SELF-INTERACTING RANDOM WALKS AND RELATED BRANCHING-LIKE PROCESSES

by

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ABSTRACT

In this thesis we study two different types of self-interacting random walks. First, we study excited random walk in a deterministic, identically-piled cookie environment under the constraint that the total drift δ contained in the cookies at each site is finite. We show that the walk is recurrent when $|\delta| < 1$ and transient when $|\delta| > 1$. In the critical case $|\delta| = 1$, we show that the walk is recurrent under mild assumptions on the environment. We also construct an environment where the total drift per site is 1, but in which the walk is transient. This behavior was not present in previously-studied excited random walk models.

Second, we study the "have your cookie and eat it" random walk proposed by Pinsky, who already proved criteria for determining when the walk is recurrent or transient and when it is ballistic. We establish limiting distributions for both the hitting times and position of the walk in the transient regime which, depending on the environment, can be either stable or Gaussian.

1. INTRODUCTION

In this chapter, we will summarize the main results of this dissertation and outline its structure. We will primarily be concerned with studying different facets of the long-term behavior of two different types of self-interacting random walks. Both are discrete stochastic processes on $\mathbb{Z}$, and in both cases the evolution of the process is influenced by the actions that the walker takes as they move around their environment.

The first part of this thesis is concerned with excited random walk, which we will describe now. First, we select a cookie environment $\omega \in \Omega = [0,1]^{\mathbb{Z}\times\mathbb{N}}$, which is comprised of a collection of cookie strengths $\omega(x, \mathrm{j})$. Once a we have fixed a cookie environment, a walker is released into it. When the walker arrives at site x for the jth time, they consume the jth cookie at that site, then step right with probability $\omega(x, \mathrm{j})$ and left with probability $1 - \omega(x, \mathrm{j})$.

Since their introduction by Benjamini and Wilson [4], excited random walks have been studied extensively. Because the model is so general, in order to prove results it is necessary to place restrictions on the types of cookie environments that we will consider. No matter what class of environments we consider, the first three questions we ask are always the same:

1. Under what conditions does excited random walk return infinitely often to its starting point, and when does it not? Put another way, when is excited random walk recurrent, and when is it transient?

2. When excited random walk is transient, when does it have a nonzero asymptotic speed?

3. What are the limiting distributions of excited random walk?

In Chapter 2, we will give a historical survey of the answers to these three questions in various types of cookie environments. One of the main difficulties in studying excited random walk is that the walk itself is not a Markov chain, and so the standard tools for analyzing random walks cannot be applied. The most fruitful approach in the literature has been to instead study two Markov chains that are associated to the walk called *branching-like processes*. This approach is classical, dating back at least to Kesten, Kozlov, and Spitzer's 1975 study of limiting distributions for random walk in random environment [14], which is a special case of excited random walk. Chapter 2 also contains a thorough introduction to the branching-like processes and a discussion of how they have been used in the literature to answer the above three questions.

In Chapter 3, we introduce one of the main objects of our study: *finite-drift cookie environments*, which are subject to the constraint that

$$\delta = \lim_{n\to\infty} \sum_{\mathrm{j}=1}^{n} (2\omega(x,\mathrm{j}) - 1)$$

exists and is finite. Our first main result is Theorem 3.2.1, where we prove that excited random walk in a finite-drift environment is recurrent when $|\delta| < 1$, transient when $|\delta| > 1$, and is recurrent when $|\delta| = 1$ and we have

$$\left| \sum_{\mathrm{j}=n}^{\infty} (2\omega(x,\mathrm{j}) - 1) \right| = o\left(\frac{1}{\log n}\right).$$

In Theorem 3.2.2, we show that excited random walk in a finite-drift environment can be transient even when $|\delta| = 1$. This behavior is interesting, because in previously-studied models excited random walk has been shown to be recurrent in the corresponding critical case. We end Chapter 3 with some discussion of why finite-drift environments admit this behavior, but other types of environments cannot.

In Chapter 4, we introduce our second self-interacting random walk: Pinsky's "have your cookie and eat it" random walk [23]. In this model, we place a single cookie of strength $p \in (0, 1)$ at each site in $\mathbb{Z}$. Whenever the walker arrives at a site with a cookie, they choose to step right with probability p and left with probability $1 - p$. The key difference between this walk and excited random walk comes in the next step: if the walker decides to step right, they do not consume the cookie at their present site (and on subsequent visits, it can be used again). However, if the walker decides to step left, they eat the cookie at their present site before they go. This constitutes a (somewhat mild) additional layer of self-interaction, since the environment that the walker experiences is not fixed before the walk begins.

Pinsky showed that the walk is transient when $p > 2/3$ and has positive limiting speed when $p > 3/4$. Our final main result, Theorem 4.2.1, identifies the limiting distributions of the “have your cookie and eat it” random walk when the walk is transient. We establish these limiting distributions using the approach that was pioneered by Kesten, Kozlov, and Spitzer in [14] and built upon by Kosygina and Mountford [15] and Kosygina and Peterson [16].

To describe the result, we will need to use the parameter $\alpha = \frac{2p-1}{1-p}$. Roughly speaking, when $p \in \left(\frac{2}{3}, \frac{5}{6}\right)$ the limiting distributions for the walk are transformations of stable laws with index $\alpha/2$, and when $p \geq 5/6$ the limiting distributions are Gaussian. The limiting distribution in the case $p = 3/4$ (where the walk transitions to positive speed) are especially delicate.

2. BACKGROUND

Excited random walk is a nearest-neighbor, self-interacting random walk on $\mathbb{Z}^d$ where the probability distribution that the walker uses to decide their next step from each site depends on the number of visits the walker has already made to that site. Excited random walks are often called *cookie random walks* due to the following intuitive interpretation: before the walk begins, a stack of cookies is placed at each site in $\mathbb{Z}^d$. When the walker visits a site for the jth time, they consumes the jth cookie in the stack at that site. Eating this cookie excites the walker, who chooses their next step according to a probability distribution encoded in that cookie.

We will primarily be concerned with the one-dimensional excited random walk in $\mathbb{Z}$. In that setting, a *cookie environment* ω is an element of $\Omega = [0,1]^{\mathbb{Z}\times\mathbb{N}}$. The number $\omega(x, \mathrm{j})$ is the *strength* of the jth cookie at site x, and upon eating this cookie the walker steps right with probability $\omega(x, \mathrm{j})$ and left with probability $1 - \omega(x, \mathrm{j})$.

Therefore, the excited random walk in the cookie environment ω is the stochastic process $\{X_n\}_{n\geq 0}$ such that $X_0 = 0$ and

$$P\left(X_{n+1} = X_n + 1 \mid X_0, \ldots, X_n\right) = \omega\left(X_n, \#\{\mathrm{j} \in \mathbb{N} : X_\mathrm{j} = X_n\}\right).$$

In this chapter, we will give a brief historical survey of results known about excited random walks and describe some of the tools that are used to prove them. Before doing so, it will be helpful to give some additional terminology. If a cookie does not give any bias to the walker's next step, i.e. $\omega(x, \mathrm{j}) = 1/2$, we will call that cookie a *placebo*. If there exists an $M \in \mathbb{N}$ such that every cookie in a stack beyond the Mth is a placebo, i.e. $\omega(x, \mathrm{j}) = 1/2$ whenever $\mathrm{j} > M$, we will say that there are M cookies at site x (imagining that the walker's default behavior is symmetric steps, and that they return to that behavior whenever cookies are not present).

2.1 Excited Random Walks on $\mathbb{Z}$

Excited random walk (ERW) in $\mathbb{Z}$ was first studied by Benjamini and Wilson [4]. The cookie environment in their original model consisted of a single cookie of strength $p \in (1/2, 1)$ at each site, and they were able to show by a short calculation that this type of ERW is always recurrent. Zerner [25] further generalized ERW to allow for any number of cookies at each site, and furthermore that the cookie stacks at each site could be chosen according to a probability distribution, subject to the constraint that $p_{\mathrm{j}}^x \in [1/2, 1]$ for all $\mathrm{j} \in \mathbb{N}$, $x \in \mathbb{Z}$, along with the mild condition that the first cookie in each stack does not have strength 1 almost surely, i.e. that $\mathbb{P}(p_1^x = 1) < 1$. Under these assumptions, Zerner identified a recurrence/transience criterion for the ERW. The key parameter for determining whether the walk will be recurrent or transient in this setting is the expected total drift δ, given by

$$\delta = \mathbb{E}\left[\sum_{\mathrm{j}=1}^{\infty}(2p_{\mathrm{j}}^x - 1)\right]. \tag{2.1}$$

We will refer to $2p_{\mathrm{j}}^x - 1$, which is the expected displacement of the walker after consuming a cookie of strength p_{j}^x, as the *drift* contained in the jth cookie at site x. When $p_{\mathrm{j}}^x > 1/2$ the drift contained in the cookie is positive, and so we will call such cookies *positive cookies*. Zerner proved that ERW in environments with only positive cookies is recurrent if the expected total drift $\delta \leq 1$ and transient to $+\infty$ when $\delta > 1$.

It is then natural to wonder about the asymptotic speed of a transient ERW:

$$\lim_{n\to\infty} \frac{X_n}{n},$$

assuming the limit exists. If this limit is nonzero, we will say that the ERW is *ballistic*. Basdevant and Singh [2] showed that ERW with finitely many positive cookies at each site is ballistic if and only if $\delta > 2$. Their approach, which relied on the analysis of branching processes associated to the walk, fundamentally changed the direction of research in ERW. Indeed, most of the results for ERW that followed this paper were established by studying these branching processes. We will discuss them in detail in the next section.

Kosygina and Zerner [18] later removed the assumption of positive cookies and established criteria for recurrence/transience and ballisticity, although they retained the assumption of finitely many cookies per site. In particular, they showed that ERW in environments with finitely many cookies per site is recurrent when $\delta \in [-1, 1]$, transient to $+\infty$ if $\delta > 1$, transient to $-\infty$ if $\delta < -1$, and ballistic if $|\delta| > 2$. Scaling limits for the "M cookies per site" model have also been established, in the nonballistic transient case $|\delta| \in (1, 2]$ by Basdevant and Singh [3], in the ballistic transient case $|\delta| \in (2, 4]$, by Kosygina and Mountford [15], and in the recurrent case $|\delta| \leq 1$ by Kosygina and Dolgopyat [8].

The natural next step was to study ERW with infinitely many cookies per site. Note that Zerner's recurrence/transience criterion for ERW in environments with only positive cookies [25] applies to environments with infinitely many positive cookies at each site, but his methods, which we discuss briefly in the next section, relied heavily on the assumption of positive cookies. Allowing cookie stacks that contain infinitely many positive and negative cookies leads to several complications that cause previous techniques and results to break down. For instance, the expected total drift at each site (2.1) may not even exist. ERW in environments that can have infinitely many positive and negative cookies have been studied by imposing special structure on the cookie stacks at each site, either by some form of regularity within the cookie stacks or by requiring cookie strengths to "taper off" as we look deeper in the stack.

In the track of "tapering off" cookie strengths, Chakthoun [5] studied cookie environments with an unbounded, but almost surely finite number of cookies at each site, subject to the constraints that δ (2.1) is well-defined and the (random) height $M(x)$ of the cookie stack at x satisfies $P(M(z) > n) \leq Cn^{-\alpha}$, where $\alpha > (|\delta| \vee 4)$. Chakthoun established that ERW in cookie environments that meet these conditions is recurrent when $|\delta| \leq 1$, and transient otherwise. We remark that the recurrence/transience criterion in this case, along with other results dealing with limit laws and functional limit theorems proved in [5], hold under the same conditions as in the case of a bounded number of cookies per site.

Along the lines of requiring regular structure within cookie stacks, Kozma, Orenshtein, and Shinkar [19] considered ERW in environments with *periodic cookie stacks*. To describe these cookie environments formally, fix $M \in \mathbb{N}$, and let $\boldsymbol{p} = (p_1, \ldots, p_M)$ be a vector of cookie strengths. Then the jth cookie at each site has strength p_{i}, where $\mathrm{j} \equiv \mathrm{i} \mod M$. Kosygina and Peterson [16] generalized the periodic model and studied ERW in environments where the cookie stacks at each site are generated by a Markov chain. In both cases, note that δ as defined in (2.1) may not exist. Recurrence/transience criteria are known for these models, but they are given in terms of parameters of the branching-like processes associated to ERW described in the next section. Although these parameters can be computed explicitly, their formulas are complicated.

2.2 Branching Processes and Branching-Like Processes

One of the main difficulties in dealing with ERW is the fact that the walk is not a Markov chain: the distribution of the walker's next step from a site depends on the number of times the walker has previously visited that site. Rather than studying the ERW directly, many of the known results for ERW were proven by analyzing two Markov chains that can be associated to ERW called *branching-like processes*. Basdevant and Singh [2] were the first to use branching processes in the study of ERW, and their technique quickly became the main tool for analyzing the long-term behavior of ERW.

