K-THEORY OF CERTAIN ADDITIVE CATEGORIES ASSOCIATED WITH VARIETIES

by

Harrison Wong

A Dissertation

Submitted to the Faculty of Purdue University

In Partial Fulfillment of the Requirements for the degree of

Doctor of Philosophy

Department of Mathematics

West Lafayette, Indiana

August 2021

THE PURDUE UNIVERSITY GRADUATE SCHOOL
STATEMENT OF COMMITTEE APPROVAL

Dr. Deepam Patel, Chair
Department of Mathematics

Dr. Donu Arapura
Department of Mathematics

Dr. Kenji Matuski
Department of Mathematics

Dr. Saugata Basu
Department of Mathematics

Approved by:
Dr. Plamen Stefanov

ACKNOWLEDGMENTS

My graduate career was an enriching and joyful time, and I will look back fondly on this privileged period of my life. I would like to thank those that made this time so special for me.

I thank my advisor Deepam Patel for his patience, attention to detail, generosity, foresight, and support. I would also like to thank Donu Arapura for accepting me as a student, guiding me through the graduate program, and putting me on the path that led to this thesis. I thank Kenji Matsuki and Jarek Włodarczyk for listening to me as I struggled to learn algebraic geometry. I recognize Jim McClure and Laszlo Lempert for teaching wonderful courses with great style and precision. I would also like to thank the Department of Mathematics at Purdue for the financial support, and for the many opportunities to teach in a variety of formats. I found teaching and personal interaction with students to be rewarding.

My family, loved ones, and friends gave me tremendous support, sincere encouragement, and good company, and they made the tough times a bit easier. I am inspired by their cheerful and positive attitude, their talents, hobbies, and skills, their love of having a good time, their camaraderie, their mathematical knowledge and enthusiasm, their perseverance, and their genuineness. These special people include Mom, Dad, Adelaide, Angelica and her family, Pockets, Graham, Michael, Dan, Matt, Monte and Paula, Roberto, Jake, and Daniel. Thank you also to all my friends from San Diego. It was always a joy to visit again and pick up right where we left off. Everyone here are amongst the best people I've met, and you all inspire me to be a better person and to discover life.

Finally, as I look towards the future, I would like to thank those who have taught me so much in preparation for a career in industry. I thank the organizers of the IMA Math-to-Industry Bootcamp and my mentors at ITM Twentyfirst for an engaging project. I would also like to thank the NSF for an internship at ORNL. I would like to recognize my mentor Jong Choi at ORNL for his kindness and diligence. I am grateful for all the opportunities I had to learn something new and exciting.

TABLE OF CONTENTS

ABSTRACT

Let $K_0(\mathsf{Var}_k)$ be the Grothendieck group of varieties over a field k. We construct an exact category, denoted $\mathrm{Add}(\mathsf{Var}_k)_S$, such that there is a surjection $K_0(\mathsf{Var}_k) \to K_0(\mathrm{Add}(\mathsf{Var}_k)_S)$. If we consider only zero dimensional varieties, then this surjection is an isomorphism. Like $K_0(\mathsf{Var}_k)$, the group $K_0(\mathrm{Add}(\mathsf{Var}_k)_S)$ is also generated by isomorphism classes of varieties, and we construct motivic measures on $K_0(\mathrm{Add}(\mathsf{Var}_k)_S)$ including the Euler characteristic if $k = \mathbb{C}$, and point counting measures and the zeta function if k is finite.

1. INTRODUCTION

Let k be a field. Let Var_k be the category of varieties over k; that is, separated reduced schemes of finite type over $\operatorname{Spec} k$. The *Grothendieck group of k-varieties*, $K_0(\mathsf{Var}_k)$, is the free abelian group generated by the isomorphism classes of k-varieties, modulo the subgroup generated by relations

$$[X] = [U] + [Z]$$

where $Z \hookrightarrow X$ is a closed immersion and $U := X \setminus Z$ is the open complement. The fiber product of varieties, with the reduced structure, induces a ring structure, and the unit is the class of $\operatorname{Spec} k$.

Let χ be a function on the isomorphism classes of varieties into an abelian group, such that $\chi(X) = \chi(U) + \chi(Z)$, for all X, U, and Z as above. Important examples of such functions include the Euler characteristic, the Hodge-Deligne polynomial, the Hasse-Weil zeta function, the Kapranov motivic zeta function, and point counting measures. The ring $K_0(\mathsf{Var}_k)$ is universal amongst all such functions χ. The Grothendieck ring is also of interest in birational geometry: Larsen and Lunts show a certain quotient of $K_0(\mathsf{Var}_k)$ detects stably birational equivalence classes ([1]).

However, the ring structure of $K_0(\mathsf{Var}_k)$ is complicated and difficult to understand. For example, Poonen shows $K_0(\mathsf{Var}_k)$ is not a domain if the characteristic of k is zero ([2]). If $k = \mathbb{C}$, then Borisov shows specifically that the class of the affine line is a zero divisor ([3]). On the other hand, if k is algebraically closed of characteristic zero, then Bittner ([4]) gives a convenient presentation of the group as generated by classes of smooth projective varieties modulo the subgroup generated by relations $[\operatorname{Bl}_Y X] - [E] = [X] - [Y]$, whenever $\operatorname{Bl}_Y X$ is the blowup of a subvariety Y at X with exceptional fiber E. We refer to [5] for an excellent introduction to the above theorems.

As with the case of the Grothendieck group of the exact category of vector bundles, we desire a higher K-theory of varieties. However, there is an immediate obstruction and we cannot proceed as usual: the category of varieties is not exact, let alone additive — there is no natural addition of morphisms, and while products and coproducts (disjoint union) exist, there are no biproducts. Others have overcome these difficulties and successfully define a higher K-theory of varieties. In [6], Zakharevich defines a object called an assembler and constructs a functor that assigns a spectrum to each assembler. It turns out the category of k-varieties is an assembler, and she constructs a spectrum whose zeroth homotopy group is $K_0(\mathsf{Var}_k)$. Campbell takes a similar approach in [7]. He defines the formalism of a SW-category and adapts the Waldhausen $S_\bullet$-construction to SW-categories. He applies this to the category of varieties and also constructs a spectrum whose zeroth homotopy group is also $K_0(\mathsf{Var}_k)$. In [8], Campbell and Zakharevich construct a more general CGW-category which generalize both exact categories and the category of varieties. This machinery is used to show there is a weak equivalence between the two spectra constructed above.

In this thesis, we pursue a different strategy to define higher K-theory of varieties. The key observation motivating our approach is: while $K_0(\mathsf{Var}_k)$ is named the "Grothendieck group of varieties", it is not the Grothendieck group associated with an exact category (see Definition 2.2.28). Therefore our strategy is to find an exact category $\mathcal{C}$ such that its Grothendieck group, $K_0(\mathcal{C})$, is isomorphic to $K_0(\mathsf{Var}_k)$. In this way, we will have access to all the existing K_0-theory framework developed by Quillen and Waldhausen (see [9], [10] for a survey). However, as we observed, there are obstructions. The first obstruction is there is no natural addition of morphisms in Var_k. The second obstruction is biproducts do not exist. We address these issues by applying the following two universal constructions from category theory. The first construction takes a category and associates to it a preadditive category. This repairs the first problem. The second construction takes a preadditive category and produces an additive category. This repairs the second problem. Both constructions also have universal properties, detailed in Section 2.2.1. We subsequently apply these constructions to Var_k and obtain an additive category $\mathrm{Add}(\mathsf{Var}_k)$, which we view with the split exact structure. We note, in $K_0(\mathrm{Add}(\mathsf{Var}_k))$, biproducts split: $[X \oplus Y] = [X] + [Y]$. However, the group $K_0(\mathrm{Add}(\mathsf{Var}_k))$ lacks the desired universal relations $[X] = [U] + [Z]$. To overcome this difficulty, we forcibly make X and $U \oplus Z$ isomorphic by localizing $\mathrm{Add}(\mathsf{Var}_k)$ at a appropriate localizing set S. We put S the smallest set of morphisms in $\mathrm{Add}(\mathsf{Var}_k)$, which

1. contains isomorphisms and morphisms $U \oplus Z \to X$ given by the open immersion and the closed immersion, and

2. is closed under direct sum and compositions.

Thus, in the Grothendieck group of the localized category $\mathrm{Add}(\mathsf{Var}_k)_S$, we have the desired relation:

$$[X] = [U \oplus Z] = [U] + [Z].$$

To establish some desirable properties of the Grothendieck group of $\mathrm{Add}(\mathsf{Var}_k)_S$, we restrict to working with the faithful subcategory of k-varieties and locally closed immersions. Lastly, we remark the construction is flexible and applies to other categories such as schemes, the category Var_k^n of at most n-dimensional varieties, topological spaces, or even sets.

The main results of this thesis are listed below.

1. The localizing set S described above is a left multiplicative system (Corollary 3.2.7). This gives the localized category $\mathrm{Add}(\mathsf{Var}_k)_S$ a more tractable construction, and it follows that $\mathrm{Add}(\mathsf{Var}_k)_S$ is additive (Proposition 2.2.24).

2. There are natural homomorphisms

$$K_0(\mathrm{Add}(\mathsf{Var}_k^{n-1})_S) \to K_0(\mathrm{Add}(\mathsf{Var}_k^n)_S)$$

induced by inclusion, and we show the cokernel is the free abelian group of n-dimensional birational classes (Theorem 4.3.5).

3. There is a surjective ring homomorphism

$$K_0(\mathsf{Var}_k) \to K_0(\mathrm{Add}(\mathsf{Var}_k)_S) \tag{1.1}$$

(Proposition 5.0.1).

4. If we restrict to zero dimensional varieties over an algebraically closed field, then the surjection in Eq. (1.1) is an isomorphism (Proposition 5.0.3).

5. We construct the following motivic measures on $K_0(\mathrm{Add}(\mathsf{Var}_k)_S)$.

   (a) If k is a finite field, then we construct the point counting measures (Example 6.0.4), the Hasse-Weil zeta function (Example 6.0.7), and compactly supported Euler characteristic given by ℓ-adic étale cohomology (Proposition 6.0.11).

   (b) If $k = \mathbb{C}$, then we construct the compactly supported Euler characteristic given by singular cohomology (Theorem 6.0.12).

This thesis is organized as follows. We begin in Chapter 2 with some of the required background from category theory. Topics we review include: the free preadditive and additive categorical constructions, localization of categories, exact categories, (symmetric) monoidal structures, and the Ax–Grothendieck theorem.

Next in Chapter 3, we apply the free additive construction to the categories of varieties and schemes, and study the resulting K_0-theory. We also construct the left multiplicative system S described above and form the localization.

In Chapter 4, we consider two direct systems, indexed by dimension, whose direct limits are $K_0(\mathsf{Var}_k)$ and $K_0(\mathrm{Add}(\mathsf{Var}_k)_S)$. We compute the cokernels of the connecting homomorphisms.

In Chapter 5, we compare $K_0(\mathsf{Var}_k)$ and $K_0(\mathrm{Add}(\mathsf{Var}_k)_S)$. We establish the homomorphism in Eq. (1.1) and show it is a surjection. If the base field is algebraically closed, then we show the surjection is an isomorphism in dimension zero. We indicate a possible strategy to prove the surjection is an isomorphism in all dimensions.

In Chapter 6, we recall the definition of motivic measure, and construct the motivic measures listed above.

2. BACKGROUND

This chapter contains the background for this thesis. We begin with some notations and definitions. Next, we proceed to a detailed review of some elements of category theory, with an emphasis on the localization of a category at a multiplicative system and free (pre)additive categories. We continue with some details on how localization of categories and free additive categories respect symmetric monoidal structures. Finally, we record the Ax-Grothendieck theorem for later use.

2.1 Notation and conventions

In this section, we fix notation and recall definitions for use throughout the rest of this thesis.

Notation 2.1.1.

1. If $X \subseteq Y$ is a locally closed subvariety, then $i_X \colon X \hookrightarrow Y$ denotes the inclusion map.

2. (a) If S is a set, then $\mathbb{Z}[S]$ denotes the free abelian group generated by the elements of S.

   (b) If $\mathcal{C}$ is a (essentially small) category, then $\mathbb{Z}[\mathcal{C}]$ denotes the free abelian group generated by the isomorphism classes of the objects of $\mathcal{C}$.

3. Let S be a set (S will usually be a set of morphisms in a category). We shall assume every element f in $\mathbb{Z}[S]$ is written as $\sum n_i f_i$ in *reduced form*, meaning $f_i \neq f_j$ for every $i \neq j$ and all n_i are nonzero integers.

4. If $\mathcal{C}$ is a category then $\mathrm{Ob}(\mathcal{C})$ denotes the objects of $\mathcal{C}$.

5. If $\mathcal{C}$ is a category and x an object in $\mathcal{C}$, then id_x or 1_x denotes the identity map on x. We occasionally drop the subscript if clear from context.

6. If X is a finite set, then $\#X$ is the cardinality of X.

7. Let k be a field. A k-variety is a separated reduced scheme of finite type over $\operatorname{Spec} k$.

Definition 2.1.2. Let $n \in \mathbb{N} \cup \{\infty\}$.

1. Let Sch_T denote the category of schemes of finite type over a noetherian base scheme T and morphisms of finite presentation. We usually suppress the base scheme and write Sch.

2. Let Var_k^n denote the category of varieties over a field k of dimension at most n and morphisms given by locally closed immersions. We usually suppress the base field k. If $n = \infty$ we simply write Var_k or Var. Note Var_k^n is a faithful subcategory of the usual category of varieties and regular maps.

3. Let $\mathsf{Var}_k^{=n}$ denote the category of varieties of dimension exactly n and locally closed immersions. We usually suppress the base field k and write $\mathsf{Var}^{=n}$.

4. Let FinSet denote the category of finite sets. For k an algebraically closed field, we identify Var_k^0 with FinSet.

5. The prefix Irr indicates the full subcategory of irreducible varieties. For instance, $\mathsf{IrrVar}^{=n}$ denotes the full subcategory of $\mathsf{Var}^{=n}$ whose objects are irreducible n-dimensional varieties.

6. Let Bir^n denote the category of irreducible varieties of dimension n and birational maps. If X is a n-dimensional irreducible variety, then $\{X\}_n$ (or just $\{X\}$ if clear for context) denotes the birational class of X. Note the set of isomorphism classes in Bir^n is the set of n-dimensional birational classes.

7. Let

$$\iota_n \colon \mathsf{IrrVar}^{=n} \to \mathsf{Bir}^n,$$

   be the functor which is the identity on objects and morphisms. Indeed, every locally closed immersion of n-dimensional irreducible varieties is an open immersion.

Definition 2.1.3. Let $n \in \mathbb{N} \cup \{\infty\}$. Let k be a field. The *(classical) Grothendieck group of (at most n-dimensional) varieties* $K_0(\mathsf{Var}_k^n)$ is the free abelian group generated by isomorphism classes of varieties in Var_k^n modulo the subgroup generated by the relations

$$[X] - [U] - [Z],$$

whenever Z is a closed subvariety of $X \in \mathsf{Var}_k^n$ and U is the open complement. Such relations are known as *scissor relations*. If $n = \infty$, we may also write $K_0(\mathsf{Var}_k)$ or $K_0(\mathsf{Var})$, depending on context.

2.2 Review of selected topics from category theory

In this section, we give a detailed review of selected topics from category theory to be used later.

2.2.1 Preadditive and additive categories

We recall the definitions and properties of preadditive and additive categories and direct sums. See [11, page 198, #5, #6] and [Stacks, Section 09SE] for a reference.

Definition 2.2.1. A *preadditive category* is a category $\mathcal{C}$ where each set of morphisms is an abelian group, and the compositions

$$\mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$

are bilinear, for all objects x, y, and z in $\mathcal{C}$.

Definition 2.2.2. Let $x_1, \cdots, x_n$ be objects in a preadditive category $\mathcal{C}$. A *(finite) biproduct* or *direct sum* of $x_1, \cdots, x_n$ is the following data:

1. an object x in $\mathcal{C}$,
2. morphisms $i_j \colon x_j \to x$, for all $1 \le j \le n$, called *inclusions*,
3. morphisms $p_j \colon x \to x_j$, for all $1 \le j \le n$, called *projections*,

such that

1. $p_j \circ i_j = \mathrm{id}_{x_j}$, for all $1 \le j \le n$,

2. $p_j \circ i_k = 0$, for all $j \ne k$ between 1 and n, and

3. $i_1 p_1 + \cdots + i_n p_n = \mathrm{id}_x$.

See [Stacks, Tag 0102, 0103].

Remark 2.2.3. Let x be the direct sum in the notation of Definition 2.2.2. The direct sum carries a product and coproduct structure as follows.

For the product structure, let z be an object and let $f_j \colon z \to x_j$ be morphisms. Then $\sum i_j \circ f_j \colon z \to x$ is the unique morphism that commutes with the f_j and p_j.

For the coproduct structure, let w be an object and let $g_j \colon x_j \to w$ be morphisms. Then $\sum g_j \circ p_j \colon x \to w$ is the unique morphism that commutes with the g_j and i_j.

Remark 2.2.4. Let $\mathcal{C}$ be a preadditive category. We show morphisms between two direct sums can be expressed as a matrix of morphisms. Let X be a direct sum of $X_1, \cdots, X_n \in \mathcal{C}$ with inclusions $i_1, \cdots, i_n$ and projections $p_1, \cdots, p_n$. Let Y be a direct sum of $Y_1, \cdots, Y_m \in \mathcal{C}$ with inclusions $j_1, \cdots, j_m$ and projections $q_1, \cdots, q_m$. The set of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and the set of m by n matrices $\{(f_{lk}) \mid f_{lk} \in \mathrm{Hom}_{\mathcal{C}}(X_k, Y_l)\}$ are in bijection. The bijection is follows.

Given a morphism $F \colon X \to Y$, we form the m by n matrix whose entries are the composites

$$X_k \xrightarrow{i_k} X \xrightarrow{F} Y \xrightarrow{q_l} Y_l.$$

Conversely, suppose we are given a matrix (f_{lk}). We use the coproduct structure on X to define F. The morphisms

$$\sum_l j_l f_{lk} \colon X_k \to Y$$

define the morphism

$$\sum_{k,l} (j_l f_{lk}) \circ p_k \colon X \to Y,$$

for each $1 \leq k \leq n$.

It follows from the definition of direct sum that these two operations are inverses. Indeed, starting with a morphism $F \colon X \to Y$, we have the matrix whose entries are $q_l \circ F \circ i_k$. This matrix defines the morphisms

$$\sum_l j_l \circ q_l \circ F \circ i_k \colon X_k \to Y.$$

In turn, these morphisms define the morphism

$$\sum_{k,l} j_l q_l \circ F \circ i_k p_k \colon X \to Y.$$

On the other hand,

$$\left(\sum_l j_l q_l \right) \circ F \circ \left(\sum_k i_k p_k \right) = \mathrm{id}_Y \circ F \circ \mathrm{id}_X = F.$$

Conversely, begin with a matrix (f_{lk}). This defines the morphism

$$\sum_{k,l} j_l \circ f_{lk} \circ p_k \colon X \to Y.$$

The $k'l'$ entry of this matrix is

$$q_{l'} \circ \left(\sum_{k,l} j_l \circ f_{lk} \circ p_k \right) \circ i_{k'} = f_{l'k'},$$

as desired.

In view of this bijection, we implicitly identify a morphism between direct sums with its associated matrix of morphisms. We refer to the entries f_{lk} of the associated matrix as *components (of f)*. Under this bijection, composition of morphisms corresponds with matrix multiplication. In particular, suppose the morphisms $F\colon X \to Y$ and $G\colon Y \to Z$ are represented by matrices A and B. Then $G \circ F\colon X \to Z$ is represented by the matrix BA.

Definition 2.2.5. An *additive category* is a preadditive category $\mathcal{C}$ that admits a zero object and all finite biproducts.

Convention 2.2.6. Let $\mathcal{C}$ be an additive category. For every finite ordered set of objects of $\mathcal{C}$, we assume $\mathcal{C}$ comes with a fixed choice of direct sum. That is, if $\{X_1, \cdots, X_n\} \subseteq \mathcal{C}$ is a finite ordered set, there is a canonical direct sum of $X_1, \cdots, X_n$, which we denote by $X_1 \oplus \cdots \oplus X_n$ or $\bigoplus_i X_i$ (leaving the set and ordering implicit).

