dissipative}},
\end{aligned}
$$

where x denotes the position, $\mathbf{u}$ denotes the velocity field, f_{inertial} is the inertial force, $f_{\text{conservative}}$ is the conservative force, and $f_{\text{dissipative}}$ is the dissipative force. And finally, the EnVarA asserts that the following balance of forces (Newton's Second Law):

$$
f_{\text{inertial}} = f_{\text{conservative}} + f_{\text{dissipative}} \tag{1.2.2}
$$

holds for the dissipative system.

1.2.1 Simplified Ericksen–Leslie system with general target

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded domain with smooth boundary, and $\mathcal{N} \subset \mathbb{R}^L$ (for $L \geq 2$) be a smooth compact Riemannian manifold without boundary. The energy dissipation law for the simplified Ericksen–Leslie system of nematic liquid crystals in which the director field $\mathbf{v}$ takes value in $\mathcal{N}$ reads

$$
\frac{d}{dt} \int_\Omega \frac{1}{2} \left(|\mathbf{u}|^2 + |\nabla \mathbf{v}|^2 \right) dx = - \int_\Omega \left(|\nabla \mathbf{u}|^2 + |\partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v}|^2 \right) dx. \tag{1.2.3}
$$

The transported director field satisfies the heat flow of harmonic maps, i.e.,

$$
\partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v} = \Delta \mathbf{v} + A(\mathbf{v})(\nabla \mathbf{v}, \nabla \mathbf{v}),
$$

where $A(\mathbf{v})(\cdot,\cdot)$ is the second fundamental form of $\mathcal{N}$ at point $\mathbf{v}\in\mathcal{N}$. For the given velocity field $\mathbf{u}(x,t)$, the corresponding flow map $x(X,t):\Omega_0\times[0,T]\to\Omega$ solves the following ODE system:

$$\begin{cases} \dfrac{d}{dt}x(X,t)=\mathbf{u}(x(X,t),t), \\ x(X,0)=X. \end{cases} \tag{1.2.4}$$

For any given one-parameter family of volume preserving flow map $\{x^\epsilon\}_{-\delta<\epsilon<\delta}$ satisfying

$$\begin{cases} \left.\dfrac{dx^\epsilon}{d\epsilon}\right|_{\epsilon=0}=\varphi, \\ x^0=x, \\ \det\left(\dfrac{\partial x^\epsilon}{\partial X}\right)\equiv 1, \text{and hence, } \nabla_x\cdot\varphi=0. \end{cases} \tag{1.2.5}$$

Then we can compute

$$\begin{aligned}
\langle f_{\text{inertial}},\varphi\rangle_{L^2(\Omega\times[0,T])} &= \left\langle \frac{\delta\int_0^T E^{\text{kinematic}}dt}{\delta x},\varphi\right\rangle_{L^2(\Omega\times[0,T])} \\
&= \frac{d}{d\epsilon}\int_0^T\int_{\Omega_0}\frac{1}{2}|x_t^\epsilon(X,t)|^2\det\left(\frac{\partial x^\epsilon}{\partial X}\right)dXdt\bigg|_{\epsilon=0} \\
&= \int_0^T\int_{\Omega_0} x_t\cdot\varphi_t dXdt \\
&= -\int_0^T\int_{\Omega_0} x_{tt}\cdot\varphi dXdt = -\int_0^T\int_{\Omega_0}\dot{\mathbf{u}}\cdot\varphi dXdt \\
&= -\int_0^T\int_{\Omega}(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u})\cdot\varphi dxdt.
\end{aligned}$$

Meanwhile, with notation $F_\epsilon=\dfrac{\partial x^\epsilon}{\partial X}$ we can derive

$$\begin{aligned}
\langle f_{\text{conservative}},\varphi\rangle_{L^2(\Omega\times[0,T])} &= \left\langle \frac{\delta\int_0^T E^{\text{Helmholtz}}dt}{\delta x},\varphi\right\rangle_{L^2(\Omega\times[0,T])} \\
&= \frac{d}{d\epsilon}\int_0^T\int_{\Omega_0}\frac{1}{2}|F_\epsilon^{-T}\nabla_X\mathbf{v}(x(X,t),t)|^2\det F^\epsilon dXdt \\
&= \int_0^T\int_{\Omega_0}F^{-T}\nabla_X\mathbf{v}(x(X,t),t):\left[\left.\frac{dF_\epsilon^{-T}}{d\epsilon}\right|_{\epsilon=0}\nabla_X\mathbf{v}(x(X,t),t)\right]dXdt \\
&= \int_0^T\int_\Omega\nabla\mathbf{v}:(-\nabla^T\varphi\nabla\mathbf{v})dxdt
\end{aligned}$$

$$= \int_0^T \int_\Omega [\nabla \cdot (\nabla \mathbf{v} \odot \nabla \mathbf{v})] \cdot \varphi dx dt.$$

On the other hand, with $\mathbf{u}^\epsilon = \mathbf{u} + \epsilon\varphi$ we can apply the MDP to $\frac{1}{2}\Delta$ to get

$$\begin{aligned}
\langle f_{\text{dissipative}}, \varphi \rangle_{L^2(\Omega)} &= \left\langle \frac{\delta(\frac{1}{2}\Delta)}{\delta \mathbf{u}}, \varphi \right\rangle_{L^2(\Omega)} \\
&= \lim_{\epsilon \to 0} \Big|_{\epsilon=0} \int_\Omega \frac{1}{2}(|\nabla \mathbf{u}^\epsilon|^2 + |\partial_t \mathbf{v} + \mathbf{u}^\epsilon \cdot \nabla \mathbf{v}|^2) dx \\
&= \int_\Omega \nabla \mathbf{u} : \nabla \varphi dx + \int_\Omega (\partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v}) \cdot (\varphi \cdot \nabla \mathbf{v}) dx \\
&= -\int_\Omega \Delta \mathbf{u} \cdot \varphi dx + \int_\Omega (\Delta \mathbf{v} + A(\mathbf{v})(\nabla \mathbf{v}, \nabla \mathbf{v})) \cdot (\varphi \cdot \nabla \mathbf{v}) dx \\
&= \int_\Omega (-\Delta \mathbf{u} + \nabla \mathbf{v} \cdot \Delta \mathbf{v}) \cdot \varphi dx \\
&= \int_\Omega (-\Delta \mathbf{u} + \nabla \cdot (\nabla \mathbf{v} \odot \nabla \mathbf{v})) \cdot \varphi dx,
\end{aligned}$$

where we use the geometry property that $A(\mathbf{v})(\nabla \mathbf{v}, \nabla \mathbf{v}) \perp T_{\mathbf{v}}\mathcal{N}$, the incompressibility of φ and the following identity

$$\nabla \mathbf{v} \cdot \Delta \mathbf{v} = \nabla \cdot (\nabla \mathbf{v} \odot \nabla \mathbf{v}) - \nabla \frac{|\nabla \mathbf{v}|^2}{2}.$$

Hence, by the EnVarA (1.2.2), we obtain the following simplified Ericksen–Leslie system

$$\begin{cases}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \Delta \mathbf{u} + \nabla \cdot (\nabla \mathbf{v} \odot \nabla \mathbf{v}), \\
\nabla \cdot \mathbf{u} = 0, \\
\partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v} = \Delta \mathbf{v} + A(\mathbf{v})(\nabla \mathbf{v}, \nabla \mathbf{v}).
\end{cases} \tag{1.2.6}$$

The generalized system (1.2.6) covers the two important cases in nematic liquid crystals:

(1) For $\mathcal{N} = \mathbb{S}^2$, the system (1.2.6) becomes the simplified, uniaxial Ericksen–Leslie system first proposed by [17]

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla P = -\nabla \cdot \Big(\nabla \mathbf{d} \odot \nabla \mathbf{d} \Big), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}, \end{cases} \tag{1.2.7}$$

for $(\mathbf{u}(x,t), \mathbf{d}(x,t), P(x,t)) : \Omega \times (0,T) \to \mathbb{R}^2 \times \mathbb{S}^2 \times \mathbb{R}$. In dimension two, the existence of a unique global weak solution has been proved in [18] [19], which satisfies the energy inequality and has at most finitely many singular times, see also [20]. Very recently, the authors in [21] have constructed example of singularity at finite time. In dimension three, a global weak solution has been constructed in [22] with initial data $d_0 \in \mathbb{S}^2_+$. Examples of finite time singularity have been constructed by [23]. Interested readers can consult the survey article [24] and the references therein.

(2) For

$$\mathcal{N} = \Big\{ (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{S}^2 \times \mathbb{S}^2 \;\Big|\; \mathbf{y}_1 \cdot \mathbf{y}_2 = 0 \Big\} \subset \mathbb{R}^6,$$

let $\mathbf{v}(x,t) = (\mathbf{n}(x,t), \mathbf{m}(x,t)) : \Omega \times (0,T) \to \mathbb{S}^2 \times \mathbb{S}^2$ with $\mathbf{n} \cdot \mathbf{m} = 0$. Then the system (1.2.6) becomes the biaxial, Ericksen–Leslie system

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla P = -\nabla \cdot \Big(\nabla \mathbf{n} \odot \nabla \mathbf{n} + \nabla \mathbf{m} \odot \nabla \mathbf{m} \Big), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{n} + \mathbf{u} \cdot \nabla \mathbf{n} = \Delta \mathbf{n} + |\nabla \mathbf{n}|^2 \mathbf{n} + \langle \nabla \mathbf{n}, \nabla \mathbf{m} \rangle \mathbf{m} \\ \partial_t \mathbf{m} + \mathbf{u} \cdot \nabla \mathbf{m} = \Delta \mathbf{m} + |\nabla \mathbf{m}|^2 \mathbf{m} + \langle \nabla \mathbf{m}, \nabla \mathbf{n} \rangle \mathbf{n} \\ \mathbf{n} \cdot \mathbf{m} = 0 \end{cases} \quad \text{in } \Omega \times (0,T). \tag{1.2.8}$$

This is a simplified version of the hydrodynamics of biaxial nematics model proposed by Grovers and Vertogen [25]–[27]. In dimensional two, the existence of a unique global weak solution has recently been shown in [28], which is smooth off at most finitely many singular times.

1.3 Nonisothermal Nematic Liquid Crystal Flows

A non-isothermal liquid crystal flow in the nematic phase can be described in terms of three physical variables: the velocity field $\mathbf{u}$ of the underlying fluid, the director field $\mathbf{d}$ representing the averaged orientation of liquid crystal molecules, and the background temperature θ. The evolution of the velocity field is governed by the incompressible Navier-Stokes system with stress tensors representing viscous and elastic effects. In the nematic case, the director field is driven by transported negative gradient flow of the Oseen–Frank energy functional which represents the internal microscopic damping [1], [29]. We consider the non-isothermal setting in which the temperature is neither spatial nor temporal homogeneous and thus contributes to total dissipation of the whole system.

A great deal of mathematical theories has been devoted to the study of nematic liquid crystals in the continuum formulation. In pioneering papers [2], [30], [31] Ericksen and Leslie have put forward a PDE model based on the principle of conservation laws and momentum balance. There has been extensive mathematical study of analytic issues of the simplified Ericksen–Leslie system. In 1989 Lin [17] first proposed a simplified Ericksen–Leslie model with one constant approximation for the Oseen–Frank energy: $(\mathbf{u}, \mathbf{d}) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n \times \mathbb{S}^2$ solves

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \mu \Delta \mathbf{u} - \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}, \end{cases} \tag{1.3.1}$$

where $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3), $P : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ denotes the pressure, $\mu > 0$ represents the viscosity constant of the fluid, and $(\nabla \mathbf{d} \odot \nabla \mathbf{d})_{\mathrm{ij}} = \sum_{k=1}^{3} \partial_{x_\mathrm{i}} \mathbf{d}^{(k)} \partial_{x_\mathrm{j}} \mathbf{d}^{(k)}$ denotes the Ericksen stress tensor. It is a system of the forced Navier–Stokes equation coupled with the transported harmonic map heat flow to $\mathbb{S}^2$. The readers can consult [32] on the study of the Navier–Stokes equations and [24] for some recent developments on harmonic map heat flow. The rigorous mathematical analysis was initiated by Lin–Liu [33], [34] in which they established the well-posedness of so-called Ginzburg–Landau approximation of (1.3.1): $(\mathbf{u}, \mathbf{d}) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n \times \mathbb{R}^3$ satisfies

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \mu \Delta \mathbf{u} - \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + \frac{1}{\varepsilon^2} \left(1 - |\mathbf{d}|^2\right) \mathbf{d}, \end{cases} \tag{1.3.2}$$

where $\varepsilon > 0$ is the parameter of approximation. They have obtained the existence of a unique, global strong solution in dimension 2 and in dimension 3 under large viscosity μ. They have also studied the existence of suitable weak solutions and their partial regularity in dimension 3, which is analogous to the celebrated regularity theorem by Caffarelli–Kohn–Nirenberg [35] (see also [36]) for the dimension 3 incompressible Navier–Stokes equation. Later on Lin–Lin–Wang [18] adopted a different approach to construct global Leray–Hopf type weak solutions (see [37]) for dimension 2 to (1.3.1) via the method of small energy regularity estimate. Huang–Lin–Wang [38] extended the works of [18] to the general Ericksen–Leslie system by a blow up argument.

The existence of global weak solution to (1.3.1) in dimension three is highly non-trivial due to the appearance of the super-critical nonlinear elastic stress term $\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d})$. Some preliminary progress was made by Lin–Wang [22], where under the assumption that an initial configuration $\mathbf{d}_0$ lies in the upper half sphere, i.e.,

$$\mathbf{d}_0(\Omega) \subset \mathbb{S}^2_+ := \left\{ y = (\mathbf{y}^1, \mathbf{y}^2, \mathbf{y}^3) \in \mathbb{R}^3 : |\mathbf{y}| = 1,\ \mathbf{y}^3 \geq 0 \right\}. \tag{1.3.3}$$

the existence of global weak solution was constructed by the Ginzburg–Laudau approximation method and a delicate blow-up analysis. See [24] for a review of recent progresses on the mathematical analysis of Ericksen–Leslie system.

Recently there has been considerable interest in the mathematical study for the hydrodynamics of non-isothermal nematic liquid crystals. Recall that a simplified, non-isothermal version of (1.3.2) can be described as follows. Let $(\mathbf{u}, \mathbf{d}, \theta) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n \times \mathbb{R}^3 \times \mathbb{R}_+$ solve

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nabla \cdot (\mu(\theta) \nabla \mathbf{u}) - \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + \frac{1}{\varepsilon^2} \left(1 - |\mathbf{d}|^2\right) \mathbf{d}, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = -\nabla \cdot \mathbf{q} + \mu(\theta) |\nabla \mathbf{u}|^2 + \left| \Delta \mathbf{d} + \frac{1}{\varepsilon^2} (1 - |\mathbf{d}|^2) \mathbf{d} \right|^2, \end{cases} \tag{1.3.4}$$

where $\mathbf{q} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n$ is the heat flux. Feireisl–Frémond–Rocca–Schimperna [39] proved the existence of a global weak solution to (1.3.4) in dimension 3. Correspondingly, non-isothermal version of (1.3.1) reads $(\mathbf{u}, \mathbf{d}, \theta) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^n \times \mathbb{S}^2 \times \mathbb{R}_+$ solves

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nabla \cdot (\mu(\theta) \nabla \mathbf{u}) - \nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta = -\nabla \cdot \mathbf{q} + \mu(\theta) |\nabla \mathbf{u}|^2 + \left| \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} \right|^2. \end{cases} \tag{1.3.5}$$

Hieber-Prüss [40] have established the existence of a unique local $L^p - L^q$ strong solution to (1.3.5), which can be extended to a global strong solution provided the initial data is close to an equilibrium state. For the general non-isothermal Ericksen–Leslie system, De Anna–Liu [41] have obtained the existence of global strong solution in Besov spaces provided the Besov norm of the initial data is sufficiently small. On $\mathbb{T}^2$, Li–Xin [42] have showed that there exists a global weak solution to (1.3.5). A natural question is that in dimension 3 whether (1.3.5) admits a global weak solution. The rest of this chapter is devoted to the thermodynamic consistency of (1.3.4) and (1.3.5).

1.3.1 Non-isothermal Ginzburg–Landau approximation

First we recall the equations of $\mathbf{u}$ and $\mathbf{d}$ in the non-isothermal Ginzburg–Laudau approximation (1.3.4):

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \operatorname{div}\left(\mu(\theta)\nabla \mathbf{u} - \nabla \mathbf{d} \odot \nabla \mathbf{d}\right), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} - \mathbf{f}_\varepsilon(\mathbf{d}), \end{cases} \tag{1.3.6}$$

where $\mathbf{f}_\varepsilon(\mathbf{d}) = \partial_{\mathbf{d}} F_\varepsilon(\mathbf{d})$, $F_\varepsilon(\mathbf{d}) = \dfrac{(|\mathbf{d}|^2 - 1)^2}{4\varepsilon^2}$.

The difference between (1.3.6) and the isothermal case (1.3.2) is that the viscosity coefficient μ is a function of temperature θ. Here the temperature plays a role as parameters both in the material coefficients and the heat conductivity coefficients, which is to be discussed later. To make the system (1.3.6) a close system, we need the evolution equation for θ. The equation of thermal dissipation is derived according to *First and Second laws of thermodynamics* [8].

First we introduce some basic concepts in thermodynamics. The internal energy density reads

$$\mathrm{e}_\varepsilon^{int} = \frac{1}{2}|\nabla \mathbf{d}|^2 + F_\varepsilon(\mathbf{d}) + \theta,$$

and the Helmholtz free energy is given by

$$\psi_\varepsilon = \frac{1}{2}|\nabla \mathbf{d}|^2 + F_\varepsilon(\mathbf{d}) - \theta \ln \theta.$$

Denote the entropy by η in the *Second law of thermodynamics*, which is determined by temperature through the Maxwell relation

$$\eta = -\frac{\partial \psi_\varepsilon}{\partial \theta} = 1 + \ln \theta. \tag{1.3.7}$$

The internal energy can be obtained by (negative) Legendre transformation of free energy with respect to η, i.e.,

$$\mathrm{e}_{\varepsilon}^{int} = \psi_{\varepsilon} + \eta\theta.$$

The heat flux $\mathbf{q}$ in the equations of both θ of (1.3.4) and (1.3.5) satisfies the generalized Fourier law:

$$\mathbf{q}(\theta) = -k(\theta)\nabla\theta - h(\theta)(\nabla\theta \cdot \mathbf{d})\mathbf{d} \tag{1.3.8}$$

where $k(\theta)$ and $h(\theta)$ represent thermal conductivities. The evolution of entropy can be written as follows.

$$\partial_t \eta + \mathbf{u} \cdot \nabla\eta = -\nabla \cdot \mathbf{g} + \Delta_{\varepsilon}, \tag{1.3.9}$$

where $\mathbf{g}$ is the entropy flux which is determined by the heat flux through the Clausius-Duhem relation

$$\mathbf{q} = \theta\mathbf{g}, \tag{1.3.10}$$

and the entropy production $\Delta_{\varepsilon} \geq 0$ is given by (1.3.13) below.

The thermal consistency of (1.3.4) is given by the following proposition.

Proposition 1.3.1. *Suppose* $(\mathbf{u}, \mathbf{d}, \theta)$ *is a strong solution to* (1.3.4)*. Then*
(1) (First law of thermodynamics). The total energy $\mathrm{e}_{\varepsilon}^{total} = \frac{1}{2}|\mathbf{u}|^2 + \mathrm{e}_{\varepsilon}^{int}$ *is conservative. More precisely, we have*

$$\frac{D}{Dt}\mathrm{e}_{\varepsilon}^{total} + \nabla \cdot (\Sigma + \mathbf{q}) = 0, \tag{1.3.11}$$

where

$$\Sigma = P\mathbf{u} - \mu(\theta)\mathbf{u} \cdot \nabla\mathbf{u} + \nabla\mathbf{d} \odot \nabla\mathbf{d} \cdot \mathbf{u} - (\nabla\mathbf{d})^T \frac{D\mathbf{d}}{Dt}, \tag{1.3.12}$$

and $\frac{D}{Dt} := \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ *denotes the material derivative.*
(2) (Second law of thermodynamics). The entropy cannot decrease during any irreversible process, which means the entropy production Δ_{ε} *is alway non-negative, i.e.,*

$$\Delta_{\varepsilon} = \frac{1}{\theta}\Big(\mu(\theta)|\nabla\mathbf{u}|^2 + \Big|\Delta\mathbf{d} + \frac{1}{\varepsilon^2}(1 - |\mathbf{d}|^2)\mathbf{d}\Big|^2 - \mathbf{q} \cdot \nabla\theta\Big) \geq 0. \tag{1.3.13}$$

Proof. We first prove (1.3.11). By direct calculations, we have

$$
\begin{aligned}
&\frac{D}{Dt}\mathrm{e}_\varepsilon^{total} = \mathbf{u}\cdot\frac{D\mathbf{u}}{Dt} + \nabla\mathbf{d} : \frac{D}{Dt}\nabla\mathbf{d} + f_\varepsilon(\mathbf{d})\cdot\frac{D\mathbf{d}}{Dt} + \frac{D\theta}{Dt}\\
&= \mathbf{u}\cdot\operatorname{div}\left(-PI + \mu(\theta)\nabla\mathbf{u} - \nabla\mathbf{d}\odot\nabla\mathbf{d}\right) + \nabla\mathbf{d} : \nabla\frac{D\mathbf{d}}{Dt} - \nabla\mathbf{d}\odot\nabla\mathbf{d} : \nabla\mathbf{u}\\
&\quad + \mathbf{f}_\varepsilon(\mathbf{d})\cdot\frac{D\mathbf{d}}{Dt} - \nabla\cdot\mathbf{q} + \mu(\theta)|\nabla\mathbf{u}|^2 + \Big|\Delta\mathbf{d} + \frac{1}{\varepsilon^2}(1-|\mathbf{d}|^2)\mathbf{d}\Big|^2\\
&= \operatorname{div}\left(-P\mathbf{u} + \mu(\theta)\mathbf{u}\cdot\nabla\mathbf{u} - \nabla\mathbf{d}\odot\nabla\mathbf{d}\cdot\mathbf{u}\right) - \mu(\theta)|\nabla\mathbf{u}|^2 + \nabla\mathbf{d}\odot\nabla\mathbf{d} : \nabla\mathbf{u}\\
&\quad + \operatorname{div}\Big((\nabla\mathbf{d})^T\frac{D\mathbf{d}}{Dt}\Big) - (\Delta\mathbf{d} - \mathbf{f}_\varepsilon(\mathbf{d}))\cdot\frac{D\mathbf{d}}{Dt} - \nabla\mathbf{d}\odot\nabla\mathbf{d} : \nabla\mathbf{u} - \nabla\cdot\mathbf{q}\\
&\quad + \mu(\theta)|\nabla\mathbf{u}|^2 + \Big|\Delta\mathbf{d} + \frac{1}{\varepsilon^2}(1-|\mathbf{d}|^2)\mathbf{d}\Big|^2\\
&= \operatorname{div}\Big(- P\mathbf{u} + \mu(\theta)\mathbf{u}\cdot\nabla\mathbf{u} - \nabla\mathbf{d}\odot\nabla\mathbf{d}\cdot\mathbf{u} + (\nabla\mathbf{d})^T\frac{D\mathbf{d}}{Dt}\Big) - \nabla\cdot\mathbf{q}\\
&= -\operatorname{div}(\Sigma + \mathbf{q}).
\end{aligned}
\tag{1.3.14}
$$

Note that (1.3.13) follows directly from (1.3.7), (1.3.9), $(1.3.4)_4$, and (1.3.8), i.e.

$$
\begin{aligned}
\Delta_\varepsilon &= \frac{1}{\theta}\Big(\mu(\theta)|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d} - f_\varepsilon(\mathbf{d})|^2 - \mathbf{q}\cdot\nabla\theta\Big)\\
&= \frac{1}{\theta}\Big(\mu(\theta)|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d} - f_\varepsilon(\mathbf{d})|^2 + k(\theta)|\nabla\theta|^2 + h(\theta)|\nabla\theta\cdot\mathbf{d}|^2\Big) \geq 0.
\end{aligned}
$$

This completes the proof. □

1.3.2 Non-isothermal simplified Ericksen–Leslie system

As ε tends to 0, due to the penalization effect of $F_\varepsilon(\mathbf{d})$, formally the equation of $\mathbf{d}$ in (1.3.6) converges to

$$\partial_t\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d} = \Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d},$$

where $|\mathbf{d}| = 1$. This is a "transported gradient flow" of the Dirichlet energy $\frac{1}{2}\int_\Omega |\nabla\mathbf{d}|^2\,dx$ for maps $\mathbf{d} : \Omega \to \mathbb{S}^2$.

As in the previous section, we introduce the total energy for (1.3.5):

$$\mathrm{e}^{total} = \frac{1}{2}(|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2) + \theta,$$

and the entropy evolution equation:

$$\partial_t \eta + \mathbf{u} \cdot \nabla \eta = -\nabla \cdot \mathbf{g} + \Delta_0, \tag{1.3.15}$$

where Δ_0 is the entropy production given by (1.3.17) below.

The thermal consistency of (1.3.5) is described by the following proposition.

Proposition 1.3.2. *Suppose* $(\mathbf{u}, \mathbf{d}, \theta)$ *is a strong solution to* (1.3.5). *Then*
(1) (First law of thermodynamics). The total energy is conservative, i.e.,

$$\frac{D}{Dt}\mathrm{e}^{total} + \nabla \cdot (\Sigma + \mathbf{q}) = 0, \tag{1.3.16}$$

where $\Sigma = P\mathbf{u} - \mu(\theta)\mathbf{u} \cdot \nabla \mathbf{u} + \nabla \mathbf{d} \odot \nabla \mathbf{d} \cdot \mathbf{u} - (\nabla \mathbf{d})^T \frac{D\mathbf{d}}{Dt}$.
(2) (Second law of thermodynamics). The entropy production Δ_0 *is non-negative, i.e.,*

$$\Delta_0 = \frac{1}{\theta}\Big(\mu(\theta)|\nabla u|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2 - \mathbf{q} \cdot \nabla \theta\Big) \geq 0. \tag{1.3.17}$$

Proof. From (1.3.5), we can compute

$$\begin{aligned}
&\frac{D\mathrm{e}^{total}}{Dt} = \frac{D}{Dt}\Big(\frac{1}{2}(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2) + \theta\Big) \\
&= \mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} + \nabla \mathbf{d} : \frac{D}{Dt}\nabla \mathbf{d} + \frac{D\theta}{Dt} \\
&= \mathbf{u} \cdot \operatorname{div}\left(-PI + \mu(\theta)\nabla \mathbf{u} - \nabla \mathbf{d} \odot \nabla \mathbf{d}\right) \\
&\quad + \nabla \mathbf{d} : \nabla \frac{D\mathbf{d}}{Dt} - \nabla \mathbf{d} \odot \nabla \mathbf{d} : \nabla \mathbf{u} - \nabla \cdot \mathbf{q} + \mu(\theta)|\nabla \mathbf{u}|^2 + \Big|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}\Big|^2 \\
&= \operatorname{div}\left(-P\mathbf{u} + \mu(\theta)\mathbf{u} \cdot \nabla \mathbf{u} - \nabla \odot \nabla \mathbf{d} \cdot \mathbf{u}\right) - \mu(\theta)|\nabla \mathbf{u}|^2 + \nabla \mathbf{d} \odot \nabla \mathbf{d} : \nabla \mathbf{u} \\
&\quad + \operatorname{div}\Big((\nabla \mathbf{d})^T \frac{D\mathbf{d}}{Dt}\Big) - (\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}) \cdot \Delta \mathbf{d} - \nabla \mathbf{d} \odot \nabla \mathbf{d} : \nabla \mathbf{u} \\
&\quad - \operatorname{div}\mathbf{q} + \mu(\theta)|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2 \\
&= -\operatorname{div}(\Sigma + \mathbf{q}),
\end{aligned}$$

where we have used the fact $|\mathbf{d}| = 1$ so that

$$(\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}) \cdot \Delta \mathbf{d} = |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}|^2.$$

This implies (1.3.16). From the entropy equation (1.3.15), Clausius–Duhem's relation (1.3.10), the temperature equation in (1.3.5), and (1.3.8), we can show

$$\begin{aligned}\Delta_0 &= \frac{1}{\theta}\Big(\mu(\theta)|\nabla \mathbf{u}|^2 + \Big|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}\Big|^2 - \mathbf{q}\cdot\nabla\theta\Big) \\ &= \frac{1}{\theta}\Big(\mu(\theta)|\nabla \mathbf{u}|^2 + \Big|\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}\Big|^2 + k(\theta)|\nabla\theta|^2 + h(\theta)|\nabla\theta\cdot\mathbf{d}|^2\Big) \geq 0.\end{aligned}$$

This yields (1.3.17). $\square$

2. WEAK COMPACTNESS OF NEMATIC LIQUID CRYSTAL FLOWS

2.1 Weak compactness of simplified Ericksen–Leslie system in 2-D

2.1.1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and $\mathcal{N} \subset \mathbb{R}^L$ (for $L \geq 2$) be a smooth compact Riemannian manifold without boundary, and $0 < T \leq \infty$. We formulate a generalized form of simplified Ericksen–Leslie system of nematic liquid crystals in which the director field takes values in $\mathcal{N}$:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla P = -\nabla \cdot \Big(\nabla \mathbf{v} \odot \nabla \mathbf{v}\Big), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v} = \Delta \mathbf{v} + A(\mathbf{v})(\nabla \mathbf{v}, \nabla \mathbf{v}), \end{cases} \quad \text{in } \Omega \times (0,T), \tag{2.1.1}$$

where $(\mathbf{u}(x,t), \mathbf{v}(x,t), P(x,t)) : \Omega \times (0,T) \to \mathbb{R}^2 \times \mathcal{N} \times \mathbb{R}$ represents the fluid velocity field, the orientation director field of nematic material (into a general Riemannian manifold), and the pressure function respectively, $\Big(\nabla \mathbf{v} \odot \nabla \mathbf{v}\Big)_{\mathrm{ij}} = \nabla_{x_\mathrm{i}} \mathbf{v} \cdot \nabla_{x_\mathrm{j}} \mathbf{v}$ for $\mathrm{i}, \mathrm{j} = 1, 2$ represents the Ericksen–Leslie stress tensor, and $A(\mathbf{v})(\cdot,\cdot)$ is the second fundamental form of $\mathcal{N}$ at the point $\mathbf{v} \in \mathcal{N}$.

A strategy to construct a weak solution of (2.1.1) and (2.1.3) is to consider a Ginzburg–Landau approximated system (cf. [33],[34]). More precisely, for any $\delta > 0$ set the δ-neighborhood of $\mathcal{N}$ by

$$\mathcal{N}_\delta = \Big\{ \mathbf{y} \in \mathbb{R}^L \ \Big| \ \mathrm{dist}(\mathbf{y}, \mathcal{N}) < \delta \Big\},$$

where $\mathrm{dist}(\mathbf{y}, \mathcal{N})$ is the distance from $\mathbf{y}$ to $\mathcal{N}$. Let $\Pi_{\mathcal{N}} : \mathcal{N}_\delta \to \mathcal{N}$ be the nearest point projection map. There exists $\delta_{\mathcal{N}} = \delta(\mathcal{N}) > 0$ such that $\mathrm{dist}(\mathbf{y}, \mathcal{N})$ and $\Pi_{\mathcal{N}}$ are smooth in $\mathcal{N}_{2\delta_{\mathcal{N}}}$. Let $\chi(s) \in C^\infty([0,\infty))$ be a monotone increasing function such that

$$\chi(s) = \begin{cases} s, & \text{if } 0 \leq s \leq \delta_{\mathcal{N}}^2, \\ 4\delta_{\mathcal{N}}^2, & \text{if } s \geq 4\delta_{\mathcal{N}}^2. \end{cases}$$

Consider the following Ginzburg–Landau energy functional for the director $\mathbf{v}$

$$E_\varepsilon(\mathbf{v}) = \int_\Omega \Big(\frac{1}{2}|\nabla \mathbf{v}|^2 + \frac{1}{\varepsilon^2}\chi\Big(\operatorname{dist}^2(\mathbf{v},\mathcal{N})\Big)\Big).$$

Then the corresponding Ginzburg–Landau approximated system of (2.1.1) can be written as

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \Delta\mathbf{u} + \nabla P = -\nabla\cdot\Big(\nabla\mathbf{v}\odot\nabla\mathbf{v}\Big), \\ \nabla\cdot\mathbf{u} = 0, \\ \partial_t\mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v} = \Delta\mathbf{v} - \dfrac{1}{\varepsilon^2}\chi\Big(\operatorname{dist}^2(\mathbf{v},\mathcal{N})\Big)\dfrac{d}{d\mathbf{v}}\Big(\operatorname{dist}^2(\mathbf{v},\mathcal{N})\Big). \end{cases} \tag{2.1.2}$$

The main purpose of this section is to study the weak compactness of solutions to the simplified Ericksen–Leslie system (2.1.1) and convergence of solutions of the Ginzburg–Landau approximation (2.1.2) to the simplified Ericksen–Leslie system (2.1.1). For this purpose, we will consider the following initial and boundary condition

$$(\mathbf{u},\mathbf{v})\,|_{\partial_p Q_T} = (\mathbf{u}_0,\mathbf{v}_0) \tag{2.1.3}$$

where $Q_T = \Omega\times(0,T)$ and $\partial_p Q_T = \Big(\Omega\times\{t=0\}\Big)\cup\Big(\partial\Omega\times[0,T]\Big)$ is the parabolic boundary of Q_T. We assume that

$$\mathbf{u}_0\Big|_{\partial\Omega} = 0, \quad \mathbf{v}_0(x)\in\mathcal{N} \ \text{ for a.e. } \ x\in\Omega, \tag{2.1.4}$$

and introduce the notations

$$\begin{aligned} \mathbf{H} &= \text{closure of } C_0^\infty\left(\Omega,\mathbb{R}^2\right)\cap\left\{f \,\middle|\, \nabla\cdot f = 0\right\} \text{ in } L^2\left(\Omega,\mathbb{R}^2\right), \\ \mathbf{J} &= \text{closure of } C_0^\infty\left(\Omega,\mathbb{R}^2\right)\cap\left\{f \,\middle|\, \nabla\cdot f = 0\right\} \text{ in } H_0^1\left(\Omega,\mathbb{R}^2\right), \\ H^1(\Omega,\mathcal{N}) &= \left\{f\in H^1(\Omega,\mathbb{R}^L) \,\middle|\, f(x)\in\mathcal{N} \text{ a.e. } x\in\Omega\right\}. \end{aligned}$$

We also assume that

$$\mathbf{u}_0\in\mathbf{H}, \quad \mathbf{v}_0\in H^1(\Omega,\mathcal{N}). \tag{2.1.5}$$

Recall the definition of weak solutions of (2.1.1).

Definition 2.1.1. *A pair of maps* $\mathbf{u} \in L^\infty([0,T],\mathbf{H})\cap L^2([0,T],\mathbf{J})$ *and* $\mathbf{v} \in L^2([0,T],H^1(\Omega,\mathcal{N}))$ *is called a weak solution to initial and boundary problem* (2.1.1), (2.1.3)-(2.1.5), *if*

$$
\begin{aligned}
&-\int_{Q_T}\langle \mathbf{u},\xi\varphi\rangle+\int_{Q_T}\langle \mathbf{u}\cdot\nabla\mathbf{u},\xi\varphi\rangle+\langle\nabla u,\xi\nabla\varphi\rangle\\
&=-\xi(0)\int_\Omega\langle\mathbf{u}_0,\varphi\rangle+\int_{Q_T}\langle\nabla\mathbf{v}\odot\nabla\mathbf{v},\xi\nabla\varphi\rangle\,,\\
&-\int_{Q_T}\langle \mathbf{v},\xi\phi\rangle+\int_{Q_T}\langle \mathbf{u}\cdot\nabla\mathbf{v},\xi\phi\rangle+\langle\nabla \mathbf{v},\xi\nabla\phi\rangle\\
&=-\xi(0)\int_\Omega\langle\mathbf{v}_0,\phi\rangle+\int_{Q_T}\langle A(\mathbf{v})(\nabla\mathbf{v},\nabla\mathbf{v}),\xi\phi\rangle\,,
\end{aligned}
\tag{2.1.6}
$$

for any $\xi \in C^\infty([0,T])$ *with* $\xi(T)=0$, $\varphi\in\mathbf{J}$ *and* $\phi\in H_0^1(\Omega,\mathbb{R}^3)$. *Moreover,* $(\mathbf{u},\mathbf{v})|_{\partial\Omega}=(\mathbf{u}_0,\mathbf{v}_0)$ *in the sense of trace. The notion of a weak solution to the system* (2.1.2) *can be defined similarly.*

Our first main theorem concerns the convergence of weak solutions of the system (2.1.2) to the system (2.1.1) as $\varepsilon\to 0$. We remark that the existence of weak solutions to (2.1.2) has been established by [33], [34] for $\mathcal{N}=\mathbb{S}^2$ by the Galerkin method, which can be easily adapted to handle the case that $\mathcal{N}$ is a compact Riemannian manifold.

Theorem 2.1.1 ([43]). *For* $\varepsilon>0$, *let* $(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon)$ *be a sequence of weak solutions to the Ginzburg–Landau approximated system* (2.1.2) *with the initial and boundary condition* (2.1.3)-(2.1.5). *Then there exists a weak solution* $(\mathbf{u},\mathbf{v})$ *of* (2.1.1) *with the initial and boundary condition* (2.1.3)-(2.1.5) *such that, after passing to subsequences,*

$$\mathbf{u}^\varepsilon \rightharpoonup \mathbf{u}\ \ in\ \ L^2([0,T],H^1(\Omega)),\quad \mathbf{v}^\varepsilon\rightharpoonup\mathbf{v}\ \ in\ \ L^2([0,T],H^1(\Omega)).$$

In particular, the initial and boundary problem (2.1.1) *and* (2.1.3)-(2.1.5) *admits at least one weak solution* $\mathbf{u}\in L^\infty([0,T],\mathbf{H})\cap L^2([0,T],\mathbf{J})$ *and* $\mathbf{v}\in L^2([0,T],H^1(\Omega,\mathcal{N}))$.

We would like to mention that when $\mathcal{N}=\mathbb{S}^2$, the convergence of solutions of system (2.1.2) to the system (1.2.7) has recently been proved in two dimensional torus $\mathbb{T}^2$ by Kortum in an interesting article [44]. In order to deal with convergence of the most difficult terms $\nabla\mathbf{d}_\varepsilon\odot\nabla\mathbf{d}_\varepsilon$ in the limit process, Kortum employed the concentration-cancellation method for the Euler equation developed by DiPerna and Majda [45] (see also [46]). Thanks to the rotational covariance of $\nabla\mathbf{d}_\varepsilon \odot \nabla\mathbf{d}_\varepsilon$, the test functions can be taken to a function of periodic one spatial variable ensuring the weak convergence of $\nabla\mathbf{d}_\varepsilon \odot \nabla\mathbf{d}_\varepsilon$ to $\nabla\mathbf{d} \odot \nabla\mathbf{d}$.

In this section, we make some new observations on the Ericksen stress tensor $\nabla\mathbf{v} \odot \nabla\mathbf{v}$, which is flexible enough to handle any smooth domain $\Omega \subset \mathbb{R}^2$. Namely, by adding $-\frac{1}{2}|\nabla\mathbf{v}^\varepsilon|^2\mathbb{I}_2$ to $\nabla\mathbf{v}^\varepsilon \odot \nabla\mathbf{v}^\varepsilon$, where $\mathbb{I}_2$ is the 2×2 identity matrix, we have

$$\nabla\mathbf{v}^\varepsilon \odot \nabla\mathbf{v}^\varepsilon - \frac{1}{2}|\nabla\mathbf{v}^\varepsilon|^2\mathbb{I}_2 = \frac{1}{2}\left(\begin{array}{ll} |\partial_{x_1}\mathbf{v}^\varepsilon|^2 - |\partial_{x_2}\mathbf{v}^\varepsilon|^2, & 2\langle\partial_{x_1}v^\varepsilon, \partial_{x_2}\mathbf{v}^\varepsilon\rangle \\ 2\langle\partial_{x_1}\mathbf{v}^\varepsilon, \partial_{x_2}\mathbf{v}^\varepsilon\rangle, & |\partial_{x_2}\mathbf{v}^\varepsilon|^2 - |\partial_{x_1}\mathbf{v}^\varepsilon|^2 \end{array}\right).$$

This is a matrix whose components constitute the Hopf differential of map $\mathbf{v}^\varepsilon$, which are $|\partial_{x_1}\mathbf{v}^\varepsilon|^2 - |\partial_{x_2}\mathbf{v}^\varepsilon|^2$ and $\langle\partial_{x_1}\mathbf{v}^\varepsilon, \partial_{x_2}\mathbf{v}^\varepsilon\rangle$. Since $\mathbf{v}^\epsilon$ is either an approximated harmonic map to $\mathcal{N}$ or a Ginzburg–Landau approximated harmonic map, we can develop its compensated compactness property by the Pohozaev type argument.

As a byproduct of the proof of Theorem 2.1.1, we obtain the following compactness for a sequence of weak solutions to the system (2.1.1).

Theorem 2.1.2 ([43]). *Let $(\mathbf{u}^k, \mathbf{v}^k) : \Omega \times (0,T) \to \mathbb{R}^2 \times \mathcal{N}$ be a sequence of weak solutions to (2.1.1), along with the initial and boundary condition $(\mathbf{u}_0^k, \mathbf{v}_0^k)$ satisfying (2.1.4), such that*

$$\sup_{k\geq 1}\left\{\int_{Q_t}\left(|\mathbf{u}^k|^2 + |\nabla\mathbf{v}^k|^2\right) + \int_{Q_t}\left(|\nabla\mathbf{u}^k|^2 + |\partial_t\mathbf{v}^k + \mathbf{u}^k\cdot\nabla\mathbf{v}^k|^2\right)\right\} < \infty, \tag{2.1.7}$$

Furthermore, if we assume that

$$(\mathbf{u}_0^k, \mathbf{v}_0^k) \rightharpoonup (\mathbf{u}_0, \mathbf{v}_0) \quad in \quad L^2(\Omega) \times H^1(\Omega),$$

then there exists a weak solution $(\mathbf{u}, \mathbf{v})$ of (2.1.1) with the initial and boundary condition $(\mathbf{u}_0, \mathbf{v}_0)$ such that, after passing to subsequences,

$$\mathbf{u}^k \rightharpoonup \mathbf{u} \ in \ L^2([0,T], H^1(\Omega)), \qquad \mathbf{v}^k \rightharpoonup v \ in \ L^2([0,T], H^1(\Omega)).$$

Since the system (2.1.1) possesses the geometric structure, i.e.,

$$A(\mathbf{v}^k)(\cdot,\cdot) \perp T_{\mathbf{v}^k}\mathcal{N}$$

where $T_{\mathbf{v}^k}\mathcal{N}$ is the tangent space of $\mathcal{N}$ at $\mathbf{v}^k$, we can show the weak convergence of $|\partial_{x_1}\mathbf{v}^k|^2 - |\partial_{x_2}\mathbf{v}^k|^2$ and $\langle \partial_{x_1}\mathbf{v}^k, \partial_{x_2}\mathbf{v}^k\rangle$ by utilizing the L^p-estimate, $1 < p < 2$, of the Hopf differential of $\mathbf{v}^k$.

2.1.2 Estimates on inhomogeneous Ginzburg–Landau equations

In this section, we will consider the inhomogeneous Ginzburg–Landau equation

$$\Delta\mathbf{v}^\varepsilon - \frac{1}{\varepsilon^2}\chi\Big(\operatorname{dist}^2(\mathbf{v}^\varepsilon,\mathcal{N})\Big)\frac{d}{dv}\Big(\operatorname{dist}^2(\mathbf{v}^\varepsilon,\mathcal{N})\Big) = \tau_\varepsilon \quad \text{in} \quad \Omega. \tag{2.1.8}$$

Suppose

$$\sup_{0<\varepsilon\le 1}\mathcal{E}_\varepsilon(\mathbf{v}^\varepsilon) = \int_\Omega \left(\frac{1}{2}|\nabla\mathbf{v}^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi(\operatorname{dist}^2(\mathbf{v}^\varepsilon,\mathcal{N}))\right) \le \Lambda_1 < \infty, \tag{2.1.9}$$

and

$$\sup_{0<\varepsilon\le 1}\|\tau^\varepsilon\|_{L^2(\Omega)} \le \Lambda_2 < \infty. \tag{2.1.10}$$

Assume that there exist $\mathbf{v} \in H^1(\Omega,\mathcal{N})$ and $\tau \in L^2(\Omega,\mathbb{R}^L)$ such that

$$\tau^\varepsilon \to \tau \text{ in } L^2(\Omega), \quad \mathbf{v}^\varepsilon \rightharpoonup \mathbf{v} \text{ in } H^1(\Omega).$$

Then we have

Lemma 2.1.3. *There exists $\delta_0 > 0$ such that if $\mathbf{v}^\varepsilon \in H^1(\Omega,\mathbb{R}^L)$ is a family of solutions to (2.1.8) satisfying (2.1.9) and (2.1.10), and for $x_0 \in \Omega$ and $0 < r_0 < \operatorname{dist}(x_0,\partial\Omega)$,*

$$\sup_{0<\varepsilon\le 1}\int_{B_{r_0}(x_0)} \left(\frac{1}{2}|\nabla\mathbf{v}^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi\Big(\operatorname{dist}^2(\mathbf{v}^\varepsilon,\mathcal{N})\Big)\right) \le \delta_0^2, \tag{2.1.11}$$

then there exists an approximated harmonic map $v \in H^1(B_{\frac{r_0}{4}}(x_0), \mathcal{N})$ *with tension filed* τ*, i.e,*

$$\Delta \mathbf{v} + A(\mathbf{v})(\nabla \mathbf{v}, \nabla \mathbf{v}) = \tau, \tag{2.1.12}$$

such that as $\epsilon \to 0$*,*

$$\mathbf{v}^\varepsilon \to \mathbf{v} \quad \textit{in} \;\; H^1(B_{\frac{r_0}{4}}(x_0)), \;\; \text{and} \;\; \frac{1}{\varepsilon^2}\chi\Big(\operatorname{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})\Big) \to 0 \;\; \text{in} \;\; L^1(B_{\frac{r_0}{4}}(x_0)). \tag{2.1.13}$$

Proof. For any fixed $x_1 \in B_{\frac{r_0}{2}}(x_0)$ and $0 < \varepsilon \leq \dfrac{r_0}{2}$, define $\widehat{\mathbf{v}}^\varepsilon(x) = \mathbf{v}^\varepsilon(x_1 + \varepsilon x) : B_1(0) \to \mathbb{R}^L$. Then we have

$$\Delta \widehat{\mathbf{v}}^\varepsilon = \chi(\operatorname{dist}^2(\widehat{\mathbf{v}}^\varepsilon, \mathcal{N}))\frac{d}{d\mathbf{v}}(\operatorname{dist}^2(\widehat{\mathbf{v}}^\varepsilon, \mathcal{N})) + \widehat{\tau}^\varepsilon \text{ in } B_1(0),$$

where $\widehat{\tau}^\varepsilon(x) = \varepsilon^2 \tau^\varepsilon(x_1 + \varepsilon x)$. Since

$$\begin{aligned}
\|\Delta \widehat{\mathbf{v}}^\varepsilon\|_{L^2(B_1(0))} &\leq \Big\|\chi(\operatorname{dist}(\widehat{\mathbf{v}}^\varepsilon, \mathcal{N}))\frac{d}{d\mathbf{v}}(\operatorname{dist}^2(\widehat{\mathbf{v}}^\varepsilon, \mathcal{N}))\Big\|_{L^2(B_1(0))} + \|\widehat{\tau}^\varepsilon\|_{L^2(B_1(0))} \\
&\leq C\Big(\int_{\Omega \cap \{\operatorname{dist}(\mathbf{v}^\epsilon, \mathcal{N}) \leq 2\delta_{\mathcal{N}}\}} |\operatorname{dist}(\mathbf{v}^\epsilon, \mathcal{N})|^2\Big)^{\frac{1}{2}} + \varepsilon \, \|\tau^\varepsilon\|_{L^2(\Omega)} \leq C + \Lambda_2.
\end{aligned}$$

Thus $\widehat{\mathbf{v}}^\varepsilon \in H^2(B_{\frac{1}{2}})$ and $\|\widehat{\mathbf{v}}^\varepsilon\|_{H^2(B_{\frac{1}{2}})} \leq C(1 + \Lambda_2)$. By Morrey's inequality, we conclude that $\widehat{\mathbf{v}}^\varepsilon \in C^{\frac{1}{2}}(B_{\frac{1}{2}})$ and

$$[\widehat{\mathbf{v}}^\varepsilon]_{C^{\frac{1}{2}}(B_{\frac{1}{2}})} \leq C\,\|\widehat{\mathbf{v}}^\varepsilon\|_{H^2(B_{\frac{1}{2}})} \leq C(1 + \Lambda_2).$$

By rescaling, we get

$$|\widehat{\mathbf{v}}^\varepsilon(x) - \widehat{\mathbf{v}}^\varepsilon(y)| \leq C(1 + \Lambda_2)\Big(\frac{|x - y|}{\varepsilon}\Big)^{\frac{1}{2}}, \qquad \forall x, y \in B_\varepsilon(x_1).$$

We claim that $\operatorname{dist}(\mathbf{v}^\varepsilon, \mathcal{N}) \leq \delta_{\mathcal{N}}$ on $B_{\frac{r_0}{2}}(x_0)$. Suppose it were false. Then there exists $x_1 \in B_{\frac{r_0}{2}}(x_0)$ such that $\operatorname{dist}(\mathbf{v}^\varepsilon(x_1), \mathcal{N}) > \delta_{\mathcal{N}}$. Then for any $\theta_0 \in (0, 1)$ and $x \in B_{\theta_0\varepsilon}(x_1)$, it holds

$$|\mathbf{v}^\varepsilon(x) - \mathbf{v}^\varepsilon(x_1)| \leq C\Big(\frac{|x - x_1|}{\varepsilon}\Big)^{\frac{1}{2}} \leq C\theta_0^{\frac{1}{2}} \leq \frac{1}{2}\delta_{\mathcal{N}},$$

provided $\theta_0 \le \frac{\delta_{\mathcal{N}}^2}{4C^2}$. It follows that

$$\operatorname{dist}(\mathbf{v}^\varepsilon(x), \mathcal{N}) \ge \frac{1}{2}\delta_{\mathcal{N}},\ \forall x \in B_{\theta_0\varepsilon}(x_1),$$

so that

$$\int_{B_{\theta_0\varepsilon}(x_1)} \frac{1}{\varepsilon^2}\chi(\operatorname{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})) \ge \boldsymbol{\pi}\delta_{\mathcal{N}}^2\theta_0^2.$$

which contradicts to the assumption that

$$\int_{B_{\theta_0\varepsilon}(x_1)} \frac{1}{\varepsilon^2}\chi(\operatorname{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})) \le \int_{B_{r_1}(0)} \left(\frac{1}{2}|\nabla \mathbf{v}^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi\left(\operatorname{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})\right)\right) \le \delta_0^2$$

for a sufficiently small $\delta_0 > 0$.

From $\operatorname{dist}(\mathbf{v}^\varepsilon, \mathcal{N}) \le \delta_{\mathcal{N}}$ in $B_{\frac{r_0}{2}}(x_0)$, we may decompose $\mathbf{v}^\varepsilon$ into

$$\mathbf{v}^\varepsilon = \Pi_{\mathcal{N}}(\mathbf{v}^\varepsilon) + \operatorname{dist}(\mathbf{v}^\varepsilon, \mathcal{N})\nu\Big(\Pi_{\mathcal{N}}(\mathbf{v}^\varepsilon)\Big) := \omega_\varepsilon + \zeta_\varepsilon\nu_\varepsilon,$$

so that the equation of $\mathbf{v}_\varepsilon$ becomes

$$\Delta\omega_\varepsilon + \Delta\zeta_\varepsilon\nu_\varepsilon + 2\nabla\zeta_\varepsilon\nabla\nu_\varepsilon + \zeta_\varepsilon\Delta\nu_\varepsilon - \frac{1}{\varepsilon^2}\chi(\zeta_\varepsilon^2)\nabla_{v_\varepsilon}\zeta_\varepsilon^2 = \tau_\varepsilon. \tag{2.1.14}$$

Multiplying (2.1.14) by ν_ε, we get

$$\Delta\zeta_\varepsilon = \langle\nabla\omega_\varepsilon, \nabla\nu_\varepsilon\rangle + \zeta_\varepsilon|\nabla\nu_\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi(\zeta_\varepsilon^2)\langle\nabla_{\mathbf{v}_\varepsilon}\zeta_\varepsilon^2, \nu_\varepsilon\rangle + \tau_\varepsilon^\perp, \tag{2.1.15}$$

where $\tau_\varepsilon^\perp = \langle\tau_\varepsilon, \nu_\varepsilon\rangle$. Plugging $\Delta\zeta_\varepsilon$ into (2.1.14), we obtain

$$\Delta\omega_\varepsilon + \langle\nabla\omega_\varepsilon, \nabla\nu_\varepsilon\rangle\nu_\varepsilon + \zeta_\varepsilon\Big(\Delta\nu_\varepsilon + |\nabla\nu_\varepsilon|^2\nu_\varepsilon\Big) + 2\langle\nabla\nu_\varepsilon, \nabla\zeta_\varepsilon\rangle = \tau_\varepsilon, \tag{2.1.16}$$

where $\tau_\varepsilon = \tau_\varepsilon - \tau_\varepsilon^\perp\nu_\varepsilon$. Here we have used the fact

$$\langle\nabla_{v_\varepsilon}\zeta_\varepsilon^2, \nu_\varepsilon\rangle\nu_\varepsilon = \nabla_{v_\varepsilon}\zeta_\varepsilon^2.$$

Let $\eta \in C_0^\infty(B_{\frac{r_0}{2}}(x_0), \mathbb{R})$ be a standard cutoff function of $B_{\frac{3r_0}{8}}(x_0)$. Since $\mathrm{dist}(\mathbf{v}^\varepsilon, \mathcal{N}) \leq \delta_{\mathcal{N}}$, we have that $\chi(\zeta_\varepsilon^2) = 1$ and hence

$$
\begin{aligned}
\left(-\Delta + \frac{2}{\varepsilon^2}\right)(\zeta_\varepsilon \eta^2) =& - \zeta_\varepsilon \Delta(\eta^2) - 2\nabla\zeta_\varepsilon \nabla(\eta^2) + \langle \nabla\omega_\varepsilon, \nabla(\nu_\varepsilon \eta^2)\rangle - \langle \nabla\omega_\varepsilon, \nu_\varepsilon \nabla(\eta^2)\rangle \\
&+ \zeta_\varepsilon \left(|\nabla(\nu_\varepsilon \eta^2)|^2 - |\nu_\varepsilon \nabla(\eta^2)|^2\right) + \tau_\varepsilon^\perp \eta^2.
\end{aligned}
\tag{2.1.17}
$$

Applying the $W^{2,\frac{4}{3}}$-estimate for $(-\Delta + \dfrac{2}{\varepsilon^2})$ (see [47]), we obtain

$$
\begin{aligned}
&\|\nabla^2(\zeta_\varepsilon \eta^2)\|_{L^{\frac{4}{3}}} \\
\lesssim& \|\zeta_\varepsilon \Delta(\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla\zeta_\varepsilon \nabla(\eta^2)\|_{L^{\frac{4}{3}}} + \|\nabla\omega_\varepsilon\|_{L^2} \|\nabla(\nu_\varepsilon \eta^2)\|_{L^4} \\
&+ \|\nabla\omega_\varepsilon\|_{L^2} + \|\zeta_\varepsilon\|_{L^\infty} \|\nabla\nu_\varepsilon\|_{L^2} \|\nabla(\nu_\varepsilon \eta^2)\|_{L^4} + \|\tau_\varepsilon^\perp\|_{L^{\frac{4}{3}}} \\
\lesssim& \|\zeta_\varepsilon\|_{L^\infty} + \|\nabla\zeta_\varepsilon\|_{L^2} + \|\nabla\omega_\varepsilon\|_{L^2}(\|\nabla(\nu_\varepsilon \eta^2)\|_{L^4} + 1) \\
&+ \|\zeta_\varepsilon\|_{L^\infty} \|\nabla\nu_\varepsilon\|_{L^2} \|\nabla(\nu_\varepsilon \eta^2)\|_{L^4} + \|\tau_\varepsilon\|_{L^2},
\end{aligned}
\tag{2.1.18}
$$

where $A \lesssim B$ stands for $A \leq CB$ for some universal positive constant C.

For ω_ε, by a similar calculation we obtain

$$
\begin{aligned}
\Delta(\omega_\varepsilon \eta^2) =& - \langle \nabla\omega_\varepsilon, \nabla(\nu_\varepsilon \eta^2)\rangle \nu_\varepsilon + \langle \nabla\omega_\varepsilon, \nu_\varepsilon \nabla(\eta^2)\rangle \nu_\varepsilon \\
&- \zeta_\varepsilon \Big[\Delta(\nu_\varepsilon \eta^2) - \nu_\varepsilon \Delta(\eta^2) - 2\nabla\nu_\varepsilon \nabla(\eta^2)\Big] + \zeta_\varepsilon \Big[|\nabla\nu_\varepsilon \eta^2|^2 - |\nu_\varepsilon \nabla(\eta^2)|^2\Big]\nu_\varepsilon \\
&- 2\Big[\langle \nabla(\nu_\varepsilon \eta^2), \nabla\zeta_\varepsilon\rangle - \langle \nabla(\eta^2), \nabla\zeta_\varepsilon\rangle \nu_\varepsilon\Big] + \tau_\varepsilon \eta^2 + \omega_\varepsilon \Delta(\eta^2) + 2\nabla\omega_\varepsilon \nabla(\eta^2).
\end{aligned}
\tag{2.1.19}
$$

Applying the $W^{2,\frac{4}{3}}$-estimate, we obtain

$$
\begin{aligned}
&\|\nabla^2(\omega_\varepsilon \eta^2)\|_{L^{\frac{4}{3}}} \\
\lesssim& \|\nabla\omega_\varepsilon\|_{L^2} \|\nabla(\nu_\varepsilon \eta^2)\|_{L^4} + \|\nabla\omega_\varepsilon\|_{L^{\frac{4}{3}}} + \|\zeta_\varepsilon\|_{L^\infty} \|\Delta(\nu_\varepsilon \eta^2)\|_{L^{\frac{4}{3}}} \\
&+ \|\zeta_\varepsilon\|_{L^\infty} \left(1 + \|\nabla\nu_\varepsilon\|_{L^{\frac{4}{3}}}\right) + \|\zeta_\varepsilon\|_{L^\infty} \|\nabla(\nu_\varepsilon \eta^2)\|_{L^2} \|\nabla(\nu_\varepsilon \eta^2)\|_{L^4} \\
&+ \|\nabla\zeta_\varepsilon\|_{L^2} \|\nabla(\nu_\varepsilon \eta^2)\|_{L^4} + \|\nabla\zeta_\varepsilon\|_{L^{\frac{4}{3}}} + \|\tau_\varepsilon\|_{L^2}.
\end{aligned}
\tag{2.1.20}
$$

Therefore, we conclude that

$$\begin{aligned} &\left\|\nabla^2(\zeta_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} + \left\|\nabla^2(\omega_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} \\ \lesssim &\|\nabla\mathbf{v}^\varepsilon\|_{L^2}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} + \|\zeta_\varepsilon\|_{L^\infty}\left\|\nabla^2(\nu_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} + \|\nabla\mathbf{v}^\varepsilon\|_{L^2} + \|\tau_\varepsilon\|_{L^2}. \end{aligned} \tag{2.1.21}$$

Since

$$v^\varepsilon\eta^2 = \omega_\varepsilon\eta^2 + \zeta_\varepsilon\nu_\varepsilon\eta^2$$

we have

$$\begin{aligned} &\left\|\nabla^2(\mathbf{v}^\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} \\ \lesssim &\left\|\nabla^2(\zeta_\varepsilon\nu_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} + \left\|\nabla^2(\omega_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} \\ \lesssim &\|\zeta_\varepsilon\|_{L^\infty}\left\|\nabla^2(\nu_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} + \|\nabla\zeta_\varepsilon\|_{L^2}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} + \left\|\nabla^2\zeta_\varepsilon\nu_\varepsilon\eta^2\right\|_{L^{\frac{4}{3}}} + \left\|\nabla^2(\omega_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} \\ \lesssim &\|\zeta_\varepsilon\|_{L^\infty}\left\|\nabla^2(\nu_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} + \|\nabla\zeta_\varepsilon\|_{L^2}\|\nabla(\nu_\varepsilon\eta^2)\|_{L^4} + \left\|\nabla^2(\zeta_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} \\ &+ \|\nabla\zeta_\varepsilon\|_{L^2} + \left\|\nabla^2(w_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} + 1. \end{aligned} \tag{2.1.22}$$

Therefore, we have

$$\begin{aligned} &\left\|\nabla^2(\mathbf{v}^\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} \\ \lesssim &\|\zeta_\varepsilon\|_{L^\infty}\left\|\nabla^2(\nu_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} + \|\nabla\mathbf{v}^\varepsilon\|_{L^2}\left[1 + \|\nabla(v^\varepsilon\eta^2)\|_{L^4} + \|\nabla(\nu_\varepsilon\eta^2)\|_{L^4}\right] + \|\tau_\varepsilon\|_{L^2} + 1 \\ \lesssim &\|\zeta_\varepsilon\|_{L^\infty}\left\|\nabla^2(\nu_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} + \|\nabla\mathbf{v}^\varepsilon\|_{L^2}\left[1 + \|\nabla(\mathbf{v}^\varepsilon\eta^2)\|_{L^4}\right] + \|\tau_\varepsilon\|_{L^2} + 1. \end{aligned} \tag{2.1.23}$$

Since $\nu_\varepsilon = \nu_\varepsilon(\mathbf{v}^\varepsilon)$, we can directly calculate and show that

$$\left\|\nabla^2(\nu_\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} \lesssim \left\|\nabla^2(\mathbf{v}^\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} + \|\nabla\mathbf{v}^\varepsilon\|_{L^2}\left[1 + \|\nabla(\mathbf{v}^\varepsilon\eta^2)\|_{L^4}\right] + 1. \tag{2.1.24}$$

Therefore, we can conclude that

$$\left(1 - C\|\zeta_\varepsilon\|_{L^\infty}\right)\left\|\nabla^2(\mathbf{v}^\varepsilon\eta^2)\right\|_{L^{\frac{4}{3}}} \lesssim \|\nabla\mathbf{v}^\varepsilon\|_{L^2}\left[1 + \|\nabla(\mathbf{v}^\varepsilon\eta^2)\|_{L^4}\right] + \|\tau_\varepsilon\|_{L^2} + 1. \tag{2.1.25}$$

By Sobolev's embedding, we have

$$\|\nabla(\mathbf{v}^\varepsilon\eta^2)\|_{L^4} \lesssim \|\nabla\mathbf{v}^\varepsilon\|_{L^2}\Big[1+\|\nabla(\mathbf{v}^\varepsilon\eta^2)\|_{L^4}\Big] + \|\tau_\varepsilon\|_{L^2} + 1. \tag{2.1.26}$$

Taking δ_0 small enough in the assumption (2.1.11), we conclude that

$$\|\nabla(\mathbf{v}^\varepsilon\eta^2)\|_{L^4} \lesssim \|\nabla\mathbf{v}^\varepsilon\|_{L^2} + \|\tau_\varepsilon\|_{L^2} + 1 \le C(\delta_0,\Lambda_2). \tag{2.1.27}$$

Substituting this into (2.19), we obtain which implies that

$$\|\nabla^2(\mathbf{v}_\varepsilon\eta^2)\|_{L^{\frac{4}{3}}} \le C(\delta_0,\Lambda_2). \tag{2.1.28}$$

Hence $\mathbf{v}^\varepsilon \to \mathbf{v}$ in $H^1(B_{\frac{r_0}{3}}(x_0))$.

By Fubini's theorem, there exists $r_1 \in [\dfrac{r_0}{4}, \dfrac{r_0}{3}]$

$$\int_{\partial B_{r_1}(x_0)} |\nabla\zeta_\varepsilon|^2 \le C\int_{B_{\frac{r_0}{3}}(x_0)} |\nabla\zeta_\varepsilon|^2 \le C, \quad \int_{\partial B_{r_1}(x_0)} |\zeta_\varepsilon|^2 \le C\int_{B_{\frac{r_0}{3}}(x_0)} |\zeta_\varepsilon|^2 \le C\varepsilon^2. \tag{2.1.29}$$

Multiplying the equation of ζ_ε by ζ_ε and integrating by parts over B_{r_2}, we obtain

$$\int_{B_{r_1}(x_0)}\Big(|\nabla\zeta_\varepsilon|^2+\frac{2}{\varepsilon^2}\chi(\zeta_\varepsilon)\zeta_\varepsilon^2+|\nabla\nu_\varepsilon|^2\zeta_\varepsilon^2+\nabla\omega_\varepsilon\nabla\nu_\varepsilon\cdot\zeta_\varepsilon\Big)-\int_{\partial B_{r_1}(x_0)}\frac{\partial\zeta_\varepsilon}{\partial\nu}\zeta_\varepsilon = \int_{B_{r_1}(x_0)}\tau_\varepsilon^\perp\zeta_\varepsilon \tag{2.1.30}$$

Then we have

$$\begin{aligned}
&\int_{B_{r_1}(x_0)}\Big(|\nabla\zeta_\varepsilon|^2+\frac{2}{\varepsilon^2}\zeta_\varepsilon^2\Big)\\
\le& C\Big(\int_{\partial B_{r_1}(x_0)}|\nabla\zeta_\varepsilon|^2\Big)^{\frac12}\Big(\int_{\partial B_{r_1}(x_0)}|\zeta_\varepsilon|^2\Big)^{\frac12} + C\Big(\int_{B_{r_1}(x_0)}|\nabla\omega_\varepsilon|^4\Big)^{\frac12}\Big(\int_{B_{r_1}(x_0)}|\zeta_\varepsilon|^4\Big)^{\frac12}\\
&+C\Big(\int_{B_{r_1}(x_0)}|\tau_\varepsilon|^2\Big)^{\frac12}\Big(\int_{B_{r_1}(x_0)}|\zeta_\varepsilon|^2\Big)^{\frac12} \le C\varepsilon.
\end{aligned} \tag{2.1.31}$$

Therefore we have that

$$\frac{\zeta_\varepsilon^2}{\varepsilon^2} \to 0 \quad\text{in}\quad L^1(B_{r_1}(x_0)). \tag{2.1.32}$$

This completes the proof. $\square$

Now we define the concentration set by

$$\Sigma := \bigcap_{r>0} \Big\{ x \in \Omega : \liminf_{k\to\infty} \int_{B_r(x)} \Big(\frac{1}{2}|\nabla v^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi\Big(\operatorname{dist}^2(v^\varepsilon, \mathcal{N})\Big)\Big) > \delta_0^2 \Big\}, \tag{2.1.33}$$

where $\delta_0 > 0$ is given in Lemma 2.1.3. We have

Lemma 2.1.4. *Σ is a finite set, and*

$$\mathbf{v}^\varepsilon \to \mathbf{v} \quad \text{in} \quad H^1_{\mathrm{loc}}(\Omega \setminus \Sigma). \tag{2.1.34}$$

The finiteness of Σ follows from a simple covering argument, see also [22] and [44].

2.1.3 Convergence of Ginzburg–Landau approximation

The section is devoted to the proof of Theorem 2.1.1. First, recall from the global energy inequality for (2.1.2) that for almost every $t \in (0, T)$,

$$\int_{\Omega\times\{t\}} \Big(|\mathbf{u}^\varepsilon|^2 + |\nabla \mathbf{v}^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi\Big(\operatorname{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})\Big)\Big) + 2\int_{Q_t} \Big(|\nabla \mathbf{u}^\varepsilon|^2 + |\partial_t \mathbf{v}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{v}^\varepsilon|^2\Big) \le E_0. \tag{2.1.35}$$

This, combined with the equation (2.1.2), implies that there exists $p > 2$ such that

$$\sup_{\varepsilon>0} \Big[\|\mathbf{u}^\varepsilon_t\|_{L^2_t H^{-1}_x + L^2_t W^{-2,p}} + \|\mathbf{v}^\varepsilon_t\|_{L^{4/3}_t L^{4/3}_x}\Big] < \infty. \tag{2.1.36}$$

Hence, by Aubin–Lions' Lemma, there exists $\mathbf{u} \in L^\infty_t L^2_x \cap L^2_t H^1_x(\Omega \times (0,T), \mathbb{R}^2)$ and $\mathbf{v} \in L^\infty_t H^1_x \cap L^\infty_t H^1_x(\Omega \times (0,T), \mathcal{N})$ such that after taking a subsequence,

$$(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon) \to (\mathbf{u}, \mathbf{v}) \text{ in } L^2(\Omega \times (0,T)), \quad (\nabla \mathbf{u}^\varepsilon, \nabla \mathbf{v}^\varepsilon) \rightharpoonup (\nabla \mathbf{u}, \nabla \mathbf{v}) \text{ in } L^2(\Omega \times (0,T)).$$

Combining this with (2.1.35), we obtain

$$\partial_t \mathbf{v}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{v}^\varepsilon \rightharpoonup \partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v} \ \text{ in } \quad L^2(\Omega \times (0,T)).$$

By the lower semi-continuity, we have

$$\int_{Q_t} (|\nabla \mathbf{u}|^2 + |\partial_t \mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v}|^2) \le \liminf_{\varepsilon\to 0} \int_{Q_t} (|\nabla \mathbf{u}^\varepsilon|^2 + |\partial_t \mathbf{v}^\varepsilon + \mathbf{u}^\varepsilon\cdot\nabla\mathbf{v}^\varepsilon|^2) < \infty. \tag{2.1.37}$$

By Fatou's Lemma, we have

$$\int_0^t \liminf_{\varepsilon\to 0} \int_\Omega (|\nabla \mathbf{u}^\varepsilon|^2 + |\mathbf{v}_t^\varepsilon + \mathbf{u}^\varepsilon\cdot\nabla\mathbf{v}^\varepsilon|^2) \le \liminf_{\varepsilon\to 0} \int_0^t \int_\Omega (|\nabla \mathbf{u}^\varepsilon|^2 + |\partial_t \mathbf{v}^\varepsilon + \mathbf{u}^\varepsilon\cdot\nabla\mathbf{v}^\varepsilon|^2) \le E_0. \tag{2.1.38}$$

Hence there exists $A \subset [0,T]$ with full Lebesgue measure T such that for any $t \in A$

$$\Big(\mathbf{u}^\varepsilon(t), \mathbf{v}^\varepsilon(t)\Big) \rightharpoonup (\mathbf{u}(t), \mathbf{v}(t)) \quad \text{in} \quad L^2 \times H^1 \tag{2.1.39}$$

and

$$\liminf_{\varepsilon\to 0^+} \int_\Omega \Big(|\nabla \mathbf{u}^\varepsilon|^2 + |\partial_t \mathbf{v}^\varepsilon + \mathbf{u}^\varepsilon\cdot\nabla\mathbf{v}^\varepsilon|^2\Big)(t) < \infty. \tag{2.1.40}$$

Now we define the concentration set at t by

$$\Sigma_t := \bigcap_{r>0} \Big\{x \in \Omega : \liminf_{\varepsilon\to 0} \int_{B_r(x)\times\{t\}} \frac{1}{2}|\nabla \mathbf{v}^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi\Big(\operatorname{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})\Big) > \delta_0^2\Big\}, \tag{2.1.41}$$

where δ_0 is given by Lemma (2.1.3). By Lemma 2.1.4, it holds $\#(\Sigma_t) \le C(E_0)$ and

$$\mathbf{v}^\varepsilon(t) \to \mathbf{v}(t) \text{ in } H^1_{\mathrm{loc}}(\Omega \setminus \Sigma(t)).$$

We would first show that v is a weak solution of $(2.1.1)_3$ by utilizing the geometric structure as in [48] (see also [49]). First notice that there exists a unit vector $\nu_\mathcal{N}^\varepsilon \perp T_{\Pi_\mathcal{N}(\mathbf{v}^\varepsilon)}\mathcal{N}$ such that

$$\frac{d}{d\mathbf{v}}\chi\Big(\operatorname{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})\Big) = 2\chi(\operatorname{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N}))\operatorname{dist}(\mathbf{v}^\varepsilon, \mathcal{N})\nu_\mathcal{N}^\varepsilon.$$

Thus for any $\phi \in C_0^\infty(\Omega, \mathbb{R}^L)$ and a.e. $t \in (0,\infty)$ it holds

$$\int_{\Omega\times\{t\}} \langle \mathbf{v}_t^\varepsilon + \mathbf{u}^\varepsilon\cdot\nabla\mathbf{v}^\varepsilon - \Delta\mathbf{v}^\varepsilon, D\Pi_\mathcal{N}(\Pi_\mathcal{N}(\mathbf{v}^\varepsilon))\phi\rangle = 0.$$

If we choose $\phi \in C_0^\infty(\Omega \setminus \Sigma_t)$, then it follows from $\nabla \mathbf{v}^\varepsilon \to \nabla \mathbf{v}$ in $H^1_{\mathrm{loc}}(\Omega \setminus \Sigma_t)$ that, after passing to the limit of the above equation,

$$\int_{\Omega\times\{t\}} \langle \partial_t \mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v}, D\Pi_{\mathcal{N}}(\mathbf{v})\phi\rangle = -\int_{\Omega\times\{t\}} \langle \nabla \mathbf{v}, \nabla(D\Pi_{\mathcal{N}}(\mathbf{v}))\phi\rangle.$$

This implies that

$$\partial_t \mathbf{v} + \mathbf{u}\cdot\nabla\mathbf{v} - \Delta\mathbf{v} = A_{\mathcal{N}}(\mathbf{v})(\nabla\mathbf{v}, \nabla\mathbf{v})$$

holds weakly in $\Omega \setminus \Sigma_t$. Since Σ_t is a finite set, it also holds weakly in Ω so that $(2.1.1)_3$ holds.

Now, we proceed to verify $\mathbf{u}$ satisfies $(2.1.1)_1$. First by the estimate (2.1.36), we have

$$\partial_t \mathbf{u}^\varepsilon \rightharpoonup \partial_t \mathbf{u}, \quad \text{in } L^2([0,T], H^{-1}) \cap L^2([0,T], W^{-2,p})$$

for some $p > 2$. For any $\xi \in C^\infty([0,T])$ with $\xi(T) = 0$, $\varphi \in \mathbf{J}$, since

$$\int_{Q_T} \partial_t \mathbf{u}^\varepsilon \xi\varphi = -\int_\Omega \mathbf{u}_0 \xi(0)\varphi - \int_{Q_T} \mathbf{u}^\varepsilon \xi\varphi,$$

which, after taking $\epsilon \to 0$, implies that

$$\int_{Q_T} \partial_t \mathbf{u} \xi\varphi = -\int_\Omega \mathbf{u}_0 \xi(0)\varphi - \int_{Q_T} \mathbf{u} \xi\varphi.$$

Claim: For any $t \in A$, it holds

$$\begin{aligned} 0 &= \int_{\Omega\times\{t\}} \langle \partial_t \mathbf{u}^\varepsilon, \varphi\rangle + \int_{\Omega\times\{t\}} \langle \mathbf{u}^\varepsilon\cdot\nabla\mathbf{u}^\varepsilon, \varphi\rangle + \int_{\Omega\times\{t\}} \langle \nabla\mathbf{u}^\varepsilon, \nabla\varphi\rangle + \int_{\Omega\times\{t\}} (\nabla\mathbf{v}^\varepsilon \odot \nabla\mathbf{v}^\varepsilon) : \nabla\varphi \\ &\to \int_{\Omega\times\{t\}} \langle \partial_t \mathbf{u}, \varphi\rangle + \int_{\Omega\times\{t\}} \langle \mathbf{u}\cdot\nabla\mathbf{u}, \varphi\rangle + \int_{\Omega\times\{t\}} \langle \nabla\mathbf{u}, \nabla\varphi\rangle + \int_{\Omega\times\{t\}} (\nabla\mathbf{v} \odot \nabla\mathbf{v}) : \nabla\varphi, \end{aligned} \tag{2.1.42}$$

for any $\varphi \in \mathbf{J}$.