2.2.1 Forward Branching-Like Process

In this section we will define the forward branching-like process (FBLP), which we will denote by $\{U_n\}_{n\geq 0}$, and explain its relationship to ERW. We can construct the FBLP and ERW from a single independent collection of Bernoulli random variables, which we will think of as coin tosses. To do this, fix a cookie environment $\boldsymbol{p} = \{p_{\mathrm{j}}^x\}_{\mathrm{j}\in\mathbb{N},\ x\in\mathbb{Z}}$, and let $\{\xi_{\mathrm{j}}^x\}_{\mathrm{j}\in\mathbb{N}, x\in\mathbb{Z}}$ be a collection of independent Bernoulli random variables, where $P(\xi_{\mathrm{j}}^x = 1) = p_{\mathrm{j}}^x$. For a fixed $x_0 \in \mathbb{Z}$, we can imagine the collection $\{\xi_{\mathrm{j}}^{x_0}\}_{\mathrm{j}\in\mathbb{N}}$ as a pile of coins sitting at $x_0 \in \mathbb{Z}$. We then "toss all the coins at each site" and record their outcomes, regarding $\xi_{\mathrm{j}}^x = 1$ as a “success” and $\xi_{\mathrm{j}}^x = 0$ as a “failure.” Then, we can construct the FBLP from this coin-tossing data: let

$$F_m^x = \inf\left\{k \in \mathbb{N} \ : \ \sum_{\mathrm{j}=1}^{k}(1-\xi_{\mathrm{j}}^x) = m\right\}, \qquad \mathcal{S}^x(m) = \sum_{\mathrm{j}=1}^{F_m^x} \xi_{\mathrm{j}}^x.$$

That is, F_m^x is the trial on which the mth failure occurs at site x, and so $S^x(m)$ counts the number of successes in the sequence of coin tosses at x before the mth failure in that sequence. The FBLP $\{U_n\}_{n\geq 0}$ is the Markov process on $\mathbb{N}_0$ with transition probabilities

$$P\left(U_n = k \mid U_{n-1} = \mathrm{j}\right) = P\left(\mathcal{S}^n(\mathrm{j}) = k\right), \qquad n \geq 1.$$

To explain why $\{U_n\}_{n\geq 0}$ is similar to a branching process, it is helpful to consider the FBLP which corresponds to ERW in the *cookieless* environment ω_0, which has $p_{\mathrm{j}}^x = 1/2$ for all $x \in \mathbb{Z}$ and all $\mathrm{j} \in \mathbb{N}$. We can regard U_n as modelling the number of organisms in the nth generation of a population. The offspring of these organisms, which will comprise generation $n+1$, can be generated from the coin-tossing data according to the following procedure. Suppose that $U_n = \mathrm{j}$. Then:

1. Count the number of successes in the sequence of coin tosses at site $n+1$ that occurred before the first failure in that sequence. This is the number of offspring that the first organism in generation n produces, which we will denote η_1. Note that in this case, $\eta_1 \sim \mathrm{Geo}(1/2)$.

2. Repeat this procedure for each of the remaining organisms in the nth generation, producing values for $\eta_2, \ldots, \eta_{\mathrm{j}}$.

3. The total number of successes in the coin tosses at site $n+1$ before the kth failure, given by $\mathcal{S}^{n+1}(\mathrm{j}) = \eta_1 + \cdots + \eta_{\mathrm{j}}$ corresponds to the total number of organisms in generation $(n+1)$. Since we assume $U_n = \mathrm{j}$, we have that $\mathcal{S}^{n+1}(\mathrm{j}) \sim \mathrm{NegBin}(\mathrm{j}, 1/2)$.

From this description, we see that $\{U_n\}_{n\geq 0}$ is in fact a Galton-Watson process with Geo(1/2) offspring distribution, and so in the case of ERW in ω_0 (which is a simple symmetric random walk), $\{U_n\}_{n\geq 0}$ truly is a branching process.

It is straightforward to construct an ERW in ω_0 from the coin-tossing data. To do so, we release a walker starting at 0. When the walker arrives at site x for the jth time, they step to $x+1$ if $\xi_{\mathrm{j}}^{x}=1$ and step to $x-1$ if $\xi_{\mathrm{j}}^{x}=0$. If we let Y_n denote the position of the walker after the nth step, then the path of the walker will be that of an ERW in ω_0 because $P(Y_n = Y_{n-1}+1|Y_0,\dots,Y_{n-1}) = P(\xi_{\mathrm{j}}^{x}=1) = p_{\mathrm{j}}^{x}$.

Intuitively, the FBLP tracks right excursions of the ERW from 0. For instance, if we set $U_0 = 1$, the walker steps right along the directed edge $(0 \to 1)$ and begins an excursion to the right. Each right step that the walker takes along the edge $(1 \to 2)$ before returning to 0 corresponds to a descendent of the initial right step along $(0 \to 1)$. Similarly, each right step along $(2 \to 3)$ before the first step along $(1 \leftarrow 2)$ corresponds to a descendent of the first right step along $(1 \to 2)$, and so on.

In addition, there is a natural correspondence between the path of the random walk and the tree that depicts the branching process. Given a tree diagram for $\{U_n\}_{n\geq 0}$, the corresponding walk path can be recovered by "tracing through" the vertices of the tree as if we were completing a depth-first search. We start from the root and move up and left along the edges of the tree whenever possible. While completing this procedure, a step "up the tree" corresponds to a right step in the walk, and a step "down the tree" corresponds to a left step. Figure 2.1 shows an example.

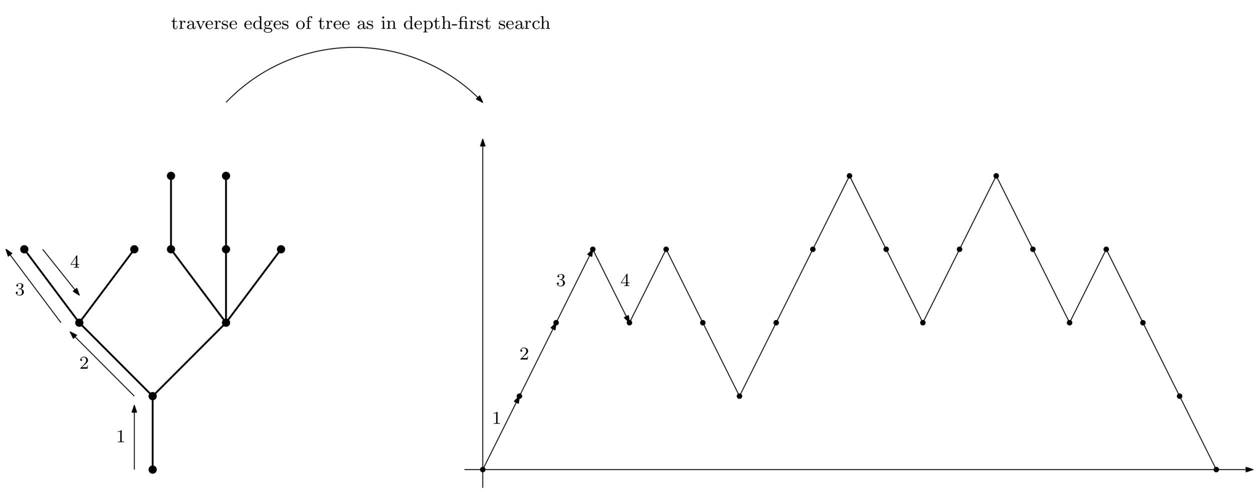

Figure 2.1. Recovering walk path from tree diagram

On the other hand, the tree diagram can be obtained from the random walk path by a "gluing" procedure. To carry it out, we imagine spreading glue on the underside of the walk path, squeezing the walk path together horizontally, and then pulling it apart again. If two edges with glue on them come into contact, they combine into a single edge. Figure 2.2 below shows an example of how to obtain the branching process's tree diagram from a walk path. For clarity, we have used colors to indicate which edges in the walk path are being glued together to form edges in the tree diagram. For instance, the blue edge in the tree diagram that connects to the root is obtained by combining together the first step of the excursion along $(0 \to 1)$ and the last step of the excursion along $(0 \leftarrow 1)$.

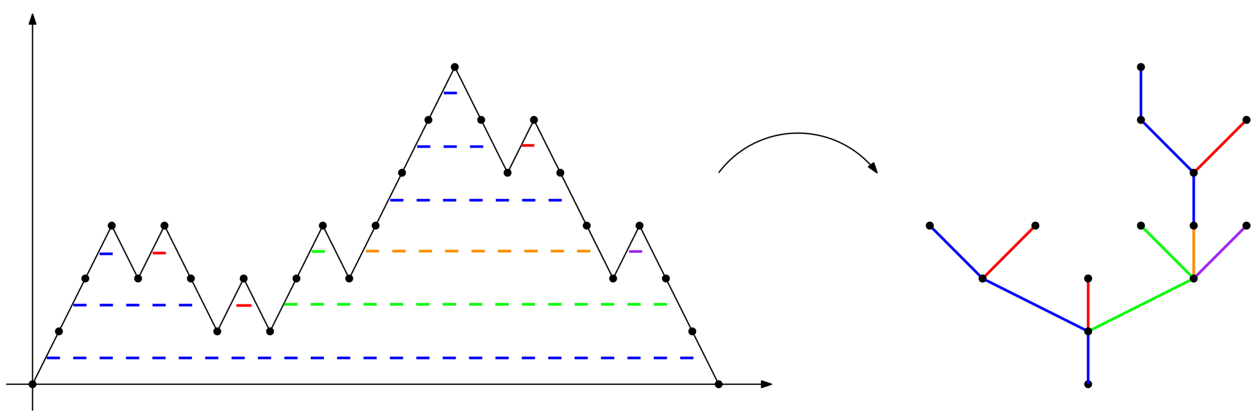

Figure 2.2. Gluing procedure that converts walk path to tree diagram

Having handled this construction in ω_0, we will now bridge the gap between that simple case and general ERW. A good first step is to consider ERW in the cookie environments studied by Basdevant and Singh [2], with M non-placebo cookies at each site. Denote by ω_M the cookie environment where $p_x^{\mathrm{j}} = p$ for all $x \in \mathbb{Z}$ and for all $\mathrm{j} \leq M$, and $p_x^{\mathrm{j}} = 1/2$ whenever $\mathrm{j} > M$. In this setting, only at most M of the organisms will use the unfair coin tosses to determine how many offspring they have (since each organism uses at least one coin toss). Therefore, we imagine the organisms reproducing according to a slightly modified procedure. Again, suppose that $U_n = \mathrm{j}$.

1. The first M organisms emigrate, taking the results of the first M coin tosses with them (if there are fewer than M organisms, they all leave). These M organisms reproduce

according to the procedure described above, supplementing with fair coin tosses as necessary.

2. The remaining $(M - \mathrm{j})_+$ organisms reproduce using the remaining coin tosses, and since they only use fair coin tosses, each has a $\eta_\mathrm{i} \sim \mathrm{Geo}(1/2)$ number of offspring, for $\mathrm{i} = 1, \ldots, (M - \mathrm{j})_+$

3. The offspring of the first M organisms, which we denote η^M, immigrate back to the population, joining the offspring of those that did not leave to form generation $n + 1$.

Then $\{U_n\}_{n\geq 0}$ is essentially a branching process, but with the additional feature that a random number of immigrants join the population in each generation. Such processes are called *branching processes with migration.*

It is more difficult to interpret $\{U_n\}_{n\geq 0}$ as a branching process if there are (potentially) infinitely many nonplacebo cookies at each site. Nevertheless, it will still be useful to think of $\{U_n\}_{n\geq 0}$ as a branching process to guide our intuition. To remind ourselves of this connection, we will refer to $\{U_n\}_{n\geq 0}$ as a *branching-like process.*

We close out this section by pointing out that there can be a slight difference in the value of U_n and the number of right steps the walk takes from n before returning to 0 which we will denote by R_n. However, $U_n = R_n$ for all n as long as $T_0 < \infty$. Formally, let $R_n = \sum_{\mathrm{j}=0}^{T_0-1} \mathbf{1}_{\{X_\mathrm{j}=n,\ X_{\mathrm{j}+1}=n+1\}}$. If $T_0 < \infty$, R_n will be the number of right steps that the walker takes from n before T_0, and if $T_0 = \infty$ then R_n is the total number of right steps that the walker takes from n.

Proposition 2.2.1. *If $T_0 < \infty$, then $U_n = R_n$ for all $n \in \mathbb{N}$. If $T_0 = \infty$, then $U_n \geq R_n$ for all $n \in \mathbb{N}$.*

The first part of the proposition is clear from the coin-toss construction of the FBLP and the walk, along with the correspondence between the FBLP trees and walk paths described above. The second part can be proved by induction, but to give intuition for when it happens that $R_n < U_n$ we will provide an example. Figure 2.3 below shows the coin tosses that are used to generate the FBLP and the path of the walk (right arrows indicate successes, left arrows indicate failures), along with the corresponding path of the ERW. In the given example, after the marked step, the walker never returns to 3.