Definition 2.2.7. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between preadditive categories. We say F is *additive* if each homomorphism $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathcal{D}}(FC_1, FC_2)$ is a group homomorphism.

There is an equivalent definition of an additive functor as follows ([Stacks, Tag 0DLP]). Let x and y be objects in $\mathcal{C}$ and let $z = x \oplus y$. There are natural morphisms $F(x) \oplus F(y) \to F(z)$ and $F(z) \to F(x) \oplus F(y)$, induced by applying F to the inclusions and projections of z. Then F is additive if and only if these natural morphisms are isomorphisms, for all such x, y, and z in $\mathcal{C}$. In particular, additive functors send direct sums to direct sums.

Definition 2.2.8. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be preadditive categories. We say a bifunctor $F\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is *additive in each factor* if,

1. for all objects $A \in \mathcal{A}$, the functor $F(A, -) : \mathcal{B} \to \mathcal{C}$ is additive, and
2. for all objects $B \in \mathcal{B}$, the functor $F(-, B) : \mathcal{A} \to \mathcal{C}$ is additive.

2.2.2 Free preadditive and additive categories

We recall the universal construction of the free preadditive category associated with a category and the free additive category associated with a preadditive category.

Definition 2.2.9. Given a category $\mathcal{A}$, the *free preadditive category* $\mathbb{Z}(\mathcal{A})$ is the following category.

1. The objects of $\mathbb{Z}(\mathcal{A})$ are the same as the objects of $\mathcal{A}$:

$$\operatorname{Ob}\mathbb{Z}(\mathcal{A}) \coloneqq \operatorname{Ob}(\mathcal{A}).$$

2. The morphisms of $\mathbb{Z}(\mathcal{A})$ are the free abelian groups generated by morphisms in $\mathcal{A}$:

$$\operatorname{Hom}_{\mathbb{Z}(\mathcal{A})}(X, Y) \coloneqq \mathbb{Z}[\operatorname{Hom}_{\mathcal{A}}(X, Y)].$$

Remark 2.2.10 (Universal property of $\mathbb{Z}(\mathcal{A})$)**.** Let $i\colon \mathcal{A} \to \mathbb{Z}(\mathcal{A})$ be the faithful functor which is the identity on objects, and on morphisms, the natural inclusion of a set into the free abelian group on that set.

The functor i admits the following universal property: given a functor $F\colon \mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ is a preadditive category, there exists a unique additive functor $G\colon \mathbb{Z}(\mathcal{A}) \to \mathcal{B}$ such that $Gi = F$. Indeed, G is same as F on objects and sends a morphism $\sum_i n_i f_i \in \operatorname{Hom}_{\mathcal{A}}(X, Y)$ to $\sum_i n_i F f_i \in \operatorname{Hom}_{\mathcal{B}}(FX, FY)$. Uniqueness is clear.

Remark 2.2.11. In view of the universal property, $\mathbb{Z}(-)$ is a functor from the category of categories and functors into the category of preadditive categories and additive functors.

Definition 2.2.12. Given a preadditive category $\mathcal{B}$, the *free additive category* $\operatorname{Add}(\mathcal{B})$ is the following category.

1. The objects of $\operatorname{Add}(\mathcal{B})$ are n-tuples of objects in $\mathcal{B}$ for each $n \geq 0$:

$$\operatorname{Ob}(\operatorname{Add}(\mathcal{B})) \coloneqq \bigcup_{n \geq 0} \{(X_1, \cdots, X_n) \mid X_1, \cdots, X_n \in \mathcal{B}\}.$$

2. The morphisms of $\operatorname{Add}(\mathcal{B})$ are matrices of morphisms in $\mathcal{B}$:

$$\operatorname{Hom}_{\operatorname{Add}(\mathcal{B})}((X_1, \cdots, X_n), (Y_1, \cdots, Y_m)) \coloneqq \{(f_{ji})_{1 \leq i \leq n, 1 \leq j \leq m} \mid f_{ji} \in \operatorname{Hom}_{\mathcal{B}}(X_i, Y_j)\}.$$

More explicitly, the jth row corresponds to the morphisms into Y_j and the ith column corresponds to the morphisms out of X_i. Composition is given by matrix multiplication, which makes sense because $\mathcal{B}$ is preadditive.

The tuple of length 0 is called the empty tuple (sometimes denoted $\emptyset$), and it is the distinguished zero object in $\mathrm{Add}(\mathcal{B})$. Given r many tuples $(X_{1,1}, \cdots, X_{1,n_1}), \cdots, (X_{r,1}, \cdots, X_{r,n_r})$ in $\mathcal{B}$, a direct sum is the concatenation of all tuples $(X_{1,1}, \cdots, X_{1,n_1}, \cdots, X_{r,1}, \cdots, X_{r,n_r})$. In view of Convention 2.2.6, we shall fix this choice of direct sum.

Remark 2.2.13 (Universal property of $\mathrm{Add}(\mathcal{B})$)**.** Let $i' \colon \mathcal{B} \to \mathrm{Add}(\mathcal{B})$ be the fully faithful functor which sends an object of $\mathcal{B}$ to the 1-tuple whose entry is that object, and sends an arrow to the 1 by 1 matrix whose entry is that arrow.

The functor i' admits the following universal property: if $\mathcal{C}$ is an additive category and $F' \colon \mathcal{B} \to \mathcal{C}$ an additive functor, then there exists an additive functor $G' \colon \mathrm{Add}(\mathcal{B}) \to \mathcal{C}$ such that $G'i' = F'$. Indeed, on objects, G' sends a tuple $(X_1, \cdots, X_n)$ to the fixed direct sum $\bigoplus_i F'X_i$ in $\mathcal{C}$. The empty tuple is sent to a zero object of $\mathcal{C}$. On morphisms, G' sends a morphism represented by the matrix (f_{ji}) to the morphism in $\mathcal{C}$ represented by the matrix $(G'(f_{ji}))$ (recall Remark 2.2.4).

Definition 2.2.14. If $\mathcal{C}$ is a category, we form the additive category $\mathrm{Add}(\mathbb{Z}(\mathcal{C}))$, or just $\mathrm{Add}(\mathcal{C})$ for short. We say $\mathrm{Add}(\mathcal{C})$ is the *additive completion* of $\mathcal{C}$.

Remark 2.2.15. Let $\mathcal{C}$ be a category. There is a functor $|{-}|$ from $\mathrm{Add}(\mathcal{C})$ into the category whose objects are Euclidean spaces and whose morphisms are matrices. For $X = (X_1, \cdots, X_n)$, declare $|X| = \mathbb{R}^n$. If $f = (f_{ji})$ is a morphism in $\mathrm{Add}(\mathcal{C})$, then $|f|$ is the integer valued matrix whose entry $|f|_{ji}$ is the sum of the coefficients of f_{ji}.

Note the functor $|{-}|$ may also be obtained in the following way. Let Vect denote the category of pairs $(V, \mathcal{B})$ where V is a real vector space with basis $\mathcal{B}$, and whose morphisms are real-valued matrices. Let $F \colon \mathcal{C} \to \mathsf{Vect}$ be the constant functor which sends every object to the pair $(\mathbb{R}, \{1\})$ and sends every morphism to the 1 by 1 identity matrix. Applying the universal properties in Remarks 2.2.10 and 2.2.13 yields the functor $|{-}|$.

Lemma 2.2.16. *Let $\mathcal{C}$ be a category. In $\mathrm{Add}(\mathcal{C})$, let $X = (X_1, \cdots, X_r)$ and $Y = (Y_1, \cdots, Y_s)$ be two tuples. If $A\colon X \to Y$ and $B\colon Y \to X$ are inverse morphisms, then $r = s$, and the matrices $|A|$ and $|B|$ are invertible.*

Proof. Apply the functor $|{-}|$ to see $|A|\colon \mathbb{R}^r \to \mathbb{R}^s$ and $|B|\colon \mathbb{R}^s \to \mathbb{R}^r$ are invertible real valued matrices, and $r = s$. $\square$

2.2.3 Localization of categories

We give an overview of localizations of categories, mainly to establish notation and record lemmas for later use. We refer the reader to [12, Chapter 7], [13, Chapter 2], [9, Chapter II, Appendix], or [Stacks, Tag 04VB] for details on localizing a category. In this section, let $\mathcal{C}$ be a category and S be a set of morphisms in $\mathcal{C}$.

We recall the following theorem:

Theorem 2.2.17 ([14, Chapter 1])**.** *There exists a category $\mathcal{C}_S$ (also written $\mathcal{C}[S^{-1}]$), and a functor $Q\colon \mathcal{C} \to \mathcal{C}_S$ with the following universal property: given a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in S$, there is a functor $G\colon \mathcal{C}_S \to \mathcal{D}$ such that $GQ = F$.*

If S is a "left multiplicative system" (defined below), then $\mathcal{C}_S$ has a particularly tractable construction, which we spend the rest of the section recalling.

Definition 2.2.18. S is a *left multiplicative system* if the following hold:

1. S is closed under composition and contains all identity morphisms.

2. S is a *left Ore set*. That is, given a diagram

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{C}$ with $s \in S$, there exist morphisms $t\colon W \to X$ and $W \to Z$ with $t \in S$ such that the diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "t"] & Z \arrow[d, "s"] \\ X \arrow[r] & Y \end{tikzcd}$$

commutes.

3. S is *left cancellative*. That is, given a commutative diagram in $\mathcal{C}$,

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{s} Z$$

with $s \in S$, there exists a morphism $t\colon W \to X$ in S such that $ft = gt$.

In [Stacks], a "right multiplicative system" is a "left multiplicative system" in our notation.

Definition 2.2.19. For $X, Y \in \mathcal{C}$, a *left roof (from X to Y)* is a pair of morphisms $X \leftarrow Z \to Y$ where $Z \to X$ lies in S. Two left roofs $X \xleftarrow{f} Z \xrightarrow{g} Y$ and $X \xleftarrow{f'} Z' \xrightarrow{g'} Y$ are *equivalent* if there is a third left roof $X \xleftarrow{f''} Z'' \xrightarrow{g''} Y$ and morphisms $u\colon Z'' \to Z$ and $v\colon Z'' \to Z'$ such that the following diagram commutes:

$$\begin{tikzcd}
 & Z' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\
X & Z'' \arrow[l, "f''"] \arrow[r, "g''"'] \arrow[u, "v"] \arrow[d, "u"'] & Y \\
 & Z \arrow[ul, "f"] \arrow[ur, "g"'] &
\end{tikzcd}$$

It is true that the above equivalence is an equivalence relation on the set of left roofs.

Definition 2.2.20. When S is a left multiplicative system, let $\mathcal{C}_S = \mathcal{C}[S^{-1}]$ be the category whose objects are the same as $\mathcal{C}$, and whose morphisms are equivalence classes of left roofs. There is a well-defined composition given by the Ore condition. Let $Q\colon \mathcal{C} \to \mathcal{C}_S$ be the functor which is the identity on objects, and sends a morphism $X \to Y$ to the class of $X \xleftarrow{\mathrm{id}} X \to Y$. Each object in $\mathcal{C}_S$ may be written as QX for an object $X \in \mathcal{C}$. If $s\colon X \to Y \in S$, then the inverse of $Q(s)$ is $Y \xleftarrow{s} X \xrightarrow{\mathrm{id}} X$. The class of a left roof $X \xleftarrow{s} Y \xrightarrow{f} Z$ may be written as $Q(f) \circ Q(s)^{-1}$.

The following lemma characterizes isomorphisms in a localization.

Lemma 2.2.21 ([12, Proposition 7.1.20])**.** *Let S be a left multiplicative system. Let $P \leftarrow Z \xrightarrow{f} W$ be a left roof. Then the class of left roof $P \leftarrow Z \xrightarrow{f} W$ is an isomorphism if and only if there exist morphisms $h\colon X \to Y$ and $g\colon Y \to Z$ such that fg and gh are in S.*

The following lemma allows us to obtain "a common denominator" of left roofs.

Lemma 2.2.22 ([Stacks, Tag 04VI])**.** *Let $\mathcal{C}$ be a category and S a left multiplicative system. Given finitely many morphisms $g_i\colon X \to Y_i$ in $\mathcal{C}_S$, there exists a morphism $s\colon X' \to X$ in S and morphisms $f_i\colon X' \to Y_i$ such that each g_i is represented by $X \xleftarrow{s} X' \xrightarrow{f_i} Y_i$.*

2.2.4 Localization of preadditive and additive categories

If $\mathcal{C}$ is (additive) preadditive and S is a left multiplicative system, then localization respects the (additive) preadditive structure of $\mathcal{C}$. We recall some details.

Proposition 2.2.23 ([Stacks, Tag 05QC])**.** *If $\mathcal{C}$ is a preadditive category, and S is a left multiplicative system, then there is a canonical preadditive structure on $\mathcal{C}_S$ such that the localization functor $Q\colon \mathcal{C} \to \mathcal{C}_S$ is additive.*

The addition is defined as follows. Suppose $g_1, g_2\colon X \to Y$ are two morphisms in $\mathcal{C}_S$. Let s, f_1, f_2 be as in Lemma 2.2.22. Define $g_1 + g_2$ to be represented by the left roof $X \xleftarrow{s} X' \xrightarrow{f_1+f_2} Y$.

Proposition 2.2.24 ([Stacks, Tag 05QE])**.** *If $\mathcal{C}$ is additive and S is a left multiplicative system, then $\mathcal{C}_S$ is an additive category, and $Q\colon \mathcal{C} \to \mathcal{C}_S$ is an additive functor. The distinguished zero object is $Q0$. The direct sum of $QX_1, \cdots, QX_n$ is $Q(X_1 \oplus \cdots \oplus X_n)$.*

2.2.5 Exact categories

We do not recall the definition of exact category and exact functor, but instead refer the reader to standard references, such as [15] or [9, Section II.7].

Example 2.2.25. Every additive category admits the structure of an exact category by declaring a sequence $X \to Y \to Z$ to be exact if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\cong"] & Y \arrow[r] \arrow[d, "\cong"] & Z \arrow[d, "\cong"] \\ A \arrow[r, "i_A"] & A \oplus B \arrow[r, "p_B"] & B \end{tikzcd} \tag{2.1}$$

where the vertical morphisms are isomorphisms, and i_A and p_B are the canonical inclusion and projection of $A \oplus B$. This exact structure is called the *split exact structure*.

Convention 2.2.26. If $\mathcal{C}$ is additive and the exact structure on $\mathcal{C}$ is unspecified, we implicitly assume $\mathcal{C}$ carries the split exact structure.

Remark 2.2.27. If $\mathcal{C}$ and $\mathcal{D}$ are additive categories with the split exact categories and $F\colon \mathcal{C} \to \mathcal{D}$ is an additive functor, then F is exact.

2.2.6 K_0 of an exact category and of an additive completion

We recall the definition of the Grothendieck group of an exact category. We also compute the Grothendieck group of the additive completion of a category with a special property. We shall see Var_k^n has the special property in Chapter 3.

Definition 2.2.28. Let $\mathcal{C}$ be an exact category. The *Grothendieck group* $K_0(\mathcal{C})$ is the free abelian group on the isomorphism classes of $\mathcal{C}$ modulo the subgroup generated by relations $[Y] - [X] - [Z]$, whenever $X \to Y \to Z$ is an exact sequence.

If $F\colon \mathcal{C} \to \mathcal{D}$ is an exact functor between exact categories, then there is a natural group homomorphism

$$K_0F\colon K_0\mathcal{C} \to K_0\mathcal{D}$$

which sends the class of X to the class of $F(X)$.

In this way, K_0 is a functor from the category of exact categories and exact functors into the category of abelian groups.

Definition 2.2.29. We say a category $\mathcal{C}$ has the *permutation isomorphism property* if, given two isomorphic tuples $(X_1, \cdots, X_r)$ and $(Y_1, \cdots, Y_s)$ in $\mathrm{Add}(\mathcal{C})$, then $r = s$ and there is a permutation σ on r letters and isomorphisms $X_i \cong Y_{\sigma(i)}$ in $\mathcal{C}$, for all $1 \leq i \leq r$.

Proposition 2.2.30. *Let $\mathcal{C}$ be a category with the permutation isomorphism property. Then there is a group isomorphism*

$$\alpha\colon K_0(\mathrm{Add}(\mathcal{C})) \to \mathbb{Z}[\mathcal{C}]$$

which sends the class of the tuple $(X_1, \cdots, X_r)$ to $\sum [X_i]$.

Proof. The homomorphism α is well-defined on the isomorphism classes of $\mathrm{Add}(\mathcal{C})$ by the permutation isomorphism property. Also, α respects the relation: given an exact sequence $X \to Y \to Z$ in $\mathrm{Add}(\mathcal{C})$, in the notation of Eq. (2.1), the permutation isomorphism property implies

$$\alpha(Y) = \alpha(A \oplus B) = \alpha(A) + \alpha(B) = \alpha(X) + \alpha(Z).$$

Hence α is a well-defined surjective map. The left inverse is the map $\beta\colon \mathbb{Z}[\mathcal{C}] \to K_0(\mathrm{Add}(\mathcal{C}))$ which sends the isomorphism class of X to the class of X in the Grothendieck group; this is independent of representative of isomorphism class. $\square$

2.3 Symmetric monoidal structures

We begin with technicalities of (symmetric) monoidal structures, and proceed to verify the free additive construction and localization at a left multiplicative system respect the (symmetric) monoidal structure. We refer to Section 1 and 7 in Chapter 7 of [11] or the monoidial category article at [nLab].

Definition 2.3.1. A category M with

1. a bifunctor $\square\colon M \times M \to M$,
2. a unit object $1 \in M$,
3. a natural isomorphism a between the functors

$$(-\square-)\square- \quad \text{and} \quad -\square(-\square-)$$

from $M \times M \times M$ to M, whose components are denoted $a_{x,y,z}\colon (x\Box y)\Box z \to x\Box(y\Box z)$,

4. a natural isomorphism λ between the functors

$$1\Box - \quad \text{and} \quad \mathrm{id}_M$$

from M to M, whose components are denoted $\lambda_x\colon 1\Box x \to x$,

5. a natural isomorphism ρ between the functors

$$-\Box 1 \quad \text{and} \quad \mathrm{id}_M$$

from M to M, whose components are denoted $\rho_x\colon x\Box 1 \to x$,

is said to be *monoidal* if the following diagrams in M commute, for all objects of M:

1. (triangle equality)

$$\begin{tikzcd}
(x\Box 1)\Box y \arrow[r, "a_{x,1,y}"] \arrow[dr, "\rho_x\Box y"'] & x\Box(1\Box y) \arrow[d, "1\Box\lambda_y"] \\
 & x\Box y
\end{tikzcd} \tag{2.2}$$

2. (pentagon equality)

$$\begin{tikzcd}
 & (w\Box x)\Box(y\Box z) \arrow[dr, "a_{w,x,y\Box z}"] & \\
((w\Box x)\Box y)\Box z \arrow[ur, "a_{w\Box x,y,z}"] \arrow[d, "a_{w,x,y}\Box 1_z"] & & w\Box(x\Box(y\Box z)) \\
(w\Box(x\Box y))\Box z \arrow[rr, "a_{w,x\Box y,z}"] & & w\Box((x\Box y)\Box z) \arrow[u, "1_w\Box a_{x,y,z}"']
\end{tikzcd} \tag{2.3}$$

Definition 2.3.2. Let M be a monoidal category. If M is additionally equipped with a natural isomorphism B (called a *braiding*) between the functors

$$x, y \mapsto x\Box y \quad \text{and} \quad x, y \mapsto y\Box x$$

from $M \times M$ to M and the following hold, then M is said to be *symmetric monoidal*:

1. The equality
$$B_{y,x} \circ B_{x,y} = 1_{x\Box y} \tag{2.4}$$
holds for all x and y in M.

2. The equality $\rho_x = \lambda_x \circ B_{x,1}$ holds for all x in M:
$$\begin{tikzcd} x\Box 1 \arrow[r, "B_{x,1}"] \arrow[dr, "\rho_x"'] & 1\Box x \arrow[d, "\lambda_x"] \\ & x \end{tikzcd} \tag{2.5}$$

3. The following diagram, known as the hexagon equation, commutes, for all x, y, and z in M.
$$\begin{tikzcd} (x\Box y)\Box z \arrow[r, "a_{x,y,z}"] \arrow[d, "B_{x,y}\Box 1_z"'] & x\Box(y\Box z) \arrow[r, "B_{x,y\Box z}"] & (y\Box z)\Box x \arrow[d, "a_{y,z,x}"] \\ (y\Box x)\Box z \arrow[r, "a_{y,x,z}"'] & y\Box(x\Box z) \arrow[r, "1_y\Box B_{x,z}"'] & y\,\Box(z\Box x) \end{tikzcd} \tag{2.6}$$

2.3.1 Symmetric monoidal structure on free additive categories

We verify taking additive completion respects (symmetric) monoidal structures, by first defining the required bifunctor and then verifying the axioms.