For this claim, it suffices to show the convergence of Ericksen stress tensors, i.e.,

$$\int_{\Omega\times\{t\}} (\nabla \mathbf{v}^\varepsilon \odot \nabla \mathbf{v}^\varepsilon) : \nabla\varphi = \int_{\Omega\times\{t\}} (\nabla v \odot \nabla \mathbf{v}) : \nabla\varphi.$$

For simplicity, we assume $\Sigma_t = \{(0,0)\} \subset \Omega$ consists of a single point at zero. Let $\varphi \in C^\infty(\Omega, \mathbb{R}^2)$ be such that $\operatorname{div}\varphi = 0$ and $(0,0) \in \operatorname{spt}(\varphi)$. Then we observe that by adding $-\dfrac{1}{2}|\nabla \mathbf{v}^\varepsilon|^2 \mathbb{I}_2$, we have

$$\int_{\Omega\times\{t\}} (\nabla \mathbf{v}^\varepsilon \odot \nabla \mathbf{v}^\varepsilon) : \nabla\varphi = \int_{\Omega\times\{t\}} \left(\nabla \mathbf{v}^\varepsilon \odot \nabla \mathbf{v}^\varepsilon - \frac{1}{2}|\nabla \mathbf{v}^\varepsilon|^2 \mathbb{I}_2\right) : \nabla\varphi.$$

While by direct computations, we have

$$\nabla \mathbf{v}^\varepsilon \odot \nabla \mathbf{v}^\varepsilon - \frac{1}{2}|\nabla \mathbf{v}^\varepsilon|^2 \mathrm{I}_2 = \frac{1}{2}\begin{pmatrix} |\partial_{x_1}\mathbf{v}^\varepsilon|^2 - |\partial_{x_2}\mathbf{v}^\varepsilon|^2, & 2\langle \partial_{x_1}\mathbf{v}^\varepsilon, \partial_{x_2}\mathbf{v}^\varepsilon\rangle \\ 2\langle \partial_{x_1}\mathbf{v}^\varepsilon, \partial_{x_2}\mathbf{v}^\varepsilon\rangle, & |\partial_{x_2}\mathbf{v}^\varepsilon|^2 - |\partial_{x_1}\mathbf{v}^\varepsilon|^2 \end{pmatrix}. \tag{2.1.43}$$

We can assume that there are two real numbers α, β such that

$$(|\partial_{x_1}\mathbf{v}^\varepsilon|^2 - |\partial_{x_2}\mathbf{v}^\varepsilon|^2)dx \rightharpoonup (|\partial_{x_1}\mathbf{v}|^2 - |\partial_{x_2}\mathbf{v}|^2)dx + \alpha\delta_{(0,0)}, \tag{2.1.44}$$

$$\langle \partial_{x_1}\mathbf{v}^\varepsilon, \partial_{x_2}\mathbf{v}^\varepsilon\rangle\, dx \rightharpoonup \langle \partial_{x_1}\mathbf{v}, \partial_{x_2}\mathbf{v}\rangle\, dx + \beta\delta_{(0,0)}, \tag{2.1.45}$$

hold as convergence of Radon measures. Next we want to show

$$\alpha = \beta = 0. \tag{2.1.46}$$

Denote

$$\Delta \mathbf{v}^\varepsilon - \frac{1}{\varepsilon^2}\chi\left(\operatorname{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})\right)\frac{d}{d\mathbf{v}}\left(\operatorname{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})\right) = \mathbf{f}^\varepsilon := \partial_t \mathbf{v}^\varepsilon + \mathbf{u}^\varepsilon \cdot \nabla \mathbf{v}^\varepsilon \tag{2.1.47}$$

and

$$\mathrm{e}_\varepsilon(\mathbf{v}^\varepsilon) := \frac{1}{2}|\nabla \mathbf{v}^\varepsilon|^2 + \frac{1}{\varepsilon^2}\chi\left(\operatorname{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})\right).$$

Now we derive the Pohozaev identity for $\mathbf{v}^\epsilon$. For any $X \in C_0^\infty(\Omega, \mathbb{R}^2)$, by multiplying the $\mathbf{v}^\varepsilon$ equation by $X \cdot \nabla \mathbf{v}^\varepsilon$ and integrating over $B_r(0)$ we get

$$\begin{aligned}&\int_{\partial B_r(0)} (X^{\mathrm{j}}\mathbf{v}^\varepsilon_{\mathrm{j}}) \cdot \Big(\mathbf{v}^\varepsilon_{\mathrm{i}} \frac{x^{\mathrm{i}}}{|x|}\Big) - \int_{B_r(0)} X^{\mathrm{j}}_{\mathrm{i}} \mathbf{v}^\varepsilon_{\mathrm{j}} \cdot \mathbf{v}^\varepsilon_{\mathrm{i}} + \int_{B_r(0)} \mathrm{div} X \mathrm{e}_\varepsilon(v^\varepsilon) - \int_{\partial B_r(0)} \mathrm{e}_\varepsilon(v^\varepsilon)(X \cdot \frac{x}{|x|})\\ &= \int_{B_r(0)} (X \cdot \nabla \mathbf{v}^\varepsilon) \cdot \mathbf{f}^\varepsilon. \end{aligned} \tag{2.1.48}$$

If we choose $X(x) = x$, then we have

$$r \int_{\partial B_r(0)} \left| \frac{\partial \mathbf{v}^\varepsilon}{\partial r} \right|^2 + \int_{B_r(0)} \frac{2}{\varepsilon^2} \chi(\mathrm{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})) - r \int_{\partial B_r(0)} \mathrm{e}_\varepsilon(\mathbf{v}^\varepsilon) = \int_{B_r(0)} |x| \frac{\partial \mathbf{v}^\varepsilon}{\partial r} \cdot \mathbf{f}^\varepsilon. \tag{2.1.49}$$

Then

$$\int_{\partial B_r(0)} \mathrm{e}_\varepsilon(\mathbf{v}^\varepsilon) = \int_{\partial B_r(0)} \left| \frac{\partial \mathbf{v}^\varepsilon}{\partial r} \right|^2 + \frac{1}{r} \int_{B_r(0)} \frac{2}{\varepsilon^2} \chi(\mathrm{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})) + O\Big(\int_{B_r(0)} |\nabla \mathbf{v}^\varepsilon| |\mathbf{f}^\varepsilon| \Big). \tag{2.1.50}$$

Integrating from r to R, we have

$$\begin{aligned}\int_{B_R(0)} \mathrm{e}_\varepsilon(\mathbf{v}^\varepsilon) - \int_{B_r(0)} \mathrm{e}_\varepsilon(\mathbf{v}^\varepsilon) &= \int_{B_R(0) \setminus B_r(0)} \left| \frac{\partial \mathbf{v}^\varepsilon}{\partial r} \right|^2 + \int_r^R \frac{1}{\tau} \int_{B_\tau(0)} \frac{2}{\varepsilon^2} \chi(\mathrm{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})) d\tau\\ &\quad + \int_r^R O\Big(\int_{B_\tau(0)} |\nabla \mathbf{v}^\varepsilon| |\mathbf{f}^\varepsilon| \Big) d\tau. \end{aligned} \tag{2.1.51}$$

Since $\Sigma_t = \{(0,0)\}$, we can assume that

$$\mathrm{e}_\varepsilon(\mathbf{v}^\varepsilon) dx \rightharpoonup \frac{1}{2} |\nabla \mathbf{v}|^2 dx + \gamma \delta_{(0,0)}, \qquad \text{in} \quad B_\delta(0) \tag{2.1.52}$$

as convergence of Radon measures, where $\gamma \geq 0$. Since $t \in A$,

$$\lim_{\varepsilon \to 0} \int_{B_\tau(0)} |\mathbf{f}^\varepsilon| |\nabla \mathbf{v}^\varepsilon| \leq \lim_{\varepsilon \to 0} \Big(\int_{B_\tau(0)} |\mathbf{f}^\varepsilon|^2 \Big)^{\frac{1}{2}} \Big(\int_{B_\tau(0)} |\nabla \mathbf{v}^\varepsilon|^2 \Big)^{\frac{1}{2}} \leq C E_0,$$

Hence, by sending $\varepsilon \to 0$ we obtain from (2.1.51) that

$$\int_{B_R(0)\setminus B_r(0)} \frac{1}{2}|\nabla \mathbf{v}|^2 \geq \int_{B_R(0)\setminus B_r(0)} \left|\frac{\partial \mathbf{v}}{\partial r}\right|^2 + \int_r^R \frac{1}{\tau} \lim_{\varepsilon\to 0} \int_{B_\tau(0)} \frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N}))d\tau + O(R).$$

Sending $r \to 0$, we have

$$\int_{B_R(0)} \frac{1}{2}|\nabla \mathbf{v}|^2 \geq \int_{B_R(0)} \left|\frac{\partial \mathbf{v}}{\partial r}\right|^2 + \int_0^R \frac{1}{\tau} \lim_{\varepsilon\to 0} \int_{B_\tau(0)} \frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N}))d\tau + O(R).$$

From this, we claim that

$$\frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})) \to 0 \quad \text{in} \quad L^1(B_\delta). \tag{2.1.53}$$

For, otherwise,

$$\frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N}))\, dx \rightharpoonup \kappa\delta_{(0,0)}$$

for some $\kappa > 0$, this implies

$$\int_0^R \frac{1}{\tau} \lim_{\varepsilon\to 0} \int_{B_\tau} \frac{2}{\varepsilon^2}\chi(\mathrm{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})) = \int_0^R \frac{\kappa}{\tau} d\tau = \infty,$$

which is impossible.

Choosing $X(x) = (x_1, 0)$ in (2.1.48), we obtain that

$$\begin{aligned} &\frac{1}{2}\int_{B_r(0)} \left(\left|\partial_{x_2}\mathbf{v}^\varepsilon\right|^2 - \left|\partial_{x_1}\mathbf{v}^\varepsilon\right|^2\right) + \int_{B_r(0)} \frac{1}{\varepsilon^2}\chi(\mathrm{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})) \\ &= \int_{B_r(0)} x_1\langle \partial_{x_1}\mathbf{v}^\varepsilon, \mathbf{f}^\varepsilon\rangle + \int_{\partial B_r(0)} \frac{x_1^2}{r} \mathrm{e}_\varepsilon(\mathbf{v}^\varepsilon) - \int_{\partial B_r(0)} x_1\langle \partial_{x_1}\mathbf{v}^\varepsilon, \frac{\partial \mathbf{v}^\varepsilon}{\partial r}\rangle. \end{aligned} \tag{2.1.54}$$

Observe that by Fubini's theorem, for a.e. $r > 0$ it holds that

$$\int_{\partial B_r(0)} x_1\langle \partial_{x_1}\mathbf{v}^\varepsilon, \frac{\partial \mathbf{v}^\varepsilon}{\partial r}\rangle \to \int_{\partial B_r(0)} x_1\langle \partial_{x_1}\mathbf{v}, \frac{\partial \mathbf{v}}{\partial r}\rangle,$$
$$\int_{\partial B_r(0)} \frac{x_1^2}{r} \mathrm{e}_\varepsilon(\mathbf{v}^\varepsilon) \to \frac{1}{2}\int_{\partial B_r} \frac{x_1^2}{r}|\nabla \mathbf{v}|^2,$$

and by (2.1.53),

$$\int_{B_r(0)} \frac{1}{\varepsilon^2}\chi(\mathrm{dist}^2(\mathbf{v}^\varepsilon, \mathcal{N})) \to 0.$$

Furthermore,

$$\Big| \int_{B_r(0)} x_1 \langle \partial_{x_1} \mathbf{v}^\varepsilon, \mathbf{f}^\varepsilon \rangle \Big| \leq Cr \left\| \mathbf{f}^\varepsilon \right\|_{L^2} \left\| \nabla \mathbf{v}^\varepsilon \right\|_{L^2} = O(r).$$

Hence, by sending $\varepsilon \to 0$ in (2.1.54), we obtain

$$\int_{B_r(0)} \left(\left| \partial_{x_2} \mathbf{v} \right|^2 - \left| \partial_{x_1} \mathbf{v} \right|^2 \right) + \alpha = O(r),$$

this further implies $\alpha = 0$ after sending $r \to 0$.

Similarly, if we choose $X(x) = (0, x_1)$ in (2.1.48) and pass the limit in the resulting equation, we can get that

$$\int_{B_r(0)} \left\langle \partial_{x_1} \mathbf{v}, \partial_{x_2} \mathbf{v} \right\rangle + \beta = O(r).$$

Hence $\beta = 0$. This proves (2.1.46) and hence completes the proof of Claim.

Multiplying (2.1.42) by $\xi \in C^\infty([0,T])$ with $\xi(T) = 0$ and integrating over $[0,T]$, we conclude that u satisfies the $(2.1.1)_1$ on Q_T. The proof of Theorem 2.1.1 is complete.

2.1.4 Compactness of simplified Ericksen–Leslie system

This section is devoted to prove Theorem 2.1.2. First notice that since the sequence of weak solutions $(\mathbf{u}^k, \mathbf{v}^k)$ satisfies the assumption (2.1.7), and

$$(\mathbf{u}_0^k, \mathbf{v}_0^k) \rightharpoonup (\mathbf{u}_0, \mathbf{v}_0), \quad \text{in } L^2(\Omega) \times H^1(\Omega),$$

there exists $(\mathbf{u}(x,t), \mathbf{v}(x,t)) : \Omega \times (0,T) \to \mathbb{R}^2 \times \mathcal{N}$ such that

$$(\mathbf{u}^k, \mathbf{v}^k) \rightharpoonup (\mathbf{u}, \mathbf{v}) \text{ in } \ L^2([0,T], H^1(\Omega)), \tag{2.1.55}$$

$$\partial_t \mathbf{v}^k + \mathbf{u}^k \cdot \nabla \mathbf{v}^k \rightharpoonup \partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v} \quad \text{in} \quad L^2([0,T], L^2(\Omega)). \tag{2.1.56}$$

Also it follows from (2.1.1) and (2.1.7) that there exists $p > 2$ that

$$\sup_k \left[\left\| \partial_t \mathbf{u}^k \right\|_{L_t^2 H_x^{-1} + L_t^2 W_x^{-2,p}} + \left\| \partial_t \mathbf{v}^k \right\|_{L_t^2 H_x^{-1}} \right] < \infty. \tag{2.1.57}$$

Hence, by Aubin–Lions' Lemma we have that

$$(\mathbf{u}^k, \mathbf{v}^k) \to (\mathbf{u}, \mathbf{v}) \text{ in } L^2(Q_T) \times L^2(Q_T).$$

By the lower semi-continuity, we have

$$\int_{Q_t} \left(|\nabla \mathbf{u}|^2 + |\partial_t \mathbf{v} + \mathbf{u} \cdot \nabla \mathbf{v}|^2 \right) \leq \liminf_{k \to \infty} \int_{Q_t} \left(|\nabla \mathbf{u}^k|^2 + |\partial_t \mathbf{v}^k + \mathbf{u}^k \cdot \nabla \mathbf{v}^k|^2 \right) \leq C_0.$$

By Fatou's Lemma and (2.1.7), we have

$$\int_0^t \liminf_{k \to \infty} \int_\Omega \left(|\nabla \mathbf{u}^k|^2 + |\partial_t \mathbf{v}^k + \mathbf{u}^k \cdot \nabla \mathbf{v}^k|^2 \right) \leq \liminf_{k \to \infty} \int_{Q_t} \left(|\nabla \mathbf{u}^k|^2 + |\partial_t \mathbf{v}^k + \mathbf{u}^k \cdot \nabla \mathbf{v}^k|^2 \right) \leq C_0.$$

Hence, there exists $A \subset [0,T]$ with full Lebesgue measure T, such that for all $t \in A$

$$\left(\mathbf{u}^k(t), \mathbf{v}^k(t) \right) \rightharpoonup (\mathbf{u}(t), \mathbf{v}(t)), \quad \text{in } L^2(\Omega) \times H^1(\Omega) \tag{2.1.58}$$

and

$$\liminf_{k \to \infty} \int_\Omega \left(|\nabla \mathbf{u}^k|^2 + |\partial_t \mathbf{v}^k + \mathbf{u}^k \cdot \nabla \mathbf{v}^k|^2 \right)(t) < \infty. \tag{2.1.59}$$

Now we define the concentration set at time $t \in (0,T]$ by

$$\Sigma_t := \bigcap_{r>0} \left\{ x \in \Omega : \liminf_{k \to \infty} \int_{B_r(x)} |\nabla v^k|^2 > \delta_0^2 \right\} \tag{2.1.60}$$

where δ_0 is small constant given by Theorem 1.2 in [50]. As in [50] (see also [22], [51]), we can show that for any $t \in A$, it holds that $\#(\Sigma_t) \le C(E_0)$ and

$$\mathbf{v}^k(t) \to \mathbf{v} \quad \text{in} \quad H^1_{\mathrm{loc}}(\Omega \setminus \Sigma_t). \tag{2.1.61}$$

Similar to the proof of Theorem 2.1.1, we can show the weak limit (u, v) satisfies the third equation of (2.1.1) in the weak sense. It remains to show that the first equation of (2.1.1) is also valid in the weak sense.

Similar to the proof of Theorem 2.1.1, to complete the proof of Theorem 2.1.2, it is suffices to show

$$\lim_{k\to\infty} \int_{\Omega\times\{t\}} \left(\nabla \mathbf{v}^k \odot \nabla \mathbf{v}^k\right) : \nabla\varphi = \int_{\Omega\times\{t\}} (\nabla \mathbf{v} \odot \nabla \mathbf{v}) : \nabla\varphi,\ \forall \varphi \in \mathbf{J}. \tag{2.1.62}$$

For simplicity, assume $\Sigma_t = \{(0,0)\} \subset \Omega$. Let $\varphi \in C^\infty(\Omega, \mathbb{R}^2)$ be such that $\operatorname{div}\varphi = 0$ and $(0,0) \in \operatorname{spt}(\varphi)$. By the same calculation as in (2.1.43), we have

$$\nabla \mathbf{v}^\varepsilon \odot \nabla \mathbf{v}^\varepsilon - \frac{1}{2}|\nabla \mathbf{v}^\varepsilon|^2 \mathbb{I}_2 = \frac{1}{2}\begin{pmatrix} |\partial_{x_1}\mathbf{v}^\varepsilon|^2 - |\partial_{x_2}\mathbf{v}^\varepsilon|^2, & 2\langle \partial_{x_1}\mathbf{v}^\varepsilon, \partial_{x_2}\mathbf{v}^\varepsilon\rangle \\ 2\langle \partial_{x_1}\mathbf{v}^\varepsilon, \partial_{x_2}\mathbf{v}^\varepsilon\rangle, & |\partial_{x_2}\mathbf{v}^\varepsilon|^2 - |\partial_{x_1}\mathbf{v}^\varepsilon|^2 \end{pmatrix}.$$

For any $t \in A$, $\mathbf{v}^k(t)$ is an approximated harmonic maps from Ω to $\mathcal{N}$:

$$\Delta \mathbf{v}^k(t) + A(\mathbf{v}^k)(\nabla \mathbf{v}^k, \nabla \mathbf{v}^k) = g^k(t) := \mathbf{v}^k_t(t) + \mathbf{u}^k \cdot \nabla \mathbf{v}^k(t) \in L^2(\Omega). \tag{2.1.63}$$

Recall the Hopf differential of $\mathbf{v}^k$ is defined by

$$\mathcal{H}^k = \left(\frac{\partial \mathbf{v}^k}{\partial z}\right)^2 = \left|\partial_{x_1}\mathbf{v}^k\right|^2 - \left|\partial_{x_2}\mathbf{v}^k\right|^2 + 2\mathrm{i}\left\langle \partial_{x_1}\mathbf{v}^k, \partial_{x_2}\mathbf{v}^k\right\rangle, \tag{2.1.64}$$

where $z = x_1 + \mathrm{i}x_2 \in \mathbb{C}$. Then

$$\frac{\partial \mathcal{H}^k}{\partial \bar{z}} = 2\frac{\partial_{x_1}\mathbf{v}^k}{\partial z}\frac{\partial^2 \mathbf{v}^k}{\partial \bar{z}\partial z} = 2\Delta \mathbf{v}^k \frac{\partial \mathbf{v}^k}{\partial z} = 2g^k(t) \cdot \frac{\partial \mathbf{v}^k}{\partial z} := G^k. \tag{2.1.65}$$

It is clear that

$$\|G^k\|_{L^1(B_r)} \le 2\|g^k(t)\|_{L^2(\Omega)}\Big\|\frac{\partial \mathbf{v}^k}{\partial z}\Big\|_{L^2(\Omega)} \le 2C_0. \tag{2.1.66}$$

Therefore, for any $z \in B_r(0)$

$$\mathcal{H}^k(z) = \int_{\partial B_{2r}(0)} \frac{\mathcal{H}^k(\omega)}{z-\omega}\, d\sigma + \int_{B_{2r}(0)} \frac{G^k(\omega)}{z-\omega}\, d\omega. \tag{2.1.67}$$

By the Young inequality of convolutions, we obtain

$$\|\mathcal{H}^k\|_{L^p(B_r)} \le C(r,p)\|\mathcal{H}^k\|_{L^1(\partial B_{2r})} + \Big\|\frac{1}{z}\Big\|_{L^p}\|G^k\|_{L^1(B_{2r})} \le C(r,p). \tag{2.1.68}$$

for any $1 < p < 2$. From this, we immediately conclude that

$$|\partial_{x_1}\mathbf{v}^k|^2 - |\partial_{x_2}\mathbf{v}^k|^2 \rightharpoonup |\partial_{x_1}\mathbf{v}|^2 - |\partial_{x_2}\mathbf{v}|^2, \quad \langle \partial_{x_1}\mathbf{v}^k, \partial_{x_2}\mathbf{v}^k\rangle \rightharpoonup \langle \partial_{x_1}\mathbf{v}, \partial_{x_2}\mathbf{v}\rangle \quad \text{in} \quad L^p(B_r(0))$$

for any $1 < p < 2$, which implies (2.1.62). This completes the proof of Theorem 2.1.2.

2.2 Weak compactness of non-isothermal simplified Ericksen–Leslie system in 3-D

In this section, we discuss the compactness of weak solutions to the non-isothermal simplified Ericksen–Leslie system with Ginzburg–Landau approximation (1.3.4), and estabilish the global existence of weak solution to (1.3.5).

2.2.1 Weak formulation for Ericksen–Leslie system

Throughout the rest of this chapter, we will assume that μ is a continuous function, and h, k are Lipschitz continuous functions, and

$$0 < \underline{\mu} \le \mu(\theta) \le \overline{\mu}, \quad 0 < \underline{k} \le k(\theta), h(\theta) \le \overline{k} \quad \text{for all } \theta > 0, \tag{2.2.1}$$

where $\underline{\mu}$, $\overline{\mu}$, $\underline{k}$, and $\overline{k}$ are positive constants. We will impose the homogeneous boundary condition for $\mathbf{u}$:

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \frac{\partial \mathbf{d}}{\partial \nu}\Big|_{\partial\Omega} = 0, \tag{2.2.2}$$

where ν is the outward unit normal vector field of $\partial\Omega$. It is readily seen that (2.2.2) implies that for Σ given by (1.3.12), it holds

$$\Sigma \cdot \nu|_{\partial\Omega} = 0. \tag{2.2.3}$$

We will also impose the non-flux boundary condition for the temperature function so that the heat flux $\mathbf{q}$ satisfies

$$\mathbf{q} \cdot \nu|_{\partial\Omega} = 0. \tag{2.2.4}$$

Set

$$\mathbf{H} = \text{ Closure of } C_0^\infty(\Omega;\mathbb{R}^3) \cap \{v : \nabla \cdot v = 0\} \text{ in } L^2(\Omega;\mathbb{R}^3),$$

$$\mathbf{J} = \text{ Closure of } C_0^\infty(\Omega;\mathbb{R}^3) \cap \{v : \nabla \cdot v = 0\} \text{ in } H^1(\Omega;\mathbb{R}^3),$$

and

$$H^1(\Omega, \mathbb{S}^2) = \left\{\mathbf{d} \in H^1(\Omega, \mathbb{R}^3) : \mathbf{d}(x) \in \mathbb{S}^2 \text{ } a.\text{e. } x \in \Omega\right\}.$$

There is some difference between the weak formulation of non-isothermal systems (1.3.4) or (1.3.5) and that of the isothermal system (1.3.2) or (1.3.1). For example, an important feature of a weak solution to (1.3.2) is the law of energy dissipation

$$\frac{d}{dt}\int_\Omega \left(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2\right) dx = -2\int_\Omega \left(\mu|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} - f_\varepsilon(\mathbf{d})|^2\right) dx \le 0, \tag{2.2.5}$$

or

$$\frac{d}{dt}\int_\Omega \left(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2\right) dx = -2\int_\Omega \left(\mu|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2\mathbf{d}|^2\right) dx \le 0 \tag{2.2.6}$$

for (1.3.1).

In contrast with (2.2.5) and (2.2.6), we need to include a weak formulation both the *first law of thermodynamics* (1.3.16) and the *second law of thermodynamics* (1.3.17) into (1.3.4) or(1.3.5). Namely, the entropy inequality for the temperature equation in (1.3.4):

$$\begin{aligned}&\partial_t H(\theta) + \mathbf{u} \cdot \nabla H(\theta)\\ &\geq -\mathrm{div}(H(\theta)\mathbf{q}) + H(\theta)\left(\mu(\theta)|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} - f_\varepsilon(\mathbf{d})|^2\right) + H(\theta)\mathbf{q} \cdot \nabla\theta,\end{aligned} \tag{2.2.7}$$

or in (1.3.5):

$$\begin{aligned}&\partial_t H(\theta) + \mathbf{u} \cdot \nabla H(\theta)\\ &\geq -\mathrm{div}(H(\theta)\mathbf{q}) + H(\theta)\left(\mu(\theta)|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} + |\nabla \mathbf{d}|^2\mathbf{d}|^2\right) + H(\theta)\mathbf{q} \cdot \nabla\theta,\end{aligned} \tag{2.2.8}$$

where H is any smooth, non-decreasing and concave function. More precisely, we have the following weak formulation to the non-isothermal system (1.3.5).

Definition 2.2.1. *For $0 < T < \infty$, a triple $(\mathbf{u}, \mathbf{d}, \theta)$ is a weak solution to (1.3.5), (2.2.8) if the following properties hold:*

i) $\mathbf{u} \in L^\infty([0,T], \mathbf{H}) \cap L^2([0,T], \mathbf{J})$, $\mathbf{d} \in L^2([0,T], H^1(\Omega, \mathbb{S}^2))$, $\theta \in L^\infty([0,T], L^1(\Omega))$.

ii) For any $\varphi \in C_0^\infty(\overline{\Omega}\times[0,T), \mathbb{R}^3)$, with $\nabla\cdot\varphi = 0$ and $\varphi\cdot\nu|_{\partial\Omega} = 0$, $\psi_1 \in C_0^\infty(\overline{\Omega}\times[0,T), \mathbb{R}^3)$, and $\psi_2 \in C^\infty(\bar{\Omega} \times [0,T))$ with $\psi_2 \geq 0$, it holds

$$\begin{aligned}&\int_0^T \int_\Omega (\mathbf{u} \cdot \partial_t\varphi + \mathbf{u} \otimes \mathbf{u} : \nabla\varphi)\\ &= \int_0^T \int_\Omega (\mu(\theta)\nabla \mathbf{u} - \nabla \mathbf{d} \odot \nabla \mathbf{d}) : \nabla\varphi - \int_\Omega \mathbf{u}_0 \cdot \varphi(\cdot, 0),\end{aligned} \tag{2.2.9}$$

$$\begin{aligned}&\int_0^T \int_\Omega (\mathbf{d} \cdot \partial_t\psi_1 + \mathbf{u} \otimes \mathbf{d} : \nabla\psi_1)\\ &= \int_0^T \int_\Omega (\nabla \mathbf{d} : \nabla\psi_1 - |\nabla \mathbf{d}|^2\mathbf{d} \cdot \psi_1) - \int_\Omega \mathbf{d}_0 \cdot \psi_1(\cdot, 0),\end{aligned} \tag{2.2.10}$$

$$\int_0^T \int_\Omega H(\theta)\partial_t\psi_2 + (H(\theta)\mathbf{u} - H(\theta)\mathbf{q}) \cdot \nabla\psi_2$$
$$\leq -\int_0^T \int_\Omega \Big[H(\theta)\left(\mu(\theta)|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d} + |\nabla\mathbf{d}|^2\mathbf{d}|^2\right) - H(\theta)\mathbf{q}\cdot\nabla\theta\Big]\psi_2$$
$$-\int_\Omega H(\theta_0)\psi_2(\cdot,0), \tag{2.2.11}$$

for any smooth, non-decreasing and concave function H.

iii) The following the energy inequality (1.3.16)

$$\int_\Omega \Big(\frac{1}{2}(|\mathbf{u}|^2 + |\nabla\mathbf{d}|^2) + \theta\Big)(\cdot,t) \leq \int_\Omega \Big(\frac{1}{2}(|\mathbf{u}_0| + |\nabla\mathbf{d}_0|^2) + \theta_0\Big) \tag{2.2.12}$$

holds for a.e. $t \in [0,T)$.

iv) The initial condition $\mathbf{u}(\cdot,0) = \mathbf{u}_0$, $\mathbf{d}(\cdot,0) = \mathbf{d}_0$, $\theta(\cdot,0) = \theta_0$ holds in the weak sense.

Now we state our main result of this section, which is the following existence theorem of global weak solutions to (1.3.5).

Theorem 2.2.1 ([52]). *For any $T > 0, \mathbf{u}_0 \in \mathbf{H}$, $\mathbf{d}_0 \in H^1(\Omega,\mathbb{S}^2)$ and $\theta_0 \in L^1(\Omega)$, if $\mathbf{d}_0(\Omega) \subset \mathbb{S}^2_+$ and $\operatorname{ess\,inf}_\Omega\theta_0 > 0$, then there exists a global weak solution $(\mathbf{u},\mathbf{d},\theta)$ to (1.3.5), (2.2.8), subject to the initial condition $(\mathbf{u},\mathbf{d},\theta) = (\mathbf{u}_0,\mathbf{d}_0,\theta_0)$ and the boundary condition (2.2.2) and (2.2.4) such that*

1. $\mathbf{u} \in L^\infty_t L^2_x \cap L^2_t H^1_x$,
2. $\mathbf{d} \in L^\infty_t H^1_x(\Omega,\mathbb{S}^2)$, *and* $\mathbf{d}(x,t) \in \mathbb{S}^2_+$ *a.e. in* $\Omega \times (0,T)$,
3. $\theta \in L^\infty_t L^1_x \cap L^p_t W^{1,p}_x$ *for* $1 \leq p < 5/4$, $\theta \geq \operatorname{ess\,inf}_\Omega\theta_0$ *a.e. in* $\Omega \times (0,T)$.

The proof of Theorem 2.2.1 is given in the sections below.

2.2.2 Maximum principle with homogeneous Neumann boundary conditions

In this section, we will sketch two a priori estimates for a drifted Ginzburg–Landau heat flow under the homogeneous Neumann boundary condition, which is similar to [22] where the Dirichlet boundary condition is considered. More precisely, for $\varepsilon > 0$, we consider

$$\begin{cases} \partial_t \mathbf{d}_\varepsilon + \mathbf{w} \cdot \nabla \mathbf{d}_\varepsilon = \Delta \mathbf{d}_\varepsilon + \dfrac{1}{\varepsilon^2}\left(1 - |\mathbf{d}_\varepsilon|^2\right) \mathbf{d}_\varepsilon & \text{in } \Omega \times (0,T), \\ \nabla \cdot \mathbf{w} = 0 & \text{in } \Omega \times (0,T), \\ \mathbf{d}_\varepsilon(x,0) = \mathbf{d}_0(x) & \text{on } \Omega, \\ \mathbf{w} = \dfrac{\partial \mathbf{d}_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{2.2.13}$$

Then we have

Lemma 2.2.1. *For $0 < T \leq \infty$, assume $\mathbf{w} \in L^2([0,T], \mathbf{J})$ and $\mathbf{d}_0 \in H^1(\Omega, \mathbb{S}^2)$. Suppose $\mathbf{d}_\varepsilon \in L^2([0,T]; H^1(\Omega, \mathbb{R}^3))$ solves (2.2.13). Then*

$$|\mathbf{d}_\varepsilon(x,t)| \leq 1 \ a.e. \ (x,t) \in \Omega \times [0,T]. \tag{2.2.14}$$

Proof. Set

$$v^\varepsilon = (|\mathbf{d}_\varepsilon|^2 - 1)_+ = \begin{cases} |\mathbf{d}_\varepsilon|^2 - 1 & if\ |\mathbf{d}_\varepsilon| \geq 1, \\ 0 & if\ |\mathbf{d}_\varepsilon| < 1. \end{cases}$$

Then v^ε is a weak solution to

$$\begin{cases} \partial_t v^\varepsilon + \mathbf{w} \cdot \nabla v^\varepsilon = \Delta v^\varepsilon - 2\Big(|\nabla \mathbf{d}_\varepsilon|^2 + \dfrac{1}{\varepsilon^2} v^\varepsilon |\mathbf{d}_\varepsilon|^2\Big) \leq \Delta v^\varepsilon & \text{in } \Omega \times (0,T), \\ \nabla \cdot \mathbf{w} = 0 & \text{in } \Omega \times (0,T), \\ v^\varepsilon(x,0) = 0 & \text{on } \Omega, \\ \mathbf{w} = \dfrac{\partial v^\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{2.2.15}$$

Multiplying $(2.2.15)_1$ by v^ε and integrating it over $\Omega \times [0,\tau]$ for any $0 < \tau \leq T$, we get

$$\int_\Omega |v^\varepsilon(\tau)|^2 + 2\int_0^\tau \int_\Omega |\nabla v^\varepsilon|^2 \leq -\int_0^\tau \int_\Omega \mathbf{w} \cdot \nabla((v^\varepsilon)^2) = 0.$$

Thus $v^\varepsilon = 0$ a.e. in $\Omega \times [0, T]$ and (2.2.14) holds. $\square$

Lemma 2.2.2. *For $0 < T \le \infty$, assume $\mathbf{w} \in L^2([0, T]; \mathbf{J})$ and $\mathbf{d}_0 \in H^1(\Omega; \mathbb{S}^2)$, with $\mathbf{d}_0(x) \in \mathbb{S}^2_+$ a.e $x \in \Omega$. If $\mathbf{d}_\varepsilon \in L^2([0, T]; H^1(\Omega; \mathbb{R}^3))$ solves (2.2.13), then*

$$\mathbf{d}^3_\varepsilon(x, t) \ge 0 \ a.e.\ (x, t) \in \Omega \times [0, T]. \tag{2.2.16}$$

Proof. Set $\varphi_\varepsilon(x, t) = \max\{-\mathrm{e}^{-\frac{t}{\varepsilon^2}} \mathbf{d}^3{}_\varepsilon(x, t), 0\}$. Then

$$\begin{cases} \partial_t \varphi_\varepsilon + \mathbf{w} \cdot \nabla \varphi_\varepsilon - \Delta \varphi_\varepsilon = \alpha_\varepsilon \varphi_\varepsilon, & \text{in } \Omega \times (0, T), \\ \nabla \cdot \mathbf{w} = 0, & \text{in } \Omega \times (0, T), \\ \varphi_\varepsilon(x, 0) = 0, & \text{on } \Omega, \\ \mathbf{w} = \dfrac{\partial \varphi_\varepsilon}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{2.2.17}$$

where

$$\alpha_\varepsilon(x, t) = \frac{1}{\varepsilon^2}(1 - |\mathbf{d}_\varepsilon(x, t)|^2) - \frac{1}{\varepsilon^2} \le 0 \ \text{ a.e. in } \Omega \times [0, T].$$

Multiplying $(2.2.17)_1$ by φ_ε and integrating over $\Omega \times [0, \tau]$ for $0 < \tau \le T$, we obtain

$$\begin{aligned} \int_\Omega |\varphi_\varepsilon|^2(\tau) + 2\int_0^\tau \int_\Omega |\nabla \varphi_\varepsilon|^2 &= -\int_0^\tau \int_\Omega \mathbf{w} \cdot \nabla(\varphi_\varepsilon^2) + 2\int_0^\tau \int_\Omega \alpha_\varepsilon |\varphi_\varepsilon|^2 \\ &= 2\int_0^\tau \int_\Omega \alpha_\varepsilon |\varphi_\varepsilon|^2 \le 0. \end{aligned}$$

Thus $\varphi_\varepsilon = 0$ a.e. in $\Omega \times [0, T]$ and (2.2.16) holds. $\square$

Finally we need the following minimum principle for the temperature which guarantees the positive lower bound of θ.

Lemma 2.2.3. *For $0 < T \le \infty$, assume $\mathbf{w} \in L^2(0,T;\mathbf{J})$, $\theta_0 \in L^1(\Omega)$ with $\operatorname{ess\,inf}_\Omega \theta_0 > 0$, and $\mathbf{d}_\varepsilon \in L^2([0,T];H^1(\Omega,\mathbb{R}^3))$. If $\theta_\varepsilon \in L_t^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega))$ solves*

$$\begin{cases} \partial_t \theta_\varepsilon + \mathbf{w} \cdot \nabla \theta_\varepsilon = -\nabla \cdot \mathbf{q}_\varepsilon + \mu(\theta_\varepsilon)|\nabla \mathbf{w}|^2 + |\Delta \mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|^2, & \text{in} \quad \Omega \times (0,T), \\ \nabla \cdot \mathbf{w} = 0, & \text{in} \quad \Omega \times (0,T), \\ \theta_\varepsilon(x,0) = \theta_0(x), & \text{on} \quad \Omega, \\ \mathbf{w} = \mathbf{q}_\varepsilon \cdot \nu = 0, & \text{on} \quad \partial\Omega \times (0,T), \end{cases} \tag{2.2.18}$$

where $\mathbf{q}_\varepsilon = -k(\theta_\varepsilon)\nabla\theta_\varepsilon - h(\theta_\varepsilon)(\nabla\theta_\varepsilon \cdot \mathbf{d}_\varepsilon)\mathbf{d}_\varepsilon$, then

$$\theta_\varepsilon(x,t) \ge \operatorname{ess\,inf}_\Omega \theta_0 \ \textit{a.e. in } \Omega \times [0,T]. \tag{2.2.19}$$

Proof. Let $\theta_\varepsilon^- = \max\{\operatorname{ess\,inf}_\Omega \theta_0 - \theta_\varepsilon, 0\}$. Then by direct computation, (2.2.18) implies that

$$\begin{cases} \partial_t \theta_\varepsilon^- + \mathbf{w} \cdot \nabla \theta_\varepsilon^- \le -\nabla \cdot \mathbf{q}_\varepsilon^-, & \text{in } \Omega \times (0,T), \\ \nabla \cdot \mathbf{w} = 0, & \text{in } \Omega \times (0,T), \\ \theta_\varepsilon^-(x,0) = 0, & \text{on } \Omega, \\ \mathbf{w} = \mathbf{q}_\varepsilon^- \cdot \nu = 0, & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{2.2.20}$$

where $\mathbf{q}_\varepsilon^- = -k(\theta_\varepsilon)\nabla\theta_\varepsilon^- - h(\theta_\varepsilon)(\nabla\theta_\varepsilon^- \cdot \mathbf{d}_\varepsilon)\mathbf{d}_\varepsilon$.

Multiplying $(2.2.20)_1$ by θ_ε^- and integrating over $\Omega \times [0,\tau]$ for $0 < \tau \le T$, we obtain

$$\int_\Omega |\theta_\varepsilon^-|^2(\tau) + 2\int_0^\tau \int_\Omega \underline{k} \left(|\nabla \theta_\varepsilon^-|^2 + |\nabla \theta_\varepsilon^- \cdot \mathbf{d}_\varepsilon|^2 \right) \le 0.$$

Therefore $\theta_\varepsilon^- = 0$ a.e. in $\Omega \times [0,T]$, which yields (2.2.19). $\square$

2.2.3 Global existence of weak solutions to simplified Ericksen–Leslie system with Ginzburg–Landau approximation

In this section we will sketch the construction of weak solutions to (2.2.21) by the Faedo-Galerkin method, which is similar to that by [39] and [33]. To simplify the presentation, we only consider the case $\varepsilon = 1$ and construct a weak solution of the following system:

$$
\begin{cases}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \operatorname{div}\left(\mu(\theta)\nabla \mathbf{u} - \nabla \mathbf{d} \odot \nabla \mathbf{d}\right), \\
\nabla \cdot \mathbf{u} = 0, \\
\partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \\
\partial_t \theta + \mathbf{u} \cdot \nabla \theta = -\operatorname{div}\mathbf{q} + \mu(\theta)|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})|^2,
\end{cases}
\tag{2.2.21}
$$

where $\mathbf{f}(\mathbf{d}) = \partial_{\mathbf{d}} F(\mathbf{d}) = (|\mathbf{d}|^2 - 1)\mathbf{d}$.

Let $\{\varphi_{\mathrm{i}}\}_{\mathrm{i}=1}^{\infty}$ be an orthonormal basis of $\mathbf{H}$ formed by eigenfunctions of the Stokes operator on Ω with zero Dirichlet boundary condition, i.e.,

$$
\begin{cases}
-\Delta \varphi_{\mathrm{i}} + \nabla P_{\mathrm{i}} = \lambda_{\mathrm{i}} \varphi_{\mathrm{i}} & \text{in } \Omega, \\
\nabla \cdot \varphi_{\mathrm{i}} = 0 & \text{in } \Omega, \\
\varphi_{\mathrm{i}} = 0 & \text{on } \partial\Omega,
\end{cases}
$$

for $\mathrm{i} = 1, 2, \cdots$, and $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$, with $\lambda_n \to \infty$.

Let $\mathbb{P}_m : \mathbf{H} \to \mathbf{H}_m = \mathit{span}\{\varphi_1, \varphi_2, \cdots, \varphi_m\}$ be the orthogonal projection operator. Consider

$$
\begin{cases}
\partial_t \mathbf{u}_m = \mathbb{P}_m \Big[- \mathbf{u}_m \cdot \nabla \mathbf{u}_m + \operatorname{div}\left(\mu(\theta_m)\nabla \mathbf{u}_m - \nabla \mathbf{d}_m \odot \nabla \mathbf{d}_m\right) \Big], \\
\mathbf{u}_m(\cdot, t) \in \mathbf{H}_m, \quad \forall t \in [0, T), \\
\mathbf{u}_m(x, 0) = \mathbb{P}_m(\mathbf{u}_0)(x), \quad \forall x \in \Omega,
\end{cases}
\tag{2.2.22}
$$

$$
\begin{cases}
\partial_t \mathbf{d}_m + \mathbf{u}_m \cdot \nabla \mathbf{d}_m = \Delta \mathbf{d}_m - f(\mathbf{d}_m), \\
\mathbf{d}_m(x, 0) = \mathbf{d}_0(x) \quad \forall x \in \Omega, \\
\dfrac{\partial \mathbf{d}_m}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega,
\end{cases}
\tag{2.2.23}
$$

$$
\begin{cases}
\partial_t \theta_m + \mathbf{u}_m \cdot \nabla \theta_m = \operatorname{div}\Big(k(\theta_m)\nabla\theta_m + h(\theta_m)(\nabla\theta_m \cdot \mathbf{d}_m)\mathbf{d}_m\Big) \\
\qquad\qquad\qquad\qquad\qquad + \mu(\theta_m)|\nabla \mathbf{u}_m|^2 + |\Delta \mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2, \\
\theta_m(x,0) = \theta_0(x) \quad \forall x \in \Omega, \\
\dfrac{\partial \theta_m}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega.
\end{cases}
\tag{2.2.24}
$$

Since $\mathbf{u}_m(\cdot,t) \in \mathbf{H}_m$, we can write

$$
\mathbf{u}_m(x,t) = \sum_{\mathrm{i}=1}^{m} g_m^{(\mathrm{i})}(t)\varphi_{\mathrm{i}}(x),
$$

so that (2.2.22) becomes the following system of ODEs:

$$
\frac{d}{dt} g_m^{(\mathrm{i})}(t) = A_{\mathrm{j}k}^{(\mathrm{i})} g_m^{(\mathrm{j})}(t) g_m^{(k)}(t) + B_{m\mathrm{j}}^{(\mathrm{i})}(t) g_m^{(\mathrm{j})}(t) + C_m^{(\mathrm{i})}(t), \tag{2.2.25}
$$

subject to the initial condition

$$
g_m^{(\mathrm{i})}(0) = \int_\Omega \langle \mathbf{u}_0, \varphi_{\mathrm{i}} \rangle, \tag{2.2.26}
$$

for $1 \le \mathrm{i} \le m$, where

$$
\begin{aligned}
A_{\mathrm{j}k}^{(\mathrm{i})} &= -\int_\Omega \langle \varphi_{\mathrm{j}} \cdot \nabla \varphi_k, \varphi_{\mathrm{i}} \rangle, \\
B_{m\mathrm{j}}^{(\mathrm{i})}(t) &= -\int_\Omega \langle \mu(\mathbf{u}_m)\nabla\varphi_{\mathrm{j}}, \nabla\varphi_{\mathrm{i}} \rangle, \\
C_m^{(\mathrm{i})}(t) &= \int_\Omega (\nabla \mathbf{d}_m \odot \nabla \mathbf{d}_m) : \nabla \varphi_{\mathrm{i}},
\end{aligned}
$$

for $1 \le \mathrm{j}, k \le m$.

For $T_0 > 0$ and $M > 0$ to be chosen later, suppose $\left(g_m^{(1)}, \cdots, g_m^{(m)}\right) \in C^1([0,T_0])$ and

$$
\sup_{0 \le t \le T_0} \sum_{\mathrm{i}=1}^{m} |g_m^{(\mathrm{i})}(t)|^2 \le M^2. \tag{2.2.27}
$$

Since $\partial_t \mathbf{u}_m, \nabla^2 \mathbf{u}_m \in C^0(\Omega \times [0,T_0])$, the standard theory of parabolic equations implies that there exists a strong solution $\mathbf{d}_m$ to (2.2.23) such that for any $\delta > 0$, $\partial_t \mathbf{d}_m, \nabla^2 \mathbf{d}_m \in$

$L^p(\Omega \times [\delta, T_0])$ for any $1 \le p < \infty$ (see [53]). Next we can solve (2.2.24) to obtain a nonnegative, strong solution θ_m. In fact, observe that

$$k(\theta_m)\nabla\theta_m + h(\theta_m)(\nabla\theta_m \cdot \mathbf{d}_m)\mathbf{d}_m = D(\theta_m)\nabla\theta_m,$$

where $(D_{\mathrm{ij}}(\theta_m)) = (k(\theta_m)\delta_{\mathrm{ij}} + h(\theta_m)\mathbf{d}_m^{\mathrm{i}}\mathbf{d}_m^{\mathrm{j}})$ is uniformly elliptic, and $\mu(\theta_m)|\nabla\mathbf{u}_m|^2 + |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 \in L^p(\Omega \times [\delta, T_0])$ holds for any $1 < p < \infty$ and $\delta > 0$. Thus by the standard theory of parabolic equations, we can first obtain a unique weak solution θ_m to (2.2.23) such that $\theta_m \in C^\alpha(\overline{\Omega} \times [\delta, T_0])$ for some $\alpha \in (0,1)$. This yields that the coefficient matrix $D(\theta_m) \in C(\overline{\Omega} \times [\delta, T_0])$ and hence by the regularity theory of parabolic equations we conclude that $\nabla\theta_m \in L^p(\Omega \times [\delta, T_0])$ for any $1 < p < \infty$ and $\delta > 0$. Now we see that θ_m satisfies

$$\partial_t\theta_m - D_{\mathrm{ij}}(\theta_m)\frac{\partial^2\theta_m}{\partial x_{\mathrm{i}}\partial x_{\mathrm{j}}} = D_{\mathrm{ij}}(\theta_m)\frac{\partial\theta_m}{\partial x_{\mathrm{i}}}\frac{\partial\theta_m}{\partial x_{\mathrm{j}}} + \mu(\theta_m)|\nabla\mathbf{u}_m|^2 + |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2,$$

where $|D_{\mathrm{ij}}(\theta_m)| \le |h(\theta_m)| + |k(\theta_m)|$ is bounded, since h and k are Lipschitz continuous. Hence by the $W_p^{2,1}$-theory of parabolic equations, $\partial_t\theta_m, \nabla^2\theta_m \in L^p(\Omega \times [\delta, T_0])$ for any $1 < p < \infty$ and $\delta > 0$.

To solve (2.2.25) and (3.1.7), we need some apriori estimates. Taking the L^2 inner product of (2.2.23) with $-\Delta\mathbf{d}_m + \mathbf{f}(\mathbf{d}_m)$ yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla\mathbf{d}_m|^2 + 2F(\mathbf{d}_m) &= -2\int_\Omega |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 + 2\int_\Omega (\mathbf{u}_m \cdot \nabla\mathbf{d}_m) \cdot (\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)) \\ &\le -\int_\Omega |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 + \int_\Omega |\mathbf{u}_m \cdot \nabla\mathbf{d}_m|^2, \quad t \in [0, T_0].\end{aligned}$$

It follows from (2.2.27) that

$$\|\mathbf{u}_m\|_{L^\infty(\Omega \times [0,T_0])} \le M \cdot \max_{1 \le \mathrm{i} \le m} \|\varphi_{\mathrm{i}}\|_{L^\infty(\Omega)} \le C_m M.$$

Therefore we get

$$\frac{d}{dt}\int_\Omega (|\nabla\mathbf{d}_m|^2 + 2F(\mathbf{d}_m)) + \int_\Omega |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 \le C_m^2 M^2 \int_\Omega |\nabla\mathbf{d}_m|^2.$$

This, combined with Gronwall's inequality and $F(\mathbf{d}_0) = 0$, implies

$$\sup_{0\le t\le T_0}\int_\Omega (|\nabla \mathbf{d}_m|^2 + F(\mathbf{d}_m)) + \int_0^{T_0}\int_\Omega |\Delta \mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 \le \mathrm{e}^{C_m^2 M^2 T_0}\int_\Omega |\nabla \mathbf{d}_0|^2,$$

so that

$$\sup_{0\le t\le T_0}\max_{1\le \mathrm{i},\mathrm{j}\le m}\left(|B_{m\mathrm{j}}^{(\mathrm{i})}(t)| + |C_m^{(\mathrm{i})}(t)|\right) \le C_0(m, M).$$

Thus we can solve (2.2.25) and (3.1.7) to obtain a unique solution $(\tilde{g}_m^{(1)}(t), \cdots, \tilde{g}_m^{(m)}(t)) \in C^1([0, T_0])$ such that for all $t \in [0, T_0]$

$$\sum_{\mathrm{i}=1}^{m} |\tilde{g}_m^{(\mathrm{i})}(t)|^2 \le \sum_{\mathrm{i}=1}^{m} |g_m^{(\mathrm{i})}(0)|^2 + C(m, M, \underline{\mu}, \overline{\mu}, \underline{k}, \overline{k})t^2. \tag{2.2.28}$$

Choose $M = 2 + 2\sum_{\mathrm{i}=1}^{m} |g_m^{(\mathrm{i})}(0)|^2$ and $T_0 > 0$ so small that the right-hand side of (2.2.28) is less than M^2 for all $t \in [0, T_0]$. Set $\tilde{\mathbf{u}}_m : \Omega \times [0, T_0] \to \mathbb{R}^3$ by

$$\tilde{\mathbf{u}}_m(x, t) = \sum_{\mathrm{i}=1}^{m} \tilde{g}_m^{(\mathrm{i})}(t)\varphi_{\mathrm{i}}(x).$$

Then $\mathcal{L}(\mathbf{u}_m) = \tilde{\mathbf{u}}_m$ defines a map from $\mathbf{U}(T_0)$ to $\mathbf{U}(T_0)$, where

$$\mathbf{U}(T_0) = \left\{\mathbf{u}_m(x, t) = \sum_{\mathrm{i}=1}^{m} g_m^{(\mathrm{i})}(t)\varphi_{\mathrm{i}}(x) : \max_{t\in[0,T_0]}\sum_{\mathrm{i}=1}^{m} |g_m^{(\mathrm{i})}(t)|^2 \le M^2, \quad \mathbf{u}_m(0) = \mathbb{P}_m\mathbf{u}_0\right\}.$$

Since $\mathbf{U}(T_0)$ is a closed, convex subset of $H_0^1(\Omega)$ and $\mathcal{L}$ is a compact operator, it follows from the Leray-Schauder theorem that $\mathcal{L}$ has a fixed point $\mathbf{u}_m \in \mathbf{U}(T_0)$ for the approximation system (2.2.22), and a classical solution $\mathbf{d}_m$ to (2.2.23) and θ_m to (2.2.24) on $\Omega \times [0, T_0]$, see [54].

Next, we will establish a priori estimates and show that the solution can be extended to $[0, T]$. To do it, taking the L^2 inner product of (2.2.22) and (2.2.23) by $\mathbf{u}_m$ and $-\Delta \mathbf{d}_m + \mathbf{f}(\mathbf{d}_m)$ respectively, and adding together these two equations, we get that for $t \in [0, T_0]$,

$$\frac{d}{dt}\int_\Omega (|\mathbf{u}_m|^2 + |\nabla \mathbf{d}_m|^2 + 2F(\mathbf{d}_m)) + 2\int_\Omega \mu(\theta_m)|\nabla \mathbf{u}_m|^2 + |\Delta \mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 = 0, \tag{2.2.29}$$

where we use the identities

$$\int_\Omega \mathbf{u}_m \cdot \operatorname{div}(\nabla \mathbf{d}_m \odot \nabla \mathbf{d}_m) = \int_\Omega (\mathbf{u}_m \cdot \nabla \mathbf{d}_m) \cdot \Delta \mathbf{d}_m,$$

$$\int_\Omega (\mathbf{u}_m \cdot \nabla \mathbf{d}_m) \cdot \mathbf{f}(\mathbf{d}_m) = \int_\Omega \mathbf{u}_m \cdot \nabla F(\mathbf{d}_m) = 0.$$

We can derive from (2.2.29) that

$$\begin{aligned}
&\sup_{0\le t\le T_0} \int_\Omega (|\mathbf{u}_m|^2 + |\nabla \mathbf{d}_m|^2 + 2F(\mathbf{d}_m)) + 2\int_0^{T_0}\int_\Omega \mu(\theta_m)|\nabla \mathbf{u}_m|^2 + |\Delta \mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2 \\
&\le \int_\Omega (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2).
\end{aligned} \tag{2.2.30}$$

Lemma 2.2.1 implies that $|\mathbf{d}_m| \le 1$ and $|\mathbf{f}(\mathbf{d}_m)| \le 1$ in $\Omega \times [0, T_0]$, so that

$$\int_0^{T_0}\int_\Omega |\Delta \mathbf{d}_m|^2 \le 2\int_0^{T_0}\int_\Omega (1 + |\Delta \mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2).$$

Hence (2.2.30) yields thqat

$$\begin{aligned}
&\sup_{0\le t\le T_0} \int_\Omega (|\mathbf{u}_m|^2 + |\nabla \mathbf{d}_m|^2) + \int_0^{T_0}\int_\Omega (\underline{\mu}|\nabla \mathbf{u}_m|^2 + |\Delta \mathbf{d}_m|^2) \\
&\le \int_\Omega (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2) + CT_0|\Omega|.
\end{aligned} \tag{2.2.31}$$

While the integration of (2.2.24) over Ω yields

$$\frac{d}{dt}\int_\Omega \theta_m = \int_\Omega (\mu(\theta_m)|\nabla \mathbf{u}_m|^2 + |\Delta \mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2). \tag{2.2.32}$$

Adding (2.2.29) together with (2.2.32) and integrating over $[0, T_0]$, we obtain

$$\sup_{0\le t\le T_0} \int_\Omega (|\mathbf{u}_m|^2 + |\nabla \mathbf{d}_m|^2 + \theta_m) \le \int_\Omega (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2 + \theta_0). \tag{2.2.33}$$

Next by choosing $H(\theta) = (1+\theta)^\alpha$, $\alpha \in (0,1)$, and multiplying the equation (2.2.24) by $H(\theta_m) = \alpha(1+\theta_m)^{\alpha-1}$, we get

$$\begin{aligned}
&\partial_t(1+\theta_m)^\alpha + \mathbf{u}_m \cdot \nabla(1+\theta_m)^\alpha \\
&= -\mathrm{div}\left(\alpha(1+\theta_m)^{\alpha-1}\mathbf{q}_m\right) + \alpha(1+\theta_m)^{\alpha-1}\left(\mu(\theta_m)|\nabla\mathbf{u}_m|^2 + |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2\right) \\
&\quad +\alpha(\alpha-1)(1+\theta_m)^{\alpha-2}\mathbf{q}_m \cdot \nabla\theta_m, \qquad (2.2.34)
\end{aligned}$$

where $\mathbf{q}_m = -h(\theta_m)\nabla\theta_m - k(\theta_m)(\nabla\theta_m \cdot \mathbf{d}_m)\mathbf{d}_m$.

Integrating (2.2.34) over $\Omega \times [0, T_0]$ yields

$$\int_0^{T_0}\int_\Omega \alpha(\alpha-1)(1+\theta_m)^{\alpha-2}\mathbf{q}_m \cdot \nabla\theta_m \le \int_{\Omega\times\{T_0\}} (1+\theta_m)^\alpha - \int_\Omega (1+\theta_0)^\alpha. \qquad (2.2.35)$$

Notice that

$$\begin{aligned}
&\int_0^{T_0}\int_\Omega \alpha(\alpha-1)(1+\theta_m)^{\alpha-2}\mathbf{q}_m \cdot \nabla\theta_m \\
&= \alpha(1-\alpha)\int_0^{T_0}\int_\Omega (1+\theta_m)^{\alpha-2}(k(\theta_m)|\nabla\theta_m|^2 + h(\theta_m)(\nabla\theta_m \cdot \mathbf{d}_m)^2) \\
&\ge \alpha(1-\alpha)\underline{k}\int_0^{T_0}\int_\Omega (1+\theta_m)^{\alpha-2}|\nabla\theta_m|^2 \\
&\ge \frac{4\alpha(1-\alpha)\underline{k}}{\alpha^2}\int_0^{T_0}\int_\Omega |\nabla\theta_m^{\frac{\alpha}{2}}|^2.
\end{aligned}$$

Thus we obtain that

$$\begin{aligned}
\int_0^{T_0}\int_\Omega \left|\nabla\theta_m^{\frac{\alpha}{2}}\right|^2 \quad &\le C(\alpha,\underline{k})\int_{\Omega\times\{T_0\}} (1+\theta_m)^\alpha \\
&\le C(\alpha,\underline{k},\Omega)\Big(\int_{\Omega\times\{T_0\}} (1+\theta_m)\Big)^\alpha \\
&\le C(\alpha,\underline{k},\Omega)\Big(1 + \int_\Omega (|\mathbf{u}_0|^2 + |\nabla\mathbf{d}_0|^2 + \theta_0)\Big)^\alpha. \qquad (2.2.36)
\end{aligned}$$

With (2.2.33) and (2.2.36), we can apply an interpolation argument, similar to (4.13) in [39], to conclude that $\theta_m \in L^q(\Omega \times [0,T_0])$ for any $1 \le q < \frac{5}{3}$, and

$$\|\theta_m\|_{L^q(\Omega\times[0,T])} \le C\Big(q, \underline{k}, \|\mathbf{u}_0\|_{L^2(\Omega)}, \|\nabla \mathbf{d}_0\|_{L^2(\Omega)}, \|\theta_0\|_{L^1(\Omega)}\Big). \tag{2.2.37}$$

This, together with (2.2.36) and Hölder's inequality:

$$\int_{\Omega\times[0,T_0]} |\nabla\theta_m|^p \le \Big(\int_{\Omega\times[0,T_0]} |\nabla\theta_m|^2 \theta_m^{\alpha-2}\Big)^{\frac{p}{2}} \Big(\int_{\Omega\times[0,T_0]} \theta_m^{(2-\alpha)\frac{p}{2-p}}\Big)^{\frac{2-p}{2}},$$

for $\alpha \in (0,1)$ and $1 \le p < 2$, implies that

$$\Big\|\nabla\theta_m\Big\|_{L^p(\Omega\times[0,T_0])} \le C\Big(p, \underline{k}, \|\mathbf{u}_0\|_{L^2(\Omega)}, \|\nabla \mathbf{d}_0\|_{L^2(\Omega)}, \|\theta_0\|_{L^1(\Omega)}\Big) \tag{2.2.38}$$

holds for all $p \in [1, 5/4)$.

Plugging the estimates (2.2.31), (2.2.33), (2.2.37), and (2.2.38) into the system (2.2.22), (2.2.23), and (2.2.24), we conclude that

$$\sup_m \Big\{ \|\partial_t \mathbf{u}_m\|_{L^{\frac{4}{3}}(0,T_0;H^{-1}(\Omega))} + \|\partial_t \mathbf{d}_m\|_{L^{\frac{4}{3}}(0,T_0;L^2(\Omega))} + \|\partial_t\theta_m\|_{L^2(0,T_0;W^{-1,4}(\Omega))} \Big\} \le C. \tag{2.2.39}$$

Therefore, by setting $\Big(\mathbf{u}_m(\cdot,T_0), \mathbf{d}_m(\cdot,T_0), \theta_m(\cdot,T_0)\Big)$ as then initial data and repeating the same argument, we can extend the solution to the interval $[0, 2T_0]$ and eventually obtain a solution $(\mathbf{u}_m, \mathbf{d}_m, \theta_m)$ to the system (2.2.22), (2.2.23), (2.2.24) in $[0,T]$ such that the estimates (2.2.31), (2.2.33), (2.2.37), (2.2.38), and (2.2.39) hold with T_0 replaced by T.

The existence of a weak solution to the original system (2.2.21) will be obtained by passing to the limit of $(\mathbf{u}_m, \mathbf{d}_m, \theta_m)$ as $m \to \infty$. In fact, by Aubin–Lions' compactness lemma [55], we know that there exists $\mathbf{u} \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega\times[0,T])$, $\mathbf{d} \in L_t^\infty H_x^1 \cap L_t^2 H_x^2(\Omega\times[0,T])$, and a nonnegative $\theta \in L_t^\infty L_x^1 \cap L_t^p W_x^{1,p}(\Omega \times [0,T])$, for $1 < p < \frac{5}{4}$, such that, after passing to a subsequence,

$$\begin{cases} \mathbf{u}_m \to \mathbf{u} & \text{in } L^2(\Omega \times [0,T]), \\ (\mathbf{d}_m, \nabla \mathbf{d}_m) \to (\mathbf{d}, \nabla \mathbf{d}) & \text{in } L^2(\Omega \times [0,T]), \\ \theta_m \to \theta & a.e. \textit{ and} \text{ in } L^{p_1}(\Omega \times [0,T]),\ \forall 1 < p_1 < \frac{5}{3}, \\ \nabla \mathbf{u}_m \rightharpoonup \nabla \mathbf{u} & \text{in } L^2(\Omega \times [0,T]), \\ \nabla^2 \mathbf{d}_m \rightharpoonup \nabla^2 \mathbf{d} & \text{in } L^2(\Omega \times [0,T]), \\ \nabla \theta_m \rightharpoonup \nabla \theta & \text{in } L^{p_2}(\Omega \times [0,T]),\ \forall 1 < p_2 < \frac{5}{4}. \end{cases}$$

Since $\mu \in C([0,\infty))$ is bounded, we have that

$$\mu(\theta_m) \to \mu(\theta) \ \text{ in } \ L^p(\Omega \times [0,T]),\ \forall 1 \leq p < \infty,$$

and

$$\mu(\theta_m)\nabla \mathbf{u}_m \rightharpoonup \mu(\theta)\nabla \mathbf{u} \ \text{ in } \ L^2(\Omega \times [0,T]).$$

After passing $m \to \infty$ in the the equations (2.2.22) and (2.2.23), we see that $(\mathbf{u}, \mathbf{d}, \theta)$ satisfies the equations $(2.2.21)_1$, $(2.2.21)_2$, and $(2.2.21)_3$ in the weak sense.

Next we want to verify that θ satisfies

$$\begin{aligned} &\int_0^T \int_\Omega \Big(H(\theta)\partial_t \psi + (H(\theta)\mathbf{u} - H(\theta)\mathbf{q}) \cdot \nabla \psi \Big) \\ &\leq - \int_0^T \int_\Omega \Big[H(\theta)(\mu(\theta)|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})|^2) - H(\theta)\mathbf{q} \cdot \nabla \theta \Big] \psi \\ &\quad - \int_\Omega H(\theta_0)\psi(\cdot, 0) \end{aligned} \tag{2.2.40}$$

holds for any smooth, non-decreasing and concave function H, and $\psi \in C_0^\infty(\overline{\Omega} \times [0,T))$ with $\psi \geq 0$. Here $\mathbf{q} = -k(\theta)\nabla\theta - h(\theta)(\nabla\theta \cdot \mathbf{d})\mathbf{d}$. Observe that by choosing $H(t) = t$, (2.2.40) yields that θ solves $(2.2.21)_4$ in the weak sense, namely,

$$\int_0^T \int_\Omega \Big(\theta \partial_t \psi + (\theta \mathbf{u} - \mathbf{q}) \cdot \nabla \psi \Big)$$

$$\leq -\int_0^T\int_\Omega (\mu(\theta)|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})|^2)\psi - \int_\Omega \theta_0\psi(\cdot,0). \tag{2.2.41}$$

In order to show (2.2.40), first observe that multiplying the equation (2.2.24) by $H(\theta_m)\psi$, integrating over $\Omega\times[0,T]$, and employing the regularity of $\theta_m, \mathbf{u}_m, \mathbf{d}_m$ implies

$$\begin{aligned}
&\int_0^T\int_\Omega \Big(H(\theta_m)\partial_t\psi + (H(\theta_m)\mathbf{u}_m - H(\theta_m)\mathbf{q}_m)\cdot\nabla\psi\Big)\\
&= -\int_0^T\int_\Omega \Big[H(\theta_m)(\mu(\theta_m)|\nabla \mathbf{u}_m|^2 + |\Delta \mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2) - H(\theta_m)\mathbf{q}_m\cdot\nabla\theta_m\Big]\psi\\
&\quad -\int_\Omega H(\theta_0)\psi(\cdot,0),
\end{aligned} \tag{2.2.42}$$

where $\mathbf{q}_m = -k(\theta_m)\nabla\theta_m - h(\theta_m)(\nabla\theta_m\cdot\mathbf{d}_m)\mathbf{d}_m$.

It follows from Lemma 2.2.3 that $\theta_m \geq \operatorname{ess\,inf}_\Omega\theta_0$ a.e.. Without loss of generality, we assume $H(0)=0$ so that $H(\theta_m)\geq H(\operatorname{ess\,inf}_\Omega\theta_0)\geq 0$ since H is nondecreasing. From $H\leq 0$, we conclude that $0\leq H(\theta_m)\leq H(\operatorname{ess\,inf}_\Omega\theta_0)$. From the concavity of H, we have

$$\frac{1}{|\Omega|}\int_\Omega H(\theta_m) \leq H(\frac{1}{|\Omega|}\int_\Omega \theta_m)$$

so that

$$\{H(\theta_m)\} \text{ is bounded in } \ L_t^\infty L_x^1 \cap L_t^p W_x^{1,p}(\Omega\times[0,T]),\ \forall 1<p<\frac{5}{4}.$$

This, combined with the bounds on $\theta_m, \mathbf{u}_m, \mathbf{d}_m$ and (2.2.42), implies that

$$\begin{aligned}
&\int_0^T\int_\Omega H(\theta_m)\mathbf{q}_m\cdot\nabla\theta_m\psi\\
&= \int_0^T\int_\Omega (|\sqrt{-H(\theta_m)k(\theta_m)\psi}\nabla\theta_m|^2 + |\sqrt{-H(\theta_m)h(\theta_m)\psi}(\nabla\theta_m\cdot\mathbf{d}_m)|^2)
\end{aligned}$$

is uniformly bounded. For any fixed $l\in\mathbb{N}^+$, since

$$\sqrt{\min\{-H(\theta_m),l\}k(\theta_m)\psi}\nabla\theta_m \rightharpoonup \sqrt{\min\{-H(\theta),l\}k(\theta)\psi}\nabla\theta,$$

and

$$\sqrt{\min\{-H(\theta_m),l\}h(\theta_m)\psi}(\nabla\theta_m\cdot\mathbf{d}_m) \rightharpoonup \sqrt{\min\{-H(\theta),l\}h(\theta)\psi}(\nabla\theta\cdot\mathbf{d})$$

in $L^p(\Omega \times [0,T]$ for $1 < p < \frac{5}{4}$, we have by the lower semicontinuity that

$$\begin{aligned}\int_0^T \int_\Omega \min\{-H(\theta), l\}\mathbf{q} \cdot \nabla\theta\psi &\leq \liminf_{m\to\infty} \int_0^T \int_\Omega \min\{-H(\theta_m), l\}\mathbf{q}_m \cdot \nabla\theta_m\psi \\ &\leq \liminf_{m\to\infty} \int_0^T \int_\Omega -H(\theta_m)\mathbf{q}_m \cdot \nabla\theta_m\psi.\end{aligned} \tag{2.2.43}$$

This, after sending $l \to \infty$, yields

$$\int_0^T \int_\Omega -H(\theta)\mathbf{q} \cdot \nabla\theta\psi \leq \liminf_{m\to\infty} \int_0^T \int_\Omega -H(\theta_m)\mathbf{q}_m \cdot \nabla\theta_m\psi. \tag{2.2.44}$$

It follows from the lower semicontinuity again that

$$\begin{aligned}&\int_0^T \int_\Omega \Big[H(\theta)(\mu(\theta)|\nabla\mathbf{u}|^2 + |\Delta\mathbf{d} - \mathbf{f}(\mathbf{d})|^2)\psi \\ &\leq \liminf_{m\to\infty} \int_0^T \int_\Omega \Big[H(\theta_m)(\mu(\theta_m)|\nabla\mathbf{u}_m|^2 + |\Delta\mathbf{d}_m - \mathbf{f}(\mathbf{d}_m)|^2)\psi.\end{aligned} \tag{2.2.45}$$

On the other hand, since

$$H(\theta_m) \to H(\theta),\ H(\theta_m)\mathbf{u}_m \to H(\theta)\mathbf{u} \ \text{ in } \ L^1(\Omega \times [0,T]),$$

and

$$H(\theta_m)\mathbf{q}_m \rightharpoonup H(\theta)\mathbf{q} \ \text{ in } \ L^1(\Omega \times [0,T]),$$

we have

$$\begin{aligned}&\int_0^T \int_\Omega \Big(H(\theta)\partial_t\psi + (H(\theta)\mathbf{u} - H(\theta)\mathbf{q}) \cdot \nabla\psi\Big) \\ &= \lim_{m\to\infty} \int_0^T \int_\Omega \Big(H(\theta_m)\partial_t\psi + (H(\theta_m)\mathbf{u}_m - H(\theta_m)\mathbf{q}_m) \cdot \nabla\psi\Big).\end{aligned} \tag{2.2.46}$$

Therefore (2.2.40) follows by passing $m \to \infty$ in (2.2.42) and applying (2.2.44), (2.2.45), and (2.2.46). This completes the construction of a global weak solution to (2.2.21). $\square$

2.2.4 Convergence and existence of global weak solutions

In this section, we will apply Lemma 2.2.1, Lemma 2.2.2, and Lemma 2.2.3 to analyze the convergence of a sequence of weak solutions $(\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon, \theta_\varepsilon)$ to the Ginzburg–Landau approximate system (1.3.4) constructed in the previous section, as $\varepsilon \to 0$, and obtain a global weak solution $(\mathbf{u}, \mathbf{d}, \theta)$ to (1.3.5).

Here we will employ the pre-compactness theorem by Lin–Wang [22] on approximated harmonic maps to show that $\mathbf{d}_\varepsilon \to \mathbf{d}$ in $L^2([0,T], H^1(\Omega))$ as $\varepsilon \to 0$.

Proof of Theorem 2.2.1. Let $(\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon, \theta_\varepsilon)$ be the weak solutions to the Ginzburg–Landau approximate system (1.3.4), under the boundary condition (2.2.2), (2.2.4), obtained from Section 5. Then there exist $C_1, C_2 > 0$ depending only on $\mathbf{u}_0$, $\mathbf{d}_0$, and θ_0 such that

$$\sup_\varepsilon \left\{ \|\mathbf{u}_\varepsilon\|_{L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega\times[0,T])} + \|\mathbf{d}_\varepsilon\|_{L_t^\infty H_x^1(\Omega\times[0,T])} \right\} \le C_1,$$
$$\sup_\varepsilon \|\theta_\varepsilon\|_{L_t^\infty L_x^1 \cap L_t^p W_x^{1,p}(\Omega\times[0,T])} \le C_2(p),\ \forall\, p \in (1, \frac{5}{4}),$$

$$\begin{aligned} &\int_{\Omega\times\{t\}} (|\mathbf{u}_\varepsilon|^2 + |\nabla \mathbf{d}_\varepsilon|^2 + \frac{2}{\varepsilon^2} F(\mathbf{d}_\varepsilon)) + 2\int_0^t \int_\Omega \left(\mu(\theta_\varepsilon)|\nabla \mathbf{u}_\varepsilon|^2 + |\Delta \mathbf{d}_\varepsilon - \frac{1}{\varepsilon^2}\mathbf{f}(\mathbf{d}_\varepsilon)|^2\right) \\ &\le \int_\Omega (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2),\ \forall t \in [0,T], \end{aligned} \tag{2.2.47}$$

$$\int_{\Omega\times\{t\}} (|\mathbf{u}_\varepsilon|^2 + |\nabla \mathbf{d}_\varepsilon|^2 + \frac{2}{\varepsilon^2} F(\mathbf{d}_\varepsilon) + \theta_\varepsilon) \le \int_\Omega (|\mathbf{u}_0|^2 + |\nabla \mathbf{d}_0|^2 + \theta_0),\ \forall t \in [0,T], \tag{2.2.48}$$

and

$$|\mathbf{d}_\varepsilon| \le 1,\ \mathbf{d}_\varepsilon^3 \ge 0,\ \theta_\varepsilon \ge \operatorname{ess\,inf}_\Omega \theta_0, \quad \text{in} \quad \Omega \times [0,T]. \tag{2.2.49}$$

Applying the equation (1.3.4), we can further deduce that

$$\sup_\varepsilon \left\{ \|\partial_t u_\varepsilon\|_{L^{\frac{4}{3}}([0,T],H^{-1}(\Omega))} + \|\partial_t \mathbf{d}_\varepsilon\|_{L^{\frac{4}{3}}([0,T],L^2(\Omega))} + \|\partial_t \theta_\varepsilon\|_{L^2([0,T],W^{-1,4}(\Omega))} \right\} < C_3. \tag{2.2.50}$$

Therefore, after passing to a subsequence, there exist $\mathbf{u} \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega \times [0,T]), \mathbf{d} \in L_t^\infty H_x^1(\Omega \times [0,T]), \theta \in L_t^\infty L_x^1 \cap L_t^p W_x^{1,p}(\Omega \times [0,T])$ for $1 < p < \frac{5}{4}$ such that

$$\begin{cases} (\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon) \to (\mathbf{u}, \mathbf{d}) & \text{in } L^2(\Omega \times (0,T)), \\ (\nabla \mathbf{u}_\varepsilon, \nabla \mathbf{d}_\varepsilon) \rightharpoonup (\nabla \mathbf{u}, \nabla \mathbf{d}) & \text{in } L^2(\Omega \times (0,T)) \end{cases} \tag{2.2.51}$$

as $\varepsilon \to 0$. Since

$$\int_{\Omega \times [0,T]} F(\mathbf{d}) \le \lim_{\varepsilon} \int_{\Omega \times [0,T]} F(\mathbf{d}_\varepsilon) = 0,$$

we conclude that $|\mathbf{d}| = 1$ a.e. in $\Omega \times [0,T]$. Sending $\varepsilon \to 0$ in the equations $(1.3.4)_{2,3}$, we obtain that

$$\nabla \cdot \mathbf{u} = 0 \; a.e. \; in \; \Omega \times [0,T],$$

and

$$(\partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d}) \times \mathbf{d} = \nabla \cdot (\nabla \mathbf{d} \times \mathbf{d}) \;\; \text{weakly in } \Omega \times [0,T],$$

which, combined with the fact that $\mathbf{d}$ is $\mathbb{S}^2$-valued, implies that

$$\partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} = \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} \;\; \text{weakly in } \Omega \times [0,T]. \tag{2.2.52}$$

Hence (2.2.10) holds.

To verify that $\mathbf{u}$ satisfies the equation $(1.3.5)_1$, we need to show that $\nabla \mathbf{d}_\varepsilon$ converges to $\nabla \mathbf{d}$ in $L^2_{\text{loc}}(\Omega \times (0,T))$. which makes sense of $\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d})$. We also need to justify the convergence of temperature equation $(1.3.5)_4$. For this purpose, we recall some basic notations and theorems in [22] that are needed in the proof.

For any $0 < a \le 2$, L_1 and $L_2 > 0$, denote by $\mathcal{X}(L_1, L_2, a)$ the space that consists of weak solutions $\mathbf{d}_\varepsilon$ of

$$\Delta \mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon) = \tau_\varepsilon \text{ in } \Omega$$

such that

1. $|\mathbf{d}_\varepsilon| \le 1$ and $\mathbf{d}_\varepsilon^{(3)} \ge -1 + a$ for x a.e. in Ω,
2. $E_\varepsilon(\mathbf{d}_\varepsilon) = \int_\Omega \frac{1}{2} |\nabla \mathbf{d}_\varepsilon|^2 + 3F_\varepsilon(\mathbf{d}_\varepsilon) dx \le L_1,$

3. $\|\tau_\varepsilon\|_{L^2(\Omega)} \leq L_2$.

The following Theorem concerning the H^1 pre-compactness of $\mathcal{X}(L_1, L_2, a)$ was shown by [22].

Theorem 2.2.2 ([22])**.** *For any $a \in (0, 2]$, $L_1 > 0$ and $L_2 > 0$, the set $\mathcal{X}(L_1, L_2, a)$ is precompact in $H^1_{\text{loc}}(\Omega; \mathbb{R}^3)$. Namely, if $\{\mathbf{d}_\varepsilon\}$ is a sequence of maps in $\mathcal{X}(L_1, L_2, a)$, then there exists a map $\mathbf{d} \in H^1(\Omega; \mathbb{S}^2)$ such that, after passing to a possible subsequence, $\mathbf{d}_\varepsilon \to \mathbf{d}$ in $H^1_{\text{loc}}(\Omega; \mathbb{R}^3)$.*

We also denote by $\mathcal{Y}(L_1, L_2, a)$ the space that consists of $\mathbf{d} \in H^1(\Omega, \mathbb{S}^2)$ that are so-called stationary approximated harmonic maps, more precisely,

$$\begin{cases} \Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} = \tau \text{ in } \Omega, \\ \displaystyle\int_\Omega (\nabla \mathbf{d} \odot \nabla \mathbf{d}) : \nabla \varphi - \frac{1}{2} |\nabla \mathbf{d}|^2 \nabla \cdot \varphi + \langle \tau, \varphi \cdot \nabla \mathbf{d} \rangle = 0, \end{cases} \tag{2.2.53}$$

for any $\varphi \in C_0^\infty(\Omega; \mathbb{R}^3)$, and

1. $\mathbf{d}^{(3)}(x) \geq -1 + a$ for x a.e. in Ω,
2. $E(\mathbf{d}) = \dfrac{1}{2} \displaystyle\int_\Omega |\nabla \mathbf{d}|^2 dx \leq L_1$,
3. $\|\tau\|_{L^2(\Omega)} \leq L_2$.

