INTEGRAL CLOSURES OF IDEALS AND COEFFICIENT IDEALS OF MONOMIAL IDEALS

by

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This dissertation is dedicated to Michael, Geoffrey, and my parents.

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ABSTRACT

The integral closure $\overline{I}$ of an ideal I in a ring R consists of all elements $x \in R$ that are integral over I. If R is an algebra over an infinite field k, one can define general elements of $I = (x_1, \ldots, x_n)$ as $x_\alpha = \sum_{i=1}^{n} \alpha_i x_i$ with $(\alpha_1, \ldots, \alpha_n)$ belonging to a Zariski-open subset of k^n.

We prove that for any ideal I of height at least 2 in a local, equidimensional excellent algebra over a field of characteristic zero, the integral closure specializes with respect to a general element of I. That is, we show that $\overline{I}/(x) = \overline{I/(x)}$.

In a Noetherian local ring (R, m) of dimension d, one has a sequence of ideals approximating the integral closure of I for I an m-primary ideal. The ideals

$$I \subseteq I_{\{d\}} \subseteq \cdots \subseteq I_{\{1\}} \subseteq I_{\{0\}} = \overline{I}$$

are the coefficient ideals of I. The i^{th} coefficient ideal $I_{\{i\}}$ of I is the largest ideal containing I and integral over I for which the first $i + 1$ Hilbert coefficients of I and $I_{\{i\}}$ coincide.

With a goal of understanding how coefficient ideals behave under specialization by general elements, we turn to the case of monomial ideals in polynomial rings over a field. A consequence of the specialization of the integral closure is that the i^{th} coefficient ideal specializes when the i^{th} coefficient ideal coincides with the integral closure. To this end, we give a formula for first coefficient ideals of m-primary monomial ideals generated in one degree in 2 variables in order to describe when $I_{\{1\}} = \overline{I}$. In the 2-dimensional case, we characterize the behavior of all coefficient ideals with respect to specialization by general elements.

In the d-dimensional case for $d \geq 3$, we give a characterization of when $I_{\{1\}} = \overline{I}$ for m-primary monomial ideals generated in one degree. In the final chapter, we give an application to the core, by characterizing when $\operatorname{core}(I) = \operatorname{adj}(I^d)$ for such ideals.

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1. INTRODUCTION

The integral closure of an ideal I in a ring R is an ideal consisting of all elements $x \in R$ that satisfy an equation of integral dependence over the ideal I, meaning

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \tag{1.1}$$

for some $a_i \in I^i$. The integral closure of an ideal is an analogue of the integral closure of a ring, which is a generalization of the algebraic closure of a field.

This dissertation focuses first on the question of whether the property of an ideal being integrally closed is preserved modulo a sufficiently general element of the ideal I, as well as a sufficiently general element of the maximal ideal m. When the property P passes when going modulo an element, we say that the property P specializes.

The specialization of the integral closure was first proved by Itoh in [Ito92] and later generalized by Hong and Ulrich in [HU14]. Itoh proved that in a Cohen-Macaulay local ring, for a parameter ideal $I = (a_1, \ldots, a_n)$, after passing from R to the faithfully flat extension $R[T]_{mR[T]}$, that the integral closure specializes with respect to a generic element of $IR[T]_{mR[T]}$. Later, Hong and Ulrich generalized the result by eliminating the Cohen-Macaulay assumption on the ring and the assumption that I is a parameter ideal to prove rather generally for ideals of height at least 2 that the integral closure specializes with respect to a generic element after passing to $R[T]$.

The main motivation for proving the specialization of the integral closure is to be able to prove results about the integral closure of the ideal by induction on the height of the ideal. Passing to a polynomial ring over R changes the dimension of the ring, and passing to $R[T]_{mR[T]}$ changes some properties of the residue field. Therefore, it is desirable to see that the integral closure specializes without extending the base ring.

In Chapter 3, we prove the following theorem.

Theorem 1.0.1. *Let (R, m) be a local equidimensional excellent k-algebra, with k is a field of characteristic 0. Let $I = (a_1, \ldots, a_n)$ be an R-ideal such that $\operatorname{ht} I \geq 2$, and let x be a general element of I. Then $\overline{I}/(x) = \overline{I/(x)}$.*

Our proof fundamentally uses that the integral closure of an ideal I can be recovered from the integral closure of the extended Rees algebra $R[It, t^{-1}]$ in $R[t, t^{-1}]$. Local Bertini Theorems proved by Flenner allow us to say that the $\overline{R[It, t^{-1}]}^{R[t,t^{-1}]}/xt\overline{R[It, t^{-1}]}^{R[t,t^{-1}]}$ is locally normal at certain primes. This is key to proving that the natural map

$$\varphi : \overline{R[It, t^{-1}]}^{R[t,t^{-1}]}/xt\overline{R[It, t^{-1}]}^{R[t,t^{-1}]} \longrightarrow \overline{\frac{R}{(x)}\left[\frac{I}{(x)}t, t^{-1}\right]}^{R/(x)[t,t^{-1}]}$$

is an isomorphism locally at certain primes. The local isomorphisms allow us to prove that the cokernel of φ sits inside a local cohomology module, which by a result of Hong and Ulrich, vanishes in the appropriate degree. We ultimately can say that the cokernel of φ vanishes in degree 1, which tells us that $\overline{I}/(x) = \overline{I/(x)}$.

Next, we consider specialization of the integral closure modulo elements of the maximal ideal of R. We give counterexamples showing that when $\dim R/I \leq 1$, the integral closure of I often does not specialize with respect to general elements of the maximal ideal. However, we do prove that if R/I is reduced and $\operatorname{depth}(R/I) \geq 2$, then I remains integrally closed when one specializes with respect to a general element of the maximal ideal.

We also consider the case of squarefree monomial ideals. It is well known that every squarefree monomial ideal is an intersection of finitely many primes generated by variables. We prove that if I is a squarefree monomial ideal that is an intersection of finitely many such primes generated by disjoint sets of variables, then the integral closure of I specializes with respect to a general linear form.

The second part of the dissertation focuses on coefficient ideals, a sequence of ideals that approximates the integral closure. We assume (R, m) is a local ring of dimension $d > 0$, or $R = k[x_1, \dots, x_d]$ is a polynomial ring over a field with $m = (x_1, \dots, x_d)$. Let I be an m-primary ideal. The Hilbert-Samuel polynomial of I can be written as

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I).$$

We define $e_0(I)$ to be the Hilbert-Samuel multiplicity of I and $e_i(I)$ to be the i^{th} Hilbert coefficient of I.

Kishor Shah proved the existence of a largest ideal $I_{\{i\}}$ containing I and integral over I such that the Hilbert coefficients $e_0(I), \ldots, e_i(I)$ coincide with $e_0(I_{\{i\}}), \ldots, e_i(I_{\{i\}})$. Thus, we have a sequence of ideals

$$I \subseteq I_{\{d\}} \subseteq \cdots \subseteq I_{\{1\}} \subseteq I_{\{0\}} = \overline{I}.$$

Rees's theorem states that in an equidimensional and universally catenary ring, the integral closure of I is the unique largest ideal containing I with the same Hilbert-Samuel multiplicity. Hence, in an equidimensional and universally catenary ring, one need not assume that that the coefficient ideals are integral over I. Ratliff and Rush previously proved that if I contains a nonzerodivisor, then there is a unique largest ideal containing I such that the entire Hilbert-Samuel polynomial coincides. Such an ideal is called the Ratliff-Rush closure, denoted $\widetilde{I}$. One has that $\widetilde{I} = I_{\{d\}}$.

Since we have proved that the integral closure specializes with respect to a general element of the ideal, it is natural to ask whether the ideals approximating $\overline{I}$ also behave well with respect to specialization by a general element.

In general, it is not true that the coefficient ideals specialize. Rossi and Swanson in [RS03] give classes of examples for which the Ratliff-Rush closure, $\widetilde{I}$, does not specialize with respect to general elements of the ideal.

We prove that for general $x \in I$, there are containments

$$I_{\{i\}}/(x) \subseteq (I/(x))_{\{i\}} \tag{1.2}$$

for $1 \leq i \leq d-1$ and

$$\widetilde{I}/(x) = I_{\{d\}}/(x) \subseteq (I/(x))_{\{d-1\}} = \widetilde{I/(x)}. \tag{1.3}$$

For $i < d$, we say that $I_{\{i\}}$ specializes with respect to general $x \in I$ if $I_{\{i\}}/(x) = (I/(x))_{\{i\}}$. For general $x \in I$, we say that the Ratliff-Rush closure specializes if $\widetilde{I}/(x) = \widetilde{I/(x)}$, i.e.

$I_{\{d\}}/(x) = (I/(x))_{\{d-1\}}$. It is easy to see that the i^{th} coefficient ideal specializes with respect to general elements of I if $I_{\{i\}} = \overline{I}$ for $i < d$ and the Ratliff-Rush closure specializes with respect to general elements of I if $\widetilde{I} = I$. Moreover, we observe that if $\widetilde{I} = I_{\{d\}} \subsetneq I_{\{d-1\}}$, then the Ratliff-Rush closure does not specialize.

This leads to several questions: When do coefficient ideals coincide with the integral closure? When does the Ratliff-Rush closure coincide with the $(d-1)^{\text{st}}$ coefficient ideal? Are there ideals for which the i^{th} coefficient ideal is not equal to the integral closure, but the i^{th} coefficient ideal still specializes?

In a polynomial ring in two variables over a field k of characteristic zero, we get a complete picture of the behavior of coefficient ideals under specialization for 0-dimensional monomial ideals generated in one degree. Since the dimension of such a ring is two, the sequence of coefficient ideals is

$$I \subseteq I_{\{2\}} \subseteq I_{\{1\}} \subseteq I_{\{0\}} = \overline{I}. \tag{1.4}$$

As previously stated, the Ratliff-Rush closure does not specialize if $I_{\{2\}} \subsetneq I_{\{1\}}$. We are also able to say that in this case, $I_{\{1\}}$ specializes regardless of whether $I_{\{1\}} = \overline{I}$, and from this it follows that the Ratliff-Rush closure specializes if $I_{\{2\}} = I_{\{1\}}$. Moreover, in the dimension 2 case, we are able to get a very concrete description of when $I_{\{2\}} = I_{\{1\}}$ and when those ideals coincide with $I_{\{0\}}$.

In a polynomial ring in $d \geq 3$ variables over a field k, for 0-dimensional monomial ideals generated in one degree, we are able to characterize when the first coefficient ideal coincides with the integral closure. We do so by utilizing a result of Corso, Polini and Vasconcelos in [CPV06], generalizing Ciupercă in [Ciu01], which characterizes $I_{\{1\}}$ as the degree 1 component of the S_2-ification of the Rees algebra of I, $R[It]$. From this result, we give a criterion to check whether a monomial belongs to the first coefficient ideal. We use this criterion to give a formula for the first coefficient ideal as a sum of finitely many colon ideals, and to characterize when $I_{\{1\}} = \overline{I}$.

Theorem 1.0.2. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k with $d \geq 2$. Let $m = (x_1, \ldots, x_d)$. Let I be an m-primary monomial ideal generated in degree n. Let A denote the matrix whose columns are the exponent vectors of monomial generators of I of degree n*

excluding the exponent vectors associated to $x_1^n, \ldots, x_d^n$. Let A_{d-1} denote the submatrix of A consisting of the first $d-1$ rows of A. Let $B_1, \ldots, B_l$ denote the $d-1$ by $d-1$ submatrices of A_{d-1}. Then $\gcd(|B_1|, \ldots, |B_l|, n) = 1$ if and only if $I_{\{1\}} = \overline{I} = m^n$. In particular, if I is generated by fewer than $2d-1$ elements, then $I_{\{1\}} \subsetneq m^n$.

This gives a class of ideals in polynomial rings in arbitrary numbers of variables whose first coefficient ideals specialize with respect to general elements of the ideal.

Lastly, we apply the results on the first coefficient ideal to the core. In a Noetherian ring, an ideal $J \subseteq I$ is a reduction of I if I is integral over J. A reduction is minimal if it is minimal with respect to containment. There are usually infinitely many reductions, even minimal reductions, of an ideal I. Therefore, one takes the intersection of all reductions, called the core of I. The core of I is a subideal of I that in some way serves as an analogue of the integral closure of I.

Since the core of I is a possibly infinite intersection of ideals, it is difficult to compute. Lipman in [Lip94] has proved that in a regular domain of dimension d, the core is related to the adjoint ideal in the following way: $\text{adj}(I^d) \subseteq \text{core}(I)$. Much work has been done to understand when this containment is an equality.

Polini, Ulrich and Vitulli in [PUV07] have shown that in a polynomial ring over a field of characteristic zero, the first coefficient ideal $I_{\{1\}}$ is the unique largest ideal containing I and integral over I for which $\text{core}(I) = \text{core}(I_{\{1\}})$. For m-primary ideals which are generated in degree n, we see that $\text{adj}(I^d) = \text{core}(m^n)$. Hence, by characterizing when $I_{\{1\}} = \overline{I} = m^n$ for m-primary monomial ideals generated in degree n, we have characterized when $\text{core}(I) = \text{adj}(I^d)$.

We now describe the contents of each chapter. In Chapter 2, we review preliminaries for Chapter 3 and 4. In Chapter 3, we prove the main theorem on the specialization of the integral closure of an ideal I with respect to a general element of the I. In Chapter 4, we give counterexamples to the specialization of the integral closure with respect to a general element of the maximal ideal, as well as classes of ideals for which the integral closure does specialize in this sense.

In Chapter 5, we review background information on coefficient ideals. Chapter 6 contains the most general results in this dissertation on how coefficient ideals behave after going modulo a general element of the ideal. Chapter 6 also contains a note that coefficient ideals respect containments of reductions, a generalization of a result in [Hei+93].

In Chapters 7 through 9, we restrict to the case of polynomial rings over a field and m-primary monomial ideals generated in one degree. In Chapter 7, we consider the 2-variable case. In a polynomial ring in two variables, we characterize how all coefficient ideals of 0-dimensional monomial ideals generated in one degree behave with respect to specialization by general elements. In Chapter 8, we describe the first coefficient ideal of m-primary monomial ideals generated in one degree in the $d \geq 2$ variable case. In Chapter 9, we apply the results on the first coefficient ideal from Chapter 8 to the core of the ideal.

2. PRELIMINARIES: PART 1

We first define the central object of this dissertation, the integral closure of an ideal.

Definition 2.0.1. *Let R be a ring and let I be an R-ideal. An element $x \in R$ is* integral *over I if I satisfies an equation of integral dependence*

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0, \tag{2.1}$$

where $a_i \in I^i$ for $1 \leq i \leq n$.

The integral closure $\overline{I}$ *of I is defined to be the set of all elements $x \in R$ which are integral over I.*

We say that an ideal I is integrally closed if $I = \overline{I}$, and normal if $I^n = \overline{I^n}$ for all $n \in \mathbb{N}$. Next, we review some of the most essential properties of the integral closure of an ideal.

Remark 2.0.1.

(i) The integral closure $\overline{I}$ of I is an integrally closed R-ideal. See [SH06, Corollary 1.3.1].

(ii) $I \subseteq \overline{I}$. Indeed, for any $a \in I$, $x - a = 0$ is an equation of integral dependence that a satisfies.

(iii) Integral closure respects containments, i.e. if $J \subseteq I$, then $\overline{J} \subseteq \overline{I}$. This is clear because $J^i \subseteq I^i$ for all i and hence an equation of integral dependence over J is also an equation of integral dependence over I.

(iv) $\sqrt{0} \subseteq \overline{I}$. Indeed, let $x \in \sqrt{0}$. Then $x^n = 0$ for some n, and this is an equation of integral dependence of x over I.

(v) $\overline{I} \subseteq \sqrt{I}$. Let $x \in \overline{I}$. Since x is integral over I, there exists $a_i \in I^i$ for $1 \leq i \leq n$ such that

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \tag{2.2}$$

for some n. Then

$$x^n = -a_1 x^{n-1} - \cdots - a_n \in I \tag{2.3}$$

and hence $x \in \sqrt{I}$.

(vi) $\operatorname{ht} I = \operatorname{ht} \overline{I}$. Since $I \subseteq \overline{I} \subseteq \sqrt{I}$ and $V(I) = V(\sqrt{I})$, it immediately follows that $V(I) = V(\overline{I}) = V(\sqrt{I})$ where $V(J) := \{p \in \operatorname{Spec}(R) \mid p \supseteq J\}$. Since the primes containing $\overline{I}$ are exactly the primes containing I, I and $\overline{I}$ have the same height.

(vii) If $\varphi : R \to S$ is a ring homomorphism, then $\varphi(\overline{I}) \subseteq \overline{\varphi(I)S}$. This property is called *persistence*. Given an equation of integral dependence of x over I, after applying φ, we have an equation of integral dependence of $\varphi(x)$ over $\varphi(I)S$.

(viii) Integral closures commute with localization. If W be a multiplicatively closed subset of R, then $\overline{I}(W^{-1}R) = \overline{IW^{-1}R}$. See [SH06, Proposition 1.1.4].

(ix) If $R \to S$ is a faithfully-flat extension or integral extension, and I is an R-ideal, then $\overline{IS} \cap R = \overline{I}$. See [SH06, Proposition 1.6.1, 1.6.2].

Next, we define reductions of an ideal.

Definition 2.0.2. *A subideal $J \subseteq I$ is a* reduction *of I if $JI^k = I^{k+1}$ for some nonnegative integer k.*

Reductions are very closely related to integral closures. If the ring R is Noetherian, or more generally, if I is finitely generated, then $J \subseteq I$ is a reduction of I if and only if $I \subseteq \overline{J}$ (see [SH06, Corollary 1.2.5]). That is, J is a reduction of I if and only if every element of I is integral over J.

2.1 Integral Closure of a Ring

The integral closure of an ideal is an analogue of the integral closure of a ring, which we now define.

Definition 2.1.1. *Let R be a ring and S an overring of R. An element $x \in S$ is* integral *over R if there exists an equation of integral dependence of the form:*

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0, \tag{2.4}$$

with $a_i \in R$ *for* $1 \leq i \leq n$.

The integral closure of R *in* S*, denoted* $\overline{R}^S$ *is the set of all elements in* S *integral over* R.

Definition 2.1.2. *An extension* $R \subseteq S$ *is called an* integral extension *if* $\overline{R}^S = S$*. We say* R *is integrally closed in* S *if* $\overline{R}^S = R$.

The integral closure $\overline{R}^S$ is an integrally closed subring of S containing R.

2.2 General Elements

A general element of an ideal I in an algebra over an infinite field is defined as follows.

Definition 2.2.1. *Let* R *be a* k*-algebra, with* k *an infinite field. Let* $I = (x_1, \ldots, x_n)$ *be an* R*-ideal. Then a* general element x_α *of* I *is* $x_\alpha = \sum_{i=1}^n \alpha_i x_i$ *where* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *is in a nonempty Zariski-open subset of* k^n.

A nonempty Zariski-open subset is dense in k^n, so we think of a general element of I as a random k-linear combination of the generators of I.

Remark 2.2.1. (i) If U is open in k^n, then U is dense if and only if it is nonempty.

(ii) If properties $P_1, \ldots, P_s$ hold for general elements $x \in I$, then $P_1 \wedge P_2 \wedge \ldots \wedge P_s$ holds for general elements $x \in I$.

If I is not contained in a given prime ideal p, then there is a nonempty open set U in k^n such that for any $(a_1, \ldots, a_n) \in U$, $a_1x_1 + \cdots + a_nx_n \notin p$. By Remark 2.2.1(ii), if I is not contained any of the primes $\{p_1, \ldots, p_n\}$, then it is a general condition to avoid $\bigcup_{i=1}^n p_i$. In particular, if R is Noetherian and $\operatorname{grade} I > 0$, then $I \not\subseteq \bigcup_{p \in \operatorname{Ass}(R)} p$, the collection of zerodivisors on R. Hence we may assume that general $x \in I$ are nonzerodivisors on the ring R.

2.3 Properties of rings

Definition 2.3.1. *A domain* R *is* normal *if* R *is integrally closed in its quotient field* $\operatorname{Quot}(R)$*. A ring* R *is normal if it is locally a normal domain at each* $m \in \text{m-Spec}(R)$.

Definition 2.3.2. *An element a of a ring is* nilpotent *if there exists $n \geq 0$ such that $a^n = 0$. A ring that has no nonzero nilpotents is called a* reduced *ring.*

The nilradical $\sqrt{0}$ of a ring R is defined to be the intersection of all prime ideals, or equivalently, the ideal consisting of all nilpotent elements of the ring. From a ring R, one can construct a reduced ring R_{red} by going modulo the nilradical.

Noetherian reduced rings have several useful properties, such as that all associated primes of a reduced ring are minimal primes. Moreover, a Noetherian ring is reduced if and only if it is reduced locally at associated primes.

Definition 2.3.3. *A ring R is* equidimensional *if $\dim R = \dim R/p$ for any minimal prime of R.*

Definition 2.3.4. *A ring R is* catenary *if for any two prime ideals p, q with $p \subseteq q$, there exists a chain of prime ideals*

$$p = p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n = q \tag{2.5}$$

that cannot be refined any further and any such chain has the same length.

Definition 2.3.5. *A Noetherian ring R is* universally catenary *if every finitely generated R-algebra is catenary.*

Remark 2.3.1. A local ring that is equidimensional and catenary has the property that for any ideal I,

$$\operatorname{ht} I + \dim R/I = \dim R. \tag{2.6}$$

The following theorem is known as the Dimension Formula.

Theorem 2.3.2. *Let R be a universally catenary Noetherian ring and let S be a domain that is essentially of finite type over R. Then for $q \in \operatorname{Spec}(S)$ and $p = q \cap R$,*

$$\dim S_q = \dim R_p + \operatorname{trdeg}_R S - \operatorname{trdeg}_{\kappa(p)} \kappa(q), \tag{2.7}$$

where $\kappa(p) = R_p/pR_p$ and $\kappa(q) = S_q/qS_q$.

2.4 Blowup algebras

Next we define two structures that are essential to the study of integral closures of ideals, the Rees algebra and the extended Rees algebra.

Definition 2.4.1. *Let R be a ring, I an R-ideal, and t a variable over R. We define the Rees algebra of I to be the subring of $R[t]$ defined as*

$$R[It] = \oplus_{n \geq 0} I^n t^n = R \oplus It \oplus I^2 t^2 \oplus \cdots . \tag{2.8}$$

Definition 2.4.2. *Let R be a ring, I an R-ideal, and t a variable over R. We define the extended Rees algebra of I to be the subring of the Laurent polynomial ring $R[t, t^{-1}]$ defined as*

$$R[It, t^{-1}] = \oplus_{n \in \mathbf{Z}} I^n t^n \tag{2.9}$$

where $I^n = R$ for $n \leq 0$.

Theorem 2.4.1 ([SH06, Theorem 5.1.4])**.** *Let R be a Noetherian ring and let I be an R-ideal. If* $\dim R$ *is finite, then:*

(i) If $I \not\subseteq p$ for any $p \in$ $\operatorname{Spec}(R)$ *with* $\dim R/p = \dim R$*, then* $\dim R[It] = \dim R + 1$*.*

(ii) If $I \subseteq p$ for some $p \in$ $\operatorname{Spec}(R)$ *with* $\dim R/p = \dim R$*, then* $\dim R[It] = \dim R$*.*

(iii) $\dim R[It, t^{-1}] = \dim R + 1$*.*

Moreover, one proves that the minimal primes of $R[It]$ and $R[It, t^{-1}]$ are precisely $pR[t] \cap R[It]$ and $pR[t, t^{-1}] \cap R[It, t^{-1}]$ for $p \in \operatorname{Min}(R)$. Moreover, one shows that

$$\dim R[It] = \max \left\{ \dim \frac{R}{p} \left[\frac{I+p}{p} t \right] \mid p \in \operatorname{Min}(R) \right\}$$

and

$$\dim R[It, t^{-1}] = \max \left\{ \dim \frac{R}{p} \left[\frac{I+p}{p} t, t^{-1} \right] \mid p \in \operatorname{Min}(R) \right\}.$$

Proposition 2.4.1. *Let R be an equidimensional ring. Then $R[It, t^{-1}]$ is equidimensional.*

Proof. This follows immediately from Section 2.4 since

$$\dim \frac{R}{p}\left[\frac{I+p}{p}t, t^{-1}\right] = \dim R/p + 1 = \dim R + 1, \tag{2.10}$$

for all $p \in \operatorname{Min}(R)$. ■

Proposition 2.4.2. *Let R be an equidimensional ring and I an R-ideal that is not contained in any minimal prime of R. Then $R[It]$ is equidimensional.*

Proof. Let $P \in \operatorname{Min}(R[It])$. Then $P = pR[t, t^{-1}] \cap R[It]$ with $p \in \operatorname{Min}(R)$. Since R is equidimensional, $\dim R/p = \dim R$. Notice that

$$R[It]/P = R[It]/(pR[t, t^{-1}] \cap R[It]) \cong \frac{R}{p}\left[\frac{I+p}{p}t\right]. \tag{2.11}$$

Since $I \not\subseteq p$, $\dim \frac{R}{p}\left[\frac{I+p}{p}t\right] = \dim R/p + 1 = \dim R + 1$ by [SH06, Theorem 5.1.4]. ■

A blowup algebra that plays an important role in the theory of reductions is the special fiber ring.

Definition 2.4.3. *Let (R, m) be a local ring and let I be an R-ideal. Then the special fiber ring of I is*

$$\mathscr{F}_R(R) = R[It]/mR[It] = R/m \oplus I/mI \oplus I^2/mI^2 \oplus \cdots .$$

The dimension of $\mathscr{F}_I(R)$ is called the analytic spread of I and denoted $\ell(I)$.

Proposition 2.4.3 ([SH06, Corollary 8.3.9])**.** *Let (R, m) be a Noetherian local ring and let I be a proper R-ideal. Then $\operatorname{ht} I \leq \ell(I) \leq \dim R$.*

2.5 The integral closure of the Rees algebra

Taking the integral closure of either the Rees algebra or the extended Rees algebra in the polynomial ring $R[t]$ or Laurent polynomial ring $R[t, t^{-1}]$, respectively, recovers the integral closure of all powers of I, as we see below.

Proposition 2.5.1 ([SH06, Proposition 5.2.1])**.** *Let R be a ring and t be a variable over R. For any ideal I in R,*

$$\overline{R[It]}^{R[t]} = R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \overline{I^3}t^3 \cdots$$

and

$$\overline{R[It, t^{-1}]}^{R[t,t^{-1}]} = \cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \overline{I^3}t^3 \oplus \cdots .$$

For this reason, to compute or study the integral closure of an ideal or its powers, we often investigate the integral closure of its Rees algebra or extended Rees algebra.

We also sometimes use the absolute integral closure of the Rees algebra in its total ring of quotients $\mathrm{Quot}(R[It])$.

Proposition 2.5.2 ([SH06, Proposition 5.2.4])**.** *Let R be a ring and $\overline{R}$ be the integral closure of R in its total ring of quotients* $\mathrm{Quot}(R)$*. The integral closure of the Rees algebra $R[It]$ in its total ring of quotients is*

$$\overline{R[It]}^{\mathrm{Quot}(R[It])} = \overline{R} \oplus \overline{I\overline{R}}t \oplus \overline{I^2\overline{R}}t^2 \oplus \overline{I^3\overline{R}}t^3 \oplus \cdots \tag{2.12}$$

and the integral closure of the extended Rees algebra $R[It, t^{-1}]$ in its total ring of quotients is

$$\overline{R[It, t^{-1}]}^{\mathrm{Quot}(R[It,t^{-1}])} = \cdots \oplus \overline{R}t^{-2} \oplus \overline{R}t^{-1} \oplus \overline{R} \oplus \overline{I\overline{R}}t \oplus \overline{I^2\overline{R}}t^2 \oplus \overline{I^3\overline{R}}t^3 \oplus \cdots . \tag{2.13}$$

Notice that the integral closure of a Rees algebra in the polynomial ring and the total ring of quotients coincide if R is normal. Likewise, the integral closure of the extended Rees algebra in the Laurent polynomial ring and its total ring of quotients coincide if R is normal.

2.6 Properties of Integral Extensions

Since we study integral closures of ideals via integral closures of Rees algebras or extended Rees algebras, properties of integral extensions are used extensively. We review some key properties of integral extensions.

Proposition 2.6.1. *Let $R \subseteq S$ be an integral extension of rings. Then* $\dim R = \dim S$*.*

Proposition 2.6.2 ([SH06, Proposition 2.1.2])**.** *Let $R \to S$ be an integral extension of rings. Let I be an S-ideal. Then $R/(I \cap R) \to S/I$ is an integral extension of rings.*

2.7 Graded rings

Definition 2.7.1. *A ring R is* graded *if it can be written as a direct sum $\oplus_{n\in\mathbb{Z}} R_n$, in which each R_n is an Abelian group and $R_m R_n \subseteq R_{m+n}$. We say R is nonnegatively graded if $R_i = 0$ for $i < 0$.*

Definition 2.7.2. *An element $x \in R$ is* homogeneous of degree *n if $x \in R_n$ for some n. An R-ideal I is* homogeneous *if I is generated by homogeneous elements of R.*

Definition 2.7.3. *A module M over a graded ring R is* graded *if it is a direct sum $M = \oplus_{n\in\mathbb{Z}} M_n$, in which each M_n is an Abelian group and $R_m M_n \subseteq M_{m+n}$.*

We define a shift of the graded module M, $M(i)$, to be the graded module whose components are $[M(i)]_n = M_{n+i}$.