Proposition 2.3.3. *Let M be a category and $\Box\colon M \times M \to M$ a bifunctor. There is a bifunctor $\Box\colon \mathbb{Z}(M) \times \mathbb{Z}(M) \to \mathbb{Z}(M)$ which extends $\Box$ on $M \times M$, and another bifunctor $\Box\colon \mathrm{Add}(M) \times \mathrm{Add}(M) \to \mathrm{Add}(M)$ which extends $\Box$ on $\mathbb{Z}(M) \times \mathbb{Z}(M)$. Both bifunctors $\Box$ on $\mathbb{Z}(M)$ and $\mathrm{Add}(M)$ are additive in each factor.*

Proof. We begin by defining $\Box\colon \mathbb{Z}(M) \times \mathbb{Z}(M) \to \mathbb{Z}(M)$ to be the same as $\Box\colon M \times M \to M$ on objects. If $\sum_{i=1}^{n} n_i f_i\colon X \to X'$ and $\sum_{j=1}^{m} m_j g_j\colon Y \to Y'$ are morphisms in $\mathbb{Z}(M)$, then declare
$$\left(\sum_{i=1}^{n} n_i f_i\right) \Box \left(\sum_{j=1}^{m} m_j g_j\right) := \sum_{i=1}^{n}\sum_{j=1}^{m} n_i m_j (f_i \Box g_j).$$

The functor $\square$ is additive on $\mathbb{Z}(M)$ in each factor: fix an object Z. Let $\sum_{j=1}^{m} m_j g_j \colon X \to X'$ be another morphism. Then, we compute:

$$\begin{aligned}\left(\sum_{i=1}^{n} n_i f_i + \sum_{j=1}^{m} m_j g_j\right) \square \operatorname{id}_Z &= \sum_{i=1}^{n} n_i (f_i \square \operatorname{id}_Z) + \sum_{j=1}^{m} m_j (g_j \square \operatorname{id}_Z) \\ &= \left(\sum_{i=1}^{n} n_i f_i\right) \square \operatorname{id}_Z + \left(\sum_{j=1}^{m} m_j g_j\right) \square \operatorname{id}_Z .\end{aligned}$$

A similar argument holds for the second factor.

To define $\square \colon \operatorname{Add}(M) \times \operatorname{Add}(M) \to \operatorname{Add}(M)$, put

$$\left(\bigoplus_i X_i\right) \square \left(\bigoplus_j Y_j\right) \coloneqq \bigoplus_{i,j} X_i \square Y_j,$$

which is the tuple whose elements are $X_i \square Y_j$ in dictionary order[1]. If either $\bigoplus_i X_i$ or $\bigoplus_j Y_j$ is the empty tuple, then we understand $\bigoplus_{i,j} X_i \square Y_j$ to be the empty tuple. Given two morphisms

$$f = (f_{ji} \colon X_i \to Y_j) \colon \bigoplus_i X_i \to \bigoplus_j Y_j \quad \text{and} \quad g = (g_{lk} \colon Z_k \to W_l) \colon \bigoplus_k Z_k \to \bigoplus_l W_l,$$

define the morphism

$$\left(\bigoplus_i X_i\right) \square \left(\bigoplus_j Y_j\right) = \bigoplus_{ij} X_i \square Y_j \to \bigoplus_{kl} Z_k \square W_l = \left(\bigoplus_k Z_k\right) \square \left(\bigoplus_l Z_l\right)$$

by the matrix whose entries are $f_{ji} \square g_{lk} \colon X_i \square Y_j \to Z_k \square W_l$, where $\square$ is on $\mathbb{Z}(M)$. The functor $\square$ is additive in each factor: fix an object V. Then the matrices of the morphisms $(f+f') \square \operatorname{id}_V$ and $(f \square \operatorname{id}_V) + (f' \square \operatorname{id}_V)$ have the entries $(f_{ji} + f'_{ji}) \square \operatorname{id}_V$ and $f_{ji} \square \operatorname{id}_V + f'_{ji} \square \operatorname{id}_V$, which are the same by definition of $\square$ on $\mathbb{Z}(M)$. $\square$

[1]↑ Given two ordered sets $(X_1, \leq_1)$ and $(X_2, \leq_2)$, the *dictionary order* $\leq_3$ on the product set $X_1 \times X_2$ is defined by $(x_1, x_2) \leq_3 (x'_1, x'_2)$ if and only either $x_1 <_1 x_2$, or $x_1 = x_2$ and $x'_1 \leq_2 x'_2$.

Proposition 2.3.4. *Let M be a (symmetric) monoidal category with product $\square\colon M \times M \to M$. The category $\mathrm{Add}(M)$, with the bifunctor $\square\colon \mathrm{Add}(M) \times \mathrm{Add}(M) \to \mathrm{Add}(M)$ defined in Proposition 2.3.3, has the structure of a (symmetric) monoidal category.*

Proof. In $\mathrm{Add}(M)$, let $x = \bigoplus_i x_i$, $y = \bigoplus_j y_j$, $z = \bigoplus_k z_k$, and $w = \bigoplus_l w_l$. The unit object is $1 \in M \subseteq \mathrm{Add}(M)$. The natural isomorphism $a\colon \mathrm{Add}(M) \times \mathrm{Add}(M) \times \mathrm{Add}(M) \to \mathrm{Add}(M)$ is defined by the components $a_{x,y,z}\colon (x\square y)\square z \to x\square(y\square z)$, which is the morphism

$$\bigoplus_{ijk} a_{x_i,y_j,z_k}\colon \bigoplus_{ijk}(x_i\square y_j)\square z_k \xrightarrow{\cong} \bigoplus_{ijk} x_i\square(y_j\square z_k).$$

The natural isomorphism $\lambda\colon \mathrm{Add}(M) \to \mathrm{Add}(M)$ is defined by the components $\lambda_x\colon 1\square x \to x$, which is the morphism

$$\bigoplus_i \lambda_{x_i}\colon \bigoplus_i 1\square x_i \xrightarrow{\cong} \bigoplus_i x_i.$$

The natural isomorphism $\rho\colon \mathrm{Add}(M) \to \mathrm{Add}(M)$ is defined by the components $\rho_x\colon x\square 1 \to x$, which is the morphism

$$\bigoplus_i \rho_{x_i}\colon \bigoplus_i x_i\square 1 \xrightarrow{\cong} \bigoplus_i x_i.$$

The following triangle diagram commutes by Eq. (2.2):

$$\begin{tikzcd}
\bigoplus_{ij}(x_i\square 1)\square y_j \arrow[r, "\bigoplus_{ij} a_{x_i,1,y_j}"] \arrow[dr, "\bigoplus_{ij} \rho_{x_i}\square y_j"'] & \bigoplus_{ij} x_i\square(1\square y_j) \arrow[d, "\bigoplus_{ij} 1_{x_i}\square\lambda_{y_j}"] \\
& \bigoplus_{ij} x_i\square y_j
\end{tikzcd}$$

The pentagon diagram commutes by Eq. (2.3):

$$\begin{tikzcd}
 & & \bigoplus_{lijk}(w_l\Box x_i)\Box(y_j\Box z_k) \arrow[drr, "\bigoplus_{lijk} a_{w_l,x_i,y_j\Box z_k}"] & & \\
\bigoplus_{lijk}((w_l\Box x_i)\Box y_j)\Box z_k \arrow[urr, "\bigoplus_{lijk} a_{w_l\Box x_i,y_j,z_k}"] \arrow[d, "\bigoplus_{lijk} a_{w_l,x_i,y_j}\Box 1_{z_k}"] & & & & \bigoplus_{lijk} w_l\Box(x_i\Box(y_j\Box z_k)) \\
\bigoplus_{lijk}(w_l\Box(x_i\Box y_j))\Box z_k \arrow[rrrr, "\bigoplus_{lijk} a_{w_l,x_i\Box y_j,z_k}"] & & & & \bigoplus_{lijk} w_l\Box((x_i\Box y_j)\Box z_k) \arrow[u, "\bigoplus_{lijk} 1_{w_l}\Box a_{x_i,y_j,z_k}"']
\end{tikzcd}$$

Hence M is a monoidal category.

Suppose M is symmetric. Define the braiding B on $\mathrm{Add}(M)$ as follows. Suppose $x = \bigoplus_i x_i$ and $y = \bigoplus_j y_j$ are in $\mathrm{Add}(M)$. Let the morphism

$$B_{x,y} \colon \bigoplus_{ij} x_i\Box y_j \to \bigoplus_{ji} y_j\Box x_i$$

be given by components $B_{x_i,y_j} \colon x_i\Box y_j \to y_j\Box x_i$. The category $\mathrm{Add}(M)$ is symmetric monoidal:

1. We see $B_{y,x}\circ B_{x,y} = 1_{x\Box y}$ by computing on components.
2. The following diagram commutes by Eq. (2.5):

$$\begin{tikzcd}
\bigoplus_i x_i\Box 1 \arrow[r, "\bigoplus_i B_{x_i,1}"] \arrow[dr, "\bigoplus_i \rho_{x_i}"'] & \bigoplus_i 1\Box x_i \arrow[d, "\bigoplus_i \lambda_{x_i}"] \\
 & x
\end{tikzcd}$$

3. The following hexagon diagram commutes by Eq. (2.6):

$$\begin{tikzcd}
\bigoplus_{ijk}(x_i\Box y_j)\Box z_k \arrow[r, "\bigoplus_{ijk} a_{x_i,y_j,z_k}"] \arrow[d, "\bigoplus_k B_{x,y}\Box 1_{z_k}"'] & \bigoplus_{ijk} x_i\Box(y_j\Box z_k) \arrow[r, "B_{x,y\Box z}"] & \bigoplus_{jki}(y_j\Box z_k)\Box x_i \arrow[d, "\bigoplus_{jki} a_{y_j,z_k,x_i}"] \\
\bigoplus_{jik}(y_j\Box x_i)\Box z_k \arrow[r, "\bigoplus_{jik} a_{y_j,x_i,z_k}"'] & \bigoplus_{jik} y_j\Box(x_i\Box z_k) \arrow[r, "\bigoplus_j 1_{y_j}\Box B_{x,z}"'] & \bigoplus_{jki} y_j\Box(z_k\Box x_i)
\end{tikzcd}$$

□

2.3.2 Symmetric monoidal structure on localized categories

In this subsection, let M be a category, and $S \subseteq M$ a left multiplicative system. We show a (symmetric) monoidal structure on M induces a (symmetric) monoidal structure on M_S such that the localization functor $M \to M_S$ respects the (symmetric) monoidal structure. We proceed by first defining the bifunctor and then verifying the axioms hold.

Proposition 2.3.5. *Let $\Box\colon M \times M \to M$ be a bifunctor such that $s_1 \Box s_2$ is in S, whenever s_1 and s_2 are in S. Then $\Box$ extends to $M_S \times M_S \to M_S$. If M is preadditive and $\Box\colon M \times M \to M$ is additive in each factor, then $\Box\colon M_S \times M_S \to M_S$ is also additive in each factor.*

Proof. Define $\Box\colon M_S \times M_S \to M_S$ on objects to be the same as $\Box$ on M. For $i = 1, 2$, let $b_i \xleftarrow{g_i} c_i \xrightarrow{f_i} a_i$ be two left roofs in M_S. Put

$$(b_1 \xleftarrow{g_1} c_1 \xrightarrow{f_1} a_1)\Box(b_2 \xleftarrow{g_2} c_2 \xrightarrow{f_2} a_2) := b_1 \Box b_2 \xleftarrow{g_1 \Box g_2} c_1 \Box c_2 \xrightarrow{f_1 \Box f_2} a_1 \Box a_2.$$

We show this definition is independent of representative of left roof equivalence class. Suppose $b_1 \xleftarrow{g_1'} c_1' \xrightarrow{f_1'} a_1$ and $b_1 \xleftarrow{g_1} c_1 \xrightarrow{f_1} a_1$ are equivalent, which is the data of a diagram

$$\begin{tikzcd}
 & c_1 \arrow[dl, "g_1"'] \arrow[dr, "f_1"] & \\
b_1 & c_1'' \arrow[l, "g_1''"'] \arrow[r, "f_1''"] \arrow[u, "u"] \arrow[d, "v"] & a_1 \\
 & c_1' \arrow[ul, "g_1'"] \arrow[ur, "f_1'"'] &
\end{tikzcd}$$

Hence, the second equality in

$$\begin{aligned}
(b_1 \xleftarrow{g_1} c_1 \xrightarrow{f_1} a_1)\Box(b_2 \xleftarrow{g_2} c_2 \xrightarrow{f_2} a_2) &= b_1 \Box b_2 \xleftarrow{g_1 \Box g_2} c_1 \Box c_2 \xrightarrow{f_1 \Box f_2} a_1 \Box a_2 \\
&= b_1 \Box b_2 \xleftarrow{g_1' \Box g_2} c_1' \Box c_2 \xrightarrow{f_1' \Box f_2} a_1 \Box a_2 \\
&= (b_1 \xleftarrow{g_1'} c_1' \xrightarrow{f_1'} a_1)\Box(b_2 \xleftarrow{g_2} c_2 \xrightarrow{f_2} a_2),
\end{aligned}$$

follows from the following diagram

$$
\begin{tikzcd}
 & c_1 \square c_2 \arrow[dl, "g_1 \square g_2"'] \arrow[dr, "f_1 \square f_2"] & \\
b_1 \square b_2 & c_1'' \square c_2 \arrow[l, "g_1'' \square g_2"'] \arrow[r, "g_1'' \square f_2"] \arrow[u, "u \square \mathrm{id}_{c_2}"'] \arrow[d, "v \square \mathrm{id}_{c_2}"] & a_1 \square a_2 \\
 & c_1' \square c_2 \arrow[ul, "g_1' \square g_2"] \arrow[ur, "f_1' \square f_2"'] &
\end{tikzcd}
$$

A similar argument shows the definition is independent of representative of the second factor.

Suppose M is preadditive and $\square\colon M \times M \to M$ is additive in each factor. Fix an object $d \in M$. The preadditive structure on M_S was given in Proposition 2.2.23. Let $F_1, F_1'\colon b_1 \to a_1$ be two morphisms in M_S. By Lemma 2.2.22, we represent them by $b_1 \xleftarrow{s} c_1 \xrightarrow{f_1} a_1$ and $b_1 \xleftarrow{s} c_1 \xrightarrow{f_1'} a_1$. Then $\square$ is additive in the first factor:

$$
\begin{aligned}
(F_1 + F_1')\square \mathrm{id}_d &= (b_1 \xleftarrow{s} c_1 \xrightarrow{f_1+f_1'} a_1)\square(d \xleftarrow{\mathrm{id}_d} d \xrightarrow{\mathrm{id}_d} d) \\
&= b_1\square d \xleftarrow{s\square \mathrm{id}_d} c_1\square d \xrightarrow{(f_1+f_1')\square \mathrm{id}_d} a_1\square d \\
&= b_1\square d \xleftarrow{s\square \mathrm{id}_d} c_1\square d \xrightarrow{(f_1\square \mathrm{id}_d)+(f_1'\square \mathrm{id}_d)} a_1\square d \\
&= (b_1\square d \xleftarrow{s\square \mathrm{id}_d} c_1\square d \xrightarrow{f_1\square \mathrm{id}_d} a_1\square d) + (b_1\square d \xleftarrow{s\square \mathrm{id}_d} c_1\square d \xrightarrow{f_1'\square \mathrm{id}_d} a_1\square d) \\
&= F_1\square \mathrm{id}_d + F_1'\square \mathrm{id}_d\,.
\end{aligned}
$$

A similar argument holds for the second factor.

□

Proposition 2.3.6. *If M is (symmetric) monoidal and S is closed under the bifunctor, then M_S is also (symmetric) monoidal under the product defined in Proposition 2.3.5.*

Proof. Apply the localization functor $Q\colon M \to M_S$ to the components of a, λ, and ρ to obtain the required natural isomorphisms a, λ, and ρ on M_S in Definition 2.3.1. The required diagrams commute because Q is a functor. Hence M is monoidal. A similar argument shows M is symmetric monoidal. □

2.3.3 Ring structure on K_0

We discuss a ring structure on K_0 induced by a bifunctor. We begin with some definitions and lemmas.

Definition 2.3.7. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be exact categories. A functor $F\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is *biexact* if:

1. for all objects $A \in \mathcal{A}$, the functor $F(A, -) : \mathcal{B} \to \mathcal{C}$ is exact, and
2. for all objects $B \in \mathcal{B}$, the functor $F(-, B) : \mathcal{A} \to \mathcal{C}$ is exact.

Remark 2.3.8. Let M be an additive category. Let $\Box\colon M \times M \to M$ be a functor additive in each factor. Then $\Box$ is biexact.

Lemma 2.3.9 ([9, Lemma 7.4])**.** *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be exact categories. A biexact functor $F\colon \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ induces a bilinear map*

$$K_0\mathcal{A} \times K_0\mathcal{B} \to K_0\mathcal{C},$$

which sends a pair $([A], [B])$ to $[F(A, B)]$.

As a consequence, we have the following.

Corollary 2.3.10. *Let $(M, \Box)$ be a (symmetric) monoidal additive category. Suppose $\Box$ is additive in each factor. Then K_0M has the structure of a (commutative) ring, where the product is given by $[x] \cdot [y] = [x\Box y]$ and the unit is the class of the unit object in M.*

Proof. The proof follows from Remark 2.3.8 and Lemma 2.3.9. $\Box$

Corollary 2.3.11. *For $i = 1, 2$, let $(M_i, \Box_i)$ be monoidal additive categories. Suppose both $\Box_i$ are additive in each factor. Let $F\colon M_1 \to M_2$ be a functor such that the diagram*

$$\begin{tikzcd} M_1 \times M_1 \arrow[r, "\Box_1"] \arrow[d, "F\times F"'] & M_1 \arrow[d, "F"] \\ M_2 \times M_2 \arrow[r, "\Box_2"'] & M_2 \end{tikzcd}$$

commutes. Then K_0F is a ring homomorphism with respect to the ring structure defined in Corollary 2.3.10.

Proof. We compute, for all objects X and Y in M_1:

$$\begin{aligned}(K_0F)([X]\cdot[Y]) &= (K_0F)([X\Box_1 Y]) \\ &= [F(X\Box_1 Y)] \\ &= [FX\Box_2 FY] \\ &= [FX]\cdot[FY] \\ &= (K_0F)([X])\cdot(K_0F)([Y])\end{aligned}$$

□

2.4 The Ax–Grothendieck theorem

We record the Ax–Grothendieck theorem and a corollary for later use.

Theorem 2.4.1 (Ax–Grothendieck, [EGA4], 17.9.6)**.** *Let S be a scheme, and X a scheme of finite presentation over S. Every S-endomorphism of X that is an monomorphism is a isomorphism.*

Corollary 2.4.2. *Let F be an endomorphism of a k-variety X. If F is a locally closed immersion, then F is an isomorphism.*

Proof. Follows from the following lemma. □

Lemma 2.4.3. *Let $i\colon Z \hookrightarrow Y$ be a locally closed immersion of k-varieties. Then i is a monomorphism in the category of schemes of finite presentation over* Spec k.