The following H^1 pre-compactness of stationary approximated harmonic maps was also shown by [22].

Theorem 2.2.3. *For any $a \in (0, 2]$, $L_1 > 0$ and $L_2 > 0$, the set $\mathcal{Y}(L_1, L_2, a)$ is pre-compact in $H^1_{\text{loc}}(\Omega; \mathbb{S}^2)$. Namely, if $\{\mathbf{d}_{\text{i}}\} \subset \mathcal{Y}(L_1, L_2, a)$ is a sequence of stationary approximated harmonic maps, with tensor fields $\{\tau_{\text{i}}\}$, then there exist $\tau \in L^2(\Omega, \mathbb{R}^3)$ and a stationary approximated harmonic map $\mathbf{d} \in \mathcal{Y}(L_1, L_2, a)$, with tensor field τ, namely,*

$$\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d} = \tau \textit{ in } \Omega,$$

such that after passing to a possible subsequence, $\mathbf{d}_{\text{i}} \to \mathbf{d}$ in $H^1_{\text{loc}}(\Omega, \mathbb{S}^2)$ and $\tau_{\text{i}} \rightharpoonup \tau$ in $L^2(\Omega; \mathbb{R}^3)$. Moreover, $\mathbf{d} \in W^{2,2}_{\text{loc}}(\Omega, \mathbb{S}^2)$.

Now we sketch the proof the compactness of $\nabla \mathbf{d}_\varepsilon$ in $L^2_{loc}(\Omega \times [0,T])$. It follows from Fatou's lemma and (2.2.47) that

$$\int_0^T \liminf_{\varepsilon \to 0} \int_\Omega |\Delta \mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|^2 \leq C_0.$$

We decompose $[0,T]$ into the sets of"good time slices" and "bad time slices". For $\Lambda \gg 1$, set

$$\mathcal{G}^T_\Lambda := \left\{ t \in [0,T] : \liminf_{\varepsilon \to 0} \int_\Omega |\Delta \mathbf{d}_\varepsilon - f_\varepsilon(\mathbf{d}_\varepsilon)|^2(t) \leq \Lambda \right\},$$

and

$$\mathcal{B}^T_\Lambda := [0,T] \setminus \mathcal{G}^T_\Lambda = \left\{ t \in [0,T] : \liminf_{\varepsilon \to 0} \int_\Omega |\Delta \mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|(t) > \Lambda \right\}.$$

From Chebyshev's inequality, we have

$$|\mathcal{B}^T_\Lambda| \leq \frac{C_0}{\Lambda}. \tag{2.2.54}$$

For any $t \in \mathcal{G}^T_\Lambda$, set $\tau_\varepsilon(t) = (\Delta \mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon))(t)$. Then Lemma 2.2.1 and 2.2.2 imply that $\{\mathbf{d}_\varepsilon(t)\} \subset \mathcal{X}(C_0, \Lambda, 1)$. Theorem 2.2.2 then implies that

$$\begin{cases} \mathbf{d}_\varepsilon(t) \to \mathbf{d}(t) & \text{in } H^1_{\text{loc}}(\Omega), \\ F_\varepsilon(\mathbf{d}_\varepsilon) \to 0 & \text{in } L^1_{\text{loc}}(\Omega), \\ \tau_\varepsilon(t) \rightharpoonup \tau(t) & \text{in } L^2(\Omega). \end{cases}$$

For any $\varphi \in C_0^\infty(\Omega; \mathbb{R}^3)$, multiplying $\tau_\varepsilon(t)$ by $\varphi \cdot \nabla \mathbf{d}_\varepsilon$ and integrating over Ω yields

$$\int_\Omega (\nabla \mathbf{d}_\varepsilon(t) \odot \nabla \mathbf{d}_\varepsilon(t)) : \nabla \varphi - \Big(\frac{1}{2}|\nabla \mathbf{d}_\varepsilon(t)|^2 + F_\varepsilon(\mathbf{d}_\varepsilon(t))\Big) \nabla \cdot \varphi + \langle \tau_\varepsilon(t), \varphi \cdot \nabla \mathbf{d}_\varepsilon(t) \rangle = 0. \tag{2.2.55}$$

Passing limit $\varepsilon \to 0$ in (2.2.55), we get

$$\int_\Omega (\nabla \mathbf{d}(t) \odot \nabla \mathbf{d}(t)) : \nabla \varphi - \frac{1}{2}|\nabla \mathbf{d}(t)|^2 \nabla \cdot \varphi + \langle \tau(t), \varphi \cdot \nabla \mathbf{d}(t) \rangle = 0.$$

Hence $\mathbf{d}(t) \in \mathcal{Y}(C_0, \Lambda, 1)$ is a stationary approximated harmonic map. Next we want to show that $\mathbf{d}_\varepsilon \to \mathbf{d}$ strongly in $L_t^2 H_x^1$. To see this, we claim that for any compact $K \subset\subset \Omega$,

$$\lim_{\varepsilon \to 0} \int_{K \times \mathcal{G}_\Lambda^T} |\nabla(\mathbf{d}_\varepsilon - \mathbf{d})|^2 = 0. \tag{2.2.56}$$

For, otherwise, there exist $\delta_0 > 0$, $K \subset\subset \Omega$ and $\varepsilon_{\rm i} \to 0$ such that

$$\int_{K \times \mathcal{G}_\Lambda^T} |\nabla(\mathbf{d}_{\varepsilon_{\rm i}} - \mathbf{d})|^2 \geq \delta_0. \tag{2.2.57}$$

From (2.2.51), we have

$$\lim_{\varepsilon_{\rm i} \to 0} \int_{K \times \mathcal{G}_\Lambda^T} |\mathbf{d}_{\varepsilon_{\rm i}} - \mathbf{d}|^2 = 0. \tag{2.2.58}$$

By Fubini's theorem, (2.2.57) and (2.2.58), there would exist $t_{\rm i} \in \mathcal{G}_\Lambda^T$ such that

$$\begin{cases} \lim\limits_{\varepsilon_{\rm i} \to 0} \int_K |\mathbf{d}_{\varepsilon_{\rm i}}(t_{\rm i}) - \mathbf{d}(t_{\rm i})|^2 = 0, \\ \int_K |\nabla(\mathbf{d}_{\varepsilon_{\rm i}}(t_{\rm i}) - \mathbf{d}(t_{\rm i}))|^2 \geq \dfrac{2\delta_0}{T}. \end{cases}$$

Thus $\{\mathbf{d}_{\varepsilon_{\rm i}}(t_{\rm i})\} \subset \mathcal{X}(C_0, \Lambda, 1)$ and $\{\mathbf{d}(t_{\rm i})\} \subset \mathcal{Y}(C_0, \Lambda, 1)$. It follows from Theorem 2.2.2 and Theorem 2.2.3 that there exist $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{Y}(C_0, \Lambda, 1)$ such that

$$\mathbf{d}_{\varepsilon_{\rm i}}(t_{\rm i}) \to \mathbf{d}_1 \text{ and } \mathbf{d}(t_{\rm i}) \to \mathbf{d}_2 \text{ strongly in } H^1(\Omega).$$

Therefore we would have

$$\int_K |\nabla(\mathbf{d}_1 - \mathbf{d}_2)|^2 = \lim_{{\rm i} \to \infty} \int_K |\nabla\left(\mathbf{d}_\varepsilon(t_{\rm i}) - \mathbf{d}(t_{\rm i})\right)|^2 \geq \frac{2\delta_0}{T},$$

and

$$\int_K |\mathbf{d}_1 - \mathbf{d}_2|^2 = \lim_{{\rm i} \to \infty} \int_K |\mathbf{d}_{\varepsilon_{\rm i}}(t_{\rm i}) - \mathbf{d}(t_{\rm i})|^2 = 0.$$

This is clearly impossible. Thus the claim is true.

We can also follow the proof of Theorem 2.2.2 in [22] to conclude that the small energy regularity criteria holds for every $(x,t) \in K \times \mathcal{G}_\Lambda^T$ so that a finite covering argument, together with estimates for Claim 4.5 in [22], yields

$$\lim_{\varepsilon \to 0} \int_{K \times \mathcal{G}_\Lambda^T} F_\varepsilon(\mathbf{d}_\varepsilon) = 0. \tag{2.2.59}$$

Hence we have that

$$\lim_{\varepsilon \to 0} \left[\|\mathbf{d}_\varepsilon - \mathbf{d}\|^2_{L_t^2 H_x^1(K \times \mathcal{G}_\Lambda^T)} + \int_{K \times \mathcal{G}_\Lambda^T} F_\varepsilon(\mathbf{d}_\varepsilon) \right] = 0.$$

On the other hand, it follows from (2.2.47) and (2.2.54) that

$$\begin{aligned}
&\|\mathbf{d}_\varepsilon - \mathbf{d}\|^2_{L_t^2 H_x^1(\Omega \times \mathcal{B}_\Lambda^T)} + \int_{\Omega \times \mathcal{B}_\Lambda^T} F_\varepsilon(\mathbf{d}_\varepsilon) \\
&\le C\Big(\sup_{t>0} \int_\Omega (|\mathbf{u}_\varepsilon|^2 + |\nabla \mathbf{d}_\varepsilon|^2 + F_\varepsilon(\mathbf{d}_\varepsilon))\Big) \big|\mathcal{B}_\Lambda^T\big| \le \frac{C}{\Lambda}.
\end{aligned}$$

Therefore, we would arrive at

$$\lim_{\varepsilon \to 0} \Big[\|\mathbf{d}_\varepsilon - \mathbf{d}\|^2_{L_t^2 H_x^1(K \times [0,T])} + \int_{K \times [0,T]} F_\varepsilon(\mathbf{d}_\varepsilon) \Big] \le \frac{C}{\Lambda}.$$

Sending $\Lambda \to \infty$ yields that

$$\lim_{\varepsilon \to 0} \Big[\|\mathbf{d}_\varepsilon - \mathbf{d}\|^2_{L_t^2 H_x^1(K \times [0,T])} + \int_{K \times [0,T]} F_\varepsilon(\mathbf{d}_\varepsilon) \Big] = 0.$$

Therefore we can conclude that $\mathbf{u}$ solves the equation (2.2.9), provided we can verify that $\mu(\theta_\varepsilon)\nabla \mathbf{u}_\varepsilon \rightharpoonup \mu(\theta)\nabla \mathbf{u}$ weakly in $L^2(\Omega \times [0,T])$, which will be verified below.

Next we turn to the convergence of θ_ε. For $\alpha \in (0,1)$, set $H(\theta_\varepsilon) = (1+\theta_\varepsilon)^\alpha$. Then from (2.2.34) we have

$$\begin{aligned}
&\partial_t (1+\theta_\varepsilon)^\alpha + \mathbf{u}_\varepsilon \cdot \nabla (1+\theta_\varepsilon)^\alpha \\
&\ge -\mathrm{div}\left(\alpha(1+\theta_\varepsilon)^{\alpha-1}\mathbf{q}_\varepsilon\right) + \alpha(1+\theta_\varepsilon)^{\alpha-1}\left(\mu(\theta_\varepsilon)|\nabla \mathbf{u}_\varepsilon|^2 + |\Delta \mathbf{d}_\varepsilon - \mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|^2\right)
\end{aligned}$$

$$+\alpha(\alpha-1)(1+\theta_\varepsilon)^{\alpha-2}\mathbf{q}_\varepsilon\cdot\nabla\theta_\varepsilon. \tag{2.2.60}$$

Integrating (2.2.60) over $\Omega\times[0,T]$, by the assumption (2.2.1) on μ, and the bound (2.2.47) on $\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon$ and θ_ε, we can derive that

$$\sup_{\varepsilon>0}\sup_{0<t<T}\int_\Omega (1+\theta_\varepsilon)^{\alpha-2}|\nabla\theta_\varepsilon|^2<\infty.$$

Therefore we conclude that $\theta_\varepsilon^{\frac{\alpha}{2}}\in L_t^2H_x^1$ and $\theta_\varepsilon\in L_t^\infty L_x^1$ are uniformly bounded. By interpolation, we would have that for $1\le p<5/4$,

$$\sup_{\varepsilon>0}\|\theta_\varepsilon\|_{L_t^pW_x^{1,p}(\Omega\times[0,T])}<\infty.$$

From the equation $(2.2.21)_4$, we have that for $1\le q<\dfrac{30}{23}$,

$$\begin{aligned}
\sup_{\varepsilon>0}\|\partial_t\theta_\varepsilon\|_{L_t^1W_x^{-1,q}} &\le \sup_{\varepsilon>0}\Big(C\|\mathbf{u}_\varepsilon\theta_\varepsilon\|_{L_t^qL_x^q}+C\|\nabla\theta_\varepsilon\|_{L_t^qL_x^q}\\
&\quad + C\Big\||\nabla\mathbf{u}_\varepsilon|^2+|\Delta\mathbf{d}_\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|^2\Big\|_{L_t^1L_x^1}\Big)\\
&\le C\sup_{\varepsilon>0}\Big(\|\mathbf{u}_\varepsilon\|_{L_t^{\frac{10}{3}}L_x^{\frac{10}{3}}}\|\theta_\varepsilon\|_{L_t^{\frac{10q}{10-3q}}L_x^{\frac{10q}{10-3q}}}+\|\nabla\theta_\varepsilon\|_{L_t^qL_x^q}\Big)+C\\
&<\infty.
\end{aligned}$$

Hence, by Aubin–Lions' compactness Lemma [55] again, up to a subsequence, there exists $\theta\in L_t^\infty L_x^1\cap L_t^pW_x^{1,p}$ for $1\le p<\dfrac{5}{4}$ such that

$$\begin{cases}\theta_\varepsilon\to\theta & \text{in } L^p(\Omega\times(0,T)),\\ \nabla\theta_\varepsilon\rightharpoonup\nabla\theta & \text{in } L^p(\Omega\times(0,T)),\end{cases}$$

as $\varepsilon\to 0$.

After taking another subsequence, we may assume that $(\mathbf{u}_\varepsilon,\mathbf{d}_\varepsilon,\theta_\varepsilon)$ converge to $(\mathbf{u},\mathbf{d},\theta)$ a.e. in $\Omega\times[0,T]$.

Since $\{\mu(\theta_\varepsilon)\}$ is uniformly bounded in $L^\infty(\Omega\times[0,T])$, $\mu(\theta_\varepsilon)\to\mu(\theta)$ a.e. in $\Omega\times[0,T]$ and $\nabla\mathbf{u}_\varepsilon\rightharpoonup\nabla\mathbf{u}$ in $L^2(\Omega\times[0,T])$, it follows that

$$\mu(\theta_\varepsilon)\nabla\mathbf{u}_\varepsilon\rightharpoonup\mu(\theta)\nabla\mathbf{u} \text{ in } L^2(\Omega\times[0,T]).$$

Thus we verify that (2.2.9) holds.

Taking the L^2 inner product of $\mathbf{u}_\varepsilon$, $\mathbf{d}_\varepsilon$, θ_ε in (2.2.21) with respect to $\mathbf{u}_\varepsilon, -\Delta\mathbf{d}_\varepsilon+\mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon), 1$, and adding the resulting equations together, we have the following energy law:

$$\frac{d}{dt}\int_\Omega\Big(\frac{1}{2}|\mathbf{u}_\varepsilon|^2+\frac{1}{2}|\nabla\mathbf{d}_\varepsilon|^2+F_\varepsilon(\mathbf{d}_\varepsilon)+\theta_\varepsilon\Big)=0. \tag{2.2.61}$$

Taking $\varepsilon\to 0$, this implies that $|\mathbf{d}|=1$ and

$$\int_\Omega\Big(\frac{1}{2}|\mathbf{u}|^2+\frac{1}{2}|\nabla\mathbf{d}|^2+\theta\Big)(t)\le\int_\Omega\Big(\frac{1}{2}|\mathbf{u}_0|^2+\frac{1}{2}|\nabla\mathbf{d}_0|+\theta_0\Big),\ \forall 0\le t\le T.$$

Hence the global energy inequality (2.2.12) holds.

It remains to show that (2.2.8) follows by passing limit $\varepsilon\to 0$ in (2.2.7). This can be done exactly as in the last part of the previous section. For any smooth, nondecreasing, concave function H, and $\psi\in C_0^\infty(\overline{\Omega}\times[0,T))$, recall from (2.2.40) that

$$\begin{aligned}&\int_0^T\int_\Omega(H(\theta_\varepsilon)\partial_t\psi+(H(\theta_\varepsilon)\mathbf{u}_\varepsilon-H(\theta_\varepsilon)\mathbf{q}_\varepsilon)\cdot\nabla\psi)\\&\le-\int_0^T\int_\Omega[H(\theta_\varepsilon)(\mu(\theta_\varepsilon)|\nabla u_\varepsilon|^2+|\Delta\mathbf{d}_\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|^2)-H(\theta_\varepsilon)\mathbf{q}_\varepsilon\cdot\nabla\theta_\varepsilon]\psi\\&\quad-\int_\Omega H(\theta_0)\psi(\cdot,0).\end{aligned} \tag{2.2.62}$$

Assume $H(0)=0$. Then the concavity of H, $0\le H(\theta_\varepsilon)\le H(\operatorname{ess\,inf}_\Omega\theta_0)$, and the uniform bound on θ_ε imply that

$$\{H(\theta_\varepsilon)\}\text{ is bounded in } L_t^\infty L_x^1\cap L_t^pW_x^{1,p}(\Omega\times[0,T]),\ \forall 1<p<\frac{5}{4}.$$

Together with the bounds on $\mathbf{u}_\varepsilon, \mathbf{d}_\varepsilon$, and (2.2.62), we have that

$$\begin{aligned}&\int_0^T\int_\Omega H(\theta_\varepsilon)\mathbf{q}_\varepsilon\cdot\nabla\theta_\varepsilon\psi\\ &=\int_0^T\int_\Omega(|\sqrt{-H(\theta_\varepsilon)k(\theta_\varepsilon)\psi}\nabla\theta_\varepsilon|^2+|\sqrt{-H(\theta_\varepsilon)h(\theta_m)\psi}(\nabla\theta_\varepsilon\cdot\mathbf{d}_\varepsilon)|^2)\end{aligned}$$

is uniformly bounded. By an argument similar to (2.2.44), we can show that

$$\int_0^T\int_\Omega -H(\theta)\mathbf{q}\cdot\nabla\theta\psi\le\liminf_{\varepsilon\to0}\int_0^T\int_\Omega -H(\theta_\varepsilon)\mathbf{q}_\varepsilon\cdot\nabla\theta_\varepsilon\psi. \tag{2.2.63}$$

Observe that

$$\Delta\mathbf{d}_\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)=\partial_t\mathbf{d}_\varepsilon+\mathbf{u}_\varepsilon\cdot\nabla\mathbf{d}_\varepsilon\rightharpoonup\partial_t\mathbf{d}+\mathbf{u}\cdot\nabla\mathbf{d}=\Delta\mathbf{d}+|\Delta\mathbf{d}|^2\mathbf{d}\quad\text{in}\quad L^2(\Omega\times[0,T]),$$

and $\{H(\theta_\varepsilon)\}$ is uniformly bounded in $L^\infty(\Omega\times[0,T])$. It follows from the lower semicontinuity that

$$\begin{aligned}&\int_0^T\int_\Omega\Big[H(\theta)(\mu(\theta)|\nabla\mathbf{u}|^2+|\Delta\mathbf{d}+|\nabla\mathbf{d}|^2\mathbf{d}|^2)\psi\\ &\le\liminf_{\varepsilon\to0}\int_0^T\int_\Omega\Big[H(\theta_\varepsilon)(\mu(\theta_\varepsilon)|\nabla\mathbf{u}_\varepsilon|^2+|\Delta\mathbf{d}_\varepsilon-\mathbf{f}_\varepsilon(\mathbf{d}_\varepsilon)|^2)\psi.\end{aligned} \tag{2.2.64}$$

On the other hand, since

$$H(\theta_\varepsilon)\to H(\theta),\ H(\theta_\varepsilon)\mathbf{u}_\varepsilon\to H(\theta)\mathbf{u}\ \text{ in }\ L^1(\Omega\times[0,T]),$$

and

$$H(\theta_\varepsilon)\mathbf{q}_\varepsilon\rightharpoonup H(\theta)\mathbf{q}\ \text{ in }\ L^1(\Omega\times[0,T]),$$

we have

$$\begin{aligned}&\int_0^T\int_\Omega\Big(H(\theta)\partial_t\psi+(H(\theta)\mathbf{u}-H(\theta)\mathbf{q})\cdot\nabla\psi\Big)\\ &=\lim_{\varepsilon\to0}\int_0^T\int_\Omega\Big(H(\theta_\varepsilon)\partial_t\psi+(H(\theta_\varepsilon)\mathbf{u}_\varepsilon-H(\theta_\varepsilon)\mathbf{q}_\varepsilon)\cdot\nabla\psi\Big).\end{aligned} \tag{2.2.65}$$

Therefore (2.2.11) follows by passing $\varepsilon \to 0$ in (2.2.62) and applying (2.2.63), (2.2.64), and (2.2.65). This completes the construction of a global weak solution to (1.3.5). $\square$

3. SUITABLE WEAK SOLUTIONS TO COROTATIONAL BERIS–EDWARDS SYSTEM IN 3-D

3.1 Introduction

In this chapter, we consider in dimension three the so-called Beris–Edwards system ([5] and [29]) that describes the hydrodynamic motion of nematic liquid crystals, with either the Landau–De Gennes bulk potential function [1] or the Maire–Saupe (Ball–Majumdar) bulk potential function [56]. Roughly speaking, this is a system that couples a forced Navier–Stokes equation for the underlying fluid velocity field $\mathbf{u}$ with a dissipative parabolic system of Q-tensors modeling nematic liquid crystal orientation fields. We are interested in establishing the existence of certain global weak solutions for such a Beris–Edwards system that enjoys partial smoothness property, analogous to the celebrated works by Cafferalli–Kohn–Nirenberg [35] on the Navier–Stokes equation and Lin–Liu [33] and [34] on the simplified Ericksen–Leslie system modeling nematic liquid crystal flows with variable degree of orientations, which was proposed by Ericksen [2], [30] and Leslie [31] in 1960's.

We begin with the description of this system. Recall that the configuration space of Q-tensors is the set of traceless, symmetric 3×3-matrices, i.e.,

$$\mathcal{S}_0^{(3)} = \Big\{ Q \in \mathbb{R}^{3\times 3} : \ Q = Q^\top, \ \mathrm{tr} Q = 0 \Big\}.$$

For technical reasons, we will consider the one constant approximate form of the Landau–De Gennes energy functional of Q-tensors, namely,

$$E(Q) = \int_\Omega \Big(\frac{L}{2} |\nabla Q|^2 + F_{\mathrm{bulk}}(Q) \Big)\, dx,$$

over the Sobolev space $H^1(\Omega, \mathcal{S}_0^{(3)})$, where Ω is a three dimensional domain that is assumed to be either $\mathbb{R}^3$ or the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ in this chapter. Here $L > 0$ denotes the elasticity constant, and $F_{\mathrm{bulk}}(Q)$ denotes the bulk potential function that usually describes the phase transition among various phase states including isotropic, uniaxial, or biaxial states. We refer interested readers to Mottram-Newton [57] and Sonnet–Virga [8] for a more detailed discussion of general Landau–De Gennes energy functionals involving multiple elasticity constants $L_{\rm i}$'s. In this chapter, we will consider two classes of bulk potential functions:

(i) (Landau–De Gennes bulk potential [1]). Here $F_{\rm bulk}(Q) = F_{\rm LdG}(Q)$, and

$$F_{\rm LdG}(Q) = \widehat{F}_{\rm LdG}(Q) - \min_{Q\in\mathcal{S}_0^{(3)}} \widehat{F}_{\rm LdG}(Q), \tag{3.1.1}$$

where

$$\widehat{F}_{\rm LdG}(Q) = \frac{a}{2}{\rm tr}(Q^2) - \frac{b}{3}{\rm tr}(Q^3) + \frac{c}{4}{\rm tr}^2(Q^2), \tag{3.1.2}$$

where $a, b, c > 0$ are temperature dependent material constants. It is a well known fact that if $0 < a < \frac{b^2}{27c}$, then $\widehat{F}_{\rm LdG}$ reaches its minimum at $Q = s_+(d\otimes d - \frac{1}{3}I_3)$, where $s_+ = \frac{b+\sqrt{b^2-24ac}}{4c}$ and $d \in \mathbb{S}^2$ is a unit vector field.

(ii) (Ball–Majumdar singular bulk potential [56]). Here $F_{\rm bulk}(Q) = F_{\rm BM}(Q)$ is a modified Maire–Saupe bulk potential introduced by Ball–Majumdar [56], which is defined as follows. $F_{\rm BM}(Q) = G_{\rm BM}(Q) - \frac{\kappa}{2}|Q|^2$ for some $\kappa > 0$, and

$$G_{\rm BM}(Q) \equiv \begin{cases} \min\limits_{\rho\in\mathcal{A}_Q} \displaystyle\int_{\mathbb{S}^2} \rho(p)\log\rho(p)\,d\sigma(p) & \text{if } -\dfrac{1}{3} < \lambda_{\rm j}(Q) < \dfrac{2}{3}, \\ \infty & \text{otherwise,} \end{cases} \tag{3.1.3}$$

where $\lambda_{\rm j}$, ${\rm j} = 1, 2, 3$, denotes the eigenvalues of $Q \in \mathcal{S}_0^{(3)}$, and

$$\begin{aligned} \mathcal{A}_Q \ \equiv\ & \Big\{0 \le \rho \in L^1(\mathbb{S}^2) : \rho(p) = \rho(-p), \int_{\mathbb{S}^2} \rho(p)\,d\sigma(p) = 1, \\ & \int_{\mathbb{S}^2} \Big(p\otimes p - \frac{1}{3}I_3\Big)\rho(p)\,d\sigma(p) = Q\Big\}. \end{aligned}$$

It was proven by [56] that $G_{\rm BM}$ is strictly convex and smooth in the interior of the convex set

$$\mathcal{D} = \Big\{Q \in \mathcal{S}_0^{(3)} :\ -\frac{1}{3} \le \lambda_{\rm i}(Q) \le \frac{2}{3},\ {\rm i} = 1, 2, 3\Big\}.$$

It is well-known that the first order variation of the Landau–De Gennes energy functional E is given by

$$H = L\Delta Q - f_{\text{bulk}}(Q), \quad f_{\text{bulk}}(Q) = \langle \nabla F_{\text{bulk}}(Q) \rangle = \nabla F_{\text{bulk}}(Q) - \frac{\operatorname{tr}(\nabla F_{\text{bulk}}(Q))}{3} I_3. \quad (3.1.4)$$

In particular, if $F_{\text{bulk}}(Q) = F_{\text{LdG}}(Q)$, then

$$f_{\text{bulk}}(Q) = \langle \nabla F_{\text{LdG}}(Q) \rangle = aQ - b\Big[Q^2 - \frac{\operatorname{tr}(Q^2)}{3} I_3\Big] + cQ \operatorname{tr}(Q^2).$$

For $0 < T \leq \infty$, denote $Q_T = \Omega \times (0, T]$. Let $\mathbf{u} : Q_T \mapsto \mathbb{R}^3$ denote the fluid velocity field and $Q : Q_T \mapsto \mathcal{S}_0^{(3)}$ denote the director field. Define

$$S(\nabla \mathbf{u}, Q) = (\xi D + \omega)\Big(Q + \frac{1}{3} I_3\Big) + \Big(Q + \frac{1}{3} I_3\Big)(\xi D - \omega) - 2\xi\Big(Q + \frac{1}{3} I_3\Big) \operatorname{tr}(Q \nabla \mathbf{u}),$$

where

$$D = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top) \ \text{ and } \ \omega = \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^\top)$$

are the symmetric part and the antisymmetric part, respectively, of the velocity gradient tensor $\nabla \mathbf{u}$, and $\xi \in \mathbb{R}$ is a rotational parameter measuring the ratio between the aligning and tumbling effects to Q by the fluid velocity field.

The Beris–Edwards Q-tensor system modeling the hydrodynamic motion of nematic liquid crystals reads [58], [59]

$$\begin{cases} \partial_t Q + \mathbf{u} \cdot \nabla Q - S(\nabla \mathbf{u}, Q) = \Gamma H \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \mu \Delta \mathbf{u} + \operatorname{div}(\tau + \sigma) \\ \operatorname{div} \mathbf{u} = 0, \end{cases} \quad (3.1.5)$$

where $\Gamma > 0$ is a relaxation time parameter, $\mu > 0$ is the fluid viscosity constant, and τ is the symmetric part of the additional stress tensor given by

$$\begin{aligned}\tau_{\alpha\beta} = &-\xi\Big(Q_{\alpha\gamma} + \frac{\delta_{\alpha\gamma}}{3}\Big)H_{\gamma\beta} - \xi H_{\alpha\gamma}\Big(Q_{\gamma\beta} + \frac{\delta_{\gamma\beta}}{3}\Big) \\ &+ 2\xi\Big(Q_{\alpha\beta} + \frac{\delta_{\alpha\beta}}{3}\Big)Q_{\gamma\delta}H_{\gamma\delta} - L\partial_\beta Q_{\gamma\delta}\partial_\alpha Q_{\gamma\delta},\ 1 \le \alpha, \beta \le 3,\end{aligned}$$

and σ is the antisymmetric part of the additional stress tensor:

$$\sigma_{\alpha\beta} = [Q, H]_{\alpha\beta} := Q_{\alpha\gamma}H_{\gamma\beta} - H_{\alpha\gamma}Q_{\gamma\beta},\ 1 \le \alpha, \beta \le 3.$$

Since both $f_{\mathrm{LdG}}(Q)$ and $f_{\mathrm{BM}}(Q)$ are isotropic functions of Q, we have

$$[Q, f_{\mathrm{bulk}}(Q)] = 0$$

so that

$$\sigma = [Q, L\Delta Q - f_{\mathrm{bulk}}(Q)] = L[Q, \Delta Q].$$

In this chapter, we will focus on the co-rotational Beris–Edwards system (3.1.5), i.e.,

$$\boxed{\xi = 0}$$

Since the exact values of L, Γ, μ don't play roles in our analysis, we will assume for simplicity

$$\boxed{L = \Gamma = \mu = 1}$$

We will also assume the domain Ω to be

$$\boxed{\Omega = \begin{cases} \mathbb{R}^3 & if \quad F_{\mathrm{bulk}}(Q) = F_{\mathrm{LdG}}(Q), \\ \mathbb{T}^3 & if \quad F_{\mathrm{bulk}}(Q) = F_{\mathrm{BM}}(Q). \end{cases}}$$

With these assumptions and the following identity:

$$\partial_\beta(\partial_\beta Q_{\gamma\delta}\partial_\alpha Q_{\gamma\delta}) = \partial_\alpha Q_{\gamma\delta}\Delta Q_{\gamma\delta} + \partial_\alpha(\frac{1}{2}|\nabla Q|^2),$$

the system (3.1.5) reduces to the following form:

$$\begin{cases} \partial_t Q + \mathbf{u}\cdot\nabla Q - [\omega, Q] = \Delta Q - f_{\text{bulk}}(Q), \\ \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \Delta\mathbf{u} - \nabla Q\cdot\Delta Q + \operatorname{div}[Q, \Delta Q], \quad \text{in } \Omega\times(0,\infty) \\ \operatorname{div}\mathbf{u} = 0, \end{cases} \tag{3.1.6}$$

subject to the initial condition

$$(\mathbf{u}, Q)|_{t=0} = (\mathbf{u}_0, Q_0)(x) \quad \text{for} \quad x\in\Omega. \tag{3.1.7}$$

A key feature of the Beris–Edwards system (3.1.6) (or (3.1.5) in general) is the energy dissipation property, which plays a fundamental role in the analysis of (3.1.6). More precisely, if $(\mathbf{u}, Q) : \Omega\times(0,\infty) \mapsto \mathbb{R}^3\times\mathcal{S}_0^{(3)}$ is a sufficiently regular solution of (3.1.5), then it satisfies the following energy inequality [59], [60]:

$$\frac{d}{dt}E(\mathbf{u}, Q)(t) = -\int_\Omega (|\nabla\mathbf{u}|^2 + |H|^2)(x,t)\,dx \tag{3.1.8}$$

where

$$E(\mathbf{u}, Q)(t) = \int_\Omega (\frac{1}{2}|\mathbf{u}|^2 + \frac{1}{2}|\nabla Q|^2 + F_{\text{bulk}}(Q))(x,t)\,dx \tag{3.1.9}$$

is the total energy of the complex fluid consisting of the elastic energy of the director field Q and the kinetic energy of the underlying fluid $\mathbf{u}$. While the right hand side of (3.1.8) denotes the dissipation rate of this system of complex fluid.

Some Notations. For $Q\in\mathcal{S}_0^{(3)}$, we use the Frobenius norm of Q, i.e.

$$|Q| = \sqrt{\operatorname{tr}(Q^2)} = \sqrt{Q_{\alpha\beta}Q_{\alpha\beta}},$$

and the Sobolev spaces of Q-tensors, $W^{l,p}\Big(\Omega,\mathcal{S}_0^{(3)}\Big)$ ($l\in\mathbb{N}_+$ and $1\le p\le\infty$), are defined by

$$W^{l,p}\Big(\Omega,\mathcal{S}_0^{(3)}\Big)=\Big\{Q=(Q_{\alpha\beta}):\Omega\mapsto\mathcal{S}_0^{(3)}:\ Q_{\alpha\beta}\in W^{l,p}(\Omega),\ \forall 1\le\alpha,\beta\le 3\Big\}.$$

When $p=2$, we denote $W^{l,2}\Big(\Omega,\mathcal{S}_0^{(3)}\Big)$ by $H^l(\Omega,\mathcal{S}_0^{(3)})$. For $A,B\in\mathbb{R}^{3\times 3}$, we denote

$$A:B=A_{\alpha\beta}B_{\alpha\beta},\ A\cdot B=\mathrm{tr}(AB),\ |\nabla Q|^2=Q_{\alpha\beta,\gamma}Q_{\alpha\beta,\gamma},\ |\Delta Q|^2=\Delta Q_{\alpha\beta}\Delta Q_{\alpha\beta},$$

and

$$(\mathbf{u}\otimes\mathbf{u})_{\alpha\beta}=u_\alpha u_\beta,\quad (\nabla Q\otimes\nabla Q)_{\alpha\beta}=\nabla_\alpha Q_{\gamma\delta}\nabla_\beta Q_{\gamma\delta}.$$

Note that $A:B=A\cdot B$ for $A,B\in\mathcal{S}_0^{(3)}$. We also use $A_{\mathrm{sym}},A_{\mathrm{anti}}$ to denote the symmetric and antisymmetric part of A respectively.

Define

$$\mathbf{H}=\text{Closure of }\Big\{\mathbf{u}\in C_0^\infty(\Omega,\mathbb{R}^3):\ \mathrm{div}\mathbf{u}=0\Big\}\text{ in }L^2(\Omega),$$

and

$$\mathbf{V}=\text{Closure of }\Big\{\mathbf{u}\in C_0^\infty(\Omega,\mathbb{R}^3):\ \mathrm{div}\mathbf{u}=0\Big\}\text{ in }H^1(\Omega).$$

For $0\le k\le 5$, $\mathcal{P}^k$ denotes the k-dimensional Hausdorff measure on $\mathbb{R}^3\times\mathbb{R}_+$ with respect to the parabolic distance:

$$\delta((x,t),(y,s))=\max\Big\{|x-y|,\sqrt{|t-s|}\Big\},\ \forall(x,t),\ (y,s)\in\mathbb{R}^3\times\mathbb{R}_+.$$

Now we would like to recall the definition of weak solutions of (3.1.6).

Definition 3.1.1. *A pair of functions* $(\mathbf{u},Q):\Omega\times(0,\infty)\mapsto\mathbb{R}^3\times\mathcal{S}_0^{(3)}$ *is a weak solution of* (3.1.6) *and* (3.3.5), *if* $\mathbf{u}\in L_t^\infty L_x^2\cap L_t^2H_x^1(\Omega\times(0,\infty))$ *and* $Q\in L_t^\infty H_x^1\cap L_t^2H_x^2(\Omega\times(0,\infty))$, *and for any* $\phi\in C_0^\infty\Big(\Omega\times[0,\infty),\mathcal{S}_0^{(3)}\Big)$ *and* $\psi\in C_0^\infty\Big(\Omega\times[0,\infty),\mathbb{R}^3\Big)$, *with* $\mathrm{div}\psi=0$ *in* $\Omega\times[0,\infty)$, *it holds*

$$\begin{aligned}&\int_{\Omega\times(0,\infty)}\Big[-Q\cdot\partial_t\phi-\Delta Q\cdot\phi-Q\cdot\mathbf{u}\otimes\nabla\phi+[Q,\omega]\cdot\phi\Big]\,dxdt\\&=-\int_{\Omega\times(0,\infty)}f_{\mathrm{bulk}}(Q)\cdot\phi\,dxdt+\int_\Omega Q_0(x)\cdot\phi(x,0)\,dx,\end{aligned}\tag{3.1.10}$$

and

$$\int_{\Omega\times(0,\infty)} \Big[-\mathbf{u}\cdot\partial_t\psi+\nabla\mathbf{u}\cdot\nabla\psi-\mathbf{u}\otimes\mathbf{u}:\nabla\psi\Big]\,dxdt=$$
$$\int_{\Omega\times(0,\infty)} \Big[-\Delta Q(\psi\cdot\nabla)Q+[\Delta Q,Q]\cdot\nabla\psi\Big]\,dxdt+\int_{\Omega}\mathbf{u}_0(x)\cdot\psi(x,0)\,dx, \quad (3.1.11)$$

Paicu-Zarnescu [59] have obtained the existence of global weak solutions to (3.1.6) and (3.3.5) in $\mathbb{R}^3$, and the existence of global strong solutions to (3.1.6) and (3.1.7) in $\mathbb{R}^2$, when the bulk potential function is $F_{\text{LdG}}(Q)$. Ding-Huang [61] have studied local strong solutions of (3.1.6). For non-corotational Beris–Edwards system (i.e. $\xi\neq 0$), Paicu-Zarnescu [60] have obtained the existence of global weak solutions to (3.1.6) and (3.1.7) in $\mathbb{R}^3$ for sufficiently small $|\xi|>0$. Later, Cavaterra-Rocca-Wu-Xu [62] have removed the smallness condition on ξ for (3.1.6) and (3.1.7) in $\mathbb{R}^2$. Wilkinson [63] has obtained the existence of global weak solutions to (3.1.6) and (3.1.7) in three dimensional torus $\mathbb{T}^3$, when the bulk potential function is the Ball–Majumdar potential $F_{\text{BM}}(Q)$. The situation of Beris–Edwards system (3.1.6) for the De Gennes potential $F_{\text{LdG}}(Q)$ on bounded domains, under the initial-boundary condition, behaves slightly different from that on $\mathbb{R}^3$. In fact, Abels-Dolzmann-Liu [64], [65] have established the well-posedness of (3.1.5) for any arbitrary constant ξ. See also [66] for related works on nonisothermal Beris–Edwards system. We also mention an interesting work on the dynamics of Q-tensor system by Wu–Xu–Zarnescu [67]. Interested readers can refer to Wang–Zhang–Zhang [68] for a rigorous derivation from Landau–De Gennes theory to Ericksen–Leslie theory. For related works on the existence of global weak solutions to the simplified Ericksen–Leslie system, see [18], [22]–[24].

These previous works mentioned above left the question open that if certain weak solutions of (3.1.5) pose either smoothness or partial smoothness properties. This motivates us to study both the existence of suitable weak solutions of (3.1.6) and their partial regularities. The notion of suitable weak solutions was first introduced by Caffarelli-Kohn-Nirenberg [35] and Scheffer [69] for the Navier–Stokes equation, and later extended by Lin–Liu [33], [34] for the simplified Ericksen–Leslie system with variable degree of orientations. Here we introduce the notion of suitable weak solutions to the Beris–Edwards system as follows.

Definition 3.1.2. *A weak solution* $(\mathbf{u}, P, Q) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(\Omega \times (0,\infty), \mathbb{R}^3) \times L^{\frac{3}{2}}(\Omega \times (0,\infty)) \times (L_t^\infty H_x^1 \cap L_t^2 H_x^2)(\Omega \times (0,\infty), \mathcal{S}_0^{(3)})$ *of* (3.1.6) *and* (3.1.7) *is a suitable weak solution of* (3.1.6)*, if, in addition,* $(\mathbf{u}, P, Q)$ *satisfies the local energy inequality:* $\forall\, 0 \le \phi \in C_0^\infty(\Omega \times (0,t])$,

$$\begin{aligned}
&\int_\Omega (|\mathbf{u}|^2 + |\nabla Q|^2)\phi(x,t)\,dx + 2\int_{Q_t} (|\nabla \mathbf{u}|^2 + |\Delta Q|^2)\phi(x,s)\,dxds \\
&\le \int_{Q_t} (|\mathbf{u}|^2 + |\nabla Q|^2)(\partial_t \phi + \Delta\phi)(x,s)\,dxds \\
&+ \int_{Q_t} [(|\mathbf{u}|^2 + 2P)\mathbf{u}\cdot\nabla\phi + 2\nabla Q \otimes \nabla Q : \mathbf{u} \otimes \nabla\phi](x,s)\,dxds \qquad (3.1.12)\\
&+2\int_{Q_t} (\nabla Q \otimes \nabla Q - |\nabla Q|^2 I_3) : \nabla^2\phi(x,s)\,dxds - 2\int_{Q_t} [Q,\Delta Q]\cdot \mathbf{u}\otimes\nabla\phi(x,s)\,dxds \\
&-2\int_{Q_t} [\omega, Q]\cdot(\nabla Q \nabla\phi) + \nabla(f_{\text{bulk}}(Q))\cdot\nabla Q\phi\Big](x,s)\,dxds.
\end{aligned}$$

The notion of suitable weak solutions turns out to be a necessary condition for the smoothness of (3.1.6). In fact, the local energy inequality (3.1.12) automatically holds for sufficiently regular solution of (3.1.5), which can be obtained by multiplying $(3.1.5)_2$ by $\mathbf{u}\phi$, and taking spatial derivative of $(3.1.5)_1$ and multiplying the resulting equation by $\nabla Q\phi$, and then applying integration by parts, see Lemma 2.2 below for the details. We would like to point out that in the process of derivation of (3.1.12), the following cancellation identity,

$$\int_\Omega [Q,\omega] : \Delta Q\phi\,dx = -\int_\Omega [Q,\Delta Q] : \nabla\mathbf{u}\phi\,dx, \qquad (3.1.13)$$

play critical roles.

Now we are ready to state our main theorem, which is valid for the Beris–Edwards system associate with both the Landau–De Gennes bulk potential $F_{\text{LdG}}(Q)$ in $\mathbb{R}^3$ and Ball–Majumdar bulk potential $F_{\text{BM}}(Q)$ in $\mathbb{T}^3$. We would like to point out that, due to the technique involving a $L^1 \to L^\infty$ estimate for the advection-diffusion equation on compact manifolds, we choose to work on the domain $\mathbb{T}^3$, instead of $\mathbb{R}^3$, for the Ball–Majumdar potential F_{BM}.

More precisely, we have

Theorem 3.1.1 ([70])**.** *For any* $\mathbf{u}_0 \in \mathbf{H}$*, if either*

(i) $\Omega = \mathbb{R}^3$*,* $F_{\text{bulk}}(\cdot) = F_{\text{LdG}}(\cdot)$ *with* $c > 0$*, and* $Q_0 \in \dot{H}^1(\mathbb{R}^3, \mathcal{S}_0^{(3)}) \cap L^\infty(\mathbb{R}^3, \mathcal{S}_0^{(3)})$*,* $F_{\text{LdG}}(Q_0) \in$

$L^1(\mathbb{R}^3)$, or

(ii) *$\Omega = \mathbb{T}^3$, $F_{\rm bulk}(\cdot) = F_{\rm BM}(\cdot)$, and $Q_0 \in H^1(\mathbb{T}^3, \mathcal{S}_0^{(3)})$ satisfies $G_{\rm bulk}(Q_0) \in L^1(\mathbb{T}^3)$,*

then there exists a global suitable weak solution $(\mathbf{u}, P, Q) : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}^3 \times \mathbb{R} \times \mathcal{S}_0^{(3)}$ of the Beris–Edwards system (3.1.6), subject to the initial condition (3.1.7). Moreover,

$$(\mathbf{u}, Q) \in C^\infty(\Omega \times (0, \infty) \setminus \Sigma),$$

where $\Sigma \subset \Omega \times \mathbb{R}_+$ is a closed subset with $\mathcal{P}^1(\Sigma) = 0$.

We would like to highlight some crucial steps of the proof for Theorem 3.1.1:

1. The existence of suitable weak solutions to (3.1.6) and (3.1.7) is obtained by modifying the retarded mollification technique, originally due to [69] and [35] in the construction of suitable weak solutions to the Navier–Stokes equation.

2. For the Landau–De Gennes potential $F_{\rm LdG}(Q)$, we establish a weak maximum principle of Q for suitable weak solutions $(\mathbf{u}, P, Q)$ of (3.1.6) and (3.1.7) that bounds the L^∞-norm of Q in $\mathbb{R}^3 \times (0, \infty)$ in terms of that of initial data Q_0, see also [58]. In particular, $\nabla_Q^l f_{\rm LdG}(Q)$ is also bounded in $\mathbb{R}^3 \times (0, \infty)$ for $l \geq 0$.

3. For the Ball–Majumdar potential $F_{\rm BM}(Q)$, we follow the approximation scheme of $G_{\rm BM}$ by Wilkinson [63] and use the convexity property of $G_{\rm BM}(Q)$ to bound

$$\|G_{\rm BM}(Q)\|_{L^\infty(\mathbb{T}^3 \times [\delta, T])}, \ \forall 0 < \delta < T < \infty,$$

in terms of $\|F_{\rm BM}(Q_0)\|_{L^1(\mathbb{T}^3)}$, δ, and T. This guarantees that Q is strictly physical in $\mathbb{T}^3 \times [\delta, T]$, i.e., there exists a small $\gamma > 0$, depending on δ, T, such that

$$-\frac{1}{3} + \gamma \leq \lambda_{\rm j}(Q(x,t)) \leq \frac{2}{3} - \gamma, \ {\rm j} = 1, 2, 3, \ \forall (x,t) \in \mathbb{T}^3 \times [\delta, T].$$

In particular, both $Q(x,t)$ and $f_{\rm BM}(Q(x,t))$ are bounded in $\mathbb{T}^3 \times [\delta, T]$ for $0 < \delta < T$.

4. Based on the local energy inequality (3.1.12), (2), and (3), we perform a blowing up argument to obtain an ε_0-regularity criteria of any suitable weak solution $(\mathbf{u}, P, Q)$ of (3.1.6), which asserts that if

$$
\begin{aligned}
&\Phi(z_0, r) := \\
&r^{-2}\int_{\mathbb{P}_r(x_0,t_0)} (|\mathbf{u}|^3 + |\nabla Q|^3)\,dxdt + \Big(r^{-2}\int_{\mathbb{P}_r(x_0,t_0)} |P|^{\frac{3}{2}}\,dxdt\Big)^2 \le \varepsilon_0^3,
\end{aligned}
\tag{3.1.14}
$$

then $(x_0, t_0) \in \Omega \times (0, \infty)$ is a smooth point of $(\mathbf{u}, Q)$. The idea is to show that $(\mathbf{u}, P, Q)$ is well approximated by a smooth solution to a linear coupling system in the parabolic neighborhood $\mathbb{P}_{\frac{r}{2}}(x_0, t_0)$ of (x_0, t_0), which heavily relies on the local energy inequality (3.1.12) and interior $L^{\frac{3}{2}}$-estimate of the pressure function P, which turns out to solve the following Poisson equation:

$$
-\Delta P = \operatorname{div}^2(\mathbf{u} \otimes \mathbf{u} + (\nabla Q \otimes \nabla Q - \frac{1}{2}|\nabla Q|^2 I_3)) \text{ in } B_r(x_0). \tag{3.1.15}
$$

Here the following simple identity plays a crucial role in the derivation of (3.1.15):

$$
\operatorname{div}^2[Q_1, \Delta Q_2 - f_{\text{bulk}}(Q_2)] = 0 \text{ in } B_r(x_0), \tag{3.1.16}
$$

for $Q_1, Q_2 \in H^2(B_r(x_0), \mathcal{S}_0^{(3)})$. See Section 2 for its proof.

This blowing up argument implies that for some $\theta \in (0, 1)$, $\Phi_{(x_*,t_*)}(r) \le Cr^{3\theta}$ for (x_*, t_*) near (x_0, t_0) , which can be used to further show that $(\mathbf{u}, \nabla Q)$ are almost bounded near (x_0, t_0) by an iterated Riesz potential estimates in the parabolic Morrey spaces, see also Huang-Wang [71], Hineman-Wang [72], and Huang–Lin–Wang [38]. Higher order regularity of $(\mathbf{u}, Q)$ near (x_0, t_0) turns out to be more involved than the usual situations, due to the special nonlinearities. Here we establish it by performing higher order energy estimates and utilizing the intrinsic cancellation property, see also [38] for a similar argument on general Ericksen–Leslie system in dimension two. It is well-known **S** that this step is sufficient to show that $(\mathbf{u}, Q)$ is smooth away from a closed set Σ which has $\mathcal{P}^{\frac{5}{3}}(\Sigma) = 0$.

5. To obtain $\mathcal{P}^1(\Sigma) = 0$ from the previous step, we adapt the argument by [35] to show that if

$$\overline{\lim}_{r\to 0} r^{-1} \int_{\mathbb{P}_r(x_0,t_0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\, dxdt < \varepsilon_1^2, \tag{3.1.17}$$

then $(\mathbf{u}, Q) \in C^\infty(\mathbb{P}_{\frac{r}{2}}(x_0, t_0))$. This will be established by extending the so called A, B, C, D Lemmas in [35] to system (3.1.6).

This chapter is organized as follows. In Section 3.2, we derive both the global and local energy inequality for sufficiently regular solutions of (3.1.6). In Section 3.3, we indicate the construction of suitable weak solutions to (3.1.6) and (3.1.7) for both Landau–De Gennes potential and Ball–Majumdar potential. In Section 3.4, we prove two weak maximum principles for suitable weak solutions to (3.1.6) and (3.1.7): one for Q and the other for $G_{\mathrm{BM}}(Q)$. In Section 3.5, we prove the first ε_0-regularity of suitable weak solutions to (3.1.6) and (3.1.7) in terms of $\Phi(z_0, r)$. In Section 3.6, we will prove the second ε_0-regularity of suitable weak solutions to (3.1.6) and (3.1.7) in terms of (3.1.17).

3.2 Global and local energy inequalities

In this section, we will present proofs for both global energy inequality and local energy inequality for sufficiently regular solutions to the Beris–Edwards system (3.1.6).

Lemma 3.2.1. *Let $(\mathbf{u}, Q) \in C^\infty(\Omega \times (0,\infty), \mathbb{R}^3 \times \mathcal{S}_0^{(3)})$ be a smooth solution of Beris–Edwards system (3.1.6). Then the global energy inequality (3.1.8) holds.*

Proof. The proof is standard, see for instance [59], [63]. □

Next we are going to present a local energy inequality for sufficiently regular solutions to the system (3.1.6).

Lemma 3.2.2. *Assume* $(\mathbf{u}, P, Q) \in C^\infty(\Omega \times (0,\infty), \mathbb{R}^3 \times \mathbb{R} \times \mathcal{S}_0^{(3)})$ *is a smooth solution of* (3.1.6). *Then for* $t > 0$ *and any nonnegative* $\phi \in C_0^\infty(\Omega \times (0,t])$, *the following inequality holds on* $Q_t = \Omega \times [0,t]$:

$$\begin{aligned}
&\int_\Omega \Big(|\mathbf{u}|^2 + |\nabla Q|^2\Big)\phi(x,t)\,dx + 2\int_{Q_t} \Big(|\nabla \mathbf{u}|^2 + |\Delta Q|^2\Big)\phi\,dxds \\
&= \int_{Q_t} \Big(|\mathbf{u}|^2 + |\nabla Q|^2\Big)(\partial_t + \Delta)\phi\,dxds \\
&+ \int_{Q_t} [(|\mathbf{u}|^2 + 2P)\mathbf{u}\cdot\nabla\phi + 2(\nabla Q \otimes \nabla Q) : \mathbf{u}\otimes\nabla\phi]\,dxds \\
&+ 2\int_{Q_t} [(\nabla Q \otimes \nabla Q - |\nabla Q|^2 I_3) : \nabla^2\phi\,dxds - 2\int_{Q_t} [Q, \Delta Q] : \mathbf{u}\otimes\nabla\phi\,dxds \\
&- 2\int_{Q_t} \Big([\omega, Q] : (\nabla Q \nabla \phi) + \nabla(f_{\text{bulk}}(Q)) \cdot \nabla Q \phi\Big)\,dxds.
\end{aligned} \tag{3.2.1}$$

Proof. Using $\text{div}\mathbf{u} = 0$, multiplying the momentum equation $(3.1.6)_2$ by $\mathbf{u}\phi$, integrating the resulting equation over Ω, and applying integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}|^2\phi\,dx + \int_\Omega |\nabla\mathbf{u}|^2\phi\,dx \\
&= \frac{1}{2}\int_\Omega |\mathbf{u}|^2(\partial_t\phi + \Delta\phi)dx + \frac{1}{2}\int_\Omega (|\mathbf{u}|^2 + 2P)\mathbf{u}\cdot\nabla\phi\,dx - \int_\Omega (\mathbf{u}\cdot\nabla)Q\cdot\Delta Q\phi\,dx \\
&\quad - \int_\Omega [Q, \Delta Q] : \nabla\mathbf{u}\phi\,dx - \int_\Omega [Q, \Delta Q] : \mathbf{u}\otimes\nabla\phi\,dx.
\end{aligned} \tag{3.2.2}$$

Taking a spatial derivative of the equation of Q $(3.1.6)_1$ yields

$$\partial_t\partial_\alpha Q + \mathbf{u}\cdot\nabla\partial_\alpha Q + \partial_\alpha\mathbf{u}\cdot\nabla Q + \partial_\alpha[Q, \omega] = \Delta\partial_\alpha Q - \partial_\alpha(f_{\text{bulk}}(Q)).$$

Using again div$\mathbf{u} = 0$, multiplying the equation above by $\partial_\alpha Q\phi$, integrating the resulting equation over Ω, and applying integration by parts, and sum over α, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla Q|^2\phi\,dx + \int_\Omega |\Delta Q|^2\phi\,dx \\
&= \frac{1}{2}\int_\Omega |\nabla Q|^2\partial_t\phi\,dx + \int_\Omega (\mathbf{u}\cdot\nabla)Q\cdot(\Delta Q\phi + \nabla Q\nabla\phi)\,dx \\
&\quad - \int_\Omega [\omega, Q] : (\Delta Q\phi + \nabla Q\nabla\phi)\,dx \\
&\quad - \int_\Omega \Delta Q\cdot\nabla Q\nabla\phi\,dx - \int_O \nabla(f_{\text{bulk}}(Q))\cdot\nabla Q\phi dx.
\end{aligned}
\tag{3.2.3}
$$

By direct calculations, there hold

$$
-\int_\Omega \Delta Q\cdot\nabla Q\nabla\phi\,dx = \int_\Omega \frac{1}{2}|\nabla Q|^2\Delta\phi\,dx + \int_\Omega (\nabla Q\otimes\nabla Q - |\nabla Q|^2 I_3) : \nabla^2\phi\,dx, \tag{3.2.4}
$$

and

$$
\int_\Omega [\omega, Q] :\Delta Q\phi\,dx = -\int_\Omega [Q, \Delta Q] : \nabla\mathbf{u}\phi\,dx. \tag{3.2.5}
$$

Hence, by adding (3.2.2) and (3.2.3) together and applying (3.2.4) and (3.2.5), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \Big(|\mathbf{u}|^2 + |\nabla Q|^2\Big)\phi\,dx + \int_\Omega \Big(|\nabla\mathbf{u}|^2 + |\Delta Q|^2\Big)\phi\,dx \\
&= \frac{1}{2}\int_\Omega \Big(|\mathbf{u}|^2 + |\nabla Q|^2\Big)(\partial_t + \Delta)\phi\,dx + \frac{1}{2}\int_\Omega (|\mathbf{u}|^2 + 2P)\mathbf{u}\cdot\nabla\phi\,dx \\
&+ \int_\Omega (\mathbf{u}\cdot\nabla)Q\cdot\nabla Q\nabla\phi\,dx - \int_\Omega [Q, \Delta Q] : \mathbf{u}\otimes\nabla\phi\,dx \\
&- \int_\Omega [\omega, Q] : \nabla Q\nabla\phi\,dx - \int_\Omega \nabla(f_{\text{bulk}}(Q))\cdot\nabla Q\phi\,dx \\
&+ \int_\Omega (\nabla Q\otimes\nabla Q - |\nabla Q|^2 I_3) : \nabla^2\phi\,dx.
\end{aligned}
$$

This, after integrating over $[0, t]$, yields the local energy inequality (3.2.1). $\square$

We close this section by giving a proof of the identity (3.1.16). More precisely, we have

Lemma 3.2.3. *For $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$, if $Q^1, Q^2 \in H^2(\Omega, \mathcal{S}_0^{(3)})$, then*

$$\operatorname{div}^2[Q^1, \Delta Q^2 - f_{\text{bulk}}(Q^2)] = 0 \quad \text{in} \quad \Omega, \tag{3.2.6}$$

in the sense of distributions.

Proof. For any $\phi \in C_0^\infty(\Omega)$, we see that

$$\int_\Omega \operatorname{div}^2[Q^1, \Delta Q^2 - f_{\text{bulk}}(Q^2)](\phi) = \int_\Omega [Q^1, \Delta Q^2 - f_{\text{bulk}}(Q^2)]_{\alpha\beta}\Big) \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta}\, dx.$$

Set

$$A_{\alpha\beta} = [Q^1, \Delta Q^2 - f_{\text{bulk}}(Q^2)]_{\alpha\beta}, \quad \forall 1 \le \alpha, \beta \le 3,$$

and

$$B_{\alpha\beta} = \frac{\partial^2 \phi}{\partial x_\alpha \partial x_\beta}, \ \forall 1 \le \alpha, \beta \le 3.$$

Since Q^1 and Q^2 are symmetric, it is easy to check that

$$A_{\alpha\beta} = -A_{\beta\alpha}, \ B_{\alpha\beta} = B_{\beta\alpha}, \ \forall 1 \le \alpha, \beta \le 3.$$

We recall the following matrix contraction:

$$A : B = A_{\text{sym}} : B_{\text{sym}} + A_{\text{anti}} : B_{\text{anti}}.$$

Hence (3.2.6) follows. □

3.3 Global existence of suitable weak solutions

This section is devoted to the construction of suitable weak solutions to the Beris–Edwards system (3.1.6). The idea is motived by the "retarded mollification technique" originally due to [69] and [35] in the context of Navier–Stokes equations. Since the procedure for Ball–Majumdar potential $F_{\text{BM}}(Q)$ is somewhat different from that for Landau–De Gennes potential $F_{\text{LdG}}(Q)$, we will describe them in two separate subsections.

We explain the construction of suitable weak solutions in the spirit of [35]. For $f:\mathbb{R}^4 \to \mathbb{R}$ and $0<\theta<1$, define the "retarded mollifier" $\Psi_\theta(f)$ of f by

$$\Psi_\theta[f](x,t) = \frac{1}{\theta^4}\int_{\mathbb{R}^4} \eta\left(\frac{y}{\theta},\frac{\tau}{\theta}\right)\tilde{f}(x-y,t-\tau)\,dyd\tau,$$

where

$$\tilde{f}(x,t) = \begin{cases} f(x,t) & t \geq 0, \\ 0 & t<0, \end{cases}$$

and the mollifying function $\eta \in C_0^\infty(\mathbb{R}^4)$ satisfies

$$\begin{cases} \eta \geq 0 \quad \text{and} \quad \displaystyle\int_{\mathbb{R}^4} \eta\,dxdt = 1, \\ \text{spt }\ \eta \subset \Big\{(x,t) : |x|^2 < t, \quad 1<t<2\Big\}. \end{cases}$$

It follows from Lemma A.8 in [35] that for $\theta \in (0,1]$ and $0<T\leq\infty$,

$$\begin{aligned} \operatorname{div}\Psi_\theta[\mathbf{u}] = 0 \quad \text{if} \quad \operatorname{div}\mathbf{u} = 0, \\ \sup_{0\leq t\leq T}\int_{\mathbb{R}^3} |\Psi_\theta[\mathbf{u}]|^2(x,t)\,dx \leq C\sup_{0\leq t\leq T}\int_{\mathbb{R}^3}|\mathbf{u}|^2(x,t)\,dx \\ \int_{\mathbb{R}^3\times[0,T]} |\nabla\Psi_\theta[\mathbf{u}]|^2(x,t)\,dxdt \leq C\int_{\mathbb{R}^3\times[0,T]}|\nabla\mathbf{u}|^2(x,t)\,dxdt. \end{aligned}$$

Now we proceed to find the existence of suitable weak solutions of (3.1.6) and (3.1.7) as follows.

3.3.1 The Landau–De Gennes potential $F_{\text{bulk}}(Q) = F_{\text{LdG}}(Q)$ and $\Omega = \mathbb{R}^3$

With the mollifier $\Psi_\theta[\mathbf{u}] \in C^\infty(\mathbb{R}^4)$, we introduce an approximate version of the Beris–Edwards system (3.1.6), namely,

$$\begin{cases} \partial_t Q^\theta + \mathbf{u}^\theta \cdot \nabla \Psi_\theta[Q^\theta] - [\omega^\theta, \Psi_\theta[Q^\theta]] = \Delta Q^\theta - f_{\text{LdG}}(Q^\theta), \\ \partial_t \mathbf{u}^\theta + \Psi_\theta[\mathbf{u}^\theta] \cdot \nabla \mathbf{u}^\theta + \nabla P^\theta \\ = \Delta \mathbf{u}^\theta - \nabla(\Psi_\theta[Q^\theta]) \cdot \Big(\Delta Q^\theta - f_{\text{LdG}}(Q^\theta)\Big) \qquad \text{in } Q_T \\ \quad + \text{div}[\Psi_\theta[Q^\theta], \Delta Q^\theta - f_{\text{LdG}}(Q^\theta)], \\ \text{div}\mathbf{u}^\theta = 0. \end{cases} \tag{3.3.2}$$

subject to the initial condition (3.3.5). Here $\omega^\theta = \omega(\mathbf{u}^\theta) = \dfrac{\nabla \mathbf{u}^\theta - (\nabla \mathbf{u}^\theta)^\top}{2}$.

The idea behind the construction of suitable weak solutions to (3.3.2) is as follows. For a fixed large $N \geq 1$, set $\theta = \dfrac{T}{N} \in (0, 1]$, we want to find $\mathbf{u} = \mathbf{u}^\theta, P = P^\theta$, and $Q = Q^\theta$ solving (3.3.2) and (3.3.5). Since $\Psi_\theta[\mathbf{u}]$ and $\Psi_\theta[Q]$ are smooth, and their values at time t depend only on the values of $\mathbf{u}$ and Q at times prior to $t - \theta$, solving (3.3.2) and (3.3.5) involves iteratively solving (3.3.2) in the interval $[m\theta, (m+1)\theta]$, subject to the initial condition

$$(\mathbf{u}, Q)\Big|_{t=m\theta} = (u^\theta, Q^\theta)(\cdot, m\theta) \text{ in } \mathbb{R}^3,$$

for $0 \leq m \leq N-1$. This amounts to solving a system that couples a semi-linear parabolic-like equation for Q and a Stokes-like equation for $\mathbf{u}$, in which all the coefficient functions are given smooth functions.

We can verify, by the classical Faedo-Garlekin method, the existence of $(\mathbf{u}^\theta, Q^\theta, P^\theta)$ inductively on each time interval $(m\theta, (m+1)\theta)$ for all $0 \leq m \leq N-1$. Indeed for $m = 0$, according to the definition of Ψ_θ, $\Psi_\theta(\mathbf{u}^\theta) = \Psi_\theta(Q^\theta) = 0$, and the system (3.3.2) reduces to a linear system

$$\begin{cases} \partial_t Q^\theta = \Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta) \\ \partial_t \mathbf{u}^\theta + \nabla P^\theta = \Delta \mathbf{u}^\theta \\ \mathrm{div}\mathbf{u}^\theta = 0 \\ (\mathbf{u}^\theta, Q^\theta)|_{t=0} = (\mathbf{u}_0, Q_0) \end{cases} \tag{3.3.3}$$

in $\mathbb{R}^3 \times [0,\theta]$. For the system (3.3.3), Q^θ and $\mathbf{u}^\theta$ are decouple, and $\mathbf{u}^\theta$ can be found according to the standard theory of Stokes equations, while the equation of Q^θ is a semi-linear parabolic equation which can be solved by the standard method for parabolic equations.

Suppose now that the system (3.3.2) has been solved for some $0 \le k < N-1$. We are going to solve the system (3.3.2)

$$\begin{cases} \partial_t Q_{\alpha\beta} + \mathbf{u}\cdot\nabla\tilde{Q}_{\alpha\beta} - [\omega, \tilde{Q}]_{\alpha\beta} = \Delta Q_{\alpha\beta} - f_{LdG}(Q)_{\alpha\beta} \\ \partial_t \mathbf{u}_\alpha + \tilde{\mathbf{u}}\cdot\nabla\mathbf{u}_\alpha + \partial_\alpha P = \Delta\mathbf{u}_\alpha - \partial_\alpha\tilde{Q}_{\beta\gamma}(\Delta Q - f_{LdG}(Q))_{\beta\gamma} \\ \qquad\qquad\qquad\qquad\qquad + \partial_\beta[\tilde{Q}, \Delta Q - f_{\mathrm{LdG}}(Q)]_{\alpha\beta} \\ \mathrm{div}\mathbf{u} = 0. \end{cases} \tag{3.3.4}$$

in the time interval $[k\theta, (k+1)\theta]$ with the initial data

$$(\mathbf{u}, Q)|_{t=k\theta} = (\mathbf{u}^\theta, Q^\theta)(\cdot, k\theta) \quad \text{in} \quad \mathbb{R}^3, \tag{3.3.5}$$

and

$$\tilde{Q} = \Psi_\theta[Q^\theta] \quad \text{and} \quad \tilde{\mathbf{u}} = \Psi_\theta[\mathbf{u}^\theta].$$

Note that $\tilde{\mathbf{u}}$ and $\tilde{Q}$ are smooth functions in $[k\theta, (k+1)\theta] \times \mathbb{R}^3$.

The existence of $(\mathbf{u}, Q)$ in (3.3.4) may be solved by using the Faedo–Galerkin method. Indeed for a pair of smooth test functions $(\psi, \phi) \in H^2(\mathbb{R}^3, \mathcal{S}_0^{(3)}) \times \mathbf{V}$, the system (3.3.4) turns to be

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3}(\nabla Q, \nabla\psi)\,dx - \int_{\mathbb{R}^3}(\mathbf{u}\cdot\nabla\tilde{Q}, \Delta\psi)\,dx - \int_{\mathbb{R}^3}([-\omega, \tilde{Q}]_{\alpha\beta}, \Delta\psi_{\alpha\beta})\,dx \\ &\quad = -\int_{\mathbb{R}^3}(\Delta Q_{\alpha\beta} - f_{LdG}(Q)_{\alpha\beta}, \Delta\psi_{\alpha\beta})\,dx, \end{aligned} \tag{3.3.6}$$

and

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3}(\mathbf{u}, \phi)\,dx + \int_{\mathbb{R}^3}(\tilde{\mathbf{u}}\cdot\nabla\mathbf{u}, \phi)\,dx + \int_{\mathbb{R}^3}(\nabla\mathbf{u}, \nabla\phi)\,dx \\ &\quad = -\int_{\mathbb{R}^3}\Big(\partial_\alpha\tilde{Q}_{\beta\gamma}(\Delta Q - f_{LdG}(Q))_{\beta\gamma}, \phi_\alpha\Big)\,dx \\ &\qquad - \int_{\mathbb{R}^3}\Big([\tilde{Q}, \Delta Q - f_{\mathrm{LdG}}(Q)]_{\alpha\beta}, \partial_\beta\phi_\alpha\Big)\,dx, \end{aligned} \tag{3.3.7}$$

in the sense of distributions. The system of first order ODE equations (3.3.6)-(3.3.7) can be solved when the test function (ψ, ϕ) are taken to be the basis of $H^2(\mathbb{R}^3, \mathcal{S}_0^{(3)}) \times \mathbf{V}$ up to a short time interval $[k\theta, k\theta + T_0]$. Performing the energy estimate for (3.3.4) as for the original system, we get that for $k\theta \le t \le k\theta + T_0$,

$$\begin{aligned} &\sup_{t\ge k\theta}\int_{\mathbb{R}^3}\Big(|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2 + F_{\mathrm{LdG}}(Q^\theta)\Big)\,dx + \int_{k\theta}^t\int_{\mathbb{R}^3}\Big(|\nabla\mathbf{u}^\theta|^2 + |\Delta Q - f_{\mathrm{LdG}}(Q^\theta)|^2\Big)\,dxds \\ &\quad \le \int_{\mathbb{R}^3}\Big(|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2 + F_{\mathrm{LdG}}(Q^\theta)\Big)(x, k\theta)\,dx. \end{aligned}$$

Hence T_0 can be extended up to θ.

Let $(\mathbf{u}^\theta, P^\theta, Q^\theta)$ be the global weak solution of (3.3.2) and (3.3.5) in Q_T. Then

$$\mathbf{u}^\theta \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T),\ \ Q^\theta \in L_t^\infty H_x^1 \cap L_t^2 H_x^2(Q_T),\ \ P^\theta \in L^2(Q_T).$$

Observe that

$$[\omega^\theta, \Psi_\theta[Q^\theta]] : (\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta)) := -[\Psi_\theta[Q^\theta], \Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta)] : \nabla\mathbf{u}^\theta.$$

Hence, by calculations similar to Lemma 3.2.1, we deduce that $(\mathbf{u}^\theta, Q^\theta)$ satisfies the global energy inequality: for $0 \le t \le T$,

$$
\begin{aligned}
&E(\mathbf{u}^\theta, Q^\theta)(t) + \int_{\mathbb{R}^3\times[0,t]} \Big(|\nabla \mathbf{u}^\theta|^2 + |\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta)|^2\Big)\,dxdt \\
&\le E(\mathbf{u}^\theta, Q^\theta)(0) = \int_{\mathbb{R}^3} (\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{LdG}}(Q_0))(x,t)\,dx.
\end{aligned} \tag{3.3.8}
$$

Direct calculations show that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \Delta Q^\theta \cdot f_{\mathrm{LdG}}(Q^\theta)\,dx \\
&= -a\int_{\mathbb{R}^3} |\nabla Q^\theta|^2\,dx - c\int_{\mathbb{R}^3} \Big(|\nabla Q^\theta|^2|Q^\theta|^2 + \frac{1}{2}|\nabla \operatorname{tr}((Q^\theta)^2)|^2\Big)\,dx \\
&\quad + b\int_{\mathbb{R}^3} \nabla\Big((Q^\theta)^2 - \frac{\operatorname{tr}((Q^\theta)^2)}{3}I_3\Big)\cdot \nabla Q^\theta\,dx \\
&\le -\frac{c}{4}\int_{\mathbb{R}^3} \Big(|\nabla Q^\theta|^2|Q^\theta|^2 + \frac{1}{2}|\nabla \operatorname{tr}((Q^\theta)^2)|^2\Big)\,dx + C(a,b,c)\int_{\mathbb{R}^3} |\nabla Q^\theta|^2\,dx.
\end{aligned}
$$

This, combined with the assumption $c > 0$ and estimate (3.3.8), gives

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} (|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2 + F_{\mathrm{LdG}}(Q^\theta))(x,t)\,dx + 2\int_{\mathbb{R}^3} \Big(|\nabla \mathbf{u}^\theta|^2 + |\Delta Q^\theta|^2\Big)\,dx \\
&+ c\int_{\mathbb{R}^3} \Big(|\nabla Q^\theta|^2|Q^\theta|^2 + \frac{1}{2}|\nabla \operatorname{tr}((Q^\theta)^2)|^2\Big)\,dx \\
&\le C(a,b,c)\int_{\mathbb{R}^3} |\nabla Q^\theta|^2\,dx.
\end{aligned} \tag{3.3.9}
$$

Therefore we deduce from (3.3.9) and Gronwall's inequality that

$$
\begin{aligned}
&\sup_{0\le t\le T}\int_{\mathbb{R}^3} (|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2 + F_{\mathrm{LdG}}(Q^\theta))(x,t)\,dx \\
&\quad + \int_{\mathbb{R}^3\times[0,T]} \Big(|\nabla \mathbf{u}^\theta|^2 + |\Delta Q^\theta|^2\Big)\,dxdt \\
&\le C(a,b,c,T)\Big(\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3)} + \|Q_0\|^2_{H^1(\mathbb{R}^3)}\Big).
\end{aligned} \tag{3.3.10}
$$

From (3.1.1), we know that there exists a $M_0 > 0$, depending on a, b, c, such that

$$
F_{\mathrm{LdG}}(Q) \ge \frac{c}{2}|Q|^4,\ \forall Q \in \mathcal{S}_0^{(3)} \text{ with } |Q| \ge M_0.
$$

This, combined with (3.3.10) and $F_{\mathrm{LdG}}(Q) \geq 0$, implies that

$$
\begin{aligned}
&\sup_{0\leq t\leq T} \int_{\{x\in\mathbb{R}^3:\ |Q^\theta(x,t)|\geq M_0\}} |Q^\theta(x,t)|^4\,dx \\
&\leq \frac{2}{c} \sup_{0\leq t\leq T} \int_{\mathbb{R}^3} F_{\mathrm{LdG}}(Q^\theta)(x,t)\,dx \\
&\leq C(a,b,c,T)\Big(\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3)} + \|Q_0\|^2_{H^1(\mathbb{R}^3)}\Big).
\end{aligned}
\tag{3.3.11}
$$

From (3.3.11), we can conclude that for any compact set $K \subset \mathbb{R}^3$,

$$
\begin{aligned}
&\sup_{0\leq t\leq T} \int_K |Q^\theta(x,t)|^4\,dx \\
&\leq \sup_{0\leq t\leq T} \Big\{ \int_{\{x\in K:\ |Q^\theta(x,t)|\leq M_0\}} |Q^\theta(x,t)|^4\,dx + \int_{\{x\in K:\ |Q^\theta(x,t)|> M_0\}} |Q^\theta(x,t)|^4\,dx \Big\} \\
&\leq |K|M_0^4 + C(a,b,c,T)\Big(\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3)} + \|Q_0\|^2_{H^1(\mathbb{R}^3)}\Big).
\end{aligned}
\tag{3.3.12}
$$

From (3.3.10) and (3.3.12), we have that $\mathbf{u}^\theta$ is uniformly bounded in $L^2_t H^1_x(\mathbb{R}^3 \times [0,T])$, Q^θ is uniformly bounded in $L^2_t H^2_x(K \times [0,T])$ for any compact set $K \subset \mathbb{R}^3$, and ∇Q^θ is uniformly bounded in $L^2_t H^1_x(\mathbb{R}^3 \times [0,T])$. Therefore, after passing to a subsequence, we may assume that as $\theta \to 0$ (or equivalently $N \to \infty$), there exist $\mathbf{u} \in L^\infty_t L^2_x \cap L^2_t H^1_x(\mathbb{R}^3 \times [0,T])$, $Q \in \cap_{R>0} L^\infty_t L^4_x(B_R \times [0,T])$, with $\nabla Q \in L^\infty_t L^2_x \cap L^2_t H^1_x(\mathbb{R}^3 \times [0,T])$, such that

$$
\begin{cases}
Q^\theta \rightharpoonup Q & \text{in} \quad L^2([0,T], L^2(\mathbb{R}^3)), \\
\nabla Q^\theta \rightharpoonup \nabla Q & \text{in} \quad L^2([0,T], H^1(\mathbb{R}^3)), \\
\mathbf{u}^\theta \rightharpoonup \mathbf{u} & \text{in} \quad L^2([0,T], H^1(\mathbb{R}^3)).
\end{cases}
\tag{3.3.13}
$$

Hence by the lower semicontinuity and (3.3.8) we have that

$$
\begin{aligned}
&E(\mathbf{u},Q)(t) + \int_{\mathbb{R}^3\times[0,t]} \Big(|\nabla\mathbf{u}|^2 + |\Delta Q - f_{\mathrm{LdG}}(Q)|^2\Big)\,dxdt \\
&\leq E(\mathbf{u},Q)(0) = \int_{\mathbb{R}^3} (\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{LdG}}(Q_0))(x,t)\,dx
\end{aligned}
\tag{3.3.14}
$$

holds for $0 \leq t \leq T$.

Now we want to estimate the pressure function P^θ. Taking divergence of (3.3.2)$_2$ gives

$$
\begin{aligned}
-\Delta P^\theta &= \operatorname{div}^2(\Psi_\theta[\mathbf{u}^\theta]\otimes\mathbf{u}^\theta) + \operatorname{div}\Big(\nabla(\Psi_\theta[Q^\theta])\cdot(\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta))\Big) \\
&\quad - \operatorname{div}^2\Big([\Psi_\theta[Q^\theta], \Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta)]\Big) \\
&= \operatorname{div}^2(\Psi_\theta[\mathbf{u}^\theta]\otimes\mathbf{u}^\theta) + \operatorname{div}\Big(\nabla(\Psi_\theta[Q^\theta])\cdot(\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta))\Big) \ \text{ in } \mathbb{R}^3.
\end{aligned}
\tag{3.3.15}
$$

Here we have used in the last step the fact that

$$
\operatorname{div}^2[\Psi_\theta[Q^\theta], \Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta)] = 0 \ \text{ in } \ \mathbb{R}^3,
$$

which follows from (3.1.16).

