P-ADIC MEASURES FOR RECIPROCALS OF L-FUNCTIONS OF TOTALLY REAL NUMBER FIELDS

by

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A Dissertation

Submitted to the Faculty of Purdue University

In Partial Fulfillment of the Requirements for the degree of

Doctor of Philosophy

Department of Mathematics

West Lafayette, Indiana

August 2021

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To my beloved family

ACKNOWLEDGMENTS

This thesis was only possible thanks to the efforts of a whole village. Undoubtedly, I will forget to mention some people who have positively influenced me throughout my career and my life.

I would like to thank my advisor Professor Freydoon Shahidi for suggesting the thesis topic and guiding me through my graduate years. Your patience, kindness, and generosity went above and beyond. This thesis could not have been finished without your help.

I would also like to thank my committee members David Goldberg, Baiying Liu, and Tong Liu for their input and suggestions. Moreover, I would like to thank Ellen Eischen, Andrei Jorza, and A. Raghuram for helpful discussions about my research topic.

I would like to thank the Department of Mathematics for supporting my professional development. I am very grateful to all the staff members for all their hard work.

I would also like to thank Donatella Danielli and the AWM Purdue Chapter. My time in the department would not have been the same without you.

I am greatly indebted to Kamal Khuri-Makdisi. Thank you for your friendship and mentorship and thank you for sharing your wealth of knowledge.

Mama, Baba, Rashad, Sana, Farah, and Alex - thank you for always believing in me. Your immense support and cheering kept me going every day. Thank you for always traveling to see me and vacationing with me during rain, sun, snow, and global pandemics.

I would like to thank Alex Barrios for being the best older mathematical sibling. Thank you for all the advice and opportunities that you have given me. I would also like to thank Lidia Mrad and Katy Yochman for helping me settle down and acclimate to life in the US.

To all my friends, thank you for sharing the journey with me.

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ABSTRACT

We generalize the work of Gelbart, Miller, Pantchichkine, and Shahidi on constructing p-adic measures to the case of totally real fields K. This measure is the Mellin transform of the reciprocal of the p-adic L-function which interpolates the special values at negative integers of the Hecke L-function of K. To define this measure as a distribution, we study the non-constant terms in the Fourier expansion of a particular Eisenstein series of the Hilbert modular group of K. Proving the distribution is a measure requires studying the structure of the Iwasawa algebra.

INTRODUCTION

Motivation

The idea that special values of L-functions encode information about the arithmetic objects they are attached to is a recurring theme in Number Theory. For example, the class number formula relates the residue at $s = 1$ of the Dedekind zeta function to the class number of the number field. The Birch and Swinnerton Dyer Conjecture, one of the most difficult problems, predicts that the value at $s = 1$ of the L-function attached to an elliptic curve relates to the rank of a certain abelian group of the elliptic curve.

In the mid nineteenth century, Kummer [1] proved that the special values of the Riemann zeta function satisfy nice congruences modulo prime powers for a given odd prime p. Kubota and Leopoldt [2] reformulated these congruences in 1964 and introduced a p-adic analogue to the Riemann zeta function. The special values of the Kubota Leopoldt p-adic zeta function at negative integers interpolate those of the classical Riemann zeta function, and furthermore the Kummer congruences translate to the continuity of this function p-adically. The family of p-adic L-functions arising from certain congruences in a similar fashion to the Kubota Leopoldt p-adic zeta function are known as analytic p-adic L-functions.

In line with the duality between the analytic and arithmetic pictures, Iwasawa [3] discovered a deeper connection between p-adic L-functions and Galois modules over towers of cyclotomic fields. The p-adic L-functions which arise following this method are called arithmetic p-adic L-functions. The main conjecture of Iwasawa theory, proven by Mazur and Wiles [4], along with several of its generalizations, essentially states that the analytic and arithmetic p-adic L-functions are the same object. These statements translate to saying that the values of p-adic L-functions at negative integers encode arithmetic information.

This significance leads us to the question of constructing p-adic L-functions in order to study the arithmetic objects. We will start by describing some known constructions in greater details.

Kubota-Leopoldt p-adic L-function

Recall that the values of the Riemann zeta function at negative integers are given by the Bernoulli numbers. For any $k \geq 1$, we have

$$\zeta(1-k) = -\frac{B_k}{k} \tag{1}$$

where the Bernoulli numbers B_k are the coefficients of the power series expansion

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}. \tag{2}$$

Kummer studied the Bernoulli numbers in order to understand when a prime p is regular - that is, when p does not divide the class number of the p^{th} cyclotomic field. He discovered the following congruences:

Theorem 0.0.1 (Kummer Congruences)**.** *([5] Corollary 5.14) Suppose that* $m \equiv n \not\equiv 0 \bmod p-1$ *are positive, even integers. Then*

$$\frac{B_m}{m} \equiv \frac{B_n}{n} \bmod p.$$

More generally, if m *and* n *are positive even integers with* $m \equiv n \bmod (p-1)p^a$ *and* $n \not\equiv 0 \bmod p-1$*, then*

$$(1-p^{m-1})\frac{B_m}{m} \equiv (1-p^{n-1})\frac{B_n}{n} \bmod p^{a+1}.$$

The values $(1-p^{k-1})\frac{B_k}{k}$ are the special values of the Riemann zeta function at negative integers with the Euler p factor removed. Kubota and Leopoldt then defined the p-adic zeta function for $k \geq 1$ to be

$$\zeta_p(1-k) = (1-p^{k-1})\zeta(1-k). \tag{3}$$

The p-adic continuity of this function depends heavily on the congruences between Bernoulli numbers which are difficult to generalize to other cases.

Mazur reinterpreted the Kummer congruences as p-adic measures. A p-adic measure is a $\mathbb{Z}_p$-linear map $\mu : C(\mathbb{Z}_p, \mathbb{Z}_p) \to \mathbb{Z}_p$, where $C(\mathbb{Z}_p, \mathbb{Z}_p)$ denotes the ring of all $\mathbb{Z}_p$-valued continuous functions on $\mathbb{Z}_p$. A key fact is that given any p-adic measure μ, if $m \equiv n \bmod (p-1)p^a$, then

$$\int_{\mathbb{Z}_p^*} x^m \, d\mu \equiv \int_{\mathbb{Z}_p^*} x^n \, d\mu \bmod p^{a+1}. \tag{4}$$

It ensures that there exists a $\mathbb{Z}_p$-valued measure that produces the values of the p-adic function when it is evaluated against the functions $f(x) = x^k$.

Theorem 0.0.2 ([6])**.** *For each integer $a \geq 2$ prime to p, there exists a $\mathbb{Z}_p$-valued measure $\mu^{(a)}$ such that for any $k \geq 0$*

$$\int_{\mathbb{Z}_p} x^k \, d\mu^{(a)} = (1 - a^{k+1})\zeta(-k).$$

Eisenstein Measures

Serre [7] gave a different construction of the p-adic zeta function as the constant term of the q-expansion of a p-adic Eisenstein series. It is well known that the holomorphic Eisenstein series on the full modular group $SL_2(\mathbb{Z})$

$$E_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{(mz+n)^k} \tag{5}$$

has the Fourier series expansion

$$E_k(z) = 2\zeta(k) + \frac{2(-2\pi \mathrm{i})^k}{(k-1)!} \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \right) \exp(nz) \tag{6}$$

for any even $k \geq 4$, where $\exp(z) = \mathrm{e}^{2\pi \mathrm{i} z}$. The sums $\sigma_k(n) = \sum_{d|n} d^k$ are called divisor functions. The divisor functions with the p factor removed

$$\sigma_k^*(n) = \sum_{\substack{d|n \\ p\nmid d}} d^k \tag{7}$$

satisfy congruences similar to the Kummer congruences, namely $\sigma_k(n)^* \equiv \sigma_{k'}(n)^* \bmod p^m$ whenever $k \equiv k' \bmod p^{m-1}(p)$.

Serre considered $\sigma_k(n)^*$ as a p-adic limit of a collection of divisor functions $\{\sigma_{k_{\mathrm{i}}}(n)\}$. There is a collection $\{E_{\mathrm{i}}\}$ of classical Eisenstein series with compatible weights k_{i} that have non-constant terms $\{\sigma_{k_{\mathrm{i}}}(n)\}$. The p-adic properties of the non-constant terms of $\{E_{\mathrm{i}}\}$ imply similar congruences for the constant term. Serre thus defined a p-adic Eisenstein series on $SL_2(\mathbb{Z})$ formally as a q-expansion which is the p-adic limit of the q-expansions of $\{E_{\mathrm{i}}\}$:

$$E_k^* = 2\zeta^*(1-k) + \frac{2(-2\pi\mathrm{i})^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}^*(n)\exp(nz). \tag{8}$$

The p-adic zeta function in this case is the constant term of the p-adic Eisenstein series.

Serre's construction is the first instance of objects we now refer to as Eisenstein measures because of the connection between Eisenstein series and L-functions. In fact, when Langlands studied Eisenstein series for general reductive groups G [8], he showed that the constant terms of these Eisenstein series are quotients of products of L-functions. This method of constructing Eisenstein measures thus has the possibility of being generalized since L-functions show up as the constant terms of a large class of Eisenstein series.

In the early 1980s, Shahidi calculated the non-constant terms of Eisenstein series and showed that L-functions occur in their denominators [9]. Around the same period of time, Langlands wrote a letter to Gelbart concerning a conversation he had with Coates about the possibility of constructiong p-adic L-functions by using the Langlands-Shahidi method. However, these ideas were forgotten until Gelbart stumbled on the letter again in 2010. At that time, Gelbart, Miller, Panchishkin, and Shahidi had already started studying p-adic congruences between the non-constant terms of Eisenstein series on $SL_2(\mathbb{Z})$.

In [10], the authors study the partial Eisenstein series

$$\varepsilon_{k,p^m}(b) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1 \\ d\equiv b \bmod p^m}} (p^m cz + d)^{-k}. \tag{9}$$

They calculate the p^mth non-constant term as

$$\frac{(-2\pi\mathrm{i})^k}{p^{mk}\Gamma(k)}\sum_{\mathrm{j}=0}^{m} p^{\mathrm{j}(k-1)}\sum_{\substack{n\neq 0 \\ p\nmid n}}\frac{\mu(|n|)}{n^k}\exp(\overline{n}p^{\mathrm{j}}b/p^m) \tag{10}$$

where $\overline{n}$ denotes the inverse of n mod p^m. Up to a correction factor which depends only on p and $\zeta(1-k)$, these terms define a p-adic measure μ_k^* whose Mellin transform is the reciprocal of the Kubota-Leopoldt p-adic zeta function

$$\int_{\mathbb{Z}_p^*} x^{k-1}\, d\mu_k^*(x) = \zeta_p(1-k)^{-1} \tag{11}$$

for all positive even integers k.

Main Result

Our main result generalizes this construction to Eisenstein series on $SL_2(\mathcal{O})$ when $\mathcal{O}$ is the ring of integers of a totally real number field K. The Hecke L-function attached to a character χ with conductor $\mathfrak{n}$ is given by

$$L(s,\chi) = \sum_{(\mathfrak{a},\mathfrak{n})=1} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{[\mathfrak{a}]} \chi([\mathfrak{a}]) \sum_{\mathfrak{b}\in[\mathfrak{a}]} \frac{1}{N(\mathfrak{b})^s} \tag{12}$$

where $[\mathfrak{a}]$ denotes the class of the ideal $\mathfrak{a}$. Siegel [11] proved that the values of the partial L-function $\sum_{\mathfrak{b}\in[\mathfrak{a}]} N(\mathfrak{b})^{k-1}$ are rational, which led to several different constructions of the p-adic L-function for a totally real field [12], [13],[14],[15].

If we denote the degree of extension of K over $\mathbb{Q}$ by r and its class number by h, then $SL_2(\mathcal{O})$ embeds componentwise into $SL_2(\mathbb{R})^r$ and has h inequivalent cusps as a Riemann surface. Let $\mathcal{C} \in C_K$ be an ideal class of K and let $\mathfrak{a}$ be any integral ideal in $\mathcal{C}$. The Eisenstein series of weight k attached to the ideal class $\mathcal{C}$ is defined by

$$E_{k,\mathcal{C}}(z) = \sum_{(c,d)\in\mathfrak{a}\times\mathfrak{a}/\mathcal{O}^*} \frac{N(\mathfrak{a})^k}{(cz+d)^k}. \tag{13}$$

In this case, $E_{k,\mathcal{C}}$ has a Fourier series expansion at each of the cusps κ. When $\kappa=\infty$, this expansion is given by

$$E_{k,\mathcal{C}} = \zeta_{K,\mathcal{C}^{-1}}(k) + \frac{(2\pi \mathrm{i})^{kr}}{(k-1)!^r} |D|^{1/2-k} \sum_{\substack{\nu\equiv 0 \bmod \mathfrak{d}^{-1} \\ \nu\gg 0}} \sigma_{k-1}(\nu\mathfrak{d}) \exp(\nu z) \tag{14}$$

where $\sigma_{k-1}(\mathfrak{n})$ is the divisor function defined by

$$\sigma_{k-1}(\mathfrak{n}) = \sum_{\mathfrak{a}|\mathfrak{n}} N(\mathfrak{a})^{k-1}. \tag{15}$$

The expansions at the other cusps κ' can be calculated by translating κ' to ∞ through matrix multiplication.

Following the ideas in [10], we study the partial Eisenstein series

$$\varepsilon_{k,\mathfrak{p}^m,\mathfrak{b}_\mathrm{j}}(\mathfrak{a}) \coloneqq (\alpha_3 A^{-1}z + \alpha_4)^k \sum_{\substack{\{c,d\}\in\mathfrak{a}\times\mathfrak{a} \\ \gcd\left(\frac{c}{\mathfrak{a}},\frac{d}{\mathfrak{a}}\right)=1 \\ d\equiv a_2 \bmod \mathfrak{p}^m\mathfrak{a}}} \frac{N(\mathfrak{a})^k}{(\mathfrak{p}^m c A^{-1}z + d)^k}. \tag{16}$$

The $\mathfrak{p}^m$th coefficient in its Fourier series expansion is given by

$$\frac{(-2\pi\mathrm{i})^{kr} N(\mathfrak{b}_\mathrm{j})^k}{(k-1)!^r |D|^{\frac{2k-1}{2}}} \sum_{u=0}^{m} N(\mathfrak{p})^{u(k-1)-mk} \sum_{\mathrm{i}=1}^{h^{\mathfrak{p}^m,+}} \sum_{\mathfrak{t}\in C_\mathrm{i}} \frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k} \exp\left(\frac{\mathfrak{t}_\mathrm{i} a_2 \mathfrak{p}^u \mathfrak{b}_\mathrm{j}}{\mathfrak{p}^m \mathfrak{d}}\right). \tag{17}$$

We can define a measure λ to be equal to this coefficient plus a correction factor. We now state our main theorem:

Theorem 0.0.3. *Let $p \in \mathbf{Q}$ be an odd prime number such that p does not divide $[K : \mathbb{Q}]$ or the class number of $K(\mathrm{e}^{2\pi\mathrm{i}/p})$. Assume further that no prime $\wp$ of the field $K(\mathrm{e}^{2\pi\mathrm{i}/p} + \mathrm{e}^{-2\pi\mathrm{i}/p})$ lying above p splits in $K(\mathrm{e}^{2\pi\mathrm{i}/p})$. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}$ lying above p and h^+ the strict class number of K.*

1. *There exists a p-adic measure λ on $G_{K,\mathfrak{p}}$ whose Mellin transform is the reciprocal of the p-adic L-function of the totally real field K*

$$\int_{G_{K,\mathfrak{p}}} N(x)^{k-1}\, d\lambda = h^+(1 - N(\mathfrak{p})^{k-1})^{-1}\zeta(1-k)^{-1}$$

   for all even positive integers k.

2. *For any non-trivial Hecke character of finite type $\chi \mod \mathfrak{p}^m$ with $m > 0$, we have*

$$\int_{G_{K,\mathfrak{p}}} \chi(x) N(x)^{k-1}\, d\lambda = h^+ L(1-k,\chi)^{-1}$$

for any positive integer k satisfying the parity condition $\chi(-1) = (-1)^k$.

Moreover, the measure λ can be expressed in terms of Fourier coefficients of the Eisenstein series defined by χ on the Hilbert upper half plane.

The conditions on p are those that ensure that Kummer's criterion holds for K [16]. Greenberg mentions in the introduction to *loc. cit.* that the condition $p \nmid [K : \mathbb{Q}]$ may be relaxed. However, the other conditions are the regularity conditions on p and cannot be removed.

We now give an outline of this thesis.

Outline

Chapter 1 presents the notation and framework of this work. We introduce the ray class groups of a number field and recall Artin's reciprocity theorem from Class Field Theory. This map forms an isomorphism between the strict ray class group of K modulo $\mathfrak{p}^\infty$ and the Galois group of the maximal unramified-outside-$\mathfrak{p}$ extension of K, which is the domain of definition of our measure.

Chapter 2 discusses the L-functions which form the core of this dissertation. In Section 2.1, we show that Hecke characters which are trivial at the infinite places are equivalent to characters on $\mathcal{O}/\mathfrak{n}$ for some integral ideal $\mathfrak{n}$ of K. We define Hecke L-functions in Section 2.2 and state their main properties, then we study the Kummer congruences that occur between their special values at negative integers in Section 2.3.

Chapter 3 describes the construction of p-adic measures for general number fields. Sections 3.1 and 3.2 introduce p-adic measures on Galois groups and Section 3.3 presents the specific case when K is totally real.

Chapter 4 describes the structure of the Iwasawa algebra. In Section 4.1, we describe its structure analytically as a power series ring by using the Mahler transform. In Section 4.2, we describe its structure algebraically. This allows us to define in Section 4.3 the p-adic L-function of K as the Mellin transform of the p-adic measure.