Given graded R-modules M, N, an R-linear map $f : M \to N$ is homogeneous of degree j if $f(M_i) \subseteq N_{i+j}$ for all $i \in \mathbb{Z}$ and homogeneous if it is homogeneous of degree 0. We discuss graded homomorphisms in more detail in Preliminaries: Part 2.

Theorem 2.7.1 ([BH93, Theorem 1.5.5])**.** *If R is a graded ring, i.e. $R = \oplus_{i\in\mathbb{Z}} R_i$, then R is Noetherian if and only if R_0 is Noetherian and R is a finitely generated R_0-algebra.*

Graded rings that are *local are an analogue of local rings.

Definition 2.7.4. *Let R be a graded ring. If R has a unique maximal homogeneous ideal m, then (R, m) is called* *local.

Notice that if (R, m) is a Noetherian local ring, then the Rees algebra $R[It]$ is a nonnegatively graded ring with maximal homogeneous ideal $mR[It] + ItR[It]$. The extended Rees algebra $R[It, t^{-1}]$ is a graded ring and if I is a proper ideal, $R[It, t^{-1}]$ has maximal homogeneous ideal $t^{-1}R[It, t^{-1}] + mR[It, t^{-1}] + ItR[It, t^{-1}]$.

2.8 Analytically Unramified Rings

Definition 2.8.1. *Let (R, m) be a Noetherian local ring. Let $\widehat{R}$ denote the m-adic completion of R. The ring R is* analytically unramified *if $\widehat{R}$ is reduced.*

Many of the results of the next sections require the analytically unramified assumption on the ring for the following purpose.

Proposition 2.8.1 ([SH06, Corollary 9.2.1])**.** *Let R be an analytically unramified local ring. Let I be an R-ideal and t a variable over R. Then the integral closure of $R[It]$ in $R[t]$ is a finite $R[It]$-module.*

Moreover, the integral closure of $R[It, t^{-1}]$ in $R[t, t^{-1}]$ is a finite $R[It, t^{-1}]$-module. This property ensures that the integral closures of the Rees algebras and extended Rees algebras are Noetherian.

2.9 Depth

We first define a regular sequence:

Definition 2.9.1. *Let R be a ring and M be an R-module. We say that $a_1, \ldots, a_n$ is an* M-regular sequence *if for $1 \leq i \leq n$, a_i is a nonzerodivisor on $M/(a_1, \ldots, a_{i-1})M$ and $M \neq (a_1, \ldots, a_n)M$.*

The following proposition indicates that if M is a finite module over a Noetherian ring R, then we have a well-defined notion of the maximal length of a regular sequence.

Proposition 2.9.1. *Let R be a Noetherian ring, M a finite R-module, and I an R-ideal such that $IM \neq M$. Then*

(a) There is a maximal M-regular sequence contained in I.

(b) The length of each such maximal M-regular sequence is

$$\min\{i \in \mathbb{Z}_{\geq 0} \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

We then define the *depth* of M with respect to I to be the maximal length of an M-regular sequence in I. If the ring (R, m) is local and the ideal is not specified, the *depth* of M is the depth of M with respect to m.

A finite module M over a Noetherian local ring R is *Cohen-Macaulay* if $\operatorname{depth}(M) = \dim M$ or if $M = 0$. If R is not local, M is *Cohen-Macaulay* if it is Cohen-Macaulay locally at all prime ideals of R or equivalently, Cohen-Macaulay locally at all maximal ideals of R.

A subset of the class of Cohen-Macaulay rings are regular rings.

Definition 2.9.2. *Let (R, m) be a Noetherian local ring. We say R is a regular local ring if the minimal number of generators of the maximal ideal m is equal to the dimension of the ring.*

Definition 2.9.3. *Let R be a Noetherian ring. We say R is regular if R_m is a regular local ring for all $m \in \text{m-Spec}(R)$.*

2.10 Serre's Conditions

Next we define Serre's Conditions S_k for finite modules over Noetherian rings and R_k for Noetherian rings. The S_k property approximates Cohen-Macaulayness of the module and the R_k property approximates regularity of the ring.

Definition 2.10.1. *Let R be a Noetherian ring and M a finite R-module. Let $k \geq 0$. We say that M satisfies S_k if $\operatorname{depth}(M)_p \geq \min\{\dim M_p, k\}$ for all $p \in \operatorname{Supp}(M)$.*

If M satisfies S_k for all $k \geq 0$, then M is Cohen-Macaulay.

Definition 2.10.2. *Let R be a Noetherian ring and $k \geq 0$. We say that R satisfies R_k if R_p is regular for every $p \in \operatorname{Spec}(R)$ such that $\dim R_p \leq k$.*

Notice that if R satisfies R_k for every $k \geq 0$, then R is regular.

Proposition 2.10.1. *A Noetherian ring R is reduced if and only if R satisfies S_1 and R_0.*

Proposition 2.10.2. *A Noetherian ring R is normal if and only if R satisfies S_2 and R_1.*

2.11 Excellent rings

Let R be a Noetherian ring. Let $\mathrm{Reg}(R) = \{p \in \mathrm{Spec}(R) \mid R_p \text{ is regular}\}$.

Definition 2.11.1. *A Noetherian ring R is $J-1$ if $\mathrm{Reg}(R)$ is open in $\mathrm{Spec}(R)$.*

Definition 2.11.2. *A Noetherian ring R is $J-2$ if any finitely generated R-algebra is $J-1$.*

Definition 2.11.3. *Let R be a Noetherian algebra over a field k. R is geometrically regular over k if for any finite field extension K of k, $R \otimes_k K$ is regular.*

Definition 2.11.4. *A homomorphism of Noetherian rings $\varphi : R \to S$ is regular if it is flat and if for each $p \in \mathrm{Spec}(R)$, $S \otimes_R \kappa(p)$ is geometrically regular over $\kappa(p)$.*

Definition 2.11.5. *A Noetherian ring R is a G-ring if for any $p \in \mathrm{Spec}(R)$, the natural map $R_p \to \widehat{R_p}$ is regular.*

Definition 2.11.6. *A Noetherian ring R is excellent if it is universally catenary, a $J-2$ ring and a G-ring.*

Classes of rings which are excellent include complete Noetherian local rings, Dedekind domains of characteristic zero, and convergent power series rings over $\mathbb{R}$ or $\mathbb{C}$. Moreover, any localization of an excellent ring is excellent and any finitely generated algebra over an excellent ring is excellent. In particular, any finitely generated algebra over a field or the integers, or any localization thereof is excellent. See [Mat80, Chapter 13].

For our purposes, one of the most useful properties of an excellent ring is the following.

Proposition 2.11.1. *Let R be an excellent reduced ring. Then R is analytically unramified.*

This follows because Serre's conditions pass from R to $\widehat{R}$ using that R is a G-ring. Therefore, since R is reduced and hence satisfies R_0 and S_1, $\widehat{R}$ is reduced as well.

2.12 Local cohomology

Let R be a ring, I an R-ideal and M an R-module. We define

$$\Gamma_I(M) = \{x \in M \mid xI^n = 0 \text{ for some } n\}$$

and call $\Gamma_I(-)$ the *section functor with respect to I*. We can see that

$$\begin{aligned}\Gamma_I(M) &= \bigcup_{n\geq 0} (0 :_M I^n) \\ &= \varinjlim (0 :_M I^n) \\ &= \varinjlim \operatorname{Hom}_R(R/I^n, M).\end{aligned}$$

From this one sees that $\Gamma_I(-)$ is a left-exact additive functor.

Definition 2.12.1. *We define the i^{th}* local cohomology functor with support in *I, $H_I^i(-)$, to be the right derived functor of $\Gamma_I(-)$.*

Remark 2.12.1. Assume R is Noetherian. Let M be an R-module and I an R-ideal. Then $I^n x = 0$ for some $x \in M$ if and only if $I^n \subseteq \operatorname{ann} x$. This is equivalent to $I \subseteq \sqrt{\operatorname{ann} x}$ because R is Noetherian. This means that $V(I) \supseteq V(\sqrt{\operatorname{ann} x}) = V(\operatorname{ann} x) = \operatorname{Supp}(Rx)$. Therefore,

$$H_I^0(M) = \{x \in M \mid \operatorname{Supp}(Rx) \subseteq V(I)\}\,. \tag{2.14}$$

Theorem 2.12.2. *Given a short exact sequence of R-modules*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

the section functor $\Gamma_I(-)$ induces a long exact sequence of local cohomology

$$0 \longrightarrow \Gamma_I(M') \longrightarrow \Gamma_I(M) \longrightarrow \Gamma_I(M'') \longrightarrow H_I^1(M') \longrightarrow H_I^1(M) \longrightarrow \cdots$$

Local cohomology with support in I is intrinsically related to the depth of a module with respect to I.

Theorem 2.12.3. *Let R be a Noetherian ring, I an R-ideal and M an R-module. Then*

$$\operatorname{depth}_I(M) = \min\{i \mid H_I^i(M) \neq 0\}. \tag{2.15}$$

One useful fact about local cohomology is that it can be computed via the Čech complex.

Definition 2.12.2. *Let $x \in R$. The* Čech complex of x *is*

$$C^{\bullet}(x): \ 0 \longrightarrow R \longrightarrow R_x \longrightarrow 0 \tag{2.16}$$

where $R \to R_x$ is given by the natural map.

If $\underline{x} = x_1, \ldots, x_n$ is a sequence of elements in R, then we define the Čech complex of $\underline{x}$ as follows

$$C^{\bullet}(\underline{x}) = C^{\bullet}(x_1) \otimes_R \cdots \otimes_R C^{\bullet}(x_n). \tag{2.17}$$

If M is an R-module, then $C^{\bullet}(\underline{x}, M) = C^{\bullet}(\underline{x}) \otimes_R M$ is the Čech complex of $\underline{x}$ with coefficients in M.

Theorem 2.12.4. *Let R be a Noetherian ring. Let $I = (x_1, \ldots, x_n)$ be an R-ideal. Let M be any R-module. Then $H^i_I(M) = H^i(C^{\bullet}(\underline{x}, M))$.*

Using that local cohomology can be computed via the Čech complex, one easily gets the following theorem.

Theorem 2.12.5. *Let $\varphi : R \to S$ be a homomorphism of Noetherian rings. Let I be an R-ideal and M an S-module. Then $H^i_I(M) \cong H^i_{IS}(M)$ for all i.*

The following two observations about local cohomology are used in the proof of the main theorem in the next chapter.

Remark 2.12.6. Let R be a Noetherian ring. Let M denote an R-module and I an R-ideal. If $M_p = 0$ for $p \notin V(I)$, then $H^0_I(M) = M$.

Proof. By Remark 2.12.1, $H^0_I(M) = \{x \in M \mid V(I) \supset \operatorname{Supp}(Rx)\}$. Notice that for any $x \in M$, $Rx \subset M$ and hence $(Rx)_p \subset M_p$ for any $p \in \operatorname{Spec}(R)$. Therefore, $(Rx)_p = 0$ for $p \notin V(I)$, and so $\operatorname{Supp}(Rx) \subset V(I)$. This implies $H^0_I(M) = M$. ■

Remark 2.12.7. Let R be a Noetherian ring, M an R-module, and I an R-ideal. If $H^0_I(M) = M$, then $H^i_I(M) = 0$ for $i \geq 1$.

Proof. Let $I = (a_1, \ldots, a_n)$. Let $C^\bullet(\underline{a}, M)$ denote the Čech complex of M with respect to $\underline{a}$.

$$C^\bullet(\underline{a}, M) : 0 \longrightarrow M \xrightarrow{\varphi} \oplus_{i=1}^{n} M_{a_i} \longrightarrow \oplus_{0\leq i\leq j\leq n} M_{a_i a_j} \longrightarrow \cdots$$

Then $H_I^i(M) \cong H^i(C^\bullet(\underline{a}, M))$ for all i. In particular, $M = H_I^0(M) = \ker\varphi$, thus $\varphi = 0$ and hence $M_{a_i} = 0$ for all i. Thus any further localization of M is also zero. Hence the Čech complex has the form

$$C^\bullet(\underline{a}, K) : 0 \longrightarrow M \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

So $H_I^i(M) = 0$ for all $i \geq 1$. ■

3. SPECIALIZATION OF THE INTEGRAL CLOSURE OF AN IDEAL

This chapter is based on joint work with Rachel Lynn.

A property P of an ideal is said to specialize with respect to an element x if after going modulo x, the property P still holds. In this chapter, we explore how the property of being integrally closed as an ideal specializes with respect to general elements of the ideal.

We first discuss the previous results in this direction: a result by Itoh for complete intersections and a generalization of this result by Hong and Ulrich.

3.1 Background

The first two approaches to proving the specialization of the integral closure involve faithfully flat extensions of the ring and generic elements. Let R be a Noetherian ring and $I = (a_1, \ldots, a_n)$ be an R-ideal. We define $T_1, \ldots, T_n$ to be variables over R. Then $R[T_1, \ldots, T_n]$ is a faithfully flat extension of R and $x = a_1T_1 + a_2T_2 + \cdots + a_nT_n$ is called a generic element of $IR[T]$. In the case where R is local with maximal ideal m, then $R(T) = R[T_1, \ldots, T_n]_{mR[T]}$ is a faithfully flat extension of R, and $x = a_1T_1 + a_2T_2 + \cdots + a_nT_n$ is a generic element of $IR(T)$. Since faithfully flat extensions preserve heights of ideals,

$$\operatorname{ht} I = \operatorname{ht} IR[T] = \operatorname{ht} IR(T). \tag{3.1}$$

By [SH06, Lemma 8.4.2],

$$\overline{I}R[T] = \overline{IR[T]} \text{ and } \overline{I}R(T) = \overline{IR(T)}. \tag{3.2}$$

Both Itoh (in [Ito92]) and Hong-Ulrich (in [HU14]) prove that after passing from the original ring to one of the faithfully flat extensions, one can specialize by the generic element x and maintain the property of being integrally closed.

More specifically, Itoh proves in [Ito92] the following theorem.

Theorem 3.1.1. *Let (R, m, k) be an analytically unramified, Cohen-Macaulay local ring of dimension $d \geq 2$ such that $|k| = \infty$. Let $I = (a_1, \ldots, a_d)$ be a parameter ideal of R. Let $x = \sum_i T_i a_i$ with $T_1, \ldots, T_d$ indeterminates. Then $\overline{IR(T)/(x)} = \overline{IR(T)}/(x)$.*

Hong and Ulrich generalized the above result to rings which are not necessarily Cohen-Macaulay and ideals that are not parameter ideals in [HU14].

Theorem 3.1.2. *Let R be a Noetherian, locally equidimensional, universally catenary ring such that R_{red} is locally analytically unramified. Let $I = (a_1, \ldots, a_n)$ be an R-ideal of height at least 2. Let $x = \sum_i T_i a_i$. Then $\overline{IR[T]/(x)} = \overline{IR[T]}/(x)$.*

The significance of these results is that it allows one to induct on the height of an integrally closed ideal. For example, Theorem 3.1.2 can be used to give a simple proof of Huneke and Itoh's notable result that for a complete intersection I, $\overline{I^{n+1}} \cap I^n = \overline{I} I^n$ for all $n \geq 0$. A limitation of this method, however, is that it requires passing to the extension $R[T]$ or $R(T)$ of R. In the case where one passes to $R[T]$, the dimension of the ring is no longer preserved. In the case where one passes to $R(T)$, properties of the residue field, such as being algebraically closed or perfect, are not preserved by the ring extension. For this reason, we sought to find a setting in which we can specialize the integral closure of an ideal without extending the base ring.

3.2 Our approach

We aim to prove that for an ideal I in a ring R and an element $x \in I$, $\overline{I}/(x)$ is integrally closed and hence equal to $\overline{I/(x)}$.

We first note that this is not true for every element of I, as the following example shows.

Example 3.2.1. Let $R = k[x, y]$ and $I = (x^2, xy, y^2)$. Then $I = \overline{I}$, but $I/(x^2)$ is not integrally closed since $x + (x^2) \in \overline{I/(x^2)} \setminus I/(x^2)$.

However, for any element $a \in I$, the containment $\overline{I}/(a) \subseteq \overline{I/(a)}$ follows immediately from the definition of the integral closure of an ideal: Let $z \in \overline{I}$. Then z satisfies an equation of integral dependence

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0, \tag{3.3}$$

for $a_i \in I^i$. Modulo (a), this yields an equation of integral dependence of $z+(a)$ over $I/(a)$, and therefore $z+(a) \in \overline{I/(a)}$.

The proof of the reverse containment requires more assumptions on the ring, that the height of the ideal is at least 2, and that the element in the ideal is sufficiently general. The strategy of proof mimics the proof of Hong and Ulrich. Let $\mathcal{A} = R[It, t^{-1}]$ denote the extended Rees algebra of I in R and let $\overline{\mathcal{A}}$ denote the integral closure of $\mathcal{A}$ in $R[t, t^{-1}]$. Let $\mathcal{B} = R/(x)[I/(x)t, t^{-1}]$ denote the extended Rees algebra of $I/(x)$ in $R/(x)$ and let $\overline{\mathcal{B}}$ denote the integral closure of $\mathcal{B}$ in $R/(x)[t, t^{-1}]$. By [SH06, Proposition 5.2.1], $[\overline{\mathcal{A}}]_1$ is $\overline{I}$ and $[\overline{\mathcal{B}}]_1 = \overline{I/(x)}$ (see also Proposition 2.5.1). The natural map $R \to R/(x)$ induces a natural map on $R[t, t^{-1}] \to R/(x)[t, t^{-1}]$ which restricts to a natural map $\overline{\mathcal{A}} \to \overline{\mathcal{B}}$. Since $xt\overline{\mathcal{A}}$ is contained in the kernel of this map, we have a natural map $\varphi : \overline{\mathcal{A}}/xt\overline{\mathcal{A}} \to \overline{\mathcal{B}}$. Notice that $[\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_1 = \overline{I}/(x)$ since

$$\overline{\mathcal{A}} = \cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus \overline{I}t \oplus \overline{I^2}t^2 \oplus \cdots \tag{3.4}$$

and

$$xt\overline{\mathcal{A}} = \cdots \oplus Rxt^{-1} \oplus Rx \oplus Rxt \oplus \overline{I}xt^2 \oplus \overline{I^2}xt^3 \oplus \cdots . \tag{3.5}$$

Let $C = \operatorname{coker}(\varphi)$. Then $[C]_1 = \left(\overline{I/(x)}\right) / \left(\overline{I}/(x)\right)$. The strategy of proof is to show that $[C]_1 = 0$ and hence $\overline{I}/(x) = \overline{I/(x)}$ by showing that

(i) φ is locally an isomorphism at certain primes, and then

(ii) $[C]_1$ embeds into a component of a local cohomology module which vanishes.

3.3 Local Bertini Theorems

We now state local Bertini theorems of Flenner which are essential to our proof. Flenner proves that when factoring out a general element, Serre's conditions are preserved locally at certain primes. These theorems allow us to prove that φ is locally an isomorphism at certain primes. Applying these theorems requires that we pass first to the case that R is normal.

Theorem 3.3.1 ([Fle77, Corollary 4.7]). *Let S be a local excellent k-algebra over the field k of characteristic 0, let $I = (x_1, \dots, x_n) \subseteq m_S$, and let $D(I) := \operatorname{Spec}(S) \setminus V(I)$. Assume that for every $p \in U = D(I)$, the ring S_p satisfies Serre's condition (S_k). For $\alpha \in k^n$ general, let $x_\alpha := \sum_{i=1}^n \alpha_i x_i$. Then for every $p \in U \cap V(x_\alpha)$ the ring $(S/x_\alpha S)_p$ also satisfies Serre's condition (S_k).*

Theorem 3.3.2 ([Fle77, Corollary 4.7]). *Let S be a local excellent k-algebra over the field k of characteristic 0, let $I = (x_1, \dots, x_n) \subseteq m_S$, and let $D(I) := \operatorname{Spec}(S) \setminus V(I)$. Assume that for every $p \in U = D(I)$, the ring S_p satisfies Serre's condition (R_k). For $\alpha \in k^n$ general, let $x_\alpha := \sum_{i=1}^n \alpha_i x_i$. Then for every $p \in U \cap V(x_\alpha)$ the ring $(S/x_\alpha S)_p$ also satisfies Serre's condition (R_k).*

Since a ring is normal if and only if it satisfies Serre's conditions S_2 and R_1, we have the following corollary:

Corollary 3.3.3 ([Fle77, Corollary 4.8]). *Let S be a local excellent k-algebra over the field k of characteristic 0 and let $(x_1, \dots, x_n) \subseteq m_S$. Let $\operatorname{Nor}(S) := \{p \in \operatorname{Spec}(S) \mid S_p \text{ is normal}\}$. For general $\alpha \in k^n$, let $x_\alpha := \sum_{i=1}^n \alpha_i x_i$, as in Theorem 3.3.1. Then*

$$\operatorname{Nor}(S) \cap V(x_\alpha) \cap D(x_1, \dots, x_n) \subseteq \operatorname{Nor}(S/x_\alpha S).$$

3.4 Vanishing of Local Cohomology

The following vanishing of local cohomology theorem of Hong and Ulrich is also essential to our proof. We prove that $[C]_1$ is contained in a component of $H^2_J(\overline{\mathcal{A}})$ which vanishes by the following theorem. We apply the theorem to $J = (It, t^{-1})\mathcal{A}$. Since the following theorem requires that J has height at least 3, this assumption forces us to assume that the height of I is at least 2.

Theorem 3.4.1 ([HU14, Theorem 1.2]). *Let R be a Noetherian, locally equidimensional, universally catenary ring such that R_{red} is locally analytically unramified. Let I be a proper R-ideal with $\operatorname{ht} I > 0$, $\mathcal{A} = R[It, t^{-1}]$ the extended Rees ring of I, and $\overline{\mathcal{A}}$ the integral closure of $\mathcal{A}$ in $R[t, t^{-1}]$. Let J be an $\mathcal{A}$-ideal of height at least 3 generated by t^{-1} and homogeneous elements of positive degree. Then $[H^2_J(\overline{\mathcal{A}})]_n = 0$ for all $n \leq 0$.*

3.5 Reducing to the normal ring case

In this section we state and prove technical lemmas necessary to reduce to the case in which the ring R is normal. Lemma 3.5.1 allows us to assume the ring is reduced, Lemma 3.5.2 allows us to assume the ring is normal, and Lemma 3.5.3 shows the height of an ideal is preserved under these reductions.

Lemma 3.5.1. *Let R be an algebra over an infinite field k, let $R_{\text{red}} := R/\sqrt{0}$, and let J be an R-ideal. Let $I = (a_1, \ldots, a_n)$ be an R-ideal. Let x be a general element of I. If the integral closure of IR_{red} specializes with respect to the image of x in R_{red}, then the integral closure of I specializes with respect to the element x. That is, if*

$$\overline{IR_{\text{red}}} + (x)R_{\text{red}}/(x)R_{\text{red}} = \overline{IR_{\text{red}} + (x)R_{\text{red}}/(x)R_{\text{red}}},$$

then $\overline{I} + (x)/(x) = \overline{I + (x)/(x)}$.

Proof. Note that $\overline{I} + (x)/(x) \subseteq \overline{I + (x)/(x)}$ by the persistence of the integral closure applied to the natural map $R \to R/(x)$. We prove the reverse containment.

Let ψ denote the natural map from $R/(x)$ to $R_{\text{red}}/(x)R_{\text{red}}$. Applying persistence to the ideal $I + (x)/(x)$ under the map ψ, we see that

$$\psi(\overline{I + (x)/(x)}) \subseteq \overline{IR_{\text{red}} + (x)R_{\text{red}}/(x)R_{\text{red}}}. \tag{3.6}$$

Taking preimages, we have that

$$\begin{aligned} \overline{I + (x)/(x)} &\subseteq \psi^{-1}(\psi(\overline{I + (x)/(x)})) && (3.7)\\ &\subseteq \psi^{-1}(\overline{IR_{\text{red}} + (x)R_{\text{red}}/(x)R_{\text{red}}}). && (3.8) \end{aligned}$$

Since $\overline{IR_{\text{red}} + (x)R_{\text{red}}/(x)R_{\text{red}}} = \overline{IR_{\text{red}}} + (x)R_{\text{red}}/(x)R_{\text{red}}$ by assumption, by taking preimages, we see that

$$\psi^{-1}(\overline{IR_{\text{red}} + (x)R_{\text{red}}/(x)R_{\text{red}}}) = \psi^{-1}(\overline{IR_{\text{red}}} + (x)R_{\text{red}}/(x)R_{\text{red}}), \tag{3.9}$$

and hence

$$\overline{I + (x)/(x)} \subseteq \psi^{-1}(\overline{IR_{\text{red}}} + (x)R_{\text{red}}/(x)R_{\text{red}}). \tag{3.10}$$

By [SH06, Proposition 1.1.5], $\overline{IR_{\text{red}}} = \overline{I}R_{\text{red}}$, which implies that

$$\psi^{-1}(\overline{IR_{\text{red}}} + (x)R_{\text{red}}/(x)R_{\text{red}}) = \psi^{-1}(\overline{I}R_{\text{red}} + (x)R_{\text{red}}/(x)R_{\text{red}}). \tag{3.11}$$

Then

$$\begin{aligned} \psi^{-1}(\overline{I}R_{\text{red}} + (x)R_{\text{red}}/(x)R_{\text{red}}) &= \{a + (x) \mid a + (x) + \sqrt{0} \in \overline{I} + (x) + \sqrt{0}\} && (3.12) \\ &= \overline{I} + (x)/(x), && (3.13) \end{aligned}$$

since $\sqrt{0} \subseteq \overline{I}$. This shows the desired containment: $\overline{I + (x)/(x)} \subseteq \overline{I} + (x)/(x)$. ■

Lemma 3.5.2. *Let R be an algebra over an infinite field k. Let $\overline{R}$ denote $\overline{R}^{\text{Quot}(R)}$. Let $I = (a_1, \dots, a_n)$ be an R-ideal, and let x be a general element of I. If the integral closure of the image of I in $\overline{R}_m$ specializes with respect to the image of x in $\overline{R}_m$ for every maximal ideal $m \in$ m-Spec$(\overline{R})$, then then the integral closure of I specializes with respect to the element x. That is, if*

$$\overline{I\overline{R}_m}/(x)\overline{R}_m = \overline{I\overline{R}_m/(x)\overline{R}_m}$$

for all $m \in$ m-Spec$(\overline{R})$, then $\overline{I}/(x) = \overline{I/(x)}$.

Proof. We first show that the integral closure of I extended to $\overline{R}$ specializes with respect to the image of x in $\overline{R}$. That is, we show that $\overline{I\overline{R}}/(x)\overline{R} = \overline{I\overline{R}/(x)\overline{R}}$. Persistence applied to the ideal $\overline{I\overline{R}}$ and the natural map $\overline{R} \to \overline{R}/(x)\overline{R}$ gives the containment $\overline{I\overline{R}}/(x)\overline{R} \subseteq \overline{I\overline{R}/(x)\overline{R}}$. Since $\left(\overline{I\overline{R}/(x)\overline{R}}\right) / \left(\overline{I\overline{R}}/(x)\overline{R}\right)$ is an $\overline{R}/(x)\overline{R}$-module, it is zero if and only if it is zero locally at all maximal ideals of $\overline{R}/(x)\overline{R}$. Therefore, to prove $\overline{I\overline{R}}/(x)\overline{R} = \overline{I\overline{R}/(x)\overline{R}}$, we check that at maximal ideals $m \in$ m-Spec$(\overline{R}/(x)\overline{R})$, $(\overline{I\overline{R}}/(x)\overline{R})_m = (\overline{I\overline{R}/(x)\overline{R}})_m$.

Identifying m-Spec$(\overline{R}/(x)\overline{R})$ with m-Spec$(\overline{R}) \cap V((x))$ and utilizing that integral closures commute with localization, we have

$$\left(\overline{I\overline{R}/(x)\overline{R}}\right)_m = \overline{I\overline{R}_m/(x)\overline{R}_m} \tag{3.14}$$

and

$$\left(\overline{I\overline{R}}/(x)\overline{R}\right)_m = \overline{I\overline{R}_m}/(x)\overline{R}_m. \tag{3.15}$$

Since $\overline{I\overline{R}_m/(x)\overline{R}_m} = \overline{I\overline{R}_m}/(x)\overline{R}_m$ by assumption, this proves the desired equality. Therefore, $\overline{I\overline{R}}/(x)\overline{R} = \overline{I\overline{R}/(x)\overline{R}}$.