For P^θ, we claim that P^θ in $L^{\frac{5}{3}}(\mathbb{R}^3\times[0,T])$ and

$$
\left\|P^\theta\right\|_{L^{\frac{5}{3}}(\mathbb{R}^3\times[0,T])} \le C\Big(a,b,c,T,\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|Q_0\|_{H^1(\mathbb{R}^3)}\Big), \ \forall\theta\in(0,1]. \tag{3.3.16}
$$

To see this, first observe that (3.3.10) implies $\nabla(\Psi_\theta[Q^\theta]) \in L^\infty_t L^2_x \cap L^2_t H^1_x(\mathbb{R}^3\times[0,T])$. Hence by the Sobolev interpolation inequality we have that

$$
\begin{aligned}
\left\|\nabla(\Psi_\theta[Q^\theta])\right\|_{L^{10}_t L^{\frac{30}{13}}_x(\mathbb{R}^3\times[0,T])} &\le C\left\|\nabla(\Psi_\theta[Q^\theta])\right\|_{L^\infty_t L^2_x\cap L^2_t H^1_x(\mathbb{R}^3\times[0,T])} \\
&\le C\Big(a,b,c,T,\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|Q_0\|_{H^1(\mathbb{R}^3)}\Big).
\end{aligned}
$$

By Hölder's inequality we then have that

$$
\begin{aligned}
&\left\|\nabla(\Psi_\theta[Q^\theta])\cdot(\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta))\right\|_{L^{\frac{5}{3}}_t L^{\frac{15}{14}}_x(\mathbb{R}^3\times[0,T])} \\
&\le \left\|\nabla(\Psi_\theta[Q^\theta])\right\|_{L^{10}_t L^{\frac{30}{13}}_x(\mathbb{R}^3\times[0,T])} \left\|\Delta Q^\theta - f_{\mathrm{LdG}}(Q^\theta)\right\|_{L^2(\mathbb{R}^3\times[0,T])} \\
&\le C\Big(a,b,c,T,\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|Q_0\|_{H^1(\mathbb{R}^3)}\Big).
\end{aligned}
\tag{3.3.17}
$$

By Calderon–Zygmund's L^p-estimate [73], we conclude that $P^\theta \in L^{\frac{5}{3}}([0,T]\times\mathbb{R}^3)$, and

$$\begin{aligned}
&\left\|P^\theta\right\|_{L^{\frac{5}{3}}([0,T]\times\mathbb{R}^3)}\\
&\le C\Big[\left\|\Psi_\theta[\mathbf{u}^\theta]\otimes\mathbf{u}^\theta\right\|_{L^{\frac{5}{3}}(\mathbb{R}^3\times[0,T])}+\left\|\nabla(\Psi_\theta[Q^\theta])\cdot(\Delta Q^\theta-f_{\mathrm{LdG}}(Q^\theta))\right\|_{L_t^{\frac{5}{3}}L_x^{\frac{15}{14}}(\mathbb{R}^3\times[0,T])}\Big]\\
&\le C\Big[\left\|\mathbf{u}^\theta\right\|^2_{L^{\frac{10}{3}}(\mathbb{R}^3\times[0,T])}+\left\|\nabla(\Psi_\theta[Q^\theta])\cdot(\Delta Q^\theta-f_{\mathrm{LdG}}(Q^\theta))\right\|_{L_t^{\frac{5}{3}}L_x^{\frac{15}{14}}(\mathbb{R}^3\times[0,T])}\Big]\\
&\le C\Big(a,b,c,T,\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)},\|Q_0\|_{H^1(\mathbb{R}^3)}\Big).
\end{aligned}$$

It follows from (3.3.16) that we may assume that there exists $P\in L^{\frac{5}{3}}(\mathbb{R}^3\times[0,T])$ such that as $\theta\to 0$,

$$P^\theta \rightharpoonup P \quad\text{in}\quad L^{\frac{5}{3}}(\mathbb{R}^3\times[0,T]). \tag{3.3.18}$$

From $(3.3.2)_2$ and the bounds (3.3.10) and (3.3.11), we have that

$$\begin{aligned}
\partial_t\mathbf{u}^\theta &= -\Psi_\theta[\mathbf{u}^\theta]\cdot\nabla\mathbf{u}^\theta-\nabla P^\theta+\Delta\mathbf{u}^\theta-\nabla(\Psi_\theta[Q^\theta])\cdot(\Delta Q^\theta-f_{\mathrm{LdG}}(Q^\theta))\\
&\qquad+\mathrm{div}([\Psi_\theta[Q^\theta],\Delta Q^\theta-f_{\mathrm{LdG}}(Q^\theta)])\\
&\in L^{\frac{5}{4}}(\mathbb{R}^3\times[0,T])+L^{\frac{5}{3}}([0,T],W^{-1,\frac{5}{3}}(\mathbb{R}^3))+\bigcap_{R>0}L^2([0,T],W^{-1,\frac{4}{3}}(B_R)),
\end{aligned}$$

and for any $0<R<\infty$,

$$\begin{aligned}
&\left\|\partial_t\mathbf{u}^\theta\right\|_{L^{\frac{5}{4}}(\mathbb{R}^3\times[0,T])+L^{\frac{5}{3}}([0,T],W^{-1,\frac{5}{3}}(\mathbb{R}^3))+L^2([0,T],W^{-1,\frac{4}{3}}(B_R))}\\
&\le C\Big(a,b,c,R,T,\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)},\|Q_0\|_{H^1(\mathbb{R}^3)}\Big),\ \ \forall\theta\in(0,1].
\end{aligned}\tag{3.3.19}$$

Similarly, it follows from $(3.3.2)_1$ and the bounds (3.3.10) and (3.3.11) that $\partial_t Q^\theta\in L^{\frac{5}{3}}(\mathbb{R}^3\times[0,T])+\bigcap_{R>0}L^2([0,T],L^{\frac{4}{3}}(B_R))$, and

$$\left\|\partial_t Q^\theta\right\|_{L^{\frac{5}{3}}(\mathbb{R}^3\times[0,T])+L^2([0,T],L^{\frac{4}{3}}(B_R))}\le C\Big(a,b,c,R,T,\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)},\|Q_0\|_{H^1(\mathbb{R}^3)}\Big), \tag{3.3.20}$$

for all $0<R<\infty$ and $\theta\in(0,1]$.

By (3.3.10), (3.3.11), (3.3.19), and (3.3.20), we can apply Aubin–Lions' compactness Lemma ([32]) to conclude that for any $0 < R < \infty$,

$$\Big(\mathbf{u}^\theta, Q^\theta, \nabla Q^\theta\Big) \to \Big(\mathbf{u}, Q, \nabla Q\Big) \quad \text{in} \quad L^3(B_R \times [0,T]), \quad \text{as} \ \ \theta \to 0. \tag{3.3.21}$$

On the other hand, it follows from $F_{\rm LdG}(Q^\theta) \ge 0$ in $\mathbb{R}^3 \times [0,T]$ and (3.3.10) that

$$\sup_{0\le t\le T} \int_{\mathbb{R}^3} |\nabla Q^\theta|^2(x,t)\, dx \le C\Big(a,b,c,T, \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|Q_0\|_{H^1(\mathbb{R}^3)}\Big).$$

Hence by (3.3.21) we also have that for any $1 < p_1 < 6$ and $1 < p_2 < \frac{10}{3}$,

$$Q^\theta \to Q \text{ in } L^{p_1}(B_R \times [0,T]);\ \ \mathbf{u}^\theta \to \mathbf{u} \ \text{ in } L^{p_2}(B_R \times [0,T]) \ \text{ as } \ \theta \to 0. \tag{3.3.22}$$

With the convergences (3.3.13), (3.3.18), and (3.3.21), it is not hard to show that the limit $(\mathbf{u}, P, Q)$ is a weak solution of (3.1.6) and (3.1.7), i.e., it satisfies the system (3.1.6) and (3.1.7) in the sense of distributions (see also [59] Proposition 3). We leave the details to interested readers, besides pointing out that in the sense of distributions, as $\theta \to 0$,

$$\nabla P^\theta - \nabla(\Psi_\theta[Q^\theta]) \cdot f_{\rm LdG}(Q^\theta) \to \nabla P - \nabla Q \cdot f_{\rm LdG}(Q) = \nabla(P - F_{\rm LdG}(Q)).$$

To show that $(\mathbf{u}, P, Q)$ is a suitable weak solution of (3.1.6), we observe that, as in Lemma 3.2.2, we can test equations of $\mathbf{u}^\theta$ in (3.3.2) by $\mathbf{u}^\theta\phi$, and take a spatial derivative of the equation of Q^θ in (3.3.2) and then test it by $\nabla Q^\theta \phi$ for any nonnegative $\phi \in C_0^\infty(\mathbb{R}^3 \times (0,t])$, to obtain the following local energy inequality

$$\begin{aligned}
&\int_{\mathbb{R}^3} \Big(|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2\Big)\phi(x,t)\, dx + 2\int_0^t \int_{\mathbb{R}^3} \Big(|\nabla \mathbf{u}^\theta|^2 + |\Delta Q^\theta|^2\Big)\phi \, dxds \\
&= \int_0^t \int_{\mathbb{R}^3} \Big[\Big(|\mathbf{u}^\theta|^2 + |\nabla Q^\theta|^2\Big)(\partial_t \phi + \Delta\phi) + 2\nabla\Psi_\theta[Q^\theta] \otimes \nabla Q^\theta : \mathbf{u}^\theta \otimes \nabla\phi\Big]\, dxds \\
&+ \int_0^t \int_{\mathbb{R}^3} (|\mathbf{u}^\theta|^2 \Psi_\theta[\mathbf{u}^\theta] \cdot \nabla\phi + 2P^\theta \mathbf{u}^\theta \cdot \nabla\phi + 2\nabla(\Psi_\theta[Q^\theta]) \cdot f_{\rm LdG}(Q^\theta)\mathbf{u}^\theta\phi)\, dxds \\
&+2\int_0^t \int_{\mathbb{R}^3} \Big([\Psi_\theta[Q^\theta], f_{\rm LdG}(Q^\theta)]\Big) : \nabla \mathbf{u}^\theta \phi \, dxds
\end{aligned}$$

$$
\begin{aligned}
&+2\int_0^t\int_{\mathbb{R}^3}\Big(\nabla Q^\theta\otimes\nabla Q^\theta-|\nabla Q^\theta|^2 I_3\Big)):\nabla^2\phi\,dxds\\
&-2\int_0^t\int_{\mathbb{R}^3}([\Psi_\theta[Q^\theta],\Delta Q^\theta-f_{\mathrm{LdG}}(Q^\theta)]):\mathbf{u}^\theta\otimes\nabla\phi\,dxds\\
&-2\int_0^t\int_{\mathbb{R}^3}[\omega^\theta,\Psi_\theta[Q^\theta]]:\nabla Q^\theta\nabla\phi\,dxds\\
&-2\int_0^t\int_{\mathbb{R}^3}\nabla(f_{\mathrm{LdG}}(Q^\theta))\cdot\nabla Q^\theta\phi\,dxds.
\end{aligned}
\tag{3.3.23}
$$

Taking the limit in (3.3.23) as $\theta\to 0$, we see by the lower semicontinuity that it holds

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\Big(|\mathbf{u}|^2+|\nabla Q|^2\Big)\phi(x,t)\,dx+2\int_0^t\int_{\mathbb{R}^3}\Big(|\nabla\mathbf{u}|^2+|\Delta Q|^2\Big)\phi\,dxds\\
&\le\liminf_{\theta\to 0}\Big[\int_{\mathbb{R}^3}\Big(|\mathbf{u}^\theta|^2+|\nabla Q^\theta|^2\Big)\phi(x,t)\,dx+2\int_0^t\int_{\mathbb{R}^3}\Big(|\nabla\mathbf{u}^\theta|^2+|\Delta Q^\theta|^2\Big)\phi\,dxds\Big].
\end{aligned}
$$

While it follows from (3.3.21) and (3.3.22) that

$$
\begin{aligned}
&\lim_{\theta\to 0}\ \text{Right hand side of (3.3.23)}\\
&=\int_0^t\int_{\mathbb{R}^3}\Big(|\mathbf{u}|^2+|\nabla Q|^2\Big)(\partial_t\phi+\Delta\phi)\,dxdt\\
&+\int_0^t\int_{\mathbb{R}^3}(|\mathbf{u}|^2+|\nabla Q|^2+2(P-F_{\mathrm{LdG}}(Q)))\mathbf{u}\cdot\nabla\phi)+2\nabla Q\otimes\nabla Q:\mathbf{u}\otimes\nabla\phi\,dxds\\
&+2\int_0^t\int_{\mathbb{R}^3}\Big[\nabla Q\otimes\nabla Q-|\nabla Q|^2 I_3\Big]:\nabla^2\phi\,dxds\\
&-2\int_0^t\int_{\mathbb{R}^3}[Q,\Delta Q]:\mathbf{u}\otimes\nabla\phi\,dxds\\
&-2\int_0^t\int_{\mathbb{R}^3}\Big(\omega Q-Q\omega\Big):\nabla Q\nabla\phi\,dxds-2\int_0^t\int_{\mathbb{R}^3}\nabla(f_{\mathrm{LdG}}(Q))\cdot\nabla Q\phi\,dxds.
\end{aligned}
$$

Here we have used the following convergence result

$$
\begin{aligned}
\int_0^t\int_{\mathbb{R}^3}\nabla(\Psi_\theta[Q^\theta])\cdot f_{\mathrm{LdG}}(Q^\theta)\mathbf{u}^\theta\phi\,dxds&\to\int_0^t\int_{\mathbb{R}^3}\nabla Q\cdot f_{\mathrm{LdG}}(Q)\mathbf{u}\phi\,dxds\\
&=\int_0^t\int_{\mathbb{R}^3}\nabla(F_{\mathrm{LdG}}(Q))\mathbf{u}\phi\,dxds\\
&=-\int_0^t\int_{\mathbb{R}^3}F_{\mathrm{LdG}}(Q)\mathbf{u}\nabla\phi\,dxds.
\end{aligned}
\tag{3.3.24}
$$

Putting these together yields the desired local energy inequality (3.1.12) for $(\mathbf{u}, P, Q)$. This completes the proof of the existence of suitable weak solution in the first case. $\square$

In the next subsection, we will indicate how to construct a suitable weak solution of (3.3.2) for the Ball–Majumdar potential function.

3.3.2 The Ball–Majumdar potential $F_{\text{bulk}}(Q) = F_{\text{BM}}(Q)$ and $\Omega = \mathbb{T}^3$

Since G_{BM}, given by (3.1.3), is singular outside the physical domain

$$\mathcal{D} = \left\{ Q \in \mathcal{S}_0^{(3)} : \ -\frac{1}{3} < \lambda_{\rm i}(Q) < \frac{2}{3}, \ {\rm i} = 1, 2, 3 \right\},$$

we need to regularize it. For this part, we follow the scheme by Wilkinson [63] (Section 3) very closely. First we regularize it by using the Yosida–Moreau regularization of convex analysis [74] [75]: For $m \in \mathbb{N}^+$, define

$$\widetilde{G}_{\text{BM}}^m(Q) := \inf_{A \in \mathcal{S}_0^{(3)}} \left\{ m|A - Q|^2 + G_{\text{BM}}(A) \right\}, \ \forall Q \in \mathcal{S}_0^{(3)}.$$

Then smoothly mollify $\widetilde{G}_{\text{BM}}^m$ through the standard mollifications:

$$G_{\text{BM}}^m(Q) := \int_{\mathcal{S}_0^{(3)}} \widetilde{G}_{\text{BM}}^m(Q - R)\Phi_m(R)\, dR,$$

where $\Phi_m(R) = m^5 \Phi\,(mR)$, and $\Phi \in C_0^\infty(\mathcal{S}_0^{(3)})$ is nonnegative and satisfies

$$\text{supp } \Phi \subset \left\{ Q \in \mathcal{S}_0^{(3)} : \ |Q| < 1 \right\}, \quad \int_{\mathcal{S}_0^{(3)}} \Phi(R)\, dR = 1.$$

As in [63] Proposition 3.1, G_{BM}^m satisfies the following properties:

- *(G0)* G_{BM}^m *is an isotropic function of* Q.
- *(G1)* $G_{\text{BM}}^m \in C^\infty(\mathcal{S}_0^{(3)})$ *is convex on* $\mathcal{S}_0^{(3)}$.
- *(G2) There exists a constant* $g_0 > 0$, *independent of* m, *such that for any* $m \in \mathbb{N}^+$, $G_{\text{BM}}^m(Q) \geq -g_0$ *holds for all* $Q \in \mathcal{S}_0^{(3)}$.

(G3) $G_{\mathrm{BM}}^m(Q) \le G_{\mathrm{BM}}^{m+1}(Q) \le G_{\mathrm{BM}}(Q)$ *on* $\mathcal{S}_0^{(3)}$ *for all* $m \ge 1$.

(G4) $G_{\mathrm{BM}}^m \to G_{\mathrm{BM}}$ *and* $\nabla_Q G_{\mathrm{BM}}^m \to \nabla_Q G_{\mathrm{BM}}$ *in* $L^\infty_{\mathrm{loc}}(\mathcal{D})$*, as* $m \to \infty$.

(G5) *There exist* $\alpha(m), \beta(m), \gamma(m) > 0$ *such that*

$$\alpha(m)|Q| - \beta(m) \le \Big|\langle \nabla_Q G_{\mathrm{BM}}^m(Q)\rangle\Big| \le \gamma(m)(1+|Q|),\ \forall Q \in \mathcal{S}_0^{(3)}.$$

(G6) *For* $k \ge 2$*, there exists* $C(m,k) > 0$ *such that*

$$\Big|\langle \nabla_Q^k G_{\mathrm{BM}}^m(Q)\rangle\Big| \le C(m,k)(1+|Q|^2),\ \forall Q \in \mathcal{S}_0^{(3)}.$$

For our purpose in this chapter, we also need the following estimate on G_{BM}^m.

Lemma 3.3.1. *For any* $m \in \mathbb{N}^+$, G_{BM}^m *satisfies*

$$G_{\mathrm{BM}}^m(Q) \ge \frac{m}{4}|Q|^2 - g_0,\ \forall Q \in \mathcal{S}_0^{(3)} \text{ with } |Q| \ge 11, \tag{3.3.25}$$

where $g_0 > 0$ *is the same constant given by* (G2).

Proof. Since $G_{\mathrm{BM}}(Q) = \infty$ for $Q \notin \mathcal{D}$, it follows from the definition of $\widetilde{G}_{\mathrm{BM}}^m$ and (G2) that

$$\begin{aligned}\widetilde{G}_{\mathrm{BM}}^m(Q) &= \inf_{A\in\mathcal{D}}\Big\{m|A-Q|^2 + G_{\mathrm{BM}}(A)\Big\}\\ &\ge \inf_{A\in\mathcal{D}}\Big\{m|A-Q|^2\Big\} - g_0\\ &= m\mathrm{dist}^2\Big(Q,\overline{\mathcal{D}}\Big) - g_0.\end{aligned}$$

Thus for any $Q \in \mathcal{S}_0^{(3)}$ with $|Q| \ge 10$, we have

$$\widetilde{G}_{\mathrm{BM}}^m(Q) \ge m(|Q| - \frac{2}{\sqrt{3}})^2 - g_0 \ge m\Big(\frac{|Q|}{\sqrt{2}}\Big)^2 - g_0 = \frac{m}{2}|Q|^2 - g_0.$$

It is not hard to see that this estimate, along with the definition of G_{BM}^m, yields (3.3.25). The proof is now complete. □

Now we set

$$F^m_{\mathrm{BM}}(Q) = G^m_{\mathrm{BM}}(Q) - \frac{\kappa}{2}|Q|^2, \ \forall Q \in \mathcal{S}^{(3)}_0,$$

and

$$f^m_{\mathrm{BM}}(Q) = \Big\langle \nabla_Q G^m_{\mathrm{BM}}(Q) \Big\rangle - \kappa Q, \ \forall Q \in \mathcal{S}^{(3)}_0.$$

Observe that the convexity of G^m_{BM} on $\mathcal{S}^{(3)}_0$ yields that

$$\mathrm{tr}\nabla_Q f^m_{\mathrm{BM}}(Q)(\nabla Q, \nabla Q) = \mathrm{tr}\nabla^2_Q F^m_{\mathrm{BM}}(Q)(\nabla Q, \nabla Q) \geq -\kappa|\nabla Q|^2, \tag{3.3.26}$$

for all $Q \in H^1(\Omega, \mathcal{S}^{(3)}_0)$.

Note that if we view a function on $\mathbb{T}^3$ as a $\mathbb{Z}^3$-periodic function on $\mathbb{R}^3$, then the "retarded" mollification procedure given in the previous subsection can be directly performed on functions defined in $\mathbb{T}^3$.

Similar to the subsection 3.1, we can introduce an approximate system of (3.3.2) for the Ball–Majumdar potential as follows. For $T > 0$ and a fixed large $N \in \mathbb{N}^+$, let $\theta = \dfrac{T}{N} \in (0, 1]$. Then we seek $(\mathbf{u}^{\theta,m}, P^{\theta,m}, Q^{\theta,m})$ that solves

$$\begin{cases} \partial_t Q^{\theta,m} + \mathbf{u}^{\theta,m} \cdot \nabla \Psi_\theta[Q^{\theta,m}] - [\omega^{\theta,m}, \Psi_\theta[Q^{\theta,m}]] \\ = \Delta Q^{\theta,m} - f^m_{\mathrm{BM}}(Q^{\theta,m}), \\ \partial_t \mathbf{u}^{\theta,m} + \Psi_\theta[\mathbf{u}^{\theta,m}] \cdot \nabla \mathbf{u}^{\theta,m} + \nabla P^{\theta,m} \\ = \Delta \mathbf{u}^{\theta,m} - \nabla(\Psi_\theta[Q^{\theta,m}]) \cdot \Big(\Delta Q^{\theta,m} - f^m_{\mathrm{BM}}(Q^{\theta,m}) \Big) \\ +\mathrm{div} \Big([\Psi_\theta[Q^{\theta,m}], \Delta Q^{\theta,m} - f^m_{\mathrm{BM}}(Q^{\theta,m})] \Big), \\ \mathrm{div}\mathbf{u}^{\theta,m} = 0, \end{cases} \tag{3.3.27}$$

in $\mathbb{T}^3 \times [0, T]$, subject to the initial condition (3.3.5). Here $\omega^{\theta,m} = \omega(\mathbf{u}^{\theta,m}) = \dfrac{\nabla \mathbf{u}^{\theta,m} - (\nabla \mathbf{u}^{\theta,m})^\top}{2}$.

Since the system (3.3.27) is simply the system (3.3.2) with f_{LdG} replaced by f^m_{BM}, we can argue as in the subsection 3.1 to find a global weak solution $(\mathbf{u}^{\theta,m}, P^{\theta,m}, Q^{\theta,m})$ of (3.3.27) and (3.3.5) in $Q_T = \mathbb{T}^3 \times [0, T]$ such that

$$\mathbf{u}^{\theta,m} \in L^\infty_t L^2_x \cap L^2_t H^1_x(Q_T), \ Q^{\theta,m} \in L^\infty_t H^1_x \cap L^2_t H^2_x(Q_T), \ P^{\theta,m} \in L^2(Q_T).$$

Moreover, by calculations similar to Lemma 3.2.1, we deduce that $(\mathbf{u}^{\theta,m}, Q^{\theta,m})$ satisfies the global energy inequality: for $0 \le t \le T$,

$$
\begin{aligned}
&E(\mathbf{u}^{\theta,m}, Q^{\theta,m})(t) + \int_{\mathbb{T}^3\times[0,t]} \Big(|\nabla \mathbf{u}^{\theta,m}|^2 + |\Delta Q^{\theta,m} - f^m_{\mathrm{BM}}(Q^{\theta,m})|^2\Big)\,dxdt \\
&= E(\mathbf{u}^{\theta,m}, Q^{\theta,m})(0) \le \int_{\mathbb{T}^3} (\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{BM}}(Q_0))(x)\,dx. \qquad (3.3.28)
\end{aligned}
$$

It follows from (3.3.28) and (3.3.26) that

$$
\begin{aligned}
&\int_{\mathbb{T}^3\times[0,t]} |\Delta Q^{\theta,m} - f^m_{\mathrm{BM}}(Q^{\theta,m})|^2\,dxdt \\
&= \int_{\mathbb{T}^3\times[0,t]} \Big(|\Delta Q^{\theta,m}|^2 + |f^m_{\mathrm{BM}}(Q^{\theta,m})|^2 - 2\Delta Q^{\theta,m}\cdot f^m_{\mathrm{BM}}(Q^{\theta,m})\Big)\,dxdt \\
&= \int_{\mathbb{T}^3\times[0,t]} \Big(|\Delta Q^{\theta,m}|^2 + |f^m_{\mathrm{BM}}(Q^{\theta,m})|^2 + 2\mathrm{tr}\nabla_Q f^m_{\mathrm{BM}}(Q^{\theta,m})(\nabla Q^{\theta,m}, \nabla Q^{\theta,m})\Big)\,dxdt \\
&\ge \int_{\mathbb{T}^3\times[0,t]} \Big(|\Delta Q^{\theta,m}|^2 + |f^m_{\mathrm{BM}}(Q^{\theta,m})|^2 - \kappa|\nabla Q^{\theta,m}|^2\Big)\,dxdt.
\end{aligned}
$$

Substituting this into (3.3.28) and applying Gronwall's inequality, we obtain that for any $0 \le t \le T$,

$$
\begin{aligned}
&E(\mathbf{u}^{\theta,m}, Q^{\theta,m})(t) + \int_{\mathbb{T}^3\times[0,t]} \Big(|\nabla \mathbf{u}^{\theta,m}|^2 + |\Delta Q^{\theta,m}|^2 + |f^m_{\mathrm{BM}}(Q^{\theta,m})|^2\Big)\,dxdt \\
&\le \mathrm{e}^{CT} \int_{\mathbb{T}^3} (\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{BM}}(Q_0))(x)\,dx. \qquad (3.3.29)
\end{aligned}
$$

It follows from (3.3.28) that

$$
\sup_{0\le t\le T} \int_{\mathbb{T}^3} F^m_{\mathrm{BM}}(Q^{\theta,m})(x,t)\,dx \le \int_{\mathbb{T}^3} (\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{BM}}(Q_0)\Big)(x)\,dx.
$$

This, combined with (G2) and (3.3.25), implies that there exists a sufficiently large $m_0 = m_0(\kappa, g_0) \in \mathbb{N}^+$ such that for all $m \geq m_0$,

$$
\begin{aligned}
&\Big(\frac{m}{8} - \frac{\kappa}{2}\Big) \int_{\{x\in\mathbb{T}^3:\ |Q^{\theta,m}(x,t)|\geq 11\}} |Q^{\theta,m}|^2(x,t)\,dx \\
&\leq \int_{\{x\in\mathbb{T}^3:\ |Q^{\theta,m}(x,t)|\geq 11\}} \Big[(\frac{m}{4}|Q^{\theta,m}|^2 - g_0) - \frac{\kappa}{2}|Q^{\theta,m}|^2\Big](x,t)\,dx \\
&\leq \int_{\{x\in\mathbb{T}^3:\ |Q^{\theta,m}(x,t)|\geq 11\}} F^m_{\mathrm{BM}}(Q^{\theta,m})(x,t)\,dx \\
&= \int_{\mathbb{T}^3} F^m_{\mathrm{BM}}(Q^{\theta,m})(x,t)\,dx - \int_{\{x\in\mathbb{T}^3:\ |Q^{\theta,m}(x,t)|\leq 11\}} F^m_{\mathrm{BM}}(Q^{\theta,m})(x,t)\,dx \\
&= \int_{\mathbb{T}^3} F^m_{\mathrm{BM}}(Q^{\theta,m})(x,t)\,dx \\
&- \int_{\{x\in\mathbb{T}^3:\ |Q^{\theta,m}(x,t)|\leq 11\}} \Big[\Big(G^m_{\mathrm{BM}}(Q^{\theta,m}) + g_0\Big) - \frac{\kappa}{2}|Q^{\theta,m}|^2 - g_0\Big](x,t)\,dx \\
&\leq \int_{\mathbb{T}^3} F^m_{\mathrm{BM}}(Q^{\theta,m})(x,t)\,dx + \int_{\{x\in\mathbb{T}^3:\ |Q^{\theta,m}(x,t)|\leq 11\}} (g_0 + \frac{\kappa}{2}|Q^{\theta,m}|^2(x,t))\,dx \\
&\leq \int_{\mathbb{T}^3} (\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{BM}}(Q_0)\Big)(x)\,dx + (g_0 + \frac{121\kappa}{2})|\mathbb{T}^3|
\end{aligned}
$$

holds for any $0 \leq t \leq T$. Therefore we conclude that for $m \geq m_0$, it holds that

$$
\begin{aligned}
&\sup_{0\leq t\leq T} \int_{\mathbb{T}^3} |Q^{\theta,m}|^2(x,t)\,dx \\
&\leq C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)}, \|Q_0\|_{H^1(\mathbb{T}^3)}, \|F_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)}, g_0, \kappa\Big).
\end{aligned}
\tag{3.3.30}
$$

As in subsection 3.1, the pressure function $P^{\theta,m}$ solves

$$
\begin{aligned}
&-\Delta P^{\theta,m} \\
&= \mathrm{div}^2\Big(\Psi_\theta[\mathbf{u}^{\theta,m}] \otimes \mathbf{u}^{\theta,m}\Big) + \mathrm{div}\Big(\nabla(\Psi_\theta[Q^{\theta,m}]) \cdot (\Delta Q^{\theta,m} - f^m_{\mathrm{BM}}(Q^{\theta,m}))\Big)
\end{aligned}
\quad \text{in } \mathbb{T}^3.
\tag{3.3.31}
$$

We can apply the same argument as in the previous subsection to conclude that $P^{\theta,m} \in L^{\frac{5}{3}}(\mathbb{T}^3 \times [0,T])$, and

$$
\Big\|P^{\theta,m}\Big\|_{L^{\frac{5}{3}}(\mathbb{T}^3\times[0,T])} \leq C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)}, \|Q_0\|_{H^1(\mathbb{T}^3)}, \|F_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)}, , g_0, \kappa\Big).
\tag{3.3.32}
$$

With estimates (3.3.32) and (3.3.29), we can utilize the system (3.3.27) to obtain that

$$
\begin{aligned}
&\left\|\partial_t \mathbf{u}^{\theta,m}\right\|_{L^2([0,T],W^{-1,4}(\mathbb{T}^3))} \\
&\le C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|Q_0\|_{H^1(\mathbb{R}^3)}, \|F_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)}, g_0, \kappa\Big),
\end{aligned}
\tag{3.3.33}
$$

$$
\left\|\partial_t Q^{\theta,m}\right\|_{L^2([0,T],L^{\frac{3}{2}}(\mathbb{T}^3))} \le C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{R}^3)}, \|Q_0\|_{H^1(\mathbb{R}^3)}, \|F_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)}, g_0, \kappa\Big), \tag{3.3.34}
$$

uniformly for $\theta \in (0,1]$ and $m \ge m_0$.

For each fixed $m \ge m_0$, we can assume without loss of generality that there exists

$$
(\mathbf{u}^m, P^m, Q^m) \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T) \times L^{\frac{5}{3}}(Q_T) \times L_t^\infty H_x^1(Q_T)
$$

such that as $\theta \to 0$,

$$
\begin{cases}
\mathbf{u}^{\theta,m} \rightharpoonup \mathbf{u}^m \quad \text{in} \quad L_t^2 H_x^1(Q_T), \\
\mathbf{u}^{\theta,m} \to \mathbf{u}^m \quad \text{in} \quad L^p(Q_T) \ \ \forall 1 < p < \dfrac{10}{3}, \\
P^{\theta,m} \rightharpoonup P^m \ \ \text{in} \ \ L^{\frac{5}{3}}(Q_T), \\
Q^{\theta,m} \rightharpoonup Q^m \ \ \text{in} \ \ L_t^2 H_x^2(Q_T), \\
Q^{\theta,m} \to Q^m \ \ \text{in} \ \ L_t^r L_x^s(Q_T), \ \forall 1 < r, s < \infty, \\
\Delta Q^{\theta,m} - f_{\mathrm{BM}}^m(Q^{\theta,m}) \rightharpoonup \Delta Q^m - f_{\mathrm{BM}}^m(Q^m) \quad \text{in} \quad L^2(Q_T), \\
F_{\mathrm{BM}}^m(Q^{\theta,m}) \to F_{\mathrm{BM}}^m(Q^m) \quad \text{in} \quad L^1(Q_T).
\end{cases}
$$

As in subsection 3.1, we can now verify that $(\mathbf{u}^m, P^m, Q^m)$ is a weak solution of

$$
\begin{cases}
\partial_t Q^m + \mathbf{u}^m \cdot \nabla Q^m - [\omega^m, Q^m] = \Delta Q^m - f_{\mathrm{BM}}^m(Q^m), \\
\partial_t \mathbf{u}^m + \mathbf{u}^m \cdot \nabla \mathbf{u}^m + \nabla(P^m - F_{\mathrm{BM}}^m(Q)) \\
= \Delta \mathbf{u}^m - \nabla Q^m \cdot \Delta Q^m + \mathrm{div}[Q^m, \Delta Q^m], \\
\mathrm{div} \mathbf{u}^m = 0,
\end{cases}
\tag{3.3.35}
$$

in $\mathbb{T}^3 \times [0,T]$, subject to the initial condition (3.3.5).

By the lower semicontinuity the following global energy inequality holds: for $0 \le t \le T$,

$$\begin{aligned}
&\int_{\mathbb{T}^3} (\frac{1}{2}|\mathbf{u}^m|^2 + \frac{1}{2}|\nabla Q^m|^2 + F^m_{\mathrm{BM}}(Q^m))(x,t)\,dx \\
&+ \int_{\mathbb{T}^3\times[0,t]} \Big(|\nabla \mathbf{u}^m|^2 + |\Delta Q^m - f^m_{\mathrm{BM}}(Q^m)|^2\Big)\,dxdt \\
&\le \int_{\mathbb{T}^3} (\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{BM}}(Q_0))(x)\,dx,
\end{aligned} \tag{3.3.36}$$

and

$$\begin{aligned}
&E(\mathbf{u}^m, Q^m)(t) + \int_{\mathbb{T}^3\times[0,t]} \Big(|\nabla \mathbf{u}^m|^2 + |\Delta Q^m|^2 + |f^m_{\mathrm{BM}}(Q^m)|^2\Big)\,dxdt \\
&\le \mathrm{e}^{CT}\int_{\mathbb{T}^3} (\frac{1}{2}|\mathbf{u}_0|^2 + \frac{1}{2}|\nabla Q_0|^2 + F_{\mathrm{BM}}(Q_0))(x)\,dx,\ \forall t \in [0,T].
\end{aligned} \tag{3.3.37}$$

Also it follows from (3.3.30), (3.3.32), (3.3.33), and (3.3.37) that

$$\begin{aligned}
&\max\Big\{\big\|Q^m\big\|_{L^\infty_t L^2(Q_T)}, \big\|P^m\big\|_{L^{\frac{5}{3}}(Q_T)}, \big\|\partial_t \mathbf{u}^m\big\|_{L^2_t W^{-1,4}_x(Q_T)}, \big\|\partial_t Q^m\big\|_{L^2_t L^{\frac{3}{2}}_x(Q_T)}\Big\} \\
&\le C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)}, \|Q_0\|_{H^1(\mathbb{T}^3)}, \|F_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)}, g_0, \kappa\Big).
\end{aligned} \tag{3.3.38}$$

Furthermore, we can check that $(\mathbf{u}^m, P^m, Q^m)$ is a suitable weak solution of (3.3.35) by verifying that it satisfies the local inequality (3.1.12) with f_{bulk} replaced by f^m_{BM}.

To show that as $m \to \infty$, $(\mathbf{u}^m, P^m, Q^m)$ gives rise to a suitable weak solution of (3.3.2), we need to first show that Q^m lies in a strictly physical subdomain of the physical domain $\mathcal{D}$, since $G_{\mathrm{BM}}(Q)$ blows up as $Q \in \mathcal{D}$ tends to $\partial\mathcal{D}$. This amounts to establishing an L^∞-estimate of $G_{\mathrm{BM}}(Q)$ in terms of the L^1-norm of $G_{\mathrm{BM}}(Q_0)$, which was previously shown by Wilkinson [63] in a slightly different setting.

More precisely, we need the following version of a generalized maximum principle.

Lemma 3.3.2. *There exist $m_0 \in \mathbb{N}^+$ and a positive constant C_0, independent of m, such that for all $m \ge m_0$,*

$$\big\|G^m_{\mathrm{BM}}(Q^m)(\cdot,t)\big\|_{L^\infty(\mathbb{T}^3)} \le C_0 t^{-\frac{5}{2}}\big\|G_{\mathrm{BM}}(Q_0)\big\|_{L^1(\mathbb{T}^3)} + C_0, \forall 0 < t < T. \tag{3.3.39}$$

For now we assume Lemma 3.3.2, which will be proved in Section 4 below. We may assume without loss of generality that there exists

$$(\mathbf{u}, P, Q) \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(Q_T) \times L^{\frac{5}{3}}(Q_T) \times L_t^\infty H_x^1 \cap L_t^2 H_x^2(Q_T)$$

such that

$$\begin{cases} \mathbf{u}^m \rightharpoonup \mathbf{u} \quad \text{in} \quad L_t^2 H_x^1(Q_T), \\ \mathbf{u}^m \to \mathbf{u} \quad \text{in} \quad L^p(Q_T), \ \forall 1 < p < \dfrac{10}{3}, \\ P^m \rightharpoonup P \quad \text{in} \quad L^{\frac{5}{3}}(Q_T), \\ Q^m \rightharpoonup Q \quad \text{in} \quad L_t^2 H_x^2(Q_T), \\ Q^m \to Q \quad \text{in} \quad L_t^r L_x^s(Q_T), \ \forall 1 < r, s < \infty. \end{cases}$$

From (3.3.39), we can also deduce that for any $0 < \delta < T$,

$$\Big\| G_{\mathrm{BM}}(Q) \Big\|_{L^\infty(\mathbb{T}^3 \times [\delta, T])} \leq (C\delta^{-\frac{5}{2}} + \mathrm{e}^T) \Big\| G_{\mathrm{BM}}(Q_0) \Big\|_{L^1(\mathbb{T}^3)} + \kappa^2 \mathrm{e}^T. \tag{3.3.40}$$

By the logarithmic divergence of G_{BM} as $Q \in \mathcal{D} \to \partial\mathcal{D}$ and (3.3.40), we conclude that for any $\delta > 0$, there exists $\varepsilon_0 = \varepsilon_0(\delta, T) > 0$ such that

$$Q(x,t) \in \mathcal{D}_{\varepsilon_0}, \ \forall (x,t) \in \mathbb{T}^3 \times [\delta, T], \tag{3.3.41}$$

where

$$\mathcal{D}_{\varepsilon_0} := \Big\{ Q \in \mathcal{D} : \ -\frac{1}{3} + \varepsilon_0 \leq \lambda_{\mathrm{i}}(Q(x,t)) \leq \frac{2}{3} - \varepsilon_0, \ \mathrm{i} = 1, 2, 3 \Big\}. \tag{3.3.42}$$

From (3.3.39) and the quadratic growth property of G_{BM}^m, we also see that there exists $C_0 > 0$, independent of m, such that for $m \geq m_0$,

$$|Q^m(x,t)| \leq C_0, \ (x,t) \in \mathbb{T}^3 \times [\delta, T]. \tag{3.3.43}$$

We now claim that

$$f_{\rm BM}^m(Q^m) \rightharpoonup f_{\rm BM}(Q) \text{ in } L^2(\mathbb{T}^3\times[\delta,T]), \text{ as } m\to\infty. \tag{3.3.44}$$

To see this, first observe that (3.3.37) yields that $f_{\rm BM}^m(Q^m)$ is uniformly bounded in $L^2(\mathbb{T}^3\times[0,T])$. Thus there exists a function $\bar f\in L^2(\mathbb{T}^3\times[0,T])$ such that

$$f_{\rm BM}^m(Q^m) \rightharpoonup \bar f \in L^2(\mathbb{T}^3\times[0,T]).$$

Now we want to identify $\bar f$. It follows from $Q^m\to Q$ in $L^2(\mathbb{T}^3\times[0,T])$ that there exists $E_m\subset \mathbb{T}^3\times[0,T]$, with $|E_m|\to 0$, such that

$$Q^m\to Q, \text{ uniformly in } \mathbb{T}^3\times[0,T]\setminus E_m,$$

which, combined with $Q(\mathbb{T}^3\times[\delta,T])\subset\mathcal{D}_{\varepsilon_0}$, yields that for sufficiently large m,

$$Q^m(\mathbb{T}^3\times[\delta,T]\setminus E_m)\subset \mathcal{D}_{\frac{\varepsilon_0}{2}}.$$

Since $f_{\rm BM}^m\to f_{\rm BM}$ in $W^{1,\infty}(\mathcal{D}_{\frac{\varepsilon_0}{2}})$, we conclude that

$$f_{\rm BM}^m(Q^m)\to f_{\rm BM}(Q), \text{ uniformly in } \mathbb{T}^3\times[\delta,T]\setminus E_m.$$

Therefore $\bar f = f_{\rm BM}(Q)$ for a.e. $(x,t)\in\mathbb{T}^3\times[0,T]$, and (3.3.44) holds.

From (3.3.44) and $\Delta Q^m\rightharpoonup \Delta Q$ in $L^2(\mathbb{T}^3\times[0,T])$, as $m\to\infty$, we see that

$$\Delta Q^m - f_{\rm BM}^m(Q^m)\rightharpoonup \Delta Q - f_{\rm BM}(Q) \text{ in } L^2(\mathbb{T}^3\times[0,T]), \text{ as } m\to\infty,$$

With all the estimates at hand, it is rather standard to show that passing to the limit in (3.3.35), as $m\to\infty$ first and $\delta\to 0$ second, yields that $(\mathbf{u},P,Q)$ is a weak solution of (3.3.2). While passing to the limit in the local inequality for $(\mathbf{u}^m,P^m,Q^m)$, as $m\to\infty$ first and then $\delta\to 0$, we can also verify that $(\mathbf{u},P,Q)$ satisfies the local energy inequality (3.1.12) with $f_{\rm bulk}(Q)$ replaced by $f_{\rm BM}(Q)$. □

3.4 Maximum principles

In this section, we will show the maximum principles for any weak solution $(\mathbf{u}, Q)$ of (3.1.6) and (3.3.5) in $\mathbb{R}^3$ with the Landau–De Gennes potential function $F_{\rm LdG}(Q)$, see also [58], [76], and in $\mathbb{T}^3$ with the Ball–Majumdar potential function $F_{\rm BM}(Q)$, see also [63]. These will play important roles in the proof of partial regularity of suitable weak solutions to (3.1.6) in the sections 5 and 6 below.

Lemma 3.4.1. *For* $(\mathbf{u}_0, Q_0) \in \mathbf{H} \times H^1(\mathbb{R}^3, \mathcal{S}_0^{(3)})$*, let* $(\mathbf{u}, Q) \in L_t^2 H_x^1(\mathbb{R}^3 \times \mathbb{R}_+, \mathbb{R}^3) \times L_t^2 H_x^2(\mathbb{R}^3 \times \mathbb{R}_+, \mathcal{S}_0^{(3)})$ *be a weak solution of* (3.1.6)-(3.3.5)*. If, in addition,* $Q_0 \in L^\infty(\mathbb{R}^3, \mathbb{S}_0^{(3)})$ *and* $c > 0$*, then there exists a constant* $C > 0$*, depending on* $\|Q_0\|_{L^\infty(\mathbb{R}^3)}$ *and* a, b, c*, such that*

$$|Q(x,t)| \le C, \ \forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+. \tag{3.4.1}$$

Proof. This is a well-known fact. The readers can find the proof in [58], [76] or [59]. □

Next we will give a proof of Lemma 3.3.2, which guarantees that Q lies inside a strictly physical subdomain $\mathcal{D}_{\varepsilon_0}$ so that $F_{\rm BM}(Q)$ becomes regular and hence $f_{\rm BM}(Q)$ is bounded.

Proof of Lemma 3.3.2. It follows from the chain rule and the equation $(3.3.35)_1$ that $G^m_{\rm BM}(Q^m)$ satisfies in the weak sense

$$\begin{aligned}
&\partial_t(G^m_{\rm BM}(Q^m)) + \mathbf{u}^m \cdot \nabla(G^m_{\rm BM}(Q^m)) \\
&= \Delta(G^m_{\rm BM}(Q^m)) - \mathrm{tr}\nabla^2_Q G^m_{\rm BM}(Q^m)(\nabla Q^m, \nabla Q^m) - f^m_{\rm BM}(Q^m)\langle \nabla_Q G^m_{\rm BM}(Q^m)\rangle, \\
&\le \Delta(G^m_{\rm BM}(Q^m)) - (\langle \nabla_Q G^m_{\rm BM}(Q^m) - \kappa Q^m)\langle \nabla_Q G^m_{\rm BM}(Q^m) \\
&\le \Delta(G^m_{\rm BM}(Q^m)) + \frac{\kappa^2}{2}|Q^m|^2,
\end{aligned} \tag{3.4.2}$$

in $\mathbb{T}^3 \times (0,T]$. Indeed, this can be obtained by multiplying $(3.3.35)_1$ by $\langle \nabla_Q G^m_{\rm BM}(Q^m)\rangle$ and using the fact $G^m_{\rm BM}$ is a smooth convex function. Therefore $G^m_{\rm BM}(Q^m) \in L_t^\infty H_x^1(\mathbb{T}^3 \times [0,T])$ satisfies in the weak sense

$$\partial_t(G^m_{\rm BM}(Q^m)) + \mathbf{u}^m \cdot \nabla(G^m_{\rm BM}(Q^m)) \le \Delta(G^m_{\rm BM}(Q^m)) + \frac{\kappa^2}{2}|Q^m|^2, \ \text{ in } \ \mathbb{T}^3 \times (0,T]. \tag{3.4.3}$$

It follows from (3.3.36) and (3.3.38) that $Q^m \in L_t^2 H_x^2(\mathbb{T}^3 \times [0,T])$. In particular, by Sobolev's embedding theorem, we have that

$$\left\|Q^m\right\|_{L_t^2 L_x^\infty(\mathbb{T}^3\times[0,T])} \le C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)}, \|Q_0\|_{H^1(\mathbb{T}^3)}, \|F_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)}, g_0, \kappa\Big). \tag{3.4.4}$$

Since the drifting coefficient $\mathbf{u}^m$ in (3.4.3) is not smooth and Q^m is not bounded in $\mathbb{T}^3 \times [0,T]$, we can not directly apply the argument of Section 8 in [63] to prove 3.3.39. Here we proceed it by first considering an auxiliary equation with mollifying $\mathbf{u}^m$ as the drifting coefficient. More precisely, let $\mathbf{u}_\epsilon^m$ be a standard ϵ-mollification on $\mathbb{T}^3 \times [0,T]$ for $0 < \epsilon < 1$. Then $\mathbf{u}_\epsilon^m \in C^\infty(\mathbb{T}^3 \times [0,T])$ satisfies $\mathrm{div}\mathbf{u}_\epsilon^m = 0$ and

$$\mathbf{u}_\epsilon^m \to \mathbf{u}^m \text{ in } L_t^2 H_x^1(\mathbb{T}^3 \times [0,T]), \text{ as } \epsilon \to 0.$$

Also let g_ϵ^m be ϵ-mollifications of $|Q^m|^2$ in $\mathbb{T}^3 \times [0,T]$, and h_ϵ^m be ϵ-mollifications of $G_{\mathrm{BM}}^m(Q_0)$ in $\mathbb{T}^3$. Then it follows from (3.4.4) that for all $m \ge m_0$,

$$\left\|g^m\right\|_{L_t^2 L_x^\infty(\mathbb{T}^3\times[0,T])} \le \left\|Q^m\right\|^2_{L_t^2 L_x^\infty(\mathbb{T}^3\times[0,T])},$$

$$\left\|h_\epsilon^m\right\|_{L^1(\mathbb{T}^3)} \le \left\|G_{\mathrm{BM}}(Q_0)\right\|_{L^1(\mathbb{T}^3)},$$

and

$$g_\epsilon^m \to |Q^m|^2 \text{ in } L^3(\mathbb{T}^3 \times [0,T]),\ h_\epsilon^m \to G_{\mathrm{BM}}^m(Q_0) \text{ in } L^1(\mathbb{T}^3), \text{ as } \epsilon \to 0.$$

Now let $v_\epsilon^m \in C^\infty(\mathbb{T}^3 \times [0,T])$ be the unique solution of

$$\begin{cases} \partial_t v_\epsilon^m + \mathbf{u}_\epsilon^m \cdot \nabla v_\epsilon^m = \Delta v_\epsilon^m + \dfrac{\kappa^2}{2} g_\epsilon^m & \text{in } \mathbb{T}^3 \times [0,T], \\ v_\epsilon^m = h_\epsilon^m & \text{on } \mathbb{T}^3 \times \{0\}. \end{cases} \tag{3.4.5}$$

For v_ϵ^m, we will modify the argument as illustrated in [63], Section 8, to achieve that for $0 < t < T$,

$$\left\|v_\epsilon^m(\cdot,t)\right\|_{L^\infty(\mathbb{T}^3)} \le C t^{-\frac{5}{2}} \left\|G_{\mathrm{BM}}(Q_0)\right\|_{L^1(\mathbb{T}^3)} + C_0. \tag{3.4.6}$$

To show (3.4.6), decompose $v_\epsilon^m = v_1 + v_2$, where v_1 solves

$$\begin{cases} \partial_t v_1 + \mathbf{u}_\epsilon^m \cdot \nabla v_1 = \Delta v_1, & \text{in} \quad \mathbb{T}^3 \times [0,T], \\ v_1 = h_\epsilon^m - \fint_{\mathbb{T}^3} h_\epsilon^m, & \text{on} \quad \mathbb{T}^3 \times \{0\}, \end{cases} \tag{3.4.7}$$

and v_2 solves

$$\begin{cases} \partial_t v_2 + \mathbf{u}_\epsilon^m \cdot \nabla v_2 = \Delta v_2 + \frac{\kappa^2}{2} g_\epsilon^m, & \text{in} \quad \mathbb{T}^3 \times [0,T], \\ v_2 = \fint_{\mathbb{T}^3} h_\epsilon^m, & \text{on} \quad \mathbb{T}^3 \times \{0\}. \end{cases} \tag{3.4.8}$$

For v_1, we can apply the $L^1 \to L^\infty$ estimate for advection-diffusion equations on compact manifold [66] as in Lemma 8.1 of [63] to conclude that

$$\left\| v_1(\cdot,t) \right\|_{L^\infty(\mathbb{T}^3)} \le Ct^{-\frac{5}{2}} \left\| h_\epsilon^m - \fint_{\mathbb{T}^3} h_\epsilon^m \right\|_{L^1(\mathbb{T}^3)} \le Ct^{-\frac{5}{2}} \left\| G_{\mathrm{BM}}(Q_0) \right\|_{L^1(\mathbb{T}^3)}, \tag{3.4.9}$$

for $0 < t < T$.

While for v_2, we can multiply $(3.4.8)_1$ by $|v_2|^{p-2} v_2$, $p > 2$, and integrate the resulting equation over $\mathbb{T}^3$ to get

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \left\| v_2(t) \right\|_{L^p(\mathbb{T}^3)}^p &\le \frac{\kappa^2}{2} \left\| g_\epsilon^m(t) \right\|_{L^p(\mathbb{T}^3)} \left\| v_2(t) \right\|_{L^p(\mathbb{T}^3)}^{p-1} \\ &\le \frac{\kappa^2}{2} \left| \mathbb{T}^3 \right|^{\frac{1}{p}} \left\| g_\epsilon^m(t) \right\|_{L^\infty(\mathbb{T}^3)} \left\| v_2(t), \right. \end{aligned}$$

so that

$$\frac{d}{dt} \left\| v_2(t) \right\|_{L^p(\mathbb{T}^3)} \le \frac{\kappa^2}{2} \left| \mathbb{T}^3 \right|^{\frac{1}{p}} \left\| g_\epsilon^m(t) \right\|_{L^\infty(\mathbb{T}^3)},$$

and hence

$$\left\| v_2(t) \right\|_{L^p(\mathbb{T}^3)} \le \left\| v_2(0) \right\|_{L^p(\mathbb{T}^3)} + \frac{\kappa^2}{2} \left| \mathbb{T}^3 \right|^{\frac{1}{p}} \int_0^T \left\| g_\epsilon^m(t) \right\|_{L^\infty(\mathbb{T}^3)} dt, \ \forall 0 < t \le T.$$

Sending $p \to \infty$ and applying (3.4.4), we obtain that for $0 < t < T$,

$$\begin{aligned}
&\Big\|v_2(t)\Big\|_{L^\infty(\mathbb{T}^3)} \\
&\le C\|h_\epsilon^m\|_{L^1(\mathbb{T}^3)} + \frac{\kappa^2}{2}\int_0^T \Big\|Q^m(t)\Big\|_{L^\infty(\mathbb{T}^3)}^2 dt \\
&\le \Big\|G_{\mathrm{BM}}(Q_0)\Big\|_{L^1(\mathbb{T}^3)} + C\Big(\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)}, \|Q_0\|_{H^1(\mathbb{T}^3)}, \|F_{\mathrm{BM}}(Q_0)\|_{L^1(\mathbb{T}^3)}, g_0, \kappa\Big).
\end{aligned} \tag{3.4.10}$$

Putting (3.4.9) and (3.4.10) together yields (3.4.6).

It is not hard to see that as $\epsilon \to 0$, there exists $v^m \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\mathbb{T}^3 \times [0,T])$ such that $v_\epsilon^m \to v^m$ in $L^2(\mathbb{T}^3 \times [0,T])$. Passing to the limit in the equation (3.4.5), we see that v^m is a weak solution of

$$\begin{cases} \partial_t v^m + \mathbf{u}^m \cdot \nabla v^m = \Delta v^m + \dfrac{\kappa^2}{2}|Q^m|^2 & \text{in } \mathbb{T}^3 \times [0,T], \\ v^m = G_{\mathrm{BM}}^m(Q_0) & \text{on } \mathbb{T}^3 \times \{0\}. \end{cases} \tag{3.4.11}$$

Moreover, passing to the limit of (3.4.6), we have that for any $0 < t < T$,

$$\Big\|v^m(\cdot,t)\Big\|_{L^\infty(\mathbb{T}^3)} \le Ct^{-\frac{5}{2}}\Big\|G_{\mathrm{BM}}(Q_0)\Big\|_{L^1(\mathbb{T}^3)} + C_0. \tag{3.4.12}$$

Now observe that by the comparison principle on (3.4.3), we know that for $m \ge m_0$, it holds.

$$G_{\mathrm{BM}}^m(Q^m)(x,t) \le v^m(\cdot,t) \le Ct^{-\frac{5}{2}}\Big\|G_{\mathrm{BM}}(Q_0)\Big\|_{L^1(\mathbb{T}^3)} + C_0,$$

for all $(x,t) \in \mathbb{T}^3 \times [0,T]$. This, combined with (G2), yields (3.3.39). $\square$

Note that passing to the limit in (3.3.39), the suitable weak solution $(\mathbf{u}, P, Q)$ to (3.3.2), constructed in Section 3.2, satisfies that for any $0 < \delta < T$,

$$\Big\|G_{\mathrm{BM}}(Q)\Big\|_{L^\infty(\mathbb{T}^3\times[\delta,T])} \le C_0\delta^{-\frac{5}{2}}\Big\|G_{\mathrm{BM}}(Q_0)\Big\|_{L^1(\mathbb{T}^3)} + C_0. \tag{3.4.13}$$

This completes the proof of Lemma 3.3.2. $\square$

3.5 Partial regularity, Part I

This section is devoted to establishing an ϵ_0-regularity for suitable weak solutions $(\mathbf{u}, Q)$ of (3.1.6) in $\Omega \times (0, \infty)$ in terms of renormalized L^3-norm of $(\mathbf{u}, Q)$. The argument we will present is based on a blowing up argument, motivated by that of Lin [36] on the Navier–Stokes equation, which works equally well for both the Landau–De Gennes potential F_{LdG} and the Ball–Majumdar potential F_{BM}. More precisely, we want to establish the following property.

Lemma 3.5.1. *For any $M > 0$, there exist $\varepsilon_0 > 0$, $0 < \tau_0 < \frac{1}{2}$, and $C_0 > 0$, depending on M, such that if $(\mathbf{u}, Q, P)$ is a suitable weak solution of (3.1.6) in $\Omega \times (0, \infty)$, which satisfies, for $z_0 = (x_0, t_0) \in \Omega \times (r^2, \infty)$ and $r > 0$,*

$$\begin{cases} |Q| \leq M & \text{if } F_{\mathrm{bulk}} = F_{\mathrm{LdG}} \text{ and } \Omega = \mathbb{R}^3, \\ |G_{\mathrm{BM}}(Q)| \leq M & \text{if } F_{\mathrm{bulk}} = F_{\mathrm{BM}} \text{ and } \Omega = \mathbb{T}^3, \end{cases} \quad \text{in } \mathbb{P}_r(z_0), \tag{3.5.1}$$

and

$$r^{-2} \int_{\mathbb{P}_r(z_0)} (|\mathbf{u}|^3 + |\nabla Q|^3) \, dxdt + \left(r^{-2} \int_{\mathbb{P}_r(z_0)} |P|^{\frac{3}{2}} \, dxdt \right)^2 \leq \varepsilon_0^3, \tag{3.5.2}$$

then

$$\begin{aligned} &(\tau_0 r)^{-2} \int_{\mathbb{P}_{\tau_0 r}(z_0)} (|\mathbf{u}|^3 + |\nabla Q|^3) \, dxdt + \left((\tau_0 r)^{-2} \int_{\mathbb{P}_{\tau_0 r}(z_0)} |P|^{\frac{3}{2}} \, dxdt \right)^2 \\ \leq\ & \frac{1}{2} \max \left\{ r^{-2} \int_{\mathbb{P}_r(z_0)} (|\mathbf{u}|^3 + |\nabla Q|^3) \, dxdt + \left(r^{-2} \int_{\mathbb{P}_r(z_0)} |P|^{\frac{3}{2}} \, dxdt \right)^2, C_0 r^3 \right\}. \end{aligned} \tag{3.5.3}$$

Proof. We prove it by contradiction. Suppose that the conclusion were false. Then there exists $M_0 > 0$ such that for any $\tau \in (0, \frac{1}{2})$, we can find $\varepsilon_{\mathrm{i}} \to 0$, $C_{\mathrm{i}} \to \infty$, and $r_{\mathrm{i}} > 0$, and $z_{\mathrm{i}} = (x_{\mathrm{i}}, t_{\mathrm{i}}) \in \mathbb{R}^3 \times (r_{\mathrm{i}}^2, \infty)$ such that

$$\begin{cases} |Q| \leq M_0 & \text{if } F_{\mathrm{bulk}} = F_{\mathrm{LdG}}, \\ |G_{\mathrm{BM}}(Q)| \leq M_0 & \text{if } F_{\mathrm{bulk}} = F_{\mathrm{BM}}, \end{cases} \quad \text{in } \mathbb{P}_{r_{\mathrm{i}}}(z_{\mathrm{i}}), \tag{3.5.4}$$

and

$$r_{\rm i}^{-2}\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})}(|\mathbf{u}|^3+|\nabla Q|^3)\,dxdt+\Big(r_{\rm i}^{-2}\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})}|P|^{\frac{3}{2}}\,dxdt\Big)^2=\varepsilon_{\rm i}^3, \tag{3.5.5}$$

but

$$\begin{aligned}&(\tau r_{\rm i})^{-2}\int_{\mathbb{P}_{\tau r_{\rm i}}(z_{\rm i})}(|\mathbf{u}|^3+|\nabla Q|^3)\,dxdt+\Big((\tau r_{\rm i})^{-2}\int_{\mathbb{P}_{\tau r_{\rm i}}(z_{\rm i})}|P|^{\frac{3}{2}}\,dxdt\Big)^2\\ &>\frac{1}{2}\max\Big\{\varepsilon_{\rm i}^3,C_{\rm i}r_{\rm i}^3\Big\}.\end{aligned} \tag{3.5.6}$$

From (3.5.6), we see that

$$\begin{aligned}C_{\rm i}r_{\rm i}^3 &\le 2(\tau r_{\rm i})^{-2}\int_{\mathbb{P}_{\tau r_{\rm i}}(z_{\rm i})}(|\mathbf{u}|^3+|\nabla Q|^3)\,dxdt+2\Big((\tau r_{\rm i})^{-2}\int_{\mathbb{P}_{\tau r_{\rm i}}(z_{\rm i})}|P|^{\frac{3}{2}}\,dxdt\Big)^2\\ &\le 2\tau^{-4}\Big\{r_{\rm i}^{-2}\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})}(|\mathbf{u}|^3+|\nabla Q|^3)\,dxdt+\Big(r_{\rm i}^{-2}\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})}|P|^{\frac{3}{2}}\,dxdt\Big)^2\Big\}\\ &= 2\tau^{-4}\varepsilon_{\rm i}^3\end{aligned}$$

so that

$$r_{\rm i}\le\Big(\frac{2\varepsilon_{\rm i}^3}{C_{\rm i}\tau^4}\Big)^{\frac{1}{3}}\to 0.$$

Also from (3.5.4), we know that there exist $C_0>0$ and $\delta_0>0$ such that in the case $F_{\rm bulk}=F_{\rm BM}$,

$$Q(z)\in\mathcal{D}_{\delta_0}\ \text{and}\ |f_{\rm BM}(Q(z))|+|\nabla_Q f_{\rm BM}(Q(z))|\le C_0,\ \forall z\in\mathbb{P}_{r_{\rm i}}(z_{\rm i}). \tag{3.5.7}$$

Define a rescaled sequence of maps

$$(\mathbf{u}_{\rm i},Q_{\rm i},P_{\rm i})(x,t)=\Big(r_{\rm i}\mathbf{u},Q,r_{\rm i}^2P\Big)(x_{\rm i}+r_{\rm i}x,t_{\rm i}+r_{\rm i}^2t),\ \forall x\in\mathbb{R}^3,\ t>-1.$$

Then $(\mathbf{u}_{\rm i}, Q_{\rm i}, P_{\rm i})$ is a weak solution of the scaled Beris–Edwards system:

$$\begin{cases} \partial_t Q_{\rm i} + \mathbf{u}_{\rm i} \cdot \nabla Q_{\rm i} - [\omega(\mathbf{u}_{\rm i}), Q_{\rm i}] = \Delta Q_{\rm i} - r_{\rm i}^2 f_{\rm bulk}(Q_{\rm i}), \\ \partial_t \mathbf{u}_{\rm i} + \mathbf{u}_{\rm i} \cdot \nabla \mathbf{u}_{\rm i} + \nabla P_{\rm i} = \Delta \mathbf{u}_{\rm i} - \nabla Q_{\rm i} \cdot \Delta Q_{\rm i} - \operatorname{div}[\Delta Q_{\rm i}, Q_{\rm i}], \\ \operatorname{div} \mathbf{u}_{\rm i} = 0, \end{cases} \tag{3.5.8}$$

where

$$\omega(\mathbf{u}_{\rm i}) = \frac{\nabla \mathbf{u}_{\rm i} - (\nabla \mathbf{u}_{\rm i})^T}{2}.$$

Moreover, $(\mathbf{u}_{\rm i}, Q_{\rm i}, P_{\rm i})$ satisfies

$$\int_{\mathbb{P}_1(0)} (|\mathbf{u}_{\rm i}|^3 + |\nabla Q_{\rm i}|^3)\, dxdt + \Big(\int_{\mathbb{P}_1(0)} |P_{\rm i}|^{\frac{3}{2}}\, dxdt \Big)^2 = \varepsilon_{\rm i}^3, \tag{3.5.9}$$

and

$$\tau^{-2} \int_{\mathbb{P}_\tau(0)} (|\mathbf{u}_{\rm i}|^3 + |\nabla Q_{\rm i}|^3)\, dxdt + \Big(\tau^{-2} \int_{\mathbb{P}_\tau(0)} |P_{\rm i}|^{\frac{3}{2}}\, dxdt \Big)^2 > \frac{1}{2} \max\big\{ \varepsilon_{\rm i}^3, C_{\rm i} r_{\rm i}^3 \big\}. \tag{3.5.10}$$

Define the blowing-up sequence $(\widehat{\mathbf{u}}_{\rm i}, \widehat{Q}_{\rm i}, \widehat{P}_{\rm i}) : \mathbb{P}_1(0) \mapsto \mathbb{R}^3 \times \mathcal{S}_0^3 \times \mathbb{R}$, of $(\mathbf{u}_{\rm i}, Q_{\rm i}, P_{\rm i})$, by letting

$$(\widehat{\mathbf{u}}_{\rm i}, \widehat{Q}_{\rm i}, \widehat{P}_{\rm i})(z) = \Big(\frac{\mathbf{u}_{\rm i}}{\varepsilon_{\rm i}}, \frac{Q_{\rm i} - \overline{Q}_{\rm i}}{\varepsilon_{\rm i}}, \frac{P_{\rm i}}{\varepsilon_{\rm i}} \Big)(z),\ \forall z = (x,t) \in \mathbb{P}_1(0),$$

where

$$\overline{Q}_{\rm i} = \frac{1}{|\mathbb{P}_1(0)|} \int_{\mathbb{P}_1(0)} Q_{\rm i}$$

denotes the average of $Q_{\rm i}$ over $\mathbb{P}_1(0)$. Then $(\widehat{\mathbf{u}}_{\rm i}, \widehat{Q}_{\rm i}, \widehat{P}_{\rm i})$ satisfies

$$\begin{cases} \int_{\mathbb{P}_1(0)} \widehat{Q}_{\rm i} = 0, \\ \int_{\mathbb{P}_1(0)} (|\widehat{\mathbf{u}}_{\rm i}|^3 + |\nabla \widehat{Q}_{\rm i}|^3)\, dxdt + \Big(\int_{\mathbb{P}_1(0)} |\widehat{P}_{\rm i}|^{\frac{3}{2}}\, dxdt \Big)^2 = 1, \\ \tau^{-2} \int_{\mathbb{P}_\tau(0)} (|\widehat{\mathbf{u}}_{\rm i}|^3 + |\nabla \widehat{Q}_{\rm i}|^3)\, dxdt + \Big(\tau^{-2} \int_{\mathbb{P}_\tau(0)} |\widehat{P}_{\rm i}|^{\frac{3}{2}}\, dxdt \Big)^2 > \frac{1}{2} \max\big\{ 1, C_{\rm i} \frac{r_{\rm i}^3}{\varepsilon_{\rm i}^3} \big\}, \end{cases} \tag{3.5.11}$$

and $(\widehat{\mathbf{u}}_{\rm i}, \widehat{Q}_{\rm i}, \widehat{P}_{\rm i})$ is a suitable weak solution of the following scaled Beris–Edwards equation:

$$
\begin{cases}
\partial_t \widehat{Q}_{\rm i} + \varepsilon_{\rm i} \widehat{\mathbf{u}}_{\rm i} \cdot \nabla \widehat{Q}_{\rm i} - [\omega(\widehat{\mathbf{u}}_{\rm i}), Q_{\rm i}] = \Delta \widehat{Q}_{\rm i} - \dfrac{r_{\rm i}^2}{\varepsilon_{\rm i}} f_{\rm bulk}(Q_{\rm i}), \\
\partial_t \widehat{\mathbf{u}}_{\rm i} + \varepsilon_{\rm i} \widehat{\mathbf{u}}_{\rm i} \cdot \nabla \widehat{\mathbf{u}}_{\rm i} + \nabla \widehat{P}_{\rm i} = \Delta \widehat{\mathbf{u}}_{\rm i} - \varepsilon_{\rm i} \nabla \widehat{Q}_{\rm i} \Delta \widehat{Q}_{\rm i} + \mathrm{div}[Q_{\rm i}, \Delta \widehat{Q}_{\rm i}] \\
\mathrm{div} \widehat{\mathbf{u}}_{\rm i} = 0,
\end{cases}
\tag{3.5.12}
$$

From (3.5.11), we assume that there exists

$$
(\widehat{\mathbf{u}}, \widehat{Q}, \widehat{P}) \in L^3(\mathbb{P}_1(0)) \times L_t^3 W_x^{1,3}(\mathbb{P}_1(0)) \times L^{\frac{3}{2}}(\mathbb{P}_1(0))
$$

such that, after passing to a subsequence,

$$
(\widehat{\mathbf{u}}_{\rm i}, \widehat{Q}_{\rm i}, \widehat{P}_{\rm i}) \rightharpoonup (\widehat{\mathbf{u}}, \widehat{Q}, \widehat{P}) \text{ in } L^3(\mathbb{P}_1(0)) \times L_t^3 W_x^{1,3}(\mathbb{P}_1(0)) \times L^{\frac{3}{2}}(\mathbb{P}_1(0)).
$$

It follows from (3.5.11) and the lower semicontinuity that

$$
\int_{\mathbb{P}_1(0)} (|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{Q}|^3) + \Big(\int_{\mathbb{P}_1(0)} |\widehat{P}|^{\frac{3}{2}} \Big)^2 \leq 1. \tag{3.5.13}
$$

Moreover, we claim that

$$
\big\| \widehat{\mathbf{u}}_{\rm i} \big\|_{L_t^\infty L_x^2(\mathbb{P}_{\frac{1}{2}}(0)) \cap L_t^2 H_x^1(\mathbb{P}_{\frac{1}{2}}(0))} + \big\| \nabla \widehat{Q}_{\rm i} \big\|_{L_t^\infty L_x^2(\mathbb{P}_{\frac{1}{2}}(0)) \cap L_t^2 H_x^1(\mathbb{P}_{\frac{1}{2}}(0))} \leq C < \infty. \tag{3.5.14}
$$

To show (3.5.14), choose a cut-off function $\phi \in C_0^\infty(\mathbb{P}_1(0))$ such that

$$
0 \leq \phi \leq 1, \ \phi \equiv 1 \text{ on } \mathbb{P}_{\frac{1}{2}}(0), \text{ and } |\partial_t \phi| + |\nabla \phi| + |\nabla^2 \phi| \leq C.
$$

Define

$$
\phi_{\rm i}(x,t) = \phi\Big(\frac{x - x_{\rm i}}{r_{\rm i}}, \frac{t - t_{\rm i}}{r_{\rm i}^2} \Big), \ \forall (x,t) \in \mathbb{R}^3 \times (0, \infty).
$$

Applying Lemma 2.2 with ϕ replaced by $\phi_{\rm i}^2$ and applying Hölder's inequality, we would arrive at

$$
\begin{aligned}
&\sup_{t_{\rm i}-\frac{r_{\rm i}^2}{4}\le t\le t_{\rm i}}\int_{B_{r_{\rm i}}(x_{\rm i})}(|\mathbf{u}|^2+|\Delta Q|^2)\phi_{\rm i}^2\,dx+\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})}(|\nabla\mathbf{u}|^2+|\nabla^2 Q|^2)\phi_{\rm i}^2\,dxdt\\
&\le C\Big[\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})}(|\mathbf{u}|^2+|\nabla Q|^2)|(\partial_t+\Delta)\phi_{\rm i}^2|\,dxdt\\
&\qquad+\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})}(|\mathbf{u}|^2+|\nabla Q|^2+|P|)|\mathbf{u}||\nabla\phi_{\rm i}^2|\,dxdt+\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})}|\nabla Q|^2|\nabla^2(\phi_{\rm i}^2)|\\
&\qquad+\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})}(|\Delta Q|+|f_{\rm bulk}(Q)|)|\mathbf{u}||\nabla\phi_{\rm i}^2|+|\nabla_Q f_{\rm bulk}(Q)||\nabla Q|^2\phi_{\rm i}^2\,dxdt\Big].
\end{aligned}
$$

Observe that

$$
\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})}|\Delta Q||\mathbf{u}||\nabla\phi_{\rm i}^2|\,dxdt\le\frac{1}{2}\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})}|\Delta Q|^2\phi_{\rm i}^2\,dxdt+C\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})}|\mathbf{u}|^2|\nabla\phi_{\rm i}|^2\,dxdt.
$$

Substituting this into the above inequality and performing rescaling, we obtain that

$$
\begin{aligned}
&\sup_{-\frac{1}{4}\le t\le 0}\int_{B_{\frac{1}{2}}(0)}(|\widehat{\mathbf{u}}_{\rm i}|^2+|\Delta\widehat{Q}_{\rm i}|^2)\,dx+\int_{\mathbb{P}_{\frac{1}{2}}(0)}(|\nabla\widehat{\mathbf{u}}_{\rm i}|^2+|\nabla^2\widehat{Q}_{\rm i}|^2)\,dxdt\\
&\le C\Big[\int_{\mathbb{P}_1(0)}(|\widehat{\mathbf{u}}_{\rm i}|^2+|\nabla\widehat{Q}_{\rm i}|^2)+(\varepsilon_{\rm i}|\widehat{\mathbf{u}}_{\rm i}|^2+\varepsilon_{\rm i}|\nabla\widehat{Q}_{\rm i}|^2+|\widehat{P}_{\rm i}|)|\widehat{\mathbf{u}}_{\rm i}|\,dxdt\Big]\\
&\quad+C\Big[\int_{\mathbb{P}_1(0)}\frac{r_{\rm i}^2}{\varepsilon_{\rm i}}|\widehat{\mathbf{u}}_{\rm i}|\,dxdt+r_{\rm i}^2\int_{\mathbb{P}_1(0)}|\nabla\widehat{Q}_{\rm i}|^2\,dxdt\Big]\\
&\le C\Big(1+\frac{r_{\rm i}^2}{\varepsilon_{\rm i}}+r_{\rm i}^2\Big)\le C.
\end{aligned}\tag{3.5.15}
$$

This yields (3.5.14). From (3.5.14), we may also assume that

$$
(\widehat{\mathbf{u}}_{\rm i},\widehat{Q}_{\rm i})\rightharpoonup(\widehat{\mathbf{u}},\widehat{Q})\ \text{in}\ L_t^2H_x^1(\mathbb{P}_{\frac{1}{2}}(0))\times L_t^2H_x^2(\mathbb{P}_{\frac{1}{2}}(0)).\tag{3.5.16}
$$

Since $r_{\rm i}\le\varepsilon_{\rm i}$ and by (3.5.7) $|Q_{\rm i}|\le M_0$ and $|f_{\rm bulk}(Q_{\rm i})|+|\nabla_Q f_{\rm bulk}(Q_{\rm i})|\le C_0$ in $\mathbb{P}_1(0)$, there exists a constant $\overline{Q}\in\mathcal{S}_0^{(3)}$, with $|\overline{Q}|\le M_0$, such that, after passing to a subsequence,

$$
Q_{\rm i}\to\overline{Q}\quad\text{in}\quad L^3(\mathbb{P}_{\frac{1}{2}}(0)),
$$

and

$$\frac{r_{\rm i}^2}{\varepsilon_{\rm i}}f_{\rm bulk}(Q_{\rm i})\to 0 \quad \text{in} \quad L^\infty(\mathbb{P}_{\frac12}(0)).$$

Hence $(\widehat{\mathbf{u}},\widehat{Q},\widehat{P}):\mathbb{P}_{\frac12}(0)\mapsto\mathbb{R}^3\times\mathcal{S}_0^{(3)}\times\mathbb{R}$ solves the linear system:

$$\begin{cases}\partial_t\widehat{Q}-\Delta\widehat{Q}=[\omega(\widehat{\mathbf{u}}),\overline{Q}],\\ \partial_t\widehat{\mathbf{u}}-\Delta\widehat{\mathbf{u}}+\nabla\widehat{P}=\operatorname{div}([\overline{Q},\Delta\widehat{Q}]),\\ \operatorname{div}\widehat{\mathbf{u}}=0,\end{cases}\tag{3.5.17}$$

Applying Lemma 3.5.2 and (3.5.13), we know that

$$(\widehat{\mathbf{u}},\widehat{Q})\in C^\infty(\mathbb{P}_{\frac14}),\ \ \widehat{P}\in L^\infty([-(\frac14)^2,0],C^\infty(B_{\frac14}(0)))$$

satisfies

$$\begin{aligned}&\tau^{-2}\int_{\mathbb{P}_\tau(0)}(|\widehat{\mathbf{u}}|^3+|\nabla\widehat{Q}|^3)\,dxdt+\Big(\tau^{-2}\int_{\mathbb{P}_\tau(0)}|\widehat{P}|^{\frac32}\,dxdt\Big)^2\\ &\le C\tau^3\int_{\mathbb{P}_{\frac12}(0)}(|\widehat{\mathbf{u}}|^3+|\nabla\widehat{Q}|^3)\,dxdt+\Big(\int_{\mathbb{P}_1(0)}|\widehat{P}|^{\frac32}\Big)^2\\ &\le C\tau^3,\ \forall\ \tau\in(0,\frac18).\end{aligned}\tag{3.5.18}$$

We now claim that

$$(\widehat{\mathbf{u}}_{\rm i},\nabla\widehat{Q}_{\rm i})\to(\widehat{\mathbf{u}},\nabla\widehat{Q}) \text{ in } L^3(\mathbb{P}_{\frac38}(0)).\tag{3.5.19}$$

To prove (3.5.19), first observe that (3.5.15) and the equation (3.5.12) imply that

$$\partial_t\widehat{\mathbf{u}}_{\rm i}\in\Big(L_t^2H^{-1}+L_t^2L_x^{\frac65}+L_t^{\frac32}W_x^{-1,\frac32}\Big)\Big(\mathbb{P}_{\frac38}(0)\Big);\ \partial_t\widehat{Q}_{\rm i}\in L_t^{\frac32}L_x^{\frac32}(\mathbb{P}_{\frac38}(0)),$$

enjoy the following uniform bounds:

$$\begin{aligned}&\Big\|\partial_t\widehat{\mathbf{u}}_{\rm i}\Big\|_{\big(L_t^2H_x^{-1}+L_t^2L_x^{\frac65}+L_t^{\frac32}W_x^{-1,\frac32}\big)(\mathbb{P}_{\frac38}(0))}\\ &\le C\Big[\|\widehat{\mathbf{u}}_{\rm i}\|_{L_t^\infty L_x^2(\mathbb{P}_{\frac12}(0))}+\|\nabla\widehat{\mathbf{u}}_{\rm i}\|_{L_t^2L_x^2(\mathbb{P}_{\frac12}(0))}+\|\nabla\widehat{Q}_{\rm i}\|_{L^3(\mathbb{P}_{\frac12}(0))}^2+\|\nabla^2\widehat{Q}_{\rm i}\|_{L^2(\mathbb{P}_{\frac12}(0))}\Big]\end{aligned}$$

$$\leq C,$$

and

$$\begin{aligned}
&\left\|\partial_t\widehat{Q}_{\rm i}\right\|_{L^{\frac{3}{2}}(\mathbb{P}_{\frac{3}{8}}(0))}\\
&\leq C\Big[\|\widehat{Q}_{\rm i}\|_{L^2_tH^1_x(\mathbb{P}_{\frac{1}{2}}(0))}+\|\nabla\widehat{\mathbf{u}}_{\rm i}\|_{L^2(\mathbb{P}_{\frac{1}{2}}(0))}+\|\nabla\widehat{Q}_{\rm i}\|_{L^3(\mathbb{P}_{\frac{1}{2}}(0))}+\|\widehat{\mathbf{u}}_{\rm i}\|_{L^3(\mathbb{P}_{\frac{1}{2}}(0))}\Big]\\
&\leq C.
\end{aligned}$$

Thus we can apply Aubin-Lions' compactness Lemma to conclude the L^3-strong convergence as in (3.5.19).