Our main approach to construct p-adic measures is by studying Eisenstein series. Chapter 5 presents the background material on the Hilbert modular group and Hilbert modular forms.

In Chapter 6, we study certain partial Eisenstein series of weight k and level $\Gamma_0(\mathfrak{n})$. We define them carefully at all the cusps in Section 5.1 and derive their Fourier series expansions in Section 5.2.

This is the main computation that leads to the definition of a reciprocal p-adic measure in Chapter 7. In Section 7.1, we state the definition of λ as a distribution and show that it satisfies the required properties. Section 7.2 gives some generalities about elements of the Iwasawa algebra which are needed to prove that λ is a measure in Section 7.3.

1. CLASS FIELD THEORY

Our first goal is to explain how to generalize p-adic integration over $\mathbb{Z}_p$ to higher dimensions. The most natural domain of integration turns out to be

$$\varprojlim_n \{\text{prime-to-}p \text{ fractional ideals of } K\} \,/\, \{< \alpha >: \alpha \in K, \alpha \gg 0, \alpha \equiv 1 \bmod p^n\}\,. \tag{1.1}$$

This turns out to be the strict ray class group of the maximal field extension of K which is unramified outside p. In this chapter, we will present describe this isomorphism in detail.

The class group C_F of a number field F is a finite abelian group that measures how much the ring of integers of F fails at being a unique factorization domain. In 1927, Artin proved his celebrated reciprocity law, which gives an isomorphism between certain ideal class groups and Galois groups. Almost 75 years prior, in 1853, Kronecker had declared that every finite abelian extension of $\mathbb{Q}$ lies in some cyclotomic field $\mathbb{Q}(\zeta_m)$, a theorem which is now known as the Kronecker-Weber theorem.

Weber was interested in generalizing Dirichlet's prime number theorem, which states that there are infinitely many prime numbers of the form $p + kq$ if p and q are two coprime numbers. A key element in the proof of Dirichlet's theorem is the non-vanishing of the Dirichlet L-function at $s = 1$. For this reason, in 1897, Weber introduced the notion of the ray class groups. These groups generalize the class group of F. Moreover, each ray class group corresponds through Artin reciprocity to the Galois group of a class field which generalizes some cyclotomic field. In fact, a generalization to the Kronecker-Weber theorem states that every finite abelian extension of F is contained in some class field. Moreover, Weber introduced an L-function attached to ray class characters and consequently proved the first inequality of class field theory.

1.1 The Base Field

We fix the following notation throughout the dissertation:

Let K be a totally real number field of degree r over $\mathbb{Q}$. This means that there are r embeddings of K into $\mathbb{C}$ that all lie in $\mathbb{R}$. We fix an ordering of the r embeddings $K \hookrightarrow \mathbb{R}$.

For any $\alpha \in K$, we denote by $\alpha^{(1)}, \cdots, \alpha^{(r)}$ the conjugates of α. We say α is totally positive, denoted $\alpha \gg 0$, if $\alpha^{(\mathrm{i})} \geq 0$ for all $1 \leq \mathrm{i} \leq r$.

Let $\mathcal{O}$ denote the ring of integers of K. $\mathcal{O}$ is a lattice in K and its dual lattice is the largest fractional ideal $\mathcal{O}^{\vee} \subset K$ consisting of elements whose trace is a rational integer. The different ideal $\mathfrak{d}$ is the inverse ideal of $\mathcal{O}^{\vee}$:

$$\mathfrak{d} = (\mathcal{O}^{\vee})^{-1} = \{\alpha \in K \mid \alpha\mathcal{O}^{\vee} \subset \mathcal{O}\}. \tag{1.2}$$

Let $N : K \to \mathbb{Q}$ denote the norm map on K given by $N(\alpha) = \prod_{\mathrm{i}=1}^{r} \alpha^{(\mathrm{i})}$ for any $\alpha \in K$. For an integral ideal $\mathfrak{a}$, its norm is the size of the quotient $N(\mathfrak{a}) = |\mathcal{O}/\mathfrak{a}|$. The discriminant D of K in absolute value is the norm of the different ideal $N(\mathfrak{d}) = |D|$.

1.2 Ray Class Groups

In this section, we will describe the ray class groups which are used in the ideal theoretic formulation of class field theory. We follow the presentation given in Chapter 3 of [17].

A modulus $\mathfrak{m} = \mathfrak{m}_0\mathfrak{m}_\infty$ is a formal product where the finite part $\mathfrak{m}_0 = \prod_{\mathfrak{p}<\infty} \mathfrak{p}^{n_\mathfrak{p}}$ is an integral ideal and the infinite part $\mathfrak{m}_\infty$ is the set of real embeddings of K into $\mathbb{C}$. Given two moduli $\mathfrak{m} = \mathfrak{m}_0\mathfrak{m}_\infty$ and $\mathfrak{n} = \mathfrak{n}_0\mathfrak{n}_\infty$, we say $\mathfrak{n}$ divides $\mathfrak{m}$, denoted $\mathfrak{n} \mid \mathfrak{m}$, if $\mathfrak{n}_0 \supset \mathfrak{m}_0$ and $\mathfrak{n}_\infty \subset \mathfrak{m}_\infty$.

A non-zero fractional ideal $\mathfrak{a}$ of K is coprime to $\mathfrak{m}$, denoted $(\mathfrak{a}, \mathfrak{m}) = 1$, if we can write $\mathfrak{a} = \frac{\mathfrak{b}}{\mathfrak{c}}$, where $\mathfrak{b}$ and $\mathfrak{c}$ are integral ideals such that $\mathfrak{b} + \mathfrak{m}_0 = \mathfrak{c} + \mathfrak{m}_0 = \mathcal{O}_K$.

Let $I_K^{\mathfrak{m}} = \{\mathfrak{a} \text{ fractional ideal of } K \mid (\mathfrak{a}, \mathfrak{m}) = 1\}$ denote the subgroup of fractional ideals of K which are coprime to $\mathfrak{m}$. We denote by $K^{\mathfrak{m}}$ the set $\{\alpha \in K^* \mid (\langle\alpha\rangle, \mathfrak{m}) = 1\}$.

An element $\alpha \in K^*$ is multiplicatively congruent to 1 modulo $\mathfrak{m}$, denoted $\alpha \equiv 1 \bmod {}^{\times}\mathfrak{m}$, if $\langle\alpha\rangle = \frac{\mathfrak{b}}{\mathfrak{c}} \in I_K^{\mathfrak{m}}$ and $\alpha \in 1 + \mathfrak{m}\mathfrak{c}^{-1}$. We denote the set of all such α by $K^{\mathfrak{m},1}$.

Let $P_K^{\mathfrak{m}} = \{\langle\alpha\rangle \mid \alpha \in K, \alpha \equiv 1 \bmod {}^{\times}\mathfrak{m}\}$ denote the subgroup of fractional ideals $\langle\alpha\rangle$ in $I_K^{\mathfrak{m}}$ which are generated by $\alpha \in K^{\mathfrak{m},1}$.

Definition 1.2.1. *The ray class group modulo* $\mathfrak{m}$ *is the quotient given by*

$$C_K^{\mathfrak{m}} = I_K^{\mathfrak{m}}/P_K^{\mathfrak{m}}.$$

If $\mathfrak{m} = \mathcal{O}$ *is the trivial modulus, this is the usual class group*

$$C_K = \{\textit{fractional ideals of } K\}/\{\textit{principal ideals of } K\}.$$

Theorem 1.2.1. *We have an exact sequence*

$$1 \to \mathcal{O}^* \cap K^{\mathfrak{m},1} \to \mathcal{O}^* \to (\mathcal{O}/\mathfrak{m})^* \to C_K^{\mathfrak{m}} \to C_K \to 1.$$

In particular, the ray class group modulo $\mathfrak{m}$ *is finite.*

We denote the cardinality of $C_K^{\mathfrak{m}}$ by $h^{\mathfrak{m}}$ and that of C_K by h.

Proof. The map $\mathcal{O}^* \cap K^{\mathfrak{m},1} \to \mathcal{O}^*$ is the natural embedding and the map $\mathcal{O}^* \to (\mathcal{O}/\mathfrak{m})^*$ is the natural surjection. Given an element $\alpha \in \mathcal{O}^*$, we denote its image in $(\mathcal{O}/\mathfrak{m})^*$ by $\overline{\alpha}$. By definition, $\mathcal{O}^* \cap K^{\mathfrak{m},1}$ is the kernel of the map $\mathcal{O}^* \to (\mathcal{O}/\mathfrak{m})^*$.

Moreover, any ideal class contains an ideal which is coprime to $\mathfrak{m}$, so the map $C_K^{\mathfrak{m}} \to C_K$ is the natural surjection.

The only non-trivial map is $(\mathcal{O}/\mathfrak{m})^* \to C_K^{\mathfrak{m}}$. This map sends any element $\overline{\alpha} \in (\mathcal{O}/\mathfrak{m})^*$ to the ideal class of $\langle\alpha\rangle$ in $C_K^{\mathfrak{m}}$. First, we need to show that it is well-defined. Suppose that $\overline{\alpha}, \overline{\beta} \in (\mathcal{O}/\mathfrak{m})^*$ are such that the class of $\langle\alpha\rangle$ and $\langle\beta\rangle$ are equal in $C_K^{\mathfrak{m}}$. This means that $\langle\alpha\rangle\langle\beta\rangle^{-1} \in P_K^{\mathfrak{m}}$, or equivalently $\frac{\alpha}{\beta} \equiv 1 \bmod {}^{\times}\mathfrak{m}$. Hence $\overline{\alpha}$ and $\overline{\beta}$ are equal in $(\mathcal{O}/\mathfrak{m})^*$.

We now prove the exactness of the sequence. We start with showing that $\mathrm{Im}(\mathcal{O}^* \to (\mathcal{O}/\mathfrak{m})^*) = \ker((\mathcal{O}/\mathfrak{m})^* \to C_K^{\mathfrak{m}})$. Suppose that $\overline{\alpha} \in (\mathcal{O}/\mathfrak{m})^*$ maps to the trivial class of $C_K^{\mathfrak{m}}$. This means that $\langle\alpha\rangle \in P_K^{\mathfrak{m}}$, so there exists $\beta \equiv 1 \bmod {}^{\times}\mathfrak{m}$ such that $\langle\alpha\rangle = \langle\beta\rangle$. Hence, $\frac{\alpha}{\beta} \in \mathcal{O}^*$. Since $\beta \equiv 1 \bmod {}^{\times}\mathfrak{m}$, then $\overline{\left(\frac{\alpha}{\beta}\right)} = \overline{\alpha}$, so $\overline{\alpha} \in \mathrm{Im}(\mathcal{O}^* \to (\mathcal{O}/\mathfrak{m})^*)$.

Lastly, we need to prove that $\mathrm{Im}((\mathcal{O}/\mathfrak{m})^* \to C_K^{\mathfrak{m}}) = \ker(C_K^{\mathfrak{m}} \to C_K)$. Let $[\mathfrak{a}] \in C_K^{\mathfrak{m}}$ be an ideal class which maps to the trivial class in C_K. This means that $\mathfrak{a} = \langle\alpha\rangle$ is a principal ideal coprime to $\mathfrak{m}$. Hence, $(\alpha, \mathfrak{m}) = 1$, so $\alpha \in (\mathcal{O}/\mathfrak{m})^*$ and $[\mathfrak{a}] \in \mathrm{Im}((\mathcal{O}/\mathfrak{m})^* \to C_K^{\mathfrak{m}})$. □

Now let us define $P_K^{\mathfrak{m},+} = \{\langle\alpha\rangle \mid \alpha \in K, \alpha \gg 0, \alpha \equiv 1 \bmod {}^{\times}\mathfrak{m}\}$.

Definition 1.2.2. *The strict (or narrow) ray class group modulo* $\mathfrak{m}$ *is the quotient*

$$C_K^{\mathfrak{m},+} = I_K^{\mathfrak{m}}/P_K^{\mathfrak{m},+}.$$

The strict class group modulo $\mathfrak{m}$ is finite and its cardinality is denoted by $h^{\mathfrak{m},+}$.

1.3 The Artin Map for Abelian Extensions

In this section, we study the canonical isomorphism between the Galois group of the maximal abelian unramified outside $\mathfrak{p}$ extension of K and its strict ray class group. We present the main ideas from the ideal theoretic point of view following Chapters 3 and 5 of [18] while skipping many proofs. An excellent treatment can be found in [19] from the idele theoretic point of view.

Let L be a finite abelian extension of K. We denote by $\mathcal{O}_L$ the ring of integers of L. Fix a prime $\mathfrak{p}$ in K which does not ramify in L and let $\mathfrak{q}$ be a prime in L which lies above $\mathfrak{p}$.

Definition 1.3.1. *The Frobenius element is the unique element* $\sigma_{\mathfrak{q}}$ *of* $Gal(L/K)$ *for which*

$$\sigma_{\mathfrak{q}}(x) \equiv x^{N(\mathfrak{p})} \bmod \mathfrak{q}$$

for all $x \in \mathcal{O}_L$.

Note that the Frobenius elements $\sigma_{\mathfrak{q}}$ for a11 $\mathfrak{q}$ lying above $\mathfrak{p}$ are conjugate. We denote them all by the Artin symbol $\left(\frac{L/K}{\mathfrak{p}}\right)$. If $\mathfrak{a}$ is any ideal of K with prime factorization $\mathfrak{a} = \prod_{\mathrm{i}} \mathfrak{p}_{\mathrm{i}}^{n_{\mathrm{i}}}$, we define the Artin symbol of $\mathfrak{a}$ by

$$\left(\frac{L/K}{\mathfrak{a}}\right) = \prod_{\mathrm{i}} \left(\frac{L/K}{\mathfrak{p}_{\mathrm{i}}}\right)^{n_{\mathrm{i}}}. \tag{1.3}$$

Theorem 1.3.1. *Suppose that L/K is a finite abelian extension of number fields. Let $\mathfrak{m}$ be a modulus for K which is divisible by all the ramified primes. Then the Artin map*

$$\psi^{\mathfrak{m}}_{L/K} : I^{\mathfrak{m}}_K \to Gal(L/K)$$
$$\prod_{\mathfrak{p}\nmid\mathfrak{m}} \mathfrak{p}^{\mathfrak{n}_\mathfrak{p}} \to \prod_{\mathfrak{p}\nmid\mathfrak{m}} \left(\frac{L/K}{\mathfrak{p}}\right)^{\mathfrak{n}_\mathfrak{p}}$$

is surjective.

Proof. Let $H \subset Gal(L/K)$ be the image of $\psi^{\mathfrak{m}}_{L/K}$ and let L^H be its fixed field. The automorphism $\psi^{\mathfrak{m}}_{L/K}(\mathfrak{p})$ acts trivially on L^H for any $\mathfrak{p} \in I^{\mathfrak{m}}_K$. Hence, $\psi^{\mathfrak{m}}_{L/K}(\mathfrak{p}) = 1$, which implies that $\mathfrak{p}$ splits completely in L^H. Hence, $[L^H : K] = 1$ and $H = Gal(L/K)$. □

Theorem 1.3.2 (Artin Reciprocity)**.** *The Artin map induces a canonical isomorphism between each subextension L/K of $K(\mathfrak{m})/K$ and $C^{\mathfrak{m}}_K/\overline{\ker\psi^{\mathfrak{m}}_{L/K}}$. In particular, it gives an order preserving bijection between the set of subgroups $P^{\mathfrak{m}}_K \subset \mathscr{C} \subset I^{\mathfrak{m}}_K$ and the set of abelian extensions L/K whose conductor divides $\mathfrak{m}$.*

The importance of Artin's Reciprocity theorem for us is that it implies the following

Proposition 1.3.1. *The strict class group $C^{\mathfrak{p}^\infty,+}_K$ of K modulo $\mathfrak{p}^\infty$ is isomorphic to the Galois group $G_{K,\mathfrak{p}}$.*

We need to define the Galois group $G_{K,\mathfrak{p}}$. As a profinite group, $G_{K,\mathfrak{p}} = \varprojlim_n G_{K,\mathfrak{p}^n}$ and each $G_{K,\mathfrak{p}^n}$ is isomorphic to $C^{\mathfrak{p}^n,+}_K$.

To describe $G_{K,\mathfrak{p}}$ as a Galois group, let $\overline{K}$ be the separable algebraic closure of K. This extension is infinite but it contains a tower $K \subset K_1 \subset K_2 \subset \cdots$ of finite Galois extensions over K. The Galois group of the extension $\overline{K}/K$ is called the absolute Galois group $G_K = Gal(\overline{K}/K)$. G_K is a profinite group, and in particular it is neither finite nor finitely generated as a topological group. Hence, it is more convenient to work with an abelian quotient of G_K. The most convenient quotient is $G_{K,\mathfrak{p}}$, the Galois group of the maximal abelian extension of K which is unramified outside $\mathfrak{p}$.

Finally, we note that $|C^{\mathfrak{p}^n,+}_K| = h^+_{\mathfrak{p}^n} = |G_{K,\mathfrak{p}^n}|$.

2. L-FUNCTIONS

In this chapter, we will discuss the complex analytic L-functions $L(s,\chi)$ which are attached to a totally real field K. We will also describe the Kummer congruences which occur between the values $L(1-k,\chi)$ at negative integers. These congruences are equivalent to the continuity of the p-adic L-function.

2.1 Characters

We review in this section the theory of characters of the strict ray class group. We closely follow the material in [20] Chapter 7, Section 6. Our main goal is to show that the strict ray class characters χ with modulus $\mathfrak{m}$ which have a trivial infinite part can be written in a simpler form as a character of $\mathcal{O}/\mathfrak{m}$.

Definition 2.1.1. *A character of a group G is a continuous group homomorphism $\chi : G \to \mathbb{C}^*$.*

We will be particularly interested in certain characters on the strict ray class group.