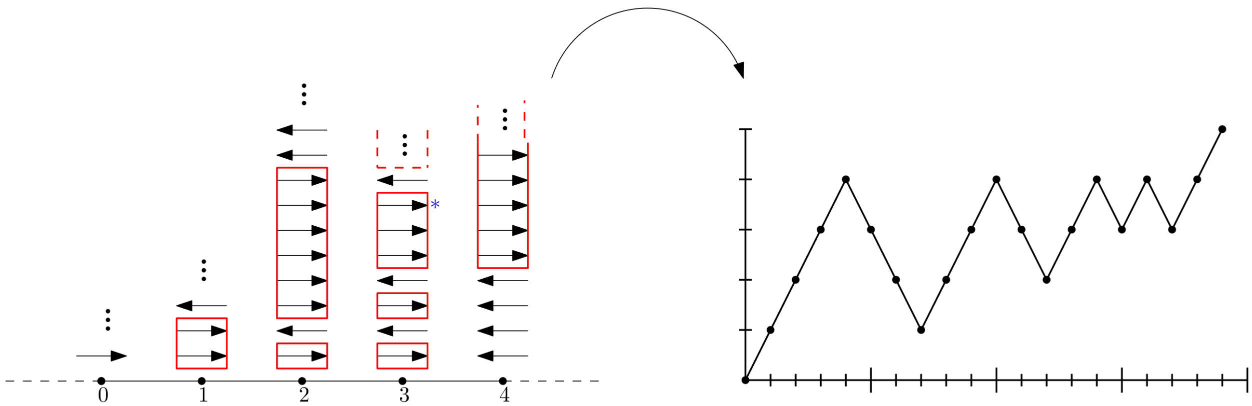

Figure 2.3. Coin tosses generate ERW path

The relevant values of U_n and R_n are:

$$R_0 = 1, \quad R_1 = 2, \quad R_2 = 3, \quad R_3 = 5, \ldots$$
$$U_0 = 1, \quad U_1 = 2, \quad U_2 = 7, \quad U_3 \geq 5, \ldots.$$

Essentially, if $T_0 = \infty$ the FBLP may count successes that the walker never uses. In this example, the walker visits the site 2 only 4 times, and so the walker never "sees" the additional 4 successes that occur at 2 before the second failure. Because $R_2 = 3$, the walker will only "see" the coin tosses up to (but not including) the third failure at site 3, but since $U_2 = 7$ the FBLP "sees" all the successes at 3 until the 7th failure there. If the next four coin-tosses at 3 are failures, then $U_3 = R_3$, but if there are any successes between the third and seventh failures at 3 we will have that $U_3 > R_3$.

2.2.2 Backward Branching-Like Process

In this section we will define the backward branching-like process (BBLP), which we will denote by $\{V_n\}_{n \geq 0}$. Intuitively, the BBLP keeps track of the amount of "backtracking" done by ERW before reaching a specified level $n > 0$. There is also a coin-toss construction of the BBLP, but it may seem artificial without proper context. For that reason, we will begin with a more intuitive construction of the BBLP and address the coin-toss construction afterward. Let $T_n = \inf\{\mathrm{j} \geq 1 : X_{\mathrm{j}} = n\}$ denote the first time the ERW reaches level n, and let D_n^x be the number of left steps from site x before the ERW reaches n for the first time,

$$D_n^x = \sum_{k=1}^{T_n} \mathbf{1}_{\{X_{k-1}=x, X_k=x-1\}}.$$

Then D_n^x tracks the amount of backtracking (i.e. moving away from level n) the walker does at site x. We will now rewrite T_n in terms of the D_n^x. In order to travel from 0 to n, the walker must take n right steps, but if the walker takes a left step from a site x during their journey from 0 to n, this left step will have to be balanced by a future right step in order for the walker to reach n. Therefore, we can rewrite T_n in the following way:

$$T_n = n + 2 \sum_{x<n} D_n^x. \tag{2.2}$$

We claim that the sequence $(D_n^n, D_n^{n-1}, \ldots, D_n^1, D_n^0)$ has a Markovian structure. To see this, consider the following example. Let $\{X_n\}_{n\geq 0}$ denote the ERW in the cookieless environment ω_0, and suppose that $D_n^{x-1} = \mathrm{j}$ for some $x \in \{1, 2, \ldots, n-1\}$, so that the walker took j left steps from x before reaching level n for the first time. Using only this information, we can determine the distribution for D_n^{x-1}. Before reaching n, the walker took j left steps from x, and therefore stepped right from $x-1$ exactly $\mathrm{j}+1$ times: once to get from $x-1$ to x for the first time, then j more times to counteract the j left steps from x to $x-1$. Let $\eta_{\mathrm{i}}, \mathrm{i} = 1, 2, \ldots, \mathrm{j}+1$ denote the number of left steps the walker took from $x-1$ before their ith right step from $x-1$. Since ERW in ω_0 always steps left from each site with probability 1/2, we have that $\eta_{\mathrm{i}} \sim \mathrm{Geo}(1/2)$ for each $\mathrm{i} \in \{1, \ldots, \mathrm{j}+1\}$, and therefore $D_n^{x-1} = \eta_1 + \cdots + \eta_{\mathrm{j}+1}$, given that $D_n^x = \mathrm{j}$ (note that $D_n^n = 0$, since at T_n the walker hasn't taken any left steps from n). To summarize,

$$P(D_n^{x-1} = k | D_n^x = \mathrm{j}) = P(\eta_1 + \cdots + \eta_{\mathrm{j}+1} = k),$$

and so $(D_n^n, \ldots, D_n^0)$ is not only Markovian, but can in fact be interpreted as a branching process with migration: in each generation, a single immigrant joins the population before reproduction, then each organism in the population has a Geo(1/2) number of offspring, and these offspring comprise the next generation.

There are two key differences between the branching process we have just described and the FBLP. First, this branching process *always* adds an extra immigrant to the population before reproduction (even when it corresponds to the ERW in ω_0). Second, because the distribution for D_n^{x-1} depends only on the value of D_n^x, this second branching process is indexed *backward*, which is why it is often referred to as the *backward branching-like process*. We will remove this troublesome notation in the coin-toss construction of the BBLP.

To that end, let $\{\xi_{\mathrm{j}}^x\}_{\mathrm{j}\in\mathbb{N}, x\in\mathbb{Z}}$ be the independent collection of Bernoulli random variables with $P(\xi_{\mathrm{j}}^x = 1) = p_{\mathrm{j}}^x$. As in the construction of the FBLP, we imagine ξ_{j}^x as the jth coin at site x, and then we "toss all the coins at each site." We then define

$$S_m^x = \inf\left\{k \in \mathbb{N} : \sum_{\mathrm{j}=1}^{k} \xi_{\mathrm{j}}^x = m\right\}, \quad \mathcal{F}^x(m) = \sum_{\mathrm{j}=1}^{S_m^x} (1 - \xi_{\mathrm{j}}^x),$$

so that S_m^x is the trial on which the mth success occurs at site x, and $\mathcal{F}_m^x$ is the number of failures at x before S_m^x. Then, we let $\{V_n\}_{n\geq 0}$ be the Markov process with transition probabilities

$$P(V_n = k | V_{n-1} = \mathrm{j}) = P(\mathcal{F}^n(\mathrm{j}+1) = k), \quad n \geq 1.$$

If we set $V_0 = 0$, we claim that $(D_n^n, D_n^{n-1}, \ldots, D_0) \stackrel{law}{=} (V_0, V_1, \ldots, V_n)$. To see this, let $V_{\mathrm{i}}^{(n)}$ be the same as the BBLP as defined above, but using the coin tosses $\{\xi_{\mathrm{j}}^{n-\mathrm{i}}\}_{\mathrm{j}\in\mathbb{N}}$ instead of $\{\xi_{\mathrm{j}}^{\mathrm{i}}\}_{\mathrm{j}\in\mathbb{N}}$. If we set $V_0^{(n)} = 0$, then we will have $V_{\mathrm{i}}^{(n)} = D_n^{n-\mathrm{i}}$ for all $\mathrm{i} = 0, 1, \ldots, n$, but then $V_{\mathrm{i}}^{(n)} \stackrel{law}{=} V_{\mathrm{i}}$ because the two collections of Bernoulli random variables used in their construction, $\{\xi_{\mathrm{j}}^{n-\mathrm{i}}\}_{\mathrm{j}\in\mathbb{N}}$ and $\{\xi_{\mathrm{j}}^{\mathrm{i}}\}_{\mathrm{j}\in\mathbb{N}}$ respectively, have the same distribution. That is, this property is a consequence of the fact that the cookie stacks at each site are either identical, as in the case of deterministic cookie stacks, or independent and identically distributed as in the case of cookie stacks generated by an independent copy of a Markov chain at each site.

2.3 Using the Branching-Like Processes

We will now explain how the two branching-like processes in the preceding sections have been used in the literature. Since we will discuss results in the literature that apply to excited random walk in particular classes of cookie environments, it will be useful to describe the different models in detail.

Example 1 – Random walk in random environment: For each site $x \in \mathbb{Z}$, select $\alpha_x \in (0, 1)$ randomly so that $\{\alpha_{\mathrm{i}}\}_{\mathrm{i}\in\mathbb{Z}}$ is a collection of i.i.d. random variables, then place infinitely many cookies of strength α_x at site x, so that $p_{\mathrm{j}}^x = \alpha_x$ for all $\mathrm{j} \in \mathbb{N}$. Let Ω_{RWRE} denote the collection of environments that can be generated in this way, with $\mathbb{P}$ the measure on environments. Given a specific environment $\omega \in \Omega_{RWRE}$, we will use P_ω to denote probabilities associated to the walk in that environment.

Example 2 – Positive cookies: For each site $x \in \mathbb{Z}$, randomly place a stack of cookies at each site, subject to the constraint that for all $\mathrm{j} \in \mathbb{N}$ and for all $x \in \mathbb{Z}$, we have $p_{\mathrm{j}}^x \geq 1/2$. Let Ω_+ denote the collection of possible cookie environments, and let $\mathbb{P}$ denote the distribution of cookie environments on Ω_+. We assume that the cookie stacks are i.i.d. spatially, but cookie strengths within the same stack may not be independent. Given a cookie environment ω, we will use P_ω to denote the law of ERW in ω.

Example 3 – M cookies per site: Fix $M \in \mathbb{N}$, then for each site $x \in \mathbb{Z}$ randomly place a cookie stack that contains at most M cookies (i.e. $p_{\mathrm{j}}^x = 1/2$ whenever $\mathrm{j} > M$) at each site according to some probability measure $\mathbb{P}$ so that the cookie stacks are i.i.d. spatially (note that cookie strengths within the same stack may not be independent). Let Ω_M denote the collection of M-cookie environments, and given a cookie environment ω, let P_ω denote the law of ERW in ω.

Example 4 – Periodic cookie stacks: Fix a finite-length vector of cookie strengths $\boldsymbol{p} = (p_1, p_2, \ldots, p_M)$, then for each $x \in \mathbb{Z}$ place a periodic cookie stack, so that for each $x \in \mathbb{Z}$ we have $p_{\mathrm{j}}^x = p_{\mathrm{i}}$ whenever $\mathrm{j} \equiv \mathrm{i} \pmod{M}$. This results in an identically piled, deterministic cookie environment which we will denote by $\omega_{\boldsymbol{p}}$.

Example 5 – Markovian cookie stacks: Fix a Markov chain on finite state space $\mathcal{R}$ which contains a unique closed irreducible set $\mathcal{R}_0$, and let K be its transition matrix. We associate to each $\mathrm{i} \in \mathcal{R}$ a probability $p_{\mathrm{i}} \in (0,1)$. Then, for each $x \in \mathbb{Z}$, we run an i.i.d. copy of this Markov chain to generate a cookie stack at each site: if the Markov chain at site x is in state i at time t, then we set $p_t^x = p_{\mathrm{i}}$. Again, let $\mathbb{P}$ denote the distribution of the cookie environment on Ω_K, and let P_ω denote the law of the walk in a given environment ω. Note that the periodic cookies model is a special case of this model.

Now that we have introduced the primary models that we will consider, we will start by discussing how the branching-like processes have been used to establish criteria for recurrence and transience of ERW.

2.3.1 Recurrence and Transience

The first use of the BLP to prove a criterion for recurrence and transience of ERW is due to Kosygina and Zerner [18]. Their result applies to ERW in environments with M cookies per site.

Theorem 2.3.1 (Theorem 1 of [18]). *Let X_n be an M-cookie excited random walk, let $\mathbb{P}$ denote the probability measure on environments $\omega \in \Omega_M$ and let P_ω denote the probability measure for the walk in ω. Let δ be the expected total drift as defined in 2.1. Then*

1. *If $\delta \in [-1,1]$, the walk is recurrent. That is, for $\mathbb{P}$-a.e. environment the walk will return to its starting point infinitely many times P_ω-almost surely.*
2. *If $\delta > 1$, the walk is transient to $+\infty$, so that for $\mathbb{P}$-a.e. environment we have that $P_\omega\left(\lim_{n\to\infty} X_n = \infty\right) = 1$.*
3. *If $\delta < -1$, the walk is transient to $-\infty$, so that for $\mathbb{P}$-a.e. environment we have that $P_\omega\left(\lim_{n\to\infty} X_n = -\infty\right) = 1$.*

Note that for ERW in M-cookie environments, whether the walk is recurrent or transient depends only on the expected total drift δ. We will now briefly describe the proof of Theorem 2.3.1. Let $\{X_n\}_{n\geq 0}$ be ERW in an environment ω chosen from Ω_M according to $\mathbb{P}$, and let $\{U_k\}_{k\geq 0}$ be the associated FBLP as defined in Section 2.2.1. Recall that this FBLP tracks the walker's right excursions, and so we will need a separate FBLP to keep track of left excursions. Let $\{\tilde{U}_k\}_{k\geq 0}$ be the FBLP associated to ERW in the *reflected environment* $\tilde{\omega}$: given an environment ω, we construct $\tilde{\omega}$ by replacing p_{j}^x by $1-p_{\mathrm{j}}^x$ for all $x \in \mathbb{Z}$ and all $\mathrm{j} \in \mathbb{N}$. Then $\{\tilde{U}_k\}_{k\geq 0}$ tracks left excursions of ERW in ω, since a left excursion in ω corresponds to a right excursion in $\tilde{\omega}$.

We will say that ERW is *recurrent from the right* if the walker's first right excursion is P_ω-a.s. finite, if there is one. We define an ERW which is *recurrent from the left* analogously. The authors quickly show that if ERW is recurrent from the right, then all of the walk's right excursions are P_ω-a.s. finite. On the other hand, if ERW is not recurrent from the right, then the walk P_ω-a.s. makes only finitely many right excursions. Similar statements hold from ERW which are recurrent from the left.