It follows from the L^3-strong convergence property (3.5.19) that for any $\tau\in(0,\frac{1}{8})$,

$$\tau^{-2}\int_{\mathbb{P}_\tau(0)}(|\widehat{\mathbf{u}}_{\rm i}|^3+|\nabla\widehat{Q}_{\rm i}|^3)=\tau^{-2}\int_{\mathbb{P}_\tau(0)}(|\widehat{\mathbf{u}}|^3+|\nabla\widehat{Q}|^3)+\tau^{-2}o(1)\leq C\tau^3+\tau^{-2}o(1),\quad (3.5.20)$$

where $o(1)$ stands for a quantity such that $\lim\limits_{{\rm i}\to\infty}o(1)=0$.

Now we need to estimate the pressure $\widehat{P}_{\rm i}$. First, by taking divergence of the second equation $(3.5.8)_2$, we see that $\widehat{P}_{\rm i}$ solves

$$\Delta\widehat{P}_{\rm i}=-\epsilon_{\rm i}{\rm div}^2\Big[\widehat{\mathbf{u}}_{\rm i}\otimes\widehat{\mathbf{u}}_{\rm i}+(\nabla\widehat{Q}_{\rm i}\otimes\nabla\widehat{Q}_{\rm i}-\frac{1}{2}|\nabla\widehat{Q}_{\rm i}|^2I_3)\Big]\ \text{in}\ B_1,\qquad (3.5.21)$$

where we have applied Lemma 3.2.3 to guarantee

$${\rm div}^2[Q_{\rm i},\Delta\widehat{Q}_{\rm i}]=0\ \text{ in }\ B_1.$$

We need to show that

$$\tau^{-2}\int_{\mathbb{P}_\tau(0)}|\widehat{P}_{\rm i}|^{\frac{3}{2}}\,dxdt\leq C\tau^{-2}(\varepsilon_{\rm i}+o(1))+C\tau,\ \forall {\rm i}\geq 1.\qquad (3.5.22)$$

To prove (3.5.22), let $\eta \in C_0^\infty(B_1(0))$ be a cut-off function such that $\eta \equiv 1$ in $B_{\frac{3}{8}}(0)$, $0 \le \eta \le 1$. For any $-(\frac{3}{8})^2 \le t \le 0$, define $\widehat{P}_{\rm i}^{(1)}(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}$ by letting

$$\widehat{P}_{\rm i}^{(1)}(x,t) = \int_{\mathbb{R}^3} \nabla_x^2 G(x-y)\eta(y)\varepsilon_{\rm i}[\widehat{\mathbf{u}}_{\rm i} \otimes \widehat{\mathbf{u}}_{\rm i} + (\nabla\widehat{Q}_{\rm i} \otimes \nabla\widehat{Q}_{\rm i} - \frac{1}{2}|\nabla\widehat{Q}_{\rm i}|^2 I_3)](y,t)\,dy, \tag{3.5.23}$$

where $G(\cdot)$ is the fundamental solution of $-\Delta$ in $\mathbb{R}^3$. Then it is easy to check that $\widehat{P}_{\rm i}^{(2)}(\cdot,t) = (\widehat{P}_{\rm i} - \widehat{P}_{\rm i}^{(1)})(\cdot,t)$ satisfies

$$-\Delta \widehat{P}_{\rm i}^{(2)}(\cdot,t) = 0 \ \text{ in } \ B_{\frac{3}{8}}(0). \tag{3.5.24}$$

For $\widehat{P}_{\rm i}^{(1)}$, we can apply the Calderon-Zygmund theory to show that

$$\big\|\widehat{P}_{\rm i}^{(1)}\big\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \le C\epsilon_{\rm i}\Big[\|\widehat{\mathbf{u}}_{\rm i}\|^2_{L^3(B_1(0))} + \|\nabla\widehat{Q}_{\rm i}\|^2_{L^3(B_1(0))}\Big] \tag{3.5.25}$$

so that

$$\begin{aligned} \big\|\widehat{P}_{\rm i}^{(1)}\big\|_{L^{\frac{3}{2}}(\mathbb{P}_{\frac{1}{3}}(0))} &\le C\varepsilon_{\rm i}(\|\widehat{\mathbf{u}}_{\rm i}\|^2_{L^3(\mathbb{P}_1(0))} + \|\nabla\widehat{Q}_{\rm i}\|^2_{L^3(\mathbb{P}_1(0))}) \\ &\le C(\varepsilon_{\rm i} + o(1)). \end{aligned} \tag{3.5.26}$$

From the standard theory on harmonic functions, $\widehat{P}_{\rm i}^{(2)}(\cdot,t) \in C^\infty(B_{\frac{1}{2}}(0))$ satisfies: for any $0 < \tau < \dfrac{1}{4}$,

$$\begin{aligned} \tau^{-2}\int_{\mathbb{P}_\tau(0)} |\widehat{P}_{\rm i}^{(2)}|^{\frac{3}{2}} &\le C\tau\int_{\mathbb{P}_{\frac{1}{3}}(0)} |\widehat{P}_{\rm i}^{(2)}|^{\frac{3}{2}} \le C\tau\Big[\int_{\mathbb{P}_{\frac{1}{3}}(0)} \big(|\widehat{P}_{\rm i}|^{\frac{3}{2}} + |\widehat{P}_{\rm i}^{(1)}|^{\frac{3}{2}}\big) \\ &\le C\tau(1 + \varepsilon_{\rm i} + o(1)). \end{aligned} \tag{3.5.27}$$

Putting (3.5.26) and (3.5.27) together, we obtain (3.5.22).

It follows from (3.5.20) and (3.5.22) that there exist sufficiently small $\tau_0 \in (0, \frac{1}{4})$ and sufficiently large $\rm i_0$, depending on τ_0, such that for any $\rm i \ge i_0$, it holds that

$$\tau_0^{-2}\int_{\mathbb{P}_{\tau_0}(0)} (|\widehat{\mathbf{u}}_{\rm i}|^3 + |\nabla\widehat{Q}_{\rm i}|^3)\,dxdt + \big(\tau_0^{-2}\int_{\mathbb{P}_{\tau_0}(0)} |\widehat{P}_{\rm i}|^{\frac{3}{2}}\,dxdt)^2 \le \frac{1}{4}.$$

This contradicts to (3.5.11). The proof of Lemma 3.5.1 is completed. $\square$

We now need to establish the smoothness of the limit equation (3.5.17), namely,

Lemma 3.5.2. *Assume that* $(\widehat{\mathbf{u}}, \widehat{Q}) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(\mathbb{P}_{\frac{1}{2}}) \times (L_t^\infty H_x^1 \cap L_t^2 H_x^2)(\mathbb{P}_{\frac{1}{2}})$ *and* $\widehat{P} \in L^{\frac{3}{2}}(\mathbb{P}_{\frac{1}{2}})$ *is a weak solution of the linear system* (3.5.17)*, then* $(\widehat{\mathbf{u}}, \widehat{Q}) \in C^\infty(\mathbb{P}_{\frac{1}{4}})$*, and the following estimate*

$$\theta^{-2} \int_{\mathbb{P}_\theta} (|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{Q}|^3 + |\widehat{P}|^{\frac{3}{2}}) \le C\theta^3 \int_{\mathbb{P}_{\frac{1}{2}}} (|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{Q}|^3 + |\widehat{P}|^{\frac{3}{2}}) \tag{3.5.28}$$

holds for any $\theta \in (0, \frac{1}{8})$*.*

Proof. The regularity of the limit equation (3.5.17) doesn't follow from the standard theory of linear parabolic equations in [53], since the source term $\operatorname{div}(\overline{Q}\Delta\widehat{Q} - \Delta\widehat{Q}\overline{Q})$ in the second equation of (3.5.17) depends on third order derivatives of $\widehat{Q}$. It is based on higher order energy methods, for which the cancellation property, as in the derivation of local energy inequality for suitable weak solutions of (3.1.6), plays a critical role.

For nonnegative multiple indices α, β, and γ such that $\alpha = \beta + \gamma$ and γ is of order 1, it is easy to see that $(\nabla^\alpha \widehat{Q}, \nabla^\beta \widehat{\mathbf{u}}, \nabla^\beta \widehat{P})$ satisfies

$$\begin{cases} \partial_t(\nabla^\alpha \widehat{Q}) - \Delta(\nabla^\alpha \widehat{Q}) = [\omega(\nabla^\alpha \widehat{\mathbf{u}}), \overline{Q}], \\ \partial_t(\nabla^\beta \widehat{\mathbf{u}}) - \Delta(\nabla^\beta \widehat{\mathbf{u}}) + \nabla(\nabla^\beta \widehat{P}) = \operatorname{div}[\overline{Q}, \Delta(\nabla^\beta \widehat{Q})], \\ \operatorname{div}(\nabla^\beta \widehat{\mathbf{u}}) = 0, \end{cases} \tag{3.5.29}$$

Now we want to derive an arbitrarily higher order local energy inequality for (3.5.29). For any given $\phi \in C_0^\infty(\mathbb{P}_{\frac{1}{2}}(0))$, multiplying the first equation of (3.5.29) by $\nabla^\alpha \widehat{Q}\phi^2$ and integrating over $\mathbb{R}^3$, we obtain that by summing over all γ,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} \frac{1}{2}|\nabla(\nabla^\beta \widehat{Q})|^2\phi^2 + \int_{\mathbb{R}^3} |\nabla^2(\nabla^\beta \widehat{Q})|^2\phi^2 \\ &= \int_{\mathbb{R}^3} \frac{1}{2}|\nabla(\nabla^\beta \widehat{Q})|^2(\partial_t + \Delta)\phi^2 \\ &\quad + \int_{\mathbb{R}^3} [\overline{Q}, \omega(\nabla^\beta \widehat{\mathbf{u}})] : (\Delta(\nabla^\beta \widehat{Q})\phi^2 + \nabla(\nabla^\beta \widehat{Q}) \cdot \nabla\phi^2). \end{aligned} \tag{3.5.30}$$

While, by multiplying the second equation of (3.5.17) by $\nabla^\beta \widehat{\mathbf{u}} \phi^2$ and integrating over $\mathbb{R}^3$, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} \frac{1}{2}|\nabla^\beta \widehat{\mathbf{u}}|^2\phi^2 + \int_{\mathbb{R}^3} |\nabla(\nabla^\beta \widehat{\mathbf{u}})|^2\phi^2 \\
&= \int_{\mathbb{R}^3} \frac{1}{2}|\nabla^\beta \widehat{\mathbf{u}}|^2(\partial_t + \Delta)\phi^2 + \int_{\mathbb{R}^3} \nabla^\beta \widehat{P} \nabla^\beta \widehat{\mathbf{u}} \cdot \nabla\phi^2 \\
&+ \int_{\mathbb{R}^3} [\overline{Q}, \Delta(\nabla^\beta \widehat{Q})] : (\nabla(\nabla^\beta \widehat{\mathbf{u}})\phi^2 + \nabla^\beta \widehat{\mathbf{u}} \otimes \nabla\phi^2).
\end{aligned} \tag{3.5.31}$$

As in above, we observe that

$$\int_{\mathbb{R}^3} [[\overline{Q}, \omega(\nabla^\beta \widehat{\mathbf{u}})] : \Delta(\nabla^\beta \widehat{Q})\phi^2 + [\overline{Q}, \Delta(\nabla^\beta \widehat{Q})] : \nabla(\nabla^\beta \widehat{\mathbf{u}})\phi^2] = 0.$$

By integration by parts we have that

$$\int_{\mathbb{R}^3} \nabla^\beta \widehat{P} \nabla^\beta \widehat{\mathbf{u}} \cdot \nabla\phi^2 = (-1)^{|\beta|} \int_{\mathbb{R}^3} \widehat{\mathbf{u}} \cdot \nabla^\beta(\nabla^\beta \widehat{P} \nabla\phi^2). \tag{3.5.32}$$

It follows from the second equation of (3.5.17) that $\widehat{P}$ solves

$$\Delta \widehat{P} = \operatorname{div}^2[\overline{Q}, \Delta\widehat{Q}] = 0, \text{ in } B_{\frac{1}{2}(0)},$$

where we have applied Lemma 3.2.3. Hence by the standard regularity theory of harmonic functions,

$$\int_{B_{\frac{3}{8}(0)}} |\nabla^l \widehat{P}|^{\frac{3}{2}} \le C \int_{B_{\frac{1}{2}}(0)} |\widehat{P}|^{\frac{3}{2}}, \quad l = k, k+1, ..., 2k, \tag{3.5.33}$$

so that by Young's inequality we can derive from (3.5.32) and (3.5.33) that

$$\left| \int_{\mathbb{R}^3} \nabla^\beta \widehat{P} \nabla^\beta \widehat{\mathbf{u}} \cdot \nabla\phi^2 \right| \le C \int_{B_{\frac{1}{2}}(0)} (|\widehat{\mathbf{u}}|^3 + |\widehat{P}|^{\frac{3}{2}}).$$

Hence, by adding (3.5.30) and (3.5.31) together and then taking summation over all β's with $|\beta| = k \geq 0$, we obtain that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} \frac{1}{2}(|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{Q}|^2)\phi^2 + \int_{\mathbb{R}^3} (|\nabla^{k+1}\widehat{\mathbf{u}}|^2 + |\nabla^{k+2}\widehat{Q}|^2)\phi^2 \\
&\leq \int_{\mathbb{R}^3} \frac{1}{2}(|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{Q}|^2)(|\partial_t(\phi^2)| + |\nabla^2(\phi^2)|) \\
&+C\int_{B_{\frac{1}{2}}(0)} (|\widehat{\mathbf{u}}|^3 + |\widehat{P}|^{\frac{3}{2}}) \\
&+C\int_{\mathbb{R}^3} \Big(|\nabla^{k+1}\widehat{\mathbf{u}}||\nabla^{k+1}\widehat{Q}| + |\nabla^k \widehat{\mathbf{u}}||\nabla^{k+2}\widehat{Q}|\Big)|\nabla\phi^2| \\
&\leq \int_{\mathbb{R}^3} \frac{1}{2}(|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{Q}|^2)(|\partial_t(\phi^2)| + |\nabla^2(\phi^2)|) \\
&+C\int_{B_{\frac{1}{2}}(0)} (|\widehat{\mathbf{u}}|^3 + |\widehat{P}|^{\frac{3}{2}}) \\
&+\frac{1}{2}\int_{\mathbb{R}^3} (|\nabla^{k+1}\widehat{\mathbf{u}}|^2 + |\nabla^{k+2}\widehat{Q}|^2)\phi^2 + C\int_{\mathbb{R}^3} \Big(|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{Q}|^2\Big)|\nabla\phi|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3} (|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{Q}|^2)\phi^2 + \int_{\mathbb{R}^3} (|\nabla^{k+1}\widehat{\mathbf{u}}|^2 + |\nabla^{k+2}\widehat{Q}|^2)\phi^2 \\
&\leq C\int_{\mathbb{R}^3} (|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{Q}|^2)(|\partial_t(\phi^2)| + |\nabla^2(\phi^2)|) \\
&+C\int_{B_{\frac{1}{2}}(0)} (|\widehat{\mathbf{u}}|^3 + |\widehat{P}|^{\frac{3}{2}}) \\
&+C\int_{\mathbb{R}^3} \Big(|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{Q}|^2\Big)|\nabla\phi|^2.
\end{aligned}
\tag{3.5.34}
$$

By choosing suitable test functions ϕ, it is not hard to see that (3.5.34) implies that for $k \geq 0$,

$$
\begin{aligned}
&\sup_{-\frac{1}{16}\leq t\leq 0}\int_{B_{\frac{1}{4}}(0)} (|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{Q}|^2) + \int_{\mathbb{P}_{\frac{1}{4}}(0)} (|\nabla^{k+1}\widehat{\mathbf{u}}|^2 + |\nabla^{k+2}\widehat{Q}|^2 \\
&\leq C\int_{\mathbb{P}_{\frac{3}{8}}(0)} (|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{Q}|^2) + C\int_{\mathbb{P}_{\frac{1}{2}}(0)} (|\widehat{\mathbf{u}}|^3 + |\widehat{P}|^{\frac{3}{2}}).
\end{aligned}
\tag{3.5.35}
$$

It is clear that with suitable adjustment of radius, applying (3.5.35 inductively on k yields that

$$\sup_{-\frac{1}{16}\le t\le 0}\int_{B_{\frac14}(0)}(|\nabla^k\widehat{\mathbf{u}}|^2+|\nabla^{k+1}\widehat{Q}|^2)+\int_{\mathbb{P}_{\frac14}(0)}(|\nabla^{k+1}\widehat{\mathbf{u}}|^2+|\nabla^{k+2}\widehat{Q}|^2)$$
$$\le C\int_{\mathbb{P}_{\frac38}(0)}(|\nabla\widehat{\mathbf{u}}|^2+|\nabla^2\widehat{Q}|^2+C\int_{\mathbb{P}_{\frac12}(0)}(|\widehat{\mathbf{u}}|^3+|\widehat{P}|^{\frac32}),\ \forall k\ge 1. \tag{3.5.36}$$

With (3.5.36), we can apply the regularity theory for both the linear Stokes equation and the linear parabolic equation to conclude that $(\widehat{\mathbf{u}},\widehat{Q})\in C^\infty(\mathbb{P}_{\frac14}(0))$. Furthermore, applying the elliptic estimate for the pressure equation (3.5.21) we see that $\nabla^k\widehat{P}\in C^0(\mathbb{P}_{\frac14}(0))$ for any $k\ge 1$. For $l\ge 1$, taking t-derivative ∂_t^l of both sides of (3.5.21), we can also see that $\nabla^k\partial_t^l\widehat{P}\in C^0(\mathbb{P}_{\frac14}(0))$. Therefore $(\widehat{\mathbf{u}},\widehat{Q},\widehat{P})\in C^\infty(\mathbb{P}_{\frac14}(0))$ and the estimate (3.5.28) holds. This completes the proof of Lemma 3.5.2. □

Now we can iterate Lemma 3.5.1 and utilize the Riesz potential estimates in Morrey spaces to obtain the following ε_0-regularity.

Lemma 3.5.3. *For any $M>0$, there exists $\varepsilon_0>0$, depending on M, such that if $(\mathbf{u},Q,P)$ is a suitable weak solution of (3.1.6) in $\Omega\times(0,\infty)$, which satisfies, for $z_0=(x_0,t_0)\in\Omega\times(r_0^2,\infty)$ and*

$$\begin{cases}|Q|\le M & \text{if } F_{\text{bulk}}=F_{\text{LdG}} \text{ and } \Omega=\mathbb{R}^3,\\ |G_{\text{BM}}(Q)|\le M & \text{if } F_{\text{bulk}}=F_{\text{BM}} \text{ and } \Omega=\mathbb{T}^3,\end{cases}\quad \text{in } \mathbb{P}_{r_0}(z_0), \tag{3.5.37}$$

and

$$r_0^{-2}\int_{\mathbb{P}_{r_0}(z_0)}(|\mathbf{u}|^3+|\nabla Q|^3)\,dxdt+\Big(r_0^{-2}\int_{\mathbb{P}_{r_0}(z_0)}|P|^{\frac32}\,dxdt\Big)^2\le\varepsilon_0^3, \tag{3.5.38}$$

then for any $1<p<\infty$, $(\mathbf{u},P,\nabla Q)\in L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))$ and

$$\Big\|(\mathbf{u},P,\nabla Q)\Big\|_{L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))}\le C(p,\varepsilon_0,M). \tag{3.5.39}$$

Proof. From (3.5.38), we have

$$\Big(\frac{r_0}{2}\Big)^{-2}\int_{\mathbb{P}_{\frac{r_0}{2}}(z)}(|\mathbf{u}|^3+|\nabla Q|^3)\,dxdt+\Big(\Big(\frac{r_0}{2}\Big)^{-2}\int_{\mathbb{P}_{\frac{r_0}{2}}(z)}|P|^{\frac{3}{2}}\,dxdt\Big)^2\le 8\varepsilon_0^3 \tag{3.5.40}$$

holds for any $z\in\mathbb{P}_{\frac{r_0}{2}}(z_0)$. By applying Lemma 3.5.1 repeatedly on $\mathbb{P}_{\frac{r_0}{2}}(z)$ for $z\in\mathbb{P}_{\frac{r_0}{2}}(z_0)$, there are $C_0>0$ and $\tau_0\in(0,\frac{1}{2})$ that for any $k\ge 1$,

$$\begin{aligned}&(\tau_0^k r_0)^{-2}\int_{\mathbb{P}_{\tau_0^k r_0}(z)}(|\mathbf{u}|^3+|\nabla Q|^3)\,dxdt+\Big((\tau_0^k r_0)^{-2}\int_{\mathbb{P}_{\tau_0^k r_0}(z)}|P|^{\frac{3}{2}}\,dxdt)^2\\&\le 2^{-k}\max\Big\{(\frac{r_0}{2})^{-2}\int_{\mathbb{P}_{\frac{r_0}{2}}(z)}(|\mathbf{u}|^3+|\nabla Q|^3)\,dxdt+\Big((\frac{r_0}{2})^{-2}\int_{\mathbb{P}_{\frac{r_0}{2}}(z)}|P|^{\frac{3}{2}}\,dxdt\Big)^2,\\&\qquad\qquad\frac{C_0r_0^3}{1-2\tau_0^3}\Big\}.\end{aligned} \tag{3.5.41}$$

Therefore for $\theta_0=\dfrac{\ln 2}{3|\ln\tau_0|}\in(0,\frac{1}{3})$, it holds that for any $0<s<\dfrac{r_0}{2}$ and $z\in\mathbb{P}_{\frac{r_0}{2}}(z_0)$

$$s^{-2}\int_{\mathbb{P}_s(z)}(|\mathbf{u}|^3+|\nabla Q|^3+|P|^{\frac{3}{2}})\,dxdt\le C(1+\varepsilon_0^3)\Big(\frac{s}{r_0}\Big)^{3\theta_0}. \tag{3.5.42}$$

By (3.5.37) and Lemma 3.3.2, there exists $C>0$, depending on M, such that

$$|Q|+|f_{\text{bulk}}(Q)|+|\nabla_Q f_{\text{bulk}}(Q)|\le C \text{ in } \mathbb{P}_{r_0}(z_0). \tag{3.5.43}$$

Now we can apply the local energy inequality (3.1.12) for $(\mathbf{u},P,Q)$ on $\mathbb{P}_{\frac{r_0}{2}}(z)$, for $z\in\mathbb{P}_{\frac{r_0}{2}}(z_0)$, to get that for $0<s<\dfrac{r_0}{2}$,

$$\begin{aligned}&s^{-1}\int_{\mathbb{P}_s(z)}(|\nabla u|^2+|\Delta Q|^2)\,dxdt\\&\le C\Big[(2s)^{-3}\int_{\mathbb{P}_{2s}(z)}(|u|^2+|\nabla Q|^2)+(2s)^{-2}\int_{\mathbb{P}_{2s}(z)}(|u|^3+|\nabla Q|^3+|P|^{\frac{3}{2}})\\&\quad+(2s)^{-2}\int_{\mathbb{P}_{2s}(z)}|u|+(2s)^{-1}\int_{\mathbb{P}_{2s}(z)}|\nabla Q|^2\Big]\\&\le C(1+\varepsilon_0^3)\Big(\frac{s}{r_0}\Big)^{2\theta_0}.\end{aligned} \tag{3.5.44}$$

Next we employ the estimate of Riesz potentials in Morrey spaces to prove the smoothness of $(\mathbf{u}, P, Q)$ near z_0, analogous to that by Huang–Wang [71], Hineman–Wang [72], and Huang–Lin–Wang [38].

For any open set $U \subset \mathbb{R}^3 \times \mathbb{R}$, $1 \le p < \infty$, and $0 \le \lambda \le 5$, define the Morrey space $M^{p,\lambda}(U)$ by

$$M^{p,\lambda}(U) := \Big\{ f \in L^p_{\mathrm{loc}}(U) : \ \big\|f\big\|^p_{M^{p,\lambda}(U)} = \sup_{z\in U, r>0} r^{\lambda-5} \int_{\mathbb{P}_r(z)} |f|^p \, dxdt < \infty \Big\}.$$

It follows from (3.5.42) and (3.5.44) that there exists $\alpha \in (0,1)$ such that

$$(\mathbf{u}, \nabla Q) \in M^{3,3(1-\alpha)}\Big(\mathbb{P}_{\frac{r_0}{2}}(z_0)\Big), \ P \in M^{\frac{3}{2},3(1-\alpha)}\Big(\mathbb{P}_{\frac{r_0}{2}}(z_0)\Big), \ (\nabla \mathbf{u}, \nabla^2 Q) \in M^{2,4-2\alpha}\Big(\mathbb{P}_{\frac{r_0}{2}}(z_0)\Big).$$

Write $(3.3.2)_1$ as

$$\partial_t Q - \Delta Q = f, \quad f \equiv -\mathbf{u}\cdot\nabla Q + [\omega, Q] - f_{\mathrm{bulk}}(Q) \in M^{\frac{3}{2},3(1-\alpha)}\Big(\mathbb{P}_{\frac{r_0}{2}}(z_0)\Big). \tag{3.5.45}$$

Let $\eta \in C_0^\infty(\mathbb{R}^4)$ be a cut off function of $\mathbb{P}_{\frac{r_0}{2}}(z_0)$ such that $0 \le \eta \le 1$, $\eta = 1$ in $\mathbb{P}_{\frac{r_0}{2}}(z_0)$, $|\partial_t \eta| + |\nabla^2 \eta| \le C r_0^{-2}$, Set $w = \eta^2(Q - Q_{z_0,r_0})$, where Q_{z_0,r_0} is the average of Q over $\mathbb{P}_{\frac{r_0}{2}}(z_0)$. Then

$$\partial_t w - \Delta w = F, \quad F := \eta^2 f + (\partial_t \eta^2 - \Delta \eta^2)(Q - Q_{z_0,r_0}) - \nabla \eta^2 \cdot \nabla Q. \tag{3.5.46}$$

We can check that $F \in M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)$ and satisfies

$$\big\|F\big\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)} \le C(1+\varepsilon_0). \tag{3.5.47}$$

Let Γ denote the heat kernel in $\mathbb{R}^3$. Then

$$|\nabla\Gamma|(x,t) \le C\delta^{-4}((x,t),(0,0)), \ \forall (x,t) \neq (0,0),$$

where $\delta(\cdot,\cdot)$ denotes the parabolic distance on $\mathbb{R}^4$. By the Duhamel formula, we have that

$$|w(x,t)| \leq \int_0^t \int_{\mathbb{R}^3} |\nabla\Gamma(x-y,t-s)||F(y,s)|\,dyds \leq C\mathcal{I}_1(|F|)(x,t), \tag{3.5.48}$$

where $\mathcal{I}_\beta$ is the Riesz potential of order β on $\mathbb{R}^4$, $\beta \in [0,4]$, defined by

$$\mathcal{I}_\beta(g)(x,t) = \int_{\mathbb{R}^4} \frac{|g(y,s)|}{\delta^{5-\beta}((x,t),(y,s))}\,dyds,\ \forall g \in L^1(\mathbb{R}^4).$$

Applying the Riesz potential estimates (see [71] Theorem 3.1), we conclude that $\nabla w \in M^{\frac{3(1-\alpha)}{1-2\alpha},3(1-\alpha)}(\mathbb{R}^4)$ and

$$\Big\|\nabla w\Big\|_{M^{\frac{3(1-\alpha)}{1-2\alpha},3(1-\alpha)}(\mathbb{R}^4)} \leq C\Big\|F\Big\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)} \leq C(1+\varepsilon_0). \tag{3.5.49}$$

Since $\lim\limits_{\alpha\uparrow\frac{1}{2}} \dfrac{3(1-\alpha)}{1-2\alpha} = \infty$, we conclude that for any $1<p<\infty$, $\nabla w \in L^p(\mathbb{P}_{r_0}(z_0))$ and

$$\Big\|\nabla w\Big\|_{L^p(\mathbb{P}_{r_0}(z_0))} \leq C(p,r_0,\varepsilon_0). \tag{3.5.50}$$

Since $Q-w$ solves

$$\partial_t(Q-w) - \Delta(Q-w) = 0 \quad \text{in} \quad \mathbb{P}_{\frac{r_0}{2}}(z_0),$$

it follows from the theory of heat equations that for any $1<p<\infty$, $\nabla Q \in \mathbb{P}_{\frac{r_0}{2}}(z_0)$ and

$$\Big\|\nabla Q\Big\|_{L^p(\mathbb{P}_{\frac{r_0}{2}}(z_0))} \leq C(p,r_0,\varepsilon_0). \tag{3.5.51}$$

We now proceed with the estimation of $\mathbf{u}$. Let $\mathbf{v}:\mathbb{R}^3\times(0,\infty)\mapsto\mathbb{R}^3$ solve the Stokes equation:

$$\begin{cases} \partial_t\mathbf{v} - \Delta\mathbf{v} + \nabla P \\ = -\mathrm{div}\Big[\eta^2\Big(\mathbf{u}\otimes\mathbf{u} + (\nabla Q\otimes\nabla Q - \frac{1}{2}|\nabla Q|^2 I_3)\Big)\Big] + \mathrm{div}\Big[\eta^2[Q,\Delta Q]\Big] & \text{in } \mathbb{R}^4_+, \\ \mathrm{div}\mathbf{v} = 0 & \text{in } \mathbb{R}^4_+, \\ \mathbf{v}(\cdot,0) = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{3.5.52}$$

By using the Oseen kernel (see Leray [37]), an estimate of $\mathbf{v}$ can be given by

$$|\mathbf{v}(x,t)| \le C\mathcal{I}_1(|X|)(x,t),\ \forall (x,t) \in \mathbb{R}^3 \times (0,\infty), \tag{3.5.53}$$

where

$$X = \eta^2\Big[\mathbf{u}\otimes\mathbf{u} + (\nabla Q \otimes \nabla Q - \frac{1}{2}|\nabla Q|^2 I_3) + [Q, \Delta Q]\Big].$$

As above, we can check that $X \in M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)$ and

$$\begin{aligned}\Big\|X\Big\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)} &\le C\Big[\|\mathbf{u}\|^2_{M^{3,3(1-\alpha)}(\mathbb{P}_{\frac{r_0}{2}}(z_0))} + \|\nabla Q\|^2_{M^{3,3(1-\alpha)}(\mathbb{P}_{\frac{r_0}{2}}(z_0))} \\ &\qquad + \|\Delta Q - f_{\mathrm{bulk}}(Q)\|_{M^{3,3(1-\alpha)}(\mathbb{P}_{\frac{r_0}{2}}(z_0))}\Big] \\ &\le C(1+\varepsilon_0).\end{aligned}$$

Hence we conclude that $\mathbf{v} \in M^{\frac{3(1-\alpha)}{1-2\alpha},3(1-\alpha)}(\mathbb{R}^4)$ and

$$\Big\|\mathbf{v}\Big\|_{M^{\frac{3(1-\alpha)}{1-2\alpha},3(1-\alpha)}(\mathbb{R}^4)} \le C\Big\|X\Big\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)} \le C(1+\varepsilon_0). \tag{3.5.54}$$

As $\alpha \uparrow \dfrac{1}{2}$, we conclude that for any $1 < p < \infty$, $\mathbf{v} \in L^p(\mathbb{P}_{r_0}(z_0))$ and

$$\Big\|\mathbf{v}\Big\|_{L^p(\mathbb{P}_{r_0}(z_0))} \le C(p, r_0, \varepsilon_0). \tag{3.5.55}$$

Note that $\mathbf{u} - \mathbf{v}$ solves the linear homogeneous Stokes equation in $\mathbb{P}_{\frac{r_0}{2}}(z_0)$:

$$\partial_t(\mathbf{u}-\mathbf{v}) - \Delta(\mathbf{u}-\mathbf{v}) + \nabla P = 0,\ \mathrm{div}(\mathbf{u}-\mathbf{v}) = 0 \ \text{ in } \ \mathbb{P}_{\frac{r_0}{2}}(z_0).$$

Then $\mathbf{u} - \mathbf{v} \in L^\infty(\mathbb{P}_{\frac{r_0}{4}}(z_0))$. Therefore for any $1 < p < \infty$, $\mathbf{u} \in L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))$ and

$$\Big\|\mathbf{u}\Big\|_{L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0)} \le C(p, r_0, \varepsilon_0). \tag{3.5.56}$$

For P, since it satisfies the Poisson equation: for $t_0 - \frac{r_0^2}{4} \le t \le t_0$,

$$-\Delta P = \mathrm{div}^2\Big[\mathbf{u}\otimes\mathbf{u} + (\nabla Q\otimes\nabla Q - \frac{1}{2}|\nabla Q|^2 I_3)\Big] \ \text{ in } \ B_{\frac{r_0}{2}}(x_0). \tag{3.5.57}$$

Hence $P \in L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))$ and satisfies the (3.5.39). The proof is now complete. □

The higher order regularity of (3.3.2) does not follow from the standard theory, since the equation for $\mathbf{u}$ involves $\nabla^3 Q$ and the equation for Q involves $\nabla\mathbf{u}$. It turns out the higher order regularity of (3.3.2) can be obtained through higher oder energy methods. Roughly speaking, if $(\mathbf{u}, P, \nabla Q)$ is in L^p for any $1 < p < \infty$, then (3.3.2) can be viewed as a perturbed version of the linear equation (3.5.17) with controllable error terms. Here higher order versions of the cancellation properties (3.1.13) and (3.1.16) in the local energy inequality (3.1.12) also play an important role. This kind of idea has been previously employed by Huang–Lin–Wang (see [38] Lemma 3.4) for general Ericksen–Leslie systems in dimension two. More precisely, we have

Lemma 3.5.4. *Under the same assumptions as Lemma 3.5.3, we have that for any* $k \ge 0$, $(\nabla^k\mathbf{u}, \nabla^{k+1}Q) \in \Big(L_t^\infty L_x^2 \cap L_t^2 H_x^1\Big)(\mathbb{P}_{\frac{1+2^{-(k+1)}}{2}r_0}(z_0))$ *and the following estimates hold*

$$\begin{aligned} &\sup_{t_0 - \left(\frac{(1+2^{-(k+1)})}{2}r_0\right)^2 \le t \le t_0} \int_{B_{\frac{1+2^{-(k+1)}}{2}r_0}(x_0)} (|\nabla^k\mathbf{u}|^2 + |\nabla^{k+1}Q|^2)\, dx \\ &\quad + \int_{\mathbb{P}_{\frac{1+2^{-(k+1)}}{2}r_0}(z_0)} (|\nabla^{k+1}\mathbf{u}|^2 + |\nabla^{k+2}Q|^2 + |\nabla^k P|^{\frac{5}{3}})\, dxdt \\ &\quad \le C(k, r_0)\varepsilon_0. \end{aligned} \tag{3.5.58}$$

In particular, $(\mathbf{u}, Q)$ *is smooth in* $\mathbb{P}_{\frac{r_0}{4}}(z_0)$.

Proof. For simplicity, assume $z_0 = (0,0)$ and $r_0 = 8$. (3.5.58) can be proved by an induction on k. It is clear that when $k = 0$, (3.5.58) follows directly from the local energy inequality

(3.1.12). Here we indicate how to prove (3.5.58) for $k=1$. First, recall from Lemma 3.5.3 that for any $\mathrm{i}\in\mathbb{N}^+$ and $1<p<\infty$,

$$\left\|Q\right\|_{L^\infty(\mathbb{P}_2)}+\left\|\nabla^{\mathrm{i}} f_{\mathrm{bulk}}(Q)\right\|_{L^\infty(\mathbb{P}_2)}\le C(\mathrm{i},\varepsilon_0),\ \left\|(\mathbf{u},P,\nabla Q)\right\|_{L^p(\mathbb{P}_2)}\le C(p)\varepsilon_0. \tag{3.5.59}$$

Taking spatial derivative of (3.1.6)[1], we have

$$\begin{cases}\partial_t Q_\alpha+\mathbf{u}\cdot\nabla Q_\alpha+\mathbf{u}_\alpha\cdot\nabla Q-[\omega_\alpha,Q]-[\omega,Q_\alpha]\\ =\Delta Q_\alpha-(f_{\mathrm{bulk}}(Q))_\alpha,\\ \partial_t\mathbf{u}_\alpha+\mathbf{u}\cdot\nabla\mathbf{u}_\alpha+\mathbf{u}_\alpha\cdot\nabla\mathbf{u}+\nabla P_\alpha & \text{in } \mathbb{P}_1.\\ =\Delta\mathbf{u}_\alpha-\nabla Q\cdot\Delta Q_\alpha-\nabla Q_\alpha\cdot\Delta Q+\mathrm{div}[Q,\Delta Q]_\alpha,\\ \mathrm{div}\mathbf{u}_\alpha=0,\end{cases} \tag{3.5.60}$$

Here $\omega_\alpha=\omega(\mathbf{u}_\alpha)$. Let $\eta\in C_0^\infty(B_2)$ be such that

$$0\le\eta\le1,\ \eta\equiv1 \text{ in } B_{1+2^{-2}},\ \eta\equiv0 \text{ out } B_{1+2^{-1}},\ |\nabla\eta|+|\nabla^2\eta|\le16.$$

Taking ∇ of $(3.5.60)_1$ and multiplying it by $\nabla Q_\alpha\eta^2$, and multiplying $(3.5.60)_2$ by $\nabla\mathbf{u}_\alpha\eta^2$, and then integrating resulting equations over B_2[2], we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla^2Q|^2\eta^2-\int_{\mathbb{R}^3}(\mathbf{u}_\alpha\cdot\nabla)Q\cdot\Delta Q_\alpha\eta^2-\int_\Omega(\mathbf{u}\cdot\nabla)Q_\alpha\cdot(\Delta Q_\alpha\eta^2+\nabla Q_\alpha\nabla\eta^2)\\ &-\int_\Omega(\mathbf{u}_\alpha\cdot\nabla)Q\cdot\nabla Q_\alpha\nabla\eta^2-\int_\Omega[Q,\omega_\alpha]\cdot(\Delta Q_\alpha\eta^2+\nabla Q_\alpha\nabla\eta^2)\\ &=\int_\Omega\Big[[Q_\alpha,\omega]-(\Delta Q_\alpha-(f_{\mathrm{bulk}}(Q))_\alpha)\Big]\cdot(\Delta Q_\alpha\eta^2+\nabla Q_\alpha\nabla\eta^2),\end{aligned}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\mathbf{u}|^2\eta^2-\int_\Omega\frac{|\nabla\mathbf{u}|^2}{2}\mathbf{u}\cdot\nabla\eta^2+\int_\Omega(\mathbf{u}_\alpha\cdot\nabla)\mathbf{u}\cdot\mathbf{u}_\alpha\eta^2-\int_\Omega P_\alpha\mathbf{u}_\alpha\cdot\nabla\eta^2$$

[1]↑Strictly speaking, we need to take finite quotient D_h^{j} of (3.1.6) ($\mathrm{j}=1,2,3$) and then sending $h\to0$

[2]↑strictly speaking, we need to multiply $\Delta(D_h^{\mathrm{j}}Q)\eta^2$ and $\nabla(D_h^{\mathrm{j}}\mathbf{u})\eta^2$ and then sending $h\to0$

$$
\begin{aligned}
&= -\int_\Omega (|\nabla^2 \mathbf{u}|^2\eta^2 - \frac{|\nabla \mathbf{u}|^2}{2}\Delta\eta^2) - \int_\Omega ((\mathbf{u}_\alpha \cdot \nabla)Q \cdot \Delta Q_\alpha \eta^2 + (\mathbf{u}_\alpha \cdot \nabla)Q_\alpha \cdot \Delta Q \eta^2) \\
&\quad - \int_\Omega [Q_\alpha, \Delta Q] \cdot (\nabla \mathbf{u}_\alpha \eta^2 + \mathbf{u}_\alpha \otimes \nabla \eta^2) - \int_\Omega [Q, \Delta Q_\alpha] \cdot (\nabla \mathbf{u}_\alpha \eta^2 + \mathbf{u}_\alpha \otimes \nabla \eta^2).
\end{aligned}
$$

Adding these two equations together and regrouping terms, and using the cancellation identity

$$
\int_\Omega [Q, \omega_\alpha] \cdot \Delta Q_\alpha \eta^2 = \int_\Omega [Q, \Delta Q_\alpha] \cdot \nabla \mathbf{u}_\alpha \eta^2,
$$

we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\eta^2 + \int_\Omega (|\nabla^2 \mathbf{u}|^2 + |\Delta \nabla Q|^2)\eta^2 \\
&= \int_\Omega [(\mathbf{u} \cdot \nabla)Q_\alpha \cdot (\Delta Q_\alpha \eta^2 + \nabla Q_\alpha \nabla \eta^2) + (\mathbf{u}_\alpha \cdot \nabla)Q \cdot \nabla Q_\alpha \nabla \eta^2] \\
&+ \int_\Omega ([Q, \omega_\alpha] - \Delta Q_\alpha) : \nabla Q_\alpha \nabla \eta^2 \\
&+ \int_\Omega \Big([Q_\alpha, \omega] + (f_{\text{bulk}}(Q))_\alpha\Big) : (\Delta Q_\alpha \eta^2 + \nabla Q_\alpha \nabla \eta^2) \\
&+ \int_\Omega [\frac{|\nabla \mathbf{u}|^2}{2}(\Delta \eta^2 + \mathbf{u} \cdot \nabla \eta^2) - \mathbf{u}_\alpha \cdot (\nabla \mathbf{u} \cdot \mathbf{u}_\alpha + \nabla Q_\alpha : \Delta Q)\eta^2 + P_\alpha \mathbf{u}_\alpha \cdot \nabla \eta^2] \\
&- \int_\Omega [Q_\alpha, \Delta Q] : (\nabla \mathbf{u}_\alpha \eta^2 + \mathbf{u}_\alpha \otimes \nabla \eta^2) - \int_\Omega [Q, \Delta Q_\alpha] : \mathbf{u}_\alpha \otimes \nabla \eta^2 \\
&:= \sum_{\mathrm{i}=1}^{6} A_{\mathrm{i}}.
\end{aligned}
$$

We can estimate A_{i}'s separately as follows.

$$
|A_6| \leq \frac{1}{16}\int_\Omega |\Delta \nabla Q|^2 \eta^2 + C\int_\Omega (|\nabla Q|^2 \eta^2 + |\nabla \mathbf{u}|^2(\eta^2 + |\nabla \eta|^2),
$$

$$
|A_5| \leq \frac{1}{16}\int_\Omega |\nabla^2 \mathbf{u}|^2 \eta^2 + C\int_\Omega |\nabla Q|^2 |\Delta Q|^2 \eta^2 + C\int_\Omega |\nabla \mathbf{u}|^2 |\nabla \eta|^2,
$$

$$
\begin{aligned}
|A_4| \leq \frac{1}{8}\int_\Omega (|\nabla^2 \mathbf{u}|^2 + |\Delta \nabla Q|^2)\eta^2 + C\int_\Omega [|\nabla \mathbf{u}|^2 |\Delta \eta^2| + |\mathbf{u}|^2(|\nabla \mathbf{u}|^2 + |\Delta Q|^2)\eta^2] \\
+ C\int_\Omega (|\nabla \mathbf{u}|^2 + |\Delta Q|^2)|\nabla \eta|^2 + C\int_\Omega (|P|^2 |\nabla \eta|^2 + |P||\nabla \mathbf{u}||\Delta \eta^2|),
\end{aligned}
$$

$$|A_3| \leq \frac{1}{16}\int_\Omega |\Delta\nabla Q|^2\eta^2 + C\int_\Omega |\nabla Q|^2(|\nabla \mathbf{u}|^2 + |\Delta Q|^2)\eta^2$$
$$+ C\int_\Omega (|\nabla Q|^2\eta^2 + |\nabla \mathbf{u}|^2|\nabla\eta|^2),$$
$$|A_2| \leq \frac{1}{16}\int_\Omega |\Delta\nabla Q|^2\eta^2 + C\int_\Omega (|\nabla \mathbf{u}|^2 + |\Delta Q|^2)|\nabla\eta|^2,$$
$$|A_1| \leq \frac{1}{16}\int_\Omega |\Delta\nabla Q|^2\eta^2 + C\int_\Omega [(|\mathbf{u}|^2| + |\nabla Q|^2)\Delta Q|^2\eta^2 + (|\nabla \mathbf{u}|^2 + |\Delta Q|^2)|\nabla\eta|^2].$$

Substituting these estimates on A_{i}'s into the above inequality, we obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\eta^2 + \int_\Omega (|\nabla^2\mathbf{u}|^2 + |\Delta\nabla Q|^2)\eta^2 \\
&\leq C\int_{B_{1+2^{-1}}} (|\mathbf{u}|^2 + |\nabla Q|^2 + |\nabla \mathbf{u}|^2 + |\Delta Q|^2 + |P|^2) \\
&\quad + C\int_\Omega (|\mathbf{u}|^2|\nabla \mathbf{u}|^2 + |\mathbf{u}|^2|\Delta Q|^2 + |\nabla Q|^2|\Delta Q|^2 + |\nabla Q|^2|\nabla \mathbf{u}|^2)\eta^2.
\end{aligned}$$

Now we want to estimate the second term in the right hand side. By Sobolev-interpolation inequalities, we have

$$\begin{aligned}
&\int_\Omega |\mathbf{u}|^2|\nabla \mathbf{u}|^2\eta^2 \\
&\leq \|\nabla \mathbf{u}\eta\|_{L^2(\Omega)}\|\nabla \mathbf{u}\eta\|_{L^3(\Omega)}\|\mathbf{u}\|^2_{L^{12}(B_{1+2^{-1}})} \\
&\leq C\|\nabla \mathbf{u}\eta\|_{L^2(\Omega)}\|\nabla \mathbf{u}\eta\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla(\nabla \mathbf{u}\eta)\|^{\frac{1}{2}}_{L^2(\Omega)}\|\mathbf{u}\|^2_{L^{12}(B_{1+2^{-1}})} \\
&\leq C\|\nabla \mathbf{u}\eta\|_{L^2(\Omega)}\|\nabla(\nabla \mathbf{u}\eta)\|_{L^2(\Omega)}\|\mathbf{u}\|^2_{L^{12}(B_{1+2^{-1}})} \\
&\leq \frac{1}{8}\int_\Omega |\nabla^2\mathbf{u}|^2\eta^2 + C\int_{B_{1+2^{-1}}} |\nabla \mathbf{u}|^2 + C\|\mathbf{u}\|^4_{L^{12}(B_{1+2^{-1}})}\int_\Omega |\nabla \mathbf{u}|^2\eta^2,
\end{aligned}$$

$$\int_\Omega |\mathbf{u}|^2|\Delta Q|^2\eta^2 \leq \frac{1}{8}\int_\Omega |\Delta\nabla Q|^2\eta^2 + C\int_{B_{1+2^{-1}}} |\Delta Q|^2$$
$$+ C\|\mathbf{u}\|^4_{L^{12}(B_{1+2^{-1}})}\int_\Omega |\Delta Q|^2\eta^2,$$
$$\int_\Omega |\nabla Q|^2|\Delta Q|^2\eta^2 \leq \frac{1}{8}\int_\Omega |\Delta\nabla Q|^2\eta^2 + C\int_{B_{1+2^{-1}}} |\Delta Q|^2$$
$$+ C\|\nabla Q\|^4_{L^{12}(B_{1+2^{-1}})}\int_\Omega |\Delta Q|^2\eta^2,$$

and

$$\int_\Omega |\nabla Q|^2|\nabla \mathbf{u}|^2\eta^2 \le \frac{1}{8}\int_\Omega |\nabla \mathbf{u}|^2\eta^2 + C\int_{B_{1+2^{-1}}} |\nabla \mathbf{u}|^2$$
$$+ C\|\nabla Q\|^4_{L^{12}(B_{1+2^{-1}})}\int_\Omega |\nabla \mathbf{u}|^2\eta^2.$$

Substituting these estimates into the above inequality, we would arrive at

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\eta^2 + \int_\Omega (|\nabla^2 \mathbf{u}|^2 + |\Delta\nabla Q|^2)\eta^2\\
&\le C\int_{B_{1+2^{-1}}} (|\mathbf{u}|^2 + |\nabla Q|^2 + |\nabla \mathbf{u}|^2 + |\Delta Q|^2 + |P|^2)\\
&\quad + C(1 + \|(\mathbf{u}, \nabla Q)\|^{12}_{L^{12}(B_{1+2^{-1}})})\int_\Omega (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\eta^2.
\end{aligned} \tag{3.5.61}$$

From (3.5.59), we can apply Gronwall's inequality to (3.5.61) to show that (3.5.58) holds for $k = 1$. For $k \ge 2$, we can perform an induction argument as in [38] Lemma 3.4. We leave the details to interested readers.

It is readily seen that by the Sobolev embedding theorem, Lemma 3.5.3 implies that $(\nabla^k u, \nabla^{k+1}Q) \in L^\infty(\mathbb{P}_{\frac{r_0}{4}}(z_0))$ for any $k \ge 1$. This, combined with the theory of linear Stokes equation and heat equation, would imply the smoothness of $(\mathbf{u}, Q)$ in $\mathbb{P}_{\frac{r_0}{4}}(z_0)$. This completes the proof. □

Applying Lemma 3.5.3, we can prove a weaker version of Theorem 2.2.1.

Proposition 3.5.1. *Under the same assumptions as in Theorem 2.2.1, there exists a closed subset* $\Sigma \subset \Omega \times (0, \infty)$, *with* $\mathcal{P}^{\frac{5}{3}}(\Sigma) = 0$, *such that* $(\mathbf{u}, Q) \in C^\infty(\Omega \times (0, \infty) \setminus \Sigma)$.

Proof. First it follows from Lemma 3.4.1 and Lemma 3.3.2 that for any $\delta > 0$, Q and $f_{\mathrm{BM}}(Q)$ are bounded in $\Omega \times (\delta, \infty)$. Define

$$\Sigma_\delta = \Big\{z \in \Omega \times (\delta, \infty) : \liminf_{r\to 0} r^{-2}\int_{\mathbb{P}_r(z)} (|\mathbf{u}|^3 + |\nabla Q|^3)\,dxdt + \Big(r^{-2}\int_{\mathbb{P}_r(z)} |P|^{\frac{3}{2}}\,dxdt\Big)^2 > \varepsilon_0^3\Big\}.$$

From Lemma 3.5.3, we know that Σ_δ is closed and $(\mathbf{u}, Q) \in C^\infty(\Omega \times (\delta, \infty) \setminus \Sigma_\delta)$. Since $\delta > 0$ is arbitrary, we have that $(\mathbf{u}, Q) \in C^\infty(\Omega \times (0, \infty) \setminus \cup_{\delta>0}\Sigma_\delta)$.

Since $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(\Omega \times (0,\infty))$ and $\nabla Q \in L_t^\infty H_x^1 \cap L_t^2 H_x^2(\Omega \times (0,\infty))$, we see that $(\mathbf{u}, \nabla Q) \in L^{\frac{10}{3}}(\Omega \times (0,\infty))$. Moreover, since P solves the Poisson equation (3.1.15) in $\Omega \times (0,\infty)$, we conclude that $P \in L^{\frac{5}{3}}(\Omega \times (0,\infty))$. By Hölder's inequality, we see that Σ_δ is a subset of

$$\mathcal{S}_\delta = \Big\{ z \in \Omega \times (\delta,\infty) : \ \liminf_{r\to 0} r^{-\frac{5}{3}} \int_{\mathbb{P}_r(z)} (|\mathbf{u}|^{\frac{10}{3}} + |\nabla Q|^{\frac{10}{3}})\, dxdt \\ + \Big(r^{-\frac{5}{3}} \int_{\mathbb{P}_r(z)} |P|^{\frac{5}{3}}\, dxdt \Big)^2 > \varepsilon_0^{\frac{10}{3}} \Big\}.$$

A simple covering argument implies that $\mathcal{P}^{\frac{5}{3}}(\mathcal{S}_\delta) = 0$, see **S**. Hence $\Sigma = \cup_{\delta>0} \Sigma_\delta$ has $\mathcal{P}^{\frac{5}{3}}(\Sigma) = 0$. This completes the proof. $\square$

3.6 Partial regularity, part II

In this section, we will utilize the results from the previous section and the Sobolev inequality to first show the so-called A-B-C-D Lemmas (see [35] and [36]) and then establish an improved ε_1-regularity property for suitable weak solutions to (3.1.6).

Theorem 3.6.1 ([70]). *Under the same assumptions as in Theorem 2.2.1, there exists $\varepsilon_1 > 0$ such that if $(\mathbf{u}, Q) : \Omega \times (0,\infty) \mapsto \mathbb{R}^3 \times \mathcal{S}_0^{(3)}$ is a suitable weak solution of (3.1.5), which satisfies, for $z_0 \in \Omega \times (0,\infty)$,*

$$\limsup_{r\to 0} \frac{1}{r} \int_{\mathbb{P}_r(z_0)} \Big(|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2 \Big) dxdt < \varepsilon_1^2, \tag{3.6.1}$$

then $(\mathbf{u}, Q)$ is smooth near z_0.

For simplicity, we assume $z_0 = (0,0) \in \Omega \times (0,\infty)$. To streamline the presentation, we introduce the following dimensionless quantities:

$$A(r) := \sup_{-r^2 \le t \le 0} r^{-1} \int_{B_r(0)\times\{t\}} (|\mathbf{u}|^2 + |\nabla Q|^2)\, dx,$$
$$B(r) := \frac{1}{r} \int_{\mathbb{P}_r(0,0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\, dxdt,$$
$$C(r) := \frac{1}{r^2} \int_{\mathbb{P}_r(0,0)} (|\mathbf{u}|^3 + |\nabla Q|^3)\, dxdt,$$

$$D(r) := r^{-2} \int_{\mathbb{P}_r(0,0)} |P|^{\frac{3}{2}} \, dxdt.$$

Also set

$$(\mathbf{u})_r(t) := \frac{1}{|B_r(0)|} \int_{B_r(0)} \mathbf{u}(x,t)\, dx, (\nabla Q)_r(t) := \frac{1}{|B_r(0)|} \int_{B_r(0)} \nabla Q(x,t)\, dx.$$

We also let $A \lesssim B$ to denote $A \leq cB$ for some universal positive constant $c > 0$.

We recall the following interpolation Lemma, whose proof can be found in [35].

Lemma 3.6.1. *For $v \in H^1(\mathbb{R}^3)$,*

$$\begin{aligned} \int_{B_r(0)} |v|^q(x,t)\, dx \;&\lesssim\; \Big(\int_{B_r(0)} |\nabla v|^2(x,t)\, dx \Big)^{\frac{q}{2}-a} \Big(\int_{B_r(0)} |v|^2(x,t)\, dx \Big)^{a} \\ &\quad + r^{3\left(1-\frac{q}{2}\right)} \Big(\int_{B_r(0)} |v|^2(x,t)\, dx \Big)^{\frac{q}{2}}. \end{aligned} \tag{3.6.2}$$

for every $B_r(0) \subset \mathbb{R}^3$, $2 \leq q \leq 6$, $a = \dfrac{3}{2}\Big(1 - \dfrac{q}{6}\Big)$.

Applying Lemma 3.6.1, we can have

Lemma 3.6.2. *For any $\mathbf{u} \in L^\infty([-\rho^2, 0], L^2(B_\rho(0))) \cap L^2([-\rho^2, 0], H^1(B_\rho(0)))$, and $Q \in L^\infty([-\rho^2, 0], H^1(B_\rho(0))) \cap L^2([-\rho^2, 0], H^2(B_\rho(0)))$, it holds that for any $0 < r \leq \rho$,*

$$C(r) \lesssim \Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho) + \Big(\frac{\rho}{r}\Big)^3 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho). \tag{3.6.3}$$

Proof. From (3.6.1) with $q = 3, a = \dfrac{3}{4}$, we obtain that for any $v \in H^1(B_\rho(0))$,

$$\begin{aligned} \int_{B_r(0)} |v|^3(x,t)\, dx \;&\lesssim\; \Big(\int_{B_r(0)} |\nabla v|^2(x,t)\, dx \Big)^{\frac{3}{4}} \Big(\int_{B_r(0)} |v|^2(x,t)\, dx \Big)^{\frac{3}{4}} \\ &\quad + r^{-\frac{3}{2}} \Big(\int_{B_r(0)} |v|^2(x,t)\, dx \Big)^{3/2}. \end{aligned} \tag{3.6.4}$$

Applying Poincaré's inequality, we obtain that for $0 < r \leq \rho$,

$$\int_{B_r(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\, dx$$

$$
\begin{aligned}
&\lesssim \int_{B_r(0)} \Big(\big| |\mathbf{u}|^2 - (|\mathbf{u}|^2)_\rho \big| + \big| |\nabla Q|^2 - (|\nabla Q|^2)_\rho \big| \Big)\, dx + \Big(\frac{r}{\rho}\Big)^3 \int_{B_\rho(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\, dx \\
&\lesssim \rho \int_{B_\rho(0)} (|\mathbf{u}||\nabla \mathbf{u}| + |\nabla Q||\nabla^2 Q|)\, dx + \Big(\frac{r}{\rho}\Big)^3 \int_{B_\rho(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\, dx \\
&\lesssim \rho^{\frac{3}{2}} \Big(\rho^{-1} \int_{B_\rho(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\, dx \Big)^{\frac{1}{2}} \Big(\int_{B_\rho(0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\, dx \Big)^{\frac{1}{2}} \\
&\quad + \Big(\frac{r}{\rho}\Big)^3 \int_{B_\rho(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)\, dx \\
&\lesssim \rho^{\frac{3}{2}} A^{\frac{1}{2}}(\rho) \Big(\int_{B_\rho(0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\, dx \Big)^{\frac{1}{2}} + \Big(\frac{r}{\rho}\Big)^3 \rho A(\rho).
\end{aligned}
$$

Substituting this estimate into the second term of the right hand side of the previous inequality, we conclude that

$$
\begin{aligned}
&\int_{B_r(0)} \Big(|\mathbf{u}|^3 + |\nabla Q|^3 \Big)\, dx \\
&\lesssim \rho^{\frac{3}{4}} \Big(\int_{B_r(0)} \Big(|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2 \Big)\, dx \Big)^{\frac{3}{4}} \Big(\rho^{-1} \int_{B_r(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)(x,t)\, dx \Big)^{\frac{3}{4}} \\
&\quad + r^{-\frac{3}{2}} \Big(\int_{B_r(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)(x,t)\, dx \Big)^{\frac{3}{2}} \\
&\lesssim \rho^{\frac{3}{4}} A^{\frac{3}{4}}(\rho) \Big(\int_{B_r(0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)(x,t)\, dx \Big)^{\frac{3}{4}} \\
&\quad + r^{-\frac{3}{2}} \Big(\int_{B_r(0)} (|\mathbf{u}|^2 + |\nabla Q|^2)(x,t)\, dx \Big)^{\frac{3}{2}} \\
&\lesssim \Big(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}} \Big) \Big(\int_{B_r(0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\, dx \Big)^{\frac{3}{4}} A^{\frac{3}{4}}(\rho) + \Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho).
\end{aligned}
$$

Integrating this inequality over $[-r^2, 0]$, by Hölder's inequality we have

$$
\begin{aligned}
C(r) &= \frac{1}{r^2} \int_{\mathbb{P}_r(0,0)} (|\mathbf{u}|^3 + |\nabla Q|^3)\, dx dt \\
&\lesssim \Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho) + r^{-2} \Big(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}} \Big) \int_{-r^2}^0 \Big(\int_{B_r(0)} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\, dx \Big)^{\frac{3}{4}} dt A^{\frac{3}{4}}(\rho) \\
&\lesssim \Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho) + r^{-\frac{3}{2}} \rho^{\frac{3}{4}} \Big(\rho^{\frac{3}{4}} + \frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}} \Big) A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) \\
&\lesssim \Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho) + \Big[\Big(\frac{\rho}{r}\Big)^{\frac{3}{2}} + \Big(\frac{\rho}{r}\Big)^3 \Big] A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho) \\
&\lesssim \Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho) + \Big(\frac{\rho}{r}\Big)^3 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho).
\end{aligned}
$$

This completes the proof of (5.2). $\square$

Next we want to estimate the pressure function.

Lemma 3.6.3. *Under the same assumption with Lemma 3.6.2, it holds for any* $0 < r \le \dfrac{\rho}{2}$

$$D(r) \lesssim \Big(\frac{r}{\rho}\Big) D(\rho) + \Big(\frac{\rho}{r}\Big)^2 A^{\frac{3}{4}}(\rho) B^{\frac{3}{4}}(\rho). \tag{3.6.5}$$

Proof. From the scaling invariance of all quantities, we only need to consider the case $\rho = 1$, $0 < r \le \dfrac{1}{2}$. By taking divergence of the equation $(3.1.5)_1$, we obtain

$$\begin{aligned}
&-\Delta P = \operatorname{div}^2 [\mathbf{u} \otimes \mathbf{u} + \nabla Q \otimes \nabla Q] \\
= \;& \operatorname{div}^2 [(\mathbf{u} - (\mathbf{u})_1) \otimes (\mathbf{u} - (\mathbf{u})_1) + \nabla Q \otimes \nabla Q] \\
= \;& \operatorname{div}^2[(\mathbf{u} - (\mathbf{u})_1) \otimes (\mathbf{u} - (\mathbf{u})_1) + (\nabla Q - (\nabla Q)_1) \otimes (\nabla Q - (\nabla Q)_1)] \\
& + \operatorname{div}^2[(\nabla Q)_1 \otimes (\nabla Q - (\nabla Q)_1) + (\nabla Q - (\nabla Q)_1) \otimes (\nabla Q)_1].
\end{aligned} \tag{3.6.6}$$

Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be a cut off function of $B_{\frac{1}{2}}(0)$ such that

$$\begin{cases}
\eta = 1, & \text{in } B_{\frac{1}{2}}(0), \\
\eta = 0, & \text{in } \mathbb{R}^3 \setminus B_1(0), \\
0 \le \eta \le 1, & |\nabla \eta| \le 8.
\end{cases} \tag{3.6.7}$$

Define the following auxillary function

$$\begin{aligned}
P_1(x,t) = -\int_{\mathbb{R}^3} \nabla_y^2 G(x-y) : \eta^2(y) \Big[& (\mathbf{u} - (\mathbf{u})_1) \otimes (\mathbf{u} - (\mathbf{u})_1) \\
& + (\nabla Q - (\nabla Q)_1) \otimes (\nabla Q - (\nabla Q)_1) + (\nabla Q - (\nabla Q)_1) \otimes (\nabla Q)_1 \\
& + (\nabla Q)_1 \otimes (\nabla Q - (\nabla Q)_1) \Big](y,t)\, dy,
\end{aligned}$$

Then we have

$$-\Delta P_1 = \operatorname{div}^2 [(\mathbf{u} - (\mathbf{u})_1) \otimes (\mathbf{u} - (\mathbf{u})_1) + \nabla Q \otimes \nabla Q] \;\text{ in } B_{\frac{1}{2}}(0),$$

and

$$-\Delta(P-P_1)=0 \text{ in } B_{\frac{1}{2}}(0).$$

For P_1, we apply the Calderon-Zygmund theory to deduce

$$\begin{aligned}
\|P_1\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{3}{2}} &\lesssim \left\|\eta^2|\mathbf{u}-(\mathbf{u})_1|^2\right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{3}{2}}+\left\|\eta^2|\nabla Q-(\nabla Q)_1|^2\right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{3}{2}} \\
&\quad+\left\|\eta^2|(\nabla Q)_1||\nabla Q-(\nabla Q)_1|\right\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}^{\frac{3}{2}} \\
&\lesssim \int_{B_1(0)}(|\mathbf{u}-(\mathbf{u})_1|^3+|\nabla Q-(\nabla Q)_1|^3)\,dx \\
&\quad+|(\nabla Q)_1|^{\frac{3}{2}}\int_{B_1(0)}|\nabla Q-(\nabla Q)_1|^{\frac{3}{2}}\,dx. \qquad (3.6.8)
\end{aligned}$$

Since $P-P_1$ is harmonic in $B_{\frac{1}{2}}(0)$, we get

$$\frac{1}{r^2}\|P-P_1\|_{L^{\frac{3}{2}}(B_r(0))}^{\frac{3}{2}}\lesssim r\,\|P-P_1\|_{L^{\frac{3}{2}}(B_1(0))}^{\frac{3}{2}}\lesssim r\Big(\|P\|_{L^{\frac{3}{2}}(B_1(0))}^{\frac{3}{2}}+\|P_1\|_{L^{\frac{3}{2}}(B_1(0))}^{\frac{3}{2}}\Big).$$

Integrating it over $[-r^2,0]$ and applying (5.8), we can show that

$$\begin{aligned}
&\frac{1}{r^2}\int_{\mathbb{P}_r(0,0)}|P|^{\frac{3}{2}}\,dxdt \\
\lesssim\ & r\int_{\mathbb{P}_1(0,0)}|P|^{\frac{3}{2}}\,dxdt+\frac{1}{r^2}\int_{\mathbb{P}_1(0,0)}(|\mathbf{u}-(\mathbf{u})_1|^3+|\nabla Q-(\nabla Q)_1|^3)\,dxdt \\
&+\frac{1}{r^2}\Big(\sup_{-1\le t\le 0}|(\nabla Q)_1(t)|\Big)^{\frac{3}{2}}\int_{\mathbb{P}_1(0,0)}|\nabla Q-(\nabla Q)_1|^{\frac{3}{2}}\,dxdt \\
\lesssim\ & r\int_{\mathbb{P}_1(0,0)}|P|^{\frac{3}{2}}\,dxdt+\frac{1}{r^2}\int_{\mathbb{P}_1(0,0)}(|\mathbf{u}-(\mathbf{u})_1|^3+|\nabla Q-(\nabla Q)_1|^3)\,dxdt \\
&+\frac{1}{r^2}A^{\frac{3}{4}}(1)\int_{\mathbb{P}_1(0,0)}|\nabla Q-(\nabla Q)_1|^{\frac{3}{2}}\,dxdt.
\end{aligned}$$

This, combined with the interpolation inequality

$$\begin{aligned}
&\int_{\mathbb{P}_1(0,0)}(|\mathbf{u}-(\mathbf{u})_1|^3+|\nabla Q-(\nabla Q)_1|^3)\,dxdt \\
\lesssim\ & \sup_{-1\le t\le 0}\Big(\int_{B_1(0)}(|\mathbf{u}|^2+|\nabla Q|^2)\,dx\Big)^{\frac{3}{4}}\times\Big(\int_{\mathbb{P}_1(0,0)}(|\nabla\mathbf{u}|^2+|\nabla^2 Q|^2)\,dxdt\Big)^{\frac{3}{4}},
\end{aligned}$$

and Hölder's inequality

$$\int_{\mathbb{P}_1(0,0)} |\nabla Q - (\nabla Q)_1|^{\frac{3}{2}}\, dxdt \lesssim \Big(\int_{\mathbb{P}_1(0,0)} |\nabla Q - (\nabla Q)_1|^2\, dxdt\Big)^{\frac{3}{4}},$$

implies that

$$D(r) \lesssim rD(1) + \frac{1}{r^2} A^{\frac{3}{4}}(1) B^{\frac{3}{4}}(1).$$

This, after scaling back to ρ, yields (3.6.5). The proof is now complete. □

Proof of Theorem 3.6.1. For $\theta \in (0, \frac{1}{2})$ and $\rho \in (0,1)$, let $\varphi \in C_0^\infty(\mathbb{P}_{\theta\rho}(0,0))$ be a function such that

$$\varphi = 1 \text{ in } \mathbb{P}_{\frac{\theta\rho}{2}}(0,0),\ |\nabla\varphi| \lesssim \frac{1}{\theta\rho},\ |\nabla^2\varphi| + |\varphi_t| \lesssim (\frac{1}{\theta\rho})^2.$$

Applying the local energy inequality in Lemma 2.2, the maximum principles Lemmas 3.4.1 and 3.3.2, and the integration by parts, we obtain that

$$\begin{aligned}
&\sup_{-(\theta\rho)^2 \le t \le 0} \int_\Omega (|\mathbf{u}|^2 + |\nabla Q|^2)\varphi^2\, dx + \int_{\Omega\times[-(\theta\rho)^2,0]} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\varphi^2\, dxdt \\
&\lesssim \int_{\Omega\times[-(\theta\rho)^2,0]} (|\mathbf{u}|^2 + |\nabla Q|^2)(|\varphi_t| + |\nabla\varphi|^2 + |\nabla^2\varphi|)\, dxdt \\
&\quad + \int_{\Omega\times[-(\theta\rho)^2,0]} [(|\mathbf{u}|^2 - (|\mathbf{u}|^2)_{\theta\rho}) + (|\nabla Q|^2 - |\nabla Q|^2)_{\theta\rho}) + |P|]|\mathbf{u}||\nabla\varphi|\, dxdt \\
&\quad + \int_{\Omega\times[-(\theta\rho)^2,0]} |\nabla Q|^2\varphi^2\, dxdt + \int_{\Omega\times[-(\theta\rho)^2,0]} (|\nabla \mathbf{u}||\nabla Q| + |\mathbf{u}||\Delta Q|)|\varphi||\nabla\varphi|\, dxdt.
\end{aligned}$$

This, with the help of Young's inequality:

$$\begin{aligned}
&\int_{\Omega\times[-(\theta\rho)^2,0]} (|\nabla \mathbf{u}||\nabla Q| + |\mathbf{u}||\Delta Q|)|\varphi||\nabla\varphi|\, dxdt \\
&\le \frac{1}{2}\int_{\Omega\times[-(\theta\rho)^2,0]} (|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2)\varphi^2\, dxdt \\
&\quad + 4\int_{\Omega\times[-(\theta\rho)^2,0]} (|\mathbf{u}|^2 + |\nabla Q|^2)|\nabla\varphi|^2\, dxdt,
\end{aligned}$$

implies that

$$A(\frac{1}{2}\theta\rho) + B(\frac{1}{2}\theta\rho)$$

$$
\begin{aligned}
&= \sup_{-(\frac{\theta\rho}{2})^2\le t\le 0} \frac{2}{\theta\rho}\int_{B_{\frac{\theta\rho}{2}}(0)} (|\mathbf{u}|^2+|\nabla Q|^2)\,dx + \frac{2}{\theta\rho}\int_{\mathbb{P}_{\frac{\theta\rho}{2}}(0,0)} (|\nabla \mathbf{u}|^2+|\nabla^2 Q|^2)\,dxdt \\
&\lesssim \sup_{-(\theta\rho)^2\le t\le 0} \frac{1}{\theta\rho}\int_{\mathbb{R}^3} (|\mathbf{u}|^2+|\nabla Q|^2)\varphi^2\,dx + \frac{1}{\theta\rho}\int_{\mathbb{R}^3\times[-(\theta\rho)^2,0]} (|\nabla \mathbf{u}|^2+|\nabla^2 Q|^2)\varphi^2\,dxdt \\
&\lesssim \frac{1}{\theta\rho}\int_{\mathbb{R}^3\times[-(\theta\rho)^2,0]} (|\mathbf{u}|^2+|\nabla Q|^2)(|\varphi_t|+|\nabla\varphi|^2+|\nabla^2\varphi|)\,dxdt \\
&\quad + \frac{1}{\theta\rho}\int_{\mathbb{R}^3\times[-(\theta\rho)^2,0]} [(|\mathbf{u}|^2-(|\mathbf{u}|^2)_{\theta\rho})+(|\nabla Q|^2-(|\nabla Q|^2)_{\theta\rho})+|P|]|\mathbf{u}||\nabla\varphi|\,dxdt \\
&\quad + \frac{1}{\theta\rho}\int_{\mathbb{R}^3\times[-(\theta\rho)^2,0]} |\nabla Q|^2\varphi^2\,dxdt \\
&\lesssim \frac{1}{(\theta\rho)^3}\int_{\mathbb{P}_{\theta\rho}(0,0)} (|\mathbf{u}|^2+|\nabla Q|^2)\,dxdt + \frac{1}{(\theta\rho)^2}\int_{\mathbb{P}_{\theta\rho}(0,0)} |P||\mathbf{u}|\,dxdt \\
&\quad + \frac{1}{(\theta\rho)^2}\int_{\mathbb{P}_{\theta\rho}(0,0)} \Big(||\mathbf{u}|^2-(|\mathbf{u}|^2)_{\theta\rho}|+||\nabla Q|^2-(|\nabla Q|^2)_{\theta\rho}|\Big)|\mathbf{u}|\,dxdt \\
&= I_1+I_2+I_3.
\end{aligned}
$$

It is not hard to see that

$$
|I_1| \lesssim \Big(\frac{1}{(\theta\rho)^2}\int_{\mathbb{P}_{\theta\rho}(0,0)} (|\mathbf{u}|^3+|\nabla Q|^3)\,dxdt\Big)^{\frac{2}{3}} \lesssim C^{\frac{2}{3}}(\theta\rho),
$$

$$
|I_2| \lesssim \Big(\frac{1}{(\theta\rho)^2}\int_{\mathbb{P}_{\theta\rho}(0,0)} |\mathbf{u}|^3\,dxdt\Big)^{\frac{1}{3}}\Big(\frac{1}{(\theta\rho)^2}\int_{\mathbb{P}_{\theta\rho}(0,0)} |P|^{\frac{3}{2}}\,dxdt\Big)^{\frac{2}{3}} \lesssim C^{\frac{1}{3}}(\theta\rho)D^{\frac{2}{3}}(\theta\rho),
$$

while, by employing Hölder's and Poincaré's inequalities,

$$
\begin{aligned}
|I_3| &\lesssim \frac{1}{(\theta\rho)^2}\int_{-(\theta\rho)^2}^{0}\Big(\int_{B_{\theta\rho}(0)} (|\mathbf{u}||\nabla\mathbf{u}|+|\nabla Q||\nabla^2 Q|)\Big)\Big(\int_{B_{\theta\rho}(0)} |\mathbf{u}|^3+|\nabla Q|^3\Big)^{\frac{1}{3}}\,dt \\
&\lesssim A^{\frac{1}{2}}(\theta\rho)B^{\frac{1}{2}}(\theta\rho)C^{\frac{1}{3}}(\theta\rho).
\end{aligned}
$$

Putting together all the estimates, we have

$$
\begin{aligned}
A(\frac{1}{2}\theta\rho)+B(\frac{1}{2}\theta\rho) &\lesssim \Big[C^{\frac{2}{3}}(\theta\rho)+A^{\frac{1}{2}}(\theta\rho)B^{\frac{1}{2}}(\theta\rho)C^{\frac{1}{3}}(\theta\rho)+C^{\frac{1}{3}}(\theta\rho)D^{\frac{2}{3}}(\theta\rho)\Big] \\
&\lesssim \Big[C^{\frac{2}{3}}(\theta\rho)+A(\theta\rho)B(\theta\rho)+D^{\frac{4}{3}}(\theta\rho)\Big]
\end{aligned}
$$

so that

$$A^{\frac{3}{2}}(\frac{1}{2}\theta\rho) \lesssim \Big[C(\theta\rho) + A^{\frac{3}{2}}(\theta\rho)B^{\frac{3}{2}}(\theta\rho) + D^2(\theta\rho)\Big].$$

While

$$D^2(\theta\rho) \lesssim \theta^2\Big[D^2(\rho) + \theta^{-6}A^{\frac{3}{2}}(\rho)B^{\frac{3}{2}}(\rho)\Big],$$

and

$$C(\theta\rho) \lesssim \theta^3 A^{\frac{3}{2}}(\rho) + \theta^{-3}A^{\frac{3}{4}}(\rho)B^{\frac{3}{4}}(\rho).$$

Also note that

$$A^{\frac{3}{2}}(\theta\rho)B^{\frac{3}{2}}(\theta\rho) \le \theta^{-3}A^{\frac{3}{2}}(\rho)B^{\frac{3}{2}}(\rho).$$

Therefore we conclude that for $0 < \theta_0 < \frac{1}{2}$,

$$\begin{aligned}
&A^{\frac{3}{2}}(\frac{1}{2}\theta_0\rho) + D^2(\frac{1}{2}\theta_0\rho)\\
&\le c[\theta_0^2 D^2(\rho) + (\theta_0^{-3} + \theta_0^{-4})A^{\frac{3}{2}}(\rho)B^{\frac{3}{2}}(\rho) + \theta_0^3 A^{\frac{3}{2}}(\rho) + \theta_0^{-3}A^{\frac{3}{4}}(\rho)B^{\frac{3}{4}}(\rho)]\\
&\le c[\theta_0^2(D^2(\rho) + A^{\frac{3}{2}}(\rho)) + \theta_0^{-8}A^{\frac{3}{2}}(\rho)B^{\frac{3}{2}}(\rho) + \theta_0^2]\\
&\le c(\theta_0^2 + \theta_0^{-8}B^{\frac{3}{2}}(\rho))(A^{\frac{3}{2}}(\rho) + D^2(\rho)) + c\theta_0^2.
\end{aligned}$$

For $\varepsilon_1 > 0$ given by Theorem 5.1, let $\theta_0 \in (0, \frac{1}{2})$ such that

$$c\theta_0^2 = \min\Big\{\frac{1}{4}, \frac{1}{2}\varepsilon_1^2\Big\}.$$

From (3.6.1), we know that

$$\limsup_{\rho\to 0} B(\rho) \le \varepsilon_1^2,$$

hence there exists $\rho_0 > 0$ such that

$$c\theta_0^{-8}B^{\frac{3}{2}}(\rho) \le \frac{1}{4},\ \forall 0 < \rho < \rho_0.$$

Therefore we conclude that there exist $\theta_0 \in (0, \frac{1}{2})$ and $\rho_0 > 0$ such that

$$A^{\frac{3}{2}}(\frac{1}{2}\theta_0\rho) + D^2(\frac{1}{2}\theta_0\rho) \le \frac{1}{2}(A^{\frac{3}{2}}(\rho) + D^2(\rho)) + \frac{1}{2}\varepsilon_1^2, \ \forall 0 < \rho < \rho_0.$$

Iterating this inequality yields that

$$A^{\frac{3}{2}}((\frac{1}{2}\theta_0)^k\rho) + D^2((\frac{1}{2}\theta_0)^k\rho) \le \frac{1}{2^k}(A^{\frac{3}{2}}(\rho) + D^2(\rho)) + \varepsilon_1^2 \tag{3.6.9}$$

holds for all $0 < \rho < \rho_0$ and $k \ge 1$.