Proof. Let $f_1, f_2 \colon X \to Z$ be morphisms such that $i \circ f_1 = i \circ f_2$. The morphism i is injective on the level of sets. Hence f_1 and f_2 are equal on the level of sets. As for the morphism of sheaves, for every $x \in X$, there is an equality of maps of stalks

$$(i \circ f_1)_x^{\#} \colon \mathcal{O}_{Y, if_1(x)} \xrightarrow{i^{\#}_{f_1(x)}} \mathcal{O}_{Z, f_1(x)} \xrightarrow{(f_1)^{\#}_x} \mathcal{O}_{X,x}$$
$$(i \circ f_2)_x^{\#} \colon \mathcal{O}_{Y, if_2(x)} \xrightarrow{i^{\#}_{f_2(x)}} \mathcal{O}_{Z, f_2(x)} \xrightarrow{(f_2)^{\#}_x} \mathcal{O}_{X,x}$$

The maps $i^{\#}_{f_1(x)}$ and $i^{\#}_{f_2(x)}$ are equal (because i is injective) and surjective (because i is a locally closed immersion). Hence $(f_1)^{\#}_x = (f_2)^{\#}_x$, for all $x \in X$. We conclude the morphisms f_1 and f_2 are equal. $\square$

3. ADDITIVE COMPLETION OF VARIETIES AND SCHEMES

In this section, we undertake the strategy outlined in the introduction. Recall the idea: we seek an exact category $\mathcal{C}$ whose Grothendieck group is isomorphic to $K_0(\mathsf{Var}_k)$. The first obstacle is the lack of a natural exact structure, or even additive structure, on Var_k. This leads us to take the additive completion of Var_k to obtain $\mathrm{Add}(\mathsf{Var}_k)$. In Section 3.1, we establish basic properties of $\mathrm{Add}(\mathsf{Var}_k)$ and compute its Grothendieck group. However, there is a second problem: the desired scissors relations do not hold in $K_0(\mathrm{Add}(\mathsf{Var}_k))$. We shall force the relation by localizing $\mathrm{Add}(\mathsf{Var}_k)$ at a certain localizing set S, which makes X and $U \oplus Z$ isomorphic, for every closed immersion $Z \hookrightarrow X$ and $U \coloneqq X \setminus Z$. We spend Section 3.2 defining S and showing it is a left multiplicative system. Finally in Section 3.3, we establish basic properties of $K_0(\mathrm{Add}(\mathsf{Var}_k)_S)$.

3.1 Properties of additive completions associated to varieties

In this section, we study the category Var^n of at most n-dimensional varieties and locally closed immersions, and show isomorphisms in $\mathbb{Z}(\mathsf{Var}^n)$ and $\mathrm{Add}(\mathsf{Var}^n)$ imply isomorphisms in Var^n. We also compute $K_0(\mathrm{Add}(\mathsf{Var}^n))$.

Theorem 3.1.1. *If $f\colon X \hookrightarrow Y$ and $g\colon Y \hookrightarrow X$ are locally closed immersions of varieties, then f and g are isomorphisms. Let $n \in \mathbb{N} \cup \{\infty\}$. If X and Y are isomorphic in $\mathbb{Z}(\mathsf{Var}^n)$, then X and Y are isomorphic in Var.*

Proof. We prove the first statement. The locally closed immersions $gf\colon X \hookrightarrow X$ and $fg\colon Y \hookrightarrow Y$ are isomorphisms by Corollary 2.4.2. Let α and β are inverses to gf and fg:

$$\alpha g f = \mathrm{id}_X = g f \alpha \quad \text{and} \quad \beta f g = \mathrm{id}_Y = f g \beta.$$

Therefore, f and g have two sided inverses.

We prove the second statement. Suppose $\sum_{i\in I} n_i f_i\colon X \to Y$ and $\sum_{j\in J} m_j g_j\colon Y \to X$ are inverses in $\mathbb{Z}(\mathsf{Var}^n)$. Neither index set I or J can be empty so there exist locally closed immersions from X to Y and from Y to X. We apply the first statement to conclude the proof. $\square$

Now we turn to $\mathrm{Add}(\mathsf{Var})$. The following definition and remark are gadgets meant to facilitate the proofs of Theorem 3.1.4 and Proposition 3.1.5.

Definition 3.1.2. Let $\mathcal{C}$ be a category. Fix a n by m matrix $N = (n_{ji})$. Let $T_N \subseteq \mathrm{Add}(\mathcal{C})$ be the set of morphisms $f = (f_{ji})$ such that f maps from a n-tuple $(X_1, \cdots, X_n)$ to a m-tuple $(Y_1, \cdots, Y_m)$ and there exists a morphism $X_i \to Y_j$ in $\mathcal{C}$ whenever $n_{ji} \neq 0$, despite f_{ji} possibly being zero. The equality $n_{ji} = 0$ does not necessarily mean that there are no locally closed immersions $X_i \to Y_j$ or that $f_{ji} = 0$.

For example, let $\mathcal{C} = \mathsf{Var}$. The morphisms (i_U, i_Z) and $(i_U, 0)$ from (U, Z) to X lie in both $T_{(\,1,1\,)}$ and $T_{(\,0,0\,)}$. The morphism $(\mathrm{id}, 0)\colon (\operatorname{Spec}\mathbb{C}, \mathbb{A}^1) \to \operatorname{Spec}\mathbb{C}$ is not in $T_{(\,1,1\,)}$, since there is no locally closed immersion $\mathbb{A}^1 \to \operatorname{Spec}\mathbb{C}$.

If $M \in T_N$ and $M' \in T_{N'}$, then $MM' \in T_{NN'}$, provided the compositions make sense. Recall the convention for composition in Remark 2.2.4.

Remark 3.1.3. If $N = (b_{ij})$ is an invertible r by r real matrix, then there exists a permutation of the rows of N such that N has no zeros along its diagonal. If not, for each permutation σ on r letters, an entry $b_{i,\sigma(i)}$ would be zero for some $1 \leq i \leq r$. Thus $\det N = \sum_{\sigma \in S_r} (\operatorname{sgn}(\sigma) \prod_{i=1}^{r} b_{i,\sigma(i)})$ would be 0.

Theorem 3.1.4. *Let $n \in \mathbb{N} \cup \{\infty\}$. Let $X = (X_1, \cdots, X_r)$ and $Y = (Y_1, \cdots, Y_s)$ be in $\mathrm{Add}(\mathsf{Var}^n)$. Suppose $A\colon X \to Y$ and $B\colon Y \to X$ are inverse morphisms. Then $r = s$, and there is a permutation σ on r letters such that X_i and $Y_{\sigma(i)}$ are isomorphic, for all i.*

Proof. Lemma 2.2.16 implies $r = s$, and $|A|$ and $|B|$ are invertible. By Ax–Grothendieck, it suffices to construct two morphisms $X \to Y$ and $Y \to X$ in T_{I_r}, where I_r is the r by r identity matrix, to conclude X and Y are coordinate wise isomorphic. By Remark 3.1.3, we may relabel $X_1, \cdots, X_r$ such that $|A|$ has no zeros along its diagonal. Hence A is in T_{I_r}. The invertibility of $|B|$ implies the morphism B is in T_E, for some permutation matrix E. Therefore $B = (BA)^{r-1}B$ is in $T_{(EI_r)^{r-1}E = I_r}$. $\square$

Proposition 3.1.5. *Let $n \in \mathbb{N} \cup \{\infty\}$. Let $X = (X_1, \cdots, X_r)$ and $Y = (Y_1, \cdots, Y_s)$ be in* $\mathrm{Add}(\mathsf{Bir}^n)$. *Suppose $A\colon X \to Y$ and $B\colon Y \to X$ are inverse morphisms. Then $r = s$, and there is a permutation σ on r letters such that X_i and $Y_{\sigma(i)}$ are birational, for each i.*

Proof. Lemma 2.2.16 implies $r = s$, and $|A|$ and $|B|$ are invertible. By Remark 3.1.3, we may relabel $X_1, \cdots, X_r$ such that $|A|$ has no zeros along its diagonal. This means there exist birational maps $X_i \to Y_i$ for each i, as desired. □

Corollary 3.1.6. *Let $n \in \mathbb{N} \cup \{\infty\}$. There is a group isomorphism*

$$K_0(\mathrm{Add}(\mathsf{Var}^n)) \to \mathbb{Z}[\mathsf{Var}^n],$$

which sends the class of the tuple $(X_1, \cdots, X_r)$ to $\sum_{i=1}^{r}[X_i]$. There is a group isomorphism

$$K_0(\mathrm{Add}(\mathsf{Bir}^n)) \to \mathbb{Z}[\mathsf{Bir}^n],$$

which sends the class of the tuple $(X_1, \cdots, X_r)$ to $\sum_{i=1}^{r}\{X_i\}$.

Proof. Theorem 3.1.4 and Proposition 3.1.5 show Var^n and Bir^n have the permutation isomorphism property (Definition 2.2.29), and the corollary follows from Proposition 2.2.30. □

3.2 A localization of additive completions associated to varieties and schemes

In this section, we define a left multiplicative system in $\mathrm{Add}(\mathsf{Var})$. The motivation is to form a new localized exact category such that a variety is identified with the direct sum of a closed subvariety and open complement in the Grothendieck group.

Definition 3.2.1. Let X be a variety. Let $i_Z\colon Z \hookrightarrow X$ be a closed immersion. Let $i_U\colon U \hookrightarrow Z$ be the complement. In $\mathrm{Add}(\mathsf{Var})$, we call the morphism $(i_U, i_Z)\colon (U, Z) \to X$ an *open closed cover* of X.

In $\mathrm{Add}(\mathsf{Var})$, define the following sets of morphisms:

1. Let S_0 be the set of all open closed covers together with the set of all isomorphisms.

2. Let S_1 be the set of direct sums of morphisms in S_0. For instance, given two open closed covers $(i_U, i_Z)\colon (U, Z) \to X$ and $(i_{U'}, i_{Z'})\colon (U', Z') \to X'$, the direct sum map $\left(\begin{smallmatrix} i_U & i_Z & 0 & 0 \\ 0 & 0 & i_{U'} & i_{Z'} \end{smallmatrix}\right)\colon (U, Z, U', Z') \to (X, X')$ is an example.

3. Let S_2 be the set of compositions of morphisms in S_1. Put $S \coloneqq S_2$.

Remark 3.2.2. The definition of S works equally well for other categories such as Var^n, Sch, or even topological spaces. All of the results in the remainder of this section hold true with similar proofs.

The remainder of this section is devoted to showing S is a left multiplicative system.

Lemma 3.2.3. *The set S is closed under direct sum.*

Proof. Let $f, g \in S$. Write $f = X_0 \xrightarrow{s_1} X_1 \xrightarrow{s_2} \cdots \xrightarrow{s_r} X_r$ and $g = Y_0 \xrightarrow{t_1} Y_1 \xrightarrow{t_2} \cdots \xrightarrow{t_m} Y_m$, where all s_i and t_j lie in S_1. Assume without loss of generality $m \geq r$. The morphism $f \oplus g\colon X_0 \oplus Y_0 \to X_r \oplus Y_m$ factors as

$$X_0 \oplus Y_0 \xrightarrow{s_1 \oplus t_1} \cdots \xrightarrow{s_r \oplus t_r} X_r \oplus Y_r \xrightarrow{\mathrm{id}_{X_r} \oplus t_{r+1}} \cdots \xrightarrow{\mathrm{id}_{X_r} \oplus t_m} X_r \oplus Y_m.$$

Because each s_i and t_i is a direct sum of morphisms in S_0, each $s_i \oplus t_i$ is in S_1. Similarly, each $\mathrm{id}_{X_r} \oplus t_i$ is in S_1. Hence the composite lies in S. ☐

Recall the category of varieties is symmetric monoidal with respect to fiber product. Let $\square\colon \mathrm{Add}(\mathsf{Var}) \times \mathrm{Add}(\mathsf{Var}) \to \mathrm{Add}(\mathsf{Var})$ denote the fiber product bifunctor obtained from Proposition 2.3.3.

Lemma 3.2.4. *The set S is closed under fiber product bifunctor $\square$ on $\mathrm{Add}(\mathsf{Var})$. That is, $s_1 \square s_2$ is in S, for all s_1 and s_2 in S.*

Proof. We proceed in steps.

Step 0: Let $s \in S$. If W is the empty tuple $\emptyset$, then $s \square \mathrm{id}_W$ and $\mathrm{id}_W \square s$ are both the zero by zero matrix on $\emptyset$, which is the identity on $\emptyset$ and lies in $S_0 \subseteq S$. For the remainder of the proof, we assume W is a nonempty tuple and write $W = (W_1, \cdots, W_n)$.

Step 1: Let $s \in S_0$. We show $s\square \mathrm{id}_W$ and $\mathrm{id}_W \square s$ are in S, for all nonempty tuples $W \in \mathrm{Add}(\mathsf{Var})$. If $s\colon X \to Y$ is an isomorphism, then $s\square \mathrm{id}_W$ is the isomorphism

$$\bigoplus_i s \times \mathrm{id}_{W_i}\colon \bigoplus_i X \times W_i \to \bigoplus_i Y \times W_i,$$

which is in $S_0 \subseteq S$. If s is the open closed cover $(i_U, i_Z)\colon (U, Z) \to X$, then $s\square \mathrm{id}_W$ is the direct sum of open closed covers:

$$\bigoplus_i (i_U \times \mathrm{id}_{W_i}, i_Z \times \mathrm{id}_{W_i})\colon (U \times W_1, \cdots, U \times W_n, Z \times W_1, \cdots, Z \times W_n) \to (X \times W_1, \cdots, X \times W_n).$$

Hence $s\square \mathrm{id}_W$ is in $S_1 \subseteq S$. Similar remarks apply to $\mathrm{id}_W \square s$.

Step 2: Let $s \in S_1$. We show $s\square \mathrm{id}_W$ and $\mathrm{id}_W \square s$ are in S, for all nonempty tuples $W \in \mathrm{Add}(\mathsf{Var})$. Write $s = \bigoplus f_i$, where each $f_i\colon X_i \to Y_i$ is in S_0. Then $s\square \mathrm{id}_W$ is the direct sum

$$\bigoplus_{i,j} f_i \times \mathrm{id}_{W_j}\colon \bigoplus_{i,j} X_i \times W_j \to \bigoplus_{i,j} Y_i \times W_j$$

of morphisms $f_i \times \mathrm{id}_{W_j}$. Step 1 and Lemma 3.2.3 show $s\square \mathrm{id}_W$ is in S. A similar argument applies to $\mathrm{id}_W \square s$.

Step 3: Suppose $s\colon X_1 \to Y_1$ and $t\colon X_2 \to Y_2$ are in S. We show $s\square t$ is in S. Write $s = s_n \circ \cdots \circ s_1$ for $s_i \in S_1$ and $t = t_m \circ \cdots \circ t_1$ for $t_j \in S_1$. Then $s\square t$ is the composite

$$(s_n\square \mathrm{id}_{Y_2}) \circ \cdots \circ (s_1\square \mathrm{id}_{Y_2}) \circ (\mathrm{id}_{X_1} \square t_m) \circ \cdots \circ (\mathrm{id}_{X_1} \square t_1).$$

By Step 2, each factor lies in S so the composite lies in S. $\square$

Proposition 3.2.5. *The set S is a left Ore set.*

Proof. Suppose we are given the diagram

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r] & Y \end{tikzcd} \tag{3.1}$$

in Add(Var) with $s \in S$. We produce two maps $W \to Z$ and $W \to X$ in S such that the diagram commutes.

Step 0: In general, X is a tuple, but we reduce to the case where X is a variety. If X is the empty tuple, then the diagram

$$\begin{tikzcd} \emptyset \arrow[r, "0"] \arrow[d, "0"] & Z \arrow[d, "s"] \\ \emptyset \arrow[r] & Y \end{tikzcd} \tag{3.2}$$

commutes. Here $0\colon \emptyset \to \emptyset$ denotes the zero by zero matrix, which is the identity map on $\emptyset$ and lies in S_0. The morphism $0\colon \emptyset \to Z$ is also a zero dimensional matrix. Otherwise write $X = (X_1, \cdots, X_n)$. Suppose we have a diagram

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ (X_1, \cdots, X_n) \arrow[r, "(f_1, \cdots, f_n)"] & Y \end{tikzcd}$$

and morphisms $t_i \in S$ and g_i such that the following diagram commutes:

$$\begin{tikzcd} W_i \arrow[r, "g_i"] \arrow[d, "t_i"] & Z \arrow[d, "s"] \\ X_i \arrow[r, "f_i"] & Y \end{tikzcd}$$

Then the diagram

$$\begin{tikzcd} (W_1, \cdots, W_n) \arrow[r, "(g_1, \cdots, g_n)"] \arrow[d, "t_1 \oplus \cdots \oplus t_n"] & Z \arrow[d, "s"] \\ (X_1, \cdots X_n) \arrow[r, "(f_1, \cdots, f_n)"] & Y \end{tikzcd}$$

commutes, and $t_1 \oplus \cdots \oplus t_n$ lies in S by Lemma 3.2.3. Assume for the remainder of the proof X is the 1-tuple of a variety.

Step 1: Suppose $s \in S_0$. If s is an isomorphism, then the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "s^{-1}f"] \arrow[d, "\mathrm{id}"] & Y' \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Assume s is an open closed cover $(U, Z) \to Y$:

$$\begin{tikzcd} & (U,Z) \arrow[d, "(i_U, i_Z)"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{3.3}$$

where $f = \sum_{i=1}^{n} n_i f_i$. Let U_i and Z_i be the pullbacks of U and Z along f_i:

$$\begin{tikzcd} U_i \arrow[r, "g_i"] \arrow[d] & U \arrow[d, "i_U"] \\ X \arrow[r, "f_i"] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Z_i \arrow[r, "h_i"] \arrow[d] & Z \arrow[d, "i_Z"] \\ X \arrow[r, "f_i"] & Y \end{tikzcd}$$

Given a multi-index $I = (i_1, \cdots, i_n) \in \{0, 1\}^n$, consider the subvarieties of X:

$$\begin{aligned} U_I &= \bigcap_{\{j | i_j = 0\}} U_j \\ Z_I &= \bigcap_{\{j | i_j = 1\}} Z_j \\ W_I &= U_I \cap Z_I = U_I \times_X Z_I. \end{aligned}$$

For instance, $U_{(1,0,1)} = U_2$, $Z_{(1,0,1)} = Z_1 \cap Z_3$, and $W_{(1,0,1)} = U_2 \cap (Z_1 \cap Z_3)$. Put $W = \bigoplus_{I \in \{0,1\}^n} W_I$ the tuple of length 2^n, written in dictionary order.

We define two morphisms $W \to (U, Z)$ in $\mathrm{Add}(\mathsf{Var})$ and $W \to X$ in S that make Eq. (3.3) commute. Fix a multi-index $I = (i_1, \cdots, i_n) \in \{0, 1\}^n$. The components of $W \to (U, Z)$ are as follows. Let $W_I \to U$ be the composite $W_I \subseteq U_I \to U$ where the morphism $U_I \to U$ is

$\sum_{\{j|i_j=0\}} n_j g_j$. Similarly, let $W_I \to Z$ be the composite $W_I \subseteq Z_I \to Z$ where the morphism $Z_I \to Z$ is $\sum_{\{j|i_j=1\}} n_j h_j$. For example, the morphism $W_{(1,0,1)} \to U$ is

$$W_{(1,0,1)} \subseteq U_{(1,0,1)} \xrightarrow{n_2 g_2} U$$

and the morphism $W_{(1,0,1)} \to Z$ is

$$W_{(1,0,1)} \subseteq Z_{(1,0,1)} \xrightarrow{n_1 h_1 + n_3 h_3} Z.$$

This defines $W \to (U, Z)$.