Therefore, if ERW is recurrent from the left but not recurrent from the right, the walker takes finitely many right excursions and each of their left excursions are P_ω-a.s. finite. The walker cannot return to 0 infinitely many times, because if they do they would P_ω-a.s. take infinitely many right excursions. It follows that the walker's final excursion must be an infinite excursion to the right, and so $\liminf_n X_n \geq 0$. Coupled with the fact that $\liminf_n X_n, \limsup_n X_n \in \{-\infty, \infty\}$ for ERW in ω_M (this follows essentially from ellipticity and the Borel-Cantelli lemma), we see that in this case $P_\omega(\lim X_n = \infty) = 1$. Similarly, one proves that if ERW is recurrent from the right but not from the left, then $P_\omega(\lim X_n = -\infty) = 1$. Finally, if the walk is recurrent from both the right and left, every excursion the walker takes is P_ω-a.s. finite, and so the walker returns to 0 infinitely many times with probability 1. Note that the case where ERW is not recurrent from the right or left is theoretically possible, but we will not encounter it here.

In light of the above discussion, if we want to establish a recurrence/transience criterion it will be enough to find conditions under which the walk is recurrent from the right or left. The key to proving Theorem 2.3.1 is to observe that the long-term behavior of the walk and the branching-like process are bound together. The following proposition relates the length of the walker's first right excursion to the survival of the associated FBLP.

Proposition 2.3.1. *Let X_n be ERW in an elliptic cookie environment, and let $\{U_n\}_{n\geq 0}$ be the associated FBLP. Assume that the walk's first step is to the right, i.e. $X_1 = 1$, and let*

$T_0 = \inf\{n \geq 1 : X_n = 0\}$ *be the time of the walker's first return to* 0*. Then* $P(T_0 = \infty) > 0$ *if and only if* $P(U_n > 0$ *for all* $n) > 0$*.*

To see why this is the case, suppose that $X_1 = 1$ and that $T_0 < \infty$. Let $n^* = \max\{X_k : k < T_0\}$ be the furthest point to the right that the walk reaches. Then $U_{n^*+1} = W_{n^*+1} = 0$ by Proposition 2.2.1, and so $U_n = 0$ for some $n \in \mathbb{N}$.

On the other hand, if $T_0 = \infty$, then we must have $W_n \geq 1$ for all $n \geq 0$: if we do not, and $W_{n^*} = 0$ for some $n^* \in \mathbb{N}$, then the walker never steps right from n^*, and is therefore trapped in the interval $[0, n^*]$. Because we assume that the cookie environment is elliptic, this would imply that $T_0 < \infty$. Now, if $U_n = 0$ for some n, by Proposition 2.2.1 we would have that $1 \leq W_n \leq U_n = 0$, a contradiction.

Therefore, $T_0 = \infty$ if and only if $U_n > 0$ for all $n > 0$, and so we must have $P(T_0 = \infty)$ if and only if $P(U_n > 0$ for all $n > 0) > 0$.

Proposition 2.3.1 reduces the problem of determining whether ERW in ω_M is recurrent or transient to identifying when the FBLPs $\{U_n\}_{n\geq 0}$ and $\{\tilde{U}_n\}_{n\geq 0}$ have a positive probability of survival and when they die out almost surely.

Recall that in the case of ERW in M-cookie environments, the FBLPs can be thought of as branching processes with migration. These processes are well-studied [11], [12], and so Kosygina and Zerner were able to appeal to known results about their long-term behavior. The long-term behavior of $\{U_n\}_{n\geq 0}$ and $\{\tilde{U}_n\}_{n\geq 0}$ are each governed by a single parameter, which we will respectively call β and $\tilde{\beta}$.

Remark: This parameter is called θ in [18], but we will use θ for a different parameter associated to the FBLP for ERW in a different class of environments. The parameter β is given by

$$\beta = \frac{\lambda}{b},$$

where λ is the average migration in the BLP and $b = \frac{1}{2}\mathbb{E}[S(S-1)]$, where S follows the offspring distribution of the BLP (in this case, S is geometric). Although β and θ are defined somewhat differently, the two parameters agree with each other in this case (see the definition θ below).

In short, if $\beta > 1$, then $P(U_k > 0 \text{ for all } k \in \mathbb{N}) > 0$, and if $\beta \leq 1$, we have that $P(U_k = 0 \text{ for some } k \in \mathbb{N}) = 1$. In this case $\beta = \mathbb{E}[S_M^{(0)}] - M$, where $S_M^{(0)}$ is the number of successes that occur at a site before M failures. By considering the number of successes that occur after the first M trials $S_M^{(0)} - (M - F)$ and conditioning on the number of failures in the first M trials, a short calculation shows that $\beta = \delta$, and by symmetry $\tilde{\beta} = -\delta$. Combining this calculation with the discussion in the preceding paragraph completes the proof of the theorem.

This general approach has become the standard technique for determining when ERW in a particular class of environments is recurrent or transient. One selects a class of environments to consider, constructs the FBLPs associated to the walk, and attempts to understand when the FBLPs die out almost surely and when they have a positive survival probability. One key result in this vein is a zero-one law for directional transience, due to Amir, Berger, and Orenshtein [1]:

Theorem 2.3.2 (Theorem 1.2 of [1])**.** *Let X_n be an ERW, and let μ be a stationary ergodic and elliptic probability measure on the space Ω of cookie environments. Then*

$$P(\lim_{n\to\infty} X_n = \infty),\ P(\lim_{n\to\infty} X_n = -\infty) \in \{0, 1\}.$$

The proof of this fact extends the ideas of the proofs of Lemma 7 and Lemma 8 of [18], but are more combinatorial than probabilistic. Theorem 2.3.2 helps to formalize the approach to proving a recurrence/transience criterion for M-cookie environments. Suppose we want to prove that X_n in a particular class of cookie environments is transient to ∞, i.e. that $P(\lim_{n\to\infty} X_n = \infty) = 1$. We have already seen that

$$P(X_n \to \infty) > 0 \iff P(T_0 = \infty) > 0,$$
$$P(T_0 = \infty) > 0 \iff P(U_n > 0 \text{ for all } n) > 0.$$

Therefore, $P(X_n \to \infty) > 0$ if and only if $P(U_n > 0 \text{ for all } n) > 0$, and in light of Theorem 2.3.2 we have that ERW is transient to ∞ if and only if $P(U_n > 0 \text{ for all } n) > 0$. Similar considerations for the left FBLP $\tilde{U}_n$ yields the following theorem.

Theorem 2.3.3. *Let ω be an elliptic cookie environment, and let X_n be ERW in ω. Let $P_n(\cdot) = P(\cdot | U_0 = n)$ Then*

1. $P_1(U_n > 0$ *for all* $n) > 0$ *if and only if* $P(X_n \to \infty) = 1$,
2. $P_1(\tilde{U}_n > 0$ *for all* $n) > 0$ *if and only if* $P(X_n \to -\infty) = 1$,
3. $P_1(U_n = 0$ *for some* $n) = P_1(\tilde{U}_n = 0$ *for some* $n) = 1$ *if and only if* $P(X_n = 0 \text{ i.}o.) = 1$.

In the case of ERW in environments with periodic cookie stacks, it is more difficult to interpret U_k and $\tilde{U}_k$ as branching processes with migration. The approach that worked in the case of M-cookie environments, which essentially treats the offspring generated from the first M coin tosses at each site as a small perturbation to a branching process with Geo(1/2) offspring distribution, fails because we cannot "use up" all the cookies at a site and then be left with a common underlying offspring distribution for the rest of reproduction. In fact, even relating the FBLPs associated to ERW in M-cookie environments to branching processes with migration (as had been previously studied in the literature) required interpolation between the two using several intermediate processes. Kozma, Orenshtein, and Shinkar [19] proved a general criteria for determining when the FBLPs die out and when they have a chance to survive. Although their result applies to any Markov chain that satisfies the conditions of the theorem, we will state it in terms of the FBLPs that we will apply it to. In order to do so, we must define a few parameters associated to the FBLPs. For ease of notation, let $P_n(\cdot) = P(\cdot | U_0 = n)$ and let $\mathbb{E}_n[\,\cdot\,]$ denote the corresponding expectation. The parameters we will be interested in are:

$$\mu(n) = \frac{\mathbb{E}_n[U_1]}{n} \qquad\qquad \rho(n) = \mathbb{E}_n[U_1 - \mu n]$$
$$\nu(n) = \frac{\mathbb{E}_n\left[(U_1 - \mu n)^2\right]}{n} \qquad\qquad \theta(n) = \frac{2\rho(n)}{\nu(n)},$$

where we define

$$\mu = \lim_{n\to\infty} \frac{\mathbb{E}_n[U_1]}{n} \qquad\qquad \rho = \lim_{n\to\infty} \mathbb{E}_n[U_1 - \mu n]$$
$$\nu = \lim_{n\to\infty} \frac{\mathbb{E}_n\left[(U_1 - \mu n)^2\right]}{n} \qquad\qquad \theta = \lim_{n\to\infty} \frac{2\rho(n)}{\nu(n)},$$

provided the limits exist. We will denote the corresponding parameters for the FBLP that tracks left excursions $\{\tilde{U}_k\}_{k\geq 0}$ by $\tilde{\mu}$, $\tilde{\rho}$, $\tilde{\nu}$, $\tilde{\theta}$ respectively, provided the limits exist. As the next theorem shows, it is possible to determine recurrence and transience by computing these parameters of the FBLP.

Theorem 2.3.4 (Theorem 1.3 of [19])**.** *Let $\{U_n\}_{n\geq 0}$ be the forward branching-like process associated to ERW. Assume that the limit $\mu = \lim_{n\to\infty} \frac{\mathbb{E}_n[U_1]}{n}$ exists, and that there is a constant C such that for all $\epsilon > 0$, it holds that*

$$P_n\left(\left|\frac{U_1}{n} - \mu\right| > \epsilon\right) \leq 2\exp\left\{-\frac{C\epsilon^2 n}{2+\epsilon}\right\}.$$

If $\mu > 1$, then $P(U_n > 0 \text{ for all } n) > 0$. If $\mu < 1$, then $P(U_n = 0 \text{ for some } n) = 1$. If $\mu = 1$, then

1. *if $\theta(n) < 1 + \frac{1}{\log n} - \alpha(n)n^{-1/2}$ for all sufficiently large n, where $\alpha(n)$ is such that $\alpha(n)\nu(n) \to \infty$ as $n \to \infty$, then $P(U_n = 0 \text{ for some } n) = 1$,*
2. *if $\theta(n) > 1 + \frac{2}{\log n} + \alpha(n)n^{-1/2}$ for all sufficiently large n, where $\alpha(n)$ is such that $\alpha(n)\nu(n) \to \infty$ as $n \to \infty$, then $P(U_n > 0 \text{ for all } n) > 0$.*

The proof of Theorem 2.3.4 relies on Lyapunov function techniques. Since the proof would constitute a major digression from our present subject, we omit it and refer the curious reader to Appendix A of [19]. Before proceeding further, we will give some explanation for the parameters of the FBLP that appear in Theorem 2.3.4. The first part of the theorem is somewhat intuitive: if $\mu > 1$, then on average the size of the population grows over time and so we expect that the FBLP has a positive probability of surviving. Similarly if $\mu < 1$, the population shrinks on average with each generation, and so we expect the FBLP dies out.

The critical case $\mu = 1$ requires more careful analysis to unpack. If we set $Y_t^{(n)} = U_{\lfloor nt \rfloor}/n$, then we should be able to approximate $Y_t^{(n)}$ with a certain squared Bessel process. Given that $Y_0^{(n)} = y$, the increment $Y_{t+\delta t} - Y_t$ is on average $\rho\delta t$, with variance $\nu \cdot y$, and so we would expect that the scaling limit Y_t will satisfy the stochastic differential equation

$$dY_t = \rho dt + \sqrt{2\nu} dB_t. \tag{2.3}$$

Therefore, Y_t is a squared Bessel process with generalized dimension 2θ. Now, a d-dimensional squared Bessel process hits 0 with probability 1 if $d < 2$, but when $d \geq 2$ this probability is 0. Since the Bessel process hitting 0 roughly corresponds to the FBLP dying out, we might expect that the FBLP dies out a.s. if $\theta < 1$ and has a chance of surviving if $\theta > 1$. If $\theta = 1$ we are in a borderline case, and it could make a difference how quickly θ converges to 1. If $\theta \to 1$ quickly, the FBLP should a.s. die out, but if $\theta \to 1$ slowly enough, the FBLP should have positive survival probability.

Combined with the zero-one law for directional transience of ERW, Theorem 2.3.4 gives another method for determining conditions under which ERW is recurrent or transient: one computes the relevant parameters and their rates of convergence.

It is important to note the slight gap in the criteria given in Theorem 2.3.4, which is an artifact of technical details in its proof. If $\theta(n) - 1 \in \left(\frac{1}{\log n}, \frac{2}{\log n}\right)$, the theorem cannot tell us anything about the long term behavior of the walk. In the cases of M-cookies per site, periodic cookie stacks, and Markovian cookie stacks, $\theta(n)$ converges quickly enough to θ that this gap doesn't come into play. However, the gap will be relevant in the next chapter when we study ERW in finite-drift cookie environments.