Employing (5.2) and (3.6.9), we obtain that

$$\begin{aligned} C((\frac{1}{2}\theta_0)^k\rho) &\le c\Big[(\frac{1}{2}\theta_0)^3 A^{\frac{3}{2}}((\frac{1}{2}\theta_0)^{k-1}\rho) + (\frac{1}{2}\theta_0)^{-3} A^{\frac{3}{4}}((\frac{1}{2}\theta_0)^{k-1}\rho) B^{\frac{3}{4}}((\frac{1}{2}\theta_0)^{k-1}\rho)\Big] \\ &\le c\Big[(\frac{1}{2}\theta_0)^3 + (\frac{1}{2}\theta_0)^{-3}\varepsilon_1^{\frac{3}{2}}\Big]\Big[\frac{1}{2^{k-1}}(A^{\frac{3}{2}}(\rho) + D^2(\rho)) + \varepsilon_1^2\Big] \end{aligned} \tag{3.6.10}$$

holds for all $0 < \rho < \rho_0$ and $k \ge 1$.

Putting (3.6.9) and (3.6.10) together, we obtain that

$$\limsup_{k\to\infty}\Big[C((\frac{1}{2}\theta_0)^k\rho) + D^2((\frac{1}{2}\theta_0)^k\rho)\Big] \le c\Big[1 + (\frac{1}{2}\theta_0)^3 + (\frac{1}{2}\theta_0)^{-3}\varepsilon_1^{\frac{3}{2}}\Big]\varepsilon_1^2 \le \frac{1}{2}\varepsilon_0^3, \tag{3.6.11}$$

holds for all $\rho \in (0, \rho_0)$, provided $\varepsilon_1 = \varepsilon_1(\theta_0, \varepsilon_0) > 0$ is chosen sufficiently small. Therefore, by Lemma 5.4 $(\mathbf{u}, Q, P)$ is smooth near $(0, 0)$. This completes the proof. $\square$

Theorem 2.2.1 can be proved by the following covering argument. Let Σ be the singular set of suitable weak solutions $(\mathbf{u}, Q, P)$. If $(x, t) \in \Sigma$, then by the theorem 3.6.1,

$$\limsup_{r\to 0}\frac{1}{r}\int_{\mathbb{P}_r(x,t)}(|\nabla\mathbf{u}|^2 + |\nabla^2 Q|^2)\,dxdt \ge \varepsilon_1^2. \tag{3.6.12}$$

Let V be a neighborhood of Σ and $\delta > 0$ such that for all $(x, t) \in \Sigma$, we can find $r < \delta$ such that $\mathbb{P}_r(x, t) \subset V$ and

$$\frac{1}{r}\int_{\mathbb{P}_r(x,t)}\Big(|\nabla\mathbf{u}|^2 + |\nabla^2 Q|^2\Big)\,dxdt \ge \varepsilon_1^2.$$

By Vitali's covering lemma, $\exists (x_{\rm i}, t_{\rm i}) \in V, 0 < r_{\rm i} < \delta$ such that $\{\mathbb{P}_{r_{\rm i}}(x_{\rm i}, t_{\rm i})\}_{\rm i=1}^{\infty}$ are pairwise disjoint and

$$\Sigma \subset \bigcup_{\rm i=1}^{\infty} \mathbb{P}_{5r_{\rm i}}(x_{\rm i}, t_{\rm i}).$$

Hence

$$\begin{aligned}
&\mathcal{P}^1_{5\delta}(\Sigma) \leq \sum_{\rm i=1}^{\infty} 5r_{\rm i} \leq \frac{5}{\varepsilon_1^2} \sum_{\rm i=1}^{\infty} \int_{\mathbb{P}_{r_{\rm i}}(x_{\rm i}, t_{\rm i})} \left(|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2\right) dxdt \\
&\leq \frac{5}{\varepsilon_1^2} \int_{\cup_{\rm i}\mathbb{P}_{r_{\rm i}}(x_{\rm i}, t_{\rm i})} \left(|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2\right) dxdt \\
&\leq \frac{5}{\varepsilon_1^2} \int_V \left(|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2\right) dxdt < \infty.
\end{aligned}$$

We can conclude that Σ is of zero Lesbegue measure. Then we can choose $|V|$ to be arbitrarily small, from the fact that

$$\int_0^{\infty} \int_{\Omega} \left(|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2\right) dxdt = \int_0^{\infty} \int_{\Omega} \left(|\nabla \mathbf{u}|^2 + |\Delta Q|^2\right) dxdt < \infty$$

and the absolute continuity of integral, we have

$$\lim_{|V| \to 0} \int_V \left(|\nabla \mathbf{u}|^2 + |\nabla^2 Q|^2\right) dxdt \to 0.$$

Hence

$$\mathcal{P}^1(\Sigma) = \lim_{\delta \to 0} \mathcal{P}^1_{5\delta}(\Sigma) = 0,$$

This completes the proof of Theorem 3.1.1. □

4. SUITABLE WEAK SOLUTIONS TO ERICKSEN–LESLIE SYSTEM WITH GINZBURG–LANDAU APPROXIMATION IN 3-D

4.1 Introduction

In this chapter we aim to investigate the Ericksen–Leslie system (1.1.3) with Ginzburg–Landau approximation on torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \nabla \cdot (\sigma^E + \sigma^L), \\ \nabla \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} - \Omega \mathbf{d} + \dfrac{\lambda_2}{\lambda_1} A\mathbf{d} = -\dfrac{1}{\lambda_1} \left(\Delta \mathbf{d} - \dfrac{1}{\varepsilon_{\mathrm{GL}}^2} \mathbf{f}(\mathbf{d}) \right) \end{cases} \tag{4.1.1}$$

which couples the incompressible flow of the liquid crystal material represented by the fluid velocity and pressure $(\mathbf{u}, P)(x,t) : \mathbb{T}^3 \times [0,\infty) \to \mathbb{R}^3 \times \mathbb{R}$, and the kinematic transported evolution of the crystal molecular director represented by the macroscopic order parameter $\mathbf{d}(x,t) : \mathbb{T}^3 \times [0,\infty) \to \mathbb{R}^3$. The Ericksen stress tensor σ^E is given by

$$\sigma^E_{\mathrm{ij}} = -\left(\frac{\partial W}{\partial \nabla \mathbf{d}} \odot \nabla \mathbf{d} \right)_{\mathrm{ij}} := -\sum_{k=1}^{3} \frac{\partial W}{\partial \mathbf{d}^k_{x_{\mathrm{i}}}} \mathbf{d}^k_{x_{\mathrm{j}}}.$$

where $W = W(\mathbf{d}, \nabla \mathbf{d})$ is the elastic distortion energy density. The relaxation form of W with Ginzburg–Landau potential $F(\mathbf{d}) = \frac{1}{4}(|\mathbf{d}|^2 - 1)^2$ reads

$$W = \frac{1}{2} |\nabla \mathbf{d}|^2 + \frac{1}{\varepsilon_{\mathrm{GL}}^2} F(\mathbf{d}). \tag{4.1.2}$$

The Leslie stress tensor σ^L has the following form:

$$\begin{aligned} \sigma^L = & \mu_1 (\mathbf{d}^T A \mathbf{d}) \mathbf{d} \otimes \mathbf{d} + \mu_2 N \otimes \mathbf{d} + \mu_3 \mathbf{d} \otimes N + \mu_4 A \\ & + \mu_5 A\mathbf{d} \otimes \mathbf{d} + \mu_6 \mathbf{d} \otimes A\mathbf{d}, \end{aligned} \tag{4.1.3}$$

where the notations

$$A = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \quad \Omega = \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^T), \quad N = \partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} - \Omega \mathbf{d}$$

represent the symmetric, skew-symmetric part of the velocity gradient, and the co-rotational derivative of the director field. The material constants λ_1 and λ_2, reflecting the molecular shape by Jeffrey's orbit, are related to the Leslie coefficients μ's through the following relations:

$$\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6, \tag{4.1.4}$$

$$\mu_2 + \mu_3 = \mu_6 - \mu_5. \tag{4.1.5}$$

Relations given in (4.1.4) are compatibility condition and (4.1.5) is called Parodi's relation, which can be derived from Onsager's reciprocal principle(cf. [4]). When it comes to the mathematical analysis of the whole system (4.1.1), these relations gives the crucial cancellations in the energy dissipation laws. We recognize that (4.1.1) is an approximation of the full Ericksen–Leslie system by assuming that the fluid density is constant in a isothermal environment without external forces, and more importantly, W is elastically isotropic via one constant approximations. Moreover, instead of dealing with the highly nonlinear constraint $|\mathbf{d}| = 1$, the Ginzburg–Landau potential only penalizes director for being away from the unit sphere. However, (physical meaning) it turns out that (4.1.1) is close to the system proposed by Leslie for anisotropic fluid with varying director length.

In this paper, we set $\varepsilon_{\mathrm{GL}} = 1$ since our result holds for any fixed $\varepsilon_{\mathrm{GL}} > 0$. By the incompressiblity of the velocity field, formally one can add a gradient field of any scalar function to $(4.1.1)_1$ which re-gauge the pressure function, for the purpose of constructing the suitable weak solutions, we subtract $\nabla(\frac{1}{2}|\nabla \mathbf{d}|^2 + F(\mathbf{d}))$ to the R.H.S. of $(4.1.1)_1$, with the help of the identity

$$\nabla \cdot (\nabla \mathbf{d} \odot \nabla \mathbf{d}) - \nabla \left(\frac{1}{2}|\nabla \mathbf{d}|^2\right) = \nabla \mathbf{d} \cdot \Delta \mathbf{d},$$

we obtain the equivalent form of (4.1.1):

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = -\nabla\mathbf{d}\cdot(\Delta\mathbf{d} - \mathbf{f}(\mathbf{d})) + \nabla\cdot\sigma^L(\mathbf{u},\mathbf{d}), \\ \nabla\cdot\mathbf{u} = 0, \\ \partial_t\mathbf{d} + \mathbf{u}\cdot\nabla\mathbf{d} - \Omega\mathbf{d} + \frac{\lambda_2}{\lambda_1}A\mathbf{d} = -\frac{1}{\lambda_1}(\Delta\mathbf{d} - \mathbf{f}(\mathbf{d})) \end{cases} \tag{4.1.6}$$

subject to the initial condition

$$(\mathbf{u},\mathbf{d})|_{t=0} = (\mathbf{u}_0,\mathbf{d}_0) \text{ in } \mathbb{T}^3. \tag{4.1.7}$$

Definition 4.1.1. *A pair of functions* $(\mathbf{u},\mathbf{d}) : \mathbb{T}^3\times\mathbb{R}_+ \to \mathbb{R}^3\times\mathbb{R}^3$ *is a weak solution solution to* (4.1.6) *and* (4.1.7)*, if* $(\mathbf{u},\mathbf{d}) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(\mathbb{T}^3\times\mathbb{R}_+,\mathbb{R}^3)\times(L_t^\infty H_x^1\cap L_t^2H_x^2)(\mathbb{T}^3\times\mathbb{R}_+,\mathbb{R}^3)$, *and for any* $\varphi\in C_0^\infty(\mathbb{T}^3\times\mathbb{R}_+,\mathbb{R}^3)$ *and* $\psi\in C_0^\infty(\mathbb{T}^3\times\mathbb{R}_+,\mathbb{R}^3)$ *with* $\nabla\cdot\varphi = 0$ *in* $\mathbb{T}^3\times\mathbb{R}_+$*, it holds that*

$$\begin{aligned} &\int_{\mathbb{T}^3\times\mathbb{R}_+}[-\mathbf{u}\cdot\partial_t\varphi + \nabla\mathbf{u} : \nabla\varphi - \mathbf{u}\otimes\mathbf{u} : \nabla\varphi]dxdt - \int_{\mathbb{T}^3\times\mathbb{R}_+}\nabla\mathbf{d}\odot\nabla\mathbf{d} : \nabla\varphi dxdt \\ &+ \int_{\mathbb{T}^3\times\mathbb{R}_+}\sigma^L(\mathbf{u},\mathbf{d}) : \nabla\varphi dxdt = \int_{\mathbb{T}^3}\mathbf{u}_0\cdot\varphi(x,0)dx, \end{aligned} \tag{4.1.8}$$

$$\begin{aligned} &\int_{\mathbb{T}^3\times\mathbb{R}_+}[-\mathbf{d}\cdot\partial_t\psi - \mathbf{u}\otimes\mathbf{d} : \nabla\psi - \frac{1}{\lambda_1}\nabla\mathbf{d} : \nabla\psi + \frac{1}{\lambda_1}\mathbf{f}(\mathbf{d})\cdot\psi]dxdt \\ &+ \int_{\mathbb{T}^3\times\mathbb{R}_+}(-\Omega\mathbf{d} + \frac{\lambda_2}{\lambda_1}A\mathbf{d})\cdot\psi dxdt = \int_{\mathbb{T}^3}\mathbf{d}_0\cdot\psi(x,0)dx. \end{aligned} \tag{4.1.9}$$

In this chapter, we assume the material coefficients satisfy the following constraint to ensure the energy dissipative structure of system:

$$\boxed{\lambda_1 < 0, \quad \mu_4 > 0, \quad \mu_1 = 0, \quad \mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} > 0.} \tag{4.1.10}$$

The global and local energy inequalities for (4.1.6)-(4.1.7) play the basic roles: for $t > 0$,

$$\begin{aligned} &\int_{\mathbb{T}^3\times\{t\}}\left(\frac{1}{2}|\mathbf{u}|^2 + \frac{1}{2}|\nabla\mathbf{d}|^2 + F(\mathbf{d})\right)dx \\ &+ \int_0^t\int_{\mathbb{T}^3}\left(\frac{\mu_4}{2}|\nabla\mathbf{u}|^2 + \frac{1}{-\lambda_1}|\Delta\mathbf{d} - \mathbf{f}(\mathbf{d})|^2\right)dxds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{T}^3}\left[\mu_1|\mathbf{d}^TA\mathbf{d}|^2+\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)|A\mathbf{d}|^2\right]dxds\\
&\le\int_{\mathbb{T}^3}\left(\frac{1}{2}|\mathbf{u}_0|^2+\frac{1}{2}|\nabla\mathbf{d}_0|^2+F(\mathbf{d}_0)\right)dx,
\end{aligned}
\tag{4.1.11}
$$

and for any $\eta\in C_0^\infty(\mathbb{T}^3\times[0,t],\mathbb{R})$ with $\eta\ge 0$, it holds

$$
\begin{aligned}
&\int_{\mathbb{T}^3\times\{t\}}\left(\frac{1}{2}|\mathbf{u}|^2+\frac{1}{2}|\nabla\mathbf{d}|^2+F(\mathbf{d})\right)\eta dx+\int_0^t\int_{\mathbb{T}^3}\left[\frac{\mu_4}{2}|\nabla\mathbf{u}|^2+\frac{1}{-\lambda_1}(|\Delta\mathbf{d}|^2+|\mathbf{f}(\mathbf{d})|^2)\right]\eta dxds\\
&+\int_0^t\int_{\mathbb{T}^3}\left[\mu_1|\mathbf{d}^TA\mathbf{d}|^2+\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)|A\mathbf{d}|^2\right]\eta dxds\\
&\le\int_{\mathbb{T}^3}\left(\frac{1}{2}|\mathbf{u}_0|^2+\frac{1}{2}|\nabla\mathbf{d}_0|^2+F(\mathbf{d}_0)\right)\eta dx\\
&+\int_0^t\int_{\mathbb{T}^3}\left[\frac{1}{2}|\mathbf{u}|^2\partial_t\eta+\left(\frac{1}{2}|\mathbf{u}|^2+P\right)\mathbf{u}\cdot\nabla\eta\right]dxds\\
&+\frac{\mu_4}{2}\int_{\mathbb{T}^3}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\eta dxds-\int_0^t\int_{\mathbb{T}^3}\sigma^L(\mathbf{u},\mathbf{d}):\mathbf{u}\otimes\nabla\eta dxds\\
&+\int_0^t\int_{\mathbb{T}^3}\left(\frac{1}{2}|\nabla\mathbf{d}|^2\partial_t\eta+\frac{1}{-2\lambda_1}|\nabla\mathbf{d}|^2\Delta\eta\right)dxds+\int_0^t\int_{\mathbb{T}^3}(\mathbf{u}\cdot\nabla\mathbf{d})\cdot(\nabla\eta\cdot\nabla\mathbf{d})dxds\\
&+\int_0^t\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}(\nabla\mathbf{d}\odot\nabla\mathbf{d}-|\nabla\mathbf{d}|^2\mathrm{I}_3):\nabla^2\eta dxds\\
&-\int_0^t\int_{\mathbb{T}^3}\left(\Omega\mathbf{d}-\frac{\lambda_2}{\lambda_1}A\mathbf{d}\right)\cdot(\nabla\eta\cdot\nabla\mathbf{d})dxds\\
&+\int_{\mathbb{T}^3}F(\mathbf{d})\partial_t\eta dx-\int_0^t\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}\left[2\nabla\mathbf{f}(\mathbf{d}):\nabla\mathbf{d}\eta+(\nabla\eta\cdot\nabla\mathbf{d})\cdot\mathbf{f}(\mathbf{d})\right]dxds.
\end{aligned}
\tag{4.1.12}
$$

We define

$$\mathbf{H}=\text{Closure of }\{\mathbf{u}\in C_0^\infty(\mathbb{T}^3,\mathbb{R}^3):\nabla\cdot\mathbf{u}=0\}\text{ in }L^2(\mathbb{T}^3),$$

and

$$\mathbf{V}=\text{Closure of }\{\mathbf{u}\in C_0^\infty(\mathbb{T}^3,\mathbb{R}^3):\nabla\cdot\mathbf{u}=0\}\text{ in }H^1(\mathbb{T}^3).$$

For $0\le k\le 5$, $\mathcal{P}^k$ denotes the k-dimensional Hausdorff measure on $\mathbb{T}^3\times\mathbb{R}$ with respect to the parabolic distance:

$$\delta((x,t),(y,s))=\max\left\{|x-y|,\sqrt{|t-s|}\right\},\forall(x,t),(y,s)\in\mathbb{T}^3\times\mathbb{R}.$$

The main theorem of this chapter concerns both the existence and partial regularity of suitable weak solutions to (4.1.6).

Theorem 4.1.1. *Assume the material coefficients satisfy the constrains* (4.1.5), (4.1.4) *and* (4.1.10). *For any* $\mathbf{u}_0 \in \mathbf{H}, \mathbf{d}_0 \in H^1(\mathbb{T}^3, \mathbb{R}^3)$, *there exists a global suitable weak solution* $(\mathbf{u}, \mathbf{d}, P) : \mathbb{T}^3 \times \mathbb{R}_+ \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ *of the Ericksen–Leslie system* (4.1.6) *and* (4.1.7) *such that*

$$(\mathbf{u}, \mathbf{d}) \in C^\infty(\mathbb{T}^3 \times (0, \infty) \setminus \Sigma),$$

where $\Sigma \subset \mathbb{T}^3 \times \mathbb{R}_+$ *is a closed subset with* $\mathcal{P}^{\frac{15}{7}+\sigma}(\Sigma) > 0$ *for all* $\sigma > 0$.

Remark. *If we look at special cases of* (4.1.6) *with*

$$\begin{aligned}
&\mu_1 = 0,\\
&\mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2), \quad \mu_3 = -\frac{1}{2}(\lambda_1 + \lambda_2),\\
&\mu_5 = \frac{1}{2}\left(\lambda_2 - \frac{\lambda_2^2}{\lambda_1}\right), \mu_6 = -\frac{1}{2}\left(\lambda_2 + \frac{\lambda_2^2}{\lambda_1}\right),
\end{aligned}$$

then (4.1.6) *can be reduced to*

$$\begin{cases}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P = \Delta \mathbf{u} - \nabla \mathbf{d} \cdot (\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})) - \nabla \cdot S_\alpha[\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}],\\
\nabla \cdot \mathbf{u} = 0,\\
\partial_t \mathbf{d} + \mathbf{u} \cdot \nabla \mathbf{d} - T_\alpha[\nabla \mathbf{u}, \mathbf{d}] = \Delta \mathbf{d} - \mathbf{f}(\mathbf{d}),
\end{cases} \tag{4.1.13}$$

where

$$\begin{aligned}
S_\alpha[\Delta \mathbf{d} - \mathbf{f}(\mathbf{d}), \mathbf{d}] &:= \alpha(\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})) \otimes \mathbf{d} - (1-\alpha)\mathbf{d} \otimes (\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})),\\
T_\alpha[\nabla \mathbf{u}, \mathbf{d}] &:= \alpha(\nabla \mathbf{u})\mathbf{d} - (1-\alpha)(\nabla \mathbf{u})^T \mathbf{d}
\end{aligned}$$

with

$$\alpha = \frac{\mu_2}{|\lambda_1|}.$$

The parameter $\alpha \in [0, 1]$ *is the shape parameter of the liquid crystal molecule. In particular,* $\alpha = 0, \frac{1}{2},$ *and* 1 *corresponds to the kinematic transport effects on disc-like, spherical and* *rod-like molecules respectively (cf. [77]). The partial regularity of suitable weak solutions to* (4.1.13) *was obtain [78]. Here Theorem 4.1.1 generalizes the result in [78].*

We would like to outline two major difficulties in the analysis of (4.1.6):

- First, as pointed out by [4], (4.1.6) suffers the loss of maximum principle for the director field $\mathbf{d}$ which plays an essential role in [34], [70]. Here, inspired by [36], [79], we will prove an ε_0-regularity result by a blowing-up argument that involves a decay estimate of renomalized L^3-norm of both $|\mathbf{u}|$ and $|\nabla \mathbf{d}|$ and the mean oscillation of $\mathbf{d}$ in L^6 as well.

- Second, the presence of the Leslie stress tensor σ^L brings an extra difficulty on the decay estimate of renormalized $L^{\frac{3}{2}}$-norm of the pressure function P. In particular, when $\mu_1 \neq 0$, the $(\mathbf{d}^T A\mathbf{d})(\mathbf{d} \otimes \mathbf{d}) : (\mathbf{u} \otimes \nabla \eta)$ term in the local energy inequality (4.1.12) does not have enough integrability to pass to the weak limit in the construction of suitable weak solutions.

We would like to mention that in a recent preprint [80], G. Koch obtained a partial regularity theorem for certain weak solutions to the Lin–Liu model (1.3.2) that may be weaker than suitable weak solutions and may not obey the maximum principle, in which a smallness condition is imposed on normalized L^6-norm of $|\mathbf{d}|$.

Remark. *Mathematically, it is a very challenging problem to ask if the set of singularity* Σ *is empty or not. Physically, the presence of potential singular set* Σ *for a solution* $(\mathbf{u}, \mathbf{d})$ *to the hydrodynamic system* (4.1.6) *may arise from the 3-D turbulence phenomenons of the underlying fluids (e.g., vortex points, lines, or filaments) as well as the defects of the liquid crystal molecular alignment field* $\mathbf{d}$ *induced by the rotating and stretching effects of fluid velocity field* $\mathbf{u}$*, see for example Chorin [81]. While Mandelbrot conjectured in [82], [83] that the self-similar nature of turbulence of the fluid may result in concentration of possible singularities of* $\mathbf{u}$ *on a set of fractional Hausdorff dimension.*

Remark. *The best known result on the set of singularities for the Navier-Stokes equation was obtained by Caffarelli–Kohn–Nirenberg [35], which asserts that it has zero* 1*-dimensional* *parabolic Hausdorff measure. For the co-rotational Beris–Edward Q-tensor system for liquid crystals, a result similar to [35] was also obtained by [70]. While our estimate on the dimension, $\frac{15}{7}$, of the singular set Σ in Theorem 4.1.1 may not be optimal, it is a natural consequence resulting from the blowup analysis (see Lemma 4.4.1) and the fractional Sobolev space regularity of the director field, i.e.* $\mathbf{d} \in W^{1,\frac{1}{2}}_{\frac{20}{7}}(Q_T)$ *(see the section 4.5 below).*

This chapter is organized as follows. In section 4.2, we will derive both the global and local energy inequality for smooth solutions of (4.1.6) and (4.1.7). In section 4.3, we will demonstrate the construction of suitable weak solution. In Section 4.4, we will prove the ε_0-regularity criteria for the suitable weak solutions. In section 4.5, we will finish the proof of the Theorem 4.1.1.

4.2 Global and local energy inequalities of the Ericksen–Leslie system with Ginzburg–Landau Approximation

In this section, we will establish both global and local energy inequality for classical solutions to the Ericksen–Leslie system (4.1.6)-(4.1.7).

Lemma 4.2.1. *Let* $(\mathbf{u}, \mathbf{d}, P) \in C^\infty(\mathbb{T}^3 \times \mathbb{R}_+, \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R})$ *be a solution to* (4.1.6)-(4.1.7)*. Then for any* $\eta \in C_0^\infty(\mathbb{T}^3 \times \mathbb{R}_+)$*, it holds*

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3}\left(\frac{1}{2}|\mathbf{u}|^2+\frac{1}{2}|\nabla\mathbf{d}|^2+F(\mathbf{d})\right)\eta dx+\int_{\mathbb{T}^3}\left[\frac{\mu_4}{2}|\nabla\mathbf{u}|^2+\frac{1}{-\lambda_1}(|\Delta\mathbf{d}|^2+|\mathbf{f}(\mathbf{d})|^2)\right]\eta dx\\
&+\int_{\mathbb{T}^3}\left[\mu_1|\mathbf{d}^TA\mathbf{d}|^2+\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)|A\mathbf{d}|^2\right]\eta dx\\
&=\int_{\mathbb{T}^3}\left[\frac{1}{2}|\mathbf{u}|^2\partial_t\eta+\left(\frac{1}{2}|\mathbf{u}|^2+P\right)\mathbf{u}\cdot\nabla\eta\right]dx+\frac{\mu_4}{2}\int_{\mathbb{T}^3}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\eta dx\\
&-\int_{\mathbb{T}^3}\sigma^L(\mathbf{u},\mathbf{d}):\mathbf{u}\otimes\nabla\eta dx\\
&+\int_{\mathbb{T}^3}\left(\frac{1}{2}|\nabla\mathbf{d}|^2\partial_t\eta+\frac{1}{-2\lambda_1}|\nabla\mathbf{d}|^2\Delta\eta\right)dx+\int_{\mathbb{T}^3}(\mathbf{u}\cdot\nabla\mathbf{d})\cdot(\nabla\eta\cdot\nabla\mathbf{d})dx\\
&+\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}(\nabla\mathbf{d}\odot\nabla\mathbf{d}-|\nabla\mathbf{d}|^2\mathrm{I}_3):\nabla^2\eta dx\\
&-\int_{\mathbb{T}^3}\left(\Omega\mathbf{d}-\frac{\lambda_2}{\lambda_1}A\mathbf{d}\right)\cdot(\nabla\eta\cdot\nabla\mathbf{d})dx\\
&+\int_{\mathbb{T}^3}F(\mathbf{d})\partial_t\eta dx-\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}[2\nabla\mathbf{f}(\mathbf{d}):\nabla\mathbf{d}\eta+(\nabla\eta\cdot\nabla\mathbf{d})\cdot\mathbf{f}(\mathbf{d})]dx. \qquad (4.2.1)
\end{aligned}
$$

Proof. For $\eta \in C_0^\infty(\mathbb{T}^3 \times \mathbb{R}_+), \eta \geq 0$, multiplying (4.1.6)$_1$ by $\mathbf{u}\eta$, integrating the resulting equation over $\mathbb{T}^3$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}|^2 \eta dx = \int_{\mathbb{T}^3} \frac{1}{2}|\mathbf{u}|^2 \partial_t \eta dx \\
&+ \int_{\mathbb{T}^3} \left(\frac{1}{2}|\mathbf{u}|^2 + P\right) \mathbf{u} \cdot \nabla \eta dx \\
&- \int_{\mathbb{T}^3} [(\mathbf{u} \cdot \nabla \mathbf{d}) \cdot \Delta \mathbf{d} - (\mathbf{u} \cdot \nabla \mathbf{d}) \cdot \mathbf{f}(\mathbf{d})]\eta dx \\
&- \int_{\mathbb{T}^3} (\sigma^L(\mathbf{u}, \mathbf{d}) : \nabla \mathbf{u}\eta + \sigma^L(\mathbf{u}, \mathbf{d}) : \mathbf{u} \otimes \nabla \eta) dx. \qquad (4.2.2)
\end{aligned}$$

By the symmetry of A, and the skew-symmetry of Ω, we have

$$\begin{aligned}
&\int_{\mathbb{T}^3} \eta \sigma^L(\mathbf{u}, \mathbf{d}) : \nabla \mathbf{u} dx \qquad (4.2.3) \\
=& \int_{\mathbb{T}^3} \eta[\mu_1(\mathbf{d}^T A \mathbf{d})(\mathbf{d} \otimes \mathbf{d}) + \mu_2 N \otimes \mathbf{d} + \mu_3 \mathbf{d} \otimes N + \mu_4 A + \mu_5 A\mathbf{d} \otimes \mathbf{d} + \mu_6 \mathbf{d} \otimes A\mathbf{d}] : (A + \Omega) dx \\
=& \int_{\mathbb{T}^3} \eta\Big[\mu_1 |\mathbf{d}^T A \mathbf{d}|^2 + \mu_4 |A|^2 + (\mu_2 + \mu_3)(N^T A \mathbf{d}) + (\mu_2 - \mu_3)(N^T \Omega \mathbf{d}) \\
&\quad + (\mu_5 + \mu_6)|A\mathbf{d}|^2 + (\mu_5 - \mu_6)(A\mathbf{d}) \cdot (\Omega \mathbf{d})\Big] dx \\
=& \int_{\mathbb{T}^3} \eta\Big[\mu_1 |\mathbf{d}^T A \mathbf{d}|^2 + \mu_4 |A|^2 + (\mu_5 + \mu_6)|A\mathbf{d}|^2 + \lambda_1(N^T \Omega \mathbf{d}) - \lambda_2(N^T A \mathbf{d}) + \lambda_2(A\mathbf{d}) \cdot (\Omega \mathbf{d})\Big] dx.
\end{aligned}$$

Now taking the gradient of (4.1.6)$_3$ yields

$$\partial_t \nabla \mathbf{d} + \nabla(\mathbf{u} \cdot \nabla \mathbf{d}) = \nabla \left[-\frac{1}{\lambda_1}(\Delta \mathbf{d} - \mathbf{f}(\mathbf{d})) + \Omega \mathbf{d} - \frac{\lambda_2}{\lambda_1} A\mathbf{d}\right].$$

Then multiplying the resulting equation by $\nabla \mathbf{d}\eta$, integrating over $\mathbb{T}^3$, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3} \frac{1}{2}|\nabla \mathbf{d}|^2 \eta dx + \int_{\mathbb{T}^3} \frac{1}{-\lambda_1}|\Delta \mathbf{d}|^2 \eta dx \\
&= \int_{\mathbb{T}^3} \frac{1}{2}|\nabla \mathbf{d}|^2 \partial_t \eta dx \\
&+ \int_{\mathbb{T}^3} (\mathbf{u} \cdot \nabla \mathbf{d}) \cdot (\Delta \mathbf{d}\eta + \nabla \eta \cdot \nabla \mathbf{d}) dx
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{T}^3}\left[\frac{1}{-\lambda_1}\Delta\mathbf{d}\cdot(\nabla\eta\cdot\nabla\mathbf{d})+\frac{1}{-\lambda_1}\nabla(\mathbf{f}(\mathbf{d})):\nabla\mathbf{d}\eta\right]dx\\
&-\int_{\mathbb{T}^3}\left(\Omega\mathbf{d}-\frac{\lambda_2}{\lambda_1}A\mathbf{d}\right)\cdot(\nabla\eta\cdot\nabla\mathbf{d})dx\\
&-\int_{\mathbb{T}^3}\left(\Omega\mathbf{d}-\frac{\lambda_2}{\lambda_1}A\mathbf{d}\right)\cdot\Delta\mathbf{d}\eta dx.
\end{aligned}
\tag{4.2.4}
$$

Moreover, multiplying $(4.1.6)_3$ by $\mathbf{f}(\mathbf{d})\eta$, integrating over $\mathbb{T}^3$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3}F(\mathbf{d})\eta dx+\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}|\mathbf{f}(\mathbf{d})|^2\eta dx\\
&=\int_{\mathbb{T}^3}[F(\mathbf{d})\partial_t\eta-(\mathbf{u}\cdot\nabla\mathbf{d})\cdot\mathbf{f}(\mathbf{d})\eta]dx\\
&-\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}[\nabla\mathbf{f}(\mathbf{d}):\nabla\mathbf{d}\eta+(\nabla\eta\cdot\nabla\mathbf{d})\cdot\mathbf{f}(\mathbf{d})]dx\\
&+\int_{\mathbb{T}^3}\left(\Omega\mathbf{d}-\frac{\lambda_2}{\lambda_1}A\mathbf{d}\right)\cdot\mathbf{f}(\mathbf{d})\eta dx.
\end{aligned}
\tag{4.2.5}
$$

The crucial cancellations among non-quadratic terms in the R.H.S. of (4.2.2) and the last terms in (4.2.4) and (4.2.5) reads

$$
\begin{aligned}
&\int_{\mathbb{T}^3}\eta\Big[\lambda_1(N^T\Omega\mathbf{d})-\lambda_2(N^TA\mathbf{d})+\lambda_2(A\mathbf{d})\cdot(\Omega\mathbf{d})\Big]dx\\
&+\int_{\mathbb{T}^3}\eta\left(\Omega\mathbf{d}-\frac{\lambda_2}{\lambda_1}A\mathbf{d}\right)\cdot(\Delta\mathbf{d}-\mathbf{f}(\mathbf{d}))dx\\
&=\int_{\mathbb{T}^3}\eta\left[-\lambda_2A\mathbf{d}-\Delta\mathbf{d}+\mathbf{f}(\mathbf{d})\right]\cdot(\Omega\mathbf{d})dx\\
&-\int_{\mathbb{T}^3}\lambda_2\eta\left[-\frac{\lambda_2}{\lambda_1}A\mathbf{d}-\frac{1}{\lambda_1}(\Delta\mathbf{d}-\mathbf{f}(\mathbf{d}))\right]\cdot(A\mathbf{d})dx\\
&+\int_{\mathbb{T}^3}\lambda_2\eta(A\mathbf{d})\cdot(\Omega\mathbf{d})dx+\int_{\mathbb{T}^3}\eta\left(\Omega\mathbf{d}-\frac{\lambda_2}{\lambda_1}A\mathbf{d}\right)\cdot(\Delta\mathbf{d}-\mathbf{f}(\mathbf{d}))dx\\
&=\int_{\mathbb{T}^3}\frac{\lambda_2^2}{\lambda_1}\eta|A\mathbf{d}|^2dx.
\end{aligned}
$$

It follows from integration by parts that

$$
\int_{\mathbb{T}^3}\mu_4|A|^2\eta dx=\frac{\mu_4}{2}\int_{\mathbb{T}^3}|\nabla\mathbf{u}|^2\eta dx-\frac{\mu_4}{2}\int_{\mathbb{T}^3}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\eta dx,
\tag{4.2.6}
$$

and

$$
\begin{aligned}
&-\int_{\mathbb{T}^3} \frac{1}{-\lambda_1}\Delta \mathbf{d}\cdot(\nabla\eta\cdot\nabla\mathbf{d})dx \\
&=\int_{\mathbb{T}^3} \frac{1}{-2\lambda_1}|\nabla\mathbf{d}|^2\Delta\eta dx \\
&+\int_{\mathbb{T}^3} \frac{1}{-\lambda_1}(\nabla\mathbf{d}\odot\nabla\mathbf{d}-|\nabla\mathbf{d}|^2\mathrm{I}_3):\nabla^2\eta dx. \qquad (4.2.7)
\end{aligned}
$$

Adding (4.2.2), (4.2.4) and (4.2.5) together, using the identities (4.2.6), (4.2.7) and (4.2.3) we get (4.2.1). □

Taking $\eta \equiv 1$, following the similar argument as above, we can show the following global energy equality for classical solutions to (4.1.6):

Lemma 4.2.2. *Let $(\mathbf{u},\mathbf{d}) \in C^\infty(\mathbb{T}^3\times\mathbb{R}_+,\mathbb{R}^3\times\mathbb{R}^3)$ be a solution to (4.1.6). Then it holds that*

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3}\left(\frac{1}{2}|\mathbf{u}|^2+\frac{1}{2}|\nabla\mathbf{d}|^2+F(\mathbf{d})\right)dx+\int_{\mathbb{T}^3}\left(\frac{\mu_4}{2}|\nabla\mathbf{u}|^2+\frac{1}{-\lambda_1}|\Delta\mathbf{d}-\mathbf{f}(\mathbf{d})|^2\right)dx \\
&+\int_{\mathbb{T}^3}\left[\mu_1|\mathbf{d}^TA\mathbf{d}|^2+\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)|A\mathbf{d}|^2\right]dx=0. \qquad (4.2.8)
\end{aligned}
$$

4.3 Existence of suitable weak solutions

In this section, we will follow the similar construction in to construct a suitable weak solution to (4.1.6). First, we introduce the so-called retarded space-time mollifier Ψ_θ for $f:\mathbb{T}^3\times\mathbb{R}_+\to\mathbb{R}$, with $0<\theta<1$, for $(x,t)\in\mathbb{T}^3\times(0,T)$, we define

$$
\Psi_\theta[f](x,t)=\frac{1}{\theta^4}\int_{\mathbb{R}^4}\psi\left(\frac{y}{\theta},\frac{\tau}{\theta}\right)\tilde{f}(x-y,t-\tau)dyd\tau,
$$

where $\psi \in C_0^\infty(\mathbb{R}^4)$ is a standard mollifier whose support is contained in the set $\{(x,t) : |x|^2 < t, 1 < t < 2\}$, and $\widetilde{f}(x,t)$ is the extension of f by zero outside $\mathbb{T}^3 \times (0,T)$. It is easy to justify that

$$\begin{aligned}
&\nabla \cdot \Psi_\theta[\mathbf{u}] = 0, \text{ if } \nabla \cdot \mathbf{u} = 0, \\
&\|\Psi_\theta[w]\|_{L^t_\infty L^2_x(Q_T)} \le C\|w\|_{L^\infty_t L^2_x(Q_T)}, \\
&\|\Psi_\theta[w]\|_{L^2_t H^1_x(Q_T)} \le C\,\|w\|_{L^2_t H^1_x(Q_T)}\,.
\end{aligned}$$

We introduce the approximation of (4.1.6):

$$\begin{cases}
\partial_t \mathbf{u}^\theta + \Psi_\theta[\mathbf{u}^\theta]\cdot\nabla\mathbf{u}^\theta + \nabla P^\theta = -\nabla\Psi_\theta[\mathbf{d}^\theta]\cdot(\Delta\mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)) + \nabla\cdot\sigma_\theta^L(\mathbf{u}^\theta,\mathbf{d}^\theta), \\
\nabla\cdot\mathbf{u}^\theta = 0, \\
\partial_t\mathbf{d}^\theta + \mathbf{u}^\theta\cdot\nabla\Psi_\theta[\mathbf{d}^\theta] - \Omega^\theta\Psi_\theta[\mathbf{d}^\theta] + \dfrac{\lambda_2}{\lambda_1}A^\theta\Psi_\theta[\mathbf{d}^\theta] = -\dfrac{1}{\lambda_1}(\Delta\mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)),
\end{cases} \tag{4.3.1}$$

where

$$\begin{aligned}
\sigma_\theta^L = {} & \mu_1(\Psi_\theta[\mathbf{d}^\theta]^T A^\theta\Psi_\theta[\mathbf{d}^\theta])\Psi_\theta[\mathbf{d}^\theta]\otimes\Psi_\theta[\mathbf{d}^\theta] + \mu_2 N^\theta\otimes\Psi_\theta[\mathbf{d}^\theta] + \mu_3\Psi_\theta[\mathbf{d}^\theta]\otimes N^\theta + \mu_4 A^\theta \\
& + \mu_5 A^\theta\Psi_\theta[\mathbf{d}^\theta]\otimes\Psi_\theta[\mathbf{d}^\theta] + \mu_6\Psi_\theta[\mathbf{d}^\theta]\otimes A^\theta\Psi_\theta[\mathbf{d}^\theta],
\end{aligned}$$

with

$$\begin{aligned}
A^\theta &= \frac{1}{2}(\nabla\mathbf{u}^\theta + (\nabla\mathbf{u}^\theta)^T), \\
\Omega^\theta &= \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u}^\theta)^T), \\
N^\theta &= \partial_t\mathbf{d}^\theta + \mathbf{u}^\theta\cdot\nabla\Psi_\theta[\mathbf{d}^\theta] - \Omega^\theta\Psi_\theta[\mathbf{d}^\theta].
\end{aligned}$$

For a fixed large integer $N \geq 1$, set $\theta = \dfrac{T}{N} \in (0,1)$. We try to find $(\mathbf{u}^\theta, \mathbf{d}^\theta, P^\theta)$ which solves (4.3.1). In $\mathbb{T}^3 \times [0,\theta]$, we have $\Psi_\theta[\mathbf{u}^\theta] = \Psi_\theta[\mathbf{d}^\theta] = 0$, and (4.3.1) reduces to a decoupled system

$$\begin{cases} \partial_t \mathbf{u}^\theta + \nabla P^\theta = \dfrac{\mu_4}{2}\Delta \mathbf{u}^\theta, \\ \nabla \cdot \mathbf{u}^\theta = 0, \\ \partial_t \mathbf{d}^\theta = \dfrac{1}{-\lambda_1}(\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)), \\ (\mathbf{u}^\theta, \mathbf{d}^\theta)\Big|_{t=0} = (\mathbf{u}_0, \mathbf{d}_0) \end{cases} \quad \text{in } \mathbb{T}^3 \times [0,\theta] \tag{4.3.2}$$

which can be solved by the standard theory. Suppose now we have solved the (4.3.1) for $\mathbb{T}^3 \times [0, k\theta]$ with $0 \leq k < N-1$, then we need to solve (4.3.1) in $\mathbb{T}^3 \times [k\theta, (k+1)\theta]$ with

$$(\mathbf{u}, \mathbf{d})|_{t=k\theta} = \lim_{t\uparrow k\theta} (\mathbf{u}^\theta, \mathbf{d}^\theta)(\cdot, t) \text{ in } \mathbb{T}^3.$$

With smooth coefficients, we can solve (4.3.1) by the Faedo–Galerkin method. For a pair of smooth test functions $(\phi, \psi) \in \mathbf{V} \times H^2(\mathbb{T}^3, \mathbb{R}^3)$, the weak formulation for (4.3.1) reads

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{T}^3} \mathbf{u}^\theta \cdot \phi dx - \int_{\mathbb{T}^3} (\Psi_\theta[\mathbf{u}^\theta] \otimes \mathbf{u}^\theta) : \nabla \phi dx \\ &= -\int_{\mathbb{T}^3} (\phi \cdot \nabla \Psi_\theta[\mathbf{d}^\theta]) \cdot (\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)) dx - \int_{\mathbb{T}^3} \sigma^L_\theta(\mathbf{u}^\theta, \mathbf{d}^\theta) : \nabla \phi dx, \end{aligned} \tag{4.3.3}$$

and

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{T}^3} \nabla \mathbf{d}^\theta : \nabla \psi dx - \int_{\mathbb{T}^3} (\mathbf{u}^\theta \cdot \nabla \Psi_\theta[\mathbf{d}^\theta]) \cdot \Delta \psi dx \\ &= -\int_{\mathbb{T}^3} \frac{1}{-\lambda_1}(\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)) \cdot \Delta \psi dx - \int_{\mathbb{T}^3} \left(\Omega^\theta \Psi_\theta[\mathbf{d}^\theta] - \frac{\lambda_2}{\lambda_1} A^\theta \Psi_\theta[\mathbf{d}^\theta] \right) \cdot \Delta \psi dx. \end{aligned} \tag{4.3.4}$$

Multiplying $(4.3.1)_1$ by $\mathbf{u}^\theta$, and $(4.3.1)_3$ by $-\Delta \mathbf{d}^\theta + \mathbf{f}(\mathbf{d}^\theta)$, integrating over $\mathbb{T}^3$, and adding them together we get

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{T}^3} \left(\frac{1}{2}|\mathbf{u}^\theta|^2 + \frac{1}{2}|\nabla \mathbf{d}^\theta|^2 + F(\mathbf{d}^\theta) \right) dx + \int_{\mathbb{T}^3} \frac{1}{-\lambda_1} |\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)|^2 dx \\ &= -\int_{\mathbb{T}^3} \sigma^L_\theta(\mathbf{u}^\theta, \mathbf{d}^\theta) : \nabla \mathbf{u}^\theta dx + \int_{\mathbb{T}^3} \left(\Omega^\theta \Psi_\theta[\mathbf{d}^\theta] - \frac{\lambda_2}{\lambda_1} A^\theta \Psi_\theta[\mathbf{d}^\theta] \right) \cdot (\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)) dx, \end{aligned} \tag{4.3.5}$$

and substituting $\sigma_\theta^L(\mathbf{u}^\theta, \mathbf{d}^\theta)$ in we get

$$\int_{\mathbb{T}^3} \sigma_\theta^L(\mathbf{u}^\theta, \mathbf{d}^\theta) : \nabla\mathbf{u}^\theta dx = \int_{\mathbb{T}^3} \sigma_\theta^L(\mathbf{u}^\theta, \mathbf{d}^\theta) : (A^\theta + \Omega^\theta) dx$$

$$= \int_{\mathbb{T}^3} \Big[\mu_1 |\Psi_\theta[\mathbf{d}^\theta]^T A^\theta \Psi_\theta[\mathbf{d}^\theta]|^2 + \mu_4 |A|^2 + (\mu_2+\mu_3)(N^\theta)^T A^\theta \Psi_\theta[\mathbf{d}^\theta] + (\mu_2-\mu_3)(N^\theta)^T \Omega^\theta \Psi_\theta[\mathbf{d}^\theta]$$
$$+ (\mu_5+\mu_6)|A^\theta \Psi_\theta[\mathbf{d}^\theta]|^2 + (\mu_5-\mu_6)(A^\theta \Psi_\theta[\mathbf{d}^\theta]) \cdot (\Omega^\theta \Psi_\theta[\mathbf{d}^\theta])\Big] dx$$
$$= \int_{\mathbb{T}^3} \Big[\mu_1 |\Psi_\theta[\mathbf{d}^\theta]^T A^\theta \Psi_\theta[\mathbf{d}^\theta]|^2 + \mu_4 |A^\theta|^2 + (\mu_5+\mu_6)|A^\theta \Psi_\theta[\mathbf{d}^\theta]|^2$$
$$+ \lambda_1 (N^\theta)^T \Omega^\theta \Psi_\theta[\mathbf{d}^\theta] - \lambda_2 (N^\theta)^T A^\theta \Psi_\theta[\mathbf{d}^\theta] + \lambda_2 (A^\theta \Psi_\theta[\mathbf{d}^\theta]) \cdot (\Omega^\theta \Psi_\theta[\mathbf{d}^\theta])\Big] dx.$$

Then we plug in

$$N^\theta = -\frac{\lambda_2}{\lambda_1} A^\theta \Psi_\theta[\mathbf{d}^\theta] + \frac{1}{-\lambda_1}(\Delta\mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta))$$

and by the fact that $\nabla \cdot \mathbf{u}^\theta = 0$ we have the following identity

$$\int_{\mathbb{T}^3} \nabla\mathbf{u}^\theta : (\nabla\mathbf{u}^\theta)^T dx = 0,$$

and hence

$$\int_{\mathbb{T}^3} \mu_4 |A^\theta|^2 dx = \int_{\mathbb{T}^3} \frac{\mu_4}{4}(|\nabla\mathbf{u}^\theta|^2 + |(\nabla\mathbf{u}^\theta)|^2 + 2\nabla\mathbf{u}^\theta : (\nabla\mathbf{u}^\theta)^T) dx$$
$$= \int_{\mathbb{T}^3} \frac{\mu_4}{2} |\nabla\mathbf{u}^\theta|^2 dx.$$

To sum up, we have

$$\frac{d}{dt}\int_{\mathbb{T}^3} \left(\frac{1}{2}|\mathbf{u}^\theta|^2 + \frac{1}{2}|\nabla\mathbf{d}^\theta|^2 + F(\mathbf{d}^\theta)\right) dx + \int_{\mathbb{T}^3} \left(\frac{\mu_4}{2}|\nabla\mathbf{u}^\theta|^2 + \frac{1}{-\lambda_1}|\Delta\mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)|^2\right) dx$$
$$+ \int_{\mathbb{T}^3} \left[\mu_1 |\Psi_\theta[\mathbf{d}^\theta]^T A^\theta \Psi_\theta[\mathbf{d}^\theta]|^2 + \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}\right) |A^\theta \Psi_\theta[\mathbf{d}^\theta]|^2\right] dx = 0. \tag{4.3.6}$$

Applying the integration by parts to $\Delta\mathbf{d}^\theta \cdot \mathbf{f}(\mathbf{d}^\theta)$ yields

$$\int_{\mathbb{T}^3} \Delta\mathbf{d}^\theta \cdot \mathbf{f}(\mathbf{d}^\theta) dx = -\int_{\mathbb{T}^3} \mathbf{d}^\theta_{\mathrm{i},x_\mathrm{j}} (\mathbf{f}(\mathbf{d}^\theta))_{\mathrm{i},x_\mathrm{j}} dx$$

$$
\begin{aligned}
&= -\int_{\mathbb{T}^3} \mathbf{d}^\theta_{\mathrm{i},x_\mathrm{j}}[(|\mathbf{d}^\theta|^2-1)\mathbf{d}^\theta_\mathrm{i}]_{,x_\mathrm{j}}dx \\
&= -\int_{\mathbb{T}^3} |\nabla \mathbf{d}^\theta|^2(|\mathbf{d}^\theta|^2-1)dx + 2\int_{\mathbb{T}^3} \mathbf{d}^\theta_{\mathrm{i},x_\mathrm{j}}\mathbf{d}^\theta_k\mathbf{d}^\theta_{k,x_\mathrm{j}}\mathbf{d}^\theta_\mathrm{i}dx \\
&= -\int_{\mathbb{T}^3} (-|\nabla \mathbf{d}^\theta|^2 + |\nabla \mathbf{d}^\theta|^2|\mathbf{d}^\theta|^2 + 2|(\nabla \mathbf{d}^\theta)^T\mathbf{d}^\theta|^2)dx.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_{\mathbb{T}^3} |\Delta \mathbf{d}^\theta|^2 = \int_{\mathbb{T}^3} (|\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)|^2 - |\mathbf{f}(\mathbf{d}^\theta)|^2 + 2\Delta \mathbf{d}^\theta \cdot \mathbf{f}(\mathbf{d}^\theta))dx \\
&= \int_{\mathbb{T}^3} (|\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)|^2 + 2|\nabla \mathbf{d}^\theta|^2)dx - \int_{\mathbb{T}^3} (|\mathbf{f}(\mathbf{d}^\theta)|^2 + 2|\nabla \mathbf{d}^\theta|^2|\mathbf{d}^\theta|^2 + 4|(\nabla \mathbf{d}^\theta)^T\mathbf{d}^\theta|^2)dx \\
&\le \int_{\mathbb{T}^3} (|\nabla \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)|^2 + 2|\nabla \mathbf{d}^\theta|^2)dx
\end{aligned}
$$

From (4.3.6), we have that

$$
\begin{aligned}
&\sup_{0<t<T}\int_{\mathbb{T}^3\times\{t\}} (|\mathbf{u}^\theta|^2 + |\nabla \mathbf{d}^\theta|^2)dx + \int_0^T\int_{\mathbb{T}^3} (|\nabla \mathbf{u}^\theta|^2 + |\Delta \mathbf{d}^\theta|^2)dxdt \\
&\le \sup_{0<t<T}\int_{\mathbb{T}^3\times\{t\}} (|\mathbf{u}^\theta|^2 + |\nabla \mathbf{d}^\theta|^2)dx + \int_0^T\int_{\mathbb{T}^3} (|\nabla \mathbf{u}^\theta|^2 + |\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)|^2 + 2|\nabla \mathbf{d}^\theta|^2)dxdt \\
&\le C(T, \lambda_1, \mu_4, \mathbf{u}_0, \mathbf{d}_0). \qquad (4.3.7)
\end{aligned}
$$

In other words, we have that $\{(\mathbf{u}^\theta, \mathbf{d}^\theta)\}_{0<\theta<1}$ is uniformly bounded in $L^\infty_t L^2_x \cap L^2_t H^1_x \times L^\infty_t H^1_x \cap L^2_t H^2_x(Q_T)$. Therefore, after passing to a subsequence, there exist $(\mathbf{u}, \mathbf{d}) \in L^\infty_t L^2_x \cap L^2_t H^1_x \times L^\infty_t H^1_x \cap L^2_t H^2_x(Q_T)$ such that

$$
\begin{cases}
\mathbf{u}^\theta \rightharpoonup \mathbf{u} & \text{in } L^\infty_t L^2_x \cap L^2_t H^1_x(Q_T), \\
\mathbf{d}^\theta \rightharpoonup \mathbf{d} & \text{in } L^\infty_t H^1_x \cap L^2_t H^2_x(Q_T).
\end{cases} \qquad (4.3.8)
$$

By the Sobolev and Hölder inequalities, we have that $\nabla \mathbf{d}^\theta \in L^{10}_t L^{\frac{30}{13}}_x, \mathbf{d}^\theta \in L^{10}_t L^{10}_x$. In fact, we can show

$$
\int_0^T \|\nabla \mathbf{d}^\theta\|^{10}_{L^{\frac{30}{13}}_x} dt \le \int_0^T \|\nabla \mathbf{d}^\theta\|^8_{L^2_x}\|\nabla \mathbf{d}^\theta\|^2_{L^6_x} dt
$$

$$
\begin{aligned}
&\leq \sup_{0<t<T} \|\nabla \mathbf{d}^\theta\|_{L_x^2}^8 \int_0^T \|\nabla \mathbf{d}^\theta\|_{L_x^6}^2 dt \\
&\leq \|\nabla \mathbf{d}^\theta\|_{L_t^\infty L_x^2}^8 \int_0^T \|\nabla \mathbf{d}^\theta\|_{H_x^1}^2 dt \\
&\leq \|\mathbf{d}^\theta\|_{L_t^\infty H_x^1}^8 \|\mathbf{d}^\theta\|_{L_t^2 H_x^2}^2, \\
\int_0^T \|\mathbf{d}^\theta\|_{L_x^{10}}^{10} dt &\leq C \int_0^T \|\mathbf{d}^\theta\|_{W_x^{1,\frac{30}{13}}}^{10} dt.
\end{aligned} \tag{4.3.9}
$$

Recall the $\mathbf{u}^\theta$ equations, we have

$$
\begin{cases}
\partial_t \mathbf{u}^\theta - \frac{\mu_4}{2}\Delta \mathbf{u}^\theta + \nabla P^\theta = -\Psi_\theta[\mathbf{u}^\theta] \cdot \nabla \mathbf{u}^\theta - \nabla \Psi_\theta[\mathbf{d}^\theta] \cdot (\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta)) + \nabla \cdot \tilde{\sigma}_\theta^L(\mathbf{u}^\theta, \mathbf{d}^\theta), \\
\nabla \cdot \mathbf{u}^\theta = 0,
\end{cases}
$$

where $\tilde{\sigma}_\theta^L(\mathbf{u}^\theta, \mathbf{d}^\theta) = \sigma_\theta^L(\mathbf{u}^\theta, \mathbf{d}^\theta) - \mu_4 A^\theta$. By the standard theory of linear Stokes system, we have

$$
\left\| \partial_t \mathbf{u}^\theta \right\|_{L^{\frac{5}{4}}(Q_T) + L^{\frac{5}{3}}([0,T], W^{-1,\frac{5}{3}}(\mathbb{T}^3))} \leq C(T, \lambda_1, \mu_4, \mathbf{u}_0, \mathbf{d}_0).
$$

And the $\mathbf{d}^\theta$ equation

$$
\partial_t \mathbf{d}^\theta - \frac{1}{-\lambda_1}\Delta \mathbf{d}^\theta = -\mathbf{u}^\theta \cdot \nabla \Psi_\theta[\mathbf{d}^\theta] + \Omega^\theta \Psi_\theta[\mathbf{d}^\theta] - \frac{\lambda_2}{\lambda_1} A^\theta \Psi_\theta[\mathbf{d}^\theta] + \frac{1}{\lambda_1}\mathbf{f}(\mathbf{d}^\theta).
$$

From the estimates for the linear parabolic system, we have

$$
\left\| \partial_t \mathbf{d}^\theta \right\|_{L^{\frac{5}{3}}(Q_T) + L^2([0,T], L^{\frac{3}{2}}(\mathbb{T}^3))} \leq C(T, \lambda_1, \lambda_2, \mu_4, \mathbf{u}_0, \mathbf{d}_0).
$$

Hence by the Sobolev embedding Theorem and Aubin–Lions's compactness Lemma, we can conclude that, after passing to a possible subsequence, as $\theta \to 0$,

$$\begin{cases} \mathbf{u}^\theta \to \mathbf{u} & \text{in } L^{p_1}(Q_T), 1 < p_1 < \dfrac{10}{3}, \\ \nabla \mathbf{u}^\theta \rightharpoonup \nabla \mathbf{u} & \text{in } L^2(Q_T), \\ \mathbf{d}^\theta \to \mathbf{d} & \text{in } L^{p_2}(Q_T), 1 < p_2 < 10, \\ \nabla \mathbf{d}^\theta \to \nabla \mathbf{d} & \text{in } L^{p_1}(Q_T), 1 < p_1 < \dfrac{10}{3}, \\ \nabla^2 \mathbf{d}^\theta \rightharpoonup \nabla^2 \mathbf{d} & \text{in } L^2(Q_T), \\ N^\theta \rightharpoonup N & \text{in } L^2(Q_T), \\ A^\theta \Psi_\theta[\mathbf{d}^\theta] \rightharpoonup A\mathbf{d} & \text{in } L^2(Q_T). \end{cases} \tag{4.3.10}$$

From (4.3.10) and the lower semicontinuity property we can justify that $(\mathbf{u}, \mathbf{d})$ solves (4.1.6) in the weak sense and the global energy inequality holds. Next we want to justify the local energy inequality. The key step is to obtain the estimate on the pressure function P^θ. Taking the divergence of $(4.3.1)_1$ yields

$$\begin{aligned} -\Delta P^\theta = &\operatorname{div}^2(\Psi_\theta[\mathbf{u}^\theta] \otimes \mathbf{u}^\theta) + \operatorname{div}\Big(\nabla \Psi_\theta[\mathbf{d}^\theta] \cdot (\Delta \mathbf{d}^\theta - \mathbf{f}(\mathbf{d}^\theta))\Big) \\ &+ \operatorname{div}^2 \widetilde{\sigma}^L_\theta(\mathbf{u}^\theta, \mathbf{d}^\theta), \text{ in } \mathbb{T}^3. \end{aligned} \tag{4.3.11}$$

By Calderon–Zygmund's L^p-theory, we can show

$$\|P^\theta\|_{L^{\frac{5}{3}}(Q_T)} \le C(\|\mathbf{u}_0\|_{L^2(\mathbb{T}^3)}, \|\mathbf{d}_0\|_{H^1(\mathbb{T}^3)}, T).$$

Hence, there exists $P \in L^{\frac{5}{3}}(Q_T)$ such that as $\theta \to 0$,

$$P^\theta \rightharpoonup P \text{ in } L^{\frac{5}{3}}(Q_T). \tag{4.3.12}$$

Now we derive the local energy inequality for $(\mathbf{u}^\theta, \mathbf{d}^\theta, P^\theta)$. For $\eta \in C_0^\infty(\mathbb{T}^3 \times \mathbb{R}_+)$, multiplying $(4.3.1)_1$ by $\mathbf{u}^\theta \eta$, integrating the resulting equation over $\mathbb{T}^3$, we obtain

$$\frac{d}{dt} \int_{\mathbb{T}^3} \frac{1}{2} |\mathbf{u}^\theta|^2 \eta dx = \frac{1}{2} \int_{\mathbb{T}^3} \frac{1}{2} |\mathbf{u}^\theta|^2 \partial_t \eta dx$$

$$
\begin{aligned}
&+\int_{\mathbb{T}^3}\frac{1}{2}|\mathbf{u}^\theta|^2\Psi_\theta[\mathbf{u}^\theta]\cdot\nabla\eta dx+\int_{\mathbb{T}^3}P^\theta\mathbf{u}^\theta\cdot\nabla\eta dx\\
&-\int_{\mathbb{T}^3}[(\mathbf{u}^\theta\cdot\nabla\Psi_\theta[\mathbf{d}^\theta])\cdot\Delta\mathbf{d}^\theta-(\mathbf{u}^\theta\cdot\nabla\Psi_\theta[\mathbf{d}^\theta])\cdot\mathbf{f}(\mathbf{d}^\theta)]\eta dx\\
&-\int_{\mathbb{T}^3}(\sigma_\theta^L(\mathbf{u}^\theta,\mathbf{d}^\theta):\nabla\mathbf{u}^\theta\eta+\sigma_\theta^L(\mathbf{u}^\theta,\mathbf{d}^\theta):\mathbf{u}^\theta\otimes\nabla\eta)dx. \qquad (4.3.13)
\end{aligned}
$$

For the Ericksen stress tensor term, we have

$$
\begin{aligned}
&\int_{\mathbb{T}^3}\eta\sigma_\theta^L(\mathbf{u}^\theta,\mathbf{d}^\theta):\nabla\mathbf{u}^\theta dx\\
=&\int_{\mathbb{T}^3}\eta\Big[\mu_1(\Psi_\theta[\mathbf{d}^\theta]^TA^\theta\Psi_\theta[\mathbf{d}^\theta])(\Psi_\theta[\mathbf{d}^\theta]\otimes\Psi_\theta[\mathbf{d}^\theta])+\mu_2N^\theta\otimes\Psi_\theta[\mathbf{d}^\theta]+\mu_3\Psi_\theta[\mathbf{d}^\theta]\otimes N^\theta+\mu_4A^\theta\\
&\quad+\mu_5A^\theta\Psi_\theta[\mathbf{d}^\theta]\otimes\Psi_\theta[\mathbf{d}^\theta]+\mu_6\Psi_\theta[\mathbf{d}^\theta]\otimes A^\theta\Psi_\theta[\mathbf{d}^\theta]\Big]:(A^\theta+\Omega^\theta)dx\\
=&\int_{\mathbb{T}^3}\eta\Big[\mu_1|\Psi_\theta[\mathbf{d}^\theta]^TA^\theta\Psi_\theta[\mathbf{d}^\theta]|^2+\mu_4|A^\theta|^2+(\mu_2+\mu_3)(N^\theta)^TA^\theta\Psi_\theta[\mathbf{d}^\theta]+(\mu_2-\mu_3)(N^\theta)^T\Omega^\theta\Psi_\theta[\mathbf{d}^\theta]\\
&\quad+(\mu_5+\mu_6)|A^\theta\Psi_\theta[\mathbf{d}^\theta]|^2+(\mu_5-\mu_6)(A^\theta\Psi_\theta[\mathbf{d}^\theta])\cdot(\Omega^\theta\Psi_\theta[\mathbf{d}^\theta])\Big]dx\\
=&\int_{\mathbb{T}^3}\eta\Big[\mu_1|\Psi_\theta[\mathbf{d}^\theta]^TA^\theta\Psi_\theta[\mathbf{d}^\theta]|^2+\mu_4|A^\theta|^2+(\mu_5+\mu_6)|A^\theta\Psi_\theta[\mathbf{d}^\theta]|^2\\
&\quad+\lambda_1(N^\theta)^T\Omega^\theta\Psi_\theta[\mathbf{d}^\theta]-\lambda_2(N^\theta)^TA^\theta\Psi_\theta[\mathbf{d}^\theta]+\lambda_2(A^\theta\Psi_\theta[\mathbf{d}^\theta])\cdot(\Omega^\theta\Psi_\theta[\mathbf{d}^\theta])\Big]dx.
\end{aligned}
$$

Now taking the gradient of (4.3.1)$_3$ yields

$$
\partial_t\nabla\mathbf{d}^\theta+\nabla(\mathbf{u}^\theta\cdot\nabla\Psi_\theta[\mathbf{d}^\theta])=\nabla\left[-\frac{1}{\lambda_1}(\Delta\mathbf{d}^\theta-\mathbf{f}(\mathbf{d}^\theta))+\Omega^\theta\Psi_\theta[\mathbf{d}^\theta]-\frac{\lambda_2}{\lambda_1}A^\theta\Psi_\theta[\mathbf{d}^\theta]\right].
$$

Now multiplying the resulting equation by $\nabla\mathbf{d}^\theta\eta$, integrating over $\mathbb{T}^3$, we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3}\frac{1}{2}|\nabla\mathbf{d}^\theta|^2\eta dx+\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}|\Delta\mathbf{d}^\theta|^2\eta dx\\
&=\int_{\mathbb{T}^3}\frac{1}{2}|\nabla\mathbf{d}^\theta|^2\partial_t\eta dx\\
&+\int_{\mathbb{T}^3}(\mathbf{u}^\theta\cdot\nabla\Psi_\theta[\mathbf{d}^\theta])\cdot(\Delta\mathbf{d}^\theta\eta+\nabla\eta\cdot\nabla\mathbf{d}^\theta)dx\\
&-\int_{\mathbb{T}^3}\left[\frac{1}{-\lambda_1}\Delta\mathbf{d}^\theta(\nabla\eta\cdot\nabla\mathbf{d}^\theta)+\frac{1}{-\lambda_1}\nabla(\mathbf{f}(\mathbf{d}^\theta)):\nabla\mathbf{d}^\theta\eta\right]dx\\
&-\int_{\mathbb{T}^3}\left(\Omega^\theta\Psi_\theta[\mathbf{d}^\theta]-\frac{\lambda_2}{\lambda_1}A^\theta\Psi_\theta[\mathbf{d}^\theta]\right)\cdot(\nabla\eta\cdot\nabla\mathbf{d}^\theta)dx
\end{aligned}
$$

$$-\int_{\mathbb{T}^3}\left(\Omega^\theta\Psi_\theta[\mathbf{d}^\theta]-\frac{\lambda_2}{\lambda_1}A^\theta\Psi_\theta[\mathbf{d}^\theta]\right)\cdot\Delta\mathbf{d}^\theta\eta dx. \tag{4.3.14}$$

Moreover, multiplying $(4.3.1)_3$ by $\mathbf{f}(\mathbf{d}^\theta)\eta$, integrating over $\mathbb{T}^3$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3}F(\mathbf{d}^\theta)\eta dx+\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}|\mathbf{f}(\mathbf{d}^\theta)|^2\eta dx\\
&=\int_{\mathbb{T}^3}[F(\mathbf{d}^\theta)\partial_t\eta-(\mathbf{u}^\theta\cdot\nabla\Psi_\theta[\mathbf{d}^\theta])\cdot\mathbf{f}(\mathbf{d}^\theta)\eta]dx\\
&-\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}[\nabla\mathbf{f}(\mathbf{d}^\theta):\nabla\mathbf{d}^\theta\eta+(\nabla\eta\cdot\nabla\mathbf{d}^\theta)\cdot\mathbf{f}(\mathbf{d}^\theta)]dx\\
&+\int_{\mathbb{T}^3}\left(\Omega^\theta\Psi_\theta[\mathbf{d}^\theta]-\frac{\lambda_2}{\lambda_1}A^\theta\Psi_\theta[\mathbf{d}^\theta]\right)\cdot\mathbf{f}(\mathbf{d}^\theta)\eta dx.
\end{aligned} \tag{4.3.15}$$

Follow a similar cancellation as in Lemma (4.2.1), we have

$$\begin{aligned}
&\int_{\mathbb{T}^3}\eta\Big[\lambda_1(N^\theta)^T\Omega^\theta\Psi_\theta[\mathbf{d}^\theta]-\lambda_2(N^\theta)^TA^\theta\Psi_\theta[\mathbf{d}^\theta]+\lambda_2(A^\theta\Psi_\theta[\mathbf{d}^\theta])\cdot(\Omega\Psi_\theta[\mathbf{d}^\theta])\Big]dx\\
&+\int_{\mathbb{T}^3}\eta\left(\Omega^\theta\Psi_\theta[\mathbf{d}^\theta]-\frac{\lambda_2}{\lambda_1}A^\theta\Psi_\theta[\mathbf{d}^\theta]\right)\cdot(\Delta\mathbf{d}^\theta-\mathbf{f}(\mathbf{d}^\theta))dx\\
&=\int_{\mathbb{T}^3}\eta[-\lambda_2A^\theta\Psi_\theta[\mathbf{d}^\theta]-\Delta\mathbf{d}^\theta+\mathbf{f}(\mathbf{d}^\theta)]\cdot(\Omega^\theta\Psi_\theta[\mathbf{d}^\theta])dx\\
&-\int_{\mathbb{T}^3}\eta\left[-\frac{\lambda_2^2}{\lambda_1}A^\theta\Psi_\theta[\mathbf{d}^\theta]-\frac{\lambda_2}{\lambda_1}(\Delta\mathbf{d}^\theta-\mathbf{f}(\mathbf{d}^\theta))\right]\cdot(A^\theta\Psi_\theta[\mathbf{d}^\theta])dx\\
&+\int_{\mathbb{T}^3}\lambda_2\eta(A^\theta\Psi_\theta[\mathbf{d}^\theta])\cdot(\Omega^\theta\Psi_\theta[\mathbf{d}^\theta])dx+\int_{\mathbb{T}^3}\eta\left(\Omega^\theta\Psi_\theta[\mathbf{d}^\theta]-\frac{\lambda_2}{\lambda_1}A^\theta\Psi_\theta[\mathbf{d}^\theta]\right)\cdot(\Delta\mathbf{d}^\theta-\mathbf{f}(\mathbf{d}^\theta))dx\\
&=\int_{\mathbb{T}^3}\frac{\lambda_2^2}{\lambda_1}\eta|A^\theta\Psi_\theta[\mathbf{d}^\theta]|^2dx.
\end{aligned}$$

Putting forward with identities

$$\int_{\mathbb{T}^3}\mu_4|A^\theta|^2\eta dx=\frac{\mu_4}{2}\int_{\mathbb{T}^3}|\nabla\mathbf{u}^\theta|^2\eta dx-\frac{\mu_4}{2}\int_{\mathbb{T}^3}(\mathbf{u}^\theta\cdot\nabla\mathbf{u}^\theta)\cdot\nabla\eta dx,$$

and

$$\begin{aligned}
&-\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}\Delta\mathbf{d}^\theta\cdot(\nabla\eta\cdot\nabla\mathbf{d}^\theta)dx\\
&=\int_{\mathbb{T}^3}\frac{1}{-2\lambda_1}|\nabla\mathbf{d}^\theta|^2\Delta\eta dx
\end{aligned}$$

$$+\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}(\nabla\mathbf{d}^\theta\odot\nabla\mathbf{d}^\theta-|\nabla\mathbf{d}^\theta|^2\mathrm{I}_3):\nabla^\theta\eta dx,$$

we can add (4.3.3), (4.3.4) and (4.3.15) together to get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^3}\left(\frac{1}{2}|\mathbf{u}^\theta|^2+\frac{1}{2}|\nabla\mathbf{d}^\theta|^2+F(\mathbf{d}^\theta)\right)\eta dx+\int_{\mathbb{T}^3}\left[\frac{\mu_4}{2}|\nabla\mathbf{u}^\theta|^2+\frac{1}{-\lambda_1}(|\Delta\mathbf{d}^\theta|^2+|\mathbf{f}(\mathbf{d}^\theta)|^2)\right]\eta dx\\
&+\int_{\mathbb{T}^3}\left[\mu_1|\Psi_\theta[\mathbf{d}^\theta]^T A^\theta\Psi_\theta[\mathbf{d}^\theta]|^2+\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)|A\Psi_\theta[\mathbf{d}^\theta]|^2\right]\eta dx\\
&=\int_{\mathbb{T}^3}\left(\frac{1}{2}|\mathbf{u}|^2\partial_t\eta+\frac{1}{2}|\mathbf{u}^\theta|^2\Psi_\theta[\mathbf{u}^\theta]\cdot\nabla\eta+P^\theta\mathbf{u}^\theta\cdot\nabla\eta\right)dx+\frac{\mu_4}{2}\int_{\mathbb{T}^3}(\mathbf{u}^\theta\cdot\nabla\mathbf{u}^\theta)\cdot\nabla\eta dx \quad (4.3.16)\\
&-\int_{\mathbb{T}^3}\sigma_\theta^L(\mathbf{u}^\theta,\mathbf{d}^\theta):\mathbf{u}^\theta\otimes\nabla\eta dx\\
&+\int_{\mathbb{T}^3}\left(\frac{1}{2}|\nabla\mathbf{d}^\theta|^2\partial_t\eta+\frac{1}{-2\lambda_1}|\nabla\mathbf{d}|^2\Delta\eta\right)dx+\int_{\mathbb{T}^3}(\mathbf{u}^\theta\cdot\nabla\Psi_\theta[\mathbf{d}^\theta])\cdot(\nabla\eta\cdot\nabla\mathbf{d}^\theta)dx\\
&+\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}(\nabla\mathbf{d}^\theta\odot\nabla\mathbf{d}^\theta-|\nabla\mathbf{d}^\theta|^2\mathrm{I}_3):\nabla^2\eta dx\\
&-\int_{\mathbb{T}^3}\left(\Omega^\theta\Psi_\theta[\mathbf{d}^\theta]-\frac{\lambda_2}{\lambda_1}A^\theta\Psi_\theta[\mathbf{d}^\theta]\right)\cdot(\nabla\eta\cdot\nabla\mathbf{d}^\theta)dx\\
&+\int_{\mathbb{T}^3}F(\mathbf{d}^\theta)\partial_t\eta dx-\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}\left[2\nabla\mathbf{f}(\mathbf{d}^\theta):\nabla\mathbf{d}^\theta\eta+(\nabla\eta\cdot\nabla\mathbf{d}^\theta)\cdot\mathbf{f}(\mathbf{d}^\theta)\right]dx. \quad (4.3.17)
\end{aligned}$$

Finally, we send $\theta\to 0$ in (4.3.17), by the lower semicontinuity we have the local energy inequality (4.2.1) holds for $(\mathbf{u},\mathbf{d},P)$, and hence, $(\mathbf{u},\mathbf{d},P)$ is a suitable weak solution to (4.1.6).