Definition 2.1.2. *A strict ray class character modulo $\mathfrak{m}$ is a character $\chi : C_K^{\mathfrak{m},+} \to \mathbb{C}^*$. We say χ is primitive if it does not factor through any $\mathfrak{n}|\mathfrak{m}$ and we call $\mathfrak{m}$ the conductor of χ.*

We need to describe a more general character, called the Hecke character, in order to study the strict ray class characters modulo $\mathfrak{m}$.

Definition 2.1.3. *A Hecke character with modulus $\mathfrak{m}$ is a character $\chi : I_K^{\mathfrak{m}} \to \mathbb{C}^*$ which can be written as a product $\chi = \chi_f \chi_\infty$ for some characters $\chi_f : (\mathcal{O}/\mathfrak{m})^* \to \mathbb{C}^*$ and $\chi_\infty : \mathbb{R}^r \to \mathbb{C}^*$.*

We denote a Hecke character with modulus $\mathfrak{m}$ by $\chi \bmod \mathfrak{m}$ and we call χ_f the finite part of χ and χ_∞ the infinite part of χ. We say $\chi \bmod \mathfrak{m}$ is primitive if χ_f does not factor through $(\mathcal{O}/\mathfrak{n})^*$ for any ideal $\mathfrak{n} \mid \mathfrak{m}$. The conductor of χ is the smallest divisor $\mathfrak{f} \mid \mathfrak{m}$ such that $\chi \bmod \mathfrak{f}$ is primitive.

Definition 2.1.4. *A Hecke character with modulus $\mathfrak{m}$ is said to be of finite type whenever χ_∞ is the trivial character. In this case, $\chi \bmod \mathfrak{m} : (\mathcal{O}/\mathfrak{m})^* \to \mathbb{C}^*$.*

Since $(\mathcal{O}/\mathfrak{m})^* \simeq \{\alpha \in K^* \mid (\langle\alpha\rangle, \mathfrak{m}) = 1\}/K^{\mathfrak{m},1}$, we can write any $\alpha \in \{\alpha \in K^* \mid (\langle\alpha\rangle, \mathfrak{m}) = 1\}$ as $\alpha = \frac{a}{b}$ with $a, b \in \{\alpha \in \mathcal{O}^* \mid (\langle\alpha\rangle, \mathfrak{m}) = 1\}$. Hence we can extend a Hecke character χ mod $\mathfrak{m}$ to $\{\alpha \in K^* \mid (\langle\alpha\rangle, \mathfrak{m}) = 1\}$ by setting $\chi(\frac{a}{b}) = \chi(a)\chi(b)^{-1}$.

Proposition 2.1.1. *The strict ray class characters modulo $\mathfrak{m}$ are exactly the Hecke characters modulo $\mathfrak{m}$ which are of finite type. Moreover, the conductors of the strict ray class characters and the corresponding Hecke characters are equal.*

Proof. First suppose that χ is a Hecke character modulo $\mathfrak{m}$ of finite type. If $a \in \mathcal{O}$ is such that $a \equiv 1 \bmod \mathfrak{m}$, then clearly $\chi_f(a) = 1$ and $\chi(\langle a\rangle) = \chi_f(a)\chi_\infty(a) = 1$. Hence χ factors through $P_K^{\mathfrak{m},+}$ and is a narrow ray class character modulo $\mathfrak{m}$.

Now suppose that χ is a narrow ray class character modulo $\mathfrak{m}$. The restriction of χ to $K^{\mathfrak{m},1}/K^{\mathfrak{m},1,+} \simeq (\pm 1)^r$ induces a character $\chi_\infty : (\mathbb{R}^*)^r \to \mathbb{C}^*$. Defining $\chi_f = \chi\overline{\chi}_\infty$ produces a character on $(\mathcal{O}/\mathfrak{m})^*$. Therefore, $\chi = \chi_f\chi_\infty$ is a Hecke character modulo $\mathfrak{m}$ of finite type. $\square$

2.2 Hecke L-functions

Let $\mathfrak{m}$ be a non-zero integral ideal of K and consider the strict ray class group $C_K^{\mathfrak{m},+}$. Let $\chi : C_K^{\mathfrak{m},+} \to \mathbb{C}^*$ be a character. We briefly define the Hecke L-functions $L(s,\chi)$ and state some of their properties, following [20] Chapter 7, Section 8.

Definition 2.2.1. *The Hecke L-function is defined by the sum*

$$L(s,\chi) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O} \\ (\mathfrak{a},\mathfrak{m})=1}} \chi(\mathfrak{a})N(\mathfrak{a})^{-s}$$

whenever $\mathrm{Re}(s) > 1$.

It is easy to see that we may decompose the L-function according to the strict ray classes modulo $\mathfrak{m}$ as

$$L(s,\chi) = \sum_{\mathcal{C}\in C_K^{\mathfrak{m},+}} \chi(\mathcal{C}) \sum_{\substack{\mathfrak{a}\subset\mathcal{O} \\ \mathfrak{a}\in\mathcal{C} \\ (\mathfrak{a},\mathfrak{m})=1}} N(\mathfrak{a})^{-s}. \tag{2.1}$$

As usual, $L(s,\chi)$ admits an Euler product over all prime ideals $\mathfrak{p} \subset \mathcal{O}$

$$L(s,\chi) = \prod_{\mathfrak{p}} (1 - \chi(\mathfrak{p})N(\mathfrak{p})^{-s}))^{-1}$$

and may be continued to a meromorphic function which is holomorphic except for a simple pole at $s = 1$. Moreover, $L(s,\chi)$ satisfies a functional equation. We will start by defining the Gauss sum.

Definition 2.2.2. *Let χ be a finite order Hecke character modulo $\mathfrak{m}$ and let $y \in \mathfrak{m}^{-1}\mathfrak{d}^{-1}$. The Gauss sum of χ is the sum*

$$\tau(\chi, y) = \sum_{x \in (\mathcal{O}/\mathfrak{m})^*} \chi(x) \exp(xy).$$

Note that the Gauss sum depends only on the coset of $y \bmod \mathfrak{d}^{-1}$. Given two representatives x and x' such that $x \equiv x' \bmod \mathfrak{m}$, then $x'y - xy \in \mathfrak{m}\mathfrak{m}^{-1}\mathfrak{d}^{-1} = \mathfrak{d}^{-1}$. This implies that $Tr(x'y) \equiv Tr(xy) \bmod \mathbb{Z}$, so $\exp(x'y) = \exp(xy)$. The argument for two representatives $y \equiv y' \bmod \mathfrak{d}^{-1}$ is similar. Hence, we will use the notation

$$\tau(\chi) = \sum_{\mathfrak{x} \in (\mathcal{O}/\mathfrak{m})^*} \chi(\mathfrak{x}) \exp(\mathfrak{x}/\mathfrak{m}\mathfrak{d}). \tag{2.2}$$

Proposition 2.2.1. *Let χ be a primitive Hecke character* $\bmod\, \mathfrak{m}$ *and $x \in \mathcal{O}$. Then the Gauss sum satisfies the transformation*

$$\sum_{\mathfrak{x} \in (\mathcal{O}/\mathfrak{m})^*} \chi(\mathfrak{x}) \exp(\mathfrak{x}x/\mathfrak{m}\mathfrak{d}) = \begin{cases} \overline{\chi}(x)\tau(\chi) & \textit{if } (x, \mathfrak{m}) = 1 \\ 0 & \textit{otherwise} \end{cases}. \tag{2.3}$$

Proof. Let $x \in \mathcal{O}, (x, \mathfrak{m}) = 1$. As $\mathfrak{x}$ runs through $(\mathcal{O}/\mathfrak{m})^*$ so does $x\mathfrak{x}$. Then

$$\sum_{\mathfrak{x} \in (\mathcal{O}/\mathfrak{m})^*} \chi(\mathfrak{x}) \exp(\mathfrak{x}x/\mathfrak{m}\mathfrak{d}) = \overline{\chi}(x) \sum_{\mathfrak{x} \in (\mathcal{O}/\mathfrak{m})^*} \chi(\mathfrak{x}x) \exp(\mathfrak{x}x/\mathfrak{m}\mathfrak{d}) \tag{2.4}$$

$$= \overline{\chi}(x)\tau(\chi). \tag{2.5}$$

When $(x,\mathfrak{m})=\mathfrak{n}\neq 1$, there exists $\mathfrak{a}\in(\mathcal{O}/\mathfrak{m})^*$ such that $\chi(\mathfrak{a})\neq 1$ and $\mathfrak{a}\equiv 1 \bmod \mathfrak{m}\mathfrak{n}^{-1}$. Hence, $x\mathfrak{a}\equiv x \bmod \mathfrak{m}$ which implies that

$$\overline{\chi}(\mathfrak{a})\sum_{\mathfrak{r}\in(\mathcal{O}/\mathfrak{m})^*}\chi(\mathfrak{r})\exp(\mathfrak{r}x/\mathfrak{m}\mathfrak{d}) = \sum_{\mathfrak{r}\in(\mathcal{O}/\mathfrak{m})^*}\chi(\mathfrak{r})\exp(\mathfrak{r}\mathfrak{a}x/\mathfrak{m}\mathfrak{d}) \tag{2.6}$$

$$= \sum_{\mathfrak{r}\in(\mathcal{O}/\mathfrak{m})^*}\chi(\mathfrak{r})\exp(\mathfrak{r}x/\mathfrak{m}\mathfrak{d}). \tag{2.7}$$

Since $\overline{\chi}(\mathfrak{a})\neq 1$, the Gauss sum is equal to 0. □

Theorem 2.2.1 ([20] Corollary 8.6)**.** *The completed L-function*

$$\Lambda(s,\chi)=|D_K|^{s/2}N(\mathfrak{m})^{s/2}L_\infty(s,\chi)L(s,\chi) \tag{2.8}$$

satisfies the functional equation

$$\Lambda(s,\chi)=W(\chi)\Lambda(1-s,\chi^{-1})$$

where $W(\chi)\in\mathbb{C}$ is a constant depending on χ such that $|W(\chi)|=1$. In particular, when $s=k$ is a positive integer, we have

$$L(k,\chi)=\frac{(2\pi \mathrm{i})^{kr}N(\mathfrak{m})^{1-k}|D|^{\frac{1-2k}{2}}}{2^r(k-1)!^r\tau(\chi^{-1})}L(1-k,\chi^{-1}) \tag{2.9}$$

The functional equation of $L(k,\chi)$ will play a crucial role in the proof of our main theorem. We will use it to show that evaluating our p-adic distribution against characters results in reciprocals of the special values of $L(s,\chi)$ at negative, odd integers.

2.3 Congruences between Special Values

The values of the Hecke L-function $L(1-k,\chi)$ for even integers k satisfy nice congruences called the generalized Kummer congruences. They were initially proven by Coates and Sinnott [14] in the case of real quadratic fields and Deligne and Ribet [15] in the case of any totally real field. In this section, we develop the Kummer congruences following [21] in a more general case for L-series attached to locally constant functions.

Let p be any (rational) prime number, $\mathbb{Z}_p$ the p-adic integers, and $\mathbb{Q}_p$ the field of fractions of $\mathbb{Z}_p$. Let $\mathbb{C}_p$ be the completion of the algebraic closure of the field of p-adic numbers $\overline{\mathbb{Q}_p}$ and let $|\cdot|_p$ denote its p-adic valuation.

Definition 2.3.1. *A function f on a topological group G is called locally constant if for every $g \in G$ there exists a neighborhood $U_g \in G$ such that $f|_{U_g}$ is constant.*

Let $\mathfrak{f}$ be a non-zero integral ideal of K and let $G_\mathfrak{f}$ be the strict ray class group of K modulo $\mathfrak{f}$. Let $\varepsilon : G_\mathfrak{f} \to \mathbb{C}$ be a locally constant function. Consider the L-series

$$L(s, \varepsilon) = \sum_{(x,\mathfrak{f})=1} \varepsilon(x) N(x)^{-s}.$$

When ε is a character, this L-series is simply the Hecke L-function. In particular, we can define the values $L(1-k, \varepsilon)$ for $k \geq 1$ by using the meromorphic continuation and functional equation of L-series. In order to p-adically interpolate the L-values, we need to know that they are rational. This was proven by Siegel.

Theorem 2.3.1 ([11])**.** *Suppose the values of ε lie in $\mathbb{Q}$. Then $L(1-k, \varepsilon)$ is a rational number for all $k \geq 1$.*

Now suppose that $\varepsilon : G_\mathfrak{f} \to \mathbb{Q}_p$. The values $L(1-k, \varepsilon)$ in $\mathbb{Q}_p$ are defined by the sum

$$L(1-k, \varepsilon) = \sum_{\mathfrak{a} \in G_\mathfrak{f}} \varepsilon(\mathfrak{a}) \zeta(1-k, \mathfrak{a}, \mathfrak{f}). \tag{2.10}$$

Proposition 2.3.1. *The L-values $L(1-k, \varepsilon)$ are well defined when we view ε as a locally constant function modulo another integral ideal $\mathfrak{f}' \subset \mathfrak{f}$.*

First, we need to define the twisted function ε_c. For $c \in G_\mathfrak{f}$, we denote by ε_c the multiplication by c map $\varepsilon_c(g) = \varepsilon(cg)$. For $c \in G = \varprojlim G_\mathfrak{f}$, we denote by $\bar{c}$ the image of c under the projection $G \to G_\mathfrak{f}$. The multiplication by c map is defined in this case by $\varepsilon_c(g) = \varepsilon(\bar{c}g)$.

Proof. Let d be a divisor of $\mathfrak{f}'$ with $(\mathfrak{f}, d) = 1$. Then there is a map $G_{\mathfrak{f}'d^{-1}} \to G_{\mathfrak{f}}$ since $\mathfrak{f}'d^{-1} \subset \mathfrak{f}$. Composing this map with ε_d gives a locally constant function ε'_d modulo $\mathfrak{f}'d^{-1}$. We can then write

$$L(1-k, \varepsilon) = \sum_d L(1-k, \varepsilon'_d) N(d)^{k-1}. \tag{2.11}$$

In particular, if $\mathfrak{f}$ and $\mathfrak{f}'$ have the same prime divisors, then $d = 1$ is the only such divisor and $L(1-k, \varepsilon) = L(1-k, \varepsilon')$, where we write ε' for the composition of ε with $G_{\mathfrak{f}'} \to G_{\mathfrak{f}}$. □

Let $\mathcal{N} : G \to \mathbb{Z}_p^*$ be the unique continuous character which extends the norm map. For $k \geq 1$, define

$$\Delta_c(1-k, \varepsilon) = L(1-k, \varepsilon) - \mathcal{N}(c)^k L(1-k, \varepsilon_c). \tag{2.12}$$

Clearly, by the previous discussion, the definition of $\Delta_c(1-k, \varepsilon)$ does not depend on the choice of $\mathfrak{f}$. Moreover, if ε is a character, then $\varepsilon(\bar{c}g) = \varepsilon(\bar{c})\varepsilon(g)$, which implies that

$$\Delta_c(1-k, \varepsilon) = \left(1 - \mathcal{N}(c)^k \varepsilon(\bar{c})\right) L(1-k, \varepsilon). \tag{2.13}$$

Now suppose that $\varepsilon_k : G_{\mathfrak{f}} \to \mathbb{Q}_p$, $k \geq 1$, is a family of functions which are almost all zero. Define a map

$$\phi(\mathfrak{a}) = \sum_{k \geq 1} \varepsilon_k(\mathfrak{a}) \mathcal{N}(\mathfrak{a})^{k-1} \tag{2.14}$$

for any integral ideal $\mathfrak{a}$ which is coprime to $\mathfrak{f}$. Apriori, $\phi(\mathfrak{a}) \in \mathbb{Q}_p$.

Theorem 2.3.2 (Generalized Kummer Congruences, [15] Theorem 8.2)**.** *If $\phi(\mathfrak{a}) \in \mathbb{Z}_p$ then $\sum_{k \geq 1} \Delta_c(1-k, \varepsilon_k) \in \mathbb{Z}_p$ for all $c \in G$.*

The proof of the generalized Kummer congruences for totally real fields depends on the q-expansion principle for Hilbert modular forms. This principle states that if the coefficients in the Fourier series expansion of a Hilbert modular form f at one cusp are all rational then the coefficients at all the cusps are rational as well. The idea then is to define an Eisenstein series whose constant term is essentially the L-series $L(1-k, \varepsilon)$ and whose non-constant terms are all p-adic integers of the form $\phi(\mathfrak{a})$ and to show that this forces the constant term to be a p-adic integer as well.

3. P-ADIC MEASURES

The generalized Kummer congruences imply that the p-adic L-function which results from interpolating the special values $L(1-k,\chi)$ is continuous p-adically. In this chapter, we will describe the construction of a p-adic measures on profinite groups and explain how to integrate over the specific group $G_{K,\mathfrak{p}}$. We then construct a measure whose Mellin transform is the p-adic L-function over the totally real field K, although this fact will be shown in the next chapter.

3.1 Integration on Profinite Groups

We will start with some generalities about integration over a profinite group $\mathcal{G}$. Our main reference for this section is [22] Chapter 3, Section 9 and Chapter 4, Section 3.

Let $\mathcal{G}$ be any profinite abelian group and let $C(\mathcal{G},\mathbb{C}_p)^{LC}$ denote the set of all locally constant functions $f:\mathcal{G}\to\mathbb{C}_p$. This set is dense in the set $C(\mathcal{G},\mathbb{C}_p)$ of all continuous functions on $\mathcal{G}$ with values in $\mathbb{C}_p$.