Next, define $s\colon W \to X$ as the composite of a direct sum of open closed covers, obtained by restricting to (U_i, Z_i):

$$\begin{aligned}
X &\leftarrow (U_1, Z_1) \\
&\leftarrow (U_1 \cap U_2, U_1 \cap Z_2, Z_1 \cap U_2, Z_1 \cap Z_2) \\
&\leftarrow (U_1 \cap U_2 \cap U_3, U_1 \cap U_2 \cap Z_3, \\
&U_1 \cap Z_2 \cap U_3, U_1 \cap Z_2 \cap Z_3, \\
&Z_1 \cap U_2 \cap U_3, Z_1 \cap U_2 \cap Z_3, \\
&Z_1 \cap Z_2 \cap U_3, Z_1 \cap Z_2 \cap Z_3) \\
&\leftarrow \cdots \\
&\leftarrow W.
\end{aligned}$$

The two defined morphisms $W \to (U, Z)$ and $W \to X$ make Eq. (3.3) commute: the morphism $W_I \to X \to Y$ is $\sum_j n_j f_j|_{W_I}$, and the morphism $W_I \to (U, Z) \to Y$ is

$$i_U \circ \left(\sum_{\{j|i_j=0\}} n_j g_j|_{W_I} \right) + i_Z \circ \left(\sum_{\{j|i_j=1\}} n_j h_j|_{W_I} \right),$$

which are the same.

Step 2: Suppose $s \in S_1$, and write s as the direct sum

$$s_1 \oplus \cdots \oplus s_n \colon Z_1 \oplus \cdots \oplus Z_n \to Y_1 \oplus \cdots \oplus Y_n$$

where each $s_i \colon Y_i \to Z_i \in S_0$. We proceed by induction on n. If $n = 1$, then we are done by Step 1. Suppose $n > 1$. We abbreviate $\tilde{Z} = Z_1 \oplus \cdots \oplus Z_{n-1}, \tilde{Y} = Y_1 \oplus \cdots \oplus Y_{n-1}$, and $\tilde{s} = s_1 \oplus \cdots \oplus s_{n-1}$. Label the arrow $X \to Y$ in Eq. (3.1) as f. We may write f as a matrix $\left(\begin{smallmatrix} \tilde{f} \\ f_n \end{smallmatrix}\right)$ where $\tilde{f} \colon X \to \tilde{Y}$ and $f_n \colon X \to Y_n$ are the components.

By inductive hypothesis, there are morphisms $s^* \colon X^* \to X$ in S and $f^* \colon X^* \to \tilde{Z}$ such that the diagram

$$\begin{tikzcd} X^* \arrow[r, "f^*"] \arrow[d, "s^*"'] & \tilde{Z} \arrow[d, "\tilde{s}"] \\ X \arrow[r, "\tilde{f}"'] & \tilde{Y} \end{tikzcd}$$

commutes. Hence the diagram

$$\begin{tikzcd} X^* \arrow[r, "\left(\begin{smallmatrix} f^* \\ f_n s^* \end{smallmatrix}\right)"] \arrow[d, "s^*"'] & \tilde{Z} \oplus Y_n \arrow[d, "M"] \\ X \arrow[r, "\left(\begin{smallmatrix} \tilde{f} \\ f_n \end{smallmatrix}\right)"'] & \tilde{Y} \oplus Y_n \end{tikzcd} \tag{3.4}$$

commutes, where $M = \left(\begin{smallmatrix} \tilde{s} & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right)$.

By Step 1, there are morphisms $s_n^* \colon X^{**} \to X^*$ in S and $f^{**} \colon X^{**} \to Z_n$ such that the diagram

$$\begin{tikzcd} X^{**} \arrow[r, "f^{**}"] \arrow[d, "s_n^*"'] & Z_n \arrow[d, "s_n"] \\ X^* \arrow[r, "f_n s^*"'] & Y_n \end{tikzcd}$$

commutes. Hence the diagram

$$\begin{tikzcd} X^{**} \arrow[r, "\left(\begin{smallmatrix} f^* s_n^* \\ f^{**} \end{smallmatrix}\right)"] \arrow[d, "s_n^*"'] & \tilde{Z} \oplus Z_n \arrow[d, "N"] \\ X^* \arrow[r, "\left(\begin{smallmatrix} f^* \\ f_n s^* \end{smallmatrix}\right)"'] & \tilde{Z} \oplus Y_n \end{tikzcd} \tag{3.5}$$

commutes, where $N = \left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & s_n \end{smallmatrix}\right)$.

Finally, stacking Eq. (3.5) over Eq. (3.4) completes the proof.

Step 3: Suppose $s \in S$. Write $s = s_n \circ \cdots \circ s_1$ where each $s_i \in S_1$. By Step 2, we obtain a commutative diagram for each s_i, and we form a tower in a similar way to the proof of Step 2 to obtain the required commutative diagram for s. □

Proposition 3.2.6. *The set S is left cancellative.*

Proof. Suppose we are given the commutative diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{s} Z \tag{3.6}$$

in Add(Var) with $s \in S$. We produce a morphism t in S such that $ft = gt$.

Step 1: Suppose $s \in S_0$. Say s is an isomorphism, and Eq. (3.6) is the diagram

$$X = (X_1, \cdots, X_n) \underset{(g_1, \cdots, g_n)}{\overset{(f_1, \cdots, f_n)}{\rightrightarrows}} Y \xrightarrow{s} Y'$$

Then $sf_i = sg_i$, for all i, so $f_i = g_i$. Hence we let t be the identity map on X.

Say s is an open closed cover, and Eq. (3.6) is the diagram

$$X = (X_1, \cdots, X_n) \underset{g}{\overset{f}{\rightrightarrows}} (U, Z) \xrightarrow{(i_U, i_Z)} Y$$

Write $f = \left(\begin{smallmatrix} f_1' & \cdots & f_n' \\ f_1'' & \cdots & f_n'' \end{smallmatrix}\right)$ and $g = \left(\begin{smallmatrix} g_1' & \cdots & g_n' \\ g_1'' & \cdots & g_n'' \end{smallmatrix}\right)$. For each $1 \leq j \leq n$, the equation

$$i_U(f_j' - g_j') = i_Z(g_j'' - f_j'')$$

holds in $\mathbb{Z}[\mathrm{Hom}(X_j, Y)]$. Both $i_U(f'_j - g'_j)$ and $i_Z(g''_j - f''_j)$ equal zero because i_U and i_Z have disjoint image. It suffices to show $f'_j - g'_j$ and $g''_j - f''_j$ are zero so that the identity map on X will do the job.

To that end, in general, if $\phi\colon X \to Z$ is in $\mathbb{Z}(\mathsf{Var})$ and $i\colon Z \hookrightarrow Y$ is a locally closed immersion with $i\phi = 0$, then ϕ is zero. Indeed, write $\phi = \sum n_p \phi_p$ in reduced form. If ϕ were nonzero, there would be indices $p \neq q$ such that $i\phi_p = i\phi_q$. But Lemma 2.4.3 would absurdly imply $\phi_p = \phi_q$.

Step 2: Suppose $s \in S_1$, and Eq. (3.6) is the diagram

$$X \underset{(g_1,\cdots,g_n)}{\overset{(f_1,\cdots,f_n)}{\rightrightarrows}} \bigoplus_{i=1}^n Y_i \xrightarrow{\bigoplus_{i=1}^n s_i} \bigoplus_{i=1}^n Z_i, \tag{3.7}$$

where all $s_i\colon Y_i \to Z_i$ are in S_0. We proceed by induction on n. If $n = 1$, we are done by Step 1. If $n \geq 2$, write $f' = f_1 \oplus \cdots \oplus f_{n-1}$, $g' = g_1 \oplus \cdots \oplus g_{n-1}$, $s' = s_1 \oplus \cdots \oplus s_{n-1}$, $Y' = Y_1 \oplus \cdots \oplus Y_{n-1}$, and $Z' = Z_1 \oplus \cdots \oplus Z_{n-1}$. The diagram

$$X \underset{g'}{\overset{f'}{\rightrightarrows}} Y' \xrightarrow{s'} Z'$$

commutes so induction grants a morphism $t'\colon W' \to X$ in S such that $f't' = g't'$. Next, the diagram

$$W' \underset{g_n t'}{\overset{f_n t'}{\rightrightarrows}} Y_n \xrightarrow{s_n} Z_n$$

commutes so Step 1 gives a morphism $t\colon W \to W'$ in S_0 such that $f_n t' t = g_n t' t$. Then the morphism $W \xrightarrow{t} W' \xrightarrow{t'} X$ in S is the desired morphism; we compute:

$$ft't = (f' \oplus f_n) \circ (t't) = f't't \oplus f_n t't = g't't \oplus g_n t't = gt't.$$

Step 3: Suppose $s \in S$. Write s as

$$Y \xrightarrow{s_1} Z_1 \xrightarrow{s_2} Z_2 \to \cdots \xrightarrow{s_n} Z_n = Z$$

where all $s_i \in S_1$. We proceed by induction on n. If $n = 1$, we are done by Step 2. If $n \geq 2$, write $s' = s_{n-1} \circ \cdots \circ s_1$. The diagram

$$X \underset{s'g}{\overset{s'f}{\rightrightarrows}} Z_{n-1} \xrightarrow{s_n} Z$$

commutes so Step 2 gives a morphism $t\colon W \to X$ in S such that $s'ft = s'gt$. Therefore, the diagram

$$W \underset{gt}{\overset{ft}{\rightrightarrows}} Y \xrightarrow{s'} Z_{n-1}$$

commutes and induction gives a morphism $t'\colon W' \to W$ in S such that $ftt' = gtt'$. Hence the morphism $W' \xrightarrow{t'} W \xrightarrow{t} X$ in S is the desired morphism. □

Corollary 3.2.7. *S is a left multiplicative system.*

Proof. The set S contains all isomorphisms and all identities, and the corollary follows from Propositions 3.2.5 and 3.2.6. □

3.3 K_0 of the localized additive completion of varieties and schemes

Let T be a base scheme. By Propositions 2.2.24 and 2.3.6 and Lemma 3.2.4, the category $\mathrm{Add}(\mathsf{Sch}_T)_S$ is additive and symmetric monoidal under fiber product. The same is true for $\mathrm{Add}(\mathsf{Var}_k^n)_S$, for all $n \in \mathbb{N} \cup \{\infty\}$. In the remainder of this thesis, we study the K_0-theory of these split exact categories where the class of a scheme is indeed identified with the direct sum of a closed subscheme and open complement.

In this section, we establish a few basic properties of $K_0(\mathrm{Add}(\mathsf{Sch}_T)_S)$. In Chapter 4, we view the family $\{K_0(\mathrm{Add}(\mathsf{Var}_k^n)_S)\}_n$ as a direct system of groups, and compute the direct limit and cokernels of the connecting morphisms. In Chapter 5, we compare this K_0-theory with $K_0(\mathsf{Var}_k)$. In Chapter 6, we show how to construct motivic measures on $K_0(\mathrm{Add}(\mathsf{Var}_k)_S)$.

Definition 3.3.1. The inclusion functor $\mathsf{Var}^{n-1} \to \mathsf{Var}^n$ induces, by universal property of additive completions and localizations, a functor

$$j_{n,n-1}\colon\ \mathrm{Add}(\mathsf{Var}^{n-1})_S \to \mathrm{Add}(\mathsf{Var}^n)_S.$$

Explicitly, it is the identity on objects and maps the class of a left roof to the same class.

Definition 3.3.2. Let

$$g_{n,n-1} = K_0(j_{n,n-1})\colon K_0(\mathrm{Add}(\mathsf{Var}^{n-1})_S) \to K_0(\mathrm{Add}(\mathsf{Var}^{n})_S).$$

be the group homomorphism obtained by applying the functor K_0 to the inclusion functors $j_{n,n-1}$.

Lemma 3.3.3. *The functor $j_{n,n-1}$ is fully faithful.*

Proof. Let X and Y be in $\mathrm{Add}(\mathsf{Var}^{n-1})_S$. Consider a morphism in $\mathrm{Hom}_{\mathrm{Add}(\mathsf{Var}^{n})_S}(X,Y)$ represented by left roof $X \leftarrow P \to Y$. It suffices to show: if a morphism $f\colon P \to X$ in S where X is in $\mathrm{Add}(\mathsf{Var}^{n-1})$, then P also is in $\mathrm{Add}(\mathsf{Var}^{n-1})$.

Let $f \in S_0$. If f is an isomorphism, then Theorem 3.1.4 shows all the entries of P are of dimension less than n. If f is the open closed cover $(U,Z) \to X$, then the entries of $P = (U,Z)$ are of dimension less than n.

Let $f \in S_1$. Write f as the direct sum of the morphisms $f_i\colon P_i \to X_i$ in S_0. The previous step shows all P_i are in $\mathrm{Add}(\mathsf{Var}^{n-1})$, hence so is their direct sum.

Let $f \in S_2$. Write f as the composite $f_{r-1} \circ \cdots \circ f_0$ where $f_i\colon Z_i \to Z_{i+1}$ is in S_1 and $Z_0 = P$ and $Z_r = X$. Starting from $f_{r-1}\colon Z_{r-1} \to Z_r = X$, we inductively conclude, from the previous step, each Z_i is in $\mathrm{Add}(\mathsf{Var}^{n-1})$. □

Proposition 3.3.4. *The abelian groups $K_0(\mathrm{Add}(\mathsf{Sch}_T))$ and $K_0(\mathrm{Add}(\mathsf{Sch}_T)[S^{-1}])$ have the structure of a commutative ring, induced by fiber product over T. Also, the localization functor $Q\colon \mathrm{Add}(\mathsf{Sch}_T) \to \mathrm{Add}(\mathsf{Sch}_T)[S^{-1}]$ induces a ring homomorphism*

$$K_0Q\colon K_0(\mathrm{Add}(\mathsf{Sch}_T)) \to K_0(\mathrm{Add}(\mathsf{Sch}_T)[S^{-1}]).$$

Proof. The category Sch_T is a symmetric monoidal category under fiber product. By Proposition 2.3.4, $\mathrm{Add}(\mathsf{Sch}_T)$ is also symmetric monoidal with bifunctor described in Proposition 2.3.3. By Proposition 2.3.6, $\mathrm{Add}(\mathsf{Sch}_T)_S$ is symmetric monoidal with bifunctor described in Proposition 2.3.5. Both bifunctors on $\mathrm{Add}(\mathsf{Sch}_T)$ and $\mathrm{Add}(\mathsf{Sch}_T)_S$ are additive in each factor. Therefore by Corollary 2.3.10, both $K_0(\mathrm{Add}(\mathsf{Sch}_T))$ and $K_0(\mathrm{Add}(\mathsf{Sch}_T)[S^{-1}])$ are rings. Corollary 2.3.11 implies K_0Q is a ring homomorphism.

□

Remark 3.3.5 (Pullback)**.** Let $f\colon T' \to T$ be a morphism of schemes. The natural pullback functor $\mathsf{Sch}_T \to \mathsf{Sch}_{T'}$ induces an additive functor $\mathrm{Add}(\mathsf{Sch}_T) \to \mathrm{Add}(\mathsf{Sch}_{T'})$, which induces a group homomorphism

$$f^*\colon K_0(\mathrm{Add}(\mathsf{Sch}_T)) \to K_0(\mathrm{Add}(\mathsf{Sch}_{T'})).$$

It is easy to verify f^* respects the ring structure.

The functor $\mathrm{Add}(\mathsf{Sch}_T) \to \mathrm{Add}(\mathsf{Sch}'_T) \to \mathrm{Add}(\mathsf{Sch}'_T)_S$ sends S into isomorphisms. Therefore there is another group homomorphism

$$f^*\colon K_0(\mathrm{Add}(\mathsf{Sch}_T)[S^{-1}]) \to K_0(\mathrm{Add}(\mathsf{Sch}_{T'})[S^{-1}]),$$

which also respects the ring structure.

The pullback maps commute with the ring homomorphism in Proposition 3.3.4.

Remark 3.3.6 (Pushforward)**.** Let $f\colon T' \to T$ be a morphism of finite presentation. There is a functor $\mathsf{Sch}_{T'} \to \mathsf{Sch}_T$ which maps a scheme X' over T' to X' with structure morphism $X' \to T' \to T$, and sends a morphism over T' to a morphism over T. Note the morphism $X' \to T' \to T$ is also of finite presentation. This induces an additive functor $\mathrm{Add}(\mathsf{Sch}_{T'}) \to \mathrm{Add}(\mathsf{Sch}_T)$, which induces a group homomorphism

$$f_!\colon K_0(\mathrm{Add}(\mathsf{Sch}_{T'})) \to K_0(\mathrm{Add}(\mathsf{Sch}_T)).$$

Usually, $f_!$ is not a ring homomorphism because it sends the unit T' to $T' \to T$.

The functor $\mathrm{Add}(\mathsf{Sch}_{T'}) \to \mathrm{Add}(\mathsf{Sch}_T) \to \mathrm{Add}(\mathsf{Sch}_T)_S$ sends S into isomorphisms. Therefore there is another group homomorphism

$$f_!\colon K_0(\mathrm{Add}(\mathsf{Sch}_{T'})[S^{-1}]) \to K_0(\mathrm{Add}(\mathsf{Sch}_T)[S^{-1}]),$$

compatible with K_0 applied to the localization functors and $f_!$ on the K_0-theory of the unlocalized category.

4. DIRECT SYSTEMS

In this chapter, we view the families $\{K_0(\mathsf{Var}_k^n)\}_n$ and $\{K_0(\mathrm{Add}(\mathsf{Var}_k^n))_S\}_n$ as a direct system of groups. In Sections 4.2 and 4.3, we show their direct limits are $K_0(\mathsf{Var}_k)$ and $K_0(\mathrm{Add}(\mathsf{Var}_k)_S)$. We also compute the cokernels of the connecting homomorphisms. In Section 4.4, we show a quotient category $\mathrm{Add}(\mathsf{Var}_k^n)_S/\,\mathrm{Add}(\mathsf{Var}_k^{n-1})_S$ exists, and we show the K_0-theory of the quotient category is the cokernel of $K_0(\mathrm{Add}(\mathsf{Var}_k^{n-1})_S) \to K_0(\mathrm{Add}(\mathsf{Var}_k^n)_S)$. We begin this chapter with some technicalities.

4.1 Preliminary setup for proofs in later sections

In this section, we set up notations and define some functors for proofs in the following sections of this chapter.

Remark 4.1.1. Let X be an n-dimensional variety. Let (U, Z) be a open closed cover of X. We set up some notation for the irreducible decomposition of U and Z in terms of the irreducible decomposition of X. Label the irreducible decomposition of X as follows:

1. Let $Y_1, \cdots, Y_t$ be the components of X of dimension less than n.
2. Let $Y_1', \cdots, Y_r'$ be the components of X of dimension n that meet U.
3. Let $Y_1'', \cdots, Y_s''$ be the components of X of dimension n that avoid U.

We have

$$U = \left(\bigcup_{i=1}^{r} U \cap Y_i'\right) \cup \left(\bigcup_{k=1}^{t} U \cap Y_k\right).$$

After deleting every empty $U \cap Y_k$, what remains is the irreducible decomposition of U because every nonempty open of an irreducible space is irreducible. The only n-dimensional components of U are $U \cap Y_i'$.

Next, we have

$$Z = \left(\bigcup_{i=1}^{r} Z \cap Y_i'\right) \cup \left(\bigcup_{j=1}^{s} Z \cap Y_j''\right) \cup \left(\bigcup_{k=1}^{t} Z \cap Y_k\right).$$

We obtain the irreducible decomposition of Z by collecting the components of $Z \cap Y_i'$, $Z \cap Y_j''$, and $Z \cap Y_k$ into a poset, ordered by inclusion, and taking the maximal elements. Notice:

1. $Z \cap Y_i'$ is a proper closed subvariety of Y_i' because U meets Y_i. Therefore the components of $Z \cap Y_i'$ are of dimension less than n.

2. $Z \cap Y_j''$ equals Y_j'' because Y_j'' avoids U.

3. The components of $Z \cap Y_k$ are of dimension less than n.

The only n-dimensional components of Z are Y_j''.

Definition 4.1.2. We define a functor

$$\operatorname{comp}_n \colon \mathsf{Var}^n \to \operatorname{Add}(\mathsf{IrrVar}^{=n}),$$

which picks out the n-dimensional components of a variety. For every variety Y of dimension n, fix, once and for all, an ordering of the n-dimensional components of Y. For $X \in \mathsf{Var}^n$, put $\operatorname{comp}_n X := \bigoplus_{i \in I} X_i$, where I is the ordered set indexing the n-dimensional components X_i of X. If I is empty (X has no n-dimensional components), then $\operatorname{comp}_n X = 0$.