4.4 Blowing up argument

Lemma 4.4.1. *For any $M>0$, there exist $\varepsilon_0=\varepsilon_0(M)>0$, $0<\tau_0(M)<\frac{1}{2}$, and $C_0=C_0(M)>0$, such that if $(\mathbf{u},\mathbf{d},P)$ is a suitable weak solution to (4.1.6) in $\mathbb{T}^3\times(0,\infty)$, which satisfies, for $r>0$ and $z_0=(x_0,t_0)\in\mathbb{T}^3\times(r^2,\infty)$,*

$$|\mathbf{d}_{z_0,r}|:=\left|\fint_{\mathbb{P}_r(z_0)}\mathbf{d}dxdt\right|\le M, \tag{4.4.1}$$

and

$$\Phi(z_0, r) := r^2 \int_{\mathbb{P}_r(z_0)} (|\mathbf{u}|^3 + |\nabla \mathbf{d}|^3) dxdt + \left(r^{-3} \int_{\mathbb{P}_r(z_0)} |P|^{\frac{3}{2}} dxdt \right)^2 \tag{4.4.2}$$

$$+ \left(⨍_{\mathbb{P}_r(z_0)} |\mathbf{d} - \mathbf{d}_{z_0,r}|^6 dxdt \right)^{\frac{1}{2}} \leq \varepsilon_0^3, \tag{4.4.3}$$

then

$$\Phi(z_0, \tau_0 r) \leq \frac{1}{2} \max \left\{ \Phi(z_0, r), C_0 r^3 \right\}. \tag{4.4.4}$$

Remark. *In the absence of maximum principle for the director field* $\mathbf{d}$*, the* L^6*-norm of the mean oscillation of* $\mathbf{d}$ *plays the role in obtaining the (local) boundedness of* $(\mathbf{u}, \nabla \mathbf{d}) \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$ *in* (4.4.11)*. By closely examining the proof of Lemma* (4.4.1)*, the* L^6*-norm can be relaxed to the* L^p*-norm of the mean oscillation of* $\mathbf{d}$ *as long as* $p > 5$*. However, this does not seem to improve the estimate of the dimension of the singular set* Σ *of* $(\mathbf{u}, \nabla \mathbf{d})$*, since we can only obtain* $\mathbf{d} \in W^{1,\frac{1}{2}}_{\frac{20}{7}}$*, which can yield the boundedness of* $L^{\frac{20}{3}}$*-norm of the mean oscillation of* $\mathbf{d}$ *(see* (4.5.4) *below).*

Proof. We proof it by contradiction. Suppose otherwise, then there exists $M_0 > 0$ such that for any $\tau \in \left(0, \frac{1}{2}\right)$, there exist $\varepsilon_{\mathrm{i}} \to 0$, $C_{\mathrm{i}} \to \infty$, $r_{\mathrm{i}} > 0$, and $z_{\mathrm{i}} = (x_{\mathrm{i}}, t_{\mathrm{i}}) \in \mathbb{T}^3 \times (r_{\mathrm{i}}^2, \infty)$ such that

$$|\mathbf{d}_{z_{\mathrm{i}}, r_{\mathrm{i}}}| \leq M_0, \tag{4.4.5}$$

and

$$\Phi(z_{\mathrm{i}}, r_{\mathrm{i}}) = \varepsilon_{\mathrm{i}}^3, \tag{4.4.6}$$

but

$$\Phi(z_{\mathrm{i}}, \tau r_{\mathrm{i}}) \geq \frac{1}{2} \max \left\{ \varepsilon_{\mathrm{i}}^3, C_{\mathrm{i}} r_{\mathrm{i}}^3 \right\}, \tag{4.4.7}$$

From which we can conclude that

$$r_{\mathrm{i}} \leq \left(\frac{\varepsilon_{\mathrm{i}}^3}{2 C_{\mathrm{i}} \max \left\{ \tau^{-4}, 8\tau^{-\frac{5}{2}} \right\}} \right)^{\frac{1}{3}} \to 0.$$

Define the translating-rescaling and blowing-up sequence by

$$(\mathbf{u}_{\mathrm{i}}, \mathbf{d}_{\mathrm{i}}, P_{\mathrm{i}}) := (r_{\mathrm{i}}\mathbf{u}, \mathbf{d}, r_{\mathrm{i}}^2 P)(x_{\mathrm{i}} + r_{\mathrm{i}}x, t_{\mathrm{i}} + r_{\mathrm{i}}^2 t), \forall x \in \mathbb{R}^3, t \geq -1,$$

and

$$(\widehat{\mathbf{u}}_{\mathrm{i}}, \widehat{\mathbf{d}}_{\mathrm{i}}, \widehat{P}_{\mathrm{i}})(z) := \left(\frac{\mathbf{u}_{\mathrm{i}}}{\varepsilon_{\mathrm{i}}}, \frac{\mathbf{d}_{\mathrm{i}} - \overline{\mathbf{d}_{\mathrm{i}}}}{\varepsilon_{\mathrm{i}}}, \frac{P_{\mathrm{i}}}{\varepsilon_{\mathrm{i}}}\right)(z), \forall z = (x,t) \in \mathbb{P}_1(0),$$

where

$$\overline{\mathbf{d}_{\mathrm{i}}} = \fint_{\mathbb{P}_1(0)} \mathbf{d}_{\mathrm{i}} dxdt.$$

It is straightforward to check that $(\widehat{\mathbf{u}}_{\mathrm{i}}, \widehat{\mathbf{d}}_{\mathrm{i}}, \widehat{P}_{\mathrm{i}})$ satisfies

$$\begin{cases} \fint_{\mathbb{P}_1(0)} \widehat{\mathbf{d}}_{\mathrm{i}} dxdt = 0, \\ |\overline{\mathbf{d}_{\mathrm{i}}}| = |\mathbf{d}_{z_{\mathrm{i}}, r_{\mathrm{i}}}| \leq M_0, \\ \int_{\mathbb{P}_1(0)} \left(|\widehat{\mathbf{u}}_{\mathrm{i}}|^3 + |\nabla \widehat{\mathbf{d}}_{\mathrm{i}}|^3\right) dxdt + \left(\int_{\mathbb{P}_1(0)} |\widehat{P}_{\mathrm{i}}|^{\frac{3}{2}} dxdt\right)^2 \\ \qquad + \left(\fint_{\mathbb{P}_1(0)} |\widehat{\mathbf{d}}_{\mathrm{i}}|^6 dxdt\right)^{\frac{1}{2}} = 1, \\ \tau^{-2} \int_{\mathbb{P}_\tau(0)} \left(|\widehat{\mathbf{u}}_{\mathrm{i}}|^3 + |\nabla \widehat{\mathbf{d}}_{\mathrm{i}}|^3\right) dxdt + \left(\tau^{-2} \int_{\mathbb{P}_\tau(0)} |\widehat{P}_{\mathrm{i}}|^{\frac{3}{2}} dxdt\right)^2 \\ \qquad + \left(\fint_{\mathbb{P}_\tau(0)} |\widehat{\mathbf{d}}_{\mathrm{i}} - (\widehat{\mathbf{d}}_{\mathrm{i}})_{0,\tau}|^6 dxdt\right)^{\frac{1}{2}} \geq \frac{1}{2} \max\left\{1, C_{\mathrm{i}} \left(\frac{r_{\mathrm{i}}}{\varepsilon_{\mathrm{i}}}\right)^3\right\}. \end{cases} \tag{4.4.8}$$

Furthermore, $(\widehat{\mathbf{u}}_{\mathrm{i}}, \widehat{\mathbf{d}}_{\mathrm{i}}, \widehat{P}_{\mathrm{i}})$ is a suitable weak solution of the blowing-up version of (4.1.6):

$$\begin{cases} \partial_t \widehat{\mathbf{u}}_{\mathrm{i}} + \varepsilon_{\mathrm{i}} \widehat{\mathbf{u}}_{\mathrm{i}} \cdot \nabla \widehat{\mathbf{u}}_{\mathrm{i}} + \nabla \widehat{P}_{\mathrm{i}} - \frac{\mu_4}{2} \Delta \widehat{\mathbf{u}}_{\mathrm{i}} = -\varepsilon_{\mathrm{i}} \nabla \widehat{\mathbf{d}}_{\mathrm{i}} \cdot \Delta \widehat{\mathbf{d}}_{\mathrm{i}} + \frac{r_{\mathrm{i}}^2}{\varepsilon_{\mathrm{i}}} \nabla \mathbf{d}_{\mathrm{i}} \cdot \mathbf{f}(\mathbf{d}_{\mathrm{i}}) + \nabla \cdot \widehat{\sigma}_{\mathrm{i}}^L, \\ \nabla \cdot \widehat{\mathbf{u}}_{\mathrm{i}} = 0, \\ \partial_t \widehat{\mathbf{d}}_{\mathrm{i}} + \varepsilon_{\mathrm{i}} \widehat{\mathbf{u}}_{\mathrm{i}} \cdot \nabla \widehat{\mathbf{d}}_{\mathrm{i}} - \widehat{\Omega}_{\mathrm{i}} \mathbf{d}_{\mathrm{i}} + \frac{\lambda_2}{\lambda_1} \widehat{A}_{\mathrm{i}} \mathbf{d}_{\mathrm{i}} = \frac{1}{|\lambda_1|} (\Delta \widehat{\mathbf{d}}_{\mathrm{i}} - \frac{r_{\mathrm{i}}^2}{\varepsilon_{\mathrm{i}}} \mathbf{f}(\mathbf{d}_{\mathrm{i}})), \end{cases} \tag{4.4.9}$$

where

$$\widehat{A}_{\mathrm{i}} = \frac{1}{2}(\nabla \widehat{\mathbf{u}}_{\mathrm{i}} + (\nabla \widehat{\mathbf{u}}_{\mathrm{i}})^T),$$
$$\widehat{\Omega}_{\mathrm{i}} = \frac{1}{2}(\nabla \widehat{\mathbf{u}}_{\mathrm{i}} - (\nabla \widehat{\mathbf{u}}_{\mathrm{i}})^T),$$

$$\widehat{N}_{\rm i} = \partial_t \widehat{\mathbf{d}}_{\rm i} + \varepsilon \widehat{\mathbf{u}}_{\rm i} \cdot \nabla \widehat{\mathbf{d}}_{\rm i} - \widehat{\Omega}_{\rm i} \mathbf{d}_{\rm i}$$

and

$$\begin{aligned}\widehat{\sigma}_{\rm i}^L = & \mu_1 (\mathbf{d}_{\rm i}^T \widehat{A}_{\rm i} \mathbf{d}_{\rm i}) \mathbf{d}_{\rm i} \otimes \mathbf{d}_{\rm i} + \mu_2 \widehat{N}_{\rm i} \otimes \mathbf{d}_{\rm i} + \mu_3 \mathbf{d}_{\rm i} \otimes \widehat{N}_{\rm i} \\ & + \mu_5 \widehat{A}_{\rm i} \mathbf{d}_{\rm i} \otimes \mathbf{d}_{\rm i} + \mu_6 \mathbf{d}_{\rm i} \otimes \widehat{A}_{\rm i} \mathbf{d}_{\rm i}.\end{aligned}$$

From (4.4.8), we assume that there exists

$$(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}, \widehat{P}) \in L^3(\mathbb{P}_1(0)) \times L_t^3 W_x^{1,3}(\mathbb{P}_1(0)) \times L^{\frac{3}{2}}(\mathbb{P}_1(0))$$

such that, after passing to a subsequence, as $\mathrm{i} \to \infty$,

$$(\widehat{\mathbf{u}}_{\rm i}, \widehat{\mathbf{d}}_{\rm i}, \widehat{P}_{\rm i}) \rightharpoonup (\widehat{\mathbf{u}}, \widehat{\mathbf{d}}, \widehat{P}) \text{ in } L^3(\mathbb{P}_1(0)) \times L_t^3 W_x^{1,3}(\mathbb{P}_1(0)) \times L^{\frac{3}{2}}(\mathbb{P}_1(0))).$$

By the lower semicontinuity we have that

$$\int_{\mathbb{P}_1(0)} \left(|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{\mathbf{d}}|^3 \right) dxdt + \left(\int_{\mathbb{P}_1(0)} |\widehat{P}|^{\frac{3}{2}} \right)^2 + \left(⨍_{\mathbb{P}_1(0)} |\widehat{\mathbf{d}}|^6 dxdt \right)^{\frac{1}{2}} \leq 1. \tag{4.4.10}$$

We claim that

$$\|\widehat{\mathbf{u}}_{\rm i}\|_{L_t^\infty L_x^2 \cap L_t^2 H_x^1 (\mathbb{P}_{\frac{1}{2}}(0))} + \|\widehat{\mathbf{d}}_{\rm i}\|_{L_t^\infty H_x^1 \cap L_t^2 H_x^2 (\mathbb{P}_{\frac{1}{2}}(0))} \leq C < \infty. \tag{4.4.11}$$

We choose a cut-off function $\phi \in C_0^\infty(\mathbb{P}_1(0))$ such that

$$0 \leq \phi \leq 1, \phi \equiv 1 \text{ on } \mathbb{P}_{\frac{1}{2}}(0), \text{ and } |\partial_t \phi| + |\nabla \phi| + |\nabla^2 \phi| \leq C.$$

Define

$$\phi_{\rm i}(x,t) := \phi \left(\frac{x - x_{\rm i}}{r_{\rm i}}, \frac{t - t_{\rm i}}{r_{\rm i}^2} \right), \quad \forall (x,t) \in \mathbb{T}^3 \times (0, \infty),$$

let $\eta = \phi_{\rm i}^2$ in (4.1.12), by Young's inequality we get

$$
\begin{aligned}
&\sup_{t_{\rm i}-\frac{r_{\rm i}^2}{4}\le t\le t_{\rm i}} \int_{B_{r_{\rm i}}(x_{\rm i})} \left(|\mathbf{u}|^2 + |\nabla \mathbf{d}|^2 + F(\mathbf{d})\right)\phi_{\rm i}^2 dx \\
&+ \int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})} (|\nabla \mathbf{u}|^2 + |\Delta \mathbf{d}|^2 + |\mathbf{f}(\mathbf{d})|^2 + |A\mathbf{d}|^2)\phi_{\rm i}^2 dxdt \\
&\le C\Big[\int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})} \left(|\mathbf{u}|^2|\partial_t \phi_{\rm i}^2| + |\mathbf{u}|^3|\nabla \phi_{\rm i}^2| + |P||\mathbf{u}||\nabla \phi_{\rm i}^2|\right) dxdt \\
&+ \int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})} (|\mathbf{u}|^2|\nabla \phi_{\rm i}|^2 + |\mathbf{u}|^2|\mathbf{d}|^2|\nabla \phi_{\rm i}|^2) dxdt \\
&+ \int_{\mathbb{P}_{r_{\rm i}}(z_{\rm i})} \left(|\nabla \mathbf{d}|^2|\partial_t \phi_{\rm i}^2| + |\nabla \mathbf{d}|^2|\Delta \phi_{\rm i}^2| + |\mathbf{u}||\nabla \mathbf{d}|^2|\nabla \phi_{\rm i}^2|\right) dxdt \\
&+ \int_{\mathbb{P}_{z_{\rm i}}(z_{\rm i})} |\mathbf{d}|^2|\nabla \mathbf{d}|^2|\nabla \phi_{\rm i}|^2 dxdt \\
&+ \int_{\mathbb{P}_{z_{\rm i}}(z_{\rm i})} \left(F(\mathbf{d})|\partial_t \phi_{\rm i}^2| + |\nabla \mathbf{f}(\mathbf{d})||\nabla \mathbf{d}|\phi_{\rm i}^2 + |\nabla \mathbf{d}|^2|\nabla \phi_{\rm i}|^2\right) dxdt\Big].
\end{aligned}
$$

By rescaling and using the estimate (4.4.8), we get

$$
\begin{aligned}
&\sup_{-\frac{1}{4}\le t\le 0} \int_{B_{\frac{1}{2}}(0)} \left(|\widehat{\mathbf{u}}_{\rm i}|^2 + |\nabla \widehat{\mathbf{d}}_{\rm i}|^2\right) dx + \int_{\mathbb{P}_{\frac{1}{2}}(0)} \left(|\nabla \widehat{\mathbf{u}}_{\rm i}|^2 + |\nabla^2 \widehat{\mathbf{d}}_{\rm i}|^2\right) dxdt \\
&\le C\Big[\int_{\mathbb{P}_1(0)} \left(|\widehat{\mathbf{u}}_{\rm i}|^2 + \varepsilon_{\rm i}|\widehat{\mathbf{u}}_{\rm i}|^3 + \varepsilon_{\rm i}|\widehat{P}_{\rm i}||\widehat{\mathbf{u}}_{\rm i}|\right) dxdt \\
&+ \int_{\mathbb{P}_1(0)} \left(|\widehat{\mathbf{u}}_{\rm i}|^2 + |\widehat{\mathbf{u}}_{\rm i}|^2|\mathbf{d}_{\rm i}|^2\right) dxdt \\
&+ \int_{\mathbb{P}_1(0)} \left(|\nabla \widehat{\mathbf{d}}_{\rm i}|^2 + \varepsilon_{\rm i}|\widehat{\mathbf{u}}_{\rm i}||\nabla \widehat{\mathbf{d}}_{\rm i}|^2\right) dxdt \\
&+ \int_{\mathbb{P}_1(0)} |\mathbf{d}_{\rm i}|^2|\nabla \widehat{\mathbf{d}}_{\rm i}|^2 dxdt \\
&+ \int_{\mathbb{P}_1(0)} \left(\frac{r_{\rm i}^2}{\varepsilon_{\rm i}^2} F(\mathbf{d}_{\rm i}) + |\nabla \widehat{\mathbf{d}}_{\rm i}|^2 + r_{\rm i}^2|\nabla \widehat{\mathbf{d}}_{\rm i}|^2|\partial_{\mathbf{d}}\mathbf{f}(\mathbf{d}_{\rm i})|\right) dxdt\Big] \\
&\le C.
\end{aligned}
$$

This yields (4.4.11). Hence we may assume that

$$
(\widehat{\mathbf{u}}_{\rm i}, \widehat{\mathbf{d}}_{\rm i}) \rightharpoonup (\widehat{\mathbf{u}}, \widehat{\mathbf{d}}) \text{ in } L_t^2 H_x^1(\mathbb{P}_{\frac{1}{2}}(0)) \times L_t^2 H_x^2(\mathbb{P}_{\frac{1}{2}}(0)). \tag{4.4.12}
$$

From the fact that $\frac{r_{\mathrm{i}}}{\varepsilon_{\mathrm{i}}} \to 0$ and $|\fint_{\mathbb{P}_1(0)} \mathbf{d}_{\mathrm{i}} dxdt| \le M_0$, we have

$$\begin{aligned}
\left| \fint_{\mathbb{P}_{\frac{1}{2}}(0)} \mathbf{d}_{\mathrm{i}} dxdt \right| &\le \left| \fint_{\mathbb{P}_{\frac{1}{2}}(0)} \left(\mathbf{d}_{\mathrm{i}} - \fint_{\mathbb{P}_1(0)} \mathbf{d}_{\mathrm{i}} dxdt \right) dxdt \right| + \left| \fint_{\mathbb{P}_1(0)} \mathbf{d}_{\mathrm{i}} dxdt \right| \\
&\le C \left(\fint_{\mathbb{P}_1(0)} |\mathbf{d}_{\mathrm{i}} - \overline{\mathbf{d}}_{\mathrm{i}}|^6 dxdt \right)^{\frac{1}{6}} + M_0 \\
&\le C\varepsilon_{\mathrm{i}} + M_0 \le C.
\end{aligned}$$

Thus by (4.4.11) and the Sobolev-interpolation, we have

$$\|\mathbf{d}_{\mathrm{i}}\|_{L^{10}(\mathbb{P}_{\frac{1}{2}}(0))} \le C,$$

and there exists a constant $\overline{\mathbf{d}} \in \mathbb{R}^3$, with $|\overline{\mathbf{d}}| \le M_0$, such that, after passing to subsequences, $\overline{\mathbf{d}}_{\mathrm{i}} \to \overline{\mathbf{d}}$,

$$\mathbf{d}_{\mathrm{i}} \to \overline{\mathbf{d}} \text{ in } L^6(\mathbb{P}_{\frac{1}{2}}(0)). \tag{4.4.13}$$

We can deduce that $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}, \widehat{P})$ solves the linear system (in the distribution sense):

$$\begin{cases}
\partial_t \widehat{\mathbf{u}} + \nabla \widehat{P} - \frac{\mu_4}{2} \Delta \widehat{\mathbf{u}} = \nabla \cdot \widehat{\sigma}^L, \\
\nabla \cdot \widehat{\mathbf{u}} = 0, \\
\partial_t \widehat{\mathbf{d}} - \widehat{\Omega} \overline{\mathbf{d}} + \frac{\lambda_2}{\lambda_1} \widehat{A} \overline{\mathbf{d}} = \frac{1}{|\lambda_1|} \Delta \widehat{\mathbf{d}},
\end{cases} \tag{4.4.14}$$

where

$$\begin{aligned}
\widehat{A} &= \frac{1}{2}(\nabla \widehat{\mathbf{u}} + (\nabla \widehat{\mathbf{u}})^T), \\
\widehat{\Omega} &= \frac{1}{2}(\nabla \widehat{\mathbf{u}} - (\nabla \widehat{\mathbf{u}})^T), \\
\widehat{N} &= \partial_t \widehat{\mathbf{d}} - \widehat{\Omega} \overline{\mathbf{d}},
\end{aligned}$$

and

$$\widehat{\sigma}^L = \mu_1 (\overline{\mathbf{d}}^T \widehat{A} \overline{\mathbf{d}}) \overline{\mathbf{d}} \otimes \overline{\mathbf{d}} + \mu_2 \widehat{N} \otimes \overline{\mathbf{d}} + \mu_3 \overline{\mathbf{d}} \otimes \widehat{N} + \mu_5 \widehat{A} \overline{\mathbf{d}} \otimes \overline{\mathbf{d}} + \mu_6 \overline{\mathbf{d}} \otimes \widehat{A} \overline{\mathbf{d}}.$$

By (4.4.10) and Lemma , we have that $(\widehat{\mathbf{u}},\widehat{\mathbf{d}})\in C^\infty(\mathbb{P}_{\frac14}(0))$, $\widehat{P}\in L^\infty([-\frac{1}{16},0],C^\infty(B_{\frac14}(0)))$ satisfies

$$
\begin{aligned}
&\tau^{-2}\int_{\mathbb{P}_\tau(0)}\left(|\widehat{\mathbf{u}}|^3+|\nabla\widehat{\mathbf{d}}|^3\right)dxdt+\left(\tau^{-2}\int_{\mathbb{P}_\tau(0)}|\widehat{P}|^{\frac32}dxdt\right)^2\\
&\le C\tau^3\left[\int_{\mathbb{P}_{\frac12}(0)}\left(|\widehat{\mathbf{u}}|^3+|\nabla\widehat{\mathbf{d}}|^3\right)dxdt+\left(\int_{\mathbb{P}_{\frac12}(0)}|\widehat{P}|^{\frac32}\right)^2\right]\\
&\le C\tau^3,\quad \forall\tau\in\left(0,\frac18\right),
\end{aligned}\tag{4.4.15}
$$

and $\exists\alpha_0\in(0,1)$ such that

$$
\left(⨏_{\mathbb{P}_\tau(0)}|\widehat{\mathbf{d}}_{0,\tau}|^6dxdt\right)^{\frac12}\le C\left(⨏_{\mathbb{P}_{\frac12}(0)}|\widehat{\mathbf{d}}|^6dxdt\right)^{\frac12}\tau^{3\alpha_0}\le C\tau^{2\alpha_0},\quad\forall\tau\in\left(0,\frac18\right).\tag{4.4.16}
$$

We now claim that

$$
\begin{cases}
(\widehat{\mathbf{u}}_{\rm i},\nabla\widehat{\mathbf{d}}_{\rm i})\to(\widehat{\mathbf{u}},\nabla\widehat{\mathbf{d}}) & \text{in } L^3(\mathbb{P}_{\frac38}(0)),\\
\widehat{\mathbf{d}}_{\rm i}\to\widehat{\mathbf{d}} & \text{in } L^6(\mathbb{P}_{\frac38}(0)).
\end{cases}\tag{4.4.17}
$$

In fact, from $(4.4.9)_{1,3}$ and (4.4.10) we can conclude that

$$
\|\partial_t\widehat{\mathbf{u}}_{\rm i}\|_{L_t^2H_x^{-1}+L_t^{\frac65}L_x^{\frac65}+L_t^{\frac32}W_x^{-1,\frac32}(\mathbb{P}_{\frac38}(0))}\le C,\tag{4.4.18}
$$

and

$$
\left\|\partial_t\widehat{\mathbf{d}}_{\rm i}\right\|_{L^{\frac32}(\mathbb{P}_{\frac38}(0))}\le C.\tag{4.4.19}
$$

Whence (4.4.17) follows from Aubin–Lions' compactness Lemma. This implies that for any $\tau\in\left(0,\frac18\right)$,

$$
\begin{aligned}
\tau^{-2}\int_{\mathbb{P}_\tau(0)}\left(|\widehat{\mathbf{u}}_{\rm i}|^3+|\nabla\widehat{\mathbf{d}}_{\rm i}|^3\right)dxdt&=\tau^{-2}\int_{\mathbb{P}_\tau(0)}(|\widehat{\mathbf{u}}|^3+|\nabla\widehat{\mathbf{d}}|^3)dxdt+\tau^{-2}o(1)\\
&\le C\tau^3+\tau^{-2}o(1).
\end{aligned}\tag{4.4.20}
$$

$$
\left(⨏_{\mathbb{P}_\tau(0)}|\widehat{\mathbf{d}}_{\rm i}-(\widehat{\mathbf{d}}_{\rm i})_{0,\tau}|^6dxdt\right)^{\frac12}\le C\tau^{3\alpha_0}+o(1),\tag{4.4.21}
$$

where $\lim_{i\to\infty} o(1) = 0$.

Next we need to estimate the pressure $\widehat{P}_i$. By taking the divergence of (4.4.9)$_1$ we get that

$$-\Delta\widehat{P}_i = \varepsilon_i \operatorname{div}^2\left[\widehat{\mathbf{u}}_i\otimes\widehat{\mathbf{u}}_i + \nabla\widehat{\mathbf{d}}_i\odot\nabla\widehat{\mathbf{d}}_i - \left(\frac{1}{2}|\nabla\widehat{\mathbf{d}}_i|^2 + \frac{r_i^2}{\varepsilon_i^2}F(\mathbf{d}_i)\right)\mathrm{I}_3\right]$$
$$+ \operatorname{div}^2\left[\mu_2\widehat{N}_i\otimes\mathbf{d}_i + \mu_3\mathbf{d}_i\otimes\widehat{N}_i + \mu_5\widehat{A}_i\mathbf{d}_i\otimes\mathbf{d}_i + \mu_6\mathbf{d}_i\otimes\widehat{A}_i\mathbf{d}_i\right] \text{ in } B_1. \tag{4.4.22}$$

We claim that

$$\tau^{-2}\int_{\mathbb{P}_\tau(0)}|\widehat{P}_i|^{\frac{3}{2}}dxdt \le C\tau + C\tau^{-2}(\varepsilon_i + o(1)). \tag{4.4.23}$$

In order to achieve (4.4.23), we will show the following strong convergence in L^2:

$$\begin{cases}(\nabla\widehat{\mathbf{u}}_i, \Delta\widehat{\mathbf{d}}_i)\to(\nabla\widehat{\mathbf{u}}, \Delta\widehat{\mathbf{d}}) \text{ in } L^2(\mathbb{P}_{\frac{3}{8}}(0)),\\ \widehat{A}_i\mathbf{d}_i \to \widehat{A\mathbf{d}} \text{ in } L^2(\mathbb{P}_{\frac{3}{8}}(0)).\end{cases} \tag{4.4.24}$$

In order to prove (4.4.24), we subtract (4.4.14) from (4.4.9) we get that

$$(\widetilde{\mathbf{u}}_i, \widetilde{\mathbf{d}}_i, \widetilde{P}_i) := \left(\widehat{\mathbf{u}}_i - \widehat{\mathbf{u}}, \widehat{\mathbf{d}}_i - \widehat{\mathbf{d}}, \widehat{P}_i - \widehat{P}\right)$$

solves the following system of equations in $\mathbb{P}_{\frac{1}{2}}(0)$:

$$\begin{cases}\partial_t\widetilde{\mathbf{u}}_i + \nabla\widetilde{P}_i - \frac{\mu_4}{2}\Delta\widetilde{\mathbf{u}}_i = -\varepsilon_i\widehat{\mathbf{u}}_i\cdot\nabla\widehat{\mathbf{u}}_i - \varepsilon_i\nabla\widehat{\mathbf{d}}_i\cdot\Delta\widehat{\mathbf{d}}_i + \frac{r_i^2}{\varepsilon_i}\nabla\mathbf{d}_i\cdot\mathbf{f}(\mathbf{d}_i) + \nabla\cdot(\widehat{\sigma}_i^L - \widehat{\sigma}^L),\\ \operatorname{div}\widetilde{\mathbf{u}}_i = 0,\\ \partial_t\widetilde{\mathbf{d}}_i - \frac{1}{|\lambda_1|}\Delta\widetilde{\mathbf{d}}_i = -\varepsilon_i\widehat{\mathbf{u}}_i\cdot\nabla\widehat{\mathbf{d}}_i - \frac{1}{|\lambda_1|}\frac{r_i^2}{\varepsilon_i}\mathbf{f}(\mathbf{d}_i) + \left(\widehat{\Omega}_i\mathbf{d}_i - \frac{\lambda_2}{\lambda_1}\widehat{A}_i\mathbf{d}_i\right) - \left(\widehat{\Omega\mathbf{d}} - \frac{\lambda_2}{\lambda_1}\widehat{A\mathbf{d}}\right).\end{cases} \tag{4.4.25}$$

It turns out that (4.4.25) enjoys a local energy inequality which leads to (4.4.24). In fact, multiplying (4.4.25)$_1$ by $\widetilde{\mathbf{u}}_i\eta$, and the gradient of (4.4.25)$_3$ by $\nabla\widetilde{\mathbf{d}}_i\eta$, integrating the resulting equation over $\mathbb{R}^3\times[0,T]$, and by integration by parts, we obtain that

$$\int_{\mathbb{R}^3}|\widehat{\mathbf{u}}_i|^2\eta(x,t)dx + \mu_4\int_0^t\int_{\mathbb{R}^3}|\nabla\widetilde{\mathbf{u}}_i|^2\eta dxds$$

$$
\begin{aligned}
&\leq \int_0^t \int_{\mathbb{R}^3} |\widetilde{\mathbf{u}}_{\mathrm{i}}|^2 (\partial_t \eta + \frac{\mu_4}{2}\Delta\eta) dxds \\
&+ \int_0^t \int_{\mathbb{R}^3} \Big[\varepsilon_{\mathrm{i}} |\widehat{\mathbf{u}}_{\mathrm{i}}|^2 \widehat{\mathbf{u}}_{\mathrm{i}} \cdot \nabla\eta + 2\varepsilon_{\mathrm{i}} (\widehat{\mathbf{u}}_{\mathrm{i}} \cdot \nabla \widehat{\mathbf{u}}_{\mathrm{i}}) \cdot \widehat{\mathbf{u}}\eta + 2\widetilde{P}_{\mathrm{i}} \widetilde{\mathbf{u}}_{\mathrm{i}} \cdot \nabla\eta\Big] dxds \\
&+ 2\int_0^t \int_{\mathbb{R}^3} \left[-(\varepsilon_{\mathrm{i}} \nabla \widehat{\mathbf{d}}_{\mathrm{i}} \cdot \Delta \widehat{\mathbf{d}}_{\mathrm{i}}) \cdot (\widehat{\mathbf{u}}_{\mathrm{i}} - \widehat{\mathbf{u}})\eta + \frac{r_{\mathrm{i}}^2}{\varepsilon_{\mathrm{i}}} (\nabla \widehat{\mathbf{d}}_{\mathrm{i}} \cdot \mathbf{f}(\mathbf{d}_{\mathrm{i}})) \cdot \widetilde{\mathbf{u}}_{\mathrm{i}}\eta\right] dxds \\
&- 2\int_0^t \int_{\mathbb{R}^3} (\widehat{\sigma}_{\mathrm{i}}^L - \widehat{\sigma}^L) : (\nabla \widehat{\mathbf{u}}_{\mathrm{i}}\eta - \nabla\widehat{\mathbf{u}}\eta + \widetilde{\mathbf{u}}_{\mathrm{i}} \otimes \nabla\eta) dxds, \qquad (4.4.26)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla \widetilde{\mathbf{d}}_{\mathrm{i}}|^2 \eta(x,t) dx + 2\int_0^t \int_{\mathbb{R}^3} \frac{1}{|\lambda_1|} |\Delta \widetilde{\mathbf{d}}_{\mathrm{i}}|^2 \eta dxds \\
&\leq \int_0^t \int_{\mathbb{R}^3} \left[|\nabla \widetilde{\mathbf{d}}_{\mathrm{i}}|^2 (\partial_t \eta + \frac{1}{|\lambda_1|}\Delta\eta) + 2\varepsilon_{\mathrm{i}} (\widehat{\mathbf{u}}_{\mathrm{i}} \cdot \nabla \widehat{\mathbf{d}}_{\mathrm{i}}) \cdot (\Delta \widehat{\mathbf{d}}_{\mathrm{i}}\eta - \Delta\widehat{\mathbf{d}}\eta + \nabla\eta \cdot \nabla \widetilde{\mathbf{d}}_{\mathrm{i}})\right] dxds \\
&+ \frac{2r_{\mathrm{i}}^2}{\varepsilon_{\mathrm{i}}} \int_0^t \int_{\mathbb{R}^3} \frac{1}{|\lambda_1|} \mathbf{f}(\mathbf{d}_{\mathrm{i}}) \cdot (\Delta \widetilde{\mathbf{d}}_{\mathrm{i}}\eta + \nabla\eta \cdot \nabla \widetilde{\mathbf{d}}_{\mathrm{i}}) dxds \\
&- 2\int_0^t \int_{\mathbb{R}^3} [(\widehat{\Omega}_{\mathrm{i}} \mathbf{d}_{\mathrm{i}} - \frac{\lambda_2}{\lambda_1} \widehat{A}_{\mathrm{i}} \mathbf{d}_{\mathrm{i}}) - (\widehat{\Omega}\overline{\mathbf{d}} - \frac{\lambda_2}{\lambda_1} \widehat{A}\overline{\mathbf{d}})] \cdot (\Delta \widehat{\mathbf{d}}_{\mathrm{i}}\eta - \Delta\widehat{\mathbf{d}}\eta + \nabla\eta \cdot \nabla \widetilde{\mathbf{d}}_{\mathrm{i}}) dxds. \qquad (4.4.27)
\end{aligned}
$$

We have the following cancellation identities:

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} \widehat{\sigma}_{\mathrm{i}}^L : \nabla \widehat{\mathbf{u}}_{\mathrm{i}} \eta dxds + \int_0^t \int_{\mathbb{R}^3} (\widehat{\Omega}_{\mathrm{i}} \mathbf{d}_{\mathrm{i}} - \frac{\lambda_2}{\lambda_1} \widehat{A}_{\mathrm{i}} \mathbf{d}_{\mathrm{i}}) \cdot \Delta \widehat{\mathbf{d}}_{\mathrm{i}} \eta dxds \\
&= \int_0^t \int_{\mathbb{R}^3} \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}\right) |\widehat{A}_{\mathrm{i}} \mathbf{d}_{\mathrm{i}}|^2 \eta dxds + \int_0^t \int_{\mathbb{R}^3} \frac{r_{\mathrm{i}}^2}{\varepsilon_{\mathrm{i}}} \mathbf{f}(\mathbf{d}_{\mathrm{i}}) \cdot (\widehat{\Omega}_{\mathrm{i}} \mathbf{d}_{\mathrm{i}} - \frac{\lambda_2}{\lambda_1} \widehat{A}_{\mathrm{i}} \mathbf{d}_{\mathrm{i}}) \eta dxds, \\
&\int_0^t \int_{\mathbb{R}^3} \widehat{\sigma}^L : \nabla \widehat{\mathbf{u}} \eta dxds + \int_0^t \int_{\mathbb{R}^3} (\widehat{\Omega}\overline{\mathbf{d}} - \frac{\lambda_2}{\lambda_1} \widehat{A}\overline{\mathbf{d}}) \cdot \Delta \widehat{\mathbf{d}} \eta dxds \\
&= \int_0^t \int_{\mathbb{R}^3} \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}\right) |\widehat{A}\overline{\mathbf{d}}|^2 \eta dxds, \\
&\int_0^t \int_{\mathbb{R}^3} \widehat{\sigma}_{\mathrm{i}}^L : \nabla \widehat{\mathbf{u}} \eta dxds + \int_0^t \int_{\mathbb{R}^3} (\widehat{\Omega}\overline{\mathbf{d}} - \frac{\lambda_2}{\lambda_1} \widehat{A}\overline{\mathbf{d}}) \cdot \Delta \widehat{\mathbf{d}}_{\mathrm{i}} \eta dxds \\
&= \int_0^t \int_{\mathbb{R}^3} \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}\right) (\widehat{A}_{\mathrm{i}} \mathbf{d}_{\mathrm{i}}) \cdot (\widehat{A} \mathbf{d}_{\mathrm{i}}) \eta dxds - \int_0^t \int_{\mathbb{R}^3} [\widehat{\Omega}(\mathbf{d}_{\mathrm{i}} - \overline{\mathbf{d}}) - \frac{\lambda_2}{\lambda_1} \widehat{A}(\mathbf{d}_{\mathrm{i}} - \overline{\mathbf{d}})] \cdot \Delta \widehat{\mathbf{d}}_{\mathrm{i}} \eta dxds \\
&+ \int_0^t \int_{\mathbb{R}^3} \frac{r_{\mathrm{i}}^2}{\varepsilon_{\mathrm{i}}} \mathbf{f}(\mathbf{d}_{\mathrm{i}}) \cdot (\widehat{\Omega} \mathbf{d}_{\mathrm{i}} - \frac{\lambda_2}{\lambda_1} \widehat{A} \mathbf{d}_{\mathrm{i}}) \eta dxds, \\
&\int_0^t \int_{\mathbb{R}^3} \widehat{\sigma}^L : \nabla \widehat{\mathbf{u}}_{\mathrm{i}} \eta dxds + \int_0^t \int_{\mathbb{R}^3} (\widehat{\Omega}_{\mathrm{i}} \mathbf{d}_{\mathrm{i}} - \frac{\lambda_2}{\lambda_1} \widehat{A}_{\mathrm{i}} \mathbf{d}_{\mathrm{i}}) \cdot \Delta \widehat{\mathbf{d}} \eta dxds
\end{aligned}
$$

$$= \int_0^t \int_{\mathbb{R}^3} \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}\right)(\widehat{A}_{\mathrm{i}}\overline{\mathbf{d}}) \cdot (\widehat{A}\overline{\mathbf{d}})\eta dxds + \int_0^t \int_{\mathbb{R}^3} [\widehat{\Omega}_{\mathrm{i}}(\mathbf{d}_{\mathrm{i}} - \overline{\mathbf{d}}) - \frac{\lambda_2}{\lambda_1}\widehat{A}_{\mathrm{i}}(\mathbf{d}_{\mathrm{i}} - \overline{\mathbf{d}})] \cdot \Delta\widehat{\mathbf{d}}\eta dxds.$$

Now we add (4.4.26) and (4.4.27) together to get

$$\begin{aligned}
&\int_{\mathbb{R}^3} (|\tilde{\mathbf{u}}_{\mathrm{i}}|^2 + |\nabla\tilde{\mathbf{d}}_{\mathrm{i}}|^2)\eta(x,t)dx + \int_0^t \int_{\mathbb{R}^3} (\mu_4|\nabla\tilde{\mathbf{u}}_{\mathrm{i}}|^2 + \frac{2}{|\lambda_1|}|\Delta\tilde{\mathbf{d}}_{\mathrm{i}}|^2)\eta dxds \\
&+ 2\int_0^t \int_{\mathbb{R}^3} \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}\right)(|\widehat{A}_{\mathrm{i}}\mathbf{d}_{\mathrm{i}} - \widehat{A}\overline{\mathbf{d}}|^2)\eta dxds \\
&\leq \int_0^t \int_{\mathbb{R}^3} \left[|\tilde{\mathbf{u}}_{\mathrm{i}}|^2(\partial_t\eta + \frac{\mu_4}{2}\Delta\eta) + |\nabla\tilde{\mathbf{d}}_{\mathrm{i}}|^2(\partial_t\eta + \frac{1}{|\lambda_1|}\Delta\eta)\right] dxds \\
&+ \int_0^t \int_{\mathbb{R}^3} \left[\varepsilon_{\mathrm{i}}|\hat{\mathbf{u}}_{\mathrm{i}}|^2\hat{\mathbf{u}}_{\mathrm{i}} \cdot \nabla\eta + 2\varepsilon_{\mathrm{i}}(\hat{\mathbf{u}}_{\mathrm{i}} \cdot \nabla\hat{\mathbf{u}}_{\mathrm{i}}) \cdot \hat{\mathbf{u}}\eta + 2\widetilde{P}_{\mathrm{i}}\tilde{\mathbf{u}}_{\mathrm{i}} \cdot \nabla\eta\right] dxds \\
&+ 2\int_0^t \int_{\mathbb{R}^3} \left[\varepsilon_{\mathrm{i}}(\hat{\mathbf{u}} \cdot \nabla\widehat{\mathbf{d}}_{\mathrm{i}}) \cdot \Delta\widehat{\mathbf{d}}_{\mathrm{i}}\eta + \frac{r_{\mathrm{i}}^2}{\varepsilon_{\mathrm{i}}}(\nabla\widehat{\mathbf{d}}_{\mathrm{i}} \cdot \mathbf{f}(\mathbf{d}_{\mathrm{i}})) \cdot \tilde{\mathbf{u}}_{\mathrm{i}}\eta\right] dxds \\
&+ 2\int_0^t \int_{\mathbb{R}^3} \varepsilon_{\mathrm{i}}(\hat{\mathbf{u}}_{\mathrm{i}} \cdot \nabla\widehat{\mathbf{d}}_{\mathrm{i}}) \cdot (-\Delta\widehat{\mathbf{d}}\eta + \eta \cdot \nabla\tilde{\mathbf{d}}_{\mathrm{i}})dxds \\
&- 2\int_0^t \int_{\mathbb{R}^3} \frac{r_{\mathrm{i}}^2}{\varepsilon_{\mathrm{i}}}\mathbf{f}(\mathbf{d}_{\mathrm{i}}) \cdot (\widehat{\Omega}_{\mathrm{i}}\mathbf{d}_{\mathrm{i}} - \frac{\lambda_2}{\lambda_1}\widehat{A}_{\mathrm{i}}\mathbf{d}_{\mathrm{i}})\eta dxds \\
&+ 2\int_0^t \int_{\mathbb{R}^3} \frac{r_{\mathrm{i}}^2}{\varepsilon_{\mathrm{i}}}\frac{1}{|\lambda_1|}\mathbf{f}(\mathbf{d}_{\mathrm{i}}) \cdot (\Delta\tilde{\mathbf{d}}_{\mathrm{i}}\eta + \nabla\eta \cdot \nabla\tilde{\mathbf{d}}_{\mathrm{i}})dxds \\
&- 2\int_0^t \int_{\mathbb{R}^3} (\widehat{\sigma}_{\mathrm{i}}^L - \widehat{\sigma}^L) : \tilde{\mathbf{u}}_{\mathrm{i}} \otimes \nabla\eta dxds, \\
&- 2\int_0^t \int_{\mathbb{R}^3} [(\widehat{\Omega}_{\mathrm{i}}\mathbf{d}_{\mathrm{i}} - \frac{\lambda_2}{\lambda_1}\widehat{A}_{\mathrm{i}}\mathbf{d}_{\mathrm{i}}) - (\widehat{\Omega}\overline{\mathbf{d}} - \frac{\lambda_2}{\lambda_1}\widehat{A}\overline{\mathbf{d}})] \cdot (\nabla\eta \cdot \nabla\tilde{\mathbf{d}}_{\mathrm{i}})dxds \\
&+ 2\int_0^t \int_{\mathbb{R}^3} \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}\right)(\widehat{A}_{\mathrm{i}}\mathbf{d}_{\mathrm{i}}) \cdot (\widehat{A}(\mathbf{d}_{\mathrm{i}} - \overline{\mathbf{d}}))\eta dxds \\
&+ 2\int_0^t \int_{\mathbb{R}^3} \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}\right)(\widehat{A}_{\mathrm{i}}(\overline{\mathbf{d}} - \mathbf{d}_{\mathrm{i}})) \cdot (\widehat{A}\overline{\mathbf{d}})\eta dxds \\
&- 2\int_0^t \int_{\mathbb{R}^3} [\widehat{\Omega}(\mathbf{d}_{\mathrm{i}} - \overline{\mathbf{d}}) - \frac{\lambda_2}{\lambda_1}\widehat{A}(\mathbf{d}_{\mathrm{i}} - \overline{\mathbf{d}})] \cdot \Delta\widehat{\mathbf{d}}_{\mathrm{i}}\eta dxds \\
&+ 2\int_0^t \int_{\mathbb{R}^3} \frac{r_{\mathrm{i}}^2}{\varepsilon_{\mathrm{i}}}\mathbf{f}(\mathbf{d}_{\mathrm{i}}) \cdot (\widehat{\Omega}\mathbf{d}_{\mathrm{i}} - \frac{\lambda_2}{\lambda_1}\widehat{A}\mathbf{d}_{\mathrm{i}})\eta dxds \\
&+ 2\int_0^t \int_{\mathbb{R}^3} [\widehat{\Omega}_{\mathrm{i}}(\mathbf{d}_{\mathrm{i}} - \overline{\mathbf{d}}) - \frac{\lambda_2}{\lambda_1}\widehat{A}_{\mathrm{i}}(\mathbf{d}_{\mathrm{i}} - \overline{\mathbf{d}})] \cdot \Delta\widehat{\mathbf{d}}\eta dxds.
\end{aligned} \tag{4.4.28}$$

From (4.4.12), (4.4.13), (4.4.18), and (4.4.19) and the Aubin-Lions lemma, we can pass the limit in the R.H.S. of (4.4.28) to zero, this implies (4.4.24).

Let $\eta \in C_0^\infty(B_{\frac{3}{8}}(0))$ be such that $\eta \equiv 1$ in $B_{\frac{5}{16}}(0)$, $0 \le \eta \le 1$. For any $-(\dfrac{3}{8}) \le t \le 0$, define $\widehat{P}_{\rm i}^{(1)}(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}$ by

$$
\begin{aligned}
\widehat{P}_{\rm i}^{(1)}(x,t) := \int_{\mathbb{R}^3} \nabla_x^2 G(x-y)\eta\Big\{\varepsilon_{\rm i}\Big[\widehat{\mathbf{u}}_{\rm i} \otimes \widehat{\mathbf{u}}_{\rm i} + \nabla\widehat{\mathbf{d}}_{\rm i} \odot \nabla\widehat{\mathbf{d}}_{\rm i} - \Big(\frac{1}{2}|\nabla\widehat{\mathbf{d}}_{\rm i}|^2 + \frac{r_{\rm i}^2}{\varepsilon_{\rm i}^2}F(\mathbf{d}_{\rm i})\Big)\mathrm{I}_3\Big] \\
+ \Big[\mu_2\widehat{N}_{\rm i} \otimes \mathbf{d}_{\rm i} + \mu_3\mathbf{d}_{\rm i} \otimes \widehat{N}_{\rm i} + \mu_5\widehat{A}_{\rm i}\mathbf{d}_{\rm i} \otimes \mathbf{d}_{\rm i} + \mu_6\mathbf{d}_{\rm i} \otimes \widehat{A}_{\rm i}\mathbf{d}_{\rm i}\Big] \\
- \Big[\mu_2\widehat{N} \otimes \overline{\mathbf{d}} + \mu_3\overline{\mathbf{d}} \otimes \widehat{N} + \mu_5\widehat{A}\overline{\mathbf{d}} \otimes \overline{\mathbf{d}} + \mu_6\overline{\mathbf{d}} \otimes \widehat{A}\overline{\mathbf{d}}\Big]\Big\}(y,t)dy, \qquad (4.4.29)
\end{aligned}
$$

and $\widehat{P}_{\rm i}^{(2)}(\cdot, t) := (\widehat{P}_{\rm i} - \widehat{P}_{\rm i}^{(1)})(\cdot, t)$. Then

$$
-\Delta\widehat{P}_{\rm i}^{(2)} = \mathrm{div}^2\left[\mu_2\widehat{N} \otimes \overline{\mathbf{d}} + \mu_3\overline{\mathbf{d}} \otimes \widehat{N} + \mu_5\widehat{A}\overline{\mathbf{d}} \otimes \overline{\mathbf{d}} + \mu_6\overline{\mathbf{d}} \otimes \widehat{A}\overline{\mathbf{d}}\right] \text{ in } B_{\frac{5}{16}}(0).
$$

For $\widehat{P}_{\rm i}^{(1)}$, by the Calderon–Zygmund theory and (4.4.24), we can show

$$
\begin{aligned}
\|\widehat{P}_{\rm i}^{(1)}\|_{L^{\frac{3}{2}}(\mathbb{P}_{\frac{1}{3}}(0))} \le C\Big[\varepsilon_{\rm i}\Big(\|\widehat{\mathbf{u}}_{\rm i}\|^2_{L^3(B_{\frac{3}{8}}(0))} + \|\nabla\widehat{\mathbf{d}}_{\rm i}\|^2_{L^3(B_{\frac{3}{8}}(0))} + \frac{r_{\rm i}^2}{\varepsilon_{\rm i}^2}\|F(\mathbf{d}_{\rm i})\|_{L^{\frac{3}{2}}(B_{\frac{3}{2}}(0))}\Big) \\
+ \|\mathbf{d}_{\rm i}\|_{L^6(B_{\frac{3}{8}}(0))}\|\widehat{N}_{\rm i} - \widehat{N}\|_{L^2(B_{\frac{3}{8}}(0))} + \|\mathbf{d}_{\rm i} - \overline{\mathbf{d}}\|_{L^6(B_{\frac{3}{8}}(0))}\|\widehat{N}\|_{L^2(B_{\frac{3}{8}}(0))} \\
+ \|\mathbf{d}_{\rm i}\|_{L^6(B_{\frac{3}{8}}(0))}\|\widehat{A}_{\rm i}\mathbf{d}_{\rm i} - \widehat{A}\overline{\mathbf{d}}\|_{L^2(B_{\frac{3}{8}}(0))} + \|\mathbf{d}_{\rm i} - \overline{\mathbf{d}}\|_{L^6(B_{\frac{3}{8}}(0))}\|\widehat{A}\overline{\mathbf{d}}\|_{L^2(B_{\frac{3}{8}}(0))}\Big] \\
\le C(\varepsilon + o(1)). \qquad (4.4.30)
\end{aligned}
$$

Form the standard theory on linear elliptic equations, $\widehat{P}_{\rm i}^{(2)} \in C^\infty(B_{\frac{5}{16}}(0))$ satisfies that for any $0 < \tau < \dfrac{9}{32}$,

$$
\begin{aligned}
\tau^{-2}\int_{\mathbb{P}_\tau(0)} |\widehat{P}_{\rm i}^{(2)}|^{\frac{3}{2}}dxdt &\le C\tau\left[\int_{\mathbb{P}_{\frac{9}{32}}(0)} |\widehat{P}_{\rm i}^{(2)}|^{\frac{3}{2}}dxdt + \|(\nabla^2\widehat{\mathbf{u}}, \nabla^3\widehat{\mathbf{d}})\|^{\frac{3}{2}}_{L^\infty(\mathbb{P}_{\frac{9}{32}}(0))}\right] \\
&\le C\tau\left[\int_{\mathbb{P}_{\frac{9}{32}}(0)} \Big(|\widehat{P}_{\rm i}|^{\frac{3}{2}} + |\widehat{P}_{\rm i}^{(1)}|^{\frac{3}{2}}\Big)dxdt + \|(\nabla^2\widehat{\mathbf{u}}, \nabla^3\widehat{\mathbf{d}})\|^{\frac{3}{2}}_{L^\infty(\mathbb{P}_{\frac{9}{32}}(0))}\right] \\
&\le C\tau(1 + \varepsilon_{\rm i} + o(1)). \qquad (4.4.31)
\end{aligned}
$$

Putting (4.4.30) and (4.4.31) together we get (4.4.23). It follows from (4.4.20), (4.4.21) and (4.4.23) that there exist sufficiently small $\tau_0 \in (0, \frac{1}{4})$ and sufficiently large i_0 depending on τ_0, such that for any $\mathrm{i} \geq \mathrm{i}_0$, it holds that

$$\tau_0^{-2} \int_{\mathbb{P}_{\tau_0}(0)} \left(|\widehat{\mathbf{u}}_{\mathrm{i}}|^3 + |\nabla \widehat{\mathbf{d}}_{\mathrm{i}}|^3\right) dxdt + \left(\tau_0^{-2} \int_{\mathbb{P}_{\tau_0}(0)} |\widehat{P}_{\mathrm{i}}|^{\frac{3}{2}} dxdt\right)^{\frac{2}{3}}$$
$$+ \left(⨍_{\mathbb{P}_{\tau_0}(0)} |\widehat{\mathbf{d}}_{\mathrm{i}} - (\widehat{\mathbf{d}}_{\mathrm{i}})_{0,\tau_0}|^6 dxdt\right)^{\frac{1}{2}} \leq \frac{1}{4}.$$

This contradicts $(4.4.8)_3$. Hence the proof of Lemma 4.4.1.

□

In the following lemma we will establish the smoothness of the limit equation (4.4.14).

Lemma 4.4.2. *Assume that* $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}, \widehat{P}) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)(\mathbb{P}_{\frac{1}{2}}(0)) \times (L_t^2 H_x^1 \cap L_t^2 H_x^2)(\mathbb{P}_{\frac{1}{2}}(0)) \times L^{\frac{3}{2}}(\mathbb{P}_{\frac{1}{2}}(0))$ *is a weak solution of the linear system* (4.4.14)*, then* $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}) \in C^\infty(\mathbb{P}_{\frac{1}{4}}(0))$*, and the following estimate*

$$\tau^{-2} \int_{\mathbb{P}_\tau(0)} \left(|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{\mathbf{d}}|^3 + |\widehat{P}|^{\frac{3}{2}}\right) dxdt \leq C\tau^3 \int_{\mathbb{P}_{\frac{1}{2}}(0)} \left(|\widehat{\mathbf{u}}|^3 + |\nabla \widehat{\mathbf{d}}|^3 + |\widehat{P}|^{\frac{3}{2}}\right) dxdt \tag{4.4.32}$$

holds for any $\tau \in (0, \frac{1}{8})$.

Proof. The smoothness of the limit equation (4.4.14) doesn't follow from the standard theory of linear equations, since the source term of $\widehat{\mathbf{u}}$ equations involve terms depedning on the third order derivatives of $\widehat{\mathbf{d}}$. It is based on higher order energy methods, for which the cancellation property, as in the derivation of local energy inequality for the suitable weak solution to (4.1.1), plays a critical role. This strategy has been adapted by Huang–Lin–Wang [38, Lemma 3.2] for the full Ericksen–Leslie in 2D. However, it is more delicate here due to the low temporal integrability of pressure. To address this issue, we split the pressure into two parts $\widehat{P}^{(1)}$ and $\widehat{P}^{(2)}$, where $\widehat{P}^{(1)}$ solves the Poisson equation involving $(\nabla \widehat{\mathbf{u}}, \Delta \widehat{\mathbf{d}})$ which belongs to L^2, and $\widehat{P}^{(2)}$, while is only $L^{\frac{3}{2}}$ in time, is harmonic space. In fact, by taking the divergence of $(4.4.14)_1$, we have $\widehat{P}$ satisfies the following Poisson equation:

$$-\Delta \widehat{P} = -\operatorname{div}^2 \widehat{\sigma}^L \text{ in } \mathbb{P}_{\frac{1}{2}}(0). \tag{4.4.33}$$

Let $\zeta \in C_0^\infty(B_{\frac{1}{2}}(0))$ be a cut-off function of $B_{\frac{3}{8}}(0)$, i.e., $\zeta \equiv 1$ on $B_{\frac{3}{8}}(0)$, $0 \leq \zeta \leq 1$. Define $\widehat{P}^{(1)}(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}$,

$$\widehat{P}^{(1)}(x,t) := -\int_{\mathbb{R}^3} \zeta(y) \nabla_x^2 G(x-y) : \widehat{\sigma}^L(y,t) dy,$$

and $\widehat{P}^{(2)}(\cdot, t) := (\widehat{P} - \widehat{P}^{(1)})(\cdot, t)$. For $\widehat{P}^{(1)}$, by Calderon–Zygmund singular integral estimate we have

$$\left\| \widehat{P}^{(1)}(\cdot, t) \right\|_{L^2(\mathbb{R}^3)} \leq C \left\| (\nabla \widehat{\mathbf{u}}, \Delta \widehat{\mathbf{d}})(\cdot, t) \right\|_{L^2(B_{\frac{1}{2}})}, \qquad -\frac{1}{4} \leq t \leq 0.$$

By integrating the inequality above in time we get

$$\int_{\mathbb{P}_{\frac{1}{2}}(0)} |\widehat{P}^{(1)}|^2 dxdt \leq C \int_{\mathbb{P}_{\frac{1}{2}}(0)} (|\nabla \widehat{\mathbf{u}}|^2 + |\Delta \widehat{\mathbf{d}}|^2) dxdt. \tag{4.4.34}$$

For $\widehat{P}^{(2)}$, it is easy to see that

$$-\Delta \widehat{P}^{(2)} = 0 \text{ in } B_{\frac{3}{8}}.$$

By the interior estimate of harmonic function we have

$$\begin{aligned}
\int_{\mathbb{P}_{\frac{5}{16}}(0)} |\nabla^l \widehat{P}^{(2)}|^{\frac{3}{2}} dxdt &\leq C \int_{\mathbb{P}_{\frac{3}{8}}(0)} |\widehat{P}^{(2)}|^{\frac{3}{2}} dxdt \\
&\leq C \int_{\mathbb{P}_{\frac{3}{8}}(0)} (|\widehat{P}|^{\frac{3}{2}} + |\widehat{P}^{(1)}|^{\frac{3}{2}}) dxdt \\
&\leq C \int_{\mathbb{P}_{\frac{1}{2}}(0)} |\widehat{P}|^{\frac{3}{2}} dxdt + C \int_{\mathbb{P}_{\frac{1}{2}}(0)} |\widehat{P}^{(1)}|^2 dxdt + C \\
&\leq C \int_{\mathbb{P}_{\frac{1}{2}}(0)} |\widehat{P}|^{\frac{3}{2}} dxdt + C \int_{\mathbb{P}_{\frac{1}{2}}(0)} (|\nabla \widehat{\mathbf{u}}|^2 + |\Delta \widehat{\mathbf{d}}|^2) dxdt + C, \quad l = 1, 2.
\end{aligned} \tag{4.4.35}$$

Taking $\partial_{x_{\rm i}}$ of the linear equation (4.4.14) yields

$$\begin{cases} \partial_t \hat{\mathbf{u}}_{x_{\rm i}} + \nabla \widehat{P}_{x_{\rm i}} - \dfrac{\mu_4}{2}\Delta \hat{\mathbf{u}}_{x_{\rm i}} = \nabla \cdot \widehat{\sigma}_{x_{\rm i}}, \\ \operatorname{div} \hat{\mathbf{u}}_{x_{\rm i}} = 0, \\ \partial_t \widehat{\mathbf{d}}_{x_{\rm i}} - \dfrac{1}{|\lambda_1|}\Delta \widehat{\mathbf{d}}_{x_{\rm i}} = (\widehat{\Omega}\overline{\mathbf{d}} - \dfrac{\lambda_2}{\lambda_1}\widehat{A}\overline{\mathbf{d}})_{x_{\rm i}}. \end{cases} \tag{4.4.36}$$

For any $\eta \in C_0^\infty(B_{\frac{5}{16}}(0))$, multiplying the equation $(4.4.36)_1$ by $\hat{\mathbf{u}}_{x_{\rm i}}\eta^2$, the $\nabla\widehat{\mathbf{d}}_{x_{\rm i}}$ equation from $(4.4.36)_3$ by $\nabla\widehat{\mathbf{d}}_{x_{\rm i}}\eta^2$, integrating the resulting equations over $B_{\frac{5}{16}}$, we obtain[1]

$$\begin{aligned} &\frac{d}{dt}\int_{B_{\frac{5}{16}}(0)} |\nabla \hat{\mathbf{u}}|^2\eta^2 dxdt + \mu_4 \int_{B_{\frac{5}{16}}(0)} |\nabla^2 \hat{\mathbf{u}}|^2 \eta^2 dx \\ &= 2\int_{B_{\frac{5}{16}}(0)} [\widehat{P}_{x_{\rm i}}\hat{\mathbf{u}}_{x_{\rm i}} \cdot \nabla(\eta^2) - \frac{\mu_4}{2}\nabla \hat{\mathbf{u}}_{x_{\rm i}} : \hat{\mathbf{u}}_{x_{\rm i}} \otimes \nabla(\eta^2)]dx \\ &- 2\int_{B_{\frac{5}{16}}(0)} [\widehat{\sigma}_{x_{\rm i}} : \hat{\mathbf{u}}_{x_{\rm i}} \otimes \nabla(\eta^2) + \widehat{\sigma}_{x_{\rm i}} : \nabla \hat{\mathbf{u}}_{x_{\rm i}}\eta^2]dx, \end{aligned} \tag{4.4.37}$$

$$\begin{aligned} &\frac{d}{dt}\int_{B_{\frac{5}{16}}(0)} |\nabla^2 \widehat{\mathbf{d}}|^2\eta^2 dx + \frac{2}{|\lambda_1|}\int_{B_{\frac{5}{16}}(0)} |\Delta\nabla \widehat{\mathbf{d}}|^2\eta^2 dx \\ &= -2\int_{B_{\frac{5}{16}}(0)} \frac{1}{|\lambda_1|}\nabla_{\rm j}\nabla\widehat{\mathbf{d}}_{x_{\rm i}} : \nabla\widehat{\mathbf{d}}_{x_{\rm i}} \otimes \nabla_{\rm j}(\eta^2)dx \\ &- 2\int_{B_{\frac{5}{16}}(0)} [(\widehat{\Omega}\overline{\mathbf{d}} - \frac{\lambda_2}{\lambda_1}\widehat{A}\overline{\mathbf{d}})_{x_{\rm i}} \cdot \nabla_{\rm j}\widehat{\mathbf{d}}_{x_{\rm i}}\nabla_{\rm j}(\eta^2) + (\widehat{\Omega}\overline{\mathbf{d}} - \frac{\lambda_2}{\lambda_1}\widehat{A}\overline{\mathbf{d}}) \cdot \Delta\widehat{\mathbf{d}}_{x_{\rm i}}\eta^2]dx. \end{aligned} \tag{4.4.38}$$

Once again, we have the following identity

$$\begin{aligned} &\int_{B_{\frac{5}{16}}(0)} \widehat{\sigma}_{x_{\rm i}} : \nabla\hat{\mathbf{u}}_{x_{\rm i}}\eta^2 dx + \int_{B_{\frac{5}{16}}(0)} (\widehat{\Omega}\overline{\mathbf{d}} - \frac{\lambda_2}{\lambda_1}\widehat{A}\overline{\mathbf{d}}) \cdot \Delta\widehat{\mathbf{d}}_{x_{\rm i}}\eta^2 dx \\ &= \int_{B_{\frac{5}{16}}(0)} \left(\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1}\right)|\widehat{A}_{x_{\rm i}}\overline{\mathbf{d}}|^2\eta^2 dx. \end{aligned}$$

[1]↑Strickly speaking, we take the finite quotient $D_h^{\rm j}$ of (4.4.36) (j=1, 2,3) (see Evans) and then send $h \to 0$.

Now we add (4.4.37) and (4.4.38) together to get

$$
\begin{aligned}
&\frac{d}{dt}\int_{B_{\frac{5}{16}}(0)}(|\nabla\widehat{\mathbf{u}}|^2+|\nabla^2\widehat{\mathbf{d}}|^2)\eta^2dx+\int_{B_{\frac{5}{16}}(0)}(\mu_4|\nabla^2\widehat{\mathbf{u}}|^2+\frac{2}{|\lambda_1|}|\Delta\nabla\widehat{\mathbf{d}}|^2)\eta^2dx\\
&+2\int_{B_{\frac{5}{16}}(0)}\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)|\widehat{A}_{x_{\mathrm{i}}}\overline{\mathbf{d}}|^2\eta^2dx\\
&=2\int_{B_{\frac{5}{16}}(0)}\widehat{P}_{x_{\mathrm{i}}}\widehat{\mathbf{u}}_{x_{\mathrm{i}}}\cdot\nabla(\eta^2)dx\\
&-2\int_{B_{\frac{5}{16}}(0)}[\frac{\mu_4}{2}\nabla\widehat{\mathbf{u}}_{x_{\mathrm{i}}}:\widehat{\mathbf{u}}_{x_{\mathrm{i}}}\otimes\nabla(\eta^2)+\frac{1}{|\lambda_1|}\nabla_{\mathrm{j}}\nabla\widehat{\mathbf{d}}_{x_{\mathrm{i}}}:\nabla\widehat{\mathbf{d}}_{x_{\mathrm{i}}}\otimes\nabla_{\mathrm{j}}(\eta^2)]dx\\
&-2\int_{B_{\frac{5}{16}}(0)}[\widehat{\sigma}_{x_{\mathrm{i}}}:\widehat{\mathbf{u}}_{x_{\mathrm{i}}}\otimes\nabla(\eta^2)+(\widehat{\Omega}\overline{\mathbf{d}}-\frac{\lambda_2}{\lambda_1}\widehat{A}\overline{\mathbf{d}})_{x_{\mathrm{i}}}\cdot\nabla_{\mathrm{j}}\widehat{\mathbf{d}}_{x_{\mathrm{i}}}\nabla_{\mathrm{j}}(\eta^2)]dx\\
&:=I_1+I_2+I_3.
\end{aligned}
\tag{4.4.39}
$$

We have the following estimates:

$$
\begin{aligned}
|I_1|&\le 2\left|\int_{B_{\frac{5}{16}}(0)}(\widehat{P}^{(1)}\widehat{\mathbf{u}}_{x_{\mathrm{i}}x_{\mathrm{i}}}\cdot\nabla(\eta^2)+\widehat{P}^{(1)}\widehat{\mathbf{u}}_{x_{\mathrm{i}}}\cdot\nabla(\eta^2)_{x_{\mathrm{i}}})dx\right|+2\left|\int_{B_{\frac{5}{16}}(0)}\widehat{\mathbf{u}}\cdot(\widehat{P}^{(2)}_{x_{\mathrm{i}}}\nabla(\eta^2))_{x_{\mathrm{i}}}dx\right|\\
&\le\frac{1}{32}\int_{B_{\frac{5}{16}}(0)}|\nabla^2\widehat{\mathbf{u}}|^2\eta^2dx+C\int_{B_{\frac{1}{2}}(0)}(|\nabla\widehat{\mathbf{u}}|^2\eta^2+|\nabla\widehat{\mathbf{u}}|^2|\nabla\eta|^2)dx+C\int_{\mathrm{spt}\,\eta}|\widehat{P}^{(1)}|^2dx\\
&\quad+C\int_{\mathrm{spt}\,\eta}(|\widehat{\mathbf{u}}|^3+|\nabla\widehat{P}^{(2)}|^{\frac{3}{2}}+|\nabla^2\widehat{P}^{(2)}|^{\frac{3}{2}})dx,\\
|I_2|&\le\frac{1}{16}\int_{B_{\frac{5}{16}}(0)}(|\nabla^2\widehat{\mathbf{u}}|^2+|\Delta\nabla\widehat{\mathbf{d}}|^2)\eta^2dx+C\int_{B_{\frac{5}{16}}(0)}(|\nabla\widehat{\mathbf{u}}|^2+|\nabla^2\widehat{\mathbf{d}}|^2)|\nabla\eta|^2dx,\\
|I_3|&\le\frac{1}{16}\int_{B_{\frac{5}{16}}(0)}(|\nabla^2\widehat{\mathbf{u}}|^2+|\Delta\nabla\widehat{\mathbf{d}}|^2)\eta^2dx+C\int_{B_{\frac{5}{16}}(0)}(|\nabla\widehat{\mathbf{u}}|^2+|\nabla^2\widehat{\mathbf{d}}|^2)|\nabla\eta|^2dx.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\frac{d}{dt}\int_{B_{\frac{5}{16}}}(|\nabla\widehat{\mathbf{u}}|^2+|\nabla^2\widehat{\mathbf{d}}|^2)\eta^2dx+\int_{B_{\frac{5}{16}}(0)}(|\nabla^2\widehat{\mathbf{u}}|^2+|\nabla^3\widehat{\mathbf{d}}|^2)\eta^2dx\\
&\le C\int_{\mathrm{spt}}(|\nabla\widehat{\mathbf{u}}|^2+|\nabla^2\widehat{\mathbf{d}}|^2+|\widehat{P}^{(1)}|^2+|\widehat{\mathbf{u}}|^3+|\nabla\widehat{P}^{(2)}|^{\frac{3}{2}}+|\nabla^2\widehat{P}^{(2)}|^{\frac{3}{2}})dx.
\end{aligned}
\tag{4.4.40}
$$

By Gronwall's inequality, we can show

$$\sup_{-(\frac{9}{32})^2\le t\le 0}\int_{B_{\frac{5}{16}}(0)\times\{t\}}(|\nabla\widehat{\mathbf{u}}|^2+|\nabla^2\widehat{\mathbf{d}}|^2)\eta^2dx+\int_{-(\frac{9}{32})^2\le t\le 0}\int_{B_{\frac{5}{16}}(0)}(|\nabla^2\widehat{\mathbf{u}}|^2+|\nabla^3\widehat{\mathbf{d}}|^2)\eta^2dxdt$$
$$\le C\int_{\mathbb{P}_{\frac{1}{2}}(0)}(|\nabla\widehat{\mathbf{u}}|^2+|\nabla^2\widehat{\mathbf{d}}|^2+|\widehat{\mathbf{u}}|^3+|\widehat{P}|^{\frac{3}{2}})dxdt+C. \tag{4.4.41}$$

For the pressure $\widehat{P}$, taking the divergence of the equation (4.4.36)$_1$ yields that for any $-\frac{1}{4}\le t\le 0$,

$$-\Delta\widehat{P}_{x_i}=-\operatorname{div}^2\widehat{\sigma}_{x_i}\text{ in }B_{\frac{5}{16}}. \tag{4.4.42}$$

We have

$$\int_{\mathbb{P}_{\frac{1}{4}}(0)}|\nabla\widehat{P}|^{\frac{3}{2}}dxdt\le C\int_{\mathbb{P}_{\frac{9}{32}}(0)}(|\widehat{\sigma}_{x_i}|^{\frac{3}{2}}+|\widehat{P}|^{\frac{3}{2}})dxdt$$
$$\le C\int_{\mathbb{P}_{\frac{9}{32}}(0)}(|\nabla^2\widehat{\mathbf{u}}|^{\frac{3}{2}}+|\nabla^3\widehat{\mathbf{d}}|^{\frac{3}{2}}+|\widehat{P}|^{\frac{3}{2}})dxdt\le C. \tag{4.4.43}$$

Now let η be a cut-off function of $B_{\frac{9}{32}}$, i.e., $\eta\equiv 1$ in $B_{\frac{3}{8}}$. Then, by combining (4.4.41) and (4.4.43), we obtain

$$\begin{aligned}&\sup_{-(\frac{1}{4})^2\le t\le 0}\int_{B_{\frac{1}{4}}}(|\nabla\widehat{\mathbf{u}}|^2+|\nabla\widehat{\mathbf{d}}|^2)dx+\int_{\mathbb{P}_{\frac{1}{4}}}(|\nabla^2\widehat{\mathbf{u}}|^2+|\nabla^3\widehat{\mathbf{d}}|^2+|\nabla\widehat{P}|^{\frac{3}{2}})dxdt\\&\le C\int_{\mathbb{P}_{\frac{1}{2}}}(|\widehat{\mathbf{u}}|^3+|\nabla\widehat{\mathbf{u}}|^2+|\nabla^2\widehat{\mathbf{d}}|^2+|\widehat{P}|^{\frac{3}{2}})dxdt+C.\end{aligned} \tag{4.4.44}$$

It turns out that we can extend the energy method above to arbitary order. Here we sketch the proof. For nonnegative multiple indices β,γ and δ such that $\gamma=\beta+\delta$ and δ is of order 1, $|\beta|=k$, then $(\nabla^\beta\widehat{\mathbf{u}},\nabla^\gamma\widehat{\mathbf{d}},\nabla^\beta\widehat{P})$ satisfies

$$\begin{cases}\partial_t(\nabla^\beta\widehat{\mathbf{u}})+\nabla(\nabla^\beta\widehat{P})-\frac{\mu_4}{2}\Delta(\nabla^\beta\widehat{\mathbf{u}})=-\nabla\cdot(\nabla^\beta\widehat{\sigma}^L),\\ \operatorname{div}(\nabla^\beta\widehat{\mathbf{u}})=0,\\ \partial_t(\nabla^\gamma\widehat{\mathbf{d}})-\frac{1}{|\lambda_1|}\Delta(\nabla^\gamma\widehat{\mathbf{d}})=\nabla^\gamma(\widehat{\Omega}\overline{\mathbf{d}}-\frac{\lambda_2}{\lambda_1}\widehat{A}\overline{\mathbf{d}}).\end{cases} \tag{4.4.45}$$

By differentiating $(\widehat{P}^{(1)}, \widehat{P}^{(2)})$ $(k-1)$ times we can estimate

$$\int_{\mathbb{P}_{\frac{1}{2}}} |\nabla^{k-1}\widehat{P}^{(1)}|^2 dxdt \le C\int_{\mathbb{P}_{\frac{1}{2}}} |(\nabla^k \hat{\mathbf{u}}, \nabla^{k+1}\widehat{\mathbf{d}})|^2 dxdt, \tag{4.4.46}$$

and

$$\int_{\mathbb{P}_{\frac{5}{16}}} |\nabla^l \widehat{P}^{(2)}|^{\frac{3}{2}} dxdt \le C\int_{\mathbb{P}_{\frac{1}{2}}} |\nabla^{k-1}\widehat{P}|^{\frac{3}{2}} dxdt + C\int_{\mathbb{P}_{\frac{1}{2}}} |(\nabla^k \hat{\mathbf{u}}, \nabla^{k+1}\widehat{\mathbf{d}})|^2 dxdt + C, \quad l = k, k+1. \tag{4.4.47}$$

Multiplying $(4.4.45)_1$ by $(\nabla^\beta \hat{\mathbf{u}})\eta^2$ and $(4.4.45)_3$ by $(\nabla^\gamma \widehat{\mathbf{d}})\eta^2$ and integrating the resulting equations over $B_{\frac{1}{2}}$, and by the same calculation and cancellation, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{B_{\frac{5}{16}}} (|\nabla^k \hat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2)\eta^2 dx + \int_{B_{\frac{5}{16}}} (|\nabla^{k+1}\hat{\mathbf{u}}|^2 + |\nabla^{k+2}\widehat{\mathbf{d}}|^2)\eta^2 dx \\
&\le C\int_{B_{\frac{5}{16}}} (|\nabla^k \hat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2 + |\nabla^{k-1}\widehat{P}^{(1)}|^2 + |\nabla^{k-1}\hat{\mathbf{u}}|^3 + |\nabla^k \widehat{P}^{(2)}|^{\frac{3}{2}} + |\nabla^{k+1}\widehat{P}^{(2)}|^{\frac{3}{2}}) dx \\
&\le C\int_{\mathbb{P}_{\frac{1}{2}}} (|\nabla^k \hat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2 + |\nabla^{k-1}\hat{\mathbf{u}}|^3 + |\nabla^{k-1}\widehat{P}|^{\frac{3}{2}}) dxdt + C.
\end{aligned} \tag{4.4.48}$$

For P, since

$$-\Delta(\nabla^\beta \widehat{P}) = \operatorname{div}^2(\nabla^\beta \widehat{\sigma}^L) \text{ in } B_{\frac{5}{16}}, \tag{4.4.49}$$

we have

$$\begin{aligned}
\int_{\mathbb{P}_{\frac{1}{4}}} |\nabla^k \widehat{P}|^{\frac{3}{2}} dxdt &\le C\int_{\mathbb{P}_{\frac{9}{32}}} |(\nabla^{k+1}\hat{\mathbf{u}}, \nabla^{k+2}\widehat{\mathbf{d}})|^{\frac{3}{2}} dxdt + C\int_{\mathbb{P}_{\frac{9}{32}}} |\nabla^{k-1}\widehat{P}|^{\frac{3}{2}} dxdt \\
&\le C\int_{\mathbb{P}_{\frac{9}{32}}} |(\nabla^{k+1}\hat{\mathbf{u}}, \nabla^{k+2}\widehat{\mathbf{d}})|^2 dxdt + C\int_{\mathbb{P}_{\frac{9}{32}}} |\nabla^{k-1}\widehat{P}|^{\frac{3}{2}} dxdt + C.
\end{aligned} \tag{4.4.50}$$

By choosing suitable t_* as above, we can integrate (4.4.48) in t to get

$$\begin{aligned}
&\sup_{-(\frac{9}{32})^2 \le t \le 0} \int_{B_{\frac{9}{32}}} (|\nabla^k \hat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2) dx + \int_{\mathbb{P}_{\frac{9}{32}}} (|\nabla^{k+1}\hat{\mathbf{u}}|^2 + |\nabla^{k+2}\widehat{\mathbf{d}}|^2) dxdt \\
&\le C\int_{\mathbb{P}_{\frac{1}{2}}} (|\nabla^k \hat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2 + |\nabla^{k-1}\hat{\mathbf{u}}|^3 + |\nabla^{k-1}\widehat{P}|^{\frac{3}{2}}) dxdt + C.
\end{aligned} \tag{4.4.51}$$

Thus, we get

$$\sup_{-(\frac{1}{4})^2 \le t \le 0} \int_{B_{\frac{1}{4}}} (|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2) dx + \int_{\mathbb{P}_{\frac{1}{4}}} (|\nabla^{k+1}\widehat{\mathbf{u}}|^2 + |\nabla^{k+2}\widehat{\mathbf{d}}|^2 + |\nabla^k \widehat{P}|^{\frac{3}{2}}) dxdt$$
$$\le C \int_{\mathbb{P}_{\frac{1}{2}}} (|\nabla^{k-1}\widehat{\mathbf{u}}|^3 + |\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2 + |\nabla^{k-1}\widehat{P}|^{\frac{3}{2}}) dxdt + C. \tag{4.4.52}$$

From Sobolev's interpolation inequality, we have

$$\int_{\mathbb{P}_{\frac{1}{2}}} |\nabla^{k-1}\widehat{\mathbf{u}}|^3 dxdt \le C \left\|\nabla^{k-1}\widehat{\mathbf{u}}\right\|^6_{L^\infty_t L^2_x(\mathbb{P}_{\frac{1}{2}})} + C \int_{\mathbb{P}_{\frac{1}{2}}} (|\nabla^{k-1}\widehat{\mathbf{u}}|^2 + |\nabla^k \widehat{\mathbf{u}}|^2) dxdt.$$

Substituting this inequality in (4.4.52) and by suitable adjusting of the radius, we can show that

$$\sup_{-(\frac{1}{4})^2 \le t \le 0} \int_{B_{\frac{1}{4}} \times \{t\}} (|\nabla^k \widehat{\mathbf{u}}|^2 + |\nabla^{k+1}\widehat{\mathbf{d}}|^2) dx + \int_{\mathbb{P}_{\frac{1}{4}}} (|\nabla^{k+1}\widehat{\mathbf{u}}|^2 + |\nabla^{k+2}\widehat{\mathbf{d}}|^2 + |\nabla^k \widehat{P}|^{\frac{3}{2}}) dxdt$$
$$\le C \left(\left\|(\widehat{\mathbf{u}}, \nabla \widehat{\mathbf{d}})\right\|_{L^\infty_t L^2_x \cap L^2_t H^2_x(\mathbb{P}_{\frac{1}{2}})}, \left\|\widehat{P}\right\|_{L^{\frac{3}{2}}(\mathbb{P}_{\frac{1}{2}})} \right). \tag{4.4.53}$$

With (4.4.53), we can apply the regularity for both the linear Stokes equations and the linear heat equation (c.f. [32], [53]) to conclude that $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}) \in C^\infty(\mathbb{P}_{\frac{1}{4}})$. Furthermore, applying the elliptic estimate for the pressure equation (4.4.33), we see that $\widehat{P} \in C^\infty(\mathbb{P}_{\frac{1}{4}})$. Therefore $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}}, \widehat{P}) \in C^\infty(\mathbb{P}_{\frac{1}{4}})$ and the estimate (4.4.32) holds. The proof is completed. □

The oscillation Lemma admits the following iterations.