Proposition 3.1.1. *A function $f:\mathcal{G}\to\mathbb{C}_p$ is locally constant if there exists an open subgroup $\mathcal{G}_{\mathrm{i}}\subset\mathcal{G}$ such that f is a function on $\mathcal{G}/\mathcal{G}_{\mathrm{i}}$.*

Proof. Since $\mathcal{G}$ is compact, there exists finitely many points $g_1,\cdots,g_k\in G$ such that

$$\mathcal{G}=\bigcup_{\mathrm{i}=1}^{k}U_{g_{\mathrm{i}}}$$

where each $U_{g_{\mathrm{i}}}$ is a neighborhood of g_{i} such that $f\mid_{U_{g_{\mathrm{i}}}}$ is constant. By the definition of the topology on $\mathcal{G}$, a basis of open sets is given by $\{g+\mathcal{G}_{\mathrm{j}}\mid g\in\mathcal{G},\mathrm{j}=0,1,\cdots\}$. Thus, there exists $\mathrm{j}\in\mathbb{N}$ such that $U_{g_{\mathrm{i}}}\supset g_{\mathrm{i}}+\mathcal{G}_{\mathrm{j}}$ for all $\mathrm{i}\in\{1,\cdots,k\}$. This implies that f induces a function $f_{\mathrm{i}}:\mathcal{G}/\mathcal{G}_{\mathrm{i}}\to\mathbb{C}_p$. □

Definition 3.1.1. *A distribution μ on $\mathcal{G}$ with values in $\mathbb{C}_p$ is a homomorphism from the set of all locally constant functions $f:\mathcal{G}\to\mathbb{C}_p$ to $\mathbb{C}_p$.*

Proposition 3.1.2. *A function $\mu : \{g + \mathcal{G}_{\mathrm{i}} \mid g \in \mathcal{G}, \mathrm{i} \geq M\} \to \mathbb{C}_p$ is induced from a distribution if and only if μ satisfies the relation*

$$\mu(h + \mathcal{G}_{\mathrm{i}}) = \sum_{g \in \mathcal{G}_{\mathrm{i}}/\mathcal{G}_{\mathrm{j}}} \mu(h + g + \mathcal{G}_{\mathrm{j}})$$

for all $h \in \mathcal{G}$ and $\mathrm{j} \geq \mathrm{i}$.

Proof. Let $char_{\mathcal{H}}$ denote the characteristic function of an open subset $\mathcal{H} \subset \mathcal{G}$. When $\mathrm{j} \geq \mathrm{i}$, we have the decomposition

$$char_{h+\mathcal{G}_{\mathrm{i}}} = \sum_{g \in \mathcal{G}_{\mathrm{i}}/\mathcal{G}_{\mathrm{j}}} char_{h+g+\mathcal{G}_{\mathrm{j}}}. \tag{3.1}$$

Given a distribution μ, we get

$$\mu(h + \mathcal{G}_{\mathrm{i}}) = \mu(char_{h+\mathcal{G}_{\mathrm{i}}}) = \sum_{g \in \mathcal{G}_{\mathrm{i}}/\mathcal{G}_{\mathrm{j}}} \mu(char_{h+g+\mathcal{G}_{\mathrm{j}}}) = \sum_{g \in \mathcal{G}_{\mathrm{i}}/\mathcal{G}_{\mathrm{j}}} \mu(h + g + \mathcal{G}_{\mathrm{j}}). \tag{3.2}$$

Conversely, suppose that μ satisfies the compatibility relation. For a locally constant function f on $\mathcal{G}$, choose i large enough so that $\mu(g+\mathcal{G}_{\mathrm{i}})$ is well-defined and put $f_{\mathrm{i}} : \mathcal{G}/\mathcal{G}_{\mathrm{i}} \to \mathbb{C}_p$. We define

$$\mu(f) = \sum_{g \in \mathcal{G}/\mathcal{G}_{\mathrm{i}}} f_{\mathrm{i}}(g)\mu(g + \mathcal{G}_{\mathrm{i}}). \tag{3.3}$$

For any other $\mathrm{j} \geq \mathrm{i}$, we have $f_{\mathrm{i}}(g) = f_{\mathrm{j}}(g)$ for all $g \in \mathcal{G}$. Hence,

$$\begin{aligned} \mu(f) &= \sum_{g \in \mathcal{G}/\mathcal{G}_{\mathrm{i}}} f_{\mathrm{i}}(g)\mu(g + \mathcal{G}_{\mathrm{i}}) \\ &= \sum_{g \in \mathcal{G}/\mathcal{G}_{\mathrm{i}}} f_{\mathrm{i}}(g) \sum_{h \in \mathcal{G}_{\mathrm{i}}/\mathcal{G}_{\mathrm{j}}} \mu(g + h + \mathcal{G}_{\mathrm{j}}) \\ &= \sum_{g \in \mathcal{G}/\mathcal{G}_{\mathrm{j}}} f_{\mathrm{j}}(g)\mu(g + \mathcal{G}_{\mathrm{j}}). \end{aligned}$$

This proves that the definition of $\mu(f)$ does not depend on the subgroup chosen. □

Definition 3.1.2. *We say a distribution μ is bounded if there exists a constant $C > 0$ such that $|\mu(f)| \leq C|f|$ for all $f \in C(\mathcal{G}, \mathbb{C}_p)$.*

Definition 3.1.3. *If a distribution μ is bounded, we call μ a measure on $\mathcal{G}$. We use the notation*

$$\mu(f) = \int_{\mathcal{G}} f \, d\mu.$$

3.2 Measures on Galois Groups

We now want to describe p-adic measures on the Galois group $G_{K,\mathfrak{p}}$. Let $C(G_{K,\mathfrak{p}}, \mathbb{C}_p)$ denote the ring of all continuous functions $f : G_{K,\mathfrak{p}} \to \mathbb{C}_p$.

Recall that we have the isomorphism $G_{K,\mathfrak{p}} = \varprojlim_n Cl_K^{\mathfrak{p}^n,+}$ and so $G_{K,\mathfrak{p}}$ is profinite. Proposition 3.1.2 implies the following construction.

Definition 3.2.1. *A $\mathbb{C}_p$-valued distribution μ on $G_{K,\mathfrak{p}}$ is a family of maps*

$$\mu_{\mathfrak{p}^n} : Cl_K^{\mathfrak{p}^n,+} \to \mathbb{C}_p$$

which satisfy the compatibility relation

$$\mu_{\mathfrak{p}^{n_1}}(x) = \sum_{y \equiv x \bmod \mathfrak{p}^{n_1}} \mu_{\mathfrak{p}^{n_2}}(y)$$

for all $\mathfrak{p}^{n_1} \mid \mathfrak{p}^{n_2}$.

This definition is not practical to use in our computations. Instead of looking at the infinite tower of ray class groups modulo $\mathfrak{p}^n$, we can view $G_{K,\mathfrak{p}}$ as the disjoint union of strict ideal classes of K, each of which is the inverse limit of one ideal class modulo $\mathfrak{p}^n$.

Let $\mathfrak{b}_1, \cdots, \mathfrak{b}_h$ represent the strict ideal classes of K, where each $\mathfrak{b}_{\mathrm{j}}$ is the ideal generated by the lower row of the matrix A_{j} associated with the distinct cusps κ_{j}. By construction, all the ideals $\mathfrak{b}_{\mathrm{j}}$ are relatively prime to $\mathfrak{p}$.

We can decompose $G_{K,\mathfrak{p}}$ as $\sqcup_{\mathrm{j}}\mathfrak{b}_{\mathrm{j}}^{-1}(\mathcal{O}_{\mathfrak{p}}^{*}/U)$, where U is the closure in $\mathcal{O}_{\mathfrak{p}}^{*}$ of the subgroup of all totally positive units in $\mathcal{O}^{*}$ ([22], Section 3.9). Then for any continuous function ϕ on $G_{K,\mathfrak{p}}$, we can define a continuous function ϕ_{j} on $\mathcal{O}_{\mathfrak{p}}$ by

$$\phi_{\mathrm{j}}(x) = \begin{cases} \phi(\mathfrak{b}_{\mathrm{j}}^{-1}x) & \text{if } x \in \mathcal{O}_{\mathfrak{p}}^{*} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

We then define p-adic measures on $G_{K,\mathfrak{p}}$ by setting

$$\int_{G_{K,\mathfrak{p}}} \phi \, d\mu = \sum_{\mathrm{j}} \int_{\mathcal{O}_{\mathfrak{p}}} \phi_{\mathrm{j}} d\mu_{\mathrm{j}}. \tag{3.5}$$

Integration on $\mathcal{O}_{\mathfrak{p}}$ is simple when we fix a strict ideal class of K. Since $\mathcal{O}_{\mathfrak{p}} = \varprojlim \mathcal{O}/\mathfrak{p}^{n}$, Proposition 3.1.2 implies that it is enough to specify the value of μ at the classes $\mathfrak{a}$ mod $\mathfrak{p}^{n}$ for some ideal $\mathfrak{a}$ in the specified strict ideal class.

3.3 p-adic Measures for Totally Real Fields

We now want to define a p-adic measure that is equivalent to the p-adic L-function of totally real fields. Let $\mathfrak{f}$ be divisible by all the primes of K lying above p. Recall the definition

$$\Delta_c(1-k,\varepsilon) = L(1-k,\varepsilon) - \mathcal{N}(c)^{k} L(1-k,\varepsilon_c).$$

We obtain for any $k \geq 1$ and $c \in G$ a distribution

$$\begin{aligned} \mu_{c,k} : \{\varepsilon : G \to \mathbb{Q}_p\} &\to \mathbb{Q}_p \\ \varepsilon &\to \Delta_c(1-k,\varepsilon). \end{aligned}$$

Theorem 3.3.1 ([21] Theorem 4.1)**.** *$\mu_{c,k}$ is a measure on G with values in $\mathbb{Z}_p$. Moreover, $\mu_{c,k} := \mathcal{N}^{k-1}\mu_{c,1}$.*

Proof. We need to show that

$$\int \varepsilon \mathcal{N}^{k-1}\, d\mu_{c,1} \equiv \Delta_c(1-k, \varepsilon) \bmod p^n \mathbb{Z}_p \tag{3.6}$$

for all $k \geq 1, n \geq 1$, and locally constant function $\varepsilon : G \to \mathbb{Z}_p$.

Let $\eta : G \to \mathbb{Z}_p$ be a locally constant function such that $\eta \equiv \mathcal{N}^{k-1} \bmod p^n$. Let $\varepsilon_k : G \to \mathbb{Z}_p$ be a family of locally constant functions defined by $\varepsilon_1 = \varepsilon\eta, \varepsilon_k = -\varepsilon$, and $\varepsilon_{\mathrm{i}} = 0$ for all other i. Then we see that for any integral ideal $\mathfrak{a}$ such that $(\mathfrak{a}, \mathfrak{f}) = 1$, we have

$$\sum_{k \geq 1} \varepsilon_k(\mathfrak{a}) \mathcal{N}(\mathfrak{a})^{k-1} = \varepsilon(\mathfrak{a})\eta(\mathfrak{a})\mathcal{N}(\mathfrak{a})^{k-1} - \varepsilon(\mathfrak{a})\mathcal{N}(\mathfrak{a})^{k-1} \in p^n \mathbb{Z}_p. \tag{3.7}$$

The generalized Kummer congruences then imply that

$$\mathbb{Z}_p \ni \sum_{k \geq 1} \Delta_c(1-k, \varepsilon_k) = \Delta_c(0, \varepsilon\eta) - \Delta_c(1-k, \varepsilon) \in p^n \mathbb{Z}_p. \tag{3.8}$$

This concludes the proof, since we then have

$$\Delta_c(1-k, \varepsilon) \equiv \Delta_c(0, \varepsilon\eta) = \int \varepsilon\eta\, d\mu_{c,1} \equiv \int \varepsilon \mathcal{N}^{k-1}\, d\mu_{c,1} \bmod p^n \mathbb{Z}_p. \tag{3.9}$$

□

4. THE IWASAWA ALGEBRA

In this chapter, we introduce the Iwasawa algebra $\Lambda(\mathcal{G})$ of a profinite group $\mathcal{G}$, which is the algebra consisting of all p-adic measures on $\mathcal{G}$. Mahler's theorem induces a power series structure on $\Lambda(\mathcal{G})$, which we will need later to define inverse measures. Moreover, $\Lambda(\mathcal{G})$ has an algebraic structure that relates p-adic measures to p-adic L-functions. Our main reference is [23] Chapter 3.

4.1 The Power Series Structure of the Iwasawa Algebra

Let $\mathbb{Z}_p[\mathcal{G}/\mathcal{G}_\mathrm{i}]$ denote the group ring over $\mathbb{Z}_p$. That is, $\mathbb{Z}_p[\mathcal{G}/\mathcal{G}_\mathrm{i}]$ is the set of all formal power series

$$\sum_{x\in\mathcal{G}/\mathcal{G}_\mathrm{i}} c_{\mathcal{G}_\mathrm{i}}(x)x; \qquad c_{\mathcal{G}_\mathrm{i}}(x)\in\mathbb{Z}_p. \tag{4.1}$$

This space is isomorphic to the space of all continuous functions $f:\mathcal{G}/\mathcal{G}_\mathrm{i}\to\mathbb{Z}_p$, the correspondence being given by

$$f:\mathcal{G}/\mathcal{G}_\mathrm{i}\to\mathbb{Z}_p \Longleftrightarrow \sum_{x\in\mathcal{G}/\mathcal{G}_\mathrm{i}} f(x)x. \tag{4.2}$$

Definition 4.1.1. *The Iwasawa algebra $\Lambda(\mathcal{G})$ is defined to be*

$$\Lambda(\mathcal{G}) = \varprojlim_{\mathcal{G}_\mathrm{i}} \mathbb{Z}_p[\mathcal{G}/\mathcal{G}_\mathrm{i}]$$

where $\mathcal{G}_\mathrm{i}$ runs over all the open subgroups of $\mathcal{G}$.

Proposition 4.1.1. *The elements of the Iwasawa algebra are p-adic measures on $\mathcal{G}$.*

Proof. Let λ be an element of $\Lambda(\mathcal{G})$ and let $f:\mathcal{G}/\mathcal{G}_\mathrm{i}\to\mathbb{C}_p$ be a locally constant function. Let $\lambda_{\mathcal{G}_\mathrm{i}}$ denote the image of λ in $\mathbb{Z}_p[\mathcal{G}/\mathcal{G}_\mathrm{i}]$. We define the integral of f against λ by

$$\int_{\mathcal{G}} f\,d\lambda = \sum_{x\in\mathcal{G}/\mathcal{G}_\mathrm{i}} c_{\mathcal{G}_\mathrm{i}}(x)f(x). \tag{4.3}$$

This definition is independent of the choice of $\mathcal{G}_\mathrm{i}$.

Now let f be any continuous $\mathbb{C}_p$-valued function on $\mathcal{G}$. There is a sequence $\{f_n\}$ of locally constant functions which converge to f. We then define the integral of f against λ by

$$\int_{\mathcal{G}} f \, d\lambda = \lim_{n\to\infty} \int_{\mathcal{G}} f_n \, d\lambda.$$

□

The elements of the Iwasawa algebra satisfy the following properties:

- Every element $g \in \mathcal{G}$ induces a Dirac measure $\mu_g \in \Lambda(\mathcal{G})$ by

$$\int_{\mathcal{G}} f \, d\mu_g = f(g). \tag{4.4}$$

- The multiplication in $\Lambda(\mathcal{G})$ is given by the convolution of measures

$$\int_{\mathcal{G}} f(x) \, d(\mu_1 * \mu_2)(x) = \int_{\mathcal{G}} \int_{\mathcal{G}} f(x+y) \, d\mu_1(x) \, d\mu_2(y). \tag{4.5}$$

- A group homomorphism $\chi : \mathcal{G} \to \mathbb{C}_p$ can be extended to an algebra homomorphism $\chi : \Lambda(\mathcal{G}) \to \mathbb{C}_p$ by setting $\chi(\mu) = \int_{\mathcal{G}} \chi \, d\mu$.

We want to find an orthonormal basis for the space of continuous functions on $\mathcal{G}$. Mahler's theorem affirms that the set of binomial polynomials forms such a basis.

Definition 4.1.2. *Define the binomial polynomial* $\binom{x}{n}$ *to be 1 if* $n = 0$ *and*

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$$

for $n \geq 1$.

Theorem 4.1.1. *(Mahler's Theorem) Let* $f : \mathbb{Z}_p \to \mathbb{C}_p$ *be any continuous function. Then* f *can be written uniquely as*

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$$

for some coefficients $a_n \in \mathbb{C}_p$ *such that* $\lim_{n\to\infty} a_n = 0$.

Proof. Define the constants a_n by

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k).$$

We show first that this definition of a_n gives the desired values $f(x)$. We calculate the sum

$$\sum_{n=0}^{\infty} a_n \binom{x}{n} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k) \right) \binom{x}{n}$$

The outer sum is in fact finite since $\binom{x}{n} = 0$ whenever $n > x$. Then we can switch the order of summation and by using properties of the binomial polynomial we get

$$\begin{aligned}
\sum_{n=0}^{x} \left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k) \right) \binom{x}{n} &= \sum_{k=0}^{x} f(k) \sum_{\mathrm{j}=0}^{x-k} (-1)^{\mathrm{j}} \binom{\mathrm{j}+k}{\mathrm{j}} \binom{x}{\mathrm{j}+k} \\
&= \sum_{k=0}^{x} f(k) \binom{x}{k} \sum_{\mathrm{j}=0}^{x-k} (-1)^{\mathrm{j}} \binom{x-k}{\mathrm{j}}.
\end{aligned}$$

The inner sum is equal to 1 when $x = k$ and 0 otherwise, so this proves the claim

$$\sum_{n=0}^{\infty} a_n \binom{x}{n} = f(x)$$

for all $x \in \mathbb{N}$. Certainly, $a_n \in \mathbb{C}_p$ and $\lim_{n\to\infty} a_n = 0$ follows from the (p-adic) continuity of the function $x \to \binom{x}{n}$. It remains to show that the a_n are unique. Since $\mathbb{N}$ is dense in $\mathbb{Z}_p$, this shows that the equality is true for all $x \in \mathbb{Z}_p$.