Let $f \colon X \hookrightarrow Y$ be a locally closed immersion and write $\operatorname{comp}_n Y = \bigoplus_{j \in J} Y_j$. Define $\operatorname{comp}_n f \colon \operatorname{comp}_n X \to \operatorname{comp}_n Y$ as follows.

1. If $\dim X$ or $\dim Y$ is less than n, then $\operatorname{comp}_n X$ or $\operatorname{comp}_n Y$ is zero, and we define $\operatorname{comp}_n f = 0$.

2. Otherwise $\dim X = n = \dim Y$, and we define the components $(\operatorname{comp}_n f)_{ji} \colon X_i \to Y_j$ of $\operatorname{comp}_n f$ as follows.

   (a) If f does not map X_i into Y_j, then put $(\operatorname{comp}_n f)_{ji} = 0$.

   (b) Otherwise f maps X_i into Y_j, and put $(\operatorname{comp}_n f)_{ji} = f$. Note in this case f maps X_i into Y_j as an open subvariety.

   In other words, $(\operatorname{comp} f)_{ji}$ is nonzero if and only if f maps X_i into Y_j.

Proposition 4.1.3. *The above assignment makes* comp_n *into a functor.*

We require a lemma before proving Proposition 4.1.3.

Lemma 4.1.4. *Let* $f\colon X \hookrightarrow Y$ *be a locally closed immersion between varieties of dimension* n*. This means* f *is an open immersion and we identify* X *with an open subvariety of* Y*. If* $X' \subseteq X$ *is a component of dimension* n*, then* X' *is contained in a unique* n*-dimensional component of* Y*.*

Proof of Lemma 4.1.4. The irreducible subvariety X' in Y must be contained in some component of Y. Also, X' cannot be contained in two different components Y_1 and Y_2 of Y, as this would mean the n-dimensional subspace X' would be contained in the lower dimensional subspace $Y_1 \cap Y_2$. □

Proof of Proposition 4.1.3. We show comp_n is functorial. Let $f\colon X \hookrightarrow Y$ and $g\colon Y \hookrightarrow Z$ be locally closed immersions. Write $h = gf\colon X \hookrightarrow Z$. If any of X, Y, or Z are of dimension less than n, then both $\mathrm{comp}_n h$ and $\mathrm{comp}_n g \circ \mathrm{comp}_n f$ are zero. Assume $n = \dim X = \dim Y = \dim Z$. Write $\mathrm{comp}_n X = \bigoplus_{i\in I} X_i$, $\mathrm{comp}_n Y = \bigoplus_{j\in J} Y_j$, and $\mathrm{comp}_n Z = \bigoplus_{k\in K} Z_k$. Fix $i \in I$ and $k \in K$. We verify the equality

$$(\mathrm{comp}_n h)_{ki} = \sum_j (\mathrm{comp}_n g)_{kj} \circ (\mathrm{comp}_n f)_{ji}.$$

By Lemma 4.1.4, there is a unique index $j' \in J$ such that $f(X_i) \subseteq Y_{j'}$, or equivalently, $(\mathrm{comp}_n f)_{j'i}$ is nonzero. Hence we are reduced to verifying

$$(\mathrm{comp}_n h)_{ki} = (\mathrm{comp}_n g)_{kj'} \circ (\mathrm{comp}_n f)_{j'i}. \tag{4.1}$$

The following are equivalent:

1. $(\mathrm{comp}_n g)_{kj'}$ is nonzero
2. g maps $Y_{j'}$ into Z_k
3. h maps X_i into Z_k

4. $(\mathrm{comp}_n\, h)_{ki}$ is nonzero

The first and second statements, and third and fourth statements, are equivalent by definition of comp_n. The second statement implies the third statement because $f(X_i) \subseteq Y_{j'}$ so

$$h(X_i) = gf(X_i) \subseteq g(Y_{j'}) \subseteq Z_k.$$

Conversely, assume the third statement. Because $f(X_i)$ is open in Y_j and g is continuous, we have

$$g(Y_j) = g(\overline{f(X_i)}) \subseteq \overline{g(f(X_i))} \subseteq \overline{Z_k} = Z_k,$$

as desired.

Finally we conclude, either

1. both $(\mathrm{comp}_n\, g)_{kj'}$ and $(\mathrm{comp}_n\, h)_{ki}$ are zero, in which case both sides of Eq. (4.1) are zero, or
2. both $(\mathrm{comp}_n\, g)_{kj'}$ and $(\mathrm{comp}_n\, h)_{ki}$ are nonzero, in which case Eq. (4.1) reduces to $h = gf$.

Thus comp_n is a functor on Var^n. $\square$

Definition 4.1.5. Consider the composite functor

$$\mathrm{bir}_n\colon\ \mathrm{Add}(\mathsf{Var}^n) \xrightarrow{\mathrm{Add}(\mathrm{comp}_n)} \mathrm{Add}(\mathsf{IrrVar}^{=n}) \xrightarrow{\mathrm{Add}(\iota_n)} \mathrm{Add}(\mathsf{Bir}^n),$$

where the functor ι_n was defined in Definition 2.1.2. We calculate bir_n explicitly. Let $X = (X_1, \cdots, X_r)$ be a tuple. Let $\{X_{ij}\}_{j\in J_i}$ be the (ordered) set of n-dimensional components of X_i. Then bir_n sends X to $\left(\bigoplus_{j\in J_1} X_{1j}\right) \oplus \cdots \oplus \left(\bigoplus_{j\in J_r} X_{rj}\right)$.

Lemma 4.1.6. *The functor* bir_n *extends to* $\mathrm{Add}(\mathsf{Var}^n)_S$*.*

Proof. We verify bir_n sends S into isomorphisms. Let $f \in S_0$. If f is an isomorphism in $\mathrm{Add}(\mathsf{Var}^n)$, then $\mathrm{bir}_n f$ is an isomorphism by Theorem 3.1.4. Otherwise f is an open closed cover $(U, Z) \to X$. Then, in the notation of Remark 4.1.1, up to reordering,

$$\mathrm{bir}_n X = \bigoplus_i Y_i' \oplus \bigoplus_j Y_j'' \quad \text{while} \quad \mathrm{bir}_n U = \bigoplus_i U \cap Y_i' \quad \text{and} \quad \mathrm{bir}_n Z = \bigoplus_j Y_j''.$$

The morphism $\mathrm{bir}_n f \colon \mathrm{bir}_n X \to \mathrm{bir}_n U \oplus \mathrm{bir}_n Z$ is a square matrix with two diagonal blocks. One block consists of open immersions $Y_i' \cap U \to Y_i'$ along the diagonal, and the other block has the identity maps $Y_j'' \to Y_j''$ along the diagonal. Hence $\mathrm{bir}_n f$ is an isomorphism in $\mathrm{Add}(\mathsf{Bir}^n)$. If f is a direct sum of morphisms in S_0, then $\mathrm{bir}_n f$ is a direct sum of isomorphisms. If f is a composition of morphisms in S_1, then $\mathrm{bir}_n f$ is a composition of isomorphisms. Therefore bir_n extends to an additive functor on $\mathrm{Add}(\mathsf{Var}^n)_S$. □

4.2 Direct system of classical K_0-theory of varieties

In this section, we consider the direct system $\{K_0(\mathsf{Var}_k^n)\}_n$, and compute the direct limit and the cokernels of the connecting morphisms.

Definition 4.2.1. Define the direct system of groups

$$F^n K \coloneqq K_0(\mathsf{Var}^n).$$

For a variety X in Var^n, let $[X]_n^K$ denote its class in $F^n K$. We may also write $[X]_n$ or $[X]$, if clear from context. For $n \leq m$, there are homomorphisms $f_{m,n} \colon F^n K \to F^m K$ given by $f_{m,n}[X]_n = [X]_m$.

Proposition 4.2.2. *The direct limit of the direct system $F^n K$ is $K_0(\mathsf{Var})$.*

Proof. There is a compatible system of homomorphisms $\phi_i \colon F^i K \to K_0(\mathsf{Var})$ where $\phi_i([X]_i) = [X]$. We show $K_0(\mathsf{Var})$ carries the universal property. Let $\psi_i \colon F^i K \to H$ be a compatible system of homomorphisms.

We define a homomorphism $u\colon K_0(\mathsf{Var}) \to H$ compatible with the ψ_k. Send the class $[X]$ in $\mathbb{Z}[\mathsf{Var}]$ to $\psi_k([X]_k)$ in H, where k is any integer larger than $\dim X$. This is independent of choice of k: if $l \geq k \geq \dim X$, then

$$\psi_k[X]_k = \psi_l f_{lk}[X]_k = \psi_l[X]_l.$$

This map on $\mathbb{Z}[\mathsf{Var}]$ respects scissors relations: the relation $[X]-[U]-[Z]$ is sent to

$$[X]-[U]-[Z] \mapsto \psi_k[X]_k - \psi_k[U]_k - \psi_k[Z]_k = \psi_k([X]_k - [U]_k - [Z]_k) = 0.$$

Therefore u is a homomorphism such that $u \circ \phi_i = \psi_i$ for all i.

Next, we show uniqueness. Suppose $u'\colon K_0(\mathsf{Var}) \to H$ is another homomorphism compatible with the ψ_k. Consider a generator $[X]$ in $K_0(\mathsf{Var})$ and let $k \geq \dim X$. Then $u([X]) = \psi_k([X]_k) = u'\phi_k([X]_k) = u'([X])$. The homomorphisms u and u' agree on the generators of $K_0(\mathsf{Var})$. □

Proposition 4.2.3. *There is a group homomorphism $\phi_n\colon F^nK \to \mathbb{Z}[\mathsf{Bir}^n]$, which maps the class of a variety X to $\sum_j \{Y_j\}$, where the Y_j are n-dimensional components of X.*

Proof. We show ϕ_n respects scissors relations. Let (U, Z) be a open closed cover of X. In the notation of Remark 4.1.1, we have:

$$\phi_n(X) = \sum_i \{Y_i'\} + \sum_j \{Y_j''\} = \sum_i \{Y_i' \cap U\} + \sum_j \{Y_j''\} = \phi_n(U) + \phi_n(Z),$$

as desired. □

Theorem 4.2.4. *There is an exact sequence of groups*

$$F^{n-1}K \xrightarrow{f_{n,n-1}} F^nK \xrightarrow{\phi_n} \mathbb{Z}[\mathsf{Bir}^n] \to 0.$$

Proof. It is clear ϕ_n is surjective, and the image of $F^{n-1}K$ is in the kernel of ϕ_n. We show the kernel of ϕ_n is in the image of $F^{n-1}K$ in two steps.

Step 1: Suppose

$$\phi_n(n_1[X_1] + \cdots + n_r[X_r]) = 0,$$

where each X_i is an irreducible variety of dimension n. We show the sum $\sum_i n_i[X_i]$ lies in the image of $F^{n-1}K$. We proceed by induction on r. If $r = 1$, then X_1 has no n-dimensional components so $n[X_1]$ is in the image of F^{n-1}. Assume $r > 1$. Fix $\{X_1\}$, and relabel $\{X_1\}, \cdots, \{X_r\}$ such that:

1. X_i is birational to X_1, for all $1 \leq i \leq s$. That is, $\{X_i\} = \{X_1\}$.

2. X_i is not birational to X_1, for all $i > s$. That is, $\{X_i\} \neq \{X_1\}$.

This implies $n_1 + \cdots + n_s = 0$. For each $1 \leq i \leq s$, let $U_i \subseteq X_1$ and $V_i \subseteq X_i$ be isomorphic opens. Let U be the open $\bigcap_{i=1}^s U_i$ contained in X_1. Let W_i be the image of U in V_i under the isomorphism $U_i \to V_i$. Put $Z_i = X_i \setminus W_i$. Then in F^nK, we write

$$[X_i] = [W_i] + [Z_i] = [U] + [Z_i].$$

Because $n_1 + \cdots + n_s = 0$,

$$\sum_{i=1}^s n_i[X_i] = \sum_{i=1}^s n_i([U] + [Z_i]) = \sum_{i=1}^s n_i[Z_i],$$

lies in the image of $F^{n-1}K$. Thus

$$0 = \phi_n(\sum_{i=1}^r n_i[X_i]) = \phi_n(\sum_{i=1}^s n_i[Z_i] + \sum_{i>s} n_i[X_i]) = \phi_n(\sum_{i>s} n_i[X_i]).$$

Since $\sum_{i>s} n_i[X_i]$ has fewer than r terms, we proceed by induction on r and conclude $\sum_{i=1}^r n_i[X_i]$ is a sum of terms in the image of $F^{n-1}K$.

Step 2: Suppose

$$\phi_n(n_1[X_1] + \cdots + n_r[X_r]) = 0,$$

where all the X_i are arbitrary (possibly reducible) varieties. We will conclude $\sum n_i[X_i]$ lies in the image of $F^{n-1}K$ by reducing to Step 1. We may assume $\dim X_i = n$, for all i.

In general, if Y is a n-dimensional variety, we may write $[Y]$ as a sum of classes of irreducible varieties of dimension n, plus terms from $F^{n-1}K$. Indeed, let $Y = Z_1 \cup \cdots \cup Z_s$ be an irreducible decomposition. For each $1 \leq i \leq s$, put $Y_i = Z_i - \bigcup_{j \neq i} Z_j$. In other words, Y_i is the open in Z_i consisting of points of Z_i that lie only in Z_i, and no other Z_j. Let $U = \bigcup_i Y_i$. Since the Y_i are disjoint, write

$$[Y] = [Y \setminus U] + [U] = [Y \setminus U] + \sum_{i=1}^{s} [Y_i].$$

Group the irreducibles Y_i of dimension n together. The rest are in the image of $F^{n-1}K$.

Now onto the proof of Step 2. Using the above argument, write each $[X_i] = A_i + B_i$, where A_i is a sum of classes of n-dimensional irreducible varieties and B_i is a sum of terms in the image of $F^{n-1}K$. Then

$$0 = \phi_n(n_1[X_1] + \cdots + n_r[X_r]) = \phi_n(n_1 A_1 + \cdots + n_r A_r).$$

Step 1 shows $\sum_i n_i A_i$ is in the image of $F^{n-1}K$. $\square$

4.3 Direct system of categorical K_0-theory of varieties

In this section, we consider the direct system $\{K_0(\mathrm{Add}(\mathsf{Var}_k^n))_S\}_n$, and compute the direct limit and the cokernels of the connecting morphisms.

Definition 4.3.1. If $X = (X_1, \cdots, X_r)$ is a tuple in $\mathrm{Add}(\mathsf{Var}^n)$, the *dimension* $\dim X$ is the integer $\max_i \dim X_i$.

Definition 4.3.2. Define the direct system of groups

$$F^n G \coloneqq K_0(\mathrm{Add}(\mathsf{Var}^n)_S).$$

For a tuple $(X_1, \cdots, X_r)$ in $\mathrm{Add}(\mathsf{Var}^n)_S$, let $[X_1, \cdots, X_r]_n^G$ denote its class in $F^n G$. We may also write $[X_1, \cdots, X_r]_n$ or $[X_1, \cdots, X_r]$, if clear from context. Let $n \leq m$. Let the connecting homomorphism $F^nG \to F^mG$ be $g_{m,n}$, which was defined in Definition 3.3.2. Explicitly, $g_{m,n}$ sends $[X_1, \cdots, X_r]_n$ to $[X_1, \cdots, X_r]_m$.

Proposition 4.3.3. *The direct limit of the direct system F^nG is $K_0(\mathrm{Add}(\mathsf{Var})_S)$.*

Proof. The proof is similar to the proof of Proposition 4.2.2. $\square$

Proposition 4.3.4. *There is a group homomorphism $\psi_n\colon F^nG \to \mathbb{Z}[\mathsf{Bir}^n]$, defined by sending the class of a tuple $(X_1, \cdots, X_r)$ to $\sum_{i,j}\{X_{ij}\}$, where $\{X_{ij}\}_{j\in J_i}$ is the set of n-dimensional components of X_i.*

Proof. The homomorphism ψ_n is the composite

$$K_0(\mathrm{Add}(\mathsf{Var}^n)_S) \xrightarrow{K_0(\mathrm{bir}_n)} K_0(\mathrm{Add}(\mathsf{Bir}^n)) \xrightarrow{\cong} \mathbb{Z}[\mathsf{Bir}^n],$$

where the isomorphism is in Corollary 3.1.6. Indeed, we compute

$$[X_1, \cdots, X_r] \mapsto \left(\bigoplus_{j\in J_1} X_{1j}\right) \oplus \cdots \oplus \left(\bigoplus_{j\in J_r} X_{rj}\right) \mapsto \sum_{\substack{1\le i\le r\\ j\in J_i}} \{X_{ij}\}.$$

$\square$

Theorem 4.3.5. *There is an exact sequence of groups*

$$F^{n-1}G \xrightarrow{g_{n,n-1}} F^nG \xrightarrow{\psi_n} \mathbb{Z}[\mathsf{Bir}^n] \to 0.$$

Proof. It is clear ψ_n is surjective and the image of $F^{n-1}G$ lands in the kernel of ψ_n. Suppose $\psi_n(\sum_{i=1}^m n_i[X_i]) = 0$ where $X_i = (X_{i,1}, \cdots, X_{i,n_i}) \in \mathrm{Add}(\mathsf{Var}^n)$. Then, in F^nG, we have

$$\sum_{i=1}^m n_i[X_i] = \sum_{\substack{1\le i\le m\\ 1\le j\le n_i}} n_i[X_{i,j}].$$

An argument similar to the proof of Theorem 4.2.4 shows $\sum_{i=1}^m n_i[X_i]$ is in the image of $F^{n-1}G$. $\square$

4.4 The quotient category $\mathrm{Add}(\mathsf{Var}^n)_S / \mathrm{Add}(\mathsf{Var}^{n-1})_S$ and its K_0-theory

We abbreviate $V^n \coloneqq \mathrm{Add}(\mathsf{Var}^n)$ and $V_S^n \coloneqq \mathrm{Add}(\mathsf{Var}^n)_S$. In this section, we show there is a quotient category V_S^n/V_S^{n-1}, relate the K_0-theories of V_S^n, V_S^{n-1}, and V_S^n/V_S^{n-1}, and compute the K_0-theory of the quotient.

Proposition 4.4.1. *There is an additive category V_S^n/V_S^{n-1} and an additive, essentially surjective functor $Q\colon V_S^n \to V_S^n/V_S^{n-1}$ with the following properties:*

1. *if X is in V_S^n, then $QX \cong 0$ if and only if $X \in V_S^{n-1}$,*
2. *(universal property) given an additive functor $F\colon V_S^n \to \mathcal{D}$ such that $F(X) \cong 0$ whenever $X \in V_S^{n-1}$, there exists an additive functor $G\colon V_S^n/V_S^{n-1} \to \mathcal{D}$ such that $F = G \circ Q$.*

$$\begin{tikzcd} V_S^n \arrow[r, "Q"] \arrow[dr, "F"'] & V_S^n/V_S^{n-1} \arrow[d, "G"] \\ & \mathcal{D} \end{tikzcd}$$

Proof. Let I be the set of morphisms of V_S^n that factor through an object of V_S^{n-1}. Let $I(X,Y) \coloneqq I \cap \mathrm{Hom}_{V_S^n}(X,Y)$. We show $I(X,Y)$ is a subgroup of $\mathrm{Hom}_{V_S^n}(X,Y)$. For $i = 1,2$, let $f_i\colon X \to Y$ be in $I(X,Y)$, and write f_i as the composite $X \xrightarrow{g_i} Z_i \xrightarrow{h_i} Y$ where $Z_i \in V_S^{n-1}$. Then $f_1 + f_2$ is the composite

$$X \xrightarrow{\binom{g_1}{g_2}} Z_1 \oplus Z_2 \xrightarrow{(h_1\ h_2)} Y,$$

and $Z_1 \oplus Z_2$ lies in V_S^{n-1}. Also the zero morphism $0\colon X \to Y$ is the composite of the zero morphism in and out of the empty tuple so 0 lies in $I(X,Y)$. Hence $I(X,Y)$ is a subgroup. The set I is an ideal; that is, if $f \in \mathrm{Hom}_{V_S^n}(X,Y)$, $g \in I(Y,Z)$, and $h \in \mathrm{Hom}_{V_S^n}(W,X)$, then $gf \in I(X,Z)$ and $fh \in I(W,Y)$.