2.3.2 Ballisticity

Basdevant and Singh [2] were the first to use branching-like processes to establish a ballisticity criteria for ERW. Their technique, which they used in the context of ERW with deterministic stacks of M identical positive cookies at each site, was generalized in [18] to apply to spatially i.i.d. stacks of M cookies per site. If the walk is recurrent, then clearly its speed is 0, so assume that the walk is transient to the right (left transience can be handled by symmetry). We will begin with the following equivalence:

$$\frac{X_n}{n} \to v \ a.s. \Longleftrightarrow \frac{T_n}{n} \to \frac{1}{v} \ a.s.$$

with the convention that if $1/v = \infty$, then $v = 0$. Therefore, if we want to establish a law of large numbers for the position of the walk, we can do so by first establishing one for the hitting times T_n. The key observation to make is that the hitting times can be rewritten in terms of the number of downcrossings the walk makes before reaching level n for the first time. Recall that D_n^k represents the number of left steps that the walker makes from k before T_n. Then we can decompose T_n:

$$T_n = n + 2\sum_{k\le n} D_n^k, \tag{2.4}$$

because the walker must take n steps to travel from 0 to n, and any left steps the walker takes must be paired with a right step in order for the walker to reach n. Since we assume that the walk is transient to ∞, the amount of time the walker spends to the left of the origin $\sum_{k<0} D_n^k$ is almost surely finite. Therefore

$$\frac{T_n}{n} \approx 1 + \frac{2}{n}\sum_{k=0}^{n} D_n^k \overset{law}{=} 1 + \frac{2}{n}\sum_{k=0}^{n} V_k,$$

where the last equality follows from the definition of the backward branching-like process. Now, one can check that $\{V_k\}_{k\ge 0}$ is an irreducible, positive recurrent Markov chain, and so it converges to its unique stationary distribution, which we will denote by V_∞. Therefore, we have that

$$\frac{1}{v} = \lim_{n\to\infty}\frac{T_n}{n} = \lim_{n\to\infty} 1 + \frac{2}{n}\sum_{k=0}^{n} V_k = 1 + 2\mathbb{E}[V_\infty].$$

Therefore $v = \frac{1}{1+2\mathbb{E}[V_\infty]}$. For that reason, we will have $v = 0$ if $\mathbb{E}[V_\infty] = \infty$, and otherwise we will have $v > 0$. In order to determine when $\mathbb{E}[V_\infty] = \infty$, Basdevant and Singh used the probability generating function $\mathbb{E}[s^{V_\infty}]$ to study the asymptotics of $\sum P(V_\infty > k)$, and determined that $v > 0$ if and only if $\delta > 2$.

Remark: If $\delta \in (1,2)$, the above results imply that ERW is transient to ∞ with asymptotic speed 0 (i.e. sublinear speed). This feature was first observed for random walk in random environment in 1975 by Solomon [24], but has also appeared in each of the commonly-studied ERW models mentioned at the beginning of this section.

Because this argument relies on the existence of the stationary distribution V_∞ but does not identify it, the formula given above cannot actually be used to calculate v, even in the relatively simple case where each site has an identical pile of M cookies each with strength p. Upper and lower bounds for the speed have been obtained in the case of 2 or 3 cookies of strength p by Madden et al. [21], but their methods, which rely on analyzing the probability generating function of V_∞, seem difficult to extend to $M > 3$ or other ERW models. The following problem therefore remains open, much to our collective embarrassment.

Problem 1. *Compute the asymptotic speed v of ERW when it is nonzero.*

Aside from the bounds mentioned above, the only progress on this problem seems to be a proof that the formula for v does not depend only on δ. For details, including an explicit construction to demonstrate this fact, we refer the reader to Theorem 4.2 of [7]. However, significant progress has been made in determining when ERW satisfies a strong law of large numbers.

Theorem 2.3.5 (Theorem 1.4 of [1], Theorem 4.1 of [17])**.** *Let X_n be an excited random walk, and suppose that μ is a stationary ergodic and elliptic probability measure on the space Ω of cookie environments. Then there exists a deterministic $v \in [-1, 1]$ such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v \quad \mathbb{P}\text{-almost surely,}$$

where $\mathbb{P}$ is the averaged measure defined by $\mathbb{P}(\cdot) = \int_\Omega P_\omega(\cdot) d\mu(\omega)$.

This theorem is an immediate consequence of combining Theorem 4.1 of [17] with Theorem 2.3.2, which was contained in [1]. The former theorem states that a strong law of large numbers holds provided that a zero-one law holds, and the latter result identifies the mild conditions under which this is the case.

Remark: Theorem 2.3.5 implies that for $\mathbb{P}$-a.e. environment ω, X_n/n converges P_ω-a.s. to a deterministic limit v.

It should be noted that although 2.3.5 describes when the limiting speed exists, it does not give a criteria for ballisticity. Instead, ballisticity criteria must be established on a model-by-model basis. Proofs of a ballisticity criteria in the literature generally rely on Basdevant and Singh's [2] main idea: that the speed v is positive when $\mathbb{E}[V_\infty] < \infty$.

The general structure of these arguments is as follows: we start by using (2.2) to rewrite T_n/n:

$$\frac{T_n}{n} = 1 + \frac{2}{n} \sum_{k \le n} V_k. \tag{2.5}$$

If we assume that the walk is transient to ∞, then the total amount of time the walk spends below 0 is almost surely finite, and so we have

$$\frac{T_n}{n} \approx 1 + \frac{2}{n} \sum_{k=0}^{n} V_k.$$

We will now decompose $\sum_{k=0}^{n} V_k$ further. Let $\sigma_0 = 0$, and define $\sigma_k = \inf\{t > \sigma_k \ : \ V_t = 0\}$ be the time of the kth extinction of the BBLP (note that the BBLP is not absorbed at 0 due to the extra immigrant in each generation). If we let $W_k = \sum_{\mathrm{j}=\sigma_{k-1}}^{\sigma_k - 1} V_{\mathrm{j}}$ be the total progeny of the BBLP between the $k-1$st and kth extinction and let $N_n = \#\{k \le n \ : \ V_k = 0\}$ be the number of extinctions that occur by time n, then we can write

$$\frac{T_n}{n} = 1 + 2 \cdot \frac{N_n}{n} \cdot \frac{1}{N_n} \left(\sum_{\mathrm{j}=1}^{N_n} W_k \right) + \frac{1}{n} \sum_{\mathrm{j}=\sigma_n}^{n} V_k. \tag{2.6}$$

The benefit of decomposing T_n/n in this way is that, while the number of offspring in each generation of the BBLP V_k are not i.i.d. random variables, the W_{j} are. Since we can bound $\sum_{\mathrm{j}=\sigma_n}^{n} V_k$ above by W_{N_n+1}, we expect that the SLLN for renewal processes will give us:

$$\lim_{n\to\infty} \frac{T_n}{n} = \frac{1}{v} = 1 + 2\frac{\mathbb{E}[W_1]}{\mathbb{E}[\sigma_1]}, \tag{2.7}$$

though in principle it could happen that $\mathbb{E}[W_1] = \infty$, that $\mathbb{E}[\sigma_1] = \infty$, or both. In the case that $\mathbb{E}[W_1] = \infty$ and $\mathbb{E}[\sigma_1] < \infty$, we have that $T_n/n \to \infty$, and so $v = 0$. If $\mathbb{E}[W_1] < \infty$ and $\mathbb{E}[\sigma_1] = \infty$, we will have that $v > 0$ (and in fact that $v = 1$). If both $\mathbb{E}[W_1]$ and $\mathbb{E}[\sigma_1]$ are finite, we will have $v > 0$. The last remaining case, where both expectations are infinite, would be delicate, and whether $v > 0$ or not would depend on the relative rates of convergence of the two terms.

In practice, we can determine when each of the above expectations is finite by studying the tail probabilities $P(W_1 > n)$ and $P(\sigma_1 > n)$. For instance, in the case of ERW in M-cookie environments, Kosygina and Zerner [18] showed that

$$P(W_1 > n) \sim Cn^{-\delta/2},$$
$$P(\sigma_1 > n) \sim Cn^{-\delta}.$$

Recall that the M-cookie ERW is transient to ∞ when $\delta > 1$. Considering the four cases above, we see that the M-cookie ERW is ballistic when $\delta > 2$ and has asymptotic speed 0 when $\delta \in (1, 2)$.

The tail probability estimates above can be obtained by using a squared Bessel process approximation similar to the one given for the FBLP in (2.3). Essentially, one shows that the BBLP is sufficiently well-approximated by a particular squared Bessel process, then one can transfer the tail decay rate for the area under the squared Bessel process to W_1 and the tail decay rate for the time between successive visits to 0 by the squared Bessel process to σ_1.

2.3.3 Limiting Distributions

We will now explain how limiting distributions for excited random walk can be obtained by using the branching-like processes. Although Basdevant and Singh [2] were the first to use branching-like processes to study ERW, the correspondence between random walk paths and branching processes was utilized at least as far back as 1975 by Kesten, Kozlov, and Spitzer [14] in the context of random walk in a random environment (RWRE).

Recall that in RWRE, a random walker is released into a random environment that is created by generating a random variable $\alpha_x \in (0, 1)$ for each $x \in \mathbb{Z}$. Whenever the walker is at site x, they step right with probability α_x and left with probability $1-\alpha_x$. We will assume that the collection of random variables $\{\alpha_x\}_{x\in\mathbb{Z}}$ are independent and identically distributed. The RWRE model can be viewed as a special case of ERW, where for each $x \in \mathbb{Z}$, we set $p^x_{\mathrm{j}} = \alpha_x$ for all $\mathrm{j} \in \mathbb{N}$. If we construct the BBLP $\{V_n\}_{n\geq 0}$ corresponding to this random walk, we see that conditional on $V_{n-1} = \mathrm{j}$, $V_n = \eta_1 + \cdots + \eta_{\mathrm{j}+1}$, where $\eta_{\mathrm{i}} \sim \mathrm{Geo}(\alpha_n)$. Then $\{V_n\}_{n\geq 0}$ is like a branching process (with an extra immigrant added before reproduction), but with a random parameter α_n that governs the reproduction rule in each generation. Note that because the α_x are i.i.d. we could obtain the same stochastic process by running a Galton-Watson process with an extra immigrant and resampling the parameter α_{i} before each generation reproduces. For that reason, the BBLP in this case is sometimes called a *branching process in a random environment*.

The general method for establishing limiting distributions for transient ERW (which is essentially the same method used by Kesten, Kozlov, and Spitzer [14] in the RWRE case) is to first prove limiting distributions for the walk's hitting times $T_n = \inf\{\mathrm{j} : X_{\mathrm{j}} = n\}$. To see how this method works, suppose we have proven that for some centering constant $a \neq 0$, scaling constant b, and scaling exponent γ

$$P\left(\frac{T_n - na}{bn^{\gamma}} \leq x\right) \to L(x), \tag{2.8}$$

where $L(\cdot)$ is the cdf of some random variable. Let $\bar{X}_n = \sup_{1\leq k\leq n} X_k$ be the running maximum of the walk, and note that

$$\{\bar{X}_n \leq m\} = \{T_m \geq n\}. \tag{2.9}$$

We will then have, for (as of yet unidentified) centering and scaling constants α and β and scaling exponent ϵ that

$$\begin{aligned}
P\left(\frac{\bar{X}_n - n\alpha}{\beta n^{\epsilon}} \leq x\right) &= P\left(\bar{X}_n \leq \lceil \beta n^{\epsilon} x + n\alpha \rceil\right) \\
&= P\left(T_{\lceil \beta n^{\epsilon} x + n\alpha\rceil} \geq n\right). \\
&= P\left(\frac{T_{\lceil \beta n^{\epsilon} x + n\alpha\rceil} - a\lceil \beta n^{\epsilon} x + n\alpha\rceil}{b\left(\lceil \beta n^{\epsilon} x + n\alpha\rceil\right)^{\gamma}} \geq \frac{n - a\lceil \beta n^{\epsilon} x + n\alpha\rceil}{b\left(\lceil \beta n^{\epsilon} x + n\alpha\rceil\right)^{\gamma}}\right).
\end{aligned}$$

Taking $\alpha = 1/a$, $\beta = ba^{\gamma-1}$, and $\epsilon = \gamma$ and using the limiting distributions from (2.8), we see that

$$P\left(\frac{\bar{X}_n - n/a}{ba^{\gamma-1}n^{\gamma}} \le x\right) \to 1 - L(-x).$$

The case where the centering constant $a = 0$ can be handled in a similar way. In that case, one can show that

$$P\left(\frac{\bar{X}_n}{(1/b)^{1/\gamma}n^{\gamma}} \le x\right) \to 1 - L(x^{-\gamma}).$$

This establishes limiting distributions for the running maximum of the walk $\bar{X}_n$, and one can extend the result to X_n by showing that $\bar{X}_n$ and X_n do not differ by too much. In light of this discussion, we will first establish limiting distributions for T_n, then use them to deduce the corresponding limiting distributions for X_n.