By persistence of integral closure applied to the ideal I and the natural map $R \to R/(x)$, we have $\overline{I}/(x) \subseteq \overline{I/(x)}$. It remains to show the reverse containment. Let ψ denote the natural map from $R/(x)$ to $\overline{R}/(x)\overline{R}$. Applying persistence to the ideal $I/(x)$ and the map ψ, we see that

$$\psi(\overline{I/(x)}) \subseteq \overline{I\overline{R}/(x)\overline{R}}. \tag{3.16}$$

Taking preimages, we have that

$$\begin{aligned}\overline{I/(x)} &\subseteq \psi^{-1}(\psi(\overline{I/(x)})) \\ &\subseteq \psi^{-1}(\overline{I\overline{R}/(x)\overline{R}}).\end{aligned}$$

Since we have shown that $\overline{I\overline{R}}/(x)\overline{R} = \overline{I\overline{R}/(x)\overline{R}}$, we can conclude that

$$\overline{I/(x)} \subseteq \psi^{-1}(\overline{I\overline{R}}/(x)\overline{R}). \tag{3.17}$$

Note that

$$\psi^{-1}((\overline{I\overline{R}}/(x)\overline{R}) = \{a + (x) \in R/(x) \mid a + (x)\overline{R} \in \overline{I\overline{R}} + (x)\overline{R}\}. \tag{3.18}$$

Since $x \in I$, and hence $(x)\overline{R} \subseteq \overline{I\overline{R}}$,

$$\psi^{-1}(\overline{I\overline{R}}/(x)\overline{R}) = \{a + (x) \in R/(x) \mid a \in \overline{I\overline{R}}\}. \tag{3.19}$$

Therefore, since $a \in R \cap \overline{I\overline{R}}$ and $R \to \overline{R}$ is an integral extension of rings, $a \in \overline{I}$ by [SH06, Proposition 1.6.1]. Therefore,

$$\psi^{-1}(\overline{I\overline{R}}/(x)\overline{R}) = \overline{I}/(x), \tag{3.20}$$

and we conclude that $\overline{I/(x)} \subseteq \overline{I}/(x)$.

$\blacksquare$

Lemma 3.5.3. *Let* (R, m) *be a local equidimensional excellent ring. Let* I *be an* R*-ideal. Then* $\operatorname{ht} I\overline{R_{\text{red}}} = \operatorname{ht} I$.

Proof. Since $\sqrt{0} \subseteq p$ for all $p \in \operatorname{Spec}(R)$, there is a one-to-one correspondence between $\operatorname{Spec}(R)$ and $\operatorname{Spec}(R_{\text{red}})$. Therefore, since R is local with maximal ideal m, its image mR_{red} is the unique maximal ideal of R_{red}. The minimal primes of R correspond to the minimal primes of R_{red}. By the correspondence of primes, we see that since R is equidimensional, R_{red} is equidimensional. Since R_{red} is a factor ring of an excellent ring, it is excellent. Furthermore, $\operatorname{ht} I = \operatorname{ht} IR_{\text{red}}$. Therefore, we assume that R is a reduced local equidimensional excellent ring, and prove that $\operatorname{ht} I\overline{R} = \operatorname{ht} I$.

Notice that if $\overline{R}$ is catenary and locally equidimensional of the same dimension at every maximal ideal $m \in$ m-$\operatorname{Spec}(\overline{R})$, then for every $p \in \operatorname{Spec}(\overline{R})$ one can easily see that,

$$\dim \overline{R}/p + \operatorname{ht} p = \dim \overline{R}. \tag{3.21}$$

and hence for any ideal I of $\overline{R}$,

$$\dim \overline{R}/I + \operatorname{ht} I = \dim \overline{R}. \tag{3.22}$$

Since $R \to \overline{R}$ is an integral extension, $\overline{I\overline{R}} \cap R = \overline{I}$ and $R/\overline{I} \to \overline{R}/\overline{I\overline{R}}$ is an integral extension. Hence $\dim \overline{R}/\overline{I\overline{R}} = \dim R/\overline{I}$. Therefore,

$$\begin{aligned}
\operatorname{ht} \overline{I} &= \dim R - \dim R/\overline{I} \\
&= \dim \overline{R} - \dim R/\overline{I} \\
&= \dim \overline{R}/\overline{I\overline{R}} + \operatorname{ht} \overline{I\overline{R}} - \dim R/\overline{I} \\
&= \dim R/\overline{I} + \operatorname{ht} \overline{I\overline{R}} - \dim R/\overline{I} \\
&= \operatorname{ht} \overline{I\overline{R}}.
\end{aligned}$$

Since any ideal and its integral closure have the same height, this shows that $\operatorname{ht} I = \operatorname{ht} I\overline{R}$, as desired.

We now show that $\overline{R}$ is catenary and locally equidimensional of the same dimension at every maximal ideal $m \in$ m-Spec$(\overline{R})$.

Since R is excellent, it is universally catenary. Since R is excellent and reduced, $\overline{R}$ is a finitely generated R-module, and therefore $\overline{R}$ is catenary.

We show that $\overline{R}$ is locally equidimensional of the same dimension at every maximal ideal $m \in$ m-Spec$(\overline{R})$. We claim that there is a one-to-one correspondence of minimal primes of R and $\overline{R}$.

Let S denote the set of nonzerodivisors on R, and let W denote the set of nonzerodivisors on $\overline{R}$. Every nonzerodivsor on R is a nonzerodivisor on $\operatorname{Quot}(R)$, and hence is a nonzerodivisor on $\overline{R}$. Therefore, $S \subseteq W$. Since $R \subseteq \overline{R}$ and $S \subseteq W$,

$$\operatorname{Quot}(R) = S^{-1}R \subseteq S^{-1}\overline{R} \subseteq W^{-1}\overline{R} = \operatorname{Quot}(\overline{R}). \tag{3.23}$$

Next, we see that every element of W is a unit in $\operatorname{Quot}(R)$: Let $w \in W$. Then since $W \subseteq \operatorname{Quot}(R)$, $w = u/v$ for $u \in R$ and $v \in S$. Moreover, since u/v is a nonzerodivisor on $\overline{R}$, u is a nonzerodivisor on R. Hence $w = u/v$ is a unit in $\operatorname{Quot}(R)$. Then since $\overline{R} \subseteq \operatorname{Quot}(R)$ and $W \subseteq (\operatorname{Quot}(R))^*$, the units of $\operatorname{Quot}(R)$, we conclude that

$$\operatorname{Quot}(\overline{R}) = W^{-1}\overline{R} \subseteq W^{-1}\operatorname{Quot}(R) = \operatorname{Quot}(R). \tag{3.24}$$

This shows that $R \to \overline{R}$ is a birational extension: $\operatorname{Quot}(R) = \operatorname{Quot}(\overline{R})$.

Note that for any Noetherian ring T, since the total ring of quotients $\operatorname{Quot}(T)$ is the localization of T with respect to the complement of the union of the associated primes of T, the minimal primes of T correspond to the minimal primes of $\operatorname{Quot}(T)$. Since the minimal primes of R and $\overline{R}$ are both in one-to-one correspondence with the minimal primes of $\operatorname{Quot}(R) = \operatorname{Quot}(\overline{R})$, we conclude that $\operatorname{Min}(R)$ is in one-to-one correspondence with $\operatorname{Min}(\overline{R})$. Therefore, every minimal prime of $\overline{R}$ contracts to a minimal prime of R.

Now let $m \in \text{m-Spec}(\overline{R})$. Let $q \in \text{Min}(\overline{R})$ be contained in m. Notice $m \cap R$ must be the unique maximal ideal of R since $R \to \overline{R}$ is an integral extension, and $q \cap R$ is a minimal prime of R by the above. Since R is equidimensional and local, $\dim R/(q \cap R) = \dim R$. Since R is universally catenary, applying the dimension formula yields:

$$\dim \left(\overline{R}/q\right)_m = \dim R/(q \cap R) + \operatorname{trdeg}_{R/(q\cap R)} \overline{R}/q - \operatorname{trdeg}_{\kappa((m\cap R)/(q\cap R))} \kappa(m/q). \tag{3.25}$$

Since $R \to \overline{R}$ is an integral extension, $R/(q \cap R) \to \overline{R}/q$ is also integral by [SH06, Proposition 2.1.2]. Therefore,

$$\operatorname{trdeg}_{R/(q\cap R)} \overline{R}/q = 0. \tag{3.26}$$

Moreover, $R/(m \cap R) \to \overline{R}/m$ is an integral extension, and therefore

$$\kappa((m \cap R)/(q \cap R)) \subseteq \kappa(m/q) \tag{3.27}$$

is integral. Therefore,

$$\operatorname{trdeg}_{\kappa((m\cap R)/(q\cap R))} \kappa(m/q) = 0. \tag{3.28}$$

Therefore,

$$\dim(\overline{R}/q)_m = \dim R/(q \cap R) = \dim R = \dim \overline{R}. \tag{3.29}$$

This shows that $\overline{R}$ is locally equidimensional of dimension equal to $\dim \overline{R}$ at every maximal ideal $m \in \text{m-Spec}(\overline{R})$. ■

The following lemma was proved by Hong and Ulrich for their proof of Theorem 3.1.2, which we are able to utilize for our proof.

Lemma 3.5.4 ([HU14, Lemma 1.1]). *Let R be a Noetherian, equidimensional, universally catenary local ring of dimension d such that $R_{\text{red}} = R/\sqrt{0}$ is analytically unramified. Let $I = (a_1, \ldots, a_n)$ be a proper R-ideal with $\operatorname{ht} I > 0$ and write $\mathcal{A} = R[It, t^{-1}]$ for the extended Rees ring of I. Let $\overline{\mathcal{A}}$ denote $\overline{\mathcal{A}}^{R[t,t^{-1}]}$, the integral closure of $\mathcal{A}$ in the Laurent polynomial ring. Then $\operatorname{grade} It(\overline{\mathcal{A}}/t^{-1}\overline{\mathcal{A}}) > 0$.*

We use the following consequence of the lemma above.

Remark 3.5.5. Assume the notation of Lemma 3.5.4. In addition, assume R contains an infinite field k. Let x be a general element of I. We may assume xt is regular on $\overline{\mathcal{A}}/t^{-1}\overline{\mathcal{A}}$. Therefore, t^{-1}, xt is a regular sequence of length 2 on $\overline{\mathcal{A}}$.

3.6 Proof of the Specialization of the Integral Closure

Theorem 3.6.1. *Let (R, m) be a local equidimensional excellent k-algebra, where k is a field of characteristic 0. Let $I = (a_1, \ldots, a_n)$ be an R-ideal such that $\operatorname{ht} I \geq 2$, and let x be a general element of I. Then $\overline{I}/(x) = \overline{I/(x)}$.*

Proof. By Lemma 3.5.1, we may pass from R to R_{red} to assume R is a reduced local equidimensional excellent k-algebra. Then by Lemma 3.5.2 we may pass from R to $\overline{R}_m$ for any $m \in \text{m-Spec}(\overline{R})$ to assume in addition that R is a local normal ring, hence also a domain, and by Lemma 3.5.3 we may still assume that I has height at least 2.

Let $\mathcal{A} = R[It, t^{-1}]$, the extended Rees algebra of I, and $\mathcal{B} = R/(x)[(I/(x))t, t^{-1}]$, the extended Rees algebra of $I/(x)$. Let $\overline{\mathcal{A}}$ denote the integral closure of $\mathcal{A}$ in the Laurent polynomial ring $R[t, t^{-1}]$ and $\overline{\mathcal{B}}$ denote the integral closure of $\mathcal{B}$ in $R/(x)[t, t^{-1}]$. Define J to be the $\mathcal{A}$-ideal $(It, t^{-1})\,\mathcal{A}$.

The natural map $R \to R/(x)$ induces a natural map of the Laurent polynomial rings $R[t, t^{-1}] \to R/(x)[t, t^{-1}]$. The image of an element of $\overline{\mathcal{A}}$ under this natural map will be integral over $\mathcal{B}$, and therefore the map restricts to a natural map

$$\overline{\mathcal{A}} \longrightarrow \overline{\mathcal{B}}. \tag{3.30}$$

Notice that $xt\overline{\mathcal{A}}$ is contained in the kernel of the above map, which implies the existence of the natural map

$$\varphi : \overline{\mathcal{A}}/xt\overline{\mathcal{A}} \longrightarrow \overline{\mathcal{B}}. \tag{3.31}$$

We show that φ_p is an isomorphism for $p \in \operatorname{Spec}(\overline{\mathcal{A}}) \setminus V(J\overline{\mathcal{A}})$. We consider two cases: $t^{-1} \notin p$ or $It \not\subseteq p$.

First suppose $t^{-1} \notin p$. Localizing at the element t^{-1}, we have

$$\overline{\mathcal{A}}_{t^{-1}} = R[t, t^{-1}], \quad (xt\overline{\mathcal{A}})_{t^{-1}} = xR[t, t^{-1}], \text{ and } \overline{\mathcal{B}}_{t^{-1}} = R/(x)[t, t^{-1}]. \tag{3.32}$$

Therefore, we see that

$$\begin{aligned}
\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_{t^{-1}} &\cong \overline{\mathcal{A}}_{t^{-1}}/(xt\overline{\mathcal{A}})_{t^{-1}} \\
&\cong R[t, t^{-1}]/xR[t, t^{-1}] \\
&\cong R/(x)[t, t^{-1}] \\
&\cong \overline{\mathcal{B}}_{t^{-1}}.
\end{aligned}$$

Since p does not contain t^{-1}, localization at p is a further localization of the rings above. Therefore, $\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p \cong \overline{\mathcal{B}}_p$.

Now let $p \in \operatorname{Spec}(\overline{\mathcal{A}}) \setminus V(It\overline{\mathcal{A}})$. We first show $\varphi_p : \overline{\mathcal{A}}_p/xt\overline{\mathcal{A}}_p \to \overline{\mathcal{B}}_p$ is injective. To do so, it suffices to show that $xt\overline{\mathcal{A}}_p$ is the kernel of the natural map

$$\psi_p : \overline{\mathcal{A}}_p \longrightarrow \overline{\mathcal{B}}_p. \tag{3.33}$$

It is clear that $xt\overline{\mathcal{A}}_p \subseteq \ker(\psi_p)$. To show that the two ideals of $\overline{\mathcal{A}}_p$ are equal, it is enough to show equality locally at associated primes of $xt\overline{\mathcal{A}}_p$.

Since R is normal, the integral closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ in $R[t, t^{-1}]$ is equal to $\overline{\mathcal{A}}^{\mathrm{Quot}(\mathcal{A})}$ and hence is normal. Since R is an excellent domain, $\overline{\mathcal{A}} = \overline{\mathcal{A}}^{\mathrm{Quot}(\mathcal{A})}$ is a finitely generated R-algebra, and therefore is also excellent. Since the properties of normality and excellence pass to a localization, $\overline{\mathcal{A}}_p$ is excellent and normal. Since xt is a general element of It, by Corollary 3.3.3, $(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})_p$ is normal and therefore a domain. Therefore, $(xt\overline{\mathcal{A}})_p$ is prime and $\operatorname{Ass}(xt\overline{\mathcal{A}}_p) = \{xt\overline{\mathcal{A}}_p\}$.

Let $q = xt\overline{\mathcal{A}}_p$. Since q is principal, $\operatorname{ht} q \leq 1$. Since t^{-1}, xt is an $\overline{\mathcal{A}}$-regular sequence by Remark 3.5.5 and any ideal with grade at least 2 has height at least 2, $t^{-1} \notin q$. By the previous case $(xt\overline{\mathcal{A}})_q = \ker \psi_q$. This shows that $(xt\overline{\mathcal{A}})_p = \ker \psi_p$. Thus φ_p is injective for all $p \in \operatorname{Spec}(\overline{\mathcal{A}}) \setminus V(It\overline{\mathcal{A}})$.

We now show that for $p \in \operatorname{Spec}(\overline{\mathcal{A}}) \setminus V(It\overline{\mathcal{A}})$, φ_p is a surjection. Since $\mathcal{A}$ surjects onto $\mathcal{B}$, the extension

$$\operatorname{im}(\mathcal{A})_{p\cap\mathcal{A}} = \mathcal{B}_{p\cap\mathcal{A}} \subset \overline{\mathcal{B}}_{p\cap\mathcal{A}} \tag{3.34}$$

is an integral extension. Since $\operatorname{im}(\mathcal{A}) \subseteq \operatorname{im}(\overline{\mathcal{A}}) \subseteq \overline{\mathcal{B}}$, the intermediate extension

$$\operatorname{im}(\overline{\mathcal{A}})_{p\cap\mathcal{A}} \subseteq \overline{\mathcal{B}}_{p\cap\mathcal{A}} \tag{3.35}$$

is an integral extension, and thus $\operatorname{im}(\overline{\mathcal{A}})_p = \operatorname{im}\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p \subseteq \overline{\mathcal{B}}_p$ is also an integral extension.

Next we show that $\operatorname{im}(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})_p \subseteq \overline{\mathcal{B}}_p$ is a birational extension. Notice that

$$(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})_{t^{-1}} \cong R/(x)[t,t^{-1}], \tag{3.36}$$

$$(\overline{\mathcal{B}})_{t^{-1}} \cong R/(x)[t,t^{-1}]\ , \tag{3.37}$$

and hence

$$(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})_{t^{-1}} \cong (\overline{\mathcal{B}})_{t^{-1}}. \tag{3.38}$$

The rings remain isomorphic after localizing at the image of $\overline{\mathcal{A}} \setminus p$ in $\overline{\mathcal{B}}$. By the argument above, $t^{-1} \notin xt\overline{\mathcal{A}}_p$. Since $(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})_p$ is a domain, t^{-1} is a nonzerodivisor on $(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})_p$. Moreover, t^{-1} is a nonzerodivisor on $R/(x)[t,t^{-1}]$, hence on $\overline{\mathcal{B}}$, and therefore is a nonzerodivisor on $\overline{\mathcal{B}}_p$. Therefore, $\operatorname{Quot}(\operatorname{im}(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})_p)$ and $\operatorname{Quot}(\overline{\mathcal{B}}_p)$ are both naturally isomorphic to the total ring of quotients of the localization of $R/(x)[t,t^{-1}]$ at the image of $\overline{\mathcal{A}} \setminus p$ in $\overline{\mathcal{B}}$. Since $\operatorname{im}(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})_p \subseteq \overline{\mathcal{B}}_p$,

$$\operatorname{Quot}(\operatorname{im}(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})_p) = \operatorname{Quot}(\overline{\mathcal{B}}_p). \tag{3.39}$$

Since $(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})_p$ is normal by Corollary 3.3.3, and $\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p \cong \operatorname{im}\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p$, $\operatorname{im}\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p$ is normal. Since $\operatorname{im}\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p \subseteq \overline{\mathcal{B}}_p$ is an integral extension in the total ring of quotients of $\operatorname{im}\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p$, we conclude that $\operatorname{im}\left(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}\right)_p = \overline{\mathcal{B}}_p$. Therefore, φ_p is surjective, hence an isomorphism.

Let K denote the kernel of φ and C denote the cokernel of φ. Recall that

$$[\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_1 = \overline{I}/(x) \quad \text{and} \quad [\overline{\mathcal{B}}]_1 = \overline{I/(x)}\,,$$

and therefore it suffices to show that $[C]_1 = (\overline{I/(x)})/(\overline{I}/(x)) = 0$. In order to do so, we identify C with a submodule of $H^2_J(\overline{\mathcal{A}})$.

Since φ_p is an isomorphism for all $p \notin V(J\overline{\mathcal{A}})$ as shown above, $K_p = C_p = 0$ for all $p \notin V(J\overline{\mathcal{A}})$. By Remark 2.12.6, $H^0_{J\overline{\mathcal{A}}}(K) = K$ and thus by Remark 2.12.7, $H^i_{J\overline{\mathcal{A}}}(K) = 0$ for all $i > 0$. Moreover, since $\mathcal{A} \to \overline{\mathcal{A}}$ is a map of Noetherian rings, $H^i_{J\overline{\mathcal{A}}}(K) = H^i_J(K)$ for all i by Theorem 2.12.5. Similarly $H^0_J(C) = H^0_{J\overline{\mathcal{A}}}(C) = C$. We note that $t^{-1} \in J$ is a regular element on $\overline{\mathcal{B}}$ and therefore $yJ^i \neq 0$ for any $y \in \overline{\mathcal{B}} \setminus \{0\}$ and any nonnegative integer i. Hence, $H^0_J(\overline{\mathcal{B}}) = 0$.

The long exact sequence of local cohomology induced by the exact sequence

$$0 \longrightarrow K \longrightarrow \overline{\mathcal{A}}/xt\overline{\mathcal{A}} \xrightarrow{\varphi} \varphi(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}) \longrightarrow 0$$

yields the exact sequences

$$H^i_J(K) \longrightarrow H^i_J(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}) \longrightarrow H^i_J(\varphi(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})) \longrightarrow H^{i+1}_J(K)$$

for all $i \geq 0$. Since $H^i_J(K) = 0$ for $i \geq 1$, we obtain $H^i_J(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}) \cong H^i_J(\varphi(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}))$ for all $i \geq 1$.

From the long exact sequence of local cohomology induced by the exact sequence

$$0 \longrightarrow \varphi(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}) \longrightarrow \overline{\mathcal{B}} \longrightarrow C \longrightarrow 0 \tag{3.40}$$

we obtain the exact sequence

$$0 = H^0_J(\overline{\mathcal{B}}) \longrightarrow H^0_J(C) \longrightarrow H^1_J(\varphi(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})).$$

Since $H^1_J(\varphi(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})) \cong H^1_J(\overline{\mathcal{A}}/xt\overline{\mathcal{A}})$, this shows that

$$C = H^0_J(C) \hookrightarrow H^1_J(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}). \tag{3.41}$$

By Remark 3.5.5, $\operatorname{depth}_J(\overline{\mathcal{A}}) \geq 2$. Thus $H^1_J(\overline{\mathcal{A}}) = 0$ by Theorem 2.12.3. Applying the long exact sequence of local cohomology to the short exact sequence

$$0 \longrightarrow xt\overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}} \longrightarrow \overline{\mathcal{A}}/xt\overline{\mathcal{A}} \longrightarrow 0$$

we obtain the exact sequence

$$0 = H^1_J(\overline{\mathcal{A}}) \longrightarrow H^1_J(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}) \longrightarrow H^2_J(xt\overline{\mathcal{A}}).$$

Therefore $C \hookrightarrow H^1_J(\overline{\mathcal{A}}/xt\overline{\mathcal{A}}) \hookrightarrow H^2_J(xt\overline{\mathcal{A}})$.

Since x is a general element of I and I is a nonzero ideal in a domain and hence has positive grade, we may assume that x is a nonzerodivisor on R. Therefore, we assume xt is a nonzerodivisor on $R[t, t^{-1}]$ and therefore on $\overline{\mathcal{A}}$. Note that $R[t, t^{-1}]$ and hence $\overline{\mathcal{A}}$ is $\mathbf{Z}$-graded by giving t degree 1. Since xt is a nonzerodivisor on $\overline{\mathcal{A}}$ with degree 1, the map given by multiplication by xt on $\overline{\mathcal{A}}$ is an injective homogeneous map of degree 1. Therefore, we have a graded isomorphism

$$\overline{\mathcal{A}}(-1) \cong xt\overline{\mathcal{A}}. \tag{3.42}$$

This shows that $C \hookrightarrow H^2_J(\overline{\mathcal{A}}(-1))$, which implies that

$$[C]_n \hookrightarrow [H^2_J(\overline{\mathcal{A}}(-1))]_n \cong [H^2_J(\overline{\mathcal{A}})]_{n-1} \tag{3.43}$$

for all n. In order to apply Theorem 3.4.1, we must now show that the height of J is at least 3.

Since R is excellent and hence universally catenary, and $\mathcal{A}$ is a finitely generated R-algebra, $\mathcal{A}$ is catenary. Since R is equidimensional, so is $\mathcal{A}$. Since R is local, $\mathcal{A}$ is *local with maximal homogeneous ideal $\mathfrak{m} = m\,\mathcal{A} + It\,\mathcal{A} + t^{-1}\,\mathcal{A}$. Since $\mathfrak{m}$ is also a maximal ideal of $\mathcal{A}$, $\dim \mathcal{A}_{\mathfrak{m}} = \dim \mathcal{A} = \dim R + 1$.

Since J is a homogeneous $\mathcal{A}$-ideal, the minimal primes of J are homogeneous. Hence $\operatorname{ht} J = \operatorname{ht} J_{\mathfrak{m}}$. Since $\mathcal{A}_{\mathfrak{m}}$ is equidimensional and catenary, we have that

$$\operatorname{ht} J_{\mathfrak{m}} = \dim \mathcal{A}_{\mathfrak{m}} - \dim(\mathcal{A}/J)_{\mathfrak{m}}.$$

Since J is homogeneous, $\dim(\mathcal{A}/J)_{\mathfrak{m}}$ is equal to $\dim(\mathcal{A}/J)$. Notice that $\mathcal{A}/J \cong R/I$ and therefore,

$$\begin{aligned}
\operatorname{ht} J &= \dim R + 1 - \dim R/I \\
&= \dim R + 1 - \dim R + \operatorname{ht} I \\
&= 1 + \operatorname{ht} I \\
&\geq 3.
\end{aligned}$$

Since J is an $\mathcal{A}$-ideal of height at least 3 generated by t^{-1} and homogeneous elements of positive degree, by Theorem 3.4.1, $[H^2_J(\overline{\mathcal{A}})]_n = 0$ for $n \leq 0$. Then since

$$[C]_1 \subseteq [H^2_J(\overline{\mathcal{A}})]_0, \tag{3.44}$$

we conclude that $[C]_1 = 0$. ■

3.7 Specialization for Powers of I

We now consider the behavior of the integral closure of powers of I with respect to specialization by general elements of I.

Itoh in [Ito92] proved that the integral closure of sufficiently large powers of I is compatible with specialization by generic elements in a faithfully flat extension of R for parameter ideals in Cohen-Macaulay rings.

Theorem 3.7.1. *[Ito92, Theorem 1(c)] Let (R, m, k) be an analytically unramified, Cohen-Macaulay local ring of dimension $d \geq 2$ such that $|k| = \infty$. Let $I = (a_1, \ldots, a_d)$ be a* *parameter ideal of* R*. Let* $x = \sum_i T_i a_i$ *with* $T_1, \ldots, T_d$ *indeterminates. Then* $\overline{I^s(R(T)/(x))} = \overline{I^s R(T)} + (x)/(x)$ *for* s *sufficiently large.*

The following result shows that for sufficiently large powers of I, and general elements $x \in I$, specialization is compatible with integral closure without extending the base ring. We require the assumptions of Theorem 3.6.1 and add the assumption that the base ring is normal. We have been unable to reduce to the normal case as in Theorem 3.6.1, due to the added complication of our general element x not belonging to the ideal I^s which we are specializing.

Proposition 3.7.1. *Let* (R, m) *be a local normal equidimensional excellent algebra over a field* k *of characteristic zero. Let* I *be an* R*-ideal such that* $\operatorname{ht} I \geq 2$*, and let* x *be a general element of* I*. Then* $\overline{I^s} + (x)/(x) = \overline{(I/(x))^s}$ *for* s *sufficiently large.*

Proof. We first show that we may assume $R/(x)$ is an excellent reduced ring. Since $R/(x)$ is a factor ring of an excellent ring, it is excellent. Since R is a domain and hence reduced, by Theorem 3.3.1, $R/(x)$ satisfies R_0 locally at primes which do not contain I. Since the primes of height zero in $R/(x)$ correspond to primes of height at most one in R, and I has height at least 2, all primes of height zero in $R/(x)$ do not contain I and hence $R/(x)$ satisfies R_0 globally. Since R is a domain and I is a nonzero ideal, we may assume x is a nonzerodivisor on R. Since R is normal and hence satisfies S_2, $R/(x)$ satisfies S_1. Therefore, $R/(x)$ is reduced.

As in Theorem 3.6.1, let $\mathcal{A}$ denote the extended Rees algebra of I and let $\mathcal{B}$ denote the extended Rees algebra of $I/(x)$. Let $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ denote the integral closures of $\mathcal{A}$ and $\mathcal{B}$ in the Laurent polynomial rings $R[t, t^{-1}]$ and $R/(x)[t, t^{-1}]$, respectively. Let J denote the $\mathcal{A}$-ideal $(It, t^{-1})\,\mathcal{A}$. Consider the natural map $\varphi : \overline{\mathcal{A}}/xt\overline{\mathcal{A}} \to \overline{\mathcal{B}}$. Denote the cokernel of φ by C.

Since $R/(x)$ is reduced, so is $R/(x)[t, t^{-1}]$. Therefore, $\mathcal{B}$ is reduced. Since $R/(x)$ is excellent and $\mathcal{B}$ is a finitely generated $R/(x)$-algebra, $\mathcal{B}$ is excellent. Since $\mathcal{B}$ is excellent and reduced, $\overline{\mathcal{B}}^{\mathrm{Quot}(\mathcal{B})}$ is a finite $\mathcal{B}$-module. Since $\mathcal{B}$ is a Noetherian ring, $\overline{\mathcal{B}}^{\mathrm{Quot}(\mathcal{B})}$ is a Noetherian $\mathcal{B}$-module. Therefore, $\overline{\mathcal{B}} \subset \overline{\mathcal{B}}^{\mathrm{Quot}(\mathcal{B})}$ is a finite $\mathcal{B}$-module, and hence a finite $\mathcal{A}$-module. Since $\overline{\mathcal{B}}$ is finitely generated as an $\mathcal{A}$-module, so is C.