Suppose by way of contradiction that there exists some coefficients $b_n \neq a_n$ such that

$$f(x) = \sum_{n=0}^{\infty} b_n \binom{x}{n}.$$

Then clearly $\sum_{n=0}^{\infty}(a_n - b_n)\binom{x}{n} = 0$. Let k be the smallest integer such that $a_k - b_k \neq 0$. Such a k exists since the sequences a_n and b_n are not equal. Then $\sum_{n=0}^{k-1}(a_n - b_n)\binom{k}{n} = 0$ because each term is 0, and

$$\sum_{n=k+1}^{\infty}(a_n - b_n)\begin{pmatrix} k \\ n \end{pmatrix} = 0 \tag{4.6}$$

because $n > k$. Hence, we must have $(a_k - b_k)\binom{k}{k} = 0 = a_k - b_k$ which contradicts the minimality of k. Therefore the coefficients a_n are unique. □

Definition 4.1.3. *The Mahler transform $\mathcal{M} : \Lambda(\mathbb{Z}_p) \to \mathbb{Z}_p[[T]]$ is defined to be*

$$\mathcal{M}(\mu) = \sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p}\begin{pmatrix} x \\ n \end{pmatrix} d\mu\right)T^n$$

where $\mu \in \Lambda(\mathbb{Z}_p)$.

Theorem 4.1.2. *The Mahler transform is an isomorphism of $\mathbb{Z}_p$ algebras.*

Proof. Consider the power series $g(T) = \sum_{n=0}^{\infty} c_n T^n$ as an element of $\mathbb{Z}_p[[T]]$. We want to explicitly define an inverse to the Mahler transform. Given a function $f(x) = \sum_{n=0}^{\infty} a_n\binom{x}{n}$, write

$$L(f) = \sum_{n=0}^{\infty} a_n c_n. \tag{4.7}$$

For every open subgroup $\mathcal{G}_{\mathrm{i}} \subset \mathcal{G}$ and every element $g \in \mathcal{G}/\mathcal{G}_{\mathrm{i}}$, set

$$\mu_{\mathcal{G}_{\mathrm{i}}} = \sum_{g \in \mathcal{G}/\mathcal{G}_{\mathrm{i}}} L(char_g)g. \tag{4.8}$$

These elements $\mu_{\mathcal{G}_{\mathrm{i}}}$ form a compatible system and their inverse limit is an element $\mu \in \Lambda(\mathcal{G})$. The map $g(T) \to \mu$ is the inverse of the Mahler transform. □

Proposition 4.1.2. *The Mahler transform of the characteristic function on $\mathbb{Z}_p$ is $1 + T$.*

This is clear by direct computation. In particular,

$$\int_{\mathbb{Z}_p}(1+T)^x\, d\mu(x) = \sum_{n \geq 0} T^n \int_{\mathbb{Z}_p}\binom{x}{n}\, d\mu(x). \tag{4.9}$$

In conclusion, note that since $\binom{x}{n}$ is a polynomial, μ can be defined by calculating the integrals of x^m against μ.

4.2 The Algebraic Structure of the Iwasawa Algebra

We now want to give a (different) algebraic structure of the Iwasawa algebra $\Lambda(\mathcal{G})$.

Proposition 4.2.1. *The group $\mathcal{G}$ decomposes as*

$$\mathcal{G} = \Delta \times \Gamma$$

where Δ is a cyclic group and Γ is isomorphic to $\mathbb{Z}_p$.

The case which is of interest to us is when $\mathcal{G} = G_{K,\mathfrak{p}}$. In this case, Δ is the Galois group of the field extension $K(\mu_{p^n})/K$, where μ_{p^n} denotes the group of roots of unity $\zeta^{p^n} = 1$, and Γ is the Galois group of the maximal cyclotomic extension of K in $K(\mu_{p^\infty}) = \cup_n K(\mu_{p^n})$.

To define this isomorphism (in the general case), let $\omega(x) = \lim_{n\to\infty} x^{p^n}$ be the Teichmuller character defined over $\mathbf{Z}_p$. $\omega \circ N$ defines a character of order $p = [(K(\mu_p) : K]$ on integral ideals of K which we will again denote by ω. Then, an arbitrary element $x \in \mathcal{G}$ factors as

$$x = \omega(x) \cdot \frac{x}{\omega(x)} \in \Delta \times \Gamma. \tag{4.10}$$

We let $\langle x \rangle = \omega(x)^{-1}x$ denote the projection of $\mathcal{G}$ to Γ.

Any measure ν on Γ can be extended to the rest of $\mathcal{G}$ by setting $\nu(aU) = \nu(U)$ for any $a \in \Delta$ and compact open $U \subset \Gamma$. Since each ω^{i} is a continuous function on $\mathcal{G}$, the product $(\omega^{\mathrm{i}}\nu)(aU)$ is a measure on $\mathcal{G}$ satisfying $(\omega^{\mathrm{i}}\nu)(aU) = a^{\mathrm{i}}(\omega^{\mathrm{i}}\nu)(U)$.

On the other hand, every measure μ on $\mathcal{G}$ can be decomposed as a sum of measures of the form $\omega^{\mathrm{i}}\nu_{\mathrm{i}}$:

$$\mu = \frac{1}{p}\sum_{\mathrm{i}=1}^{p} \omega^{\mathrm{i}}\nu_{\mathrm{i}} \tag{4.11}$$

by setting $\omega^{\mathrm{i}}\nu_{\mathrm{i}}(U) = \sum_{a\in\Delta} \omega(a)^{-\mathrm{i}}\mu(aU)$.

For any continuous homomorphism $\chi : \mathcal{G} \to \mathbb{C}_p^*$, let $\chi|_\Gamma$ denote the restriction of χ to Γ. The values of $\chi(\omega(x))$ are p^{th} roots of unity in $\mathbb{C}_p^*$, and hence $\chi|_\Gamma$ has the form $x \to x^{-\mathrm{j}}$ for some j mod p.

Fix a topological generator γ of Γ. If the isomorphism $\Gamma \cong \mathbb{Z}_p$ is given by α, then for any $x \in \Gamma$, we can write $x = \gamma^{\alpha(x)}$. Moreover, the homomorphism $\chi : \mathcal{G} \to \mathbb{C}_p^*$ can be written as $\chi = \chi_\Delta \chi_\Gamma$, where χ_Δ is a character which is trivial on Γ and $\chi_\Gamma : x \to \chi(\gamma)^{\alpha(x)}$.

Proposition 4.2.2. *There is a unique isomorphism $\Lambda(\mathcal{G}) \to \mathbb{Z}_p[\mu_p][[T]]$ given by $\gamma \to 1+T$.*

This isomorphism is given by setting the integral

$$\int \chi \, d\mu = \sum_{n \geq 0} a_n (\chi(\gamma) - 1)^n$$

where the coefficients a_n are given by

$$\int_{\mathcal{G}} \chi_\Delta(x) \binom{x}{n} \, d\mu(x).$$

In particular, when the image of γ in $\mathbb{Z}_p$ is 1, this map is simply the Mahler transform.

4.3 p-adic L-functions for Totally Real Fields

We want to show that the measure $\mu_{c,1}$ gives rise to a p-adic L-function $L_p(s, \varepsilon)$ when ε is an even continuous character on G. We show that the function $L_p(s, \varepsilon)$ exists by defining a measure which interpolates the values $L(1-k, \varepsilon)$. The fact that $L_p(s, \varepsilon)$ is analytic follows from the structure of the Iwasawa algebra.

Theorem 4.3.1. *Choose $c \in G$ such that at least one of c and $\varepsilon(c)$ are not equal to 1. Let $\mu = \frac{1}{1-c} \mathcal{N}^{-1} \mu_{c,1}$. For $k \geq 1$, we have*

$$\int \mathcal{N}^k \varepsilon \, d\mu = L(1-k, \varepsilon).$$

Proof. This is true by direct computation. We have

$$\begin{aligned}\int \mathcal{N}^k \varepsilon \, d\mu &= \frac{\int \mathcal{N}^k \varepsilon \, d\left(\mathcal{N}^{-1}\mu_{c,1}\right)}{1-\mathcal{N}(c)^k\varepsilon(c)} = \frac{\int \mathcal{N}^{k-1}\varepsilon \, d\mu_{c,1}}{1-\mathcal{N}(c)^k\varepsilon(c)} \\ &= \frac{\left(1-\mathcal{N}(c)^k\varepsilon(c)\right) L(1-k,\varepsilon)}{1-\mathcal{N}(c)^k\varepsilon(c)} = L(1-k,\varepsilon)\end{aligned}$$

$\square$

Definition 4.3.1. *A character $\chi : \mathbb{Z}_p \to \mathbb{Z}_p$ is called Iwasawa analytic if there exists a power series $f \in \mathbb{Z}_p[[T]]$ such that $f(\gamma^{1-s}-1) = \chi(s)$.*

By the structure theory described in the previous section, this power series is unique and does not depend on the choice of γ.

Theorem 4.3.2 ([21], (4.8)-(4.9))**.** *For each $c \in \mathcal{G}$, both $\int \mathcal{N}^k\varepsilon \, d\left(\mathcal{N}^{-1}\mu_{c,1}\right)$ and $1-\mathcal{N}(c)^k\varepsilon(c)$ are Iwasawa analytic. Moreover, if ε_Δ is non-trivial on Δ, then the p-adic L-function $L_p(s,\epsilon)$ extends to an Iwasawa analytic function on $\mathbb{Z}_p$*

$$L_p(s,\varepsilon) = \int \varepsilon\langle x\rangle^{1-s} \, d\mu.$$

This integral is called the Mellin transform of $L_p(s,\varepsilon)$.

5. HILBERT MODULAR FORMS

Having described the construction of the p-adic L-function of a totally real field K both as a p-adic measure and as an Iwasawa function, we wish to proceed in defining our reciprocal measure. This definition is based on the study of the non-constant terms of Hilbert Eisenstein series.

In this chapter, we introduce the Hilbert modular group and the modular forms which act on it. We follow Chapter 1 of [24].

5.1 Structure of the Hilbert Modular Group

In this section, we want to study the structure of the Hilbert modular group and some of its discrete subgroups.

Given any ring F, $SL_2(F)$ will denote the group

$$SL_2(F) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in F, \det(A) = 1 \right\}. \tag{5.1}$$

Definition 5.1.1. *The Hilbert modular group is the subgroup $SL_2(\mathcal{O})$ of $SL_2(K)$.*

Embedding the totally real field K into $\mathbb{R}^r$ induces an embedding of $SL_2(K)$ into $SL_2(\mathbb{R})^r$ componentwise. Thus we can realize $SL_2(\mathcal{O})$ as the subgroup $SL_2(\mathbb{Z})^r \subset SL_2(\mathbb{R})^r$ via the map $A \to (A^{(1)}, \cdots, A^{(r)})$, where $A^{(\mathrm{i})}$ denotes the i^{th} embedding

$$A^{(\mathrm{i})} = \begin{pmatrix} a^{(\mathrm{i})} & b^{(\mathrm{i})} \\ c^{(\mathrm{i})} & d^{(\mathrm{i})} \end{pmatrix}. \tag{5.2}$$

Recall that the special linear group $SL_2(\mathbb{Z})$ acts on the upper half plane

$$\mathfrak{h} = \{z = x + \mathrm{i}y \in \mathbb{C} \mid y > 0\} \tag{5.3}$$

by fractional linear transformations

$$A \cdot z = \frac{az+b}{cz+d}. \tag{5.4}$$

This action induces an action of $SL_2(\mathcal{O})$ on the product of r copies of $\mathfrak{h}$ by Mobius transformations componentwise: for any $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathcal{O})$ and any $z = (z_1, \cdots, z_r) \in \mathfrak{h}^r$,

$$A \cdot z = \left(\frac{a^{(1)}z_1 + b^{(1)}}{c^{(1)}z_1 + d^{(1)}}, \cdots, \frac{a^{(r)}z_r + b^{(r)}}{c^{(r)}z_r + d^{(r)}} \right). \tag{5.5}$$

The quotient space $SL_2(\mathcal{O})\backslash\mathfrak{h}^r$ is not compact. However, attaching a finite number of points $\{\alpha_{\mathrm{j}}\}$ to $SL_2(\mathcal{O})\backslash\mathfrak{h}^r$ results in a compact space. We will now explain this precisely.

We start by considering the action of $SL_2(\mathcal{O})$ on the larger space $\overline{\mathfrak{h}^r}^r$, where

$$\overline{\mathfrak{h}^r} = \mathfrak{h}^r \cup K \cup \{\infty\} = \mathfrak{h}^r \cup \mathbb{P}^1(K). \tag{5.6}$$

Since

$$\lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c}, \tag{5.7}$$

defining the action of $SL_2(\mathcal{O})$ on rational tuples $q = \left(\frac{s_1}{t_1}, \cdots, \frac{s_r}{t_r}\right) \in \mathbb{Q}^r$ by

$$A \cdot q = \left(\frac{a^{(1)}s_1 + b^{(1)}t_1}{c^{(1)}s_1 + d^{(1)}t_1}, \cdots, \frac{a^{(r)}s_r + b^{(r)}t_r}{c^{(r)}s_r + d^{(r)}t_r} \right) \tag{5.8}$$

guarantees its continuity.

Definition 5.1.2. *The orbits of $\mathbb{P}^1(K) \subset \mathbb{P}^1(\mathbb{R})^n$ under $SL_2(\mathcal{O})$ are called the cusps of $SL_2(\mathcal{O})$. We also call the representative element κ of an orbit the cusp.*

Proposition 5.1.1. *There is a bijection*

$$f : SL_2(\mathcal{O})\backslash\mathbb{P}^1(K) \to C_K$$
$$(\alpha : \beta) \to \langle \alpha, \beta \rangle.$$

Proof. First, we need to show that this map is well defined. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathcal{O})$ and suppose that

$$\frac{\alpha}{\beta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{\gamma}{\delta}. \tag{5.9}$$

We have $f((\alpha : \beta)) = \langle \alpha, \beta \rangle = \langle a\gamma + b\delta, c\gamma + d\delta \rangle \subset f((\gamma : \delta))$. Moreover,

$$\frac{\gamma}{\delta} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{\alpha}{\beta} \tag{5.10}$$

which implies similarly that $f((\gamma : \delta)) \subset f((\alpha : \beta))$. Hence, $f((\alpha : \beta)) = f((\gamma : \delta))$.

Now, we show that f is injective. Let $\mathfrak{a}$ be a fractional ideal of K. Suppose that $\langle \alpha, \beta \rangle$ and $\langle \gamma, \delta \rangle$ are two different sets of generators of $\mathfrak{a}$. We need to show that there exists $A \in SL_2(\mathcal{O})$ such that $(\alpha, \beta) = (\gamma, \delta)A$.

The set of fractional ideals of K forms a group under multiplication, so there exists an ideal $\mathfrak{b}$ such that $\mathfrak{ab} = \mathcal{O}$. Since $1 \in \mathcal{O}$, there exists elements $a_1, a_2, b_1, b_2 \in \mathfrak{b}$ satisfying $a_1\beta - b_1\alpha = a_2\delta - b_2\gamma = 1$. In particular, the matrices

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ \alpha & \beta \end{pmatrix}, A_2 = \begin{pmatrix} a_2 & b_2 \\ \gamma & \delta \end{pmatrix} \tag{5.11}$$

are integral and have determinant 1, so they belong to $SL_2(\mathcal{O})$. Moreover,

$$(\gamma, \delta)A_2^{-1}A_1 = (\gamma, \delta) \begin{pmatrix} \delta & -b_2 \\ -\gamma & a_2 \end{pmatrix} A_1 = (0, 1)A_1 = (\alpha, \beta). \tag{5.12}$$

Finally, f is surjective since every integral ideal can be generated by two elements. $\square$

Proposition 5.1.2 ([24], Corollary 3.5)**.** *The equivalence classes of cusps κ under the action of $SL_2(\mathcal{O})$ are in one-to-one correspondence with the ideal classes of K. In particular, the number of inequivalent cusps is h.*

In particular, the cusp $\kappa = \infty = (\infty, \cdots, \infty)$, which is represented by $(1 : 0) \in \mathbb{P}^1(K)$, corresponds to the trivial ideal class consisting of all principal ideals of K.

Definition 5.1.3. *Let $\mathfrak{n}$ be a non-zero integral ideal of K. The principal congruence subgroup of $SL_2(\mathcal{O})$ of level $\mathfrak{n}$ is defined to be*

$$\Gamma(\mathfrak{n}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{n} \right\}. \tag{5.13}$$

A congruence subgroup $SL_2(\mathcal{O})$ is any discrete subgroup Γ containing some $\Gamma(\mathfrak{n})$.

We will be most interested in the congruence subgroups

$$\Gamma_0(\mathfrak{n}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}) \,\middle|\, \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod \mathfrak{n} \right\} \tag{5.14}$$

and

$$\Gamma_1(\mathfrak{n}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{n} \right\}. \tag{5.15}$$

The number of inequivalent cusps of $\Gamma_0(\mathfrak{n})$ and $\Gamma_1(\mathfrak{n})$ are related in a similar way to the number of strict ideal classes $h^{+,\mathfrak{n}}$ of K modulo $\mathfrak{n}$.