The proof of limiting distributions for T_n will again make use of the decomposition from the preceding section, namely that

$$T_n \overset{law}{=} n + 2\sum_{k \le n} V_k.$$

Centering (assuming for now $T_n/n \to 1/v > 0$) and dividing through by n^{γ}, we see that

$$\begin{aligned}\frac{T_n - n/v}{n^{\gamma}} &\overset{law}{=} \frac{2\sum_{\mathrm{j}=0}^{n} V_{\mathrm{j}} - (1/v-1)n}{n^{\gamma}} + \frac{2}{n^{\gamma}}\sum_{k<0} V_k \\ &= \frac{2\sum_{\mathrm{j}=1}^{N_n} W_{\mathrm{j}}}{n^{\gamma}} - \frac{(1/v-1)n}{n^{\gamma}} + \frac{2}{n^{\gamma}}\sum_{k=\sigma_n}^{n} V_k + \frac{2}{n^{\gamma}}\sum_{k<0} V_k.\end{aligned}$$

Moreover, the $\{W_{\mathrm{j}}\}_{\mathrm{j}\ge 1}$ are i.i.d. random variables, and by (2.7) we have that $\mathbb{E}[W_1] = \frac{\mathbb{E}[\sigma_1]}{2}(1/v - 1)$. Centering the W_{j} by this value, we obtain

$$\frac{T_n - n/v}{n^{\gamma}} \overset{law}{=} \frac{2\sum_{\mathrm{j}=1}^{N_n}(W_{\mathrm{j}} - \mathbb{E}[W_1])}{n^{\gamma}} + 2\mathbb{E}[W_1]\frac{N_n - n/\mathbb{E}[\sigma_1]}{n^{\gamma}} + \frac{2}{n^{\gamma}}\sum_{k=\sigma_n}^{n} V_k + \frac{2}{n^{\gamma}}\sum_{k<0} V_k. \quad (2.10)$$

Since we assume X_n is transient to ∞, the last two terms will converge to 0 in probability.

The second term in (2.10) requires some care. If $\mathbb{E}[\sigma_1^2] < \infty$, then by the renewal central limit theorem we can show that for any $\varepsilon > 0$ there exists a constant C so that $P\left(|N_n - n/\mathbb{E}[\sigma_1]| > C\sqrt{n}\right) < \varepsilon$, and so this term will tend to 0 in probability (see, for instance, Theorems II.5.1 and II.5.2 of [13]). If σ_1 does not have finite second moment, this term would require further analysis. Fortunately, in RWRE $\mathbb{E}[\sigma_1^2]$ is always finite, because in that case $P(\sigma_1 > n) \sim \mathrm{e}^{-Cn}$ for some constant $C > 0$. Therefore, for RWRE the limiting distribution for T_n is determined solely by the term

$$\frac{\sum_{\mathrm{j}=1}^{N_n} (W_{\mathrm{j}} - \mathbb{E}[W_1])}{n^{\gamma}}.$$

It remains to identify the scaling exponent γ and the corresponding limiting distribution. We can do so by establishing tail asymptotics for W_1. For instance, in the case of RWRE Kesten, Kozlov, and Spitzer [14] show that $P(W_1 > n) \sim n^{-\kappa}$ and that $P(\sigma_1 > n) \sim \mathrm{e}^{-Cn}$, where κ is the unique value such that $\mathbb{E}\left[\left(\frac{1-\alpha_0}{\alpha_0}\right)^{\kappa}\right] = 1$. In fact, these tail asymptotics show that W_{i} is in the domain of attraction for a κ-stable random variable (when $\kappa \geq 2$, W_{i} is in the domain of attraction for a Gaussian random variable).

To identify the stable limits when they arise, we will need some notation. Let $L_{\alpha,b}$ denote the α-stable distribution whose characteristic function satisfies

$$\log \mathbb{E}\left[\mathrm{e}^{\mathrm{i}uL_{\alpha,b}}\right] = \begin{cases} -b|u|^{\alpha}\left(1 - \mathrm{i}\frac{u}{|u|}\tan\left(\frac{\pi\alpha}{2}\right)\right) & \text{for } \alpha \neq 1, \\ -b|u|\left(1 + \frac{2\mathrm{i}}{\pi}\frac{u}{|u|}\log|u|\right) & \text{for } \alpha = 1. \end{cases}$$

where $\alpha \in (0, 2]$ and $b > 0$. Note that the limiting distributions for the hitting times are totally skewed to the right. We summarize the limiting distributions for transient RWRE in the table below. The boundary cases where $\kappa = 1$ and $\kappa = 2$ are delicate, since RWRE changes from nonballistic to ballistic across $\kappa = 1$ and from stable to Gaussian limiting distributions across $\kappa = 2$. We will not give the details in this section, and instead refer the reader to the last chapter where similar considerations are given to the boundary cases of the “have your cookie and eat it” random walk model.

Table 2.1. Limiting distributions for transient RWRE

Parameter	Limiting Dist. for T_n	Limiting Dist. for X_n
$\kappa \in (0,1)$	$\mathbb{P}\left(\frac{T_n}{n^{1/\kappa}} \le x\right) \to L_{\kappa,b}(x)$	$\mathbb{P}\left(\frac{X_n}{n^{\kappa}} \le x\right) \to 1 - L_{\kappa,b}(x^{-1/\kappa})$
$\kappa \in (1,2)$	$\mathbb{P}\left(\frac{T_n - n/v}{n^{1/\kappa}} \le x\right) \to L_{\kappa,b}(x)$	$\mathbb{P}\left(\frac{X_n - nv}{v^{1+1/\kappa}n^{1/\kappa}} \le x\right) \to 1 - L_{\kappa,b}(-x)$
$\mathbb{E}\left[\left(\frac{1-\alpha_0}{\alpha_0}\right)^2\right] < 1$	$\mathbb{P}\left(\frac{T_n - n/v}{\sigma\sqrt{n}} \le x\right) \to \Phi(x)$	$\mathbb{P}\left(\frac{X_n - nv}{v^{3/2}\sqrt{n}} \le x\right) \to \Phi(x)$

Limiting distributions for ERW in various types of cookie environments have been established using this same method. In the M-cookies case for instance, one has that $P(W_1 > n) \sim n^{-\delta/2}$ and $P(\sigma_1 > n) \sim n^{-\delta}$. Recall that the M-cookie ERW is transient to ∞ when $\delta > 1$ and has positive speed when $\delta > 2$. By the discussion above, we expect that the M-cookie ERW has limiting distributions which are transformations of $\delta/2$-stable distributions when $\delta \in (1,4)$. When $\delta \ge 4$, the M-cookie ERW admits Gaussian limiting distributions. Basdevant and Singh [3] handled the case $\delta \in (1,2)$ (using generating function methods instead of the technique described above), Kosygina and Mountford [15] handled the case $\delta \in (2,4)$, and Kosygina and Zerner [18] dealt with the case $\delta \ge 4$. Their results are summarized in the table below. For a self-contained reference, we recommend Kosygina and Zerner's thorough survey paper [17].

In the above table $D(n)$ and $\Gamma(n)$ are functions such that $D(n) \sim \log n$ and $\Gamma(n) \sim 1/\log n$, and c is the constant such that (when $\delta = 2$) we have

$$\frac{T_n}{n \log n} \to \frac{1}{c}, \qquad \frac{X_n}{n/\log n} \to c.$$

Remark: The borderline case $\delta = 2$ (which corresponds to $\kappa = 1$ in RWRE) is especially delicate. From the tail asymptotics on W and σ we know that the centering term for T_n should be on the order of $n \log n$, and therefore the centering term X_n should be on the order of $n/\log n$. However, it turns out that significantly more work is required to translate the limiting distribution for T_n into a limiting distribution for X_n in this case. To see

Table 2.2. Limiting Distributions for transient M-cookie ERW

Parameter	Limiting Dist. for T_n	Limiting dist. for X_n
$\delta \in (1,2)$	$\frac{T_n}{n^{2/\delta}} \Rightarrow L_{\delta/2,b}$	$\frac{X_n}{n^{\delta/2}} \Rightarrow \left(L_{\delta/2,b}\right)^{-\delta/2}$
$\delta = 2^*$	$\frac{T_n - nD(n)/c}{n} \Rightarrow L_{1,b}$	$\frac{X_n - cn\Gamma(n)}{c^2 n \log^{-2} n} \Rightarrow -L_{1,b}$
$\delta \in (2,4)$	$\frac{T_n - n/v}{n^{2/\delta}} \Rightarrow L_{\delta/2,b}$	$\frac{X_n - nv}{v^{1+2/\delta} n^{2/\delta}} \Rightarrow -L_{\delta/2,b}$
$\delta = 4$	$\frac{T_n - n/v}{\sqrt{n \log n}} \Rightarrow L_{2,b}$	$\frac{X_n - nv}{v^{3/2}\sqrt{n \log n}} \Rightarrow -L_{2,b}$
$\delta > 4$	$\frac{T_n - n/v}{\sqrt{n}} \Rightarrow L_{2,b}$	$\frac{X_n - nv}{v^{3/2}\sqrt{n}} \Rightarrow -L_{2,b}$

the complete details of the argument worked out in the case of Markovian cookie stacks, we refer the reader to Appendix B of [16]. Our proof of the limiting distribution in the corresponding case for the "have your cookie and eat it" random walk, which follows very closely to Appendix B of [16], can be found in Section 4.3.3.

We will not give an exhaustive list of ERW limiting distribution results here, but should point out that the limiting distributions for M-cookie ERW also arise in the case of other ERW models, including the Markov cookie stacks model (of which periodic cookie stacks is a special case) which was studied by Kosygina and Peterson [16]. The long-term behavior of ERW with Markov cookie stacks is determined by the parameters $\bar{p}$, θ, and $\tilde{\theta}$. In short, if $\bar{p} \neq 1/2$ the walk admits Gaussian limiting distributions, and if $\bar{p} = 1/2$ the limiting distributions (along with recurrence/transience and ballisticity) depend on the parameters θ and $\tilde{\theta}$ as defined in Section 2.3.1. In fact, the particular limiting distributions in the critical case $\bar{p} = 1/2$ of the Markov cookies model matches those in the table above, with δ replaced by θ.

3. EXCITED RANDOM WALK IN FINITE-DRIFT ENVIRONMENTS

This section contains material that has been posted as part of a preprint to arXiv.org by the author [20] and submitted for publication. The preprint version can be accessed at https://arxiv.org/abs/2103.05570.

3.1 Description of model

In this section, we will study excited random walk in environments with deterministic, identical cookie stacks at each site. Let $\boldsymbol{p} = (p_1, p_2, p_3, \ldots)$ be a vector of cookie strengths, and for each $x \in \mathbb{Z}$ set $\omega_x^{\mathrm{j}} = p_{\mathrm{j}}$. Throughout this section, we will refer to the environment with cookie stack $\boldsymbol{p}$ at each site as the cookie environment $\boldsymbol{p}$. We will assume that the environment is elliptic, so that $p_{\mathrm{i}} \in (0,1)$ for all $\mathrm{i} \in \mathbb{N}$, and that the total drift

$$\delta(\boldsymbol{p}) = \lim_{n\to\infty} \sum_{\mathrm{j}=1}^{n} (2p_{\mathrm{j}} - 1) \tag{3.1}$$

exists and is finite. In particular, $\boldsymbol{p}$ can potentially contain infinitely many non-placebo cookies at each site, and so this model can be viewed as an extension of the M-cookies model studied in [2], [15], [18]. Previously-studied ERW models that allowed for infinitely many cookies at each site required special structure within the cookie stack: that the cookie stacks be periodic [19], generated by independent copies of a Markov chain at each site [16], or that all of the cookies be non-negative [25], i.e. $\omega_x^{\mathrm{j}} \geq 1/2$ for all $x \in \mathbb{Z}$ and $\mathrm{j} \in \mathbb{N}$. In the first two cases, the parameter $\delta = \lim_{n\to\infty} \mathbb{E}\left[\sum_{\mathrm{j}=1}^{n}(2p_{\mathrm{j}} - 1)\right]$ may not exist, and all three could have $\delta \in \{-\infty, \infty\}$. Before stating the main results of this chapter, we define some additional notation. We will frequently need to refer to the total drift contained in the first m cookies of $\boldsymbol{p}$, which we will represent by δ_m :

$$\delta_m = \sum_{\mathrm{j}=1}^{m} (2p_{\mathrm{j}} - 1).$$

Remark: In this chapter, we will use T_n to represent the trial in the sequence of Bernoulli trials $\{\xi_{\mathrm{j}}\}_{\mathrm{j}\geq 1}$ on which the nth failure occurs, i.e.

$$T_n = \inf\{m : \sum_{k=1}^{m}(1-\xi_k) = n\}.$$

This differs from the other chapters where T_n denotes the time the walk reaches level n for the first time. We hope that no confusion arises from this difference in notation. Before stating our main results, we collect a few examples of finite-drift cookie environments that we will refer to throughout the chapter.

Example 1 (Positive Cookie Environment)**.** Zerner [25] studied ERW with infinitely many *positive* cookies at each site. For instance, let

$$p_{\mathrm{j}} = \frac{1}{2} + \left(\frac{1}{2}\right)^{\mathrm{j}+1}.$$

Then, we have that $\delta = \sum \left(\frac{1}{2}\right)^{\mathrm{j}} = 1$. According to [25], ERW in this environment is recurrent.

Example 2 (Alternating Environments)**.** Environments where the p_{j} are chosen such that $\delta = \sum(2p_{\mathrm{j}} - 1)$ is an alternating series will be useful as examples. Suppose $f : \mathbb{N} \to (0, 1)$ with the property that $f(n)$ decreases monotonically to 0 as $n \to \infty$. Let

$$p_{\mathrm{j}} = \frac{1}{2} + \frac{(-1)^{\mathrm{j}} f(\mathrm{j})}{2},$$

so that we have

$$\delta = \sum_{\mathrm{j}=1}^{\infty}(-1)^{\mathrm{j}} f(\mathrm{j}).$$

Note that $f(\mathrm{j}) = |2p_{\mathrm{j}} - 1|$. This class of examples will be useful because δ necessarily exists and is finite, but $|2p_{\mathrm{j}} - 1|$ can tend to 0 arbitrarily slowly.