We claim that since C is a finite $\mathcal{A}$-module, then $[C]_s = 0$ for s sufficiently large. Note that C inherits a grading from $\overline{\mathcal{B}}$. Let $z_1, \ldots, z_r$ be a set of homogeneous generators of C as an $\mathcal{A}$-module. As in the proof of Theorem 3.6.1, $H^0_J(C) = C$. Therefore, by the definition of the section functor, there exists k_i such that $J^{k_i} z_i = 0$ for $1 \leq i \leq r$. Let $k = \max\{k_i \mid 1 \leq i \leq r\}$. Then $J^k z_i = 0$ for all i and since $It \subseteq J$, $(It)^k z_i = 0$ for all i.

Let $s > \max\{\deg(z_i) \mid 1 \leq i \leq r\}$. Notice that

$$\begin{aligned} [C]_s &= [\,\mathcal{A}\, z_1 + \cdots + \mathcal{A}\, z_r]_s \\ &= (It)^{s-\deg(z_1)} z_1 + \cdots + (It)^{s-\deg(z_r)} z_r \end{aligned}$$

If $s \geq k + \max\{\deg(z_i) \mid 1 \leq i \leq r\}$, then $s - \deg(z_i) \geq k$ for $1 \leq i \leq r$. Hence $(It)^{s-\deg(z_i)} z_i = 0$ for $1 \leq i \leq r$ and we conclude that $[C]_s = 0$.

Since $[\overline{\mathcal{A}}/xt\overline{\mathcal{A}}]_s = \overline{I^s} + (x)/(x)$ and $[\overline{\mathcal{B}}]_s = \overline{I^s + (x)/(x)}$, we see that

$$\overline{I^s} + (x)/(x) = \overline{I^s + (x)/(x)}$$

for $s > k + \max\{\deg(z_i) \mid 1 \leq i \leq r\}$. ■

4. SPECIALIZATION OF THE INTEGRAL CLOSURE OF AN IDEAL BY A GENERAL LINEAR FORM

This chapter is based on joint work with Rachel Lynn.

Next, we consider the case in which we specialize the integral closure of an ideal by a general element of the unique maximal ideal rather than a general element of the ideal I. The integral closure of an ideal does not behave as well with respect to specialization by a general element of the maximal ideal as it does with respect to a general element of the ideal. We give an example of an integrally closed monomial ideal of height 2 which does not specialize with respect to a general linear form, and give classes of ideals for which the integral closure does specialize with respect to a general linear form.

Example 4.0.1. Let $R = \mathbb{Q}[x, y, z]$. Let $I = (x^2, y\,z)$. Note that I is an integrally closed height 2 ideal of R. Let

$$a = \alpha\, x + \beta\, y + \gamma\, z$$

with α, β, γ nonzero. We see below that $I + (a)/(a)$ is not an integrally closed ideal of $R/(a)$.

We first show that $z^2 + (a)$ satisfies an equation of integral dependence over $I + (a)/(a)$ in $R/(a)$. Since $yz \in I$, $\beta\, y\, z \in I$. Therefore, $\alpha\, x\, z + \gamma\, z^2 \in I + (a)$. Notice that

$$\begin{aligned}
&(z^2)^2 + (\frac{2\,\beta}{\gamma} y\, z - \frac{\alpha^2}{\gamma^2}\, x^2)(z^2) + \frac{1}{\gamma^2}(\alpha x\, z + \gamma\, z^2)^2 \\
&= z^4 + \frac{2\,\beta}{\gamma} y\, z^3 - \frac{\alpha^2}{\gamma^2} x^2\, z^2 + \frac{1}{\gamma^2}(\alpha^2\, x^2\, z^2 + 2\,\alpha\,\gamma x\, z^3 + \gamma^2\, z^4) \\
&= 2\, z^4 + \frac{2\,\beta}{\gamma} y z^3 + \frac{2\,\alpha}{\gamma} x\, z^3 - \frac{\alpha^2}{\gamma^2} x^2\, z^2 + \frac{\alpha^2}{\gamma^2} x^2\, z^2 \\
&= 2\, z^4 + \frac{2\,\beta}{\gamma} y z^3 + \frac{2\,\alpha}{\gamma} x\, z^3 \\
&= \frac{2}{\gamma} z^3(\gamma\, z + \beta\, y + \alpha\, x) \in (a).
\end{aligned}$$

Since $2\frac{\beta}{\gamma} yz - \frac{\alpha^2}{\gamma^2} x^2 \in I \subseteq I + (a)$ and $\frac{1}{\gamma^2}(\alpha\, x\, z + \gamma\, z^2)^2 \in (I + (a))^2$, this shows that $z^2 + (a) \in \overline{I + (a)/(a)}$.

We show that $z^2 + (a) \notin I + (a)/(a)$. Suppose toward contradiction that z^2 is in the image of the ideal I in $R/(a)$. Let $r \in I$ such that $z^2 + r \in (a)$. Since (a) is a homogeneous ideal, we may assume that $z^2 + r$ is a homogeneous element of degree 2. Therefore, we may assume that there exists $s, t \in \mathbb{Q}$ such that

$$p(x, y, z) = z^2 + s\,x^2 + t\,y\,z \in (a). \tag{4.1}$$

It follows that

$$p(x, y, z) = (d\,x + e\,y + f\,z)(\alpha\,x + \beta\,y + \gamma\,z) \tag{4.2}$$

for some $d, e, f \in \mathbb{Q}$. Since the coefficient of z^2 in $p(x, y, z)$ is 1, $f = \frac{1}{\gamma}$. Since the coefficient of x^2 in $p(x, y, z)$ is s, $d = \frac{s}{\alpha}$. Since the coefficient of y^2 in $p(x, y, z)$ is 0, we conclude that $e\beta = 0$. Since $\beta \neq 0$ by assumption, $e = 0$.

Therefore,

$$\begin{aligned} p(x, y, z) &= \left(\frac{s}{\alpha}x + \frac{1}{\gamma}z\right)(\alpha\,x + \beta\,y + \gamma\,z) \\ &= s\,x^2 + \frac{s\,\beta}{\alpha}x\,y + \left(\frac{s\gamma}{\alpha} + \frac{\alpha}{\gamma}\right)x\,z + \frac{\beta}{\alpha}y\,z + z^2. \end{aligned}$$

If $s = 0$, then the above expression has a nonzero $x\,z$ term, a contradiction. If $s \neq 0$, then the above expression has a nonzero $x\,y$ term, a contradiction. Therefore, $z^2 + (a) \in \overline{I + (a)/(a)} \setminus I + (a)/(a)$.

Next, we give a class of ideals whose integral closures do not specialize with respect to a general linear form.

Example 4.0.2. Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over an infinite field k. Let $\mathfrak{m} = (x_1, \ldots, x_d)$ denote the homogeneous maximal ideal of R. Let I be an integrally closed ideal of height $d-1$ generated by t forms of degree n with $t < \binom{n+d-2}{n}$. Let $a = \alpha_1 x_1 + \cdots + \alpha_d x_d$ be a general linear form. Then $I + (a)/(a)$ is not an integrally closed $R/(a)$-ideal.

Note that we may assume $a \notin I$.

Since $\operatorname{ht} I = d - 1$ and hence

$$\dim R/I = \dim R - \operatorname{ht} I = 1, \tag{4.3}$$

$\dim R/(I+(a)) = 0$. Hence $I+(a)/(a)$ is an $\mathfrak{m}/(a)$-primary ideal in $R/(a)$.

Note that we have an isomoprhism

$$\varphi : R/(a) \longrightarrow k[x_1, \ldots, x_{d-1}], \tag{4.4}$$

defined by mapping $x_d \mapsto \frac{1}{\alpha_d}(-\alpha_1 x_1 - \ldots - \alpha_{d-1}x_{d-1})$. Let $\mathfrak{n} = (x_1, \ldots, x_{d-1})$ denote the homogeneous maximal ideal of $k[x_1, \ldots, x_{d-1}]$. Let $J = \varphi(I+(a)/(a))$. Since $I+(a)/(a)$ is $\mathfrak{m}/(a)$-primary, J is $\mathfrak{n}$-primary. Moreover, J is generated by forms of degree n.

Suppose that J is integrally closed. Since J is generated by forms of degree n, $J \subseteq \mathfrak{n}^n$. Since J is $\mathfrak{n}$-primary, there is $\mathfrak{n}^s \subseteq J \subseteq \mathfrak{n}^n$ for some $s \geq n$. Therefore, there exists $t \geq 1$ such that

$$\mathfrak{n}^{nt} \subseteq \mathfrak{n}^s \subseteq J \subseteq \mathfrak{n}^n. \tag{4.5}$$

Since $J \subseteq \mathfrak{n}^n$, $J(\mathfrak{n}^n)^{t-1} \subseteq (\mathfrak{n}^n)^t$. Since $(\mathfrak{n}^n)^t \subseteq J$, and J is generated in degree n, $(\mathfrak{n}^n)^t \subseteq J(\mathfrak{n}^n)^{t-1}$. Hence $J(\mathfrak{n}^n)^{t-1} = \mathfrak{n}^{nt}$, and therefore $J = \mathfrak{n}^n$.

Therefore, the minimal number of generators of J must be $\binom{n+d-2}{n}$. Since I is generated by $t < \binom{n+d-2}{n}$ forms, so is $\varphi(I+(a)/(a))$. We conclude that $\varphi(I+(a)/(a))$ is not integrally closed. Therefore, $I+(a)/(a)$ is not integrally closed.

However, there are cases in which going modulo a general element of the maximal ideal does preserve the property of being integrally closed, such as when R/I is reduced and $\operatorname{depth}(R/I) \geq 2$:

Proposition 4.0.1. *Let (R, m) be a local excellent algebra over an infinite field k. Let $\mathfrak{m} = (x_1, \ldots, x_n)$. Let I be an R-ideal such that R/I is reduced and* $\operatorname{depth}(R/I) \geq 2$. *Let $a = \sum_{i=1}^n \alpha_i x_i$ be a general element of $\mathfrak{m}$. Then $I+(a)/(a)$ is an integrally closed ideal of $R/(a)$.*

Proof. We claim that $R/(I+(a))$ is reduced. Since R/I is reduced and a is a general element of the maximal ideal $\mathfrak{m}$, by [Fle77, Corollary 4.2], $R/(I+(a))$ is reduced locally on the punctured spectrum. We will show that $R/(I+(a))$ is reduced globally.

Since $\operatorname{depth}(R/I) \geq 2$ and a is a general element of $\mathfrak{m}$, $\operatorname{depth}(R/(I+(a))) \geq 1$. Therefore, the maximal ideal $\mathfrak{m}/(I+(a))$ contains a nonzerodivisor on $R/(I+(a))$ and hence $\mathfrak{m}/(I+(a))$

is not an associated prime of $R/(I+(a))$. Since a ring is reduced if and only if it is reduced locally at associated primes, $R/(I+(a))$ is reduced.

Therefore, $I+(a)/(a) = \sqrt{I+(a)/(a)}$, and hence

$$I+(a)/(a) = \overline{I+(a)/(a)}.$$

■

Let I be a squarefree monomial ideal. Then by [HH11, Corollary 1.3.4], I has a primary decomposition

$$I = \bigcap_{j=1}^{n} p_j, \tag{4.6}$$

where $p_j = (x_{j_1}, \ldots, x_{j_k})$ with $1 \leq j_1 < \ldots < j_k \leq d$ for each $1 \leq j \leq n$. Note that if the generating sets of variables of the ideals p_j are disjoint, then $I = \prod_{j=1}^{n} p_j$ and is generated in degree n.

The following result shows that the integral closures of squarefree monomial ideals which are intersections of prime ideals generated by disjoint sets of variables specialize with respect to a general linear form.

Proposition 4.0.2. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over an infinite field k. Let $I = \bigcap_{i=1}^{n} p_i$, where each p_i is generated by a disjoint set of variables. Let $a = \sum_{i=1}^{d} \alpha_i x_i$ be a general linear form. Then $I+(a)/(a)$ is an integrally closed ideal of $R/(a)$.*

Proof. Since $I = \bigcap_{i=1}^{n} p_i$ and the ideals p_i for $1 \leq i \leq n$ are generated by disjoint sets of variables, every monomial generator of I is in $\prod_{i=1}^{n} p_i$. Since $\prod_{i=1}^{n} p_i \subseteq \bigcap_{i=1}^{n} p_i = I$, $I = \prod_{i=1}^{n} p_i$.

We prove the result by induction on n.

Base case: We first prove the result for $n = 1$. Notice that if $I = p$, a prime ideal generated by variables, then $I+(a)$ is a prime ideal of R, and hence $I+(a)/(a)$ is a prime ideal of $R/(a)$. Since any prime ideal is integrally closed, $\overline{I+(a)/(a)} = I+(a)/(a)$.

Next we prove the result for $n = 2$. Let $I = p \cap q$. For ease of notation, we redefine R to be $k[x_1, \ldots, x_m, y_1, \ldots, y_n]$ and let $p = (x_1, \ldots, x_j)$ and $q = (y_1, \ldots, y_l)$, with $j \leq m$ and $l \leq n$. Let $m = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ denote the homogeneous maximal ideal of R.

Case 1: Suppose $p+q \subsetneq m$, or equivalently $0 < j < m$ or $0 < l < n$. Note that $I+(a)/(a)$ is integrally closed if and only if it is integrally closed after passing to the localization at the homogeneous maximal ideal $m/(a)$. Therefore, we may assume R is local. Since I is equal to the intersection of its minimal primes, R/I is reduced.

Next, we show that $\operatorname{depth}(R/I) \geq 2$. Notice that $\dim R/I = \max\{\dim R/p, \dim R/q\}$. Since $p+q \subsetneq m$ and p and q are generated by nonempty disjoint sets of variables we have that $p \subsetneq p+(y_1) \subsetneq m$ and $q \subsetneq q+(x_1) \subsetneq m$. These chains of primes ideals indicate that $\dim R/p \geq 2$ and $\dim R/q \geq 2$, and hence $\dim R/I \geq 2$.

Without loss of generality, assume $y_n \notin q$. Since all generators of I do not involve y_n, the image of I under the natural map $R \to R/(y_n)$ is a squarefree monomial ideal with the same generators as I, and is therefore reduced. Since $\dim R/I \geq 2$ and hence $\dim R/(I, y_n) \geq 1$, we conclude that $\operatorname{depth}(R/(I, y_n)) \geq 1$. Therefore, $\operatorname{depth}(R/I) \geq 2$. Applying Proposition 4.0.1, we conclude that $I+(a)/(a)$ is integrally closed.

Case 2: Suppose $p+q = m$. That is, assume $j = m$ and $l = n$. Let $\omega + (a) \in \overline{I+(a)/(a)}$. Then there exists $n \in \mathbb{N}$ and $a_i \in (I+(a))^i$ such that

$$(\omega+(a))^n + a_1(\omega+(a))^{n-1} + \ldots + a_n = 0 \tag{4.7}$$

in $R/(a)$. This is equivalent to the existence of $n \in \mathbb{N}$ and $a_i \in I^i$ such that

$$\omega^n + a_1\omega^{n-1} + \ldots + a_n \in (a) \tag{4.8}$$

in R. We may assume that ω is homogeneous of degree at least 2 since (a) is a homogeneous ideal and I is generated in degree 2.

Notice that since $I+(a)/(a) \subseteq \overline{I+(a)/(a)}$, we may assume that $\omega = \omega_1 + \omega_2$, in which $\operatorname{Supp}\omega_1 \subseteq p \setminus q$ and $\operatorname{Supp}\omega_2 \subseteq q \setminus p$ since any term containing both an x_j and y_i belongs to I.

We notice that

$$(\omega_1+\omega_2)^n + a_1(\omega_1+\omega_2)^{n-1} + \ldots + a_n \in (\sum_{i=1}^{m} \alpha_i x_i + \sum_{j=1}^{n} \beta_j y_j), \tag{4.9}$$

modulo $q = (y_1, \ldots, y_n)$ yields

$$(\omega_1)^n \in (\sum_{i=1}^{m} \alpha_i x_i) + (y_1, \ldots, y_n) \tag{4.10}$$

Since $(\sum_{i=1}^{m} \alpha_i x_i, y_1, \ldots, y_n)$ is a prime ideal in R, then we see that

$$\omega_1 \in (\sum_{i=1}^{m} \alpha_i x_i) + (y_1, \ldots, y_n). \tag{4.11}$$

Since $\operatorname{Supp} \omega_1 \subseteq p \setminus q$, we see that

$$\omega_1 \in (\sum_{i=1}^{m} \alpha_i x_i). \tag{4.12}$$

Moreover, by assumption ω_1 has degree at least 2 and therefore

$$\omega_1 \in (\sum_{i=1}^{m} \alpha_i x_i) \cap p^2 \subseteq (\sum_{i=1}^{m} \alpha_i x_i) p. \tag{4.13}$$

Let $\omega_1 = s(\sum_{i=1}^{m} \alpha_i x_i)$ with $s \in p$. Then

$$\begin{aligned} \omega_1 + (a) &= s(\sum_{i=1}^{m} \alpha_i x_i) + (a) \\ &= s(-\sum_{j=1}^{n} \beta_i y_i) + (a) \\ &\in I + (a). \end{aligned}$$

Repeating this argument modulo $p = (x_1, \ldots, x_m)$ shows that $\omega_2 + (a) \in I + (a)$ and therefore, $\overline{I + (a)/(a)} = I + (a)/(a)$.

Now let $n \geq 3$, and suppose the result holds for any intersection of $n-1$ disjoint primes generated by variables.

Let $I = \bigcap_{i=1}^{n} p_i$. We consider two cases.

Case 1: If $\sum_{i=1}^{n} p_i \subsetneq m$, the proof follows as in the $n = 2$ case.

Case 2: Suppose $\sum_{i=1}^{n} p_i = m$. Let $\omega + (a) \in \overline{I + (a)/(a)}$. Since $I = \prod_{i=1}^{n} p_i \subseteq \prod_{i=1}^{n-1} p_i$, one has that

$$\omega + (a) \in \overline{\prod_{i=1}^{n-1} p_i + (a)/(a)}. \tag{4.14}$$

By the inductive hypothesis,

$$\omega + (a) \in \prod_{i=1}^{n-1} p_i + (a)/(a). \tag{4.15}$$

Since $\omega + (a) \in \overline{I + (a)/(a)}$, there exists $n \in \mathbb{N}$ and $a_i \in (I + (a))^i$ such that

$$(\omega + (a))^n + a_1(\omega + (a))^{n-1} + \ldots + a_n = 0 \in R/(a). \tag{4.16}$$

This is equivalent to the existence of $n \in \mathbb{N}$ and $a_i \in I^i$ such that

$$\omega^n + a_1\omega^{n-1} + \ldots + a_n \in (a). \tag{4.17}$$

Without loss of generality, since $I + (a)/(a) \subseteq \overline{I + (a)/(a)}$, we may assume that

$$\operatorname{Supp} \omega \subseteq \prod_{j=1}^{n-1} p_j \setminus p_n. \tag{4.18}$$

Denote

$$\begin{aligned}
p_1 &= (x_{1,1}, \ldots, x_{1,i_1}) \\
p_2 &= (x_{2,1}, \ldots, x_{2,i_2}) \\
&\vdots \\
p_n &= (x_{n,1}, \ldots, x_{n,i_n}).
\end{aligned}$$

Since $\sum_{i=1}^{n} p_i = m$, let

$$a = \sum_{j=1}^{n} \sum_{k=1}^{i_j} \alpha_{j,k} x_{j,k} \tag{4.19}$$

denote the general linear form. Then since

$$\omega^n + a_1\omega^{n-1} + \ldots + a_n \in (\sum_{j=1}^{n}\sum_{k=1}^{i_j}\alpha_{j,k}x_{j,k}), \tag{4.20}$$

and each $a_i \in I \subseteq p_n$, we conclude that modulo p_n,

$$\omega^n \in (\sum_{j=1}^{n-1}\sum_{k=1}^{i_j}\alpha_{j,k}x_{j,k}) + p_n. \tag{4.21}$$

Notice that $(\sum_{j=1}^{n-1}\sum_{k=1}^{i_j}\alpha_{j,k}x_{j,k}) + p_n$ is a prime ideal of R, and hence

$$\omega \in (\sum_{j=1}^{n-1}\sum_{k=1}^{i_j}\alpha_{j,k}x_{j,k}) + p_n. \tag{4.22}$$

Since $\operatorname{Supp}\omega \subseteq \prod_{j=1}^{n} p_j \setminus p_n$, we conclude that

$$\omega \in (\sum_{j=1}^{n-1}\sum_{k=1}^{i_j}\alpha_{j,k}x_{j,k}). \tag{4.23}$$

Let $s \in R$ such that

$$\omega = s(\sum_{j=1}^{n-1}\sum_{k=1}^{i_j}\alpha_{j,k}x_{j,k}). \tag{4.24}$$

Then since we assume $\omega \in \prod_{j=1}^{n-1} p_j$ and $(\sum_{j=1}^{n-1}\sum_{k=1}^{i_j}\alpha_{j,k}x_{j,k}) \notin p_j$ for any $1 \leq j \leq n-1$, we conclude that $s \in \prod_{j=1}^{n-1} p_j$. Therefore,

$$\begin{aligned}\omega + (a) &= s(\sum_{j=1}^{n-1}\sum_{k=1}^{i_j}\alpha_{j,k}x_{j,k}) + (a)\\ &= s(-\sum_{k=1}^{i_n}\alpha_{n,k}x_{n,k}) + (a)\\ &\in I + (a).\end{aligned}$$

This completes the proof that $\overline{I + (a)/(a)} = I + (a)/(a)$. ■

5. PRELIMINARIES: PART 2

Let (R, m) be a Noetherian local ring of dimension $d > 0$ or $R = k[x_1, \ldots, x_d]$ with $d > 0$ be a polynomial ring over a field k with homogeneous maximal ideal $m = (x_1, \ldots, x_d)$.

For an m-primary ideal I, the Hilbert-Samuel function $H_I(i)$ of I, is defined as $H_I(i) = \lambda(R/I^i)$. The Hilbert-Samuel function is a polynomial in n of degree d for sufficiently large n. This polynomial is called the Hilbert-Samuel polynomial of I. The Hilbert-Samuel polynomial can be written in the form

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I).$$

The coefficient $e_0(I)$ is referred to as the Hilbert-Samuel multiplicity of I, and $e_i(I)$ the i^{th} Hilbert coefficient of I. See [Mat86, Section 5.13] for more information on the Hilbert-Samuel polynomial.

Kishor Shah proved in [Sha91] that in an equidimensional and universally catenary Noetherian local ring (R, m) with infinite residue field, there exists a largest ideal containing I and contained in $\overline{I}$ for which the first several Hilbert-Samuel coefficients coincide. This was generalized in [CPV06] for any m-primary ideal in a Noetherian local ring (R, m). These ideals are defined below.

Definition 5.0.1. *Let (R, m) be a Noetherian local ring of dimension d. Let I be an m-primary ideal. For $0 \leq k \leq d$, we define $I_{\{k\}}$ to be the largest ideal in $\overline{I}$ containing I for which $e_i(I) = e_i(I_{\{k\}})$ for $i = 0, \ldots, k$. We call $I_{\{k\}}$ the k^{th} coefficient ideal of I.*

This definition yields the following sequence of ideals

$$I \subseteq I_{\{d\}} \subseteq \cdots \subseteq I_{\{1\}} \subseteq I_{\{0\}} = \overline{I}. \tag{5.1}$$

Note that $I_{\{d\}}$ is the largest ideal integral over I whose entire Hilbert-Samuel polynomial coincides with the Hilbert-Samuel polynomial of I, and $I_{\{0\}}$ is the largest ideal integral over I whose Hilbert-Samuel multiplicity coincides with I.

The following theorem of Rees implies that in an equidimensional and universally catenary ring, it is not necessary to assume that the coefficient ideals are contained in $\overline{I}$.

Theorem 5.0.1 ([Ree61, Theorem 3.2]). *Let (R,m) be an equidimensional and universally catenary local ring. Let I, J be two m-primary ideals with $I \subseteq J$. Then $J \subseteq \overline{I}$ if and only if $e_0(J) = e_0(I)$.*

When I contains a nonzerodivisor, Ratliff and Rush proved that there is a unique largest ideal $\widetilde{I}$ containing I for which $(\widetilde{I})^n = I^n$ for n sufficiently large [RR78, Theorem 2.1]. Such an ideal is referred to as the Ratliff-Rush closure of I and Ratliff and Rush showed that $\widetilde{I} = \bigcup_{n\geq 0}(I^{n+1} : I^n)$. Since large powers of I and $\widetilde{I}$ coincide, the Hilbert-Samuel polynomial of I is equal to the Hilbert-Samuel polynomial of $\widetilde{I}$. Moreover, Ratliff and Rush showed that $\widetilde{I}$ is the unique largest ideal I containing I and integral over I for which the Hilbert-Samuel polynomials coincide, and hence $\widetilde{I} = I_{\{d\}}$.

Both the integral closure and the Ratliff-Rush closure are significantly more well-understood than any of the intermediate coefficient ideals. However, the first coefficient ideal is connected to the S_2-ification of the Rees algebra and arises naturally in the study of the core.

5.1 Corso-Polini-Vasconcelos Characterization of Coefficient Ideals

In [CPV06], Corso, Polini and Vasconcelos give a characterization of the j^{th} coefficient ideal as the ideal of degree 1 forms in a subalgebra of $\overline{R[It,t^{-1}]}^{R[t,t^{-1}]}$, generalizing work of Ciupercă in [Ciu01].

Lemma 5.1.1 ([CPV06, Lemma 4.1]). *Let (R,m) be a Noetherian local ring of dimension $d > 0$. Let $I \subseteq L$ be m-primary ideals integral over I with Hilbert coefficients $e_i(I)$ and $e_i(L)$. For any integer $0 \leq j \leq d$, one has $e_i(I) = e_i(L)$ for all $0 \leq i \leq j$ if and only if the cokernel C in the exact sequence of $R[It,t^{-1}]$-modules*

$$0 \longrightarrow R[It,t^{-1}] \longrightarrow R[Lt,t^{-1}] \longrightarrow C \longrightarrow 0$$

has dimension at most $d-j$.

Let R be a Noetherian ring of dimension d. Let I be an R-ideal, and $\mathcal{A} = R[It, t^{-1}]$ the extended Rees algebra of I. Recall that $\dim \mathcal{A} = d+1$. For $0 \leq j \leq d+1$, define $B^{(j)}$ to be

$$B^{(j)} = \{h \in \overline{\mathcal{A}}^{R[t,t^{-1}]} \mid \dim(\mathcal{A}/\mathcal{A} :_{\mathcal{A}} h) \leq d-j\}.$$

We first observe that $B^{(j)}$ is a subalgebra of $\overline{R[It,t^{-1}]}^{R[t,t^{-1}]}$. Notice that $1 \in B^{(j)}$ since $\mathcal{A} :_{\mathcal{A}} 1 = \mathcal{A}$ and hence $\dim(\mathcal{A}/\mathcal{A} :_{\mathcal{A}} 1) \leq -1$. Let $h, k \in B^{(j)}$. Then $(\mathcal{A} :_{\mathcal{A}} (h-k)) \supseteq (\mathcal{A} :_{\mathcal{A}} h) \cap (\mathcal{A} :_{\mathcal{A}} k)$. Since $V((\mathcal{A} :_{\mathcal{A}} h) \cap (\mathcal{A} :_{\mathcal{A}} k)) = V((\mathcal{A} :_{\mathcal{A}} h)) \cup V((\mathcal{A} :_{\mathcal{A}} k))$, $\dim(\mathcal{A}/(\mathcal{A} :_{\mathcal{A}} h)) \leq d-j$ and $\dim(\mathcal{A} : (\mathcal{A} :_{\mathcal{A}} k)) \leq d-j$, we see that

$$\dim(\mathcal{A}/((\mathcal{A} :_{\mathcal{A}} h) \cap (\mathcal{A} :_{\mathcal{A}} k))) \leq d-j. \tag{5.2}$$

Since $(\mathcal{A} :_{\mathcal{A}} (h-k)) \supseteq ((\mathcal{A} :_{\mathcal{A}} h) \cap (\mathcal{A} :_{\mathcal{A}} k))$,

$$\dim(\mathcal{A}/(\mathcal{A} :_{\mathcal{A}} (h-k))) \leq d-j. \tag{5.3}$$

Furthermore, since $\mathcal{A} :_{\mathcal{A}} h \subseteq \mathcal{A} :_{\mathcal{A}} hk$,

$$\dim(\mathcal{A}/(\mathcal{A} :_{\mathcal{A}} hk)) \leq \dim(\mathcal{A}/(\mathcal{A} :_{\mathcal{A}} h)) \leq d-j. \tag{5.4}$$

Hence, $hk \in B^{(j)}$. Hence $B^{(j)}$ is a subalgebra of $\overline{R[It,t^{-1}]}^{R[t,t^{-1}]}$ for $1 \leq j \leq d+1$.