5.2 Hilbert Modular Forms

Recall that we have fixed an ordering of the r embeddings $K \hookrightarrow \mathbb{R}$. For a point $z = (z_1, \cdots, z_r) \in \mathbb{C}^r$, a tuple $k = (k_1, \cdots, k_r) \in \mathbb{Z}^r$, and elements $c, d \in K$, we write

$$Tr(k) = \sum_{\mathrm{i}=1}^{r} k_{\mathrm{i}} \qquad Tr(z) = \sum_{\mathrm{i}=1}^{r} z_{\mathrm{i}} \tag{5.16}$$

$$(cz+d)^k = \prod_{\mathrm{i}=1}^{r} (c^{(\mathrm{i})} z_{\mathrm{i}} + d^{(\mathrm{i})})^{k_{\mathrm{i}}} \tag{5.17}$$

Definition 5.2.1. *Let $f : \mathfrak{h}^r \to \mathbb{C}$ be a complex valued function and let $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$ be an element of $SL_2(\mathcal{O})$. The slash operator $|_k$ is defined by*

$$(f \mid_k \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right))(z) = (cz+d)^{-k} f(\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \cdot z).$$

Definition 5.2.2. *Let Γ be a congruence subgroup of $SL_2(\mathcal{O})$. We say that f is a Hilbert modular form of weight $k = (k_1, \cdots, k_r)$ and level Γ if*

1. *f is a holomorphic function on $\mathfrak{h}^r$*
2. *f satisfies the modularity condition $f \mid_k \gamma = f$ for all $\gamma \in \Gamma$.*

When $k_1 = k_2 = \cdots = k_r = k$, we say the Hilbert modular form is of parallel weight k. We denote the space of all the Hilbert modular forms of parallel weight k and level Γ by $\mathcal{M}_k(\Gamma)$.

Theorem 5.2.1. *A Hilbert modular form $f \in \mathcal{M}_k(\Gamma)$ has a Fourier expansion at the cusp ∞ of the form*

$$f(z) = \sum_{\substack{\nu \equiv 0 \bmod \mathfrak{d}^{-1} \\ \nu \gg 0}} a(\nu) \exp(\nu z)$$

where $\exp(\nu z) = \mathrm{e}^{2\pi \mathrm{i} Tr(\nu z)}$.

Proof. By the modularity condition, it is easy to see that $f(z + a) = f(z)$ for any $a \in \mathcal{O}$. Since $\mathcal{O}$ is a lattice in $\mathbb{R}^r$, f has a Fourier series of the form

$$f(z) = \sum_{\nu \in \hat{\mathcal{O}}} a(\nu) \exp(\nu z) \tag{5.18}$$

where $\hat{\mathcal{O}}$ is the dual lattice to $\mathcal{O}$. By definition, $\hat{\mathcal{O}} = \mathfrak{d}^{-1}$.

It remains to show that $a(\nu) = 0$ unless ν is totally positive or $\nu = 0$. Suppose without loss of generality that $a(\nu) \neq 0$ and $\nu^{(1)} < 0$. There exists a unit $\epsilon \gg 0$ such that $\epsilon^{(1)} > 1$ and $0 < \epsilon^{(\mathrm{i})} < 1$ for $2 \leq \mathrm{i} \leq r$ such that $Tr(\epsilon\nu) < 0$. This implies that $Tr(\epsilon^m \nu) \to -\infty$ as $m \to \infty$, which contradicts the holomorphicity condition. □

For any arbitrary cusp $\kappa \neq \infty$, there is a matrix $A = \left(\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix}\right)$ of determinant 1 such that $A\kappa = \infty$ ([25] pg. 181). Moreover, A can be chosen so that

$$\kappa = -\frac{\alpha_4}{\alpha_3}, \quad \mathfrak{b} = \langle \alpha_3, \alpha_4 \rangle, \quad \mathfrak{b}^{-1} = \langle \alpha_1, \alpha_2 \rangle, \quad \gcd(\mathfrak{n}, \mathfrak{b}) = 1. \tag{5.19}$$

In this case, f admits a Fourier expansion at $\kappa = A^{-1}\infty$ of the form

$$(\alpha_3 A^{-1}z + \alpha_4)^k f(A^{-1}z) = \sum_{\substack{\nu \equiv 0 \bmod \mathfrak{b}^2(\mathfrak{d})^{-1} \\ \nu \gg 0}} a_A(\nu) \exp(\nu z). \tag{5.20}$$

Definition 5.2.3. *A Hilbert modular form f of weight k is a cusp form if the constant term a_0 vanishes in the Fourier series expansion of $f \mid_k A$ for any $A \in SL_2(\mathcal{O})$. Otherwise, f is an Eisenstein series.*

Proposition 5.2.1. *Suppose that $k = (k_1, \cdots, k_r)$ is not a parallel weight - that is, $k_{\mathrm{i}} \neq k_{\mathrm{j}}$ for some $1 \leq \mathrm{i} \neq \mathrm{j} \leq r$. Then f is a cusp form.*

Proof. Let $\epsilon \in \mathcal{O}^*$. Then f transforms under the matrix $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{smallmatrix}\right)$ as

$$\sum_{\substack{\nu \equiv 0 \bmod \mathfrak{d}^{-1} \\ \nu \gg 0}} a(\nu) \exp(\nu z) = \epsilon^{k_1} \cdots \epsilon^{k_r} \sum_{\substack{\nu \equiv 0 \bmod \mathfrak{d}^{-1} \\ \nu \gg 0}} a(\nu) \exp(\epsilon^2 \nu z) \tag{5.21}$$

Comparing coefficients, the result follows. □

5.3 Eisenstein Series

The Eisenstein series provide some basic examples of Hilbert modular forms. We keep the notation $(cz+d)^k = \prod_{\mathrm{i}=1}^{r} (c^{(\mathrm{i})} z_{\mathrm{i}} + d^{(\mathrm{i})})^{k_{\mathrm{i}}}$ as before and we assume that $k = (k, k, \cdots, k)$.

The most natural definition for the Eisenstein series is an $r-$dimensional analog to the elliptic Eisenstein series:

$$E_{\text{naive}}(z) = \sum_{c,d \in \mathcal{O}} \frac{1}{(cz+d)^k}. \tag{5.22}$$

However, this sum cannot converge, for given a unit $\epsilon \in \mathcal{O}^*$ and $k \in \mathbb{N}$ even, we have

$$(\epsilon cz + \epsilon d)^{-k} = \prod_{\mathrm{i}=1}^{r} (\epsilon^{(\mathrm{i})} c^{(\mathrm{i})} z_{\mathrm{i}} + \epsilon^{(\mathrm{i})} d^{(\mathrm{i})})^{-k} = N(\epsilon)^{-k} (cz+d)^{-k} = (cz+d)^{-k} \tag{5.23}$$

and the unit group $\mathcal{O}^*$ is infinite. Thus, we must quotient the summand by the action of the unit group. This action is given on an integral ideal $\mathfrak{a}$ in an ideal class $\mathcal{C} \in C_K$ by $(c, d) \to (\epsilon c, \epsilon d)$ for $(c, d) \in \mathfrak{a} \times \mathfrak{a}$ and $\epsilon \in \mathcal{O}^*$.

Definition 5.3.1. *Let $k \geq 2$ be an even integer. Let $\mathcal{C} \in C_K$ be an ideal class of K and let $\mathfrak{a}$ be any integral ideal in $\mathcal{C}$. The Eisenstein series of weight k attached to the ideal class $\mathcal{C}$ is defined by*

$$E_{k,\mathcal{C}}(z) = \sum_{(c,d) \in \mathfrak{a} \times \mathfrak{a}/\mathcal{O}^*} \frac{N(\mathfrak{a})^k}{(cz+d)^k}.$$

Proposition 5.3.1. *The Eisenstein series $E_{k,\mathcal{C}}(z)$ does not depend on the choice of a representative ideal $\mathfrak{a} \in \mathcal{C}$.*

Proof. Let $\mathfrak{a}$ and $\mathfrak{a}$' be two distinct ideals in $\mathcal{C}$. There exists an element $\alpha \in K^*$ such that $\mathfrak{a}' = \alpha\mathfrak{a}$. Thus $N(\mathfrak{a}') = |N(\alpha)|^k N(\mathfrak{a}) = N(\alpha)^k N(\mathfrak{a})$ since k is even. Moreover, there is a one-to-one correspondence between tuples $(c, d) \in \mathfrak{a} \times \mathfrak{a}$ and $(c', d') \in \mathfrak{a}' \times \mathfrak{a}'$ given by multiplication by α. Hence,

$$\begin{aligned} \sum_{(c',d') \in \mathfrak{a}' \times \mathfrak{a}'/\mathcal{O}^*} \frac{N(\mathfrak{a}')^k}{(c'z+d')^k} &= \sum_{(c,d) \in \mathfrak{a} \times \mathfrak{a}/\mathcal{O}^*} \frac{N(\alpha)^k N(\mathfrak{a})}{(\alpha cz+\alpha d)^k} \\ &= \sum_{(c,d) \in \mathfrak{a} \times \mathfrak{a}/\mathcal{O}^*} \frac{N(\alpha)^k N(\mathfrak{a})}{N(\alpha)^k (cz+d)^k} = \sum_{(c,d) \in \mathfrak{a} \times \mathfrak{a}/\mathcal{O}^*} \frac{N(\mathfrak{a})^k}{(cz+d)^k}. \end{aligned} \tag{5.24}$$

□

Theorem 5.3.1. *Let $k > 2$ be an even integer. Then the Eisenstein series $\{E_{k,\mathcal{C}} \mid \mathcal{C} \in C_K\}$ are linearly independent.*

As in the elliptic case, the Eisenstein series has an explicit Fourier expansion. Recall the partial Dedekind zeta function $\zeta_{K,\mathcal{C}}(s)$ of K is given by

$$\zeta_{K,\mathcal{C}}(s) = \sum_{\substack{\mathfrak{n} \in \mathcal{C} \\ \mathfrak{n} \subset \mathcal{O}}} N(\mathfrak{n})^{-s}. \tag{5.25}$$

Theorem 5.3.2 ([26] Theorem 15.4.17)**.** *Let $k > 2$ be an even integer. Then the Eisenstein series $E_{k,\mathcal{C}}$ admits the Fourier series expansion*

$$E_{k,\mathcal{C}} = \zeta_{K,\mathcal{C}^{-1}}(k) + \frac{(2\pi \mathrm{i})^{kr}}{(k-1)!^r} |D|^{1/2-k} \sum_{\substack{\nu \equiv 0 \bmod \mathfrak{d}^{-1} \\ \nu \gg 0}} \sigma_{k-1}(\nu\mathfrak{d}) \exp(\nu z)$$

where $\sigma_{k-1}(\mathfrak{n})$ is the divisor function defined by $\sigma_{k-1}(\mathfrak{n}) = \sum_{\mathfrak{a} \mid \mathfrak{n}} N(\mathfrak{a})^{k-1}$.

6. EISENSTEIN SERIES

In the previous chapter, we defined Eisenstein series over the full modular group $SL_2(\mathcal{O})$. We are now interested in defining Eisenstein series over a congruence subgroup $\Gamma_0(\mathfrak{n})$ and studying their Fourier series expansions.

6.1 Eisenstein Series of Weight k and Level $\Gamma_0(\mathfrak{n})$

Let $\mathfrak{n}$ be an integral ideal of K. Given a Hecke character χ mod $\mathfrak{n}$ of finite type, an integer $k \geq 3$, and a cusp κ of $\Gamma_0(\mathfrak{n})$, the weight k holomorphic Eisenstein series for $\Gamma_0(\mathfrak{n})$ at κ is given by

$$E_{k,\mathfrak{n}}(\chi, z, \kappa) = \sum_{\gamma \in \Gamma_0(\mathfrak{n})_\kappa \backslash \Gamma_0(\mathfrak{n})} \chi(\gamma_d) \mathrm{Im}(\gamma z)^k \tag{6.1}$$

for any $z \in \mathfrak{h}^r$, where γ_d denotes the $(2,2)$ entry of the matrix γ. Here, for any congruence subgroup Γ, Γ_κ denotes the stabilizer of κ inside Γ.

The Eisenstein series $E_{k,\mathfrak{n}}$ is a linear combination of Eisenstein series $E_{k,\mathfrak{n}}^{\Gamma_1}$ of level $\Gamma_1(\mathfrak{n})$ at the cusps κ_1 lying above κ. To see this, let γ be a representative element of $\Gamma_0(\mathfrak{n})_\kappa \backslash \Gamma_0(\mathfrak{n})$. Then γ^{-1} belongs to a set $R \subset \Gamma_0(\mathfrak{n})$ which contains one element $\gamma\kappa$ of each orbit $\Gamma_0(\mathfrak{n})\kappa$. If we consider the $\Gamma_1(\mathfrak{n})$ orbits of κ, we see that there are only finitely many such orbits in $\Gamma_0(\mathfrak{n})\kappa$. Let $\gamma_1, \cdots, \gamma_{\mathrm{j}} \in \Gamma_0(\mathfrak{n})$ denote the representatives of the $\Gamma_1(\mathfrak{n})$ orbits. Then there exists subsets $R_{\mathrm{j}} \subset \Gamma_1(\mathfrak{n})$ such that $R = \sqcup_{\mathrm{j}} R_{\mathrm{j}} \gamma_{\mathrm{j}}$. The elements of R_{j} represent the orbits $\Gamma_1(\mathfrak{n})\gamma_{\mathrm{j}}\kappa$. Thus, we have

$$E_{k,\mathfrak{n}}(\chi, z, \kappa) = \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(\mathfrak{n}) \backslash \Gamma_0(\mathfrak{n})} \chi(\langle d \rangle) E_{k,\mathfrak{n}}^{\Gamma_1}(\chi, z, \kappa_1). \tag{6.2}$$

We can describe the Eisenstein series explicitly at the cusp $\kappa = \infty$. This cusp corresponds to the trivial ideal class so all the ideals are principal. Two matrices in $SL_2(\mathcal{O})$ are left-equivalent under Γ_∞ if and only if they have the same bottom row $(c, d) \in \mathcal{O} \times \mathcal{O}$. Fix a principal ideal $\mathfrak{a}$ which is not necessarily integral. A tuple (c, d) is the bottom row of a matrix in $\Gamma_0(\mathfrak{n})$ or $\Gamma_1(\mathfrak{n})$ precisely when $c, d \in \mathfrak{a}$, $\gcd\left(\frac{c}{\mathfrak{a}}, \frac{d}{\mathfrak{a}}\right) = 1$, and the tuple satisfies the congruence condition (5.14) and (5.15) of the respective subgroup (**BMP** pgs. 706 Equation

(46)). Since $\mathfrak{a} = \langle \alpha \rangle$ is principal, every element $a \in \mathfrak{a}$ can be written as $a = a^*\alpha$, and we use the notation $\frac{a}{\mathfrak{a}}$ to denote a^*. Choose a set of representatives $a_2 \in \mathfrak{a}$ which runs over all the congruence classes modulo $\mathfrak{n}\mathfrak{a}$. Now the sum in Equation (6.1) at the cusp $\kappa = \infty$ can be written as

$$E_{k,\mathfrak{n}}(\chi, z) \coloneqq E_{k,\mathfrak{n}}(\chi, z, \infty) = \sum_{a_2 \in (\mathfrak{a}/\mathfrak{n}\mathfrak{a})^\times} \chi(\langle a_2 \rangle) \sum_{\substack{\{c,d\} \in \mathfrak{a} \times \mathfrak{a} \\ \gcd\left(\frac{c}{\mathfrak{a}}, \frac{d}{\mathfrak{a}}\right) = 1 \\ \mathfrak{n} | c \\ d \equiv a_2 \bmod \mathfrak{n}\mathfrak{a}}} \frac{N(\mathfrak{a})^k}{(cz + d)^k} \tag{6.3}$$

$$= \sum_{a_2 \in (\mathfrak{a}/\mathfrak{n}\mathfrak{a})^\times} \chi(\langle a_2 \rangle) \sum_{\substack{\{c,d\} \in \mathfrak{a} \times \mathfrak{a} \\ \gcd\left(\frac{c}{\mathfrak{a}}, \frac{d}{\mathfrak{a}}\right) = 1 \\ d \equiv a_2 \bmod \mathfrak{n}\mathfrak{a}}} \frac{N(\mathfrak{a})^k}{(\mathfrak{n}cz + d)^k}. \tag{6.4}$$

The notation $\{c, d\}$ means that two tuples (c, d) and (c', d') are identified together if there exists $\varepsilon \in \mathcal{O}^*, \varepsilon \equiv 1 \bmod \mathfrak{n}$ such that $c = \varepsilon c'$ and $d = \varepsilon d'$.

We now want to describe the Eisenstein series at a cusp $\kappa \neq \infty$, which corresponds to a (fixed) non-trivial ideal class. The idea is to translate the cusp κ to ∞ where an explicit expression of the Eisenstein series is easy to describe.

By Equation (5.20), the Fourier expansion of $E_{k,\mathfrak{n}}(\chi, z, \kappa)$ at κ is equal to the Fourier expansion of $E_{k,\mathfrak{n}}(\chi, z)|_k A^{-1} = (-\alpha_3 z + \alpha_1)^{-k} E_{k,\mathfrak{n}}(\chi, A^{-1}z)$ at ∞. Since $\det A = 1$, a direct computation shows that $(-\alpha_3 z + \alpha_1) = (\alpha_3 A^{-1} z + \alpha_4)^{-1}$. Hence, we have

$$E_{k,\mathfrak{n}}(\chi, z, \kappa) = (\alpha_3 A^{-1} z + \alpha_4)^k \sum_{a_2 \in (\mathfrak{a}/\mathfrak{n}\mathfrak{a})^\times} \chi(\langle a_2 \rangle) \sum_{\substack{\{c,d\} \in \mathfrak{a} \times \mathfrak{a} \\ \gcd\left(\frac{c}{\mathfrak{a}}, \frac{d}{\mathfrak{a}}\right) = 1 \\ d \equiv a_2 \bmod \mathfrak{n}\mathfrak{a}}} \frac{N(\mathfrak{a})^k}{(\mathfrak{n}cA^{-1}z + d)^k}. \tag{6.5}$$

6.2 Fourier Coefficients

We are interested in calculating the non-constant Fourier coefficients of (6.5). Indeed, this will be the main calculation needed to construct our p-adic measure λ. We start by defining a more general Eisenstein series by

$$G_k(z, a_1, a_2, \mathfrak{n}, \mathfrak{a}) = \sum_{\substack{\{c,d\}^+ \in \mathfrak{a}\times\mathfrak{a} \\ (c,d)\equiv(a_1,a_2) \bmod \mathfrak{n}\mathfrak{a}}} \frac{N(\mathfrak{a})^k}{(cz+d)^k}. \tag{6.6}$$

The notation $\{c, d\}^+$ means that two tuples (c, d) and (c', d') are identified together if there exists a unit $\varepsilon \in \mathcal{O}^*$ which is totally positive and congruent to 1 mod $\mathfrak{n}$ such that $c = \varepsilon c'$ and $d = \varepsilon d'$.