Example 3 (Spaced Out Cookies)**.** When constructing finite-drift cookie environments, we can manipulate how quickly the series that defines δ converges by adding blocks of placebo cookies between the cookies in the stack that contain nonzero drift. For instance, we could define

$$p_{\mathrm{j}} = \begin{cases} \frac{5}{6} & k = 1, 2, 3 \\ \frac{1}{2} - \left(\frac{1}{2}\right)^{m+1} & k = 4^{4^m},\ m = 1, 2, \ldots \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$

In this case, we will have that

$$\delta = \sum_{\mathrm{j}=1}^{\infty}(2p_{\mathrm{j}} - 1) = 2 + \sum_{m=1}^{\infty} -\left(\frac{1}{2}\right)^m = 1.$$

Note, however, that $\delta_m \to 1$ very slowly. Indeed, for $m > 3$ we have

$$|\delta_m - \delta| = 1 + \sum_{\mathrm{j}=4}^{m}(2p_{\mathrm{j}} - 1),$$

and the value of $\sum_{\mathrm{j}=4}^{m}(2p_{\mathrm{j}} - 1)$ remains stable for (increasingly) long periods of time. For reference, the sum first becomes nonzero when $m = 256$, then next changes when $m \approx 4 \times 10^9$. However, with the placebo cookies removed the series converges exponentially fast, since in that case (for $m > 3$):

$$|\delta_m - \delta| = 1 - \sum_{n=4}^{m}\left(\frac{1}{2}\right)^{m-3} = 2^{-m}.$$

We will conclude this chapter by showing that ERW in the environment of Example 3 is transient even though the total drift at each site $\delta = 1$.

3.2 Main Results

Our first result is a criteria for recurrence/transience of ERW in $\boldsymbol{p}$.

Theorem 3.2.1. *Let $\{X_n\}_{n\geq 0}$ be an excited random walk in a deterministic, identically-piled elliptic cookie environment $\boldsymbol{p}$ with finite total drift $\delta(\boldsymbol{p})$.*

1. *If $\delta > 1$, then $P(\lim_{n\to\infty} X_n = \infty) = 1$.*

2. *If* $\delta < -1$*, then* $P(\lim_{n\to\infty} X_n = -\infty) = 1$*.*

3. *If* $|\delta| < 1$*, then* $P(X_n = 0$ *infinitely often*$) = 1$*.*

4. *If* $|\delta| = 1$ *and it holds that*

$$\left|\sum_{\mathrm{j}=n}^{\infty}(2p_{\mathrm{j}} - 1)\right| = o\left(\frac{1}{\log n}\right), \tag{3.2}$$

   then $P(X_n = 0$ *infinitely often*$) = 1$*.*

We should note that Theorem 3.2.1 does not characterize the behavior of ERW in $\boldsymbol{p}$ where $\delta = 1$ and the tail of the series in (3.1) tends to 0 slower than $\frac{1}{\log n}$. It is worthwhile to discuss what happens in the corresponding boundary cases in similar models: for ERW in M-cookie environments or environments with only (potentially infinitely many) positive cookies at each site, when $\delta = 1$ the walk is recurrent. In the cases of periodic or Markovian cookie stacks, the parameter $\delta = \lim_{n\to\infty} \mathbb{E}\left[\sum_{\mathrm{j}=1}^{n}(2p_{\mathrm{j}} - 1)\right]$ may not exist or could be $\pm\infty$. The corresponding borderline case in those models occurs when $\theta = 1$, where θ is the parameter of the FBLP associated to the walk defined in Section 2.2.1 above. In fact, we will soon show that for ERW in $\boldsymbol{p}$, we have $\theta = \delta(\boldsymbol{p})$. For ERW in periodic or Markovian cookie stacks, the walk is recurrent when $\theta = 1$. In light of these facts, it would be natural to suspect that ERW in $\boldsymbol{p}$ is *always* recurrent when $\delta = 1$. We will not prove that the condition in (3.2) is necessary for ERW in $\boldsymbol{p}$ to be recurrent, but we do show that it is possible for ERW in $\boldsymbol{p}$ to be transient even when $|\delta| = 1$.

Theorem 3.2.2. *There exist cookie environments* $\boldsymbol{p}$ *such that* $|\delta| = 1$ *and excited random walk in* $\boldsymbol{p}$ *is transient.*

Theorem 3.2.2 shows that ERW in $\boldsymbol{p}$ can contain behavior that was not present in previously-studied ERW models with infinitely many cookies per site. Indeed, determing recurrence/transience for ERW in $\boldsymbol{p}$ depends not only on the FBLP parameters μ, θ, and $\tilde{\theta}$, but also on the rate of convergence of the series that defines δ. We will prove Theorem 3.2.2 by showing that ERW in the cookie environment from Example 3 is transient.

Our main tool for proving Theorem 3.2.1 will be Theorem 2.3.4, which was used by Kozma, Orenshtein, and Shinkar [19] to prove a recurrence/transience criteria for ERW with periodic cookie stacks and by Kosygina and Peterson [16] to do the same for Markovian cookie stacks.

Before proceeding, it will be useful to recall that parameters of the FBLP that we are interested in. If we use the notation $P_n(\cdot) := P(\cdot|U_0 = n)$ with expectation $\mathbb{E}_n[\cdot] := \mathbb{E}[\cdot|U_0 = n]$, then we define the parameters of interest by

$$\mu(n) = \frac{\mathbb{E}_n[U_1]}{n} \qquad \rho(n) = \mathbb{E}_n[U_1 - \mu n]$$
$$\nu(n) = \frac{\mathrm{Var}(U_1|U_0 = n)}{n} \qquad \theta(n) = \frac{2\rho(n)}{\nu(n)},$$

where μ, ρ, ν, and θ denote the respective limits as $n \to \infty$, provided they exist. Recall also that we will use a tilde to mark the parameters of the left-excursion FBLP $\tilde{U}$: $\tilde{\mu}$, $\tilde{\rho}$, $\tilde{\nu}$, and $\tilde{\theta}$.

Remark: Note that the way we define $\nu(n)$ above is slightly different from the definition given in Section 2.3.1. As we will see later, defining $\nu(n)$ in this way does not make a difference for our purposes. To see exactly why this is the case, see the proof of Lemma 2 below.

We will show in the next section that, under the assumptions of our model, we always have $\mu = 1$, and are therefore always in the "critical case" of Theorem 2.3.4. Therefore, we will give a version of Theorem 2.3.4 specialized to that particular case.

Theorem 3.2.3 (Theorem 1.3 of [19])**.** *Let $\{U_n\}_{n\geq 0}$ be the forward branching-like process associated to ERW in $\boldsymbol{p}$. Assume that $\mu = \lim_{n\to\infty} \frac{\mathbb{E}_n[U_1]}{n} = 1$, and that there is a constant C such that for all n sufficiently large and for all $\epsilon > 0$ it holds that*

$$P_n\left(\left|\frac{U_1}{n} - 1\right| > \epsilon\right) \leq 2\exp\left\{-\frac{C\epsilon^2 n}{2+\epsilon}\right\}.$$

Then

1. *if* $\theta(n) < 1 + \frac{1}{\log n} - \alpha(n)n^{-1/2}$ *for all sufficiently large* n*, where* $\alpha(n)$ *is such that* $\alpha(n)\nu(n) \to \infty$ *as* $n \to \infty$*, then* $P(U_n = 0 \text{ for some } n) = 1$*, and*

2. *if* $\theta(n) > 1 + \frac{2}{\log n} + \alpha(n)n^{-1/2}$ *for all sufficiently large* n*, where* $\alpha(n)$ *is such that* $\alpha(n)\nu(n) \to \infty$ *and* $n \to \infty$*, then* $P(U_n > 0 \text{ for all } n) > 0$.

In short, we will prove Theorem 3.2.1 by showing that the FBLPs are concentrated about their means in the necessary way, then compute the parameters of the FBLPs and their respective rates of convergence. We close this section by verifying that the FBLPs are concentrated. The proof is taken directly from [20].

Theorem 3.2.4 (Concentration of FBLP)**.** *Let* U_n *be the forward branching-like process associated to ERW in* $\boldsymbol{p}$*. Then*

$$P_n\left(\left|\frac{U_1}{n} - 1\right|\right) \leq 2\exp\left\{-\frac{C_1\epsilon^2 n}{2+\epsilon}\right\}. \tag{3.3}$$

Proof. First, we note that

$$P_n(U_1 > m) = P\left(\sum_{\mathrm{i}=1}^{m+n} \xi_{\mathrm{i}} > m\right),$$

where $\{\xi_{\mathrm{i}}\}_{\mathrm{i}\geq 1}$ are the "coin-toss" Bernoulli random variables, $\xi_{\mathrm{i}} \sim \mathrm{Ber}(p_{\mathrm{i}})$. By centering, we see that

$$\begin{aligned}
P\left(\sum_{\mathrm{i}=1}^{m+n} \xi_{\mathrm{i}} > m\right) &= P\left(\sum_{\mathrm{i}=1}^{m+n} (\xi_{\mathrm{i}} - p_{\mathrm{i}}) > m - \sum_{\mathrm{i}=1}^{m+n} p_{\mathrm{i}}\right) \\
&= P\left(\sum_{\mathrm{i}=1}^{m+n} (\xi_{\mathrm{i}} - p_{\mathrm{i}}) > \frac{m-n}{2} - \frac{1}{2}\sum_{\mathrm{i}=1}^{m+n} (2p_{\mathrm{i}} - 1)\right) \\
&= P\left(\sum_{\mathrm{i}=1}^{m+n} (\xi_{\mathrm{i}} - p_{\mathrm{i}}) > \frac{m-n-\delta_{m+n}}{2}\right).
\end{aligned}$$

Note that when $m > n$ and is sufficiently large, $(m - n - \delta_{n+m}/2 > 0$ because the sequence $\{\delta_k\}_{k\geq 1}$ is bounded. Applying Hoeffding's inequality yields

$$\begin{aligned}
P_n(U_1 > m) &= P\left(\sum_{\mathrm{i}=1}^{m+n}(\xi_{\mathrm{i}} - p_{\mathrm{i}}) > \frac{m-n-\delta_{m+n}}{2}\right) \\
&\leq \exp\left\{-\frac{((m-n)-\delta_{m+n})^2}{2(m+n)}\right\} \\
&= \exp\left\{-\frac{(m-n)^2 - 2(m-n)\delta_{m+n} + \delta_{m+n}^2}{2(m+n)}\right\} \\
&\leq \exp\left\{-\frac{(m-n)^2}{2(m+n)}\right\}\exp\left\{\frac{(m-n)\delta_{m+n}}{m+n}\right\} \qquad (3.4)
\end{aligned}$$

as long as m is large enough. A corresponding bound on $P_n(U_1 < m)$ can be obtained in a similar way. We can use the bound in (3.4) to obtain the necessary concentration bound:

$$\begin{aligned}
P_n\left(\left|\frac{U_1}{n} - 1\right| > \epsilon\right) &= P_n\left(|U_1 - n| > n\epsilon\right) \\
&= P_n\left(U_1 > n(1+\epsilon)\right) + P_n\left(U_1 < n(1-\epsilon)\right) \\
&\leq P_n\left(U_1 > \lfloor n(1+\epsilon)\rfloor\right) + P_n\left(U_1 < \lceil n(1-\epsilon)\rceil\right) \\
&\leq 2\exp\left\{-\frac{C_2 n^2\epsilon^2}{(2+\epsilon)n}\right\}\exp\left\{\frac{C_3\epsilon\delta_{\lfloor(2+\epsilon)n\rfloor}}{2+\epsilon}\right\} \\
&\leq 2\exp\left\{-\frac{C_4 n\epsilon^2}{2+\epsilon}\right\},
\end{aligned}$$

where the constant C_4 depends only on the sequence $\{\delta_k\}_{k\geq 1}$. □

As a final note, observe that since conditional on the event $\{U_0 = n\}$

$$T_n := T_n = \inf\{m : \sum_{k=1}^{m}(1 - \xi_k) = n\} = U_1 + n,$$

since in that case U_1 counts the number of successes in $\{\xi_{\mathrm{j}}\}_{\mathrm{j}\geq 1}$ before the nth failure. Therefore, we can use Theorem 3.2.4 to put a concentration bound on T_n as well:

$$P\left(\left|\frac{T_n}{n} - 2\right| > \epsilon\right) \leq 2\exp\left\{-\frac{C_4\epsilon^2 n}{2+\epsilon}\right\}. \qquad (3.5)$$

3.3 Calculating Parameters of the Forward Branching-Like Process

We will now compute the parameters of the forward branching-like process and establish bounds on their rates of convergence where needed. Before we do so, note that by symmetry the left-excursion FBLP $\{\tilde{U}_n\}_{n\geq 0}$ for ERW in $\boldsymbol{p}$ has the same distribution as $\{U_n\}_{n\geq 0}$ in the *reflected environment* $\mathbf{1}-\boldsymbol{p}=(1-p_1,1-p_2,\ldots)$. Therefore, it will suffice to compute the parameters μ, ρ, ν, and θ, then use this symmetry to obtain the parameters of $\tilde{U}$. Furthermore, any rates of convergence we establish for the parameters of U will carry over to $\tilde{U}$ due to this symmetry. With this in mind, we will compute the parameters of U.