Notice that if $h \in B^{(j)}$, then $\dim(\mathcal{A}/\mathcal{A} :_{\mathcal{A}} h) \leq d-j < d-(j-1)$. Therefore, $h \in B^{(j-1)}$. Hence we have containments $B^{(j)} \subseteq B^{(j-1)}$ for $1 \leq j \leq d+1$.

Notice also that

$$\begin{aligned} h \in \mathcal{A} &\Longleftrightarrow \mathcal{A} :_{\mathcal{A}} h = \mathcal{A} \\ &\Longleftrightarrow \mathcal{A}/(\mathcal{A} :_{\mathcal{A}} h) = 0 \\ &\Longleftrightarrow \dim(\mathcal{A}/(\mathcal{A} :_{\mathcal{A}} h)) \leq -1\,. \end{aligned}$$

Hence $\mathcal{A} = B^{(d+1)}$.

Notice that for any $h \in \overline{\mathcal{A}}^{R[t,t^{-1}]}$, there exists $n \in \mathbb{N}$ such that $t^{-n}h \in \mathcal{A}$ and hence $t^{-n} \in \mathcal{A} :_{\mathcal{A}} h$. Therefore $\operatorname{ht}(\mathcal{A} :_{\mathcal{A}} h) \geq 1$. Then for any $h \in \overline{\mathcal{A}}^{R[t,t^{-1}]}$,

$$\begin{aligned}\dim(\mathcal{A}/\mathcal{A} :_{\mathcal{A}} h) &\leq \dim \mathcal{A} - \operatorname{ht}(\mathcal{A} :_{\mathcal{A}} h)\\ &\leq (d+1) - 1\\ &= d.\end{aligned}$$

This shows $B^{(0)} = \overline{\mathcal{A}}^{R[t,t^{-1}]}$.

Therefore we have a containment of subalgebras of $\overline{\mathcal{A}}^{R[t,t^{-1}]}$:

$$\mathcal{A} = B^{(d+1)} \subseteq B^{(d)} \subseteq \ldots \subseteq B^{(1)} \subseteq B^{(0)} = \overline{\mathcal{A}}^{R[t,t^{-1}]}. \tag{5.5}$$

The following theorem of Corso, Polini and Vasconcelos says that the ideals generated by forms of degree 1 in the subalgebras $B^{(j)}$ of $R[t,t^{-1}]$ are the coefficient ideals of Shah.

Theorem 5.1.2 ([CPV06, Theorem 4.2])**.** *Let (R,m) be a Noetherian local ring of dimension $d > 0$ and let I be an m-primary ideal. Let $0 \leq j \leq d$. Then $I_{\{j\}}$ is the R-ideal consisting of all forms of degree 1 in $B^{(j)}$.*

Corso, Polini and Vasconcelos note that $[B^{(1)}]_{\geq 0}$ is the S_2-ification of the Rees algebra $R[It]$ provided R is S_2 and universally catenary.

Remark 5.1.3. The results of Corso, Polini and Vasconcelos hold if one takes a graded *local ring rather than a Noetherian local ring as the base ring.

5.2 Graded Hom, Graded Ext and the Graded Canonical Module

Let R be a nonnegatively graded ring. Given graded R-modules M, N, an R-linear map $f : M \to N$ is homogeneous of degree j if $f(M_i) \subseteq N_{i+j}$ for all $i \in \mathbb{Z}$ and homogeneous if it is homogeneous of degree 0. Let ${}^*\operatorname{Mod} R$ denote the category whose objects are graded R-modules and morphisms are homogeneous R-linear maps. The free objects in ${}^*\operatorname{Mod} R$ are $\oplus_{i\in\mathcal{I}} R(-n_i)$ with $n_i \in \mathbb{Z}$. We notice that if M is a graded R-module, we can write

$M = \sum_{i \in \mathcal{I}} Rz_i$ with z_i homogeneous of degree n_i. Then M is the image of the homogeneous R-linear map

$$\oplus_{i \in \mathcal{I}} R(-n_i) \to M$$

defined by sending $e_i \mapsto z_i$. Kernels and images of homogeneous R-linear maps are graded. Therefore, it follows that every graded module M has a homogeneous free resolution. Moreover, homology modules remain in the category $^*\operatorname{Mod} R$.

Definition 5.2.1. *Let R be a graded ring. Let M, N be graded R-modules. We define $\underline{\operatorname{Hom}}_R(M,N)$ to be the direct sum $\oplus_{n\in\mathbb{Z}} \operatorname{Hom}_i(M,N)$ where $\operatorname{Hom}_i(M,N)$ denotes the R-linear maps from M to N which are homogeneous of degree i.*

We see that $\underline{\operatorname{Hom}}_R(M,N) \subseteq \operatorname{Hom}_R(M,N)$. Furthermore, $\underline{\operatorname{Hom}}_R(M,N)$ is graded and a submodule of $\operatorname{Hom}_R(M,N)$ with $[\underline{\operatorname{Hom}}_R(M,N)]_i = \operatorname{Hom}_i(M,N)$. Given $r \in R_i$ and $\varphi \in \operatorname{Hom}_j(M,N)$, $r\varphi$ is homogeneous of degree $i+j$. One can check that $\underline{\operatorname{Hom}}_R(-,N)$ is an additive contravariant functor from $^*\operatorname{Mod} R$ to $^*\operatorname{Mod} R$.

Definition 5.2.2. *Let R be a graded ring and N a graded R-module. We define*

$\underline{\operatorname{Ext}}^i_R(-,N) = R^i(\underline{\operatorname{Hom}}_R(-,N))$

Remark 5.2.1. If M is a finite R-module, $\underline{\operatorname{Hom}}_R(M,N) = \operatorname{Hom}_R(M,N)$. Let $x_1, \ldots, x_n$ be homogeneous elements of M such that $M = Rx_1 + \cdots + Rx_n$. If $\varphi : M \to N$ is a homomorphism of R-modules, then $\varphi(x_i) = y_1 + \cdots + y_s \in \oplus_{j=k_1}^{k_s} N_j$ for $1 \leq i \leq n$. This implies that φ is a finite sum of homogeneous maps in finitely many degrees. Hence $\operatorname{Hom}_R(M,N) \subseteq \underline{\operatorname{Hom}}_R(M,N)$.

It follows that if M is finite and R is Noetherian, then $\underline{\operatorname{Ext}}^i_R(M,N) = \operatorname{Ext}^i_R(M,N)$ for all i.

Definition 5.2.3. *Let R be a Gorenstein ring. Let $S = R[X_1, \ldots, X_n]$ be a polynomial ring over R with $\deg(X_i) > 0$, and I be a homogeneous S-ideal of height g. Let $T = S/I$. Then we define*

$$\underline{\omega}_T = \underline{\operatorname{Ext}}^g_S(T,S)(-\sum_{i=1}^n \deg(X_i)).$$

to be a graded canonical module of T.

One can show that $\underline{\omega}_T$ is uniquely determined up to homogeneous isomorphisms.

Remark 5.2.2. Since T is a finite S-module, $\underline{\omega}_T = \operatorname{Ext}_S^g(T,S)(-\sum_{i=1}^n \deg(X_i))$.

Proposition 5.2.1. *Let T be as in Definition 5.2.3. The graded canonical module $\underline{\omega}_T$ satisfies Serre's condition S_2 as a T-module.*

Proof. It suffices to check that $\underline{\omega}_T$ satisfies S_2 locally at maximal homogeneous ideals. Let m be a maximal homogeneous ideal of T. Then $\operatorname{ht} IT_m \geq \operatorname{ht} I$. If $\operatorname{ht} IT_m > \operatorname{ht} I = g$, then $(\underline{\omega}_T)_m$ is zero, and hence satisfies S_2 as a T-module.

Assume $\operatorname{ht} IT_m = \operatorname{ht} I = g$. Since $g = \operatorname{ht} IT_m$, IT_m contains a regular sequence of length g, say $f_1, \ldots, f_g$. Let $\mathfrak{a} = (f_1, \ldots, f_g)$. Then since $f_1, \ldots, f_g \subseteq \operatorname{ann}(T_m)$,

$$\begin{aligned}
\operatorname{Ext}^g_{S_m}(T_m, S_m)(-\sum_{i=1}^n \deg(X_i)) &\cong \operatorname{Hom}_{S_m}(T_m, S_m/(f_1, \ldots, f_g))(-\sum_{i=1}^n \deg(X_i)) && (5.6)\\
&\cong \operatorname{Hom}_{S_m/\mathfrak{a}}(T_m, S_m/\mathfrak{a})(-\sum_{i=1}^n \deg(X_i)). && (5.7)
\end{aligned}$$

Notice that $S_m/\mathfrak{a}$ is a Cohen-Macaulay ring and hence satisfies S_2. Therefore, $\operatorname{Hom}_{S_m/\mathfrak{a}}(T_m, S_m/\mathfrak{a})$ satisfies S_2. ■

Definition 5.2.4. *A ring R is generically Gorenstein ring if R_p is Gorenstein for all $p \in \operatorname{Ass}(R)$.*

If R is reduced, then R is generically Gorenstein ring. Therefore, any Rees algebra of an ideal in a reduced ring is generically Gorenstein ring.

Theorem 5.2.3 ([NV93, Theorem 1.3])**.** *Let T be as in Definition 5.2.3. Assume T is generically Gorenstein ring. Assume all minimal primes of I have the same height. Then $\operatorname{Hom}_T(\underline{\omega}_T, \underline{\omega}_T)$ is a commutative ring naturally containing T which satisfies S_2, is a finite T-module and satisfies S_2 as a T-module. Moreover, it can be identified with a subring of $\operatorname{Quot}(T)$. Furthermore, it is minimal with this property. Given a T-module $W \subseteq \operatorname{Quot}(T)$ such that $T \subseteq W$, W is finite as a T-module, and W satisfies S_2 as a T-module, then $\operatorname{Hom}_T(\underline{\omega}_T, \underline{\omega}_T) \subseteq W$.*

5.3 The S_2-ification of a ring

Definition 5.3.1. *Let R be a Noetherian ring with total ring of quotients* $\mathrm{Quot}(R)$*. We say that an overring S of R is an S_2-ification of R if:*

(i) $R \subseteq S \subseteq \mathrm{Quot}(R)$ *and S is a finite R-module;*

(ii) S is S_2 as an R-module;

(iii) S is minimal with respect to having properties (i) and (ii).

Remark 5.3.1. When an S_2-ification of R exists, it must be unique, and is equal to

$$\{t \in \overline{R}^{\mathrm{Quot}(R)} \mid \mathrm{ht}\,(R :_R t) \geq 2\}.$$

We can see this by following the proof of [HH94, Proposition 2.4].

By Theorem 5.2.3, $\mathrm{Hom}_T(\underline{\omega}_T, \underline{\omega}_T)$ is an S_2-ification of T with T as in Theorem 5.2.3.

Proposition 5.3.1. *Let $A \to B$ be a faithfully flat extension of Gorenstein rings. Let $X_1, \ldots, X_n$ be variables such that* $\deg(X_i) > 0$*. Let I be a homogeneous $A[X_1, \ldots, X_n]$-ideal of height g. Let $T = A[X_1, \ldots, X_n]/I$ and let $U = B[X_1, \ldots, X_n]/IB[X_1, \ldots, X_n] = T \otimes_A B$. Then*

$$\underline{\omega}_U = \underline{\omega}_T \otimes_A B.$$

Proof. Notice that since $A \to B$ is a faithfully flat extension of Gorenstein rings, so is $A[X_1, \ldots, X_n] \to B[X_1, \ldots, X_n]$. Since faithfully flat extensions preserve heights of ideals, $\mathrm{ht}\, I = \mathrm{ht}\, IB[X_1, \ldots, X_n] = g$. Then by the definition of the graded canonical module of U, we see that

$$\underline{\omega}_U = \mathrm{Ext}^g_{B[X_1,\ldots,X_n]}(U, B[X_1, \ldots, X_n])(-\sum_{i=1}^{n} \deg(X_i)). \tag{5.8}$$

Since $B[X_1, \ldots, X_n]$ is a flat $A[X_1, \ldots, X_n]$-module, and $T = A[X_1, \ldots, X_n]/I$ is a finite $A[X_1, \ldots, X_n]$-module, one has

$$
\begin{aligned}
&\mathrm{Ext}^g_{B[X_1,\dots,X_n]}(U, B[X_1,\dots,X_n])(-\sum_{i=1}^n \deg(X_i)) \\
&= \mathrm{Ext}^g_{A[X_1,\dots,X_n]}(T, A[X_1,\dots,X_n])(-\sum_{i=1}^n \deg(X_i)) \otimes_{A[X_1,\dots,X_n]} B[X_1,\dots,X_n] \\
&= \mathrm{Ext}^g_{A[X_1,\dots,X_n]}(T, A[X_1,\dots,X_n])(-\sum_{i=1}^n \deg(X_i)) \otimes_A B \\
&= \underline{\omega}_T \otimes_A B.
\end{aligned}
$$

■

Proposition 5.3.2. *Let $A \to B$ be a faithfully flat extension of Gorenstein rings. Let $X_1, \dots, X_n$ be variables such that $\deg(X_i) > 0$. Let I be a homogeneous $A[X_1, \dots, X_n]$ ideal of height g. Let $T = A[X_1, \dots, X_n]/I$ and let $U = B[X_1, \dots, X_n]/IB[X_1, \dots, X_n] = T \otimes_A B$. Then there is a graded isomorphism*

$$
\mathrm{Hom}_T(\underline{\omega}_T, \underline{\omega}_T) \otimes_A B \cong \mathrm{Hom}_U(\underline{\omega}_U, \underline{\omega}_U). \tag{5.9}
$$

Proof. Since B is a flat A-module, $U = T \otimes_A B$ is a flat T-module. Since $\underline{\omega}_T$ is a finite T-module,

$$
\mathrm{Hom}_T(\underline{\omega}_T, \underline{\omega}_T) \otimes_T U \cong \mathrm{Hom}_U(\underline{\omega}_T \otimes_T U, \underline{\omega}_T \otimes_T U). \tag{5.10}
$$

Notice that

$$
\begin{aligned}
\underline{\omega}_T \otimes_T U &= \underline{\omega}_T \otimes_T (T \otimes_A B) \\
&= (\underline{\omega}_T \otimes_T T) \otimes_A B \\
&= \underline{\omega}_T \otimes_A B.
\end{aligned}
$$

By Proposition 5.3.1, $\underline{\omega}_T \otimes_A B \cong \underline{\omega}_U$. Hence, $\mathrm{Hom}_T(\underline{\omega}_T, \underline{\omega}_T) \otimes_T U \cong \mathrm{Hom}_U(\underline{\omega}_U, \underline{\omega}_U)$.

■

Remark 5.3.2. Let T and U be defined as in Proposition 5.3.1. We claim that if T is generically Gorenstein, then U is generically Gorenstein.

Let q be an associated prime of U, and p be the contraction of q to T. We must show that U_q is Gorenstein to see that U is generically Gorenstein. Recall that a flat local extension of Noetherian rings $(R, m) \to (S, n)$ has the property that S if Gorenstein if and only if R and S/mS are Gorenstein [BH93, Corollary 3.3.15].

Notice that since A and B are Gorenstein, so are $A[X_1, \ldots, X_n]$ and $B[X_1, \ldots, X_n]$. Localizing at the primes p and q of $A[X_1, \ldots, X_n]$ and $B[X_1, \ldots, X_n]$ as above, one has $A[X_1, \ldots, X_n]_p$ and $B[X_1, \ldots, X_n]_q$ are Gorenstein, and the map

$$(A[X_1, \ldots, X_n]_p, pA[X_1, \ldots, X_n]_p) \longrightarrow (B[X_1, \ldots, X_n]_q, qB[X_1, \ldots, X_n]_q) \tag{5.11}$$

is a flat local extension of Gorenstein rings. Hence $B[X_1, \ldots, X_n]_q/pB[X_1, \ldots, X_n]_q \cong U_q/pU_q$ is Gorenstein.

Therefore, applying [BH93, Corollary 3.3.15] to the flat local map

$$(T_p, pT_p) \longrightarrow (U_q, qU_q) \tag{5.12}$$

we see that since $B[X_1, \ldots, X_n]_q/pB[X_1, \ldots, X_n]_q$ is Gorenstein and $B[X_1, \ldots, X_n]_q/pB[X_1, \ldots, X_n]_q \cong U_q/pU_q$, U_q/pU_q is Gorenstein. Since T_p is Gorenstein, then we conclude that U_q is Gorenstein.

Next, we claim that if the minimal primes of T have the same height, then the minimal primes of U have the same height. Let q be a minimal prime of $IB[X_1, \ldots, X_n]$ in $B[X_1, \ldots, X_n]$. Let $p = q \cap A[X_1, \ldots, X_n]$. Notice that p is an associated prime of I, and hence minimal. Notice that

$$A[X_1, \ldots, X_n]_p \longrightarrow B[X_1, \ldots, X_n]_q$$

is a flat local homomorphism. Notice that $I \subseteq p$ and hence $IB[X_1, \dots, X_n]_q \subseteq pB[X_1, \dots, X_n]_q$. Therefore, the fiber of the flat local homomorphism is an epimorphic image of

$$B[X_1, \dots, X_n]_q / IB[X_1, \dots, X_n]_q,$$

which is a zero-dimensional ring since q is a minimal prime of $IB[X_1, \dots, X_n]$. This implies that $\dim A[X_1, \dots, X_n]_p = \dim B[X_1, \dots, X_n]_q$. Since we assume all minimal primes of I have the same height, it follows that all minimal primes of $IB[X_1, \dots, X_n]$ have the same height.

This shows that if T satisfies assumptions of Theorem 5.2.3, then so does U. Hence $\mathrm{Hom}_U(\underline{\omega_U}, \underline{\omega_U}) \cong \mathrm{Hom}_T(\underline{\omega_T}, \underline{\omega_T}) \otimes_A B$ is an S_2-ification of U.

6. SPECIALIZATION OF COEFFICIENT IDEALS

This chapter is based on joint work with Rachel Lynn.

In Chapter 3, we have shown that for ideals I of height at least 2 in a large class of rings, $\overline{I}/(x) = \overline{(I/(x))}$ for general $x \in I$. Given that for m-primary ideals I in a local ring R, the sequence of coefficient ideals approximates $\overline{I} = I_{\{0\}}$, a natural question arises. Is $I_{\{j\}}/(x) = (I/(x))_{\{j\}}$ for $1 \leq j \leq d-1$? In the case where $j = d$ and $\operatorname{grade} I > 0$, we note that $(I/(x))_{\{d\}}$ is undefined since the dimension of $R/(x)$ is $d-1$. Since $I_{\{d\}}$ coincides with the Ratliff-Rush closure of I, and $(I/(x))_{\{d-1\}}$ coincides with the Ratliff-Rush closure of $I/(x)$, the natural question in this case is: When does the Ratliff-Rush closure specialize? When is $I_{\{d\}}/(x) = (I/(x))_{\{d-1\}}$?

Rossi and Swanson consider the behavior of the Ratliff-Rush closure under specialization in [RS03]. Note that an element a of an ideal I is said to be superficial if there exists $c \in \mathbb{N}$ such that for all $n \geq c$,

$$(I^n : a) \cap I^c = I^{n-1}. \tag{6.1}$$

By [SH06, Proposition 8.5.7], when R is a local ring with an infinite residue field, an element being superficial is a general condition. Rossi and Swanson give two classes of examples of Ratliff-Rush closed ideals which are not Ratliff-Rush closed after specialization by superficial elements:

Proposition 6.0.1 ([RS03, Proposition 2.3]). *Let $R = k[[x, y]]$ be a power series ring in 2 variables over a field. Then for $l \geq 3$, $I = \left(x^l, xy^{l-1}, y^l\right)$ is Ratliff-Rush closed and so is each power of I, yet $I/(a)$ is not Ratliff-Rush closed for any superficial element $a \in I$.*

Proposition 6.0.2 ([RS03, Proposition 2.4]). *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring in $d \geq 2$ variables over a field. Let $l \geq 3$ and let I be the ideal generated by all monomials of degree l except $x_1^{l-1}x_2$. Then I and all powers of I are Ratliff-Rush closed, yet $I/(a)$ is not Ratliff-Rush closed for any superficial element $a \in I$.*

In this section, we prove that

$$I_{\{i\}}/(x) \subseteq (I/(x))_{\{i\}} \text{ for } 1 \leq i \leq d-1 \tag{6.2}$$

and

$$I_{\{d\}}/(x) \subseteq (I/(x))_{\{d-1\}}\,. \tag{6.3}$$

These containments allow us to say that whenever a coefficient ideal coincides with the integral closure, the coefficient ideal specializes in the sense described above. Moreover, we prove that if the d^{th} coefficient ideal does not coincide with the $(d-1)^{\text{st}}$ coefficient ideal, then the Ratliff-Rush closure does not specialize.

The following lemma is needed for the proof that $I_{\{i\}}/(x) \subseteq (I/(x))_{\{i\}}$ for $0 \le i \le d-1$.

Lemma 6.0.1. *Let (R, m) be a Noetherian local k-algebra, with k an infinite field. Let I be an R-ideal and x a general element of I. Let M be a nonzero finitely generated R-module. Then*

$$\dim(M/xM) = \max\{\dim(M/IM), \dim M - 1\}.$$

Proof. Suppose $\dim(M/IM) = \dim M$. Since $\operatorname{ann}(M/xM) \subseteq \operatorname{ann}(M/IM)$,

$$\dim(M/xM) \ge \dim(M/IM) = \dim M. \tag{6.4}$$

Since $\operatorname{ann}(M) \subseteq \operatorname{ann}(M/xM)$, $\dim M \ge \dim(M/xM)$. Hence

$$\dim(M/xM) = \dim M = \max\{\dim(M/IM), \dim M - 1\}. \tag{6.5}$$

Now suppose $\dim(M/IM) < \dim M$. We claim that I is not contained in any $p \in \operatorname{Min}(()\operatorname{ann} M)$ with $\dim R/p = \dim M$. Suppose toward contradiction $I \subseteq p$ with $p \in \operatorname{Min}(()\operatorname{ann} M)$ such that $\dim R/p = \dim M$. Then $(\operatorname{ann} M, I) \subseteq p$. Since M is a finite R-module, $\sqrt{(\operatorname{ann} M, I)} = \sqrt{\operatorname{ann}(M/IM)}$. Hence $\operatorname{ann}(M/IM) \subseteq p$. Therefore,

$$\dim M = \dim R/p \le \dim(M/IM), \tag{6.6}$$

a contradiction. Since I is not contained in any prime $p \in \operatorname{Min}(() \operatorname{ann} M)$ with $\dim R/p = \dim M$, we may assume x is not contained in any such prime. Since $\sqrt{\operatorname{ann}(M/xM)} = \sqrt{(\operatorname{ann} M, x)}$,

$$\dim(M/xM) = \dim R/(\operatorname{ann} M, x) = \dim R/(\operatorname{ann} M) - 1 = \dim M - 1. \tag{6.7}$$

■

We are now ready to prove that $I_{\{i\}}/(x) \subseteq (I/(x))_{\{i\}}$ for $1 \le i \le d-1$.

Proposition 6.0.2. *Let (R, m) be a Noetherian local k-algebra of dimension $d > 0$, with k an infinite field. Let I be an m-primary ideal of R, and let x be a general element of I.*

Then $I_{\{i\}}/(x) \subseteq (I/(x))_{\{i\}}$ for all $1 \le i \le d-1$, where $I_{\{i\}}$ is the i^{th} coefficient ideal of I.

Proof. By [CPV06, Lemma 4.1], $I_{\{i\}}$ is the unique largest ideal integral over I containing I such that $\dim M \le d - i$, where M is the cokernel of the natural inclusion of $R[It, t^{-1}]$-modules

$$0 \longrightarrow R[It, t^{-1}] \longrightarrow R[I_{\{i\}}t, t^{-1}] \longrightarrow M \longrightarrow 0$$

(see Lemma 5.1.1). Since $I_{\{i\}}/(x) \subseteq \overline{I}/(x) \subseteq \overline{I/(x)}$, $I_{\{i\}}/(x)$ is integral over $I/(x)$. Since R is local with positive dimension and I is m-primary, we may assume that x is not contained in any minimal primes of maximal dimension, and hence $\dim R/(x) = d - 1$. Thus, to show the desired containment, it suffices to show that the cokernel N in the short exact sequence

$$0 \longrightarrow \tfrac{R}{(x)}\left[\tfrac{I}{(x)}t, t^{-1}\right] \longrightarrow \tfrac{R}{(x)}\left[\tfrac{I_{\{i\}}}{(x)}t, t^{-1}\right] \longrightarrow N \longrightarrow 0$$

satisfies $\dim N \le (d-1) - i$. Note that the result is trivial if $I = I_{\{\mathrm{i}\}}$, and hence we may assume $M \neq 0$.

Notice that the natural maps

$$R[It, t^{-1}] \longrightarrow \frac{R}{(x)}\left[\frac{I}{(x)}t, t^{-1}\right] \quad \text{and} \quad R[I_{\{i\}}t, t^{-1}] \longrightarrow \frac{R}{(x)}\left[\frac{I_{\{i\}}}{(x)}t, t^{-1}\right] \tag{6.8}$$

are surjections. These maps induce a natural map $M \to N$. Applying the Snake Lemma to the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R[It, t^{-1}] & \longrightarrow & R[I_{\{i\}}t, t^{-1}] & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \frac{R}{(x)}\left[\frac{I}{(x)}t, t^{-1}\right] & \longrightarrow & \frac{R}{(x)}\left[\frac{I_{\{i\}}}{(x)}t, t^{-1}\right] & \longrightarrow & N & \longrightarrow & 0\,, \end{array} \tag{6.9}$$

we see that the natural map $M \to N$ is surjective. Moreover, xtM is contained in the kernel of the natural map $M \to N$ and hence N is the epimorphic image of M/xtM. Therefore, $\dim N \le \dim(M/xtM)$. To show $\dim N \le d - i - 1$, it suffices to show $\dim(M/xtM) \le d - i - 1$.

By Lemma 6.0.1,

$$\dim(M/xtM) = \max\{\dim(M/ItM), \dim M - 1\}. \tag{6.10}$$

We claim that $\dim(M/ItM) \le 0$. It is clear that It annihilates M/ItM. Notice

$$R[It, t^{-1}] \subseteq R[I_{\{i\}}t, t^{-1}] \subseteq R[\overline{I}t, t^{-1}] \subseteq \overline{R[It, t^{-1}]}^{R[t,t^{-1}]}. \tag{6.11}$$

Hence $R[I_{\{i\}}t, t^{-1}]$ is a finite $R[It, t^{-1}]$-module, as it is an integral extension of $R[It, t^{-1}]$ and a finitely generated $R[It, t^{-1}]$-algebra. Therefore M is a finite $R[It, t^{-1}]$-module. Furthermore, M is concentrated in positive degrees and therefore, there exists $n \in \mathbb{N}$ such that $t^{-n}M = 0$. Hence

$$(It, t^{-1}) \subseteq \sqrt{\operatorname{ann}_{R[It,t^{-1}]}(M/ItM)}. \tag{6.12}$$

Therefore,

$$\dim M/ItM \le \dim R[It, t^{-1}]/(It, t^{-1}) = \dim R/I = 0. \tag{6.13}$$

Since $\dim M \leq d-i$, we conclude that

$$\begin{aligned}\dim N &\leq \dim M/xtM \\ &= \max\{\dim M/ItM, \dim M - 1\} \\ &\leq \max\{0, d-i-1\} \\ &= d-i-1.\end{aligned}$$

This completes the proof that $I_{\{i\}}/(x) \subseteq (I/(x))_{\{i\}}$. ■

The following result gives the analogous containment for the d^{th} coefficient ideal.

Proposition 6.0.3. *Let (R,m) be a Noetherian local k-algebra of dimension d with k an infinite field with* $\operatorname{depth}(R) \geq 2$. *Let I be an m-primary ideal of R. Let x be a general element of I. Then $I_{\{d\}}/(x) \subseteq (I/(x))_{\{d-1\}}$. Moreover, if $\widetilde{I} = I_{\{d\}} \subsetneq I_{\{d-1\}}$, then the Ratliff-Rush closure $\widetilde{I}$ does not specialize with respect to general elements of I.*

Proof. Recall that in a d-dimensional ring, when I contains a nonzerodivisor,

$$I_{\{d\}} = \widetilde{I} = \bigcup_{n\geq 0}\left(I^{n+1} : I^n\right).$$

Notice that since $\operatorname{depth}(R) \geq 2$, $\operatorname{grade} m \geq 2$. Since I is m-primary, this implies that $\operatorname{grade} I \geq 2$.