Define the additive category V_S^n/V_S^{n-1} as follows. The objects of V_S^n/V_S^{n-1} are as the same as V_S^n. Let $\mathrm{Hom}_{V_S^n/V_S^{n-1}}(X,Y)$ be the quotient group $\mathrm{Hom}_{V_S^n(X,Y)}(X,Y)/I(X,Y)$. Composition is well-defined because I is an ideal. The functor $Q\colon V_S^n \to V_S^n/V_S^{n-1}$ is given by the identity on objects and the quotient map on morphisms. It is clear Q is additive and essentially surjective.

Let $X \in V_S^n$. Suppose $QX \cong 0$. The isomorphism must be given by the classes of the maps $0 \to X$ and $X \to 0$. Hence the zero map $X \to 0 \to X$ is equivalent to the identity map id_X so $\mathrm{id}_X \in I(X,X)$. This means X has no terms of dimension n because we cannot decompose the identity map of a n-dimensional variety into locally closed immersions through a lower dimensional variety. Conversely, suppose $X \in V_S^{n-1}$. Then id_X is in $I(X,X)$ so id_X and the zero map $X \to 0 \to X$ are equivalent.

We verify the universal property. Given an object $QX \in V_S^n/V_S^{n-1}$, put $G(QX) = FX$. For a morphism in $\mathrm{Hom}_{V_S^n/V_S^{n-1}}(X,Y)$ represented by $f \in \mathrm{Hom}_{V_S^n}(X,Y)$, put $G(\overline{f}) = F(f)$. This is independent of representative: if $f' - f \in I(X,Y)$, then $f' - f$ factors through some $Z \in V_S^{n-1}$, so $F(f'-f)$ factors through $F(Z) \cong 0$, and $F(f') = F(f'-f) + F(f) = F(f)$. It is routine to verify G is an additive functor. □

Proposition 4.4.2 ([Stacks, Tag 02MX])**.** *There is an exact sequence*

$$K_0(V_S^{n-1}) \xrightarrow{K_0 i} K_0(V_S^n) \xrightarrow{K_0 Q} K_0(V_S^n/V_S^{n-1}) \to 0$$

of abelian groups, where the functor $i\colon V_S^{n-1} \to V_S^n$ is the localization of the additive functor $V^{n-1} \to V^n \to V_S^n$.

Proof. The composition $K_0(V_S^{n-1}) \to K_0(V_S^n/V_S^{n-1})$ is zero by construction of V_S^n/V_S^{n-1}, and the functor $K_0 Q$ is surjective because the objects of V_S^n and V_S^n/V_S^{n-1} are the same. We show the kernel of $K_0 Q$ is contained in the image of $K_0(V_S^{n-1})$.

Each element in $K_0(V_S^n)$ may be written as $[A] - [A']$, for some objects A and A' in V_S^n. Indeed, let $X = \sum_{i \in I} n_i [X_i] \in K_0(V_S^n)$. Then the objects $A = \bigoplus_{\{i \in I | n_i > 0\}} \bigoplus_{j=1}^{n_i} X_i$ and $A' = \bigoplus_{\{i \in I | n_i < 0\}} \bigoplus_{j=1}^{n_i} X_i$ in V_S^n fit the bill.

Let $[A] - [A']$ be in the kernel of $K_0 Q$, and put $B = QA$ and $B' = QA'$. Since $K_0 Q([A] - [A']) = 0$, there exists

1. a finite set $I = I^+ \sqcup I^-$, and

2. for each $i \in I$, an exact sequence $B_i \to B_i' \to B_i''$ in V_S^n/V_S^{n-1},

such that the equality

$$[B]-[B'] = \sum_{i\in I^+}([B_i']-[B_i]-[B_i'']) - \sum_{i\in I^-}([B_i']-[B_i]-[B_i''])$$

holds in the free abelian group of isomorphism classes of objects of V_S^n/V_S^{n-1}. Rewrite this as

$$[B]+\sum_{i\in I^+}([B_i]+[B_i''])+\sum_{i\in I^-}[B_i'] = [B']+\sum_{i\in I^-}([B_i]+[B_i''])+\sum_{i\in I^+}[B_i'].$$

Thus, there is a bijection of sets

$$\tau\colon \{B\}\sqcup\bigsqcup_{i\in I^+}\{B_i\}\sqcup\bigsqcup_{i\in I^+}\{B_i''\}\sqcup\bigsqcup_{i\in I^-}\{B_i'\}\to\{B'\}\sqcup\bigsqcup_{i\in I^-}\{B_i\}\sqcup\bigsqcup_{i\in I^-}\{B_i''\}\sqcup\bigsqcup_{i\in I^+}\{B_i'\}$$

such that, for all M in the domain of τ, there are isomorphisms $M\cong\tau(M)$ in V_S^n/V_S^{n-1}.

If $F_1\to F_2\to F_3$ is an exact sequence in V_S^n/V_S^{n-1}, then there is an exact sequence $E_1\to E_2\to E_3$ in V_S^n such that $QE_i\cong F_i$, for $i=1,2,3$. We are not asserting $QE_1\to QE_2\to QE_3$ is isomorphic to $F_1\to F_2\to F_3$ as an exact sequence, but rather there are only term-wise isomorphisms. Since V_S^n/V_S^{n-1} is split exact, there are objects X and Y in V_S^n/V_S^{n-1} such that $X\cong F_1$, $X\oplus Y\cong F_2$, and $Y\cong F_3$ in V_S^n/V_S^{n-1}. The sequence $X\to X\oplus Y\to Y$ in V_S^n will do the job.

For each $i\in I$, let $A_i\to A_i'\to A_i''$ be such an exact sequence in V_S^n corresponding to $B_i\to B_i'\to B_i''$. Then there is a bijection of sets

$$\pi\colon \{A\}\sqcup\bigsqcup_{i\in I^+}\{A_i\}\sqcup\bigsqcup_{i\in I^+}\{A_i''\}\sqcup\bigsqcup_{i\in I^-}\{A_i'\}\to\{A'\}\sqcup\bigsqcup_{i\in I^-}\{A_i\}\sqcup\bigsqcup_{i\in I^-}\{A_i''\}\sqcup\bigsqcup_{i\in I^+}\{A_i'\}$$

such that, for all N in the domain of π, the objects N and $\pi(N)$ are isomorphic after passing to V_S^n/V_S^{n-1} via Q. In $K_0(V_S^n)$, we have the equality

$$\begin{aligned}\sum_{N\in\text{ domain of }\pi}[N]-[\tau(N)] &= \left([A]+\sum_{i\in I^+}([A_i]+[A_i''])+\sum_{i\in I^-}[A_i']\right)-\left([A']+\sum_{i\in I^-}([A_i]+[A_i''])+\sum_{i\in I^+}[A_i']\right)\\ &= [A]-[A'],\end{aligned}$$

where the cancellations are due to the exact sequences $A_i \to A'_i \to A''_i$. It remains to see each $[N] - [\pi(N)]$ lies in the image of $K_0(V_S^{n-1})$. This will be proved in Lemma 4.4.4. $\square$

We record the following lemma for use in the proof of Lemma 4.4.4.

Lemma 4.4.3. *The functor* $\mathrm{bir}_n\colon\ \mathrm{Add}(\mathsf{Var}^n)_S \to \mathrm{Add}(\mathsf{Bir}^n)$ *descends to the quotient* V_S^n/V_S^{n-1}.

Proof. This follows from the universal property of the quotient. $\square$

Lemma 4.4.4. *Let X and Y be in V_S^n. If QX and QY are isomorphic in V_S^n/V_S^{n-1}, then the element $[X] - [Y]$ in $K_0(V_S^n)$ is in the image of $K_0(V_S^{n-1})$.*

Proof. Let I denote the image of $K_0(V_S^{n-1})$ in $K_0(V_S^n)$. We begin with two claims.

Claim 1. *If M and N are n-dimensional birational varieties, then $[M] - [N]$ lies in I.*

Proof of Claim 1. If $M \supseteq U \subseteq N$ is an isomorphic open, then

$$[M] - [N] = [U, M \setminus U] - [U, N \setminus U] = [M \setminus U] - [N \setminus U] \in I.$$

$\square$

Claim 2. *Let W be a n-dimensional variety with irreducible decomposition*

$$W = \left(\bigcup_i W_i\right) \cup \left(\bigcup_j W'_j\right),$$

where all $\dim W_i = n$ and all $\dim W'_j < n$. Then $[W] - \sum_i [W_i]$ lies in I.

Proof of Claim 2. Let

$$Z = \left(\bigcup_{i,i'} W_i \cap W_{i'}\right) \cup \left(\bigcup_{i,j} W_i \cap W_j\right) \cup \left(\bigcup_{j,j'} W_j \cap W_{j'}\right)$$

be the closed set in W consisting of points lying in at least two components of W. So $W \setminus Z$ is the disjoint union of the opens $W_i \setminus Z$ and $W'_j \setminus Z$. Then the element

$$\begin{aligned}
[W] - \sum_{i=1}^{r}[W_i] &= [W \setminus Z] + [Z] - \sum_{i=1}^{r}[W_i] \\
&= \left(\sum_{i=1}^{r}[W_i \setminus Z] + \sum_{j=1}^{s}[W'_j \setminus Z]\right) + [Z] - \sum_{i=1}^{r}[W_i]
\end{aligned}$$

of $K_0(V_S^n)$ lies in I because $W_i \setminus Z$ and W_i are birational for all i, and Z and each $W'_j \setminus Z$ are of dimension less than $n-1$. $\square$

Now onto the proof of Lemma 4.4.4. Write X' for the tuple consisting of terms of X of dimension n, and X'' for the tuple consisting of terms of X of dimension less than n; similiarly for Y' and Y''. There is a permutation isomorphism $X \cong X' \oplus X''$. In $K_0(V_S^n)$, we have

$$[X] - [Y] = [X'] - [Y'] + [X''] - [Y''],$$

so it suffices to show $[X'] - [Y']$ lies in I. There are isomorphisms

$$QX \cong Q(X' \oplus X'') \cong QX' \quad \text{and} \quad QY \cong Q(Y' \oplus Y'') \cong QY' \tag{4.2}$$

in V_S^n / V_S^{n-1}. Therefore the hypothesis gives an isomorphism $QX' \cong QY'$ in V_S^n / V_S^{n-1}. Write $X' = (X'_1, \cdots, X'_p)$ and $Y' = (Y'_1, \cdots, Y'_q)$, and let $X'_{i,1}, \cdots, X'_{i,n_i}$ be the n-dimensional components of X'_i, and similarly for $Y'_{i,1}, \cdots, Y'_{i,m_i}$. Let

$$M = \{(i,j) \mid 1 \leq i \leq p, 1 \leq j \leq n_i\} \quad \text{and} \quad N = \{(i,j) \mid 1 \leq i \leq q, 1 \leq j \leq m_i\}$$

be sets indexing the n-dimensional components of all the summands of X' and Y'. By Eq. (4.2) and Lemma 4.4.3, the tuples

$$\mathrm{bir}_n(QX') = \bigoplus_{(i,j)\in M} X'_{i,j} \quad \text{and} \quad \mathrm{bir}_n(QY') = \bigoplus_{(i,j)\in N} Y'_{i,j}$$

in $\mathrm{Add}(\mathsf{Bir}^n)$ are isomorphic. By Proposition 3.1.5, there is a bijection $\tau\colon M \to N$ of index sets such that $X'_{i,j}$ is birational to $Y'_{\tau(i,j)}$, for all $(i,j) \in M$. Lastly, in $K_0(V_S^n)/I$,

$$\begin{aligned} [X'] - [Y'] &= \sum_{(i,j)\in M} [X'_{ij}] - \sum_{(i,j)\in N} [Y'_{ij}] && \text{(Claim 2)} \\ &= \sum_{(i,j)\in M} [\tau(X'_{ij})] - \sum_{(i,j)\in N} [Y'_{ij}] && \text{(Claim 1)} \\ &= 0. && (\tau \text{ is bijective}) \end{aligned}$$

□

Proposition 4.4.5. *Consider the homomorphism*

$$\alpha\colon K_0(V_S^n/V_S^{n-1}) \to \mathbb{Z}[\mathsf{Bir}^n],$$

which sends the class of a tuple $(X_1, \cdots, X_r)$ in V_S^n/V_S^{n-1} to $\sum_{i,j}\{X_{ij}\}$, where $\{X_{ij}\}_{j\in J_i}$ is the set of n-dimensional components of X_i. Then α is a group isomorphism.

Proof. We may write α as the composite

$$K_0(V_S^n/V_S^{n-1}) \xrightarrow{K_0(\mathrm{bir}_n)} K_0(\mathrm{Add}(\mathsf{Bir}^n)) \overset{\cong}{\to} \mathbb{Z}[\mathsf{Bir}^n],$$

where the last isomorphism is from Corollary 3.1.6. Hence α is well-defined.

Let $\beta\colon \mathbb{Z}[\mathsf{Bir}^n] \to K_0(V_S^n/V_S^{n-1})$ be the homomorphism which sends the birational class $\{X\}$ of a variety X to the class of X in the Grothendieck group. It follows from Claim 1 in the proof of Lemma 4.4.4 that β is well-defined on birational classes.

It is clear $\alpha \circ \beta$ is the identity on $\mathbb{Z}[\mathsf{Bir}^n]$. Conversely, given a tuple $(X_1, \cdots, X_r)$ in V_S^n/V_S^{n-1}, we have

$$\beta\alpha([X_1, \cdots, X_r]) = \beta(\sum_{i,j}\{X_{ij}\}) = \sum_{i,j}[X_{ij}] = \sum_i [X_i],$$

where the last equality holds by Claim 2 in the proof of Lemma 4.4.4. □

5. COMPARISONS BETWEEN $K_0(\mathsf{Var}_k)$ and $K_0(\mathrm{Add}(\mathsf{Var}_k)_S)$

In this chapter, we show there is a surjection from the classical K_0-theory of at most n-dimensional varieties onto the K_0-theory of the localized additive completion of at most n-dimensional varieties. If $n = 0$ and the base field k is algebraically closed, then the surjection is an isomorphism, and we identify the K_0-theory of zero dimensional varieties with $\mathbb{Z}$.

Proposition 5.0.1. *For all $n \in \mathbb{N} \cup \{\infty\}$, there are surjective group homomorphisms*

$$\alpha_n \colon K_0(\mathsf{Var}^n) \to K_0(\mathrm{Add}(\mathsf{Var}^n)_S),$$

which sends the class of a variety X to the class of 1-tuple of X. Both α_∞ and α_0 respect the ring structure.

Proof. Let $[X] - [U] - [Z]$ in $K_0(\mathsf{Var}^n)$ be a relation in $K_0(\mathsf{Var}^n)$. This relation maps to $[X] - [U] - [Z]$ in $K_0(\mathrm{Add}(\mathsf{Var}^n)_S)$. But

$$[X] - [U] - [Z] = [U, Z] - [U] - [Z] = [U] + [Z] - [U] - [Z] = 0.$$

It is clear α_0 and α_∞ respect the ring structure. $\square$

Note the first equality in the proof of Proposition 5.0.1 motivated the construction of S.

Remark 5.0.2. The diagram

$$\begin{tikzcd}
K_0(\mathsf{Var}^{n-1}) \arrow[r, "f_{n-1,n}"] \arrow[d, "\alpha_{n-1}"] & K_0(\mathsf{Var}^n) \arrow[r, "\phi_n"] \arrow[d, "\alpha_n"] & \mathbb{Z}[\mathsf{Bir}^n] \arrow[r] \arrow[d, equal] & 0 \\
K_0(\mathrm{Add}(\mathsf{Var}^{n-1})_S) \arrow[r, "g_{n-1,n}"] & K_0(\mathrm{Add}(\mathsf{Var}^n)_S) \arrow[r, "\psi_n"] & \mathbb{Z}[\mathsf{Bir}^n] \arrow[r] & 0
\end{tikzcd} \tag{5.1}$$

commutes where the rows are from Theorems 4.2.4 and 4.3.5.

Proposition 5.0.3. *Suppose the base field k is algebraically closed. In Eq. (5.1), the morphisms ϕ_0, ψ_0, and α_0 are isomorphisms.*

Proof. Since k is algebraically closed, the only object in the category Bir^0 is the point $\operatorname{Spec} k$. Hence $\mathbb{Z}[\mathsf{Bir}^0]$ is isomorphic to $\mathbb{Z}$.

The surjectivity of ϕ_0 was shown in Theorem 4.2.4. We begin with the injectivity of ϕ_0. Suppose $\phi_0(\sum n_i[X_i]) = 0$. Recalling $\#X$ denotes the cardinality of a finite set X, we have

$$\sum_i n_i[X_i] = \sum_i n_i \cdot \#X_i[\operatorname{Spec} k]$$

and

$$0 = \phi_0(\sum_i n_i \cdot \#X_i[\operatorname{Spec} k]) = \sum_i n_i \cdot \#X_i.$$

This concludes the proof of injectivity of ϕ_0.

A similar argument shows ψ_0 is an isomorphism. The commutativity of Eq. (5.1) shows α_0 is an isomorphism. $\square$

Fix an algebraically closed field k. There is an equivalence of categories between the category of finite sets FinSet and the category of zero dimensional varieties Var_k^0. Under this equivalence of categories, let $S \subseteq \mathsf{FinSet}$ correspond to the left multiplicative system defined for Var_k^0 in Section 3.2.

Corollary 5.0.4. *There is a ring isomorphism*

$$\#\colon K_0(\mathrm{Add}(\mathsf{FinSet})_S) \to \mathbb{Z},$$

which sends the class of a tuple $(X_1, \cdots, X_r)$ *to* $\sum_i \#X_i$.

Proof. The map $\#$ is the composite of ring isomorphisms

$$K_0(\mathrm{Add}(\mathsf{FinSet})_S) = K_0(\mathrm{Add}(\mathsf{Var}_k^0)_S) \xrightarrow{\alpha_0} \mathbb{Z}[\mathsf{Bir}^0] \cong \mathbb{Z}.$$

$\square$

Question 5.0.5. We wish to show the surjective map $K_0(\mathsf{Var}_k) \to K_0(\mathrm{Add}(\mathsf{Var}_k)_S)$ is an isomorphism. Here is a possible approach. It suffices to show the surjective maps in Proposition 5.0.1 are isomorphisms because we could pass to the direct limit, and conclude the isomorphism from Propositions 4.2.2 and 4.3.3. We have already seen α_0 is an isomorphism if k is algebraically closed. Consider Eq. (5.1). If we could show the kernel of $f_{n,n-1}$ surjects onto the kernel of $g_{n,n-1}$, then the Five Lemma would show us inductively each α_n is an isomorphism, completing the proof.

6. MOTIVIC MEASURES

Invariants such as the Euler characteristic or Hasse-Weil zeta function are important tools for understanding the geometry of an algebraic variety. The classical Grothendieck group of varieties $K_0(\mathsf{Var}_k)$ is interesting because these invariants factor through $K_0(\mathsf{Var}_k)$. In this chapter, we show how to construct analogous functions on $K_0(\mathrm{Add}(\mathsf{Var}_k)_S)$. If k is a finite field, we construct the point-counting measures, the Hasse-Weil zeta function, and ℓ-adic Euler characteristic. If $k = \mathbb{C}$, we construct the Euler characteristic given by singular cohomology. We treat étale cohomology as a black box and refer the reader to standard references such as [16].

Definition 6.0.1. Let G be an abelian group. A *(motivic) measure (with values in G)* is a group homomorphism $K_0(\mathrm{Add}(\mathsf{Var}_k)_S) \to G$.

Notation 6.0.2. Let X be a variety over a field k. Let ℓ a prime different from the characteristic of k. Let $\bar{X}$ denote the base change of X to an algebraic closure of k.