Proposition 6.2.1. *We have*

$$\sum_{\substack{\{c,d\}\in\mathfrak{a}\times\mathfrak{a} \\ \gcd\left(\frac{c}{\mathfrak{a}},\frac{d}{\mathfrak{a}}\right)=1 \\ d\equiv a_2 \bmod \mathfrak{n}\mathfrak{a}}} \frac{N(\mathfrak{a})^k}{(cz+d)^k} = \sum_{a_1\in\mathfrak{a}/\mathfrak{a}\mathfrak{n}} \sum_{\mathfrak{i}=1}^{h^{\mathfrak{n},+}} \sum_{\mathfrak{t}\in C_{\mathfrak{i}}} \frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k} G_k(z, \tau a_1, \tau a_2, \mathfrak{n}, \mathfrak{a}\mathfrak{t})$$

where $\mu(\mathfrak{m})$ denotes the Mobius function for any integral ideal $\mathfrak{m}$ given by

$$\mu(\mathfrak{m}) = \begin{cases} 1 & \text{if } \mathfrak{m} = \mathcal{O} \\ (-1)^r & \text{if } \mathfrak{m} = \mathfrak{p}_1\cdots\mathfrak{p}_r \\ 0 & \text{if } \mathfrak{p}^2|\mathfrak{m} \text{ for some prime ideal } \mathfrak{p} \end{cases} \tag{6.7}$$

and $h(\mathfrak{n})$ denotes the strict ideal class number of K mod $\mathfrak{n}$.

Proof. Recall the following property of the Mobius function:

$$\sum_{\mathfrak{t}|\mathfrak{n}} \mu(\mathfrak{t}) = \begin{cases} 1 & \text{if } \mathfrak{n} = \mathcal{O} \\ 0 & \text{otherwise} \end{cases}. \tag{6.8}$$

We can use this property to remove the coprimeness condition. Recall that $\frac{c}{\mathfrak{a}}, \frac{d}{\mathfrak{a}} \in \mathcal{O}$, which implies that the ideal $\gcd\left(\frac{c}{\mathfrak{a}}, \frac{d}{\mathfrak{a}}\right)$ is an integral ideal. We now have

$$\sum_{\substack{\{c,d\}\in\mathfrak{a}\times\mathfrak{a} \\ \gcd\left(\frac{c}{\mathfrak{a}},\frac{d}{\mathfrak{a}}\right)=1 \\ (c,d)\equiv(a_1,a_2) \bmod \mathfrak{na}}} \frac{N(\mathfrak{a})^k}{(cz+d)^k} = \sum_{\substack{\{c,d\}\in\mathfrak{a}\times\mathfrak{a} \\ (c,d)\equiv(a_1,a_2) \bmod \mathfrak{na}}} \frac{N(\mathfrak{a})^k}{(cz+d)^k} \sum_{\mathfrak{t}\mid \gcd(\frac{c}{\mathfrak{a}},\frac{d}{\mathfrak{a}})} \mu(\mathfrak{t}) \tag{6.9}$$

$$= \sum_{\gcd(\mathfrak{t},\mathfrak{n})=1} \mu(\mathfrak{t}) \sum_{\substack{\{c,d\}\in\mathfrak{a}\times\mathfrak{a} \\ (c,d)\equiv(a_1,a_2) \bmod \mathfrak{na} \\ (c,d)\equiv(0,0) \bmod \mathfrak{at}}} \frac{N(\mathfrak{a})^k}{(cz+d)^k}. \tag{6.10}$$

Let $C_1, \cdots, C_{h(\mathfrak{n})}$ denote the strict ideal classes of K modulo $\mathfrak{n}$. Choose integral ideals $\mathfrak{t}_1, \cdots, \mathfrak{t}_{h(\mathfrak{n})}$ such that $\mathfrak{t}_{\mathrm{i}} \in C_{\mathrm{i}}^{-1}$ for all $\mathrm{i} \in \{1, \cdots h(\mathfrak{n})\}$. For each $\mathfrak{t} \in C_{\mathrm{i}}$, let $(\tau) = \mathfrak{t}\mathfrak{t}_{\mathrm{i}}$ where $\tau \equiv 1 \bmod \mathfrak{n}$. Then the previous sum is equal to

$$\sum_{\mathrm{i}=1}^{h^{\mathfrak{n},+}} \sum_{\substack{\mathfrak{t}\in C_{\mathrm{i}} \\ \gcd(\mathfrak{t},\mathfrak{n})=1}} \mu(\mathfrak{t}) \sum_{\substack{\{c,d\}\in\mathfrak{a}\times\mathfrak{a} \\ (c,d)\equiv(\tau a_1,\tau a_2) \bmod \mathfrak{nat}}} \frac{N(\mathfrak{a})^k}{(cz+d)^k} \tag{6.11}$$

$$= \sum_{\mathrm{i}=1}^{h^{\mathfrak{n},+}} \sum_{\mathfrak{t}\in C_{\mathrm{i}}} \frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k} G_k(z, \tau a_1, \tau a_2, \mathfrak{n}, \mathfrak{at}). \tag{6.12}$$

Summing up over the equivalence classes of a_1 modulo $\mathfrak{na}$, we get the desired result. $\square$

The Fourier expansion of the Eisenstein series in Equation (6.6) has been computed by Klingen ([25], pg. 181, 182) as follows:

Theorem 6.2.1. *The Fourier expansion of $G_k(z, a_1, a_2, \mathfrak{n}, \mathfrak{a})$ at a general cusp κ is given by*

$$\delta\left(\frac{(a_1^*)}{\mathfrak{nab}}\right) N(\mathfrak{a})^k \sum_{\substack{d\equiv a_2^* \bmod \mathfrak{nab}^{-1} \\ \{d\}^+}} \frac{sgn^k N(d)}{|N(d)|^k}$$
$$+ \frac{(-2\pi \mathrm{i})^{kr} N(\mathfrak{a})^{k-1} N(\mathfrak{b})}{(k-1)!^r N(\mathfrak{n}) |D|^{1/2}} \sum_{\substack{c\equiv a_1^* \bmod \mathfrak{nab} \\ \nu\equiv 0 \bmod \mathfrak{b}/\mathfrak{nad} \\ c\nu\gg 0, \{c\}^+}} \operatorname{sgn} N(\nu) N(\nu)^{k-1} \exp\left(a_2^*\nu + c\nu z\right) \tag{6.13}$$

where $a_1^* = \alpha_4 a_1 - \alpha_1 a_2 \in \mathfrak{ab}$ and $a_2^* = \alpha_1 a_2 - \alpha_2 a_1 \in \mathfrak{ab}^{-1}$.

We use this Fourier expansion to prove the following:

Theorem 6.2.2. *The Fourier expansion of the partial Eisenstein series*

$$(\alpha_3 A^{-1}z+\alpha_4)^k \sum_{\substack{\{c,d\}\in\mathfrak{a}\times\mathfrak{a}\\ \gcd\left(\frac{c}{\mathfrak{a}},\frac{d}{\mathfrak{a}}\right)=1\\ d\equiv a_2 \bmod \mathfrak{n}\mathfrak{a}}} \frac{N(\mathfrak{a})^k}{(\mathfrak{n}cA^{-1}z+d)^k}$$

at the cusp ∞ is given by

$$\sum_{l\equiv a_2^* \bmod \mathfrak{n}\mathfrak{a}\mathfrak{b}^{-1}} \frac{N(\mathfrak{a})^k}{N(l)^k}\sum_{\substack{\mathfrak{t}\in K\\ \mathfrak{t}|\langle l\rangle\\ \mathfrak{n}\nmid\mathfrak{t}}}\mu(\mathfrak{t})$$

$$+\frac{(-2\pi\mathrm{i})^{kr}N(\mathfrak{b})^k}{(k-1)!^r N(\mathfrak{n})^k|D|^{\frac{2k-1}{2}}}\sum_{\mathrm{i}=1}^{h^{\mathfrak{n},+}}\sum_{\mathfrak{t}\in C_\mathrm{i}}\frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k}\sum_{\substack{\nu\in\mathcal{O}\\ c\nu\gg 0,\{c\}^+}} \operatorname{sgn} N(\nu)N(\nu)^{k-1}\exp\left(\frac{\mathfrak{t}_\mathrm{i}a_2'\nu\mathfrak{b}}{\mathfrak{n}\mathfrak{d}}+\frac{c\nu\mathfrak{b}^2 z}{\mathfrak{d}}\right).$$

Proof. Lemma 6.2.1 implies that we need to calculate the Fourier expansion of

$$(\alpha_3 A^{-1}z+\alpha_4)^k\sum_{a_1\in\mathfrak{a}/\mathfrak{a}\mathfrak{n}}\sum_{\mathrm{i}=1}^{h^{\mathfrak{n},+}}\sum_{\mathfrak{t}\in C_\mathrm{i}}\frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k}G_k(\mathfrak{n}A^{-1}z,\tau a_1,\tau a_2,\mathfrak{n},\mathfrak{a}\mathfrak{t}). \tag{6.14}$$

Theorem 6.2.1 along with Equation (5.20) now show that the constant term of (6.14) is non-zero only when $a_1^*\equiv 0 \bmod \mathfrak{n}\mathfrak{a}\mathfrak{t}\mathfrak{b}$. This implies that the contribution from the terms of the outermost sum are all zero except for one equivalence class of a_1. The nonzero term remaining is

$$\sum_{\mathrm{i}=1}^{h^{\mathfrak{n},+}}\sum_{\mathfrak{t}\in C_\mathrm{i}}\frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k}N(\mathfrak{a}\mathfrak{t})^k\sum_{\substack{d\equiv\tau a_2^* \bmod \mathfrak{n}\mathfrak{a}\mathfrak{t}\mathfrak{b}^{-1}\\ \{d\}^+}}\frac{\operatorname{sgn}^k N(d)}{|N(d)|^k}. \tag{6.15}$$

The ideals $\mathfrak{n}$ and $\mathfrak{t}$ being coprime, we can rewrite $d\equiv\tau a_2^* \bmod \mathfrak{n}\mathfrak{a}\mathfrak{t}\mathfrak{b}^{-1}$ as two equivalences modulo $\mathfrak{n}\mathfrak{a}\mathfrak{b}^{-1}$ and $\mathfrak{a}\mathfrak{t}\mathfrak{b}^{-1}$. We now obtain the constant term to be

$$\sum_{\mathrm{i}=1}^{h^{\mathfrak{n},+}}\sum_{\mathfrak{t}\in C_\mathrm{i}}\frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k}\sum_{\substack{d\equiv a_2^* \bmod \mathfrak{n}\mathfrak{a}\mathfrak{b}^{-1}\\ d\equiv 0 \bmod \mathfrak{a}\mathfrak{b}^{-1}\mathfrak{t}\\ \{d\}^+}} N(\mathfrak{a}\mathfrak{t})^k\frac{\operatorname{sgn}^k N(d)}{|N(d)|^k}. \tag{6.16}$$

Switching the order of summation and writing $\langle l\rangle$ for the ideal generated by l, the constant term becomes

$$\sum_{\substack{\mathfrak{t}\in K\\ \mathfrak{p}\nmid\mathfrak{t}}}\frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k}\sum_{\substack{\mathfrak{t}d\equiv a_2^* \bmod \mathfrak{n}\mathfrak{a}\mathfrak{b}^{-1}\\ \{d\}^+}} N(\mathfrak{a})^k\,\frac{1}{N(d)^k} \tag{6.17}$$

$$= \sum_{l\equiv a_2^* \bmod \mathfrak{n}\mathfrak{a}\mathfrak{b}^{-1}}\frac{N(\mathfrak{a})^k}{N(l)^k}\sum_{\substack{\mathfrak{t}\in K\\ \mathfrak{t}|\langle l\rangle\\ \mathfrak{n}\nmid\mathfrak{t}}}\mu(\mathfrak{t}). \tag{6.18}$$

We consider next the non-constant terms. Note that as a_1 varies over the equivalence classes of $\mathfrak{a}/\mathfrak{a}\mathfrak{n}$, a_1^* varies over the equivalence classes of $\mathfrak{a}\mathfrak{b}/\mathfrak{a}\mathfrak{b}\mathfrak{n}$. It is clear then that the non-constant terms are

$$\frac{(-2\pi\mathrm{i})^{kr}N(\mathfrak{b})}{(k-1)!^r N(\mathfrak{n})|D|^{1/2}}\sum_{a_1^*\in\mathfrak{a}\mathfrak{b}/\mathfrak{a}\mathfrak{b}\mathfrak{n}}\sum_{\mathrm{i}=1}^{h^{\mathfrak{n},+}}\sum_{\mathfrak{t}\in C_{\mathrm{i}}}\frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k}N(\mathfrak{a}\mathfrak{t})^{k-1}$$
$$\sum_{\substack{c\equiv\tau a_1^* \bmod \mathfrak{n}\mathfrak{a}\mathfrak{b}\mathfrak{t}\\ \nu\equiv 0 \bmod \mathfrak{b}/\mathfrak{n}\mathfrak{a}\mathfrak{t}\mathfrak{d}\\ c\nu\gg 0,\{c\}^+}} \operatorname{sgn} N(v)N(\nu)^{k-1}\exp(\tau a_2^*\nu + c\nu\mathfrak{n}z). \tag{6.19}$$

As before, we can split the equivalence modulo $\mathfrak{n}\mathfrak{a}\mathfrak{b}\mathfrak{t}$ into two equivalences modulo $\mathfrak{n}\mathfrak{a}\mathfrak{b}$ and $\mathfrak{a}\mathfrak{b}\mathfrak{t}$. Moreover, we make a change of variables $\nu\to\frac{\nu\mathfrak{b}}{\mathfrak{n}\mathfrak{a}\mathfrak{d}\mathfrak{t}}$ and $c\to\mathfrak{a}\mathfrak{b}\mathfrak{t}c$. Since $N(\mathfrak{d})=|D|$, we get

$$\frac{(-2\pi\mathrm{i})^{kr}N(\mathfrak{b})^k}{(k-1)!^r N(\mathfrak{n})^k|D|^{\frac{2k-1}{2}}}\sum_{\mathrm{i}=1}^{h^{\mathfrak{n},+}}\sum_{\mathfrak{t}\in C_{\mathrm{i}}}\frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k}\sum_{\substack{\nu\in\mathcal{O}\\ c\nu\gg 0,\{c\}^+}}\operatorname{sgn} N(\nu)N(\nu)^{k-1}\exp\left(\frac{\mathfrak{t}_{\mathrm{i}}a_2'\nu\mathfrak{b}}{\mathfrak{n}\mathfrak{d}}+\frac{c\nu\mathfrak{b}^2z}{\mathfrak{d}}\right). \tag{6.20}$$

□

7. THE RECIPROCAL MEASURE

We are now ready to use the non-constant Fourier coefficients of the partial Eisenstein series to define a distribution λ whose Mellin transform is the reciprocal of $L_p(1-k,\chi)$. We will show that λ is a measure by proving that the measure associated to $L_p(1-k,\chi)$ is invertible in the Iwasawa algebra $\Lambda(G_{K,\mathfrak{p}})$ and that its inverse is related to λ.

7.1 Definition of λ

The structure of the Galois group $G_{K,\mathfrak{p}}$ allows us to define a distribution λ on $G_{K,\mathfrak{p}}$ by defining it on each piece $\mathfrak{b}_{\mathrm{j}}^{-1}(\mathcal{O}_{\mathfrak{p}}^{*}/U)$. We will denote the restriction of λ to each $\mathfrak{b}_{\mathrm{j}}^{-1}(\mathcal{O}_{\mathfrak{p}}^{*}/U)$ by λ_{j}.