Let $y \in I_{\{d\}} = \widetilde{I}$. Then $y \in I^{n+1} : I^n$ for some $n \geq 0$. Since $yI^n \subseteq I^{n+1}$,

$$(y + (x))(I/(x))^n \subseteq (I/(x))^{n+1}. \tag{6.14}$$

Hence

$$y + (x) \in (I/(x))^{n+1} : (I/(x))^n \subseteq (\widetilde{I/(x)}). \tag{6.15}$$

Hence $\widetilde{I}/(x) \subseteq \widetilde{I/(x)}$. Since we may assume x is a nonzerodivisor on the ring R, $R/(x)$ is a ring of dimension $d-1$ and $\operatorname{depth}(R/(x)) \geq 1$. Therefore, $I/(x)$ contains a nonzerodivisor. Hence $\widetilde{I/(x)} = (I/(x))_{\{d-1\}}$. This proves the containment $I_{\{d\}}/(x) \subseteq (I/(x))_{\{d-1\}}$.

Since $I \subseteq I_{\{d\}} \subseteq I_{\{d-1\}}$ and $x \in I$,

$$I_{\{d\}}/(x) \subseteq I_{\{d-1\}}/(x). \tag{6.16}$$

Moreover, if $I_{\{d\}} \subsetneq I_{\{d-1\}}$, then $I_{\{d\}}/(x) \subsetneq I_{\{d-1\}}/(x)$.

By 6.0.2, $I_{\{d-1\}}/(x) \subseteq (I/(x))_{\{d-1\}}$. Therefore, if $\widetilde{I} = I_{\{d\}} \subsetneq I_{\{d-1\}}$, then

$$\widetilde{I}/(x) = I_{\{d\}}/(x) \subsetneq I_{\{d-1\}}/(x) \subseteq (I/(x))_{\{d-1\}} = \widetilde{I/(x)}. \tag{6.17}$$

■

In the next chapter, we will see that in a polynomial ring in two variables, the ideals Rossi and Swanson gave as counterexamples to the specialization of the Ratliff-Rush closure in [RS03] are ideals for which $\widetilde{I} = I_{\{2\}} \subsetneq I_{\{1\}}$.

The following corollaries say that if any coefficient ideal of I coincides with the integral closure of I, then the coefficient ideal specializes with respect to general $x \in I$.

Corollary 6.0.4. *Let (R, m) is a local excellent k-algebra of dimension $d \geq 2$ with k a field of characteristic zero. Let I be an m-primary R-ideal. Let x be a general element of I. If $I_{\{i\}} = \overline{I}$ for some $i \in \{1, \dots, d-1\}$, then*

$$I_{\{j\}}/(x) = (I/(x))_{\{j\}} \text{ for } 0 \leq j \leq i.$$

Proof. Let $1 \leq i \leq d-1$. Then

$$I_{\{i\}}/(x) \subseteq (I/(x))_{\{i\}} \tag{6.18}$$
$$\subseteq \overline{I/(x)} \tag{6.19}$$
$$= \overline{I}/(x). \tag{6.20}$$

Note that Eq. (6.18) follows from Proposition 6.0.2, Eq. (6.19) follows from the definition of coefficient ideals, and Eq. (6.20) follows from Theorem 3.6.1. Since $I_{\{i\}}/(x) = \overline{I}/(x)$, the above containments are equalities.

Furthermore, the coefficient ideals between the integral closure and the i^{th} coefficient ideal specialize, since if $I_{\{i\}} = \overline{I}$, then $I_{\{j\}} = \overline{I}$ for all $j < i$. ■

Corollary 6.0.5. *Let (R, m) is a local excellent k-algebra of dimension $d \geq 2$ with k a field of characeristic zero and* $\operatorname{depth}(R) \geq 2$. *Let I be an m-primary R-ideal. Let x be a general element of I. If $I_{\{d\}} = \overline{I}$, then*

$$\widetilde{I}/(x) = I_{\{d\}}/(x) = (I/(x))_{\{d-1\}} = \widetilde{I/(x)}. \tag{6.21}$$

and

$$I_{\{i\}}/(x) = (I/(x))_{\{i\}} \text{ for all } 0 \leq i \leq d-1. \tag{6.22}$$

Proof. Since $\operatorname{depth}(R) \geq 2$ and I is m-primary, I contains a nonzerodivisor. Therefore, $\widetilde{I} = I_{\{d\}}$. Hence

$$\begin{aligned} \widetilde{I}/(x) &= I_{\{d\}}/(x) && (6.23)\\ &\subseteq (I/(x))_{\{d-1\}} && (6.24)\\ &\subseteq \overline{I/(x)} && (6.25)\\ &= \overline{I}/(x) && (6.26) \end{aligned}$$

Note that Eq. (6.24) follows from Proposition 6.0.3, Eq. (6.25) follows from the definition of coefficient ideals, and Eq. (6.26) follows from Theorem 3.6.1. Since $\widetilde{I} = \overline{I}$, all containments are equalities.

Furthermore, since $\widetilde{I} = \overline{I}$, $I_{\{i\}} = \overline{I}$ for all $1 \leq i \leq d-1$, and the specialization of the i^{th} coefficient ideal follows from Corollary 6.0.4. ■

6.1 Containment Preservation Property for Coefficient Ideals

Recall that the integral closure preserves containments for all ideals: if $J \subseteq I$, then $\overline{J} \subseteq \overline{I}$. Heinzer, Johnston, Lantz and Shah in [Hei+93] gave an example of $J \subseteq I$ for which $\widetilde{J} \not\subseteq \widetilde{I}$. In Chapter 5, we give an example showing that the first coefficient ideal does not preserve containments in general. While coefficient ideals do not preserve containments in general, coefficient ideals preserve containments of reductions.

The next proposition was proved by Heinzer, Johnston, Lantz, and Shah with some additional assumptions on the ring (see [Hei+93, Corollary 3.20]).

Proposition 6.1.1. *Let (R, m) be a Noetherian local ring of dimension $d > 0$. Let $J \subseteq I$ be m-primary ideals such that J is a reduction of I. Then $J_{\{k\}} \subseteq I_{\{k\}}$ for $0 \leq k \leq d$.*

Proof. Define $D_J(h) = R[Jt, t^{-1}] :_{R[Jt,t^{-1}]} h$ and $D_I(h) = R[It, t^{-1}] :_{R[It,t^{-1}]} h$. By [CPV06], the coefficient ideal $I_{\{j\}}$ is the degree one component of the algebra

$$B^{(j)} = \{h \in \overline{R[It, t^{-1}]}^{R[t,t^{-1}]} \mid \dim(R[It, t^{-1}]/D_I(h)) \leq d - j\}.$$

Let $b \in J_{\{k\}}$. Then

$$\dim(R[Jt, t^{-1}]/D_J(bt)) \leq d - k. \tag{6.27}$$

We show that $\dim(R[It, t^{-1}]/D_I(bt)) \leq d - k$ to see that $b \in I_{\{k\}}$.

Since J is a reduction of I, the extension $R[Jt, t^{-1}] \subseteq R[It, t^{-1}]$ is integral. Therefore $R[Jt, t^{-1}]/(D_I(bt) \cap R[Jt, t^{-1}]) \subseteq R[It, t^{-1}]/D_I(bt)$ is integral. Hence

$$\dim(R[Jt, t^{-1}]/(D_I(bt) \cap R[Jt, t^{-1}])) = \dim(R[It, t^{-1}]/D_I(bt)). \tag{6.28}$$

It is clear that $D_I(bt) \cap R[Jt, t^{-1}] = D_J(bt)$. Hence

$$\dim(R[It, t^{-1}]/D_I(bt)) = \dim(R[Jt, t^{-1}]/D_J(bt)) \leq d - k. \tag{6.29}$$

Hence $b \in I_{\{k\}}$.

$\blacksquare$

7. COEFFICIENT IDEALS IN A POLYNOMIAL RING IN TWO VARIABLES

In this chapter, we restrict to polynomial rings in two variables over an infinite field k. We give a formula for the first coefficient ideal for all m-primary monomial ideals generated in one degree. We use a formula for the Ratliff-Rush closure due to Veronica Crispin Quiñonez to characterize when the Ratliff-Rush closure coincides with the first coefficient ideal. With the additional assumption that the field k has characteristic zero, we give a complete description of how coefficient ideals for m-primary monomial ideals generated in one degree behave with respect to specialization by a general element of the ideal.

7.1 The First Coefficient Ideal

To begin, we compute the first coefficient ideal of 0-dimensional monomial ideals generated in one degree.

Lemma 7.1.1. *Let $(R,m) \to (S,n)$ be a faithfully flat extension of Gorenstein local rings of dimension $d \geq 1$. Let I be an m-primary R-ideal. Then*

$$I_{\{1\}}S = (IS)_{\{1\}}.$$

Proof. Recall that $I_{\{1\}}$ is the degree 1 component of the S_2-ification of $R[It]$.

Let $I = (a_1, \ldots, a_n)$. Then the Rees algebra of I, $R[It]$, is naturally isomorphic to

$$R[X_1, \ldots, X_n]/J$$

with $\deg(X_i) = 1$ and J a homogeneous ideal, referred to as the defining ideal of the Rees algebra $R[It]$. We now see that $R[It]$ is generically a Gorenstein ring. Let $p \in \operatorname{Ass}(R[It])$. Then p contracts to an associated prime of R. Since I is an m-primary ideal in a Cohen-Macaulay ring of dimension at least 1, $I \not\subseteq p \cap R$. This implies that $R[It]_p$ is a polynomial ring over the Gorenstein ring $R_{p\cap R}$. Hence $R[It]_p$ is Gorenstein.

We must show that all minimal primes of J have the same height. Notice that R is local, equidimensional and universally catenary. Since R is equidimensional and I is not contained in any minimal primes of maximal dimension, $R[It]$ is equidimensional. Let q be a minimal prime of J. To show that $\operatorname{ht} q = \operatorname{ht} J$, since $R[X_1, \ldots, X_n]$ is equidimensional and universally catenary, it suffices to show that

$$\dim R[X_1, \ldots, X_n]/J = \dim R[X_1, \ldots, X_n]/q.$$

This equality is clear since $R[X_1, \ldots, X_n]/J$ is equidimensional and q/J is a minimal prime of $R[X_1, \ldots, X_n]/J$.

Likewise, $S[ISt]$ is generically Gorenstein, the associated primes of the defining ideal of the Rees algebra have the same height.

This shows that we can use Theorem 5.2.3 to compute the S_2-ifications of the Rees algebras $R[It]$ and $S[ISt]$. By Theorem 5.2.3, the S_2-ification of $R[It]$ is $\operatorname{Hom}_{R[It]}(\omega_{R[It]}, \omega_{R[It]})$. By Proposition 5.3.2, there is a graded isomorphism,

$$\operatorname{Hom}_{R[It]}(\omega_{R[It]}, \omega_{R[It]}) \otimes_R S \cong \operatorname{Hom}_{S[ISt]}(\omega_{S[ISt]}, \omega_{S[ISt]}). \tag{7.1}$$

Therefore, the degree 1 components are naturally isomorphic, so $I_{\{1\}}S = (IS)_{\{1\}}$. ■

Proposition 7.1.1. *Let $R = k[x, y]$ be a polynomial ring over a field k, and $m = (x, y)$. Let I be an m-primary monomial ideal generated in degree n. Write*

$$I = \left(x^n, y^n, x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r}\right)$$

with $a_i + b_i = n$ for all i. Let $a = \gcd(n, a_1, \ldots, a_r)$. Then

$$I_{\{1\}} = \left(x^n, y^n, x^a y^{n-a}, x^{2a}y^{n-2a}, \ldots, x^{\beta a}y^{n-\beta a}\right),$$

for $\beta = \frac{n}{a} - 1$. In particular, $I_{\{1\}} = \overline{I}$ if and only if $\gcd(n, a_1, \ldots, a_r) = 1$.

Proof. We note that $S = k[x^a, y^a] \hookrightarrow R = k[x, y]$ is a free extension of Gorenstein domains. Given the R-ideal I, there is an S-ideal $J = ((x^a)^{n/a}, (y^a)^{n/a}, (x^a)^{a_1/a}(y^a)^{b_1/a}, \ldots, (x^a)^{a_r/a}(y^a)^{b_r/a})$.

Notice that $JR = I$, and $\gcd(n/a, a_1/a, \ldots, a_r/a) = 1$. By Lemma 7.1.1, $J_{\{1\}}R = I_{\{1\}}$. Therefore, we may assume that $\gcd(n, a_1, \ldots, a_r) = 1$ and show that $I_{\{1\}} = m^n$.

We now show that $x^k y^{n-k} \in I_{\{1\}}$ for $1 \leq k \leq n-1$. Let $\mathcal{A} = R[It, t^{-1}]$.

As in [CPV06], define

$$B^{(1)} = \{h \in \overline{\mathcal{A}}^{R[t,t^{-1}]} \mid \dim(\mathcal{A} / \mathcal{A} :_{\mathcal{A}} h) \leq d-1\},$$

or equivalently, in this case,

$$B^{(1)} = \{h \in \overline{\mathcal{A}}^{R[t,t^{-1}]} \mid \operatorname{ht} \mathcal{A} :_{\mathcal{A}} h \geq 2\}.$$

By [CPV06], to show that $x^k y^{n-k} \in I_{\{1\}}$, we show that $x^k y^{n-k} t \in B^{(1)}$. Let $\mathcal{A} = R[It, t^{-1}]$. To show $x^k y^{n-k} t \in B^{(1)}$, we must show that $\operatorname{ht}(\mathcal{A} :_{\mathcal{A}} x^k y^{n-k} t) \geq 2$.

First, notice that $(x^n, y^n, t^{-1}) \subset (\mathcal{A} :_{\mathcal{A}} x^k y^{n-k} t)$, since

$$\begin{aligned} x^n x^k y^{n-k} t &= x^k y^{n-k}(x^n t) \in It \\ y^n x^k y^{n-k} t &= x^k y^{n-k}(y^n t) \in It \\ t^{-1} x^k y^{n-k} t &= x^k y^{n-k} \in R \end{aligned}$$

We show that there exist nonnegative integers $\alpha_1, \ldots, \alpha_r$ such that

$$(x^{a_1} y^{b_1} t)^{\alpha_1} \ldots (x^{a_r} y^{b_r} t)^{\alpha_r} \in (\mathcal{A} :_{\mathcal{A}} x^k y^{n-k} t). \tag{7.2}$$

To do so, we show that there exist nonnegative integers $\alpha_1, \ldots, \alpha_r$ such that

$$(x^{a_1} y^{b_1} t)^{\alpha_1} \ldots (x^{a_r} y^{b_r} t)^{\alpha_r} x^k y^{n-k} t \in (x^n t, y^n t)^{(\sum_{i=1}^{r} \alpha_i)+1}. \tag{7.3}$$

Since $\gcd(n, a_1, \ldots, a_r) = 1$, there exist integers $\beta_0, \ldots, \beta_r$ such that

$$\beta_0 n + \beta_1 a_1 + \cdots + \beta_r a_r = 1.$$

Multiplying through by k, we have coefficients $\gamma_0, \dots, \gamma_r$ such that

$$\gamma_0 n + \gamma_1 a_1 + \cdots + \gamma_r a_r = k.$$

Modulo (n), we see that

$$k - \gamma_1 a_1 - \cdots - \gamma_r a_r \equiv 0.$$

Adding multiples of n if necessary to make the coefficients positive, we see that there are positive integers $\alpha_1, \dots, \alpha_r$ such that

$$k + \alpha_1 a_1 + \cdots + \alpha_r a_r \equiv 0 \mod (n).$$

It follows immediately that

$$n - k + \alpha_1(n - a_1) + \cdots + \alpha_r(n - a_r) \equiv 0 \mod (n).$$

Hence,

$$(x^{a_1} y^{b_1} t)^{\alpha_1} \dots (x^{a_r} y^{b_r} t)^{\alpha_r} x^k y^{n-k} t = x^{nu} y^{nv} t^{(\sum_{j=1}^{r} \alpha_j)+1}.$$

We claim that this is an element of $R[It, t^{-1}]$. Notice that the total degree in x and y on the left-hand side is $n((\sum_{i=1}^{r} \alpha_i) + 1)$. Hence

$$nu + nv = n((\sum_{i=1}^{r} \alpha_i) + 1). \tag{7.4}$$

Therefore, $u + v = (\sum_{i=1}^{r} \alpha_i) + 1$. Hence

$$x^{nu} y^{nv} = (x^n)^u (y^n)^v \in I^{(\sum_{i=1}^{r} \alpha_i)+1}. \tag{7.5}$$

This shows that $(x^{a_1} y^{b_1} t)^{\alpha_1} \dots (x^{a_r} y^{b_r} t)^{\alpha_r} \in \mathcal{A} :_{\mathcal{A}} x^k y^{n-k} t$.

We now show that $\operatorname{ht}(\mathcal{A} :_{\mathcal{A}} x^k y^{n-k} t) \geq 2$. Notice that $(x, y, t^{-1}) \subseteq \sqrt{(\mathcal{A} :_{\mathcal{A}} x^k y^{n-k} t)}$. Since

$$R[It, t^{-1}]/(x, y, t^{-1}) \cong \mathscr{F}_I(R),$$

the special fiber ring of I, and $\ell(I) = 2$ since $\operatorname{ht} I \leq \ell(I) \leq \dim R$, we see that $\operatorname{ht}(x, y, t^{-1}) = 1$. Let $p, q \in \mathscr{F}_I(R)$ be nonzero. Let $p_i \in I^i/mI^i$ be the first nonzero component of p and $q_j \in I^j/mI^j$ be the first nonzero component of q. Since p_i and q_j are nonzero, there exists nonzero P_i and Q_j consisting of sums of elements of degree ni and nj, respectively, in $I^i \setminus mI^i$ and $I^j \setminus mI^j$ such that $P_i + mI^i = p_i$ and $Q_j + mI^j = q_j$. Then since R is a domain, P_iQ_j is nonzero and contained in I^{i+j}. Since P_i is a sum of elements of degree ni and Q_j is a sum of elements of degree nj, P_iQ_j is a sum of elements of degree $ni+nj$. Hence $p_iq_j = (P_i+mI^i)(Q_j+mI^j) = P_iQ_j+mI^{i+j}$ is nonzero, since P_iQ_j must have degree exactly $n(i+j)$. Since this term cannot cancel with any other terms, pq is nonzero.

Thus, (x, y, t^{-1}) is a prime ideal of height 1 in $R[It, t^{-1}]$. We now observe that

$$(x^{a_1}y^{b_1}t)^{\alpha_1} \ldots (x^{a_r}y^{b_r}t)^{\alpha_r} \notin (x, y, t^{-1})\,\mathcal{A}\,.$$

It suffices to see that

$$(x^{a_1}y^{b_1}t)^{\alpha_1} \ldots (x^{a_r}y^{b_r}t)^{\alpha_r} \notin (x, y)I^{\sum_{j=1}^{r}\alpha_j},$$

which is clear by degree considerations. Therefore, $\sqrt{(\mathcal{A} :_{\mathcal{A}} x^k y^{n-k} t)}$ properly contains a height one prime. This shows that $\operatorname{ht}(\mathcal{A} :_{\mathcal{A}} x^k y^{n-k} t) = \operatorname{ht}\sqrt{\mathcal{A} :_{\mathcal{A}} x^k y^{n-k} t} \geq 2$. Hence, $x^k y^{n-k} \in I_{\{1\}}$ for $1 \leq k \leq \beta$.

■

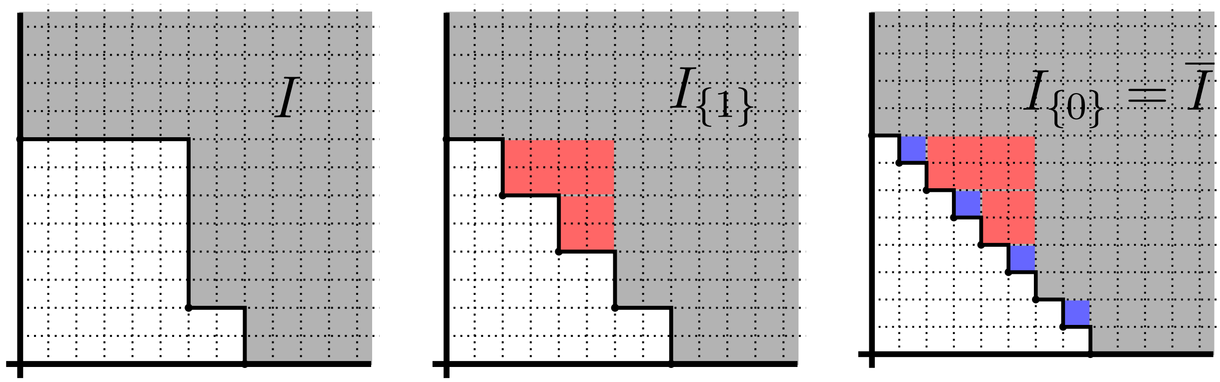
We note that the result above was known in the case where $\gcd(n, a_1, \ldots, a_r) = 1$ by Polini, Ulrich, and Vitulli. Their proof in [PUV07, Corollary 6.5] uses the core of ideals to compute the first coefficient ideal.

Remark 7.1.2. Notice that Proposition 7.1.1 demonstrates that in for all m-primary monomial ideals I generated in one degree, that the first coefficient ideal $I_{\{1\}}$ is generated in the same degree. This is not true in higher dimensions. We will see a counterexample in a polynomial ring in 3 variables in the next chapter.

Recall that we saw in the previous chapter that first coefficient ideals preserve containments when $J \subseteq I$ is a reduction (see Proposition 6.1.1). We use Proposition 7.1.1 to

Figure 7.1.

(a) I (b) The first coefficient ideal of I (c) The integral closure of I



construct an example showing that the first coefficient ideal does not respect containments in general.

Example 7.1.1. Let $R = k[x,y]$ be a polynomial ring over a field k and $m = (x,y)$. Let $J = (x^9, x^5y^4, y^9)$ and $I = (x^8, x^4y^4, y^8)$. Then I, J are m-primary ideals with $J \subseteq I$. By Proposition 7.1.1, the first coefficient ideals are

$$J_{\{1\}} = m^9$$

and

$$I_{\{1\}} = \left(x^8, x^4y^4, y^8\right).$$

In particular, $x^7y^2 \in J_{\{1\}} \setminus I_{\{1\}}$.

Example 7.1.2. One can visualize the relationship between the ideal itself, the first coefficient ideal $I_{\{1\}}$ and the integral closure $I_{\{0\}}$ for the ideal $I = (x^8, x^6y^2, y^8)$ in the following graphs. The lattice points (a,b) in the shaded regions correspond to elements x^ay^b contained in the ideal.

Remark 7.1.3. Combining Proposition 6.1.1 and Proposition 7.1.1, we have a larger class of ideals for which $I_{\{1\}} = \overline{I} = m^n$. Let J be an m-primary monomial ideal generated in degree n for which $J_{\{1\}} = \overline{J} = m^n$. That is, $J = (x^n, y^n, x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r})$ with $a_i + b_i = n$ for all i and $\gcd(n, a_1, \ldots, a_r) = 1$.

Let K be any ideal contained in m^n.

Then we claim that $I = J + K$ is an ideal for which $I_{\{1\}} = \overline{I} = m^n$. Notice that $\overline{I} = m^n$ since

$$J \subseteq I \subseteq m^n$$

implies

$$\overline{J} \subseteq \overline{I} \subseteq \overline{m^n}$$

and $\overline{J} = m^n$ and $\overline{m^n} = m^n$. Notice that J is a reduction of $I = J + K$ since $I = J + K \subseteq \overline{J} = m^n$. Hence, by Proposition 6.1.1,

$$J_{\{1\}} \subseteq I_{\{1\}} \, .$$

Since $J_{\{1\}} = m^n$ and $I_{\{1\}} \subseteq \overline{I} = m^n$, we conclude that $I_{\{1\}} = \overline{I}$. This extends the class of examples whose first coefficient ideal coincides with the integral closure to monomial ideals that are not generated in the same degree and even non-monomial ideals.

Example 7.1.3. Let $I = (x^4, y^4, xy^3, x^3y^2)$. Then the above remark implies that $I_{\{1\}} = \overline{I}$. Notice that I is generated in degrees 4 and 5.

Let $I = (x^5, y^5, x^4y, x^3y^2 + xy^4)$. Then I is a non-monomial ideal whose first coefficient ideal coincides with its integral closure.

7.2 The Ratliff-Rush Closure

Recall that the Ratliff-Rush closure of an ideal I containing a nonzerodivisor is defined to be

$$\widetilde{I} := \bigcup_{n \geq 0} I^{n+1} : I^n, \tag{7.6}$$

and that the 2nd coefficient ideal coincides with the Ratliff-Rush closure for m-primary ideals in 2-dimensional rings.

Veronica Crispin Quiñonez has given the following description for the Ratliff-Rush closure of m-primary monomial ideals generated in one degree in $k[x, y]$.

Proposition 7.2.1 ([Qui06, Proposition 3.7])**.** *Let $R = k[x,y]$ be a polynomial ring over a field k and $m = (x,y)$. Let I be an m-primary monomial ideal generated in degree n. Write $I = \left(x^n, y^n, x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r}\right)$ with $a_i + b_i = n$ for $1 \leq i \leq r$ ordered so that $a_i < a_{i+1}$. Let S denote the numerical semigroup $\langle a_i \rangle_{i=1}^r$ and T denote the numerical semigroup $\langle b_i \rangle_{i=1}^r$. Define $I_S = (x^s y^{n-s} \mid s \in S, s \leq n)$ and $I_T = (x^{n-t}y^t \mid t \in T, t \leq n)$. Then*

$$\widetilde{I} = I_S \cap I_T. \tag{7.7}$$

This result along with the computation of the first coefficient ideal for these ideals allows us to characterize when $\widetilde{I} = I_{\{1\}}$.

Proposition 7.2.2. *Let $R = k[x,y]$ be a polynomial ring over a field k and $m = (x,y)$. Let I be an m-primary monomial ideal generated in degree n. Write $I = \left(x^n, y^n, x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r}\right)$ with $a_i + b_i = n$ for $1 \leq i \leq r$ ordered so that $a_i < a_{i+1}$. Let $\gcd(n, a_1, \ldots, a_r) = a$. Then $\widetilde{I} = I_{\{1\}}$ if and only if $a_1 = b_r = a$. Moreover, $\widetilde{I} = I_{\{1\}} = \overline{I}$ if and only if $a_1 = b_r = 1$.*

Proof. First suppose $a_1 = b_r = a$. Since $a = \gcd(n, a_1, \ldots, a_r)$, $S = \langle a \rangle$. Since

$$\gcd(n, b_1, \ldots, b_r) = \gcd(n, n - a_1, \ldots, n - a_r) = \gcd(n, a_1, \ldots, a_r) = a,$$

$T = \langle a \rangle$. Therefore, $I_S = I_T$. By Proposition 7.1.1, $I_S = I_T = I_{\{1\}}$ and therefore, $\widetilde{I} = I_{\{1\}}$.

Next, suppose that $a_1 \neq a$. Then $a \notin S$ and hence $x^a y^{n-a} \notin I_S$. Therefore, $x^a y^{n-a} \notin \widetilde{I}$, but $x^a y^{n-a} \in I_{\{1\}}$ by Proposition 7.1.1. Similarly, suppose $b_r \neq a$. Then $a \notin T$ and hence $x^{n-a}y^a \notin I_T$. Therefore, by Proposition 7.2.1, $x^{n-a}y^a \notin \widetilde{I}$, but $x^{n-a}y^a \in I_{\{1\}}$.

Furthermore, we note that from Proposition 7.1.1, $I_{\{1\}} = \overline{I}$ if and only if $a = 1$. Hence $\widetilde{I} = \overline{I}$ if and only if $a_1 = b_r = 1$. ∎

Example 7.2.1. Let $R = k[x,y]$ be a polynomial ring over a field k. Let $I = (x^7, y^7, x^2y^5, x^5y^2)$.

Then

$$I_{\{2\}} = (x^7, y^7, x^2y^5, x^5y^2, x^4y^4) \tag{7.8}$$

$$I_{\{1\}} = \overline{I} = m^7. \tag{7.9}$$

Note that Eq. (7.8) follows from Proposition 7.2.1 and Eq. (7.9) follows from Proposition 7.1.1. Notice that this example shows that when I is generated in one degree, the Ratliff-Rush closure need not be generated in one degree.

Example 7.2.2. Let $R = k[x, y]$ be a polynomial ring over a field k. Let $I = (x^8, x^6y^2, x^4y^4, y^8)$.

Then

$$I_{\{2\}} = I \tag{7.10}$$

$$I_{\{1\}} = (x^8, x^6y^2, x^4y^4, x^2y^6, y^8) \tag{7.11}$$

$$\overline{I} = m^8. \tag{7.12}$$

Note that Eq. (7.10) follows from Proposition 7.2.1 and Eq. (7.13) follows from Proposition 7.1.1.