1. Let $H^i_{\text{ét},c}(\bar{X}, \mathbb{Q}_\ell)$ denote the degree i compactly supported l-adic cohomology.
2. Let $\chi_c^{\text{ét},\ell}(X)$ denote the (compactly supported) l-adic Euler characteristic:

$$\chi_c^{\text{ét},\ell}(X) \coloneqq \sum_i (-1)^i \dim_{\mathbb{Q}_\ell} H^i_{\text{ét},c}(\bar{X}, \mathbb{Q}_\ell).$$

Notation 6.0.3. Let X be a variety over $\mathbb{C}$.

1. Let $H^i_c(X^{\text{an}}, \mathbb{Q})$ denote the degree i compactly supported singular cohomology of the analytic space X^{an} with rational coefficients.
2. Let $\chi_c(X)$ denote the (compactly supported) Euler characteristic:

$$\chi_c(X) \coloneqq \sum_i (-1)^i \dim_{\mathbb{Q}} H^i_c(X^{\text{an}}, \mathbb{Q}).$$

Example 6.0.4. Let k be a finite field and K/k a finite extension field. There is a functor

$$-(K) \colon \mathsf{Var}_k \to \mathsf{FinSet},$$

which assigns a variety X to the finite set $X(K)$ of its K-valued points. The *point counting motivic measure (associated to K_0)* is the group homomorphism

$$\#_K \colon K_0(\mathrm{Add}(\mathsf{Var}_k)_S) \to K_0(\mathrm{Add}(\mathsf{FinSet})_S) \xrightarrow{\#} \mathbb{Z},$$

which sends the class of the tuple $(X_1, \cdots, X_r)$ to $\#X_1(K) + \cdots + \#X_r(K)$. The first arrow is obtained by applying the functor $K_0(\mathrm{Add}(-))$ to $-(K)$. The second arrow is the isomorphism in Corollary 5.0.4.

Definition 6.0.5. Let $k = \mathbb{F}_q$. Let X be a variety over k. Let $\#_{\mathbb{F}_{q^m}}(X)$ be the cardinality of $X(\mathbb{F}_{q^m})$. The *Hasse-Weil zeta function* is the formal power series

$$Z(X,t) = \exp\left(\sum_{m\geq 1} \frac{\#_{\mathbb{F}_{q^m}}(X)}{m} t^m\right) \in 1 + t\mathbb{Q}[[t]].$$

Remark 6.0.6. If X is a variety over a finite field, the zeta function $Z(X,t)$ is rational. See, for instance, [17, Theorem 7.4.1].

Example 6.0.7. Let $k = \mathbb{F}_q$. We extend the zeta function to a motivic measure. Assemble the above point counting measures into the group homomorphism

$$Z(-,t) \colon K_0(\mathrm{Add}(\mathsf{Var}_k)_S) \to (1 + t\mathbb{Q}[[t]], \cdot),$$

which sends the class of $(X_1, \cdots, X_r)$ to the power series

$$\begin{aligned} \exp\left(\sum_{m\geq 1} \frac{\#_{\mathbb{F}_{q^m}}([X_1, \cdots, X_r])}{m} t^m\right) &= \exp\left(\sum_{i=1}^r \sum_{m\geq 1} \frac{\#_{\mathbb{F}_{q^m}}([X_i])}{m} t^m\right) \\ &= \prod_{i=1}^r \exp\left(\sum_{m\geq 1} \frac{\#_{\mathbb{F}_{q^m}}([X_i])}{m} t^m\right) \\ &= \prod_{i=1}^r Z(X_i, t). \end{aligned}$$

Remark 6.0.8. Let $f(t)$ be a rational function (a quotient of polynomials over $\mathbb{C}$) such that $f(0) = 1$. Then we may uniquely write f as a finite product corresponding to the zeros and poles of f:

$$f(t) = \prod_{z \in \mathbb{C}^\times} (1 - zt)^{-n_f(z)},$$

where $-n_f(z)$ is the order of zero or pole of f at $1/z$. The minus sign convention is required for Proposition 6.0.10 to hold.

If k is a finite field, we will define a family of motivic measures, indexed by the integers, which interpolate between the point counting measures and l-adic Euler characteristic.

Proposition 6.0.9. *Fix an integer m. Let $(X_1, \cdots, X_r)$ be a tuple in $\mathrm{Add}(\mathsf{Var}_k)_S$. Let $n_i(z)$ be the integers obtained from the decomposition of the rational function $Z(X_i, t)$ in Remark 6.0.8. Then there is a group homomorphism*

$$\Psi_m \colon K_0(\mathrm{Add}(\mathsf{Var})_S) \to \mathbb{C},$$

which maps the class of a tuple $(X_1, \cdots, X_r)$ to the finite sum

$$\sum_{z \in \mathbb{C}^\times} (n_1(z) + \cdots + n_r(z)) z^m.$$

Proof. We will decompose Ψ_m into two group homomorphisms.

Let $G \subseteq (1+t\mathbb{Q}[[t]], \cdot)$ be the multiplicative subgroup of rational power series (power series that can be written as a quotient of polynomials). The measure $Z(-,t)$ in Example 6.0.7 lands in G by Remark 6.0.6. Define the group homomorphism

$$\tau \colon (G, \cdot) \to (\mathbb{C}, +)$$

as follows. Given a rational power series f in G, let $n_f(z)$ be the integers from the decomposition in Remark 6.0.8. Define $\tau(f) := \sum_{z\in\mathbb{C}^\times} n_f(z)z^m$. This is clearly a group homomorphism. Therefore Ψ_m is the composite

$$K_0(\mathrm{Add}(\mathsf{Var})_S) \xrightarrow{Z(-,t)} G \xrightarrow{\tau} \mathbb{C}.$$

□

Proposition 6.0.10. *Let $k = \mathbb{F}_q$. Let $m \geq 1$ be an integer. Then Ψ_m is equal to the point counting measure $\#_{\mathbb{F}_{q^m}}$ in Example 6.0.4.*

Proof. It is enough to show both measures agree on the class of a 1-tuple (X), since these generate $K_0(\mathrm{Add}(\mathsf{Var})_S)$. Write $Z(X,t) = \prod_{z\in\mathbb{C}^\times}(1-zt)^{-n(z)}$. Recall the power series expansion $-\log(1-x) = \sum_{m\geq 1} x^m/m$. We compute:

$$\begin{aligned}
\sum_{m\geq 1} \#_{\mathbb{F}_{q^m}}(X)\frac{t^m}{m} = \log Z(X,t) &= \sum_{z\in\mathbb{C}^\times} -n(z)\log(1-zt) \\
&= \sum_{z\in\mathbb{C}^\times} n(z) \sum_{m\geq 1} \frac{(zt)^m}{m} \\
&= \sum_{m\geq 1}\left(\sum_{z\in\mathbb{C}^\times} n(z)z^m\right)\frac{t^m}{m} \\
&= \sum_{m\geq 1} \Psi_m(X)\frac{t^m}{m}.
\end{aligned}$$

□

Note we introduce the minus sign in Remark 6.0.8 to make the calculation in the proof of Proposition 6.0.10 work.

Proposition 6.0.11. *Let k be a finite field. Fix a prime ℓ different from the characteristic of k. Then Ψ_0 sends the class of a tuple $(X_1, \cdots, X_r)$ to $\sum_i \chi_c^{\text{ét},\ell}(X_i)$. In view of this result, we sometimes denote Ψ_0 by $\chi_c^{\text{ét},\ell}$.*

Proof. It is suffices to verify the claim on the class of a 1-tuple (X). Abbreviate $n(z) = n_{Z(X,t)}(z)$. By [16, Theorem 13.1], we have

$$\prod_{z\in\mathbb{C}^\times}(1-zt)^{-n(z)} = Z(X,t) = \prod_i \det(\mathrm{id} - tF \mid H^i_{\text{ét},c}(\bar{X},\mathbb{Q}_\ell))^{(-1)^{i+1}},$$

where $\bar{X}$ is the base change of X to an algebraic closure of k, and $\bar{F}$ is the Frobenius on $\bar{X}$. The degree of the polynomial $\det(\mathrm{id} - tF \mid H^i_{\text{ét},c}(\bar{X},\mathbb{Q}_\ell))$ in variable t is $\dim_{\mathbb{Q}_\ell} H^i_{\text{ét},c}(\bar{X},\mathbb{Q}_\ell)$. Therefore,

$$\Psi_0(X) = \sum_{z\in\mathbb{C}^\times} n(z) = \sum_i (-1)^i \dim_{\mathbb{Q}_\ell} H^i_{\text{ét},c}(\bar{X},\mathbb{Q}_\ell) = \chi_c^{\text{ét},\ell}(X).$$

$\square$

The remainder of this section is devoted to constructing a measure on $K_0(\mathrm{Add}(\mathsf{Var}_{\mathbb{C}})_S)$ analogous to the Euler characteristic.

Theorem 6.0.12. *Let $\tilde{X} = (\tilde{X}_1, \cdots, \tilde{X}_r)$ and $\tilde{Y} = (\tilde{Y}_1, \cdots, \tilde{Y}_s)$ be isomorphic tuples in $\mathrm{Add}(\mathsf{Var}_{\mathbb{C}})_S$. Then $\sum_i \chi_c(\tilde{X}_i^{\mathrm{an}}) = \sum_j \chi_c(\tilde{Y}_j^{\mathrm{an}})$. In other words, there is a group homomorphism*

$$\chi_c \colon K_0(\mathrm{Add}(\mathsf{Var}_{\mathbb{C}})_S) \to \mathbb{Z},$$

which maps the class of a tuple $(X_1, \cdots, X_n)$ to $\sum_i \chi_c(X_i^{\mathrm{an}})$.

Proof. We collect two claims and then prove the proposition.

Claim 1: *Let A be a finitely generated $\mathbb{Z}$-algebra contained in $\mathbb{C}$. Let U in $\operatorname{Spec} A$ be nonempty open. Fix a prime ℓ. Then there exists a closed point in U whose residue characteristic is different from ℓ.*

Proof of Claim 1: The structure morphism $F\colon \operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ maps a point to the prime ideal generated by its residue characteristic. Hence the set of points in U with residue characteristic different from ℓ is the open set $V \coloneqq F^{-1}(\operatorname{Spec}\mathbb{Z} \setminus \{\ell\}) \cap U$. We first argue V is nonempty, and V has a closed point.

The $\mathbb{Z}$-module A is flat by [Stacks, Tag 00HD]: the homomorphisms $n\mathbb{Z} \otimes_{\mathbb{Z}} A \to A$, are injective, for all n, because A is torsion free. By [18, Exercise III.9.1], F is an open map. Hence $F^{-1}(\ell) \cap U$ is a proper subset of U (otherwise U would map into $\{(\ell)\}$, which is not open in $\operatorname{Spec}\mathbb{Z}$.). In other words, V is nonempty.

$\operatorname{Spec} A$ is a Jacobson space ([Stacks, Tag 01P1, 00GB]) so the set of closed points of $\operatorname{Spec} A$ is dense. Hence there is a closed point in the open set V. □

Claim 2: *Let X be a variety over $\mathbb{C}$. Then $\chi_c^{\text{ét},\ell}(X) = \chi_c(X^{\text{an}})$.*

Proof of Claim 2. By Artin comparison ([19, Section 4.2]), there is an isomorphism

$$H^i_{\text{ét},c}(X, \mathbb{Q}_\ell) \cong H^i_c(X^{\text{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$

of $\mathbb{Q}_\ell$-vector spaces. Since $H^i_c(X^{\text{an}}, \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$, we have

$$\dim_{\mathbb{Q}_\ell} H^i_{\text{ét},c}(X, \mathbb{Q}_\ell) = \dim_{\mathbb{Q}} H^i_c(X^{\text{an}}, \mathbb{Q}).$$

The equality follows. □

Now onto the proof of the proposition. By Lemma 2.2.21, there is a diagram

$$\begin{tikzcd} \tilde{Z}_1 \arrow[r] & \tilde{Z}_2 \arrow[r] & \tilde{Z}_3 \arrow[r] \arrow[d] & \tilde{Y} \\ & & \tilde{X} & \end{tikzcd} \tag{6.1}$$

in $\mathrm{Add}(\mathsf{Var}_{\mathbb{C}})$ such that $\tilde{Z}_1 \to \tilde{Z}_3$ and $\tilde{Z}_2 \to \tilde{Y}$ are in S. There exists a finitely generated $\mathbb{Z}$-algebra A such that the diagram

$$\begin{tikzcd} Z_1 \arrow[r] & Z_2 \arrow[r] & Z_3 \arrow[r] \arrow[d] & Y \\ & & X & \end{tikzcd} \tag{6.2}$$

lies in $\mathrm{Add}(\mathsf{Sch}_{\operatorname{Spec} A})$, and whose base change to $\mathbb{C}$ is Eq. (6.1). Indeed, we can take A to be $\mathbb{Z}$ adjoined with all the coefficients of equations for all varieties appearing in the diagram.

Write $X = (X_1, \cdots, X_r)$ and $Y = (Y_1, \cdots, Y_s)$; in other words, $X_i \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbb{C} = \tilde{X}_i$ and similarly for Y_j.

Step 1: Fix a prime ℓ. Suppose W is a variety over $\operatorname{Spec} A$. By [20, Proposition 10.1.17, 10.1.5], there exists a dense open set $U \subseteq \operatorname{Spec} A$ such that for every geometric point $\operatorname{Spec} L \to U$, the étale cohomologies $H^k_{\text{ét},c}(W_L, \mathbb{Z}_\ell)$ are isomorphic, for all k. Let U_i and V_j be such open sets corresponding to X_i and Y_j. Put

$$U = \left(\bigcap_{i=1}^{r} U_i \right) \cap \left(\bigcap_{j=1}^{s} V_j \right).$$

By Claim 1, select a closed point $P \in U$ such that the characteristic of A/P is not ℓ. Put $K = A/P$. Then $\operatorname{Spec} \bar{K} \to \operatorname{Spec} K \to U$ is a geometric point in U. Also $\operatorname{Spec} \mathbb{C} \to U$ is a geometric point in U ($\operatorname{Spec} \mathbb{C}$ maps to the generic point of $\operatorname{Spec} A$, which lies in every nonempty open). Hence, for every summand W of X and Y, the groups $H^i_{\text{ét},c}(W_{\bar{K}}, \mathbb{Z}_\ell)$ and $H^i_{\text{ét},c}(\tilde{W}, \mathbb{Z}_\ell)$ are isomorphic, and we conclude

$$\chi_c^{\text{ét},\ell}(W_{\bar{K}}) = \chi_c^{\text{ét},\ell}(\tilde{W}).$$

Step 2: Claim 2 and Step 1 give the equalities

$$\sum_{i=1}^{r} \chi_c(\tilde{X}_i^{\text{an}}) = \sum_{i=1}^{r} \chi_c^{\text{ét},\ell}(\tilde{X}_i) = \sum_{i=1}^{r} \chi_c^{\text{ét},\ell}((X_i)_K) = \chi_c^{\text{ét},\ell}(X_K)$$

and

$$\sum_{j=1}^{s} \chi_c(\tilde{Y}_j^{\text{an}}) = \sum_{j=1}^{s} \chi_c^{\text{ét},\ell}(\tilde{Y}_j) = \sum_{j=1}^{s} \chi_c^{\text{ét},\ell}((Y_j)_K) = \chi_c^{\text{ét},\ell}(Y_K).$$

To complete the proof, it remains to see the right hand sides are equal. Reducing modulo P, we have the diagram

$$\begin{tikzcd}
(Z_1)_K \arrow[r] & (Z_2)_K \arrow[r] & (Z_3)_K \arrow[r] \arrow[d] & Y_K \\
& & X_K &
\end{tikzcd}$$

in $\mathrm{Add}(\mathsf{Var}_K)$. The morphisms $(Z_1)_K \to (Z_3)_K$ and $(Z_2)_K \to Y_K$ are in S because open and closed immersions, isomorphisms, direct sums, and compositions are closed under base change. Therefore X_K and Y_K are isomorphic in $\mathrm{Add}(\mathsf{Var}_K)$. Since P is a closed point of a finitely generated $\mathbb{Z}$-algebra, K is a finite field. Therefore we may apply Proposition 6.0.11 to conclude $\chi_c^{\text{ét},\ell}(X_K) = \chi_c^{\text{ét},\ell}(Y_K)$. □

Remark 6.0.13. In Chapter 5, we only established a surjection from the classical K_0-theory of varieties onto the K_0-theory of our localized category. A priori, there is the possibility that our K_0-theory is zero. However, in view of the measures we construct in this chapter, our K_0-theory is nonzero, at least in the case where k is the complex numbers or a finite field.

REFERENCES

[1] M. Larsen and V. Lunts, "Motivic measures and stable birational geometry," *Moscow Mathematical Journal*, vol. 3, no. 1, pp. 85–95, 2003.

[2] B. Poonen, "The grothendieck ring of varieties is not a domain," *Mathematical Research Letters*, vol. 9, no. 4, pp. 493–497, 2002. DOI: 10.4310/mrl.2002.v9.n4.a8. [Online]. Available: https://doi.org/10.4310%2Fmrl.2002.v9.n4.a8.

[3] L. A. Borisov, "The class of the affine line is a zero divisor in the grothendieck ring," *Journal of Algebraic Geometry*, vol. 27, no. 2, pp. 203–209, Jun. 2017. DOI: 10.1090/jag/701. [Online]. Available: https://doi.org/10.1090%2Fjag%2F701.

[4] F. Bittner, "The universal euler characteristic for varieties of characteristic zero," *Compositio Mathematica*, vol. 140, no. 04, pp. 1011–1032, Jul. 2004.

[5] M. Mustaţă, *Zeta functions in algebraic geometry*. [Online]. Available: http://www-personal.umich.edu/~mmustata/.

[6] I. Zakharevich, "The k-theory of assemblers," *Advances in Mathematics*, vol. 304, pp. 1176–1218, Jan. 2017.

[7] J. A. Campbell, "The K-theory spectrum of varieties," *Transactions of the American Mathematical Society*, vol. 371, no. 11, pp. 7845–7884, Feb. 2019.

[8] J. A. Campbell and I. Zakharevich, *Devissage and localization for the grothendieck spectrum of varieties*, 2018.

[9] C. Weibel, *The K-book: An Introduction to Algebraic K-theory*. American Mathematical Society (AMS), 2013.

[10] V. Srinivas, *Algebraic K-Theory*. Birkhäuser Boston, 1991.

[11] S. Mac Lane, *Categories for the Working Mathematician*. Springer New York, 1971.

[Stacks] Stacks project authors. "The stacks project." (2021), [Online]. Available: https://stacks.math.columbia.edu.

[12] M. Kashiwara and P. Schapira, *Categories and Sheaves*. Springer Berlin Heidelberg, 2006.

[13] A. Neeman, *Triangulated Categories*. Princeton University Press, Dec. 2001.

[14] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*. Springer Berlin Heidelberg, 1967. DOI: 10.1007/978-3-642-85844-4.

[15] M. Schlichting, "Higher algebraic k-theory (after quillen, thomason and others)," in *Topics in Algebraic and Topological K-Theory*, Springer Berlin Heidelberg, Sep. 2010, pp. 167–241.

[nLab] nLab authors, *Nlab*, 2021. [Online]. Available: https://ncatlab.org.

[EGA4] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique IV*, ser. Publications Mathématiques. Institute des Hautes Études Scientifiques., 1964–1967, vol. 20, 24, 28, 32.

[16] J. S. Milne, *Etale Cohomology (PMS-33)*. Princeton University Press, Dec. 1980.

[17] B. Poonen, *Rational Points on Varieties*. American Mathematical Society, Dec. 2017.

[18] R. Hartshorne, *Algebraic Geometry*. Springer New York, 1977.

[19] J.-P. Serre, *Lectures on $N_X(p)$*. CRC Press, 2011.

[20] L. Fu, *Etale Cohomology Theory*. World Scientific, Jan. 2011.

VITA

Harrison was born in Los Angeles, CA. He graduated in 2012 from the University of California, San Diego with a Bachelor of Science in Mathematics. In 2013, he began the mathematics graduate program at Purdue University. He participated in the 2021 Winter IMA Math-to-Industry Bootcamp. He is currently an intern at Oak Ridge National Laboratory, where he uses deep learning techniques to compress fusion energy data.