Fix a strict ideal class $\mathfrak{b}_{\mathrm{j}}$. Define a map

$$\varepsilon_{k,\mathfrak{p}^m,\mathfrak{b}_{\mathrm{j}}}(\mathfrak{a}) : \mathcal{O}/\mathfrak{p}^m\mathcal{O} \to (M_k(\Gamma_1(\mathfrak{p}^m))) \tag{7.1}$$

by

$$\varepsilon_{k,\mathfrak{p}^m,\mathfrak{b}_{\mathrm{j}}}(\mathfrak{a}) := (\alpha_3 A^{-1}z+\alpha_4)^k \sum_{\substack{\{c,d\}\in\mathfrak{a}\times\mathfrak{a} \\ \gcd\left(\frac{c}{\mathfrak{a}},\frac{d}{\mathfrak{a}}\right)=1 \\ d\equiv a_2 \bmod \mathfrak{p}^m\mathfrak{a}}} \frac{N(\mathfrak{a})^k}{(\mathfrak{p}^m cA^{-1}z+d)^k}. \tag{7.2}$$

Let $\mathcal{C}_n$ denote the map which sends a holomorphic Hilbert modular form to the coefficient of $\mathrm{e}^{2\pi\mathrm{i}Tr(nz)}$ in its Fourier expansion. Theorem 6.2.2 states that

$$\mathcal{C}_{\mathfrak{p}^m}(\varepsilon_{k,\mathfrak{p}^m,\mathfrak{b}_{\mathrm{j}}}) = \frac{(-2\pi\mathrm{i})^{kr}N(\mathfrak{b}_{\mathrm{j}})^k}{(k-1)!^r|D|^{\frac{2k-1}{2}}} \sum_{u=0}^{m} N(\mathfrak{p})^{u(k-1)-mk} \sum_{\mathrm{i}=1}^{h^{\mathfrak{p}^m,+}} \sum_{\mathfrak{t}\in C_{\mathrm{i}}} \frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k} \exp\left(\frac{\mathfrak{t}_{\mathrm{i}}a_2\mathfrak{p}^u\mathfrak{b}_{\mathrm{j}}}{\mathfrak{p}^m\mathfrak{d}}\right). \tag{7.3}$$

Define a distribution on $\mathfrak{b}_{\mathrm{j}}^{-1}(\mathcal{O}_{\mathfrak{p}}^{*}/U)$ by

$$\lambda_{\mathrm{j}}(\mathfrak{a}+\mathfrak{p}^m\mathcal{O}_{\mathfrak{p}}) = \frac{1}{2^r N(\mathfrak{b}_{\mathrm{j}})^k}\mathcal{C}_{\mathfrak{p}^m}(\varepsilon_{k,\mathfrak{p}^m,\mathfrak{b}_{\mathrm{j}}}) + \gamma(k)\lambda_{Haar} \tag{7.4}$$

where

$$\gamma(k) = \begin{cases} \frac{N(\mathfrak{p})^{2k-1}}{N(\mathfrak{p})^{k}-1}\frac{(1-N(\mathfrak{p})^{k-1})^{-1}}{\zeta(1-k)} & \text{for k even} \\ 0 & \text{otherwise} \end{cases}. \tag{7.5}$$

Note that since $a_2 \in \mathfrak{a}\mathfrak{b}_{\mathrm{j}}^{-1}$, the term inside the exponential function, and hence the definition of λ, does not depend on the ideals $\mathfrak{b}_{\mathrm{j}}$. Henceforth we will write $\mathfrak{a}$ instead of $a_2\mathfrak{b}_{\mathrm{j}}$.

Theorem 7.1.1. *Let χ be a Hecke character of finite type with conductor $\mathfrak{p}^m$ for some $m \geq 1$. Let k be an integer of the same parity as χ. Then we have*

$$\int_{G_{K,\mathfrak{p}}} \chi^{-1}(x)\, d\lambda(x) = \frac{h^{+}\left(1-\chi(\mathfrak{p})N(\mathfrak{p})^{k-1}\right)^{-1}}{L(1-k,\chi)} \tag{7.6}$$

where h^{+} denotes the strict class number of K.

Proof. First, we consider the case when χ is the trivial character and k is even. By finite additivity, we have

$$\lambda_{\mathrm{j}}(\mathcal{O}_{\mathfrak{p}}^{*}) = \sum_{\mathfrak{a}\in\mathcal{O}/\mathfrak{p}\mathcal{O}} \lambda_{\mathrm{j}}(\mathfrak{a}+\mathfrak{p}\mathcal{O}_{\mathfrak{p}}) \tag{7.7}$$

$$= \sum_{\mathfrak{a}\in\mathcal{O}/\mathfrak{p}\mathcal{O}} \frac{1}{2^r}\mathcal{C}_{\mathfrak{p}}(\varepsilon_{k,\mathfrak{p},\mathfrak{b}_{\mathrm{j}}}) + \sum_{\mathfrak{a}\in\mathcal{O}/\mathfrak{p}\mathcal{O}} \gamma(k)\lambda_{Haar}. \tag{7.8}$$

We calculate each sum separately and start with the first sum. Since the sum of roots of unity modulo $\mathfrak{p}$ is 0, we have

$$\sum_{\mathfrak{a}\in\mathcal{O}/\mathfrak{p}\mathcal{O}} \exp\left(\frac{\mathfrak{t}_{\mathrm{i}}\mathfrak{a}}{\mathfrak{p}\mathfrak{d}}\right) = -1. \tag{7.9}$$

Moreover, $\exp\left(\frac{\mathfrak{t}_{\mathrm{i}}\mathfrak{a}}{\mathfrak{d}}\right) = \mathrm{e}^{2\pi\mathrm{i}Tr(\mathfrak{t}_{\mathrm{i}}\mathfrak{a}/\mathfrak{d})} = 1$ for our choice of $\mathfrak{t}_{\mathrm{i}}$ and $\mathfrak{a}$, so

$$\sum_{\mathfrak{a}\in\mathcal{O}/\mathfrak{p}\mathcal{O}} \exp\left(\frac{\mathfrak{t}_{\mathrm{i}}\mathfrak{a}}{\mathfrak{d}}\right) = N(\mathfrak{p}) - 1. \tag{7.10}$$

These comments directly imply that the first sum is equal to

$$\frac{(-2\pi i)^{kr}}{2^r(k-1)!^r|D|^{\frac{2k-1}{2}}}\sum_{\substack{\mathfrak{t}\in K\\ \mathfrak{p}\nmid\mathfrak{t}}}\frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k}\sum_{u=0}^{1}N(\mathfrak{p})^{u(k-1)-k}\sum_{\mathfrak{a}\in\mathcal{O}/\mathfrak{p}\mathcal{O}}\exp\left(\frac{\mathfrak{t}_i\mathfrak{p}^u\mathfrak{a}}{\mathfrak{p}\mathfrak{d}}\right) \tag{7.11}$$

$$=\frac{(-2\pi i)^{kr}}{2^r(k-1)!^r|D|^{\frac{2k-1}{2}}}\sum_{\substack{\mathfrak{t}\in K\\ \mathfrak{p}\nmid\mathfrak{t}}}\frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k}\left(-N(\mathfrak{p})^{-k}+\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})}\right). \tag{7.12}$$

The term $\sum_{\substack{\mathfrak{t}\in K\\ \mathfrak{p}\nmid\mathfrak{t}}}\mu(\mathfrak{t})N(\mathfrak{t})^{-k}$ is the reciprocal of the Dedekind zeta function with the Euler factor at $\mathfrak{p}$ removed, so the previous sum is equal to

$$\frac{(-2\pi i)^{kr}}{2^r(k-1)!^r|D|^{\frac{2k-1}{2}}}\frac{(1-N(\mathfrak{p})^{-k})^{-1}}{\zeta(k)}\left(-N(\mathfrak{p})^{-k}+\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})}\right). \tag{7.13}$$

The functional equation of the Dedekind zeta function is given by

$$\frac{1}{\zeta(1-k)}=\frac{(-2\pi i)^{kr}}{2^r(k-1)!^r|D|^{\frac{2k-1}{2}}}\frac{1}{\zeta(k)}. \tag{7.14}$$

It directly implies the first sum is equal to

$$\frac{(1-N(\mathfrak{p})^{-k})^{-1}}{\zeta(1-k)}\left(\frac{N(\mathfrak{p})^k-N(\mathfrak{p})^{k-1}-1}{N(\mathfrak{p})^k}\right). \tag{7.15}$$

The second sum is

$$\frac{N(\mathfrak{p})^{2k-1}}{N(\mathfrak{p})^k-1}\frac{(1-N(\mathfrak{p})^{k-1})^{-1}}{\zeta(1-k)}\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})}. \tag{7.16}$$

Adding Equations (7.15) and (7.16) gives

$$\lambda_{\mathrm{j}}(\mathcal{O}_{\mathfrak{p}}^*)=\frac{(1-N(\mathfrak{p})^{-k})^{-1}}{\zeta(1-k)}. \tag{7.17}$$

Summing up over j equivalence classes we get the required result.

Next we consider the case when χ is a non-trivial Hecke character of finite type with conductor $\mathfrak{p}^m$ and k is an integer of the same parity as χ. Since χ is orthogonal to Haar measure, the integral of χ against λ_{Haar} is zero. The integral of χ against λ_{j} is

$$\int_{\mathcal{O}_{\mathfrak{p}}^*} \chi^{-1}(x)\, d\lambda_{\mathrm{j}}(x) = \frac{(-2\pi\mathrm{i})^{kr}}{(k-1)!^r |D|^{\frac{2k-1}{2}}} \sum_{u=0}^{m} N(\mathfrak{p})^{u(k-1)-mk} \sum_{\mathrm{i}=1}^{h^{\mathfrak{p}^m,+}} \sum_{\mathfrak{t}\in C_{\mathrm{i}}} \frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k} \sum_{\mathfrak{a}\in(\mathcal{O}/\mathfrak{p}^m)^\times} \chi(\mathfrak{a}) \exp\left(\frac{\mathfrak{t}_{\mathrm{i}}\overline{\mathfrak{a}}\mathfrak{p}^u}{\mathfrak{p}^m\mathfrak{d}}\right). \quad (7.18)$$

If χ is primitive and $x \in \mathcal{O}$, then

$$\sum_{\mathfrak{x}\in\mathcal{O}/\mathfrak{p}^m} \chi(\mathfrak{x}) \mathrm{e}^{2\pi\mathrm{i}Tr(\mathfrak{x}x/\mathfrak{d})} = \begin{cases} \overline{\chi}(x)\tau(\chi) & \text{if } \gcd(x,\mathfrak{p}^m)=1 \\ 0 & \text{otherwise} \end{cases}. \quad (7.19)$$

Then the term with $u = 0$ is the only non-zero term in Equation (7.18), so we have

$$\frac{(-2\pi\mathrm{i})^{kr}}{2^r(k-1)!^r N(\mathfrak{p}^m)^k |D|^{\frac{2k-1}{2}}} \sum_{\mathrm{i}=1}^{h^{\mathfrak{p}^m,+}} \sum_{\mathfrak{t}\in C_{\mathrm{i}}} \frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k} \sum_{\mathfrak{a}\in(\mathcal{O}/\mathfrak{p}^m)^\times} \chi(\mathfrak{a}) \exp\left(\frac{\mathfrak{t}_{\mathrm{i}}\overline{\mathfrak{a}}}{\mathfrak{p}^m\mathfrak{d}}\right) \quad (7.20)$$

$$= \frac{(-2\pi\mathrm{i})^{kr}}{2^r(k-1)!^r N(\mathfrak{p}^m)^k |D|^{\frac{2k-1}{2}}} \sum_{\substack{\mathfrak{t}\in K \\ \mathfrak{p}\nmid\mathfrak{t}}} \frac{\mu(\mathfrak{t})}{N(\mathfrak{t})^k} \chi(\mathfrak{t})^{-1}\tau(\overline{\chi}) \quad (7.21)$$

$$= \frac{(-2\pi\mathrm{i})^{kr}}{2^r(k-1)!^r N(\mathfrak{p}^m)^k |D|^{\frac{2k-1}{2}}} \tau(\overline{\chi}) \frac{1}{L(k,\overline{\chi})}. \quad (7.22)$$

The functional equation of the Hecke L-function of a totally real field along with the relation $\tau(\chi)\tau(\overline{\chi}) = \chi(-1)N(\mathfrak{p}^m)$ gives the integral against λ_{j} to be $L(1-k,\chi)^{-1}$. Summing over the strict ideal classes of K proves the result. $\square$

7.2 Invertibility of Measures

Recall that the group $G_{K,\mathfrak{p}}$ decomposes as $G_{K,\mathfrak{p}} = \Delta \times \Gamma$, where Δ is the Galois group of the field extension $K(\mu_{p^n})/K$ and Γ is the Galois group of the maximal cyclotomic extension of K in $K(\mu_{p^\infty})$. We keep the same notation as Chapter 4.

Since χ transforms under Δ by $\omega^{-\mathrm{j}}$, the integral of χ against μ only involves the term for $\mathrm{i} = \mathrm{j}$ and equals

$$\int_{G_{K,\mathfrak{p}}} \chi(x)\, d\mu = \frac{1}{q} \int_{G_{K,\mathfrak{p}}} \chi(x)\, d(\omega^{\mathrm{j}} \nu_{\mathrm{j}}) = \int_{\Gamma} \chi|_{\Gamma}\, d\nu_{\mathrm{j}}, \tag{7.23}$$

reducing the integration of the character χ on $G_{K,\mathfrak{p}}$ to that of $\chi|_{\Gamma}$ on Γ.

Now assume that $\nu_{\mathrm{i}} \equiv 0$ whenever i is even and that $\nu_{\mathrm{i}}|_{\Gamma}$ is a unit in the Iwasawa algebra $\Lambda(G_{K,\mathfrak{p}})$ whenever i is odd. Since $\Lambda(G_{K,\mathfrak{p}}) \cong \mathcal{O}[[T]]$ has a formal power structure, the invertibility condition is equivalent to the constant term of the power series

$$(\omega^{\mathrm{i}} \nu_{\mathrm{i}})(\Gamma) = \sum_{a \in \Delta} \omega(a)^{-\mathrm{i}} \mu(a\Gamma) = \int_{G_{K,\mathfrak{p}}} \omega(x)^{-\mathrm{i}}\, d\mu \tag{7.24}$$

being a p-adic unit. In this case, for each odd value of $\mathrm{i} \bmod q$, there exists an inverse measure ν_{i}^{-1} such that $\nu_{\mathrm{i}} * \nu_{\mathrm{i}}^{-1}$ is equal to the Dirac distribution μ_1 at the identity, or in other words

$$\int_{\Gamma} f(x)\, d(\nu_{\mathrm{i}} * \nu_{\mathrm{i}}^{-1})(x) = \int_{\Gamma} \int_{\Gamma} f(xy)\, d\nu_{\mathrm{i}}(x)\, d\nu_{\mathrm{i}}^{-1}(y) = f(1) \tag{7.25}$$

for any continuous $\mathbb{C}_p$-valued function on Γ. In particular, when i is odd,

$$\int_{\Gamma} \chi|_{\Gamma}\, d\nu_{\mathrm{i}}^{-1} = \left(\int_{\Gamma} \chi|_{\Gamma}\, d\nu_{\mathrm{i}} \right)^{-1}. \tag{7.26}$$

We can then define an inverse measure μ^{-1} on $G_{K,\mathfrak{p}}$ by

$$\mu^{-1} = \frac{1}{q} \sum_{\substack{1 \le \mathrm{i} \le q \\ \mathrm{i}\ \mathrm{odd}}} \omega^{\mathrm{i}} \nu_{\mathrm{i}}^{-1}. \tag{7.27}$$

Moreover, evaluating the inverse measure μ^{-1} against odd characters χ,

$$\int_{G_{K,\mathfrak{p}}} \chi(x)\, d\mu^{-1} = \int_{\Gamma} \chi|_{\Gamma}\, d\nu_{\mathrm{j}}^{-1} = \left(\int_{\Gamma} \chi|_{\Gamma}\, d\nu_{\mathrm{j}} \right)^{-1} = \left(\int_{G_{K,\mathfrak{p}}} \chi(x)\, d\mu \right)^{-1}, \tag{7.28}$$

we see that the integrals of μ and μ^{-1} are reciprocals of each other.

7.3 Continuity

We will prove that λ is a bounded measure by showing that the p-adic measure on totally real fields $\mu_{1,\mathfrak{c}}$ is a unit in the Iwasawa algebra and that the regularization μ^* of its inverse $\mu_{1,\mathfrak{c}}^{-1}$ is equal as a distribution to a multiple of λ.

We start by considering the p-adic measure on totally real fields $\mu_{1,\mathfrak{c}}$. By Equation (7.24), ν_{i} is invertible whenever the integral

$$\int_G \omega^{-\mathrm{i}}(x)\, d\mu_{1,\mathfrak{c}} \tag{7.29}$$

is a p-adic unit. Using the property

$$\int_G \chi(x)N(x)^{n-1}\, d\mu_{1,\mathfrak{c}}(x) = (1-\chi(\mathfrak{c})N(\mathfrak{c})^n)\, L(1-n,\chi), \tag{7.30}$$

the integral in (7.29) is equal to $(1-\omega^{-\mathrm{i}}(\mathfrak{c})N(\mathfrak{c}))\, L(0,\omega^{-\mathrm{i}})$. Kummer's criterion in the case of totally real fields ([16]) implies that p divides $L(0,\omega^{-\mathrm{i}})$ for any odd $1 \le \mathrm{i} \le q$ if and only if p divides the class number of $K(\mathrm{e}^{2\pi\mathrm{i}/p})$. Since p is regular by assumption, this does not happen, so the ν_{i} are p-adic units.

Therefore, we conclude that the measure $\mu_{1,\mathfrak{c}}^{-1}$ satisfies the property

$$\int_G \chi(x)N(x)^{k-1}\, d\mu_{1,\mathfrak{c}}^{-1} = (1-\chi(\mathfrak{c})N(\mathfrak{c})^k)^{-1}L(1-k,\chi)^{-1} \tag{7.31}$$

for any Hecke character χ and non-negative integer k of the same parity. Moreover, by direct computation, the regularization

$$\mu^*(U) = \mu_{1,\mathfrak{c}}^{-1}(U) - N(\mathfrak{c})\mu_{1,\mathfrak{c}}^{-1}(\mathfrak{c}U) \tag{7.32}$$

satisfies

$$\int_G \chi(x)N(x)^{k-1}\, d\mu^* = L(1-k,\chi)^{-1}. \tag{7.33}$$

We are now ready to prove that λ is a measure.

Theorem 7.3.1. *The distribution λ defined in Equation 7.4 is a p-adic measure.*

Proof. Define a distribution $\overline{\lambda}$ on G by $\overline{\lambda}(U) = \lambda(U^{-1})$. The integral in Equation (7.6) is then equal to $\int_G \chi(x)\, d\overline{\lambda}$. Moreover, since μ^* is a measure, so is $N(x)^{k-1}\mu^*$ for any k. Comparing Equations (7.6) and (7.33), we see that $\overline{\lambda} - h^+ N(x)^{k-1}\mu^*$ vanishes when integrated against any Hecke character of finite type whenever $k \geq 3$. Since these span the space of locally constant functions on G, the distribution $\overline{\lambda} - h^+ N(x)^{k-1}\mu^*$ is identically zero. This implies that $\overline{\lambda}$ is equal to $h^+ N(x)^{k-1}\mu^*$ as distributions and so it is bounded. This implies that λ is a measure. $\square$

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