Example 7.2.3. Let $R = k[x, y]$ be a polynomial ring over a field k. Let $I = (x^{12}, x^2\, y^{10}, x^8\, y^4, y^{12})$. Then

$$I_{\{2\}} = (x^{12}, x^2\, y^{10}, x^4\, y^8, x^8\, y^4, y^{12}) \tag{7.13}$$

$$I_{\{1\}} = (x^{12}, x^2\, y^{10}, x^4\, y^8, x^6\, y^6, x^8\, y^4, x^{10}\, y^2, y^{12}) \tag{7.14}$$

$$\overline{I} = m^{12}. \tag{7.15}$$

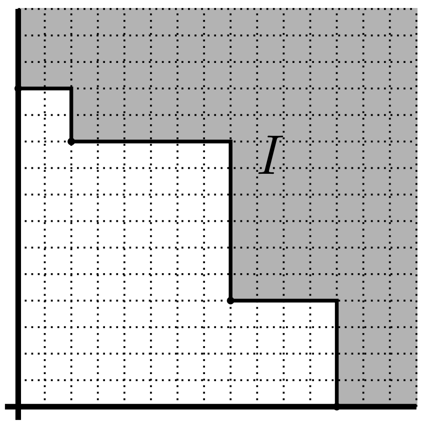
Note that Eq. (7.13) follows from Proposition 7.2.1 and Eq. (7.14) follows from Proposition 7.1.1. This is an example of an ideal for which

$$I \subsetneq I_{\{2\}} \subsetneq I_{\{1\}} \subsetneq \overline{I}.$$

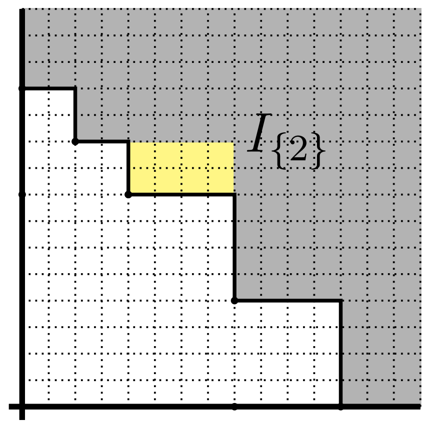
We depict the relationship between the coefficient ideals of I in the figures below.

Figure 7.2.

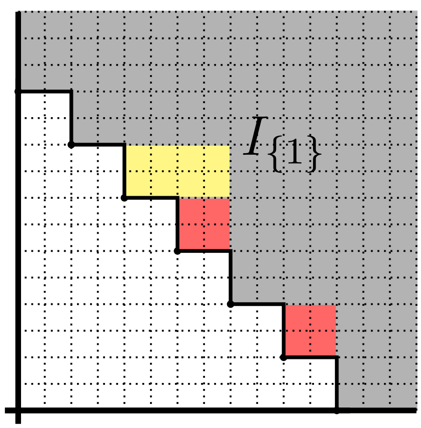
(a) I



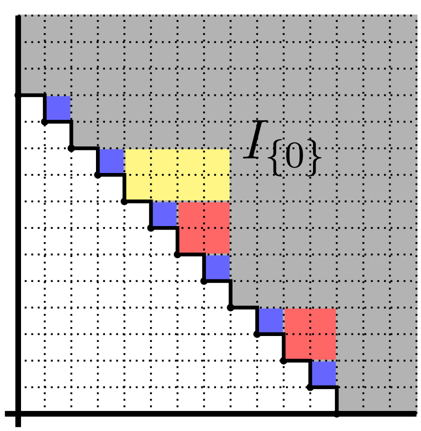
(b) The Ratliff-Rush closure of I



(c) The first coefficient ideal of I



(d) The integral closure of I



7.3 Behavior of Coefficient Ideals Under Specialization

Proposition 7.3.1. *Let $S = k[x,y]$ be a polynomial ring over a field k of characteristic zero. Let I be an m-primary monomial ideal generated in degree n. Write*

$$I = \left(x^n, y^n, x^{a_1}y^{b_1}, \ldots, x^{a_r}y^{b_r}\right)$$

with $a_i + b_i = n$ for all i. Let a be a general element of I. Then

$$I_{\{1\}}/(a) = (I/(a))_{\{1\}}.$$

Proof. Note that if $\gcd(n, a_1, \ldots, a_r) = 1$, then $I_{\{1\}} = \overline{I}$ by Proposition 7.1.1 and hence $I_{\{1\}}$ specializes by Corollary 6.0.4.

Suppose $\gcd(n, a_1, \ldots, a_r) = \delta > 1$. Let $R = k[x^\delta, y^\delta]$. Then $R \to S$ is a free extension of Gorenstein rings. Let $J = \left((x^\delta)^{n/\delta}, (y^\delta)^{n/\delta}, (x^\delta)^{a_1/\delta}(y^\delta)^{b_1/\delta}, \ldots, (x^\delta)^{a_r/\delta}(y^\delta)^{b_r/\delta}\right)$. Notice that

$$\gcd(n/\delta, a_1/\delta, \ldots, a_r/\delta) = 1.$$

Hence, by the previous case, $J_{\{1\}} = \overline{J} = (x^\delta, y^\delta)^{n/\delta}$ and hence as $R/(a)$-ideals, $J_{\{1\}}/(a) = (J/(a))_{\{1\}}$.

We want to show that the first coefficient ideal of JS specializes with respect to aS. Recall that $R \to S$ is a free and hence faithfully flat extension of Gorenstein rings. Since we may assume a is a nonzerodivisor on R and S, $R/(a)$ and S/aS are Gorenstein. Moreover, $R/(a) \to S/aS$ is a faithfully flat extension because $R \to S$ is.

Then

$$\begin{aligned}
((J/(a))S/(a)S)_{\{1\}} &= (J/(a))_{\{1\}} \otimes_{R/(a)} S/(a)S && (7.16)\\
&= J_{\{1\}}/(a) \otimes_{R/(a)} S/(a)S && (7.17)\\
&= J_{\{1\}}S/(a)S && (7.18)\\
&= (JS)_{\{1\}}/(a)S. && (7.19)
\end{aligned}$$

Notice that Eq. (7.16) and Eq. (7.19) follow from Lemma 7.1.1, Eq. (7.17) is shown above, and Eq. (7.18) follows from $S/(a)S$ being flat over $R/(a)$. ■

Next, we restate the result from the Specialization of Coefficient Ideals chapter in the dimension $d = 2$ case.

Proposition 7.3.2. *Let (R, m) be a Noetherian local Cohen-Macaulay k-algebra of dimension 2 with k an infinite field. Let I be an m-primary ideal of R. Let x be a general element of I. Then $I_{\{2\}}/(x) \subseteq I_{\{1\}}/(x) \subseteq (I/(x))_{\{1\}}$. In particular, if $\widetilde{I} = I_{\{2\}} \subsetneq I_{\{1\}}$, then the Ratliff-Rush closure $\widetilde{I}$ does not specialize with respect to general elements of I.*

Let $R = k[x, y]$ be a polynomial ring over a field of characteristic zero, $m = (x, y)$ and I an m-primary ideal. The above proposition implies that the Ratliff-Rush closure does not specialize if $\widetilde{I} = I_{\{2\}} \subsetneq I_{\{1\}}$. Assume I is an m-primary monomial ideal generated in one degree. Since we have shown that $I_{\{1\}}/(a) = (I/(a))_{\{1\}}$ for general $a \in I$ in Proposition 7.3.1, if $I_{\{2\}} = I_{\{1\}}$, then

$$\widetilde{I}/(a) = I_{\{2\}}/(a) = I_{\{1\}}/(a) = (I/(a))_{\{1\}} = \widetilde{I/(a)}. \tag{7.20}$$

Hence, the Ratliff-Rush closure specializes if and only if $\widetilde{I} = I_{\{1\}}$.

Therefore, to summarize, if I is an m-primary monomial ideal generated in one degree and (a) is a general element of I. Then

(i) $I_{\{0\}} = \overline{I}$ specializes with respect to (a) by Theorem 3.6.1

(ii) $I_{\{1\}}$ specializes with respect to (a) by Proposition 7.3.1

(iii) $I_{\{2\}} = \widetilde{I}$ specializes with respect to (a) if and only if $I_{\{2\}} = I_{\{1\}}$ by the discussion above.

8. FIRST COEFFICIENT IDEALS

In this chapter, we describe the first coefficient ideal of 0-dimensional monomial ideals generated in one degree in a polynomial ring over a field.

Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field, let $m = (x_1, \ldots, x_d)$ denote the homogeneous maximal ideal of R, and let I be m-primary. Notice that the Hilbert-Samuel polynomial of I is equal to the Hilbert-Samuel polynomial of I_m in R_m, and hence we may apply the results of Corso, Polini and Vasconcelos in [CPV06] to compute $I_{\{1\}}$.

Heinzer and Lantz proved that coefficient ideals of monomial ideals in a polynomial ring over an infinite field are monomial ideals (see [HL97, Observation 3.3]).

The first result gives a criterion for a monomial to be contained in the first coefficient ideal of a 0-dimensional monomial ideal generated in one degree.

Proposition 8.0.1. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k. Let $m = (x_1, \ldots, x_d)$. Let I be an m-primary monomial ideal generated in degree n. Write $I = (x_1^n, \ldots, x_d^n, v_1, \ldots, v_s)$ with $v_1, \ldots, v_s$ monomials of degree n. Let $J = (v_1, \ldots, v_s)$.*

The following are equivalent:

(i.) $\underline{x}^{\underline{a}} \in I_{\{1\}}$

(ii.) There exists a monomial $\omega \in I^k \setminus mI^k$ for some $k \geq 0$ such that $\underline{x}^{\underline{a}}\omega \in I^{k+1}$.

If I is generated by at least $d+1$ elements, then (i) and (ii) are equivalent to:

(iii.) There exists a monomial $\rho \in J^k \setminus mJ^k$ for some $k \geq 0$ such that $\underline{x}^{\underline{a}}\rho \in (x_1^n, \ldots, x_d^n)^{k+1}$.

Proof. We first prove $(i) \iff (ii)$. We then prove $(ii) \iff (iii)$ with the additional assumption that I is generated by at least $d+1$ elements.

$(i) \implies (ii)$: Let $\underline{x}^{\underline{a}}$ be a monomial in R. Suppose that for every $k \geq 0$, and for all monomials $\omega \in I^k \setminus mI^k$, $\underline{x}^{\underline{a}}\omega \notin I^{k+1}$. Let

$$b_{-n}t^{-n} + \ldots + b_0 + b_1 t + \ldots + b_s t^s \in R[It, t^{-1}] :_{R[It,t^{-1}]} \underline{x}^{\underline{a}}t. \tag{8.1}$$

Then since

$$b_{-n}\underline{x}^{\underline{a}}t^{-n+1} + \ldots + b_0\underline{x}^{\underline{a}}t + b_1\underline{x}^{\underline{a}}t^2 + \ldots + b_s\underline{x}^{\underline{a}}t^{s+1} \in R[It, t^{-1}]\,, \tag{8.2}$$

we see that $b_i\underline{x}^{\underline{a}} \in I^{i+1}$ for $0 \leq i \leq s$. By assumption, since $b_i \in I^i$ for $0 \leq i \leq s$ and $b_i\underline{x}^{\underline{a}} \in I^{i+1}$, $b_i \in mI^i$. Hence,

$$b_{-n}t^{-n} + \ldots + b_0 + b_1t + \ldots + b_st^s \in t^{-1}R[It, t^{-1}] + mR[It, t^{-1}]. \tag{8.3}$$

This shows that $R[It, t^{-1}] :_{R[It,t^{-1}]} \underline{x}^{\underline{a}}t \subseteq t^{-1}R[It, t^{-1}] + mR[It, t^{-1}]$. We will now show that $t^{-1}R[It, t^{-1}] + mR[It, t^{-1}]$ is a prime ideal of $R[It, t^{-1}]$ with height one, so that the height of $\left(R[It, t^{-1}] :_{R[It,t^{-1}]} \underline{x}^{\underline{a}}t\right)$ is at most one.

Notice that

$$R[It, t^{-1}]/(t^{-1}R[It, t^{-1}] + mR[It, t^{-1}]) \cong R/m \oplus I/mI \oplus I^2/mI^2 \oplus \ldots\,, \tag{8.4}$$

the special fiber ring of I, $\mathscr{F}_I(R)$. Since I is generated in degree n, we can see that $\mathscr{F}_I(R)$ is a domain: Let $p, q \in \mathscr{F}_I(R)$ be nonzero. Let $p_i \in I^i/mI^i$ be the first nonzero component of p and $q_j \in I^j/mI^j$ be the first nonzero component of q. Since p_i and q_j are nonzero, there exists nonzero P_i and Q_j consisting of sums of elements of degree ni and nj, respectively, in $I^i \setminus mI^i$ and $I^j \setminus mI^j$ such that $P_i + mI^i = p_i$ and $Q_j + mI^j = q_j$. Then since R is a domain, P_iQ_j is nonzero and contained in I^{i+j}. Since P_i is a sum of elements of degree ni and Q_j is a sum of elements of degree nj, P_iQ_j is a sum of elements of degree $ni+nj$. Hence $p_iq_j = (P_i+mI^i)(Q_j+mI^j) = P_iQ_j+mI^{i+j}$ is nonzero, since P_iQ_j must have degree exactly $n(i+j)$. Since this term cannot cancel with any other terms, pq is nonzero.

Since $\mathscr{F}_I(R)$ is a domain, by Eq. (8.4), $(t^{-1}R[It,t^{-1}]+mR[It,t^{-1}])$ is a prime ideal of $R[It,t^{-1}]$. Since $\operatorname{ht} I \leq \ell(I) \leq \dim R$, $\ell(I)=d$. Since R is a universally catenary domain, and hence so is $R[It,t^{-1}]$,

$$\begin{aligned}\operatorname{ht}(t^{-1}R[It,t^{-1}]+mR[It,t^{-1}]) &= \dim R[It,t^{-1}]-\dim \mathscr{F}_I(R)\\ &= d+1-d\\ &= 1.\end{aligned}$$

Therefore, $R[It,t^{-1}] :_{R[It,t^{-1}]} \underline{x}^{\underline{a}}t$ has height at most one. Therefore, $\underline{x}^{\underline{a}}t \notin B^{(1)}$ and hence $\underline{x}^{\underline{a}} \notin I_{\{1\}}$.

$(ii) \implies (i)$: Suppose there exists a monomial $\omega \in I^k \setminus mI^k$ for some $k \geq 0$ such that $\underline{x}^{\underline{a}}\omega \in I^{k+1}$. We show that $\underline{x}^{\underline{a}} \in I_{\{1\}}$ by showing that $\underline{x}^{\underline{a}}t \in B^{(1)}$. To do so, we must show that $\operatorname{ht} R[It,t^{-1}] :_{R[It,t^{-1}]} \underline{x}^{\underline{a}}t \geq 2$.

It is clear that $(x_1^n,\ldots,x_d^n,t^{-1},\omega t^k) \subseteq R[It,t^{-1}] :_{R[It,t^{-1}]} \underline{x}^{\underline{a}}t$. It immediately follows that

$$mR[It,t^{-1}]+t^{-1}R[It,t^{-1}] \subseteq \sqrt{R[It,t^{-1}] :_{R[It,t^{-1}]} \underline{x}^{\underline{a}}t}\,.$$

As in the proof of $(i) \implies (ii)$, $mR[It,t^{-1}]+t^{-1}R[It,t^{-1}]$ is a prime ideal of height 1 in $R[It,t^{-1}]$.

Since $\omega \in I^k \setminus mI^k$ and I is generated in degree n, ω has degree exactly kn. Then $\omega t^k \notin mR[It,t^{-1}]+t^{-1}R[It,t^{-1}]$, since $[mR[It,t^{-1}]+t^{-1}R[It,t^{-1}]]_k$ is generated by elements of the form zt^k, where z is a monomial in R of degree at least $nk+1$. Therefore, $\sqrt{R[It,t^{-1}] :_{R[It,t^{-1}]} \underline{x}^{\underline{a}}t}$ properly contains a height one prime. This shows that

$$\operatorname{ht}(R[It,t^{-1}] :_{R[It,t^{-1}]} \underline{x}^{\underline{a}}t) = \operatorname{ht}\sqrt{R[It,t^{-1}] :_{R[It,t^{-1}]} \underline{x}^{\underline{a}}t} \geq 2.$$

Hence, $\underline{x}^{\underline{a}} \in I_{\{1\}}$.

Now assume I is generated by at least $d+1$ elements.

$(ii) \implies (iii)$ Let $\omega \in I^k \setminus mI^k$ be a monomial such that $\underline{x}^{\underline{a}}\omega \in I^{k+1}$. Then

$$\underline{x}^{\underline{a}}\omega = (x_1^n)^{\alpha_1}(x_2^n)^{\alpha_2}\cdots(x_d^n)^{\alpha_d}(v_1)^{\beta_1}\cdots(v_s)^{\beta_s}u \tag{8.5}$$

where $\sum_{i=1}^d \alpha_i+\sum_{i=1}^s \beta_i \geq k+1$, and $u \in R$. Notice that we may assume that u is a monomial of the form $x_1^{a_1}\cdots x_d^{a_d}$ with $0 \leq a_i < n$ for $1 \leq i \leq d$. We first show that we can assume $\beta_i = 0$ for $1 \leq i \leq s$. Notice that v_i^n can be written as products of $x_1^n, \ldots, x_d^n$. Therefore, we may assume that $\beta_i < n$ for $1 \leq i \leq s$.

Now assume that $0 < \beta_i < n$ for $1 \leq i \leq s$. Then we multiply both sides of Eq. (8.5) by $v_i^{n-\beta_i}$ to see that

$$\underline{x}^{\underline{a}}\omega v_1^{n-\beta_1}\cdots v_s^{n-\beta_s} = (x_1^n)^{\alpha_1}(x_2^n)^{\alpha_2}\cdots(x_d^n)^{\alpha_d}(v_1)^{n}\cdots(v_s)^{n}u. \tag{8.6}$$

Since each v_i has degree exactly n, we maintain that $\omega v_1^{n-\beta_1}\cdots v_s^{n-\beta_s} \in I^K \setminus mI^K$ and $(x_1^n)^{\alpha_1}(x_2^n)^{\alpha_2}\ldots(x_d^n)^{\alpha_d}(v_1)^n\ldots(v_s)^n \in I^{K+1}$, for $K = k+sn-\sum_{i=1}^s \beta_s$.

Let $\rho = \omega\, v_1^{n-\beta_1}\ \cdots\ v_s^{n-\beta_s}$. Since each v_i^n can be rewritten as products of $x_1^n, \ldots, x_d^n$, this shows that

$$\underline{x}^{\underline{a}}\rho = (x_1^n)^{\gamma_1}\cdots(x_d^n)^{\gamma_d}u, \tag{8.7}$$

where $(\sum_{i=1}^d \alpha_i) + sn = \sum_{i=1}^d \gamma_i$. Therefore, $(x_1^n)^{\gamma_1}\cdots(x_d^n)^{\gamma_d} \in I^{K+1}$. Notice that if $\omega \notin J^k$, then factors of $(x_j)^n$ for $1 \leq j \leq d$ appear on both left-hand and right-hand sides of Eq. (8.7) which will cancel. Therefore, we may assume ω has no factors of x_j^n for $1 \leq j \leq d$. Therefore, we may assume $\omega \in J^k$. With ρ defined as above, we have $\rho \in J^K \setminus mJ^{K+1}$ such that $\rho\underline{x}^{\underline{a}} \in (x_1^n, \ldots, x_d^n)^{K+1}$.

$(iii) \implies (ii)$: This is immediate because $J^k \subseteq I^k$, $(x_1^n, \ldots, x_d^n)^k \subseteq I^k$ for all k and for degree reasons $(J^k \setminus mJ^k) \cap I^k \subseteq I^k \setminus mI^k$.

■

The following result can easily be seen in other ways since the Rees algebra of a complete intersection is known to be Cohen-Macaulay and hence S_2, but we record it as an immediate consequence of Proposition 8.0.1.

Corollary 8.0.1. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k. Let $I = (x_1^n, \ldots, x_d^n)$. Then $I = I_{\{1\}}$.*

Proof. This follows immediately from Proposition 8.0.1 because

$$I^k = \left((x_1^n)^{a_1} \cdots (x_d^n)^{a_d} \mid \sum_{i=1}^{d} a_i = k \right) \tag{8.8}$$

for all k. Hence if $\omega \in I^k \setminus mI^k$ multiplies $\underline{x}^{\underline{a}}$ into I^{k+1}, $\underline{x}^{\underline{a}}$ must be in $I = (x_1^n, \ldots, x_d^n)$. ∎

The following corollary of Proposition 8.0.1 gives a formula for the first coefficient ideal of an almost complete intersection as a sum of finitely many ideals.

Corollary 8.0.2. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k with $d \geq 2$. Let I be an m-primary almost complete intersection monomial ideal generated in degree n. That is, $I = (x_1^n, \ldots, x_d^n) + J$ where $J = (x_1^{e_1} \cdots x_d^{e_d})$ with $e_i \geq 0$ for $1 \leq i \leq d$ and $\sum_{i=1}^{d} e_i = n$. Then the first coefficient ideal of I is*

$$I_{\{1\}} = \sum_{k=0}^{n-1} \left((x_1^n, \ldots, x_d^n)^{k+1} : J^k \right).$$

Proof. Let $\underline{x}^{\underline{e}}$ denote the element $x_1^{e_1} \cdots x_d^{e_d}$. Let $K = (x_1^n, \ldots, x_d^n)$. Since I is a monomial ideal, so is $I_{\{1\}}$. By Proposition 8.0.1, $\underline{x}^{\underline{a}} \in I_{\{1\}}$ if and only if there is a power $(\underline{x}^{\underline{e}})^k$ such that

$$\underline{x}^{\underline{a}} \cdot (\underline{x}^{\underline{e}})^k \in K^{k+1}. \tag{8.9}$$

Therefore, $\underline{x}^{\underline{a}} \in I_{\{1\}}$ if and only if $\underline{x}^{\underline{a}} \in K^{k+1} : J^k$ for some $k \geq 0$. It now suffices to show that

$$\sum_{k \geq 0} \left(K^{k+1} : J^k \right) = \sum_{k=0}^{n-1} \left(K^{k+1} : J^k \right). \tag{8.10}$$

Assume $s \geq n$. Then $s = qn + r$, for some integer r with $0 \leq r < n$. We show that $K^{s+1} : J^s \subseteq K^{r+1} : J^r$. Suppose

$$(\underline{x}^{\underline{a}})(\underline{x}^{\underline{e}})^{qn+r} \in K^{qn+r+1}. \tag{8.11}$$

Then

$$(\underline{x}^{\underline{a}})(\underline{x}^{\underline{e}})^r (x_1^n)^{qe_1} \cdots (x_d^n)^{qe_d} \in K^{qn+r+1}. \tag{8.12}$$

Hence $(\underline{x}^{\underline{a}})(\underline{x}^{\underline{e}})^r \in K^{qn+r+1} : (x_1^n)^{qe_1} \cdots (x_d^n)^{qe_d}$. We claim that

$$K^{qn+r+1} : (x_1^n)^{qe_1} \cdots (x_d^n)^{qe_d} = K^{r+1}.$$

It is clear that $K^{r+1} \subseteq K^{qn+r+1} : (x_1^n)^{qe_1} \cdots (x_d^n)^{qe_d}$, since

$$(x_1^n)^{qe_1} \cdots (x_d^n)^{qe_d} \in K^{q(e_1+\ldots+e_d)} = K^{qn}. \tag{8.13}$$

Let $w \in K^{qn+r+1} : (x_1^n)^{qe_1} \cdots (x_d^n)^{qe_d}$. Then

$$w(x_1^n)^{qe_1} \cdots (x_d^n)^{qe_d} = r(x_1^n)^{k_1} \cdots (x_d^n)^{k_d}, \tag{8.14}$$

for some $r \in R$ and $\sum_{i=1}^{d} k_i \geq qn + r + 1$. We may assume $r \notin (x_1^n) \cup (x_2^n) \cup \cdots \cup (x_d^n)$. Then $(x_i^n)^{qe_i} \mid r(x_1^n)^{k_1} \cdots (x_d^n)^{k_d}$ and $r \notin (x_i^n)$. Hence $qe_i \leq k_i$.

Hence

$$w = r(x_1^n)^{k_1 - qe_1} \cdots (x_d^n)^{k_d - qe_d}. \tag{8.15}$$

Since

$$\begin{aligned}
\sum_{i=1}^{d} (k_i - qe_i) &= \sum_{i=1}^{d} k_i - q \sum_{i=1}^{d} e_i && (8.16)\\
&\geq qn + r + 1 - qn && (8.17)\\
&= r + 1, && (8.18)
\end{aligned}$$

we conclude that $w \in K^{r+1}$.

This proves the claim. Therefore, since $(\underline{x}^{\underline{a}})(\underline{x}^{\underline{e}})^r \in K^{qn+r+1} : (x_1^n)^{qe_1} \cdots (x_d^n)^{qe_d} = K^{r+1}$,

$$\underline{x}^{\underline{a}} \in K^{r+1} : J^r. \tag{8.19}$$

Therefore, $I_{\{1\}} = \sum_{k=0}^{n-1} K^{k+1} : J^k$. ■

Remark 8.0.3. By the same reasoning, if I is an m-primary monomial ideal generated in degree n, one has a formula for $I_{\{1\}}$ as a sum of finitely many ideals. Write $I = (x_1^n, \dots, x_d^n) + J$ where $J = (v_1, \dots, v_s)$ with each v_i a monomial of degree n. Then $I_{\{1\}}$ is the sum of ideals $(x_1^n, \dots, x_d^n)^{k+1} : (v_1^{a_1} \dots v_s^{a_s})$ with $0 \le a_i \le n-1$ for $1 \le i \le s$ and $\sum_{i=1}^s a_i = k$.

Proposition 8.0.4. *Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field k with $d \ge 2$. Let I be an m-primary ideal generated in degree n. Write $I = (x_1^n, \dots, x_d^n, v_1, \dots, v_k)$ with each v_i for $1 \le i \le k$ a monomial of degree n. Let $v_i = x_1^{a_{1,i}} \dots x_d^{a_{d,i}}$ for $1 \le i \le k$. Let*

$$b_1 = \gcd(n, a_{1,j} \mid 1 \le j \le k)$$

$$b_2 = \gcd(n, a_{2,j} \mid 1 \le j \le k)$$

$$\vdots$$

$$b_d = \gcd(n, a_{d,j} \mid 1 \le j \le k).$$

Let $S = k[x_1^{b_1}, x_2^{b_2}, \dots, x_d^{b_d}]$. Then $[I_{\{1\}}]_n \subseteq S_n$. In particular, if some $b_i > 1$, then $I_{\{1\}} \subsetneq m^n$.

Proof. Let $\underline{x}^{\underline{a}}$ be a monomial of degree n. By Proposition 8.0.1, $\underline{x}^{\underline{a}} \in I_{\{1\}}$ if and only if there exist $\alpha_1, \dots, \alpha_k \ge 0$, $\delta_1, \dots, \delta_d \ge 0$ and $u \in R$ such that

$$\underline{x}^{\underline{a}} v_1^{\alpha_1} \dots v_k^{\alpha_k} = (x_1^n)^{\delta_1} \dots (x_d^n)^{\delta_d} u \,,$$

with $(\sum_{i=1}^k \alpha_i) + 1 = \sum_{i=1}^d \delta_d$. Comparing degrees of the left-hand and right-hand sides, we conclude that $u = 1$. Hence

$$\underline{x}^{\underline{a}} v_1^{\alpha_1} \dots v_k^{\alpha_k} = (x_1^n)^{\delta_1} \dots (x_d^n)^{\delta_d} \,.$$

In matrix form, this equation is equivalent to:

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} + \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k} \\ a_{2,1} & a_{2,2} & \dots & a_{2,k} \\ & \vdots & & \\ a_{d,1} & a_{d,2} & \dots & a_{d,k} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{bmatrix} = \begin{bmatrix} \delta_1 n \\ \delta_2 n \\ \vdots \\ \delta_d n \end{bmatrix}. \tag{8.20}$$

Then for $1 \leq i \leq d$,

$$a_i = \delta_i n - (a_{i,1}\alpha_1 + a_{i,2}\alpha_2 + \ldots + a_{i,k}\alpha_k) \tag{8.21}$$

and therefore b_i must divide a_i for each $1 \leq i \leq d$. Hence, $\underline{x}^{\underline{a}} \in S_n$. ■

Example 8.0.1. Note that even if each $b_i = 1$, then we may have a proper inclusion of $I_{\{1\}} \subsetneq m^n$. Let $R = k[x, y, z]$. Consider $I = (x^5, y^5, z^5, xyz^3, x^2y^2z)$. By Remark 8.0.3,

$$I_{\{1\}} = I + \left(x^3y^3, x^4z^2, y^4z^2, y^3z^4, x^3z^4\right) \subsetneq m^5. \tag{8.22}$$

Notice also that this shows the first coefficient ideal of an m-primary monomial ideal generated in degree n may have generators in degrees greater than n.

The next result characterizes when the first coefficient ideal coincides with the integral closure for m-primary monomial ideals generated in degree n.

Theorem 8.0.5. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k with $d \geq 2$. Let $m = (x_1, \ldots, x_d)$. Let I be an m-primary monomial ideal generated in degree n. Let A denote the matrix whose columns are the exponent vectors of monomial generators of I of degree n excluding the exponent vectors associated to $x_1^n, \ldots, x_d^n$. Let A_{d-1} denote the submatrix of A consisting of the first $d-1$ rows of A. Let $B_1, \ldots, B_k$ denote the $d-1$ by $d-1$ submatrices of A_{d-1}. Then $\gcd(|B_1|, \ldots, |B_k|, n) = 1$ if and only if $I_{\{1\}} = \overline{I} = m^n$. In particular, if I is generated by fewer than $2d-1$ elements, then $I_{\{1\}} \subsetneq m^n$.*

Proof. We first prove that if $\gcd(|B_1|, \ldots, |B_k|, n) = 1$, then $I_{\{1\}} = \overline{I} = m^n$.