Lemma 4.4.3. *Let* $(\mathbf{u}, \mathbf{d}, P), M, \varepsilon_0(M), \tau_0(M), C_0(M), z_0$ *be as in Lemma 4.4.2. Then there exist* $r_0 = r_0(M), \varepsilon_1 = \varepsilon_1(M) > 0$ *such that for* $0 < r \le r_0$*, if*

$$|\mathbf{d}_{z_0,r}| \le \frac{M}{2}, \qquad \Phi(z_0, r) \le \varepsilon_1^3,$$

then for any $k = 1, 2, \dots,$ we have

$$\begin{aligned} &|\mathbf{d}_{z_0, \tau_0^{k-1} r}| \le M, \\ &\Phi(z_0, \tau_0^{k-1} r) \le \varepsilon_1^3, \\ &\Phi(z_0, \tau_0^k r) \le \frac{1}{2} \max \left\{ \Phi(z_0, \tau_0^{k-1} r), C_0 (\tau_0^{k-1} r)^3 \right\}. \end{aligned} \tag{4.4.54}$$

Proof. We prove it by an induction on k. By translational invariance we may assume that $z_0 = 0$, and we abbreviate $\mathbf{d}_{0,r}$ to $\mathbf{d}_r$ for simplicity.

For $k = 1$, the conclusion follows from Lemma 4.4.2, if we choose ε_1 such that $\varepsilon_1 < \varepsilon_0$. Suppose the conclusion is true for all $k \le k_0, k_0 \ge 1$, we show it remains true for $k = k_0 + 1$. By the inductive hypothesis

$$\begin{aligned} &|\mathbf{d}_{\tau_0^{k-1} r}| \le M, \\ &\Phi(0, \tau_0^{k-1} r) \le \varepsilon_1^3, \\ &\Phi(0, \tau_0^k r) \le \frac{1}{2} \max \left\{ \Phi(0, \tau_0^{k-1} r), C_0 (\tau_0^{k-1} r)^3 \right\} \le \frac{1}{2} \max \left\{ \varepsilon_1^3, C_0 (\tau_0^{k-1} r)^3 \right\} \end{aligned}$$

for all $k \le k_0$. Thus,

$$\begin{aligned} \Phi(0, \tau_0^k r) &\le \frac{1}{2} \max \left\{ \Phi(0, \tau_0^{k-1} r), C_0 (\tau_0^{k-1} r)^3 \right\} \\ &\le \frac{1}{2} \max \left\{ \frac{1}{2} \max \left\{ \Phi(0, \tau_0^{k-2} r), C_0 (\tau_0^{k-2} r)^3 \right\}, C_0 (\tau_0^{k-1} r)^3 \right\} \\ &\le \cdots \le 2^{-k} \max \left\{ \Phi(0, r), \frac{C_0 r^3}{1 - 2\tau_0^3} \right\} \\ &\le 2^{-k} \max \left\{ \varepsilon_1^3, \frac{C_0 r_0^3}{1 - 2\tau_0^3} \right\}, \quad \forall k \le k_0. \end{aligned}$$

Then

$$\begin{aligned} |\mathbf{d}_{\tau_0^{k_0} r}| &\le |\mathbf{d}_r| + \sum_{k=1}^{k_0} \left| \mathbf{d}_{\tau_0^k} - \mathbf{d}_{\tau_0^{k-1} r} \right| \\ &\le \frac{M}{2} + \sum_{k=1}^{k_0} \left(\fint_{\mathbb{P}_{\tau_0^k r}(0)} |\mathbf{d} - \mathbf{d}_{\tau_0^{k-1} r}|^6 \right)^{\frac{1}{6}} \end{aligned}$$

$$
\begin{aligned}
&\le \frac{M}{2} + \sum_{k=1}^{k_0} \Phi(0, \tau_0^{k-1} r)^{\frac{1}{3}} \\
&\le \frac{M}{2} + \sum_{k=1}^{k_0} 2^{-\frac{1}{3}(k-1)} \max\Big\{\varepsilon_1, \Big(\frac{C_0 r_0^3}{1-2\tau_0^3}\Big)^{\frac{1}{3}}\Big\} \\
&\le \frac{M}{2} + \frac{1}{1-2^{-\frac{1}{3}}} \max\Big\{\varepsilon_1, \Big(\frac{C_0 r_0^3}{1-2\tau_0^3}\Big)^{\frac{1}{3}}\Big\}.
\end{aligned}
$$

If we choose sufficiently small $r_0 = r_0(M)$, $\varepsilon_1 = \varepsilon_1(M)$, we see

$$
\begin{aligned}
&|\mathbf{d}_{\tau_0^{k_0} r}| \le M, \\
&\Phi(0, \tau_0^{k_0} r) \le \varepsilon_1^3 \le \varepsilon_0^3.
\end{aligned}
$$

It follows directly from Lemma 4.4.2 with r replaced by $\tau_0^k r$ that

$$
\Phi(0, \tau_0^{k+1} r) \le \frac{1}{2} \max\Big\{\Phi(0, \tau_0^k r), C_0(\tau_0^k r)^3\Big\}.
$$

This completes the proof. $\square$

The local boundedness of the solutions can be obtained by utilizing the Riesz potential estimates between Morrey spaces as in the following lemma.

Lemma 4.4.4. *For any $M > 0$, there exists $\varepsilon_2 > 0$, depending on M, such that if $(\mathbf{u}, \mathbf{d}, P)$ is a suitable weak solution of (4.1.6) in $\mathbb{R}^3 \times (0, \infty)$, which satisfies, for $z_0 = (x_0, t_0) \in \mathbb{R}^3 \times (r_0^2, \infty)$*

$$
|\mathbf{d}_{z_0, r_0}| \le \frac{M}{4}, \text{ and } \Phi(z_0, r_0) \le \varepsilon_2^3, \tag{4.4.55}
$$

then for any $1 < p < \infty$, $(\mathbf{u}, \nabla \mathbf{d}) \in L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))$, $\mathbf{d} \in C^\theta(\mathbb{P}_{\frac{r_0}{2}}(z_0))$ and

$$
|\mathbf{d}| \le M \text{ in } \mathbb{P}_{\frac{r_0}{2}}(z_0), \quad [\mathbf{d}]_{C^\theta(\mathbb{P}_{\frac{r_0}{2}}(z_0))} \le C(\theta, M)(\varepsilon_1 + r_0). \tag{4.4.56}
$$

$$
\|(\mathbf{u}, \nabla \mathbf{d})\|_{L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))} \le C(p, M)(\varepsilon_1 + r_0), \tag{4.4.57}
$$

where ε_1 is the constant in Lemma 4.4.3.

Proof. Let $\varepsilon_2 = \min\Big\{\Big(\dfrac{M}{4}\Big), 2^{-\frac{11}{6}}\varepsilon_1(M)\Big\}$. For any $z \in \mathbb{P}_{\frac{r_0}{2}}(z_0)$,

$$\begin{aligned}|\mathbf{d}_{z,\frac{r_0}{2}}| &\leq \Big|\mathbf{d}_{z,\frac{r_0}{2}} - \mathbf{d}_{z_0,r_0}\Big| + |\mathbf{d}_{z_0,r_0}| \\ &\leq \fint_{\mathbb{P}_{\frac{r_0}{2}}(z)} |\mathbf{d} - \mathbf{d}_{z_0,r_0}| + \frac{M}{4} \leq \varepsilon_2 + \frac{M}{4} \leq \frac{M}{2}.\end{aligned}$$

Meanwhile,

$$\begin{aligned}&\Big(\fint_{\mathbb{P}_{\frac{r_0}{2}}(z)} |\mathbf{d} - \mathbf{d}_{z,\frac{r_0}{2}}|^6 dxdt\Big)^{\frac{1}{2}} \\ &\leq \Big(2^5 \fint_{\mathbb{P}_{\frac{r_0}{2}}(z)} |\mathbf{d} - \mathbf{d}_{z_0,r_0}|^6 dxdt + 2^5 |\mathbf{d}_{z_0,r_0} - \mathbf{d}_{z,\frac{r_0}{2}}|^6\Big)^{\frac{1}{2}} \\ &\leq \Big(2^{10} \fint_{\mathbb{P}_{r_0}(z_0)} |\mathbf{d} - \mathbf{d}_{z_0,r_0}|^6 dxdt + 2^5 \fint_{\mathbb{P}_{\frac{r_0}{2}}(z)} |\mathbf{d} - \mathbf{d}_{z_0,r_0}|^6 dxdt\Big)^{\frac{1}{2}} \\ &\leq 2^{\frac{11}{2}} \Big(\fint_{\mathbb{P}_{r_0}(z_0)} |\mathbf{d} - \mathbf{d}_{z_0,r_0}|^6 dxdt\Big)^{\frac{1}{2}},\end{aligned}$$

Hence we get that

$$\Phi(z, \frac{r_0}{2}) \leq 2^{\frac{11}{2}} \Phi(z_0, r_0) \leq 2^{\frac{11}{2}} \varepsilon_2^3 \leq \varepsilon_1^3.$$

Then we deduce from Lemma 4.4.3 that for any $k = 1, 2, \ldots,$

$$\begin{aligned}&|\mathbf{d}_{z,\tau_0^{k-1}\frac{r_0}{2}}| \leq M, \\ &\Phi(z, \tau_0^k \frac{r_0}{2}) \leq \frac{1}{2} \max\Big\{\Phi(z, \tau_0^{k-1}\frac{r_0}{2}), C_0(\tau_0^{k-1} r)^3\Big\}.\end{aligned} \tag{4.4.58}$$

By Lebesgue's differentiation theorem, we have $|\mathbf{d}| \leq M$ a.e. in $\mathbb{P}_{\frac{r_0}{2}}(z_0)$. Furthermore, we have

$$\Phi(z, \frac{\tau_0^k r_0}{2}) \leq 2^{-k} \max\Big\{\Phi(z, \frac{r_0}{2}), \frac{C_0 r_0^3}{1 - 2\tau_0^3}\Big\}.$$

Therefore for $\theta_0 = \dfrac{\ln 2}{3|\ln \tau_0|} \in (0, \dfrac{1}{3})$, it holds for any $0 < s < \dfrac{r_0}{2}$ and $z \in \mathbb{P}_{\frac{r_0}{2}}(z_0)$,

$$\Phi(z, s) \leq C(r_0^3 + \varepsilon_1^3)\Big(\frac{s}{r_0}\Big)^{3\theta_0}. \tag{4.4.59}$$

By the Campanato theory, $\mathbf{d} \in C^{\theta}(\mathbb{P}_{\frac{r_0}{4}}(z_0))$ and (4.4.57) holds. Now for $\phi \in C_0^{\infty}(\mathbb{P}_{\frac{r_0}{2}})(z_0)$, from (4.2.2), (4.2.4) we can derive the following local energy inequality:

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{T}^3}(|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)\phi(x,t)dx+\int_0^t\int_{\mathbb{T}^3}(\frac{\mu_4}{2}|\nabla\mathbf{u}|^2+\frac{1}{|\lambda_1|}|\Delta\mathbf{d}|^2)\phi(x,s)dxds\\
&+\int_0^t\int_{\mathbb{T}^3}\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)|A\mathbf{d}|^2\phi(x,s)dxds \qquad (4.4.60)\\
&\leq \int_0^t\int_{\mathbb{T}^3}(\frac{1}{2}|\mathbf{u}|^2+P)\mathbf{u}\cdot\nabla\phi(x,s)dxds+\int_0^t\frac{1}{2}(|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)\partial_t\phi(x,s)dxds\\
&+\int_0^t\int_{\mathbb{T}^3}[(\mathbf{u}\cdot\nabla\mathbf{d})\cdot\mathbf{f}(\mathbf{d})-\frac{1}{-\lambda_1}\nabla(\mathbf{f}(\mathbf{d})):\nabla\mathbf{d}]\phi(x,s)dxds\\
&+\int_0^t\int_{\mathbb{T}^3}[-\sigma^L(\mathbf{u},\mathbf{d}):\mathbf{u}\otimes\nabla\phi+(\mathbf{u}\cdot\nabla\mathbf{d})\cdot(\nabla\phi\cdot\nabla\mathbf{d})](x,s)dxds\\
&-\int_0^t\int_{\mathbb{T}^3}(\Omega\mathbf{d}-\frac{\lambda_2}{\lambda_1}A\mathbf{d})\cdot(\nabla\phi\cdot\nabla\mathbf{d})(x,s)dxds+\frac{\mu_4}{2}\int_0^t\int_{\mathbb{T}^3}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla\phi(x,s)dxds\\
&+\int_0^t\int_{\mathbb{T}^3}\frac{1}{-2\lambda_1}|\nabla\mathbf{d}|^2\Delta\phi(x,s)dxds+\int_0^t\int_{\mathbb{T}^3}\frac{1}{-\lambda_1}(\nabla\mathbf{d}\odot\nabla\mathbf{d}-|\nabla\mathbf{d}|^2\mathrm{I}_3):\nabla^2\phi(x,s)dxds.
\end{aligned}
$$

Let $\phi \in C_0^{\infty}(\mathbb{P}_{2s}(z))$ be a cut-off function of $\mathbb{P}_s(z)$. Replacing ϕ by ϕ^2 in (4.4.60), we can show that for $0 < s < \dfrac{r_0}{2}$,

$$
\begin{aligned}
&s^{-1}\int_{\mathbb{P}_s(z)}(|\nabla\mathbf{u}|^2+|\Delta\mathbf{d}|^2)dxdt\\
&\leq C[(2s)^{-3}\int_{\mathbb{P}_{2s}(z)}(|\mathbf{u}|^2+|\nabla\mathbf{d}|^2)dxdt+(2s)^{-2}\int_{\mathbb{P}_{2s}(z)}(|\mathbf{u}|^3+|\nabla\mathbf{d}|^3+|P|^{\frac{3}{2}})dxdt] \qquad (4.4.61)\\
&\leq C(r_0^3+\varepsilon_1^3)\Big(\frac{s}{r_0}\Big)^{2\theta_0}.
\end{aligned}
$$

Now we are ready to perform the Riesz potential estimate. For any open set $U \subset \mathbb{T}^3\times\mathbb{R}, 1 \leq p < \infty$, define the Morrey space $M^{p,\lambda}(U)$ by

$$
M^{p,\lambda}(U) := \left\{ f \in L^p_{\mathrm{loc}}(U) : \|f\|^p_{M^{p,\lambda}(U)} = \sup_{z\in U, r>0} r^{\lambda-5}\int_{\mathbb{P}_r(z)}|f|^p dxdt < \infty \right\}.
$$

It follows from (4.4.59) and (4.4.61) that there exists $\alpha \in (0,1)$ such that

$$
(\mathbf{u},\nabla\mathbf{d}) \in M^{3,3(1-\alpha)}\Big(\mathbb{P}_{\frac{r_0}{2}}(z_0)\Big), P \in M^{\frac{3}{2},3(1-\alpha)}\Big(\mathbb{P}_{\frac{r_0}{2}}(z_0)\Big), (\nabla\mathbf{u},\nabla^2\widehat{\mathbf{d}}) \in M^{2,4-2\alpha}\Big(\mathbb{P}_{\frac{r_0}{2}}(z_0)\Big).
$$

Write $\mathbf{d}$ equation in (4.1.6) as

$$\partial_t \mathbf{d} - \frac{1}{|\lambda_1|}\Delta \mathbf{d} = -\mathbf{u}\cdot\nabla\mathbf{d} + \Omega\mathbf{d} - \frac{\lambda_2}{\lambda_1}A\mathbf{d} + \frac{1}{|\lambda_1|}\mathbf{f}(\mathbf{d}) \in M^{\frac{3}{2},3(1-\alpha)}(\mathbb{P}_{\frac{r_0}{2}}(z_0)). \tag{4.4.62}$$

Let $\eta \in C_0^\infty(\mathbb{T}^3\times\mathbb{R})$ be such that $0\le\eta\le 1$, $\eta = 1$ in $\mathbb{P}_{\frac{r_0}{2}}(z_0)$, $|\partial_t\eta| + |\nabla^2\eta| \le Cr_0^2$. Set $w = \eta^2(\mathbf{d} - \mathbf{d}_{z_0,\frac{r_0}{2}})$. Then

$$\partial_t w - \Delta w = F, \quad F := \eta^2(\partial_t\mathbf{d} - \Delta\mathbf{d}) + (\partial_t\eta^2 - \Delta\eta^2)(\mathbf{d} - \mathbf{d}_{z_0,\frac{r_0}{2}}) - 2\nabla\eta^2\cdot\nabla\mathbf{d}. \tag{4.4.63}$$

We can check that $F \in M^{\frac{3}{2},3(1-\alpha)}(\mathbb{T}^3\times\mathbb{R})$ and satisfies

$$\|F\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{R}^4)} \le C(r_0 + \varepsilon_1). \tag{4.4.64}$$

Let Γ denote the heat kernel in $\mathbb{R}^3$. Then

$$|\nabla\Gamma(x,t)| \le C\delta^{-4}((x,t),(0,0)), \forall (x,t)\neq(0,0),$$

where $\delta(\cdot,\cdot)$ denotes the parabolic distance on $\mathbb{R}^4$. By the Duhamel formula, we have that

$$|w(x,t)| \le \int_0^t\int_{\mathbb{R}^3} |\nabla\Gamma(x-y,t-s)||F(y,s)|dyds \le C\mathcal{I}_1(|F|)(x,t), \tag{4.4.65}$$

where $\mathcal{I}_\beta$ is the parabolic Riesz potential of order β on $\mathbb{R}^4$, $0\le\beta\le 5$, defined by

$$\mathcal{I}_\beta(g)(x,t) = \int_{\mathbb{R}^4} \frac{|g(y,s)|}{\delta^{5-\beta}((x,t),(y,s))}dyds, \forall g \in L^2(\mathbb{R}^4).$$

Applying the Riesz potential estimates, we conclude that $\nabla w \in M^{\frac{3(1-\alpha)}{1-2\alpha},3(1-\alpha)}(\mathbb{T}^3\times\mathbb{R})$ and

$$\|\nabla w\|_{M^{\frac{3(1-\alpha)}{1-2\alpha},3(1-\alpha)}(\mathbb{T}^3\times\mathbb{R})} \le C\|F\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{T}^3\times\mathbb{R})} \le C(r_0+\varepsilon_1). \tag{4.4.66}$$

Since $\lim_{\alpha\uparrow\frac{1}{2}} \frac{3(1-\alpha)}{1-2\alpha} = \infty$, we conclude that for any $1 < p < \infty$, $\nabla w \in L^p(\mathbb{P}_{r_0}(z_0))$ and

$$\|\nabla w\|_{L^p(\mathbb{P}_{\frac{r_0}{2}}(z_0))} \leq C(p)(r_0 + \varepsilon_1).$$

Since $\mathbf{d} - w$ solves

$$\partial_t(\mathbf{d} - w) - \Delta(\mathbf{d} - w) = 0 \text{ in } \mathbb{P}_{\frac{r_0}{4}(z_0)},$$

it follows from the theory of heat equations that $\nabla(\mathbf{d} - w) \in L^\infty(\mathbb{P}_{\frac{r_0}{4}}(z_0))$. Therefore for any $1 < p < \infty$, $\mathbf{d} \in L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0)$, and

$$\|\nabla \mathbf{d}\|_{L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))} \leq C(p)(r_0 + \varepsilon_1).$$

We now proceed with the estimation of $\mathbf{u}$. Let $\mathbf{v} : \mathbb{T}^3 \times (0, \infty) \mapsto \mathbb{R}^3$ solve the Stokes equation:

$$\begin{cases} \partial_t \mathbf{v} - \frac{\mu_4}{2}\Delta \mathbf{v} + \nabla P = -\operatorname{div}[\eta^2(\mathbf{u}\otimes\mathbf{u} + \nabla\mathbf{d}\odot\nabla\mathbf{d} - \frac{1}{2}|\nabla\mathbf{d}|^2\mathrm{I}_3)] \\ \qquad\qquad\qquad\qquad + \operatorname{div}\{\eta^2(F(\mathbf{d}) - F(\mathbf{d})_{z_0,\frac{r_0}{2}})\mathrm{I}_3\} \\ \qquad\qquad\qquad\qquad + \operatorname{div}\{\eta^2(\sigma^L - (\frac{\mu_2}{\lambda_1}\mathbf{f}(\mathbf{d})\otimes\mathbf{d} + \frac{\mu_3}{\lambda_1}\mathbf{d}\otimes\mathbf{f}(\mathbf{d}))_{z_0,\frac{r_0}{2}})\}, \\ \nabla\cdot\mathbf{v} = 0, \\ \mathbf{v}(\cdot, 0) = 0. \end{cases} \tag{4.4.67}$$

By using the Oseen kernel, an estimate of $\mathbf{v}$ can be given by

$$|\mathbf{v}(x,t)| \leq C\mathcal{I}_1(|X|)(x,t), \forall (x,t) \in \mathbb{T}^3 \times (0,\infty), \tag{4.4.68}$$

where

$$\begin{aligned} X = \eta^2\Big[\mathbf{u}\otimes\mathbf{u} + \Big(\nabla\mathbf{d}\odot\nabla\mathbf{d} - \frac{1}{2}|\nabla\mathbf{d}|^2\mathrm{I}_3\Big) - (F(\mathbf{d}) - F(\mathbf{d})_{z_0,\frac{r_0}{2}})\mathrm{I}_3 \\ -(\sigma^L - (\frac{\mu_2}{\lambda_1}\mathbf{f}(\mathbf{d})\otimes\mathbf{d} + \frac{\mu_3}{\lambda_1}\mathbf{d}\otimes\mathbf{f}(\mathbf{d}))_{z_0,\frac{r_0}{2}})\Big]. \end{aligned}$$

As above, we can check that $X \in M^{\frac{3}{2},3(1-\alpha)}(\mathbb{T}^3 \times \mathbb{R})$ and

$$\|X\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{T}^3\times\mathbb{R})} \leq C(r_0 + \varepsilon_1).$$

Hence we conclude that $\mathbf{v} \in M^{\frac{3(1-\alpha)}{1-2\alpha},3(1-\alpha)}(\mathbb{T}^3 \times \mathbb{R})$, and

$$\|\mathbf{v}\|_{M^{\frac{3(1-\alpha)}{1-2\alpha},3(1-\alpha)}(\mathbb{T}^3\times\mathbb{R})} \leq C\,\|X\|_{M^{\frac{3}{2},3(1-\alpha)}(\mathbb{T}^3\times\mathbb{R})} \leq C(r_0 + \varepsilon_1). \tag{4.4.69}$$

As $\alpha \uparrow \frac{1}{2}$, $\frac{3(1-\alpha)}{1-2\alpha} \to \infty$, we conclude that for any $1 < p < \infty$, $\mathbf{v} \in L^p(\mathbb{P}_{\frac{r_0}{2}}(z_0))$. Since

$$\partial_t(\mathbf{u}-\mathbf{v}) - \frac{\mu_4}{2}\Delta(\mathbf{u}-\mathbf{v}) + \nabla P = 0, \nabla \cdot (\mathbf{u}-\mathbf{v}) = 0 \text{ in } \mathbb{P}_{\frac{r_0}{2}(z_0)},$$

we have that $\mathbf{u}-\mathbf{v} \in L^\infty(\mathbb{P}_{\frac{r_0}{4}}(z_0))$. Therefore for any $1 < p < \infty$, $\mathbf{u} \in L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))$ and

$$\|\mathbf{u}\|_{L^p(\mathbb{P}_{\frac{r_0}{4}}(z_0))} \leq C(p)(r_0 + \varepsilon_1).$$

$\square$

For the rest of this section, we will establish the higher order regularity of (4.1.6). Again we prove it via a high order energy method which has been employed by Huang–Lin–Wang [38] for general Ericksen–Leslie systems in dimension two, and Du–Hu–Wang [70] for co-rotational Beris–Edwards model in dimension three.

Lemma 4.4.5. *Under the same assumption as Lemma 4.4.4, we have that for any $k \geq 0$, $(\nabla^k\mathbf{u}, \nabla^{k+1}\mathbf{d}) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1)\Big(\mathbb{P}_{\frac{1+2^{-(k+1)}}{2}r_0}(z_0)\Big)$ and the following estimates hold*

$$\begin{aligned}
&\sup_{t_0-\left(\frac{1+2^{-(k+1)}}{2}r_0\right)^2 \leq t \leq t_0} \int_{B_{\frac{1+2^{-(k+1)}}{2}r_0}(x_0)} (|\nabla^k\mathbf{u}|^2 + |\nabla^{k+1}\mathbf{d}|^2)dx \\
&\quad + \int_{\mathbb{P}_{\frac{1+2^{-(k+1)}}{2}r_0}(z_0)} \left(|\nabla^{k+1}\mathbf{u}|^2 + |\nabla^{k+2}\mathbf{d}|^2 + |\nabla^k P|^{\frac{3}{2}}\right) dxdt \\
&\leq C(k, r_0)\varepsilon_1.
\end{aligned} \tag{4.4.70}$$

In particular, $(\widehat{\mathbf{u}}, \widehat{\mathbf{d}})$ *is smooth in* $\mathbb{P}_{\frac{r_0}{4}}(z_0)$.

Proof. For simplicity, assume $z_0 = (0,0)$ and $r_0 = 2$. (4.4.70) can be proved by an induction on k. It is clear that when $k = 0$, (4.4.70) follows directly from the local energy inequality (4.4.60). Here we indicate to how to proof (4.4.70) for $k \geq 1$. Suppose that (4.4.70) holds for $k \leq l-1$, we want to show that (4.4.70) also holds for $k = l$. From the induction hypothesis, we have that for $0 \leq k \leq l-1$,

$$\begin{aligned}
&\sup_{-\left(1+2^{-(k+1)}\right)^2 \leq t \leq 0} \int_{B_{1+2^{-(k+1)}}} (|\nabla^k \mathbf{u}|^2 + |\nabla^{k+1}\mathbf{d}|^2) dx \\
&+ \int_{\mathbb{P}_{1+2^{-(k+1)}}} \left(|\nabla^{k+1}\mathbf{u}| + |\nabla^{k+2}\mathbf{d}|^2 + |\nabla^k P|^{\frac{3}{2}}\right) dxdt \leq C(l)\varepsilon_1.
\end{aligned} \tag{4.4.71}$$

Hence by the Sobolev embedding we have

$$\int_{\mathbb{P}_{1+2^{-l}}} (|\nabla^{l-1}\mathbf{u}|^{\frac{10}{3}} + |\nabla^l \mathbf{d}|^{\frac{10}{3}}) dxdt \leq C(l)\varepsilon_1, \tag{4.4.72}$$

and for $0 \leq k \leq l-2$, by the Sobolev-interpolation inequality as in (4.3.9) we have

$$\int_{\mathbb{P}_{1+2^{-(k+1)}}} (|\nabla^k \mathbf{u}|^{10} + |\nabla^{k+1}\mathbf{d}|^{10}) dxdt \leq C(l)\varepsilon_1. \tag{4.4.73}$$

Also, for $1 \leq \mathrm{j} \leq l-1$, we have

$$\begin{aligned}
&\int_{-(1+2^{-\mathrm{j}})^2}^0 \left\|(\nabla^{\mathrm{j}}\mathbf{u}, \nabla^{\mathrm{j}+1}\mathbf{d})\right\|_{L^3(B_{1+2^{-\mathrm{j}}})}^4 dt \\
&\leq \int_{-(1+2^{-\mathrm{j}})^2}^0 \left\|(\nabla^{\mathrm{j}}\mathbf{u}, \nabla^{\mathrm{j}+1}\mathbf{d})\right\|_{L^2(B_{1+2^{-\mathrm{j}}})}^2 \left\|(\nabla^{\mathrm{j}}\mathbf{u}, \nabla^{\mathrm{j}+1}\mathbf{d})\right\|_{L^6(B_{1+2^{-\mathrm{j}}})}^2 dt \\
&\leq \left\|(\nabla^{\mathrm{j}}\mathbf{u}, \nabla^{\mathrm{j}+1}\mathbf{d})\right\|_{L_t^\infty L_x^2(\mathbb{P}_{1+2^{-\mathrm{j}}})}^2 \left\|(\nabla^{\mathrm{j}}\mathbf{u}, \nabla^{\mathrm{j}+1}\mathbf{d})\right\|_{L_t^2 H_x^1(\mathbb{P}_{1+2^{-\mathrm{j}}})}^2 \leq C(l)\varepsilon_1
\end{aligned} \tag{4.4.74}$$

By Lemma 4.4.4 we also have that any $\mathrm{i} \in \mathbb{N}^+$ and $1 < p < \infty$,

$$\begin{aligned}
&\|\mathbf{d}\|_{L^\infty(\mathbb{P}_2)} \leq M, \quad \left[\mathbf{d}\right]_{C^\theta(\mathbb{P}_2)} + [D^{\mathrm{i}}_{\mathbf{d}}\mathbf{f}(\mathbf{d})]_{C^\theta(\mathbb{P}_2)} \leq C(\mathrm{i}, M)\varepsilon_0, \\
&\left\|(\mathbf{u}, \nabla\mathbf{d})\right\|_{L^p(\mathbb{P}_2)} \leq C(p)\varepsilon_1.
\end{aligned} \tag{4.4.75}$$

Notice that $\nabla^{l-1}P$ satisfies

$$\begin{aligned}
-\Delta\nabla^{l-1}P = \operatorname{div}^2\Big[\nabla^{l-1}\Big(&\mathbf{u}\otimes\mathbf{u}+\nabla\mathbf{d}\odot\nabla\mathbf{d}-\frac{1}{2}|\nabla\mathbf{d}|^2\mathrm{I}_3 \\
&-(F(\mathbf{d})\mathrm{I}_3-\fint_{\mathbb{P}_2}F(\mathbf{d})\mathrm{I}_3)-\sigma^L+\fint_{\mathbb{P}_2}(\frac{\mu_2}{\lambda_1}\mathbf{f}(\mathbf{d})\otimes\mathbf{d}+\frac{\mu_3}{\lambda_1}\mathbf{d}\otimes\mathbf{f}(\mathbf{d}))\Big)\Big],
\end{aligned} \tag{4.4.76}$$

Now let $\zeta\in C_0^\infty(B_{1+2^{-l}})$ be a cut-off function of $B_{1+2^{-(l+1)}+3^{-(l+1)}}$, and $P^{(1)}(\cdot,t):\mathbb{R}^3\to\mathbb{R}$, $-(1+2^{-1})^2\le t\le 0$,

$$\begin{aligned}
P^{(1)}(x,t):=\int_{\mathbb{R}^3}\nabla_x^2G(x-y)\zeta(y)\Big[&\mathbf{u}\otimes\mathbf{u}+\nabla\mathbf{d}\odot\nabla\mathbf{d}-\frac{1}{2}|\nabla\mathbf{d}|^2\mathrm{I}_3 \\
&-(F(\mathbf{d})\mathrm{I}_3-\fint_{\mathbb{P}_2}F(\mathbf{d})\mathrm{I}_3)-\sigma^L+\fint_{\mathbb{P}_2}(\frac{\mu_2}{\lambda_1}\mathbf{f}(\mathbf{d})\otimes\mathbf{d}+\frac{\mu_3}{\lambda_1}\mathbf{d}\otimes\mathbf{f}(\mathbf{d}))\Big](y)dy,
\end{aligned} \tag{4.4.77}$$

and $P^{(2)}(\cdot,t):=(P-P^{(1)})(\cdot,t)$. For $P^{(1)}$, we have that

$$\begin{aligned}
\nabla^{l-1}P^{(1)}(x)=\int_{\mathbb{R}^3}\nabla_x^2G(x-y)\nabla^{l-1}\Big[\eta\Big(&\mathbf{u}\otimes\mathbf{u}+\nabla\mathbf{d}\odot\nabla\mathbf{d}-\frac{1}{2}|\nabla\mathbf{d}|^2\mathrm{I}_3 \\
&-(F(\mathbf{d})\mathrm{I}_3-\fint_{\mathbb{P}_2}F(\mathbf{d})\mathrm{I}_3)-\sigma^L+\fint_{\mathbb{P}_2}(\frac{\mu_2}{\lambda_1}\mathbf{f}(\mathbf{d})\otimes\mathbf{d}+\frac{\mu_3}{\lambda_1}\mathbf{d}\otimes\mathbf{f}(\mathbf{d}))\Big](y)dy.
\end{aligned}$$

By Calderon-Zygmund's singular integral estimate, with bounds (4.4.71)-(4.4.75) we can show that

$$\int_{\mathbb{P}_{1+2^{-l}}}|\nabla^{l-1}P^{(1)}|^2dxdt\le C(l)\varepsilon_1. \tag{4.4.78}$$

We see that $P^{(2)}$ satisfies

$$-\Delta P^{(2)}=0 \text{ in } B_{1+2^{-(l+1)}+3^{-(l+1)}}. \tag{4.4.79}$$

Then we derive from the regularity of harmonic function that for $1\le \mathrm{j}\le 2l$,

$$\begin{aligned}
\int_{\mathbb{P}_{1+2^{-(l+1)}+5^{-(l+1)}}}|\nabla^{\mathrm{j}}P^{(2)}|^{\frac{3}{2}}dxdt &\le C\int_{\mathbb{P}_{1+2^{-(l+1)}+4^{-(l+1)}}}|\nabla^{l-1}P^{(2)}|^{\frac{3}{2}}dxdt \\
&\le C\int_{\mathbb{P}_{1+2^{-l}}}|\nabla^{l-1}P|^{\frac{3}{2}}dxdt+C\int_{\mathbb{P}_{1+2^{-l}}}|P^{(1)}|^{\frac{3}{2}}dxdt \\
&\le C(l)\varepsilon_1.
\end{aligned}$$

Now take $l-$th order spatial derivative of the equation $(4.1.6)_1$, we have[2]

$$\begin{aligned}
&\partial_t(\nabla^l \mathbf{u}) + \nabla^l\nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \nabla^l\nabla P \\
&\quad = -\nabla^l\nabla\cdot\left[\nabla\mathbf{d}\odot\nabla\mathbf{d} - \frac{1}{2}|\nabla\mathbf{d}|\mathrm{I}_3 - F(\mathbf{d})\mathrm{I}_3 - \sigma^L\right].
\end{aligned} \tag{4.4.80}$$

Let $\eta\in C_0^\infty(B_{1+2^{-l}})$. Multiplying (4.4.80) by $\nabla^l\mathbf{u}\eta^2$ and integrating over B_2, we obtain[3]

$$\begin{aligned}
&\frac{d}{dt}\int_{B_2}\frac{1}{2}|\nabla^l\mathbf{u}|^2\eta^2dx \\
&= \int_{B_2}[\nabla^l(\mathbf{u}\otimes\mathbf{u}) : \nabla\nabla^l\mathbf{u}\eta^2 + \nabla^l(\mathbf{u}\otimes\mathbf{u}) : \nabla^l\mathbf{u}\otimes\nabla(\eta^2)]dx \\
&\quad + \int_{B_2}\nabla^l P\cdot\nabla^l\mathbf{u}\cdot\nabla(\eta^2)dx - \int_{B_2}\nabla\nabla^l\mathbf{u} : \nabla^l\mathbf{u}\otimes\nabla(\eta^2)dx \\
&\quad + \int_{B_2}\nabla^l\left[\nabla\mathbf{d}\odot\nabla\mathbf{d} - \frac{1}{2}|\nabla\mathbf{d}|^2 - F(\mathbf{d})\mathrm{I}_3\right] : \nabla(\nabla^l\mathbf{u}\eta^2)dx \\
&\quad - \int_{B_2}\nabla^l\sigma^L : \nabla(\nabla^l\mathbf{u}\eta^2)dx \\
&:= I_1 + I_2 + I_3 + I_4 - I_5.
\end{aligned} \tag{4.4.81}$$

Now we have the following estimate:

$$\begin{aligned}
|I_1| &\lesssim \int_{B_2}\left[|\mathbf{u}||\nabla^l\mathbf{u}| + \sum_{\mathrm{j}=1}^{l-1}|\nabla^{\mathrm{j}}\mathbf{u}||\nabla^{l-\mathrm{j}}\mathbf{u}|\right](|\nabla^{l+1}\mathbf{u}|\eta^2 + |\nabla^l\mathbf{u}|\eta|\nabla\eta|)dx \\
&\le \frac{1}{32}\int_{B_2}|\nabla^{l+1}\mathbf{u}|^2\eta^2dx + C\int_{B_2}|\mathbf{u}|^2|\nabla^l\mathbf{u}|^2\eta^2dx \\
&\quad + C\int_{B_2}\sum_{\mathrm{j}=1}^{l-1}|\nabla^{\mathrm{j}}\mathbf{u}|^2|\nabla^{l-\mathrm{j}}\mathbf{u}|^2\eta^2dx + C\int_{\operatorname{spt}\eta}|\nabla^l\mathbf{u}|^2dx, \\
|I_2| &\lesssim \int_{B_2}[|\nabla^{l-1}P^{(1)}|(|\nabla^{l+1}\mathbf{u}|\eta|\nabla\eta| + |\nabla^l\mathbf{u}||\nabla^2(\eta^2)|) + |\mathbf{u}||\nabla^l(\nabla^l P^{(2)}\nabla\eta^2)|)]dx \\
&\le \frac{1}{32}\int_{B_2}|\nabla^{l+1}\mathbf{u}|^2\eta^2dx + C\int_{\operatorname{spt}\eta}(|\nabla^{l-1}P^{(1)}|^2 + |\nabla^l\mathbf{u}|^2)dx \\
&\quad + C\int_{\operatorname{spt}\eta}(|\mathbf{u}|^3 + |P^{(2)}|^{\frac{3}{2}})dx \\
|I_3| &\lesssim \int_{B_2}|\nabla^{l+1}\mathbf{u}|\eta|\nabla^l\mathbf{u}||\nabla\eta|dx \le \frac{1}{32}\int_{B_2}|\nabla^{l+1}\mathbf{u}|^2\eta^2dx + C\int_{\operatorname{spt}\eta}|\nabla^l\mathbf{u}|^2dx,
\end{aligned}$$

[2]↑Strictly speaking, we need to take finite difference quotient $D_h^{\mathrm{i}}\nabla^{l-1}$ of $(4.1.6)_1$ and then sending $h\to 0$.
[3]↑Strictly speaking, we need to multiply the equation by $D_h^{\mathrm{i}}\nabla^{l-1}\mathbf{u}\eta^2$.

$$
\begin{aligned}
|I_4| \le & \int_{B_2} \Big(|\nabla^{l+1}\mathbf{d}||\nabla\mathbf{d}| + \sum_{\mathrm{j}=1}^{l-1} |\nabla^{\mathrm{j}+1}\mathbf{d}||\nabla^{l+1-\mathrm{j}}\mathbf{d}| + |\nabla^l F(\mathbf{d})|\Big) \\
& \times (|\nabla^{l+1}\mathbf{u}|\eta^2 + |\nabla^l\mathbf{u}||\nabla(\eta^2)|)dx \\
\le & \frac{1}{32}\int_{B_2} |\nabla^{l+1}\mathbf{u}|^2\eta^2 dx + C\int_{B_2} (|\nabla^{l+1}\mathbf{d}|^2|\nabla\mathbf{d}|^2\eta^2 + \sum_{\mathrm{j}=1}^{l-1} |\nabla^{\mathrm{j}+1}\mathbf{d}|^2|\nabla^{l+1-\mathrm{j}}\mathbf{d}|^2\eta^2)dx \\
& + C\int_{B_2} |\nabla^l F(\mathbf{d})|^2\eta^2 dx + C\int_{\operatorname{spt}\eta} |\nabla^l\mathbf{u}|^2 dx.
\end{aligned}
$$

For I_5, let

$$
\begin{aligned}
A^l := & \mu_2(\frac{1}{|\lambda_1|}\Delta\nabla^l\mathbf{d} - \frac{\lambda_2}{\lambda_1}\nabla^l A\mathbf{d}) \otimes \mathbf{d} + \mu_3\mathbf{d} \otimes (\frac{1}{|\lambda_1|}\Delta\nabla^l\mathbf{d} - \frac{\lambda_2}{\lambda_1}\nabla^l A\mathbf{d}) \\
& + \mu_4\nabla^l A + \mu_5\nabla^l A\mathbf{d} \otimes \mathbf{d} + \mu_6\mathbf{d} \otimes \nabla^l A\mathbf{d},
\end{aligned}
$$

and $B^l := \nabla^l\sigma^L - A^l$, then we have

$$
\begin{aligned}
I_5 = & \int_{B_2} [A^l : \nabla\nabla^l\mathbf{u}\eta^2 + B^l : \nabla\nabla^l\mathbf{u}\eta^2 + A^l : \nabla^l\mathbf{u} \otimes \nabla(\eta^2) + B^l : \nabla^l\mathbf{u} \otimes \nabla(\eta^2)]dx \\
=: & I_{51} + I_{52} + I_{53} + I_{54}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
|I_{52}| \le & \frac{1}{32}\int_{B_2} |\nabla^{l+1}\mathbf{u}|^2\eta^2 dx + C\int_{B_2} |\nabla\mathbf{d}|^2|\nabla^{l+1}\mathbf{d}|^2\eta^2 dx \\
& + C\int_{B_2} \sum_{\mathrm{j}=1}^{l-1} |\nabla^{\mathrm{j}+1}\mathbf{d}|^2|\nabla^{l+1-\mathrm{j}}\mathbf{d}|^2\eta^2 dx + C\int_{B_2} \sum_{\mathrm{j}=1}^{l-1} |\nabla^{\mathrm{j}}\mathbf{u}|^2|\nabla^{l-\mathrm{j}+1}\mathbf{d}|^2\eta^2 dx \\
& + C\int_{B_2} |\nabla^l(\mathbf{f}(\mathbf{d}) \otimes \mathbf{d})|^2\eta^2 dx, \\
|I_{53}| \le & \frac{1}{32}\int_{B_2} |\nabla^{l+2}\mathbf{d}|^2\eta^2 dx + \frac{1}{32}\int_{B_2} |\nabla^{l+1}\mathbf{u}|^2\eta^2 dx + C\int_{\operatorname{spt}\eta} |\nabla^l\mathbf{u}|^2 dx, \\
|I_{54}| \lesssim & \int_{\operatorname{spt}\eta} |\nabla^l\mathbf{u}|^2 dx + \int_{B_2} |\nabla\mathbf{d}|^2|\nabla^{l+1}\mathbf{d}|^2\eta^2 dx + \int_{B_2} \sum_{\mathrm{j}=1}^{l-1} |\nabla^{\mathrm{j}+1}\mathbf{d}|^2|\nabla^{l+1-\mathrm{j}}\mathbf{d}|^2\eta^2 dx \\
& + \int_{B_2} \sum_{\mathrm{j}=1}^{l-1} |\nabla^{\mathrm{j}}\mathbf{u}|^2|\nabla^{l-\mathrm{j}+1}\mathbf{d}|^2\eta^2 dx + \int_{B_2} |\nabla^l(\mathbf{f}(\mathbf{d}) \otimes \mathbf{d})|^2\eta^2 dx.
\end{aligned}
$$

Now we take $(l+1)$-th order spatial derivative of the equation $(4.1.6)_3$, we have

$$\partial_t(\nabla\nabla^l\mathbf{d}) + \nabla\nabla^l(\mathbf{u}\cdot\nabla\mathbf{d}) - \nabla\nabla^l(\Omega\mathbf{d} - \frac{\lambda_2}{\lambda_1}A\mathbf{d}) = \frac{1}{-\lambda_1}[\Delta\nabla\nabla^l\mathbf{d} - \nabla\nabla^l\mathbf{f}(\mathbf{d})]. \tag{4.4.82}$$

Multiplying (4.4.82) by $\nabla\nabla^l\mathbf{d}\eta^2$ and integrating over B_2, we obtain[4]

$$\begin{aligned}
&\frac{d}{dt}\int_{B_2}\frac{1}{2}|\nabla^{l+1}\mathbf{d}|^2\eta^2dx + \frac{1}{|\lambda_1|}\int_{B_2}|\nabla^{l+2}\mathbf{d}|^2\eta^2dx \\
&= \int_{B_2}\nabla^l(\mathbf{u}\cdot\nabla\mathbf{d})\cdot\nabla\cdot(\nabla\nabla^l\mathbf{d}\eta^2)dx \\
&\quad - \int_{B_2}[\nabla^l(\Omega\mathbf{d} - \frac{\lambda_2}{\lambda_1}A\mathbf{d})\cdot\Delta\nabla^l\mathbf{d}\eta^2 + \nabla^l(\Omega\mathbf{d} - \frac{\lambda_2}{\lambda_1}A\mathbf{d})\cdot(\nabla(\eta^2)\cdot\nabla\nabla^l\mathbf{d})]dx \\
&\quad - \frac{1}{|\lambda_1|}\int_{B_2}\nabla\nabla^l\mathbf{f}(\mathbf{d}):\nabla\nabla^l\mathbf{d}\eta^2dx =: K_1 - K_2 + K_3.
\end{aligned} \tag{4.4.83}$$

Then we have the following estimates:

$$\begin{aligned}
|K_1| &\lesssim \int_{B_2}\Big[|\nabla\mathbf{d}||\nabla^l\mathbf{u}| + |\mathbf{u}||\nabla^{l+1}\mathbf{d}| + \sum_{\mathrm{j}=1}^{l-1}|\nabla^{\mathrm{j}}\mathbf{u}||\nabla^{l-\mathrm{j}+1}\mathbf{d}|\Big]\Big(|\nabla^{l+2}\mathbf{d}|\eta^2 + |\nabla^{l+1}\mathbf{d}|\eta|\nabla\eta|\Big)dx \\
&\le \frac{1}{32}\int_{B_2}|\nabla^{l+2}\mathbf{d}|^2\eta^2dx + C\int_{B_2}|\nabla\mathbf{d}|^2|\nabla^l\mathbf{u}|^2\eta^2dx + C\int_{B_2}|\mathbf{u}|^2|\nabla^{l+1}\mathbf{d}|^2\eta^2dx \\
&\qquad + C\int_{B_2}\sum_{\mathrm{j}=1}^{l-1}|\nabla^{\mathrm{j}}\mathbf{u}|^2|\nabla^{l-\mathrm{j}+1}\mathbf{d}|^2\eta^2dx + C\int_{\operatorname{spt}\eta}|\nabla^{l+1}\mathbf{d}|^2dx, \\
|K_3| &\lesssim \int_{B_2}|\nabla^{l+1}\mathbf{d}|^2\eta^2dx + \int_{B_2}|\nabla^{l+1}\mathbf{f}(\mathbf{d})|^2\eta^2dx.
\end{aligned}$$

For K_2, let

$$C^l =: (\nabla^l\Omega)\mathbf{d} - \frac{\lambda_2}{\lambda_1}(\nabla^l A)\mathbf{d}, \quad D^l := \nabla^l(\Omega\mathbf{d} - \frac{\lambda_2}{\lambda_1}A\mathbf{d}) - C^l,$$

then we have

$$\begin{aligned}
K_2 &= \int_{B_2}[C^l\cdot\Delta\nabla^l\mathbf{d}\eta^2 + D^l\cdot\Delta\nabla^l\mathbf{d}\eta^2 + C^l\cdot(\nabla(\eta^2)\cdot\nabla\nabla^l\mathbf{d}) + D^l\cdot(\nabla(\eta^2)\cdot\nabla\nabla^l\mathbf{d})]dx \\
&=: K_{21} + K_{22} + K_{23} + K_{24}.
\end{aligned}$$

[4] ↑Strictly speaking, we need to multiply the equation by $D_h^{\mathrm{i}}\nabla^l\mathbf{d}\eta^2$.

Now we estimate

$$
\begin{aligned}
|K_{22}| &\leq \frac{1}{32}\int_{B_2}|\nabla^{l+2}\mathbf{d}|^2\eta^2dx + C\int_{B_2}|\nabla\mathbf{d}|^2|\nabla^l\mathbf{u}|^2\eta^2dx + C\int_{B_2}\sum_{\mathrm{j}=1}^{l-1}|\nabla^{\mathrm{j}}\mathbf{u}|^2|\nabla^{l+1-\mathrm{j}}\mathbf{d}|^2\eta^2dx,\\
|K_{23}| &\leq \frac{1}{32}\int_{B_2}|\nabla^{l+1}\mathbf{u}|^2\eta^2dx + C\int_{\operatorname{spt}\eta}|\nabla^{l+1}\mathbf{d}|^2dx,\\
|K_{24}| &\lesssim \int_{\operatorname{spt}\eta}|\nabla^{l+1}\mathbf{d}|^2dx + \int_{B_2}|\nabla\mathbf{d}|^2|\nabla^l\mathbf{u}|^2\eta^2dx + \int_{B_2}\sum_{\mathrm{j}=1}^{l-1}|\nabla^{\mathrm{j}}\mathbf{u}|^2|\nabla^{l+1-\mathrm{j}}\mathbf{d}|^2\eta^2dx.
\end{aligned}
$$

Combine all estimate above, and with the identity that

$$
\begin{aligned}
I_{51}+K_{21} &= \int_{B_2}(A^l:\nabla\nabla^l\mathbf{u} + C^l\cdot\Delta\nabla^l\mathbf{d})\eta^2dx\\
&= \int_{B_2}\mu_4|\nabla^lA|\eta^2dx + \int_{B_2}(\mu_5+\mu_6)|\nabla^lA\mathbf{d}|^2\eta^2dx\\
&\quad + \int_{B_2}(-\lambda_2\nabla^lA\mathbf{d} - \Delta\nabla^l\mathbf{d})\cdot(\nabla^l\Omega\mathbf{d})\eta^2dx\\
&\quad - \int_{B_2}\lambda_2(-\frac{\lambda_2}{\lambda_1}\nabla^lA\mathbf{d} + \frac{1}{|\lambda_1|}\Delta\nabla^l\mathbf{d})\cdot(\nabla^lA\mathbf{d})\eta^2dx\\
&\quad + \int_{B_2}\lambda_2(\nabla^lA\mathbf{d})\cdot(\nabla^l\Omega\mathbf{d})\eta^2dx + \int_{B_2}(\nabla^l\Omega\mathbf{d} - \frac{\lambda_2}{\lambda_1}\nabla^lA\mathbf{d})\cdot\Delta\nabla^l\mathbf{d}\eta^2dx\\
&= \int_{B_2}\mu_4|\nabla^lA|^2\eta^2dx + \int_{B_2}\left(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1}\right)|\nabla^lA\mathbf{d}|^2\eta^2dx,
\end{aligned}
$$

we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\int_{B_2}\left(|\nabla^l\mathbf{u}|^2+|\nabla^{l+1}\mathbf{d}|^2\right)\eta^2dx + \int_{B_2}\left(|\nabla^{l+1}\mathbf{u}|^2+|\nabla^{l+2}\mathbf{d}|^2\right)\eta^2dx\\
&\leq C\int_{B_2}(|\mathbf{u}|^2|\nabla^l\mathbf{u}|^2\eta^2 + \sum_{\mathrm{j}=1}^{l-1}|\nabla^{\mathrm{j}}\mathbf{u}|^2|\nabla^{l-\mathrm{j}}\mathbf{u}|^2\eta^2)dx + C\int_{\operatorname{spt}\eta}(|\nabla^l\mathbf{u}|^2+|\nabla^{l+1}\mathbf{d}|^2)dx\\
&+ C\int_{\operatorname{spt}\eta}(|\nabla^l\mathbf{u}|^2+|\nabla^{l+1}\mathbf{d}|^2+|\nabla^{l-1}P^{(1)}|^2+|\mathbf{u}|^3+|P^{(2)}|^{\frac{3}{2}})dx\\
&+ C\int_{B_2}(|\nabla\mathbf{d}|^2|\nabla^{l+1}\mathbf{d}|^2\eta^2 + \sum_{\mathrm{j}=1}^{l-1}|\nabla^{\mathrm{j}+1}\mathbf{d}|^2|\nabla^{l+1-\mathrm{j}}\mathbf{d}|^2\eta^2)dx\\
&+ C\int_{B_2}(|\nabla^lF(\mathbf{d})|^2\eta^2 + |\nabla^l(\mathbf{f}(\mathbf{d})\otimes\mathbf{d})|^2\eta^2 + |\nabla^{l+1}\mathbf{f}(\mathbf{d})|^2\eta^2)dx\\
&+ C\int_{B_2}(|\nabla\mathbf{d}|^2|\nabla^l\mathbf{u}|^2\eta^2 + |\mathbf{u}|^2|\nabla^{l+1}\mathbf{d}|^2\eta^2 + \sum_{\mathrm{j}=1}^{l-1}|\nabla^{\mathrm{j}}\mathbf{d}|^2|\nabla^{l+1-\mathrm{j}}\mathbf{d}|^2\eta^2)dx
\end{aligned}
$$

$$+ C\int_{B_2}\sum_{\mathrm{j}=1}^{l-1}|\nabla^{\mathrm{j}}\mathbf{u}|^2|\nabla^{l+1-\mathrm{j}}\mathbf{d}|^2\eta^2dx. \tag{4.4.84}$$

By Sobolev-interpolation inequality, we have

$$\begin{aligned}
&\int_{B_2}|\mathbf{u}|^2|\nabla^l\mathbf{u}|^2\eta^2dx\\
&\le \left\|\nabla^l\mathbf{u}\eta\right\|_{L^6(B_2)}\left\|\nabla^l\mathbf{u}\eta\right\|_{L^2(B_2)}\|\mathbf{u}\|^2_{L^6(\operatorname{spt}\eta)}\\
&\le C\left\|\nabla(\nabla^l\mathbf{u}\eta)\right\|_{L^2(B_2)}\left\|\nabla^l\mathbf{u}\eta\right\|_{L^2(B_2)}\|\mathbf{u}\|^2_{L^6(\operatorname{spt}\eta)}\\
&\le \frac{1}{32}\int_{B_2}|\nabla^{l+1}\mathbf{u}|^2\eta^2dx + C\int_{\operatorname{spt}\eta}|\nabla^l\mathbf{u}|^2dx + C\|\mathbf{u}\|^4_{L^6(\operatorname{spt}\eta)}\int_{B_2}|\nabla^l\mathbf{u}|^2\eta^2dx,
\end{aligned}$$

$$\begin{aligned}
&\int_{B_2}|\mathbf{u}|^2|\nabla^{l+1}\mathbf{d}|^2\eta^2dx\\
&\le \frac{1}{32}\int_{B_2}|\nabla^{l+2}\mathbf{d}|^2\eta^2dx + C\int_{\operatorname{spt}\eta}|\nabla^{l+1}\mathbf{d}|^2dx + C\|\mathbf{u}\|^4_{L^6(\operatorname{spt}\eta)}\int_{B_2}|\nabla^{l+1}\mathbf{d}|^2\eta^2dx,
\end{aligned}$$

$$\begin{aligned}
&\int_{B_2}|\nabla\mathbf{d}|^2|\nabla^l\mathbf{u}|^2\eta^2dx\\
&\le \frac{1}{32}\int_{B_2}|\nabla^{l+1}\mathbf{u}|\eta^2dx + C\int_{\operatorname{spt}\eta}|\nabla^l\mathbf{u}|^2dx + C\|\nabla\mathbf{d}\|^4_{L^6(\operatorname{spt}\eta)}\int_{B_2}|\nabla^l\mathbf{u}|^2\eta^2dx,
\end{aligned}$$

$$\begin{aligned}
&\int_{B_2}|\nabla\mathbf{d}|^2|\nabla^{l+1}\mathbf{d}|^2\eta^2dx\\
&\le \frac{1}{32}\int_{B_2}|\nabla^{l+2}\mathbf{d}|^2\eta^2dx + C\int_{\operatorname{spt}\eta}|\nabla^{l+1}\mathbf{d}|^2dx + C\|\nabla\mathbf{d}\|^4_{L^6(\operatorname{spt}\eta)}\int_{B_2}|\nabla^{l+1}\mathbf{d}|^2\eta^2dx.
\end{aligned}$$

For lower order terms, we have that for $1\le \mathrm{j}\le l-1$,

$$\begin{aligned}
\int_{B_2}|\nabla^{l-1}\mathbf{u}|^2|\nabla^{\mathrm{j}}\mathbf{u}|^2\eta^2dx &\le \left\|\nabla^{l-1}\mathbf{u}\eta\right\|^2_{L^6(B_2)}\left\|\nabla^{\mathrm{j}}\mathbf{u}\right\|^2_{L^3(\operatorname{spt}\eta)}\\
&\le C\left\|\nabla(\nabla^{l-1}\mathbf{u}\eta)\right\|^2_{L^2(B_2)}\left\|\nabla^{\mathrm{j}}\mathbf{u}\right\|^2_{L^3(\operatorname{spt}\eta)}\\
&\le C\left\|\nabla^{\mathrm{j}}\mathbf{u}\right\|^2_{L^3(\operatorname{spt}\eta)}\int_{B_2}|\nabla^l\mathbf{u}|^2\eta^2dx
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + C \left\|\nabla^{l-1}\mathbf{u}\right\|^2_{L^3(\operatorname{spt}\eta)} \left\|\nabla^{\mathrm{j}}\mathbf{u}\right\|^2_{L^3(\operatorname{spt}\eta)}, \\
\int_{B_2} |\nabla^l \mathbf{d}|^2 |\nabla^{\mathrm{j}}\mathbf{u}|^2 \eta^2 dx &\le C \left\|\nabla^{\mathrm{j}}\mathbf{u}\right\|^2_{L^3(\operatorname{spt}\eta)} \int_{B_2} |\nabla^{l+1}\mathbf{d}|^2 \eta^2 dx \\
&\quad + C \left\|\nabla^{l}\mathbf{d}\right\|^2_{L^3(\operatorname{spt}\eta)} \left\|\nabla^{\mathrm{j}}\mathbf{u}\right\|^2_{L^3(\operatorname{spt}\eta)}, \\
\int_{B_2} |\nabla^{l-1} \mathbf{u}| |\nabla^{\mathrm{j}+1}\mathbf{d}|^2 \eta^2 dx &\le C \left\|\nabla^{\mathrm{j}+1}\mathbf{d}\right\|^2_{L^3(\operatorname{spt}\eta)} \int_{B_2} |\nabla^{l}\mathbf{u}| \eta^2 \\
&\quad + C \left\|\nabla^{l-1}\mathbf{u}\right\|^2_{L^3(\operatorname{spt}\eta)} \left\|\nabla^{\mathrm{j}+1}\mathbf{d}\right\|^2_{L^3(\operatorname{spt}\eta)}, \\
\int_{B_2} |\nabla^l \mathbf{d}|^2 |\nabla^{\mathrm{j}+1}\mathbf{d}|^2 \eta^2 dx &\le C \left\|\nabla^{\mathrm{j}+1}\mathbf{d}\right\|^2_{L^3(\operatorname{spt}\eta)} \int_{B_2} |\nabla^{l+1}\mathbf{d}|^2 \eta^2 dx \\
&\quad + C \left\|\nabla^{l}\mathbf{d}\right\|^2_{L^3(\operatorname{spt}\eta)} \left\|\nabla^{\mathrm{j}+1}\mathbf{d}\right\|^2_{L^3(\operatorname{spt}\eta)},
\end{aligned}
$$

and for $1 \le \mathrm{j}, k \le l-2$ that

$$
\int_{B_2} |\nabla^{\mathrm{j}}\mathbf{u}|^2 |\nabla^{k+1}\mathbf{d}|^2 \eta^2 dx \le C \int_{\operatorname{spt}\eta} |\nabla^{\mathrm{j}}\mathbf{u}|^4 dx + C \int_{\operatorname{spt}\eta} |\nabla^{k+1}\mathbf{d}|^4 dx.
$$

Since $|\mathbf{d}| \le M$ in $\mathbb{P}_2$, by the calculus inequality for H^s (c.f. [84, Appendix]), we have for $-4 \le t \le 0$,

$$
\begin{aligned}
\left\|\nabla^l F(\mathbf{d})\right\|_{L^2(\operatorname{spt}\eta)} &\lesssim \left\|\nabla^l \mathbf{d}\right\|_{L^2(\operatorname{spt}\eta)}, \\
\left\|\nabla^l (\mathbf{f}(\mathbf{d}) \otimes \mathbf{d})\right\|_{L^2(\operatorname{spt}\eta)} &\lesssim \left\|\nabla^l \mathbf{d}\right\|_{L^2(\operatorname{spt}\eta)}, \\
\left\|\nabla^{l+1} \mathbf{f}(\mathbf{d})\right\|_{L^2(\operatorname{spt}\eta)} &\lesssim \left\|\nabla^{l+1} \mathbf{d}\right\|_{L^2(\operatorname{spt}\eta)}.
\end{aligned}
$$

Put all these estimates together, we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\int_{B_2}\Big(|\nabla^l\mathbf{u}|^2+|\nabla^{l+1}\mathbf{d}|^2\Big)\eta^2dx+\int_{B_2}\Big(|\nabla^{l+1}\mathbf{u}|^2+|\nabla^{l+2}\mathbf{d}|^2\Big)\eta^2dx\\
&\le C\int_{\operatorname{spt}\eta}[|\nabla^l\mathbf{u}|^2+|\nabla^{l+1}\mathbf{d}|^2+|\nabla^l\mathbf{d}|^2+\sum_{\mathrm{j}=1}^{l-2}(|\nabla^{\mathrm{j}}\mathbf{u}|^4+|\nabla^{\mathrm{j}+1}\mathbf{d}|^4]dx)\\
&+C\int_{\operatorname{spt}\eta}(|\mathbf{u}|^3+|\nabla^{l-1}P^{(1)}|^2+|P^{(2)}|^{\frac{3}{2}})dx\\
&+C\Big(\left\|\nabla^{l-1}\mathbf{u}\right\|^4_{L^3(\operatorname{spt}\eta)}+\left\|\nabla^l\mathbf{d}\right\|^4_{L^3(\operatorname{spt}\eta)}+\sum_{\mathrm{j}=1}^{l-1}\Big(\left\|\nabla^{\mathrm{j}}\mathbf{u}\right\|^4_{L^3(\operatorname{spt}\eta)}+\left\|\nabla^{\mathrm{j}+1}\mathbf{d}\right\|^4_{L^3(\operatorname{spt}\eta)}\Big)\Big)\\
&+C\Big(\|(\mathbf{u},\nabla\mathbf{d})\|^4_{L^6(B_2)}+\sum_{\mathrm{j}=1}^{l-1}\|(\nabla^{\mathrm{j}}\mathbf{u},\nabla^{\mathrm{j}+1}\mathbf{d})\|^2_{L^3(B_2)}\Big)\int_{B_2}(|\nabla^l\mathbf{u}|^2+|\nabla^{l+1}\mathbf{d}|^2)\eta^2dx.
\end{aligned}
\tag{4.4.85}
$$

Now let $\eta\in C_0^\infty(B_{1+2^{-(l+1)}+5^{-(l+1)}})$ be a cut-off function of $B_{1+2^{-(l+1)}+10^{-(l+1)}}$. We can apply the Gronwall's inequality to (4.4.85), together with (4.4.71)-(4.4.75) to get

$$
\begin{aligned}
&\sup_{-\left(1+2^{-(l+1)}+10^{-(l+1)}\right)^2\le t\le 0}\int_{B_{1+2^{-(l+1)}+10^{-(l+1)}}}(|\nabla^l\mathbf{u}|^2+|\nabla^{l+1}\mathbf{d}|^2)dx\\
&\quad+\int_{\mathbb{P}_{1+2^{-(l+1)}+10^{-(l+1)}}}\Big(|\nabla^{l+1}\mathbf{u}|^2+|\nabla^{l+2}\mathbf{d}|^2\Big)dxdt\\
&\le C(l)\varepsilon_1.
\end{aligned}
\tag{4.4.86}
$$

Recall that ∇^lP satisfies

$$
\begin{aligned}
-\Delta\nabla^lP=\operatorname{div}^2\Big[\nabla^l\Big(&\mathbf{u}\otimes\mathbf{u}+\nabla\mathbf{d}\odot\nabla\mathbf{d}-\frac{1}{2}|\nabla\mathbf{d}|^2\mathrm{I}_3\\
&-(F(\mathbf{d})\mathrm{I}_3-⨍_{\mathbb{P}_2}F(\mathbf{d})\mathrm{I}_3)-\sigma^L+⨍_{\mathbb{P}_2}(\frac{\mu_2}{\lambda_1}\mathbf{f}(\mathbf{d})\otimes\mathbf{d}+\frac{\mu_3}{\lambda_1}\mathbf{d}\otimes\mathbf{f}(\mathbf{d}))\Big)\Big].
\end{aligned}
\tag{4.4.87}
$$

Then by the Calderón-Zygmund theory and (4.4.71)-(4.4.75), (4.4.86) we can show

$$
\int_{\mathbb{P}_{1+2^{-(l+1)}}}|\nabla^lP|^{\frac{3}{2}}dxdt\le C(l)\varepsilon_1. \tag{4.4.88}
$$

This yields that the conclusion holds for $k=l$. Thus the proof is complete. $\square$

4.5 Partial regularity

As a consequence of Lemma 4.4.5, we get the following regularity criteria for (4.1.6):

Corollary 4.5.1. *For a suitable weak solution* $(\mathbf{u}, \mathbf{d}, P)$ *to* (4.1.6)*, if* $z \in \mathbb{T}^3 \times (0, \infty)$ *satisfies*

$$\begin{cases} \sup_{0<r<\delta} |\mathbf{d}_{z,r}| < \infty, \\ \liminf_{r \to 0+} \Phi(z, r) = 0, \end{cases} \tag{4.5.1}$$

Then there exists $\delta_1 > 0$ *such that* $(\mathbf{u}, \mathbf{d}) \in C^\infty(\mathbb{P}_{\delta_1}(z))$.

The following Lemma is well-known, see [79].

Lemma 4.5.1. *Let* $\mathbf{d}$ *be a function in* $L^6(\mathbb{T}^3 \times (0, \infty))$*, and let* $z = (x, t) \in \mathbb{T}^3 \times (0, \infty)$ *such that*

$$⨍_{\mathbb{P}_r(z)} |\mathbf{d} - \mathbf{d}_{z,r}|^6 dxdt \le Cr^\delta \tag{4.5.2}$$

for some $\delta > 0$ *and some* C *depending on* $\mathbf{d}$ *and* z*. Then* $\lim_{r \to 0} \mathbf{d}_{z,r}$ *exists, and is finite.*

Next we will control the oscillation of $\mathbf{d}$. For $0 < T \le \infty$, denote $Q_T = \mathbb{T}^3 \times (0, T)$. Recall the fractional parabolic Sobolev space $W_p^{1,\frac{1}{2}}(Q_T)$, $1 \le p < \infty$, contains all f's satisfying

$$\|f\|_{W_p^{1,\frac{1}{2}}(Q_T)} = \|f\|_{L^p(Q_T)} + \|f\|_{\dot{W}_p^{1,\frac{1}{2}}(Q_T)} < \infty,$$

where

$$\|f\|_{\dot{W}_p^{1,\frac{1}{2}}(Q_T)} := \Big(\int_{Q_T} |\nabla f|^p dtdx + \int_{\mathbb{T}^3} \int_0^T \int_0^T \frac{|f(x,t) - f(x,s)|^p}{|t-s|^{1+\frac{p}{2}}} dtdsdx \Big)^{\frac{1}{p}}.$$

From the global energy estimate (4.1.11) and the Sobolev embedding theorem, we have

$$(\mathbf{u}, \nabla \mathbf{d}) \in (L_t^\infty L_x^2 \cap L_t^2 H_x^1 \cap L_t^{\frac{10}{3}} L_x^{\frac{10}{3}})(Q_T), \ \mathbf{d} \in L_t^{10} L_x^{10}(Q_T). \tag{4.5.3}$$

It follows that

$$\partial_t \mathbf{d} - \frac{1}{|\lambda_1|} \Delta \mathbf{d} = -\mathbf{u} \cdot \nabla \mathbf{d} + \Omega \mathbf{d} - \frac{\lambda_2}{\lambda_1} A \mathbf{d} + \frac{1}{|\lambda_1|} \mathbf{f}(\mathbf{d}) \in L^{\frac{5}{3}}(Q_T).$$

From the fractional Galiardo-Nirenberg inequality [85], [86], we get $\mathbf{d} \in W^{1,\frac{1}{2}}_{\frac{20}{7}}(Q_T)$, and

$$\|\mathbf{d}\|^2_{W^{1,\frac{1}{2}}_{\frac{20}{7}}(Q_T)} \le C\,\|\mathbf{d}\|_{L^{10}(Q_T)}\,\|(\partial_t\mathbf{d},\nabla\mathbf{d})\|_{L^{\frac{5}{3}}(Q_T)} + C\,\|\mathbf{d}\|^2_{L_t^{\frac{20}{7}}W_x^{1,\frac{20}{7}}(Q_T)} < \infty.$$

Then the parabolic Sobolev-Poincaré inequality yields

$$\begin{aligned}&\Big(\fint_{\mathbb{P}_r(z)} |\mathbf{d}-\mathbf{d}_{z,r}|^p dxdt\Big)^{\frac{1}{p}} \\ &\le C\Big[r^{\frac{20}{7}-5}\int_{\mathbb{P}_r(z)}|\nabla\mathbf{d}|^{\frac{20}{7}} + r^{\frac{20}{7}-5}\int_{B_r(x)}\int_{t-r^2}^t\int_{t-r^2}^t \frac{|\mathbf{d}(x,s_1)-\mathbf{d}(x,s_2)|^{\frac{20}{7}}}{|s_1-s_2|^{1+\frac{10}{7}}}ds_1ds_2dx\Big]^{\frac{7}{20}}.\end{aligned}$$

where $p = \dfrac{5\cdot\frac{20}{7}}{5-\frac{20}{7}} = \dfrac{20}{3} > 6$. Hence by Hölder inequality we have that

$$\begin{aligned}&\Big(\fint_{\mathbb{P}_r(z)} |\mathbf{d}-\mathbf{d}_{z,r}|^6 dxdt\Big)^{\frac{1}{6}} \le \Big(\fint_{\mathbb{P}_r(z)} |\mathbf{d}-\mathbf{d}_{z,r}|^{\frac{20}{3}} dxdt\Big)^{\frac{3}{20}} \\ &\le C\Big[r^{\frac{20}{7}-5}\int_{\mathbb{P}_r(z)}|\nabla\mathbf{d}|^{\frac{20}{7}} + r^{\frac{20}{7}-5}\int_{B_r(x)}\int_{t-r^2}^t\int_{t-r^2}^t \frac{|\mathbf{d}(x,s_1)-\mathbf{d}(x,s_2)|^{\frac{20}{7}}}{|s_1-s_2|^{1+\frac{10}{7}}}ds_1ds_2dx\Big]^{\frac{7}{20}}.\end{aligned} \tag{4.5.4}$$

Proof of Theorem 4.1.1. Define

$$\Sigma = \Big\{z\in\mathbb{T}^3\times(0,\infty) : \liminf_{r\to 0}\Phi(z,r) > \varepsilon_2^6 \text{ or } \liminf_{r\to0}|\mathbf{d}_{z,r}| = \infty\Big\}.$$

It follows from Corollary 4.5.1 that Σ is closed and $(\mathbf{u},\mathbf{d}) \in C^\infty(\mathbb{T}^3\times(0,\infty)\setminus\Sigma)$. From (4.5.4) and Lemma 4.5.1, we know that $\Sigma \subset \cap_{\sigma>0}\mathcal{S}_\sigma$, where $\mathcal{S}_\sigma$ is defined by

$$\begin{aligned}\mathcal{S}_\sigma = \Big\{z\in Q_T : &\liminf_{r\to0}\Big[r^{-\frac{5}{3}}\int_{\mathbb{P}_r(z)}\Big(|\mathbf{u}|^{\frac{10}{3}}+|\nabla\mathbf{d}|^{\frac{10}{3}}\Big)dxdt + \Big(r^{-\frac{5}{3}}\int_{\mathbb{P}_r(z)}|P|^{\frac{5}{3}}dxdt\Big)^2\Big] > 0, \text{ or} \\ &\liminf_{r\to0} r^{-\frac{15}{7}-\sigma}\Big(\int_{\mathbb{P}_r(z)}|\nabla\mathbf{d}|^{\frac{20}{7}}dxdt + \int_{B_r(x)}\int_{t-r^2}^t\int_{t-r^2}^t\frac{|\mathbf{d}(x,s_1)-\mathbf{d}(x,s_2)|^{\frac{20}{7}}}{|s_1-s_2|^{1+\frac{10}{7}}}ds_1ds_2dx\Big) > 0\Big\}.\end{aligned}$$

For the last integral, we have that

$$f(x,s_1,s_2) = \frac{|\mathbf{d}(x,s_1)-\mathbf{d}(x,s_2)|^{\frac{20}{7}}}{|s_1-s_2|^{1+\frac{10}{7}}} \in L^1(\mathbb{T}^3\times(0,T)\times(0,T)).$$

Let $\tilde{\delta}$ be the metric on $\mathbb{T}^3 \times \mathbb{R} \times \mathbb{R}$:

$$\tilde{\delta}(\xi_1, \xi_2) = \max\left\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}, \sqrt{|s_1 - s_2|}\right\}, \quad \forall \xi_{\mathrm{i}} = (x_{\mathrm{i}}, t_{\mathrm{i}}, s_{\mathrm{i}}) \in \mathbb{T}^3 \times \mathbb{R} \times \mathbb{R}.$$

A standard covering argument implies that

$$\widetilde{\mathcal{P}}^{\frac{15}{7}+\sigma}\Big\{(x,s,t) \in \mathbb{T}^3 \times (0,T) \times (0,T) : \liminf_{r \to 0+} r^{-\frac{15}{7}-\sigma} \int_{B_r(x)} \int_{s-r^2}^{s} \int_{t-r^2}^{t} f(\xi) d\xi > 0\Big\} = 0,$$

where $\widetilde{\mathcal{P}}^k$ denotes the k-dimensional Hausdorff measure on $\mathbb{T}^3 \times \mathbb{R}_+ \times \mathbb{R}_+$ with respect to the metric $\tilde{\delta}$.

Since the map $T(x,t) = (x,t,t) : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{T}^3 \times \mathbb{R} \times \mathbb{R}$ is an isometric embedding of $(\mathbb{T}^3 \times \mathbb{R}, \delta)$ into $(\mathbb{T}^3 \times \mathbb{R} \times \mathbb{R}, \tilde{\delta})$, we have that

$$\begin{aligned}
&\mathcal{P}^{\frac{15}{7}+\sigma}\left(\left\{(x,t) \in Q_T : \liminf_{r \to 0+} r^{-\frac{15}{7}-\sigma} \int_{B_r(x)} \int_{t-r^2}^{t} \int_{t-r^2}^{t} f(\xi) d\xi > 0\right\}\right) \\
&= \widetilde{\mathcal{P}}^{\frac{15}{7}+\sigma}\left(T\left[\left\{(x,t) \in Q_T : \liminf_{r \to 0+} r^{-\frac{15}{7}-\sigma} \int_{B_r(x)} \int_{t-r^2}^{t} \int_{t-r^2}^{t} f(\xi) d\xi > 0\right\}\right]\right) \\
&= \widetilde{\mathcal{P}}^{\frac{15}{7}+\sigma}\left(\left\{(x,t,t) \in Q_T \times (0,T) : \liminf_{r \to 0+} r^{-\frac{15}{7}-\sigma} \int_{B_r(x)} \int_{t-r^2}^{t} \int_{t-r^2}^{t} f(\xi) d\xi > 0\right\}\right) \qquad (4.5.5) \\
&\leq \widetilde{\mathcal{P}}^{\frac{15}{7}+\sigma}\left(\left\{(x,s,t) \in Q_T \times (0,T) : \liminf_{r \to 0+} r^{-\frac{15}{7}-\sigma} \int_{B_r(x)} \int_{s-r^2}^{s} \int_{t-r^2}^{t} f(\xi) d\xi > 0\right\}\right) \\
&= 0.
\end{aligned}$$

Again, by a simple covering argument we can show

$$\mathcal{P}^{\frac{15}{7}+\sigma}\Big(\Big\{z \in Q_T : r^{-\frac{15}{7}-\sigma} \int_{\mathbb{P}_r(z)} |\nabla \mathbf{d}|^{\frac{20}{7}} dxdt > 0\Big\}\Big) = 0, \qquad (4.5.6)$$

and

$$\mathcal{P}^{\frac{5}{3}}\Big(\Big\{z \in Q_T : \lim_{r \to 0} r^{-\frac{5}{3}} \int_{\mathbb{P}_r(z)} (|\mathbf{u}|^{\frac{10}{3}} + |\nabla \mathbf{d}|^{\frac{10}{3}}) dxdt + \Big(r^{-\frac{5}{3}} \int_{\mathbb{P}_r(z)} |P|^{\frac{5}{3}}\Big)^2 > 0\Big\}\Big) = 0. \qquad (4.5.7)$$

It follows from (4.5.5), (4.5.6) and (4.5.7) that $\mathcal{P}^{\frac{15}{7}+\sigma}(\mathcal{S}_\sigma) = 0$ so that $\mathcal{P}^{\frac{15}{7}+\sigma}(\Sigma) = 0, \forall \sigma > 0$. $\square$

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