There exist integers $\alpha_1, \ldots, \alpha_l, \beta$ such that

$$\sum_{i=1}^{k} \alpha_i \, |B_i| + \beta n = 1. \tag{8.23}$$

Without loss of generality, we may assume that $|B_i|$ is nonzero for each i.

Let $x_1^{a_1} \ldots x_d^{a_d} \in m^n$ with $\sum_{i=1}^{d} b_i = n$. We show that $x_1^{a_1} \ldots x_d^{a_d} \in I_{\{1\}}$. By Proposition 8.0.1, it suffices to show that there exists a monomial of $I^k \setminus mI^k$ which multiplies $x_1^{a_1} \ldots x_d^{a_d}$ into I^{k+1} for some k.

To do so, we show that there exists a vector v with nonnegative entries and nonnegative integers $\delta_1, \ldots, \delta_{d-1}$ such that

$$A_{d-1} \, v = \begin{bmatrix} \delta_1 n - a_1 \\ \delta_2 n - a_2 \\ \vdots \\ \delta_{d-1} n - a_{d-1} \end{bmatrix}.$$

For each B_i with $|B_i| \neq 0$, we notice that the equation

$$B_i \, x_i = \begin{bmatrix} -a_1 \\ \vdots \\ -a_{d-1} \end{bmatrix}$$

has a solution given by

$$x_i = \frac{1}{|B_i|} \operatorname{adj}(B_i) \begin{bmatrix} -a_1 \\ \vdots \\ -a_{d-1} \end{bmatrix}.$$

Let $y_i = |B_i| \, x_i$. Then

$$B_i \, y_i = |B_i| \begin{bmatrix} -a_1 \\ \vdots \\ -a_{d-1} \end{bmatrix}. \tag{8.24}$$

We may extend y_i to $\widetilde{y_i}$ by inserting zeros into entries corresponding to columns of A_{d-1} not included in B_i. Then one has

$$A_{d-1}\,\widetilde{y_i} = B\,y_i = |B_i| \begin{bmatrix} -a_1 \\ \vdots \\ -a_{d-1} \end{bmatrix}. \tag{8.25}$$

Then

$$\begin{aligned} A_{d-1}(\sum_{i=1}^{k} \alpha_i \widetilde{y_i}) &= (\sum_{i=1}^{k} \alpha_i\,|B_i|) \begin{bmatrix} -a_1 \\ \vdots \\ -a_{d-1} \end{bmatrix} \\ &= (1-\beta n) \begin{bmatrix} -a_1 \\ \vdots \\ -a_{d-1} \end{bmatrix} \\ &= \begin{bmatrix} \beta a_1 n - a_1 \\ \vdots \\ \beta a_{d-1} n - a_{d-1} \end{bmatrix}. \end{aligned}$$

Notice that if any entry of the vector $\sum_{i=1}^{l} \alpha_i \widetilde{y_i}$ is negative, one may add n sufficiently many times to any such entry yielding a vector z with nonnegative entries. This vector z is a solution to

$$A_{d-1}\,z = \begin{bmatrix} \delta_1 n - a_1 \\ \delta_2 n - a_2 \\ \vdots \\ \delta_{d-1} n - a_{d-1} \end{bmatrix}, \tag{8.26}$$

where the vector $\delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_{d-1} \end{bmatrix}$ is the sum of vectors $\begin{bmatrix} \beta a_1 \\ \vdots \\ \beta a_{d-1} \end{bmatrix}$ and positive scalar multiples of columns of A_{d-1}. Since A_{d-1} and z have all nonnegative entries, so does

$$\begin{bmatrix} \delta_1 n - a_1 \\ \delta_2 n - a_2 \\ \vdots \\ \delta_{d-1} n - a_{d-1} \end{bmatrix}.$$

Therefore, each δ_i is nonnegative.

Let

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_l \end{bmatrix}.$$

Let the i^{th} column of A be the exponent vector of the generator $x_1^{a_{1,i}} \dots x_d^{a_{d,i}}$. By Eq. (8.26), we have that

$$a_i + a_{i,1} z_1 + a_{i,2} z_2 + \cdots + a_{i,l} z_l = \delta_i n$$

for $1 \leq i \leq d-1$. This directly implies that

$$(x_1^{a_1} \cdots x_{d-1}^{a_{d-1}})(x_1^{a_{1,1}} \cdots x_{d-1}^{a_{d-1,1}})^{z_1} \cdots (x_1^{a_{1,l}} \cdots x_{d-1}^{a_{d-1,l}})^{z_l} = (x_1^n)^{\delta_1} \cdots (x_{d-1}^n)^{\delta_{d-1}} . \tag{8.27}$$

Notice that

$$(x_1^{a_1} \cdots x_d^{a_d})(x_1^{a_{1,1}} \cdots x_d^{a_{d,1}})^{z_1} \cdots (x_1^{a_{1,l}} \cdots x_d^{a_{d,l}})^{z_l} = (x_1^n)^{\delta_1} \cdots (x_{d-1}^n)^{\delta_{d-1}} (x_d)^w \tag{8.28}$$

for some $w \geq 0$. The monomial on the left-hand side has degree $n(1 + \alpha_1 + \ldots + \alpha_{d-1})$. Therefore, since the exponents of the $x_1, \ldots, x_{d-1}$ on the right-hand side are divisible by n, we conclude that w must be divisible by n. Let $w = \delta_d n$. Therefore, $z_1 + \ldots + z_l + 1 = \delta_1 + \ldots + \delta_d$. This shows that there is a monomial in $I^{z_1+\ldots+z_l} \setminus mI^{z_1+\ldots+z_l}$ which multiplies $x_1^{a_1} \ldots x_d^{a_d}$ into $I^{z_1+\ldots+z_l+1}$. Therefore, $x_1^{a_1} \ldots x_d^{a_d} \in I_{\{1\}}$ and hence $I_{\{1\}} = m^n$.

We now show the reverse implication. Let A denote the matrix whose columns are the exponent vectors associated to the generators of I other than $x_1^n, \ldots, x_d^n$. If I has fewer than $2d-1$ generators, append zero vectors to make A a d by $d-1$ matrix. Let A_{d-1} denote the submatrix of A consisting of the first $d-1$ rows. Let $B_1, \ldots, B_k$ be the $d-1$ by $d-1$ submatrices of A_{d-1}. Suppose $\gcd(|B_1|, \ldots, |B_k|, n) \neq 1$.

We have two cases: $\operatorname{rank}(A_{d-1}) < d-1$ or $\operatorname{rank}(A_{d-1}) = d-1$ as matrices over $\mathbb{Q}$.

Case 1: Assume $\operatorname{rank}(A_{d-1}) < d-1$. Then the column space of A contains fewer than $d-1$ independent vectors. We show that $I_{\{1\}}$ does not contain at least one of $x_1 x_d^{n-1}, \ldots, x_{d-1} x_d^{n-1}$ so that $I_{\{1\}} \subsetneq m^n$. Suppose toward contradiction that for all $1 \leq i \leq d-1$, $x_i x_d^{n-1} \in I_{\{1\}}$. Let $I = (x_1^n, \ldots, x_d^n, v_1, \ldots, v_s)$ with v_i monomials of degree n. By Proposition 8.0.1, this implies that there exist nonnegative integers $\alpha_{i,1}, \ldots, \alpha_{i,s}$ such that

$$v_1^{\alpha_{i,1}} \cdots v_s^{\alpha_{i,s}} x_i x_d^{n-1} \in (x_1^n, \ldots, x_d^n)^{\sum_{j=1}^{s} \alpha_{i,j}+1}. \tag{8.29}$$

This implies the existence of vectors α_i and δ_i in $\mathbb{N}^{d-1}$ such that

$$A_{d-1} \cdot \alpha_i = n\delta_i - e_i \tag{8.30}$$

where e_i denotes the standard basis vector in $\mathbb{Z}^{d-1}$ with 1 in the i^{th} component and 0 elsewhere.

Therefore, for $1 \leq i \leq d-1$, $n\delta_i - e_i$ is in the column space of A_{d-1}.

Let C denote the $d-1$ by $d-1$ matrix with vectors $n\delta_i - e_i$ as columns. For any prime divisor p of n, over $\mathbb{Z}/(p)$,

$$C = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & & \cdots & \\ 0 & 0 & \cdots & -1 \end{bmatrix}$$

and $\det_{\mathbb{Z}/(p)}(C) = (-1)^{d-1}$. Therefore, as a matrix over $\mathbb{Q}$, the determinant of C is nonzero. This implies that the columns of C are linearly independent and the rank of C is $d-1$, a contradiction. Therefore, for some $1 \leq j \leq d-1$, $x_j x_d^{n-1} \notin I_{\{1\}}$. Hence $I_{\{1\}} \subsetneq m^n$.

Case 2: Assume $\mathrm{rank}(A_{d-1}) = d-1$. Suppose toward contradiction that $x_1x_d^{n-1}, \ldots, x_{d-1}x_d^{n-1}$ are elements of $I_{\{1\}}$. Let M denote a $d-1$ by $d-1$ submatrix of A_{d-1} whose columns generate the column space of A_{d-1} over $\mathbb{Z}$. Then $\gcd(\det(M), n)$ is not one by assumption. Therefore, there is a prime p that divides $\det(M)$ and n. As a matrix over $\mathbb{Z}/(p)$, $\det(M) = 0$. Let C be defined as in Case 1. Since we are assuming $x_ix_d^{n-1} \in I_{\{1\}}$ for $1 \leq i \leq d-1$, the columns of C are contained in the column space of A_{d-1} and hence of M. But since the $d-1$ columns of C are linearly independent over $\mathbb{Z}/(p)$ and $\mathrm{rank}_{\mathbb{Z}/(p)}(M) < d-1$, this is not possible. Therefore, for some $1 \leq j \leq d-1$, $x_jx_d^{n-1} \notin I_{\{1\}}$. Hence $I_{\{1\}} \subsetneq m^n$. ■

We will see an application of Theorem 8.0.5 to the core of an ideal in the next chapter. However, now we can state a consequence of Theorem 8.0.5 regarding the behavior of the first coefficient ideal under specialization by general elements.

Corollary 8.0.6. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k of characteristic zero with $d \geq 2$. Let I be an m-primary monomial ideal generated in degree n. Let x be a general element of I. Let A be the matrix whose columns are the exponent vectors of monomial generators of I of degree n excluding the exponent vectors associated to $x_1^n, \ldots, x_d^n$. Let A_{d-1} denote the submatrix of A consisting of the first $d-1$ rows. Let $B_1, \ldots, B_k$ denote the $d-1$ by $d-1$ submatrices of A_{d-1}. If $\gcd(|B_1|, \ldots, |B_k|, n) = 1$, then $I_{\{1\}}/(x) = (I/(x))_{\{1\}}$.*

Proof. This is an immediate consequence of Theorem 8.0.5 and Corollary 6.0.4. ■

Remark 8.0.7. As in Remark 7.1.3, we can use Proposition 6.1.1 to produce a larger class of ideals whose first coefficient ideal is equal to its integral closure.

Let J be an m-primary monomial ideal generated in degree n for which $J_{\{1\}} = \overline{J} = m^n$. Let K be any ideal contained in m^n.

By the argument in Remark 7.1.3, $I = J + K$ has first coefficient ideal equal to its integral closure.

Example 8.0.2. Let $J = (x^6, y^6, z^6, xyz^4, x^2yz^3)$. Then with notation as in Theorem 8.0.5,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 4 & 3 \end{bmatrix}. \tag{8.31}$$

The 2 by 2 minor $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = -1$, hence is relatively prime to 6. Therefore $J_{\{1\}} = \overline{J} = m^n$ by Theorem 8.0.5.

Moreover, by Remark 8.0.7, the ideal $I = J + (x^2z^5, y^3z^4)$ is a monomial ideal such that $I_{\{1\}} = \overline{I} = m^n$.

9. THE CORE AND THE FIRST COEFFICIENT IDEAL

The core of an ideal I is the intersection of all reductions of I. The core of I is in general difficult to compute, and it is desirable to know when it is equal to an adjoint ideal, since the adjoint ideal has an explicit combinatorial description. In this section, we classify when the core of an ideal I is equal to an adjoint ideal for 0-dimensional monomial ideals generated in one degree by computing first coefficient ideals.

9.1 Background

Definition 9.1.1. *Let R be a ring and I an ideal of R. An ideal J such that $J \subseteq I$ is called a* reduction *of I if $JI^k = I^{k+1}$ for some integer $k \geq 0$. A reduction is said to be minimal if it is minimal with respect to containment.*

If R is Noetherian or I is finitely generated, then $J \subseteq I$ is a reduction if and only if I is integral over J. Therefore, if J is a reduction of I, one has

$$J \subseteq I \subseteq \overline{I}, \tag{9.1}$$

where I is integral over J and $\overline{I}$ is integral over I. However, unlike the integral closure of an ideal, a reduction of an ideal is highly non-unique.

Theorem 9.1.1 ([NR54, Theorem 1])**.** *Let (R, m) be a Noetherian local ring with infinite residue field. Then any ideal generated by $\ell(I)$ general elements of I is a minimal reduction of I.*

This theorem indicates that even when minimal reductions are known to exist, there are usually infinitely many minimal reductions. In order to find a unique subideal of the ideal I to act as an analogue of the integral closure of the ideal I, the core of I was defined by Sally and Rees in [RS88].

Definition 9.1.2. *The* core *of an ideal I is the intersection of all reductions of I.*

Since the core is by definition an intersection of possibly infinitely many ideals, it is in general difficult to compute. Much work has been done to understand what the core is in local rings and to a lesser extent in graded rings.

We are primarily interested in the classifying when the core of I coincides with an adjoint ideal. We now give some background on adjoint ideals.

9.2 Adjoint Ideals

In order to define the adjoint ideal, we first define valuations, valuation rings and divisorial valuations.

Definition 9.2.1. *Let K be a field. A* valuation *on K is a group homomorphism V from the multiplicative group $K \setminus \{0\}$ to a totally ordered abelian group G such that for all $x, y \in K$,*

$$v(x+y) \geq \min\{v(x), v(y)\}.$$

Definition 9.2.2. *Let K be a field. A domain V with field of fractions K is called a* valuation ring *if for every nonzero $x \in K$, $x \in V$ or $x^{-1} \in V$.*

A valuation ring is local with unique maximal ideal $m_V = \{x \in V \mid x = 0 \text{ or } x^{-1} \notin V\}$. Given a valuation, one has a valuation ring $R_v = \{x \in K^* \mid v(x) \geq 0\} \cup \{0\}$.

By [SH06, Proposition 6.2.3], every valuation ring comes from a valuation.

Definition 9.2.3. *Let R be a Noetherian domain with field of fractions K. Let (V, m_V) be a valuation ring of K containing R and let $p = m_V \cap R$. If $\operatorname{trdeg}_{\kappa(p)} \kappa(m_V) = \operatorname{ht} p - 1$, then V is said to be a* divisorial valuation ring of K with respect to R.

If R is a Noetherian domain which is locally analytically unramified, then every divisorial valuation V with respect to R is essentially of finite type over R. See [SH06, Theorem 9.3.2].

Next, we record the definition of an adjoint ideal due to Lipman ([Lip94]).

Definition 9.2.4. *Let R be a regular domain. Let $D(R)$ denote the set of divisorial valuations with respect to R. Let $J_{V/R}$ denote the Jacobian ideal of V with respect to R. Then the adjoint ideal of an R-ideal I is*

$$\operatorname{adj}(I) = \bigcap_{V \in D(R)} \{r \in R \mid rJ_{V/R} \subseteq IV\}. \tag{9.2}$$

The integral closure of I also has a description in terms of divisorial valuations.

Proposition 9.2.1 ([SH06, Proposition 6.8.2])**.** *Let R be a Noetherian domain. Let $D(R)$ denote the set of divisorial valuations with respect to R. Then*

$$\overline{I} = \bigcap_{V \in D(R)} \{r \in R \mid r \in IV\}.$$

From this description of the integral closure, one immediately sees that $\overline{I} \subseteq \operatorname{adj}(I)$. Furthermore, the adjoint ideal is itself integrally closed.

Craig Huneke and Irena Swanson in [HS95] were the first to obtain a description of the core for a large class of ideals by relating it to an adjoint ideal.

Theorem 9.2.1 ([HS95, Theorem 3.14])**.** *Let (R, m) be a regular local ring of dimension 2 with an infinite residue field. Let I be an integrally closed m-primary ideal. Then* $\operatorname{core}(I) = \operatorname{adj}(I^2)$.

The result above does not generalize to higher dimensions. Kohlhaas in [Koh10] gives an example of an integrally closed ideal m-primary ideal in a 3-dimensional regular ring such that $\operatorname{adj}(I^3) \subsetneq \operatorname{core}(I)$.

However, it has been shown by Lipman in ([Lip94]) that the following containment is true for arbitrary ideals in regular domains.

Theorem 9.2.2. *Let R be a regular domain of dimension d and I an R-ideal. Then* $\operatorname{adj}(I^d) \subseteq \operatorname{core}(I)$.

The Briançon-Skoda Theorem implies that $\overline{I^d} \subseteq \operatorname{core}(I)$, and Lipman's theorem is a strengthening of this fact since

$$\overline{I^d} \subseteq \operatorname{adj}(I^d) \subseteq \operatorname{core}(I)\,.$$

Hence $\operatorname{adj}(I^d)$ is an integrally closed ideal which is closer to the core of I.

9.3 Cores of Monomial Ideals

Definition 9.3.1. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k. Let I be a monomial ideal of R. We define the* exponent set *of I to be*

$$\Gamma(I) = \{(a_1, \ldots, a_d) \mid x_1^{a_1} \cdots x_d^{a_d} \in I\}\,. \tag{9.3}$$

Definition 9.3.2. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k. Let I be a monomial ideal. The* Newton polyhedron *of I, denoted $NP(I)$ is the convex hull in $\mathbb{Q}^d$ or $\mathbb{R}^d$ of the exponent set of I.*

It is well known that the integral closure of a monomial ideal is determined by its Newton polyhedron.

Proposition 9.3.1 ([SH06, Proposition 1.4.6])**.** *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k. The exponent set of $\overline{I}$ consists of all integer lattice points in the Newton polyhedron of I.*

If I is a monomial ideal, the adjoint ideal of Lipman can be described in terms of the interior of the Newton polyhedron of I, denoted $\mathrm{NP}^o(I)$.

Theorem 9.3.1 ([How01, Main Theorem],[HS08, Theorem 4.1])**.** *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k. Let I be a monomial ideal. The adjoint ideal of I is*

$$\operatorname{adj}(I) = (\underline{x}^{\underline{a}} \mid \underline{a} + (1, 1, \ldots, 1) \in NP^o(I))\ .$$

By Lipman's result, we know that $\operatorname{adj}(I^d) \subseteq \operatorname{core}(I)$. It is desirable to know when this containment is an equality, which would give a nice combinatorial description of the core. We now review the known results about when the core of a monomial ideal coincides with the adjoint ideal of I^d.

The following theorem says that if I is a 0-dimensional monomial ideal with a reduction generated by a regular sequence of monomials, then $\operatorname{core}(I) = \operatorname{adj}(I^d)$ if powers of I are close to being integrally closed.

We note that if J is a reduction of I, then the reduction number of J with respect to I, denoted $r_J(I)$, is the smallest integer such that $JI^k = I^{k+1}$.

Theorem 9.3.2 ([PUV07, Theorem 4.11])**.** *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over an infinite field k. Let I be a 0-dimensional monomial ideal with a reduction J generated by a regular sequence of monomials. Assume the characteristic of k is zero, greater than $r_J(I)$ or I is generated by monomials of the same degree. Let $J^{\langle t+1\rangle}$ denote the ideal generated by $t+1$ powers of minimal monomial generators of J. If $\overline{I^{dt}} \subseteq \left(I^{dt}, J^{\langle t+1\rangle}\right)$ for some $t \geq \max\{r_J(I), d-1\}$, then* $\operatorname{core}(I) = \operatorname{adj}(I^d)$.

The next result of Polini, Ulrich and Vitulli gives two classes of ideals for which $\operatorname{core}(I) = \operatorname{adj}(I^d)$.

Theorem 9.3.3 ([PUV07, Corollary 6.6],[PUV07, Corollary 7.11])**.** *(i.) Let $R = k[x, y]$ with k an infinite field. Let*

$$I = \left(x^n, y^n, x^{n-k_1}y^{k_1}, \ldots, x^{n-k_s}y^{k_s}\right)$$

with $\gcd(k_1, \ldots, k_s, n) = 1$. *Then* $\operatorname{core}(I) = \operatorname{adj}(I^2)$.

(ii.) Let $R = k[x, y, z]$ with k an infinite field. Let

$$I = \left(x^n, y^n, z^n, \{x^{n-k_i}y^{k_i}\}, \{x^{n-l_i}z^{l_i}\}, \{y^{n-m_i}z^{m_i}\}\right)$$

with $\gcd(n, k_i, l_i) = 1$, $\gcd(n, k_i, m_i) = 1$, *and* $\gcd(n, l_i, m_i) = 1$. *Then* $\operatorname{core}(I) = \operatorname{adj}(I^3)$.

Kohlhaas in [Koh10] has given equivalent conditions to $\mathrm{core}(I) = \mathrm{adj}(I^d)$ with the assumption that I is a 0-dimensional monomial ideal with a reduction generated by a regular sequence of monomials.

Theorem 9.3.4 ([Koh10, Theorem 4.1.3])**.** *Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field k of characteristic zero. Let I be a 0-dimensional monomial ideal with a reduction generated by a regular sequence of monomials. Then the following are equivalent:*

(i.) $\mathrm{core}(I) = \mathrm{adj}(I^d)$

(ii.) $\mathrm{core}(I)$ *is integrally closed.*

(iii.) $R[It]$ *has Serre's condition* R_1 *.*

In this chapter, we will give an additional equivalence with the additional assumption that I is generated in degree n. Notice that when I is a 0-dimensional monomial ideal generated in degree n, then the condition that I has a reduction generated by a regular sequence of monomials is satisfied. In this case, I contains $(x_1^n, \dots, x_d^n)$ and $I \subseteq \overline{(x_1^n, \dots, x_d^n)}$, and hence $(x_1^n, \dots, x_d^n)$ is a reduction of I generated by a regular sequence of monomials.

In order to obtain an additional equivalence, we need the following result which relates the core to the first coefficient ideal of I.

Theorem 9.3.5 ([PUV07, Corollary 4.9(b)])**.** *Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field k of characteristic 0. Let I be a 0-dimensional monomial ideal with a reduction generated by a regular sequence of monomials. Then $I_{\{1\}}$ is the largest ideal containing I and integral over I for which* $\mathrm{core}(I) = \mathrm{core}(I_{\{1\}})$.

Remark 9.3.6. Note that by [PUV07, Corollary 4.9], we only need k infinite to see that $\mathrm{core}(I) = \mathrm{core}(I_{\{1\}})$.

Polini, Ulrich and Vitulli also computed the core for powers of the homogeneous maximal ideal, which we now state.

Proposition 9.3.2 ([PUV07, Proposition 5.2])**.** *Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over an infinite field. Let $m = (x_1, \dots, x_d)$ denote the homogeneous maximal ideal. Then* $\mathrm{core}(m^n) = m^{dn-(d-1)}$.

We now combine the results above to give a corollary to Theorem 8.0.5.

Corollary 9.3.7. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k of characteristic zero. Let $m = (x_1, \ldots, x_d)$ denote the homogeneous maximal ideal of R. Let I be an m-primary monomial ideal generated in degree n. Let A denote the matrix whose columns are the exponent vectors associated to monomial generators of degree n of I excluding exponent vectors associated to $x_1^n, \ldots, x_d^n$. Let A_{d-1} denote the submatrix of A consisting of the first $d-1$ rows of A. Let $B_1, \ldots, B_k$ denote the $d-1$ by $d-1$ submatrices of A_{d-1}.*

Then the following are equivalent:

(i) $\gcd(|B_1|, \ldots, |B_k|, n) = 1$

(ii) $\operatorname{core}(I) = \operatorname{adj}(I^d)$

Proof. By Theorem 9.3.1,

$$\operatorname{adj}(I^d) = \left(x_1^{a_1} \cdots x_d^{a_d} \mid (a_1, \ldots, a_d) + (1, \ldots, 1) \in NP^o(I^d)\right). \tag{9.4}$$

Since I^d contains $(x_1^{dn}, \ldots, x_d^{dn})$, the Newton polyhedron of I^d contains all lattice points associated to monomials of degree dn. Hence $\underline{a}+\underline{1} \in \mathrm{NP}^o(I^d)$ if and only if $\sum_{i=1}^d (a_i+1) > dn$, which is equivalent to $\sum_{i=1}^d a_i \geq dn - (d-1)$. Therefore, $\operatorname{adj}(I^d) = m^{dn-(d-1)}$.

We now prove the equivalence of (i) and (ii).

$(i) \implies (ii)$: Suppose A has a $d-1$ by $d-1$ minor relatively prime to n. By Theorem 8.0.5, $I_{\{1\}} = m^n$. Hence,

$$\begin{aligned}
\operatorname{core}(I) &= \operatorname{core}(I_{\{1\}}) && (9.5)\\
&= \operatorname{core}(m^n) && (9.6)\\
&= m^{dn-(d-1)} && (9.7)\\
&= \operatorname{adj}(I^d) && (9.8)
\end{aligned}$$

where Eq. (9.5) follows from Theorem 9.3.5, Eq. (9.7) follows from Proposition 9.3.2, and Eq. (9.8) follows from the argument above.

$(ii) \implies (i)$: Suppose that A does not have a $d-1$ by $d-1$ minor relatively prime to n. By Theorem 8.0.5, $I_{\{1\}} \subsetneq m^n$. By Theorem 9.3.5, since $I_{\{1\}}$ is the largest ideal containing I and integral over I for which $\operatorname{core}(I) = \operatorname{core}(I_{\{1\}})$, we conclude that

$$\operatorname{core}(I) \neq \operatorname{core}(m^n) = m^{dn-(d-1)} = \operatorname{adj}(I^d)\,.$$

■

By [Koh10, Theorem 4.1.3], we immediately get the following corollary.

Corollary 9.3.8. *Let $R = k[x_1, \ldots, x_d]$ be a polynomial ring over a field k of characteristic zero. Let $m = (x_1, \ldots, x_d)$ denote the homogenous maximal ideal. Let I be an m-primary monomial ideal generated in degree n. Let A denote the matrix whose columns are exponent vectors of monomial generators of degree n of I other than exponent vectors associated to $x_1^n, \ldots, x_d^n$. Let A_{d-1} denote the submatrix of A consisting of the first $d-1$ rows of A. Let $B_1, \ldots, B_k$ denote the $d-1$ by $d-1$ submatrices of A_{d-1}. Then the following are equivalent:*

(i) $\gcd(|B_1|, \ldots, |B_k|, n) = 1$

(ii) *$R[It]$ satisfies Serre's condition R_1.*

We now give a few examples demonstrating how Corollary 9.3.7 and Corollary 9.3.8 can be used to detect whether some Rees algebras have Serre's condition R_1 or how to build examples of Rees algebras with or without Serre's condition R_1 which are not normal.

Example 9.3.1. Let $R = k[x, y, z]$, with k a field of characteristic zero. Let $I = (x^5, y^5, z^5, xyz^3, x^2y^2z)$. Using the notation of Corollary 9.3.7,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 3 & 1 \end{bmatrix}.$$

Notice that the 2 by 2 minors are 0 or -5, and hence $\operatorname{core}(I) \neq \operatorname{adj}(I^3)$ by Corollary 9.3.7. By Corollary 9.3.8, $R[It]$ does not satisfy R_1.

Example 9.3.2. Let $R = k[x, y, z]$ with k a field of characteristic zero. Let $I = (x^5, y^5, z^5, xyz^3, x^2yz^2)$. Using the notation of Corollary 9.3.7,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 2 \end{bmatrix}.$$

Notice that $\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$ is relatively prime to 5, and hence by Corollary 9.3.7, $\operatorname{core}(I) = \operatorname{adj}(I^3)$. By Corollary 9.3.8, $R[It]$ satisfies R_1. Notice that $R[It]$ is not normal since I is not integrally closed.

Example 9.3.3. Let $R = k[x_1, \ldots, x_d]$ with k a field of characteristic zero. Let

$$I = \left(x_1^n, x_2^n, \ldots, x_d^n, x_1x_d^{n-1}, x_2x_d^{n-1}, \ldots, x_{d-1}x_d^{n-1}\right). \tag{9.9}$$

Then

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ n-1 & n-1 & n-1 & \cdots & n-1 \end{bmatrix}. \tag{9.10}$$

By Corollary 9.3.7 and Corollary 9.3.8, $\operatorname{core}(I) = \operatorname{adj}(I^d)$ and $R[It]$ satisfies R_1.

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