Theorem 3.3.1 (Parameters of FBLP)**.** *Let $\boldsymbol{p}$ be a cookie environment with finite total drift $\delta(\boldsymbol{p})$, and let $\{U_n\}_{n\geq 0}$ be the forward branching-like process associated to the excited random walk in $\boldsymbol{p}$. Then*

1. *We have $\mu=1$, $\rho=\delta(\boldsymbol{p})$, $\nu=2$, and $\theta=\delta(\boldsymbol{p})$.*
2. *We have $|\nu(n)=2|=\mathcal{O}\left(n^{-1/2}\log^4 n+b_n\log^4 b_n\right)$, where $b_n=\frac{1}{4n}\sum_{j=1}^{n}(2p_j-1)^2$. Under the assumption in (3.2), we have that $|\nu(n)-2|=o\left(\frac{1}{\log n}\right)$.*
3. *Under the assumption in (3.2), $|\rho(n)-\delta|=o\left(\frac{1}{\log n}\right)$.*
4. *Under the assumption in (3.2), $|\theta(n)-\delta|=o\left(\frac{1}{\log n}\right)$.*

Before proving Theorem 3.3.1, we will show how to prove the recurrence/transience criteria for ERW in finite drift environments. Our proof is taken directly from [20].

Proof of Theorem 3.2.1 assuming Theorem 3.3.1. Let $\boldsymbol{p}$ be a cookie environment with finite total drift $\delta(\boldsymbol{p})$. Let $\{U_n\}_{n\geq 0}$ and $\{\tilde{U}\}_{n\geq 0}$ be the forward branching-like processes associated to the ERW in $\boldsymbol{p}$.

Intuitively, while the branching-like process U is positive the ERW takes an excursion to the right, and when the branching-like process dies out the ERW returns to 0. In a similar way, $\tilde{U}$ tracks left excursions of the ERW (by symmetry, a left excursion in the cookie environment $\boldsymbol{p}$ corresponds to a right excursion in the "reflected cookie environment" $\mathbf{1}-\boldsymbol{p}$). We showed in Theorem 3.2.4 that both the branching-like processes are concentrated in the way that we need to apply Theorem 3.2.3, and a quick calculation shows that $\delta(\mathbf{1}-\boldsymbol{p}) = -\delta(\boldsymbol{p})$. We consider each of the cases described in Theorem 3.2.1 in turn:

- Suppose that $\delta(\boldsymbol{p}) > 1$, and therefore $\delta(\mathbf{1}-\boldsymbol{p}) < -1$. By Theorem 3.2.3, $P_1(U_n > 0 \text{ for all } n) > 0$ and $\tilde{P}_1(\tilde{U}_n = 0 \text{ for some } n) = 1$. Therefore the ERW in $\boldsymbol{p}$ is transient to ∞ a.s. by Theorem 2.3.3.

- Similarly, when $\delta(\boldsymbol{p}) < -1$, we have $\delta(\mathbf{1}-\boldsymbol{p}) > 1$. It then follows from Theorem 3.2.3 that we have $P_1(U_n = 0 \text{ for some } n) = 1$ and $\tilde{P}_1(\tilde{U}_n > 0 \text{ for all } n) > 0$. Therefore ERW in $\boldsymbol{p}$ is transient to $-\infty$ a.s. in this case.

- In the case that $\delta(\boldsymbol{p}) \in (-1, 1)$, we also have that $\delta(\mathbf{1}-\boldsymbol{p}) \in (-1, 1)$. By Theorem 3.2.3, we see that $P_1(U_n = 0 \text{ for some } n) = \tilde{P}_1(\tilde{U}_n = 0 \text{ for some } n) = 1$, and so in this case the ERW in $\boldsymbol{p}$ is recurrent a.s.

- In the critical case where $\delta = \pm 1$, extra care is needed. In this case, we must rely on the fact that the assumption in (3.2) is sufficient to guarantee that part I. of Theorem 3.2.3 is satisfied. Under that assumption, we can apply the theorem to show that the ERW in $\boldsymbol{p}$ is recurrent a.s. when $\delta = \pm 1$.

$\square$

To complete our proof of Theorem 3.2.1, we only need to compute the parameters of the FBLPs and determine their rates of convergence.

3.3.1 Computing ρ and μ

Conditional on $\{U_0 = n\}$, we can express U_1 as

$$U_1 = \sum_{k=1}^{T_n} \xi_k,$$

where $T_n = \inf\{k : \sum_{k=1}^{m}(1-\xi_k) = n\}$ is the trial in the sequence $\{\xi_{\mathrm{j}}\}_{\mathrm{j}\geq 1}$ on which the nth failure occurs. We will now related T_n and U_1 in two different ways. First, we have that

$$\mathbb{E}_n[U_1] = \mathbb{E}\left[\sum_{k=1}^{T_n} p_k\right] = \mathbb{E}\left[\frac{1}{2}\sum_{k=1}^{T_n}(2p_k-1) + \frac{T_n}{2}\right] = \frac{1}{2}\left(\mathbb{E}\left[\delta_{T_n}\right] + \mathbb{E}[T_n]\right),$$

where the first equality follows from Lemma 7, a version of Wald's identity (see Section 3.5 below for details). We then have that $\mathbb{E}[T_n] = 2\mathbb{E}_n[U_1] - \mathbb{E}[\delta_{T_n}]$. To relate T_n and U_1 another way, recall that on the event $\{U_0 = n\}$, $T_n = U_1 + n$, and so $\mathbb{E}[T_n] = \mathbb{E}_n[U_1] + n$. Equating these expressions and solving for $\mathbb{E}_n[U_1]$ gives

$$\mathbb{E}_n[U_1] = n + \mathbb{E}[\delta_{T_n}],$$

and therefore

$$\rho = \lim_{n\to\infty} \mathbb{E}_n[U_1 - n] = \lim_{n\to\infty} \mathbb{E}_n[\delta_{T_n}] = \delta(\boldsymbol{p}),$$

where for the final equality we have used the fact that $T_n \geq n$, the sequence $\{\delta_k\}_{k\geq 1}$ is bounded, and the dominated convergence theorem. Using the assumption in (3.2), we can bound the rate at which $\rho_n \to \rho$:

$$|\rho(n) - \rho| = |\mathbb{E}[\delta_{T_n} - \delta]| = \left|\mathbb{E}\left[\sum_{\mathrm{j}=T_n+1}^{\infty}(2p_{\mathrm{j}}-1)\right]\right| \leq \sup_{m\geq n}\left|\sum_{\mathrm{j}=m+1}^{\infty}(2p_{\mathrm{j}}-1)\right| = o\left(\frac{1}{\log n}\right).$$

We can also use this expression to compute μ:

$$\mu = \lim_{n\to\infty}\frac{\mathbb{E}[U_1]}{n} = 1.$$

We now turn our attention to calculating ν, which will prove to be much more involved.

3.3.2 Computing ν

We begin by defining two additional parameters associated to the cookie environment $\boldsymbol{p}$: the average cookie strength contained in the first n cookies given by $\bar{p}_n = \frac{1}{n}\sum_{\mathrm{j}=1}^{n}$ and the “average sample variance” in cookie strengths experienced by the walker in the first n visits to a site given by $A_n = \frac{1}{n}\sum_{\mathrm{j}=1}^{n} p_{\mathrm{j}}(1-p_{\mathrm{j}})$. Since we assume that $\delta(\boldsymbol{p})$ exists and is finite, we expect that $\bar{p}_n \to 1/2$ and $A_n \to 1/4$. The next lemma shows that this is indeed the case.

Lemma 1. *Let $\boldsymbol{p}$ be a deterministic, identically-piled elliptic cookie environment such that $\delta(\boldsymbol{p})$ exists and is finite, and let $\bar{p}_n = \frac{1}{n}\sum_{\mathrm{j}=1}^{n} p_{\mathrm{j}}$ and $A_n = \frac{1}{n}\sum_{\mathrm{j}=1}^{n} p_{\mathrm{j}}(1-p_{\mathrm{j}})$. Then*

(a) $\bar{p}_n \to \frac{1}{2}$ as $n \to \infty$, and there exists a constant C_5 such that $|\bar{p}_n - \frac{1}{2}| \le \frac{C_5}{n}$ for all $n \in \mathbb{N}$.

(b) $A_n \to \frac{1}{4}$ as $n \to \infty$.

Proof of Lemma 1. To prove part (a), we observe that

$$\left|\bar{p}_n - \frac{1}{2}\right| = \frac{1}{n}\left|\sum_{\mathrm{j}=1}^{n}\left(p_{\mathrm{j}} - \frac{1}{2}\right)\right| = \frac{|\delta_n|}{2n}.$$

The fact that $\{\delta_k\}_{k\ge 1}$ is convergent, hence bounded, proves the claim. Similar considerations take care of part (b):

$$\left|A_n - \frac{1}{4}\right| = \frac{1}{n}\left|\sum_{\mathrm{j}=1}^{n}(p_{\mathrm{j}}(1-p_{\mathrm{j}}) - 1/4)\right| = \frac{1}{n}\left|\sum_{\mathrm{j}=1}^{n}\left(p_{\mathrm{j}} - \frac{1}{2}\right)\left(\frac{1}{2} - p_{\mathrm{j}}\right)\right| = \frac{1}{4n}\left|\sum_{\mathrm{j}=1}^{n}(2p_{\mathrm{j}} - 1)^2\right|.$$

We assume that $\sum(2p_{\mathrm{j}} - 1)$ converges, and so $(2p_{\mathrm{j}} - 1) \to 0$ as $\mathrm{j} \to \infty$. Therefore $\sum_{\mathrm{j}=1}^{n}(2p_{\mathrm{j}} - 1)^2 = o(n)$, and so $A_n \to \frac{1}{4}$ as $n \to \infty$. $\square$

Remark: Note that although we have good control over how fast $\bar{p}_n \to \frac{1}{2}$, we could not expect the same for A_n. For instance, if we consider Example 2 with

$$p_{\mathrm{j}} = \frac{1}{2} + \frac{(-1)^{\mathrm{j}}}{2\sqrt{\mathrm{j}+1}},$$

then clearly $\delta(\boldsymbol{p}) = \sum_{\mathrm{j}=1}^{\infty}(-1)^{\mathrm{j}}(\mathrm{j}+1)^{-1/2}$ converges since it is an alternating series and $(\mathrm{j}+1)^{-1/2} \searrow 0$. However,

$$\left|A_n - \frac{1}{4}\right| = \frac{1}{4n}\sum_{\mathrm{j}=1}^{n}\frac{1}{\mathrm{j}+1} \approx \frac{\log(n)}{4n}.$$

As we shall see below, this issue crops up when we attempt to calculate ν.

We will now introduce our main tool for calculating ν, which is a modified version of Lemma 2.5 of [19].

Lemma 2 (Modified from Lemma 2.5 of [19])**.** *Let $\boldsymbol{p}$ be a deterministic, identically-piled elliptic cookie environment such that $\delta(\boldsymbol{p})$ exists and is finite, and let $b_n = |A_n - \frac{1}{4}|$. Then*

$$\nu = \lim_{n\to\infty} \nu(n) = \lim_{n\to\infty} \frac{1}{n}\mathrm{Var}(U_1|U_0 = n) = 2.$$

Furthermore, we can bound the rate of convergence:

$$|\nu(n) - 2| = \left|\frac{1}{n}\mathrm{Var}(U_1|U_0 = n) - 2\right| = \mathcal{O}\left(\frac{\log^4 n}{\sqrt{n}} + b_n \log^4 b_n\right),$$

where the constant implied by $\mathcal{O}(\cdot)$ depends only on the cookie environment $\boldsymbol{p}$.

Remark: Note that $n^{-1/2}\log^4 n$ will be the dominant term when $b_n \ll n^{-1/2}$ (in the example given above, for instance), but the other term will dominate when $b_n \gg n^{-1/2}$. Consider Example 2 where the cookie strengths decay to 1/2 very slowly, say with

$$p_{\mathrm{j}} = \frac{1}{2} + \frac{(-1)^{\mathrm{j}}\sqrt{(\mathrm{j}+1)^{2/3} - \mathrm{j}^{2/3}}}{2}.$$

Then we will have $b_n = \frac{1}{4n}\sum_{\mathrm{j}=1}^{n}\left((\mathrm{j}+1)^{2/3} - \mathrm{j}^{2/3}\right) = \frac{(n+1)^{2/3}}{4n} \gg n^{-1/2}$.

We will now prove Lemma 2. The proof we present below is taken directly from [20], though we make the necessary adjustments to notation and include a few additional details.

Proof of Lemma 2. The proof is the same in spirit as the proof of Lemma 2.5 in [19]. The key difference comes in obtaining the rate of convergence for $\frac{1}{n}\mathbb{E}_n\left[(U_1 - n)^2\right]$, and we will point out adjustments that need to be made along the way. First, we can use the fact that $\mathbb{E}_n[U_1] = n + \mathbb{E}[\delta_{T_n}]$ to see that

$$\mathrm{Var}(U_1|U_0 = n) = \mathbb{E}\left[(U_1 - n)^2\right] - \mathbb{E}[\delta_{T_n}]^2,$$

and so there exists a constant C such that $\frac{1}{n}\left|\mathrm{Var}(U_1|U_0=n)-\mathbb{E}[(U_1-n)^2]\right| \leq \frac{C}{n}$. It will therefore suffice to work with $\mathbb{E}[(U_1-n)^2]$. We will start by rewriting this quantity:

$$\begin{aligned}
\mathbb{E}_n\left[(U_1-n)^2\right] &= \sum_{t=0}^{\infty}(2t+1)P_n\left(|U_1-n|>t\right) \\
&= 2\sum_{t=0}^{\infty} t\cdot P_n\left(|U_1-n|>t\right) + \sum_{t=0}^{\infty} P_n(|U_1-n|)>t \\
&= 2\sum_{t=0}^{\infty} t\cdot P_n(|U_1-n|>t) + \mathbb{E}_n\left[|U_1-n|\right].
\end{aligned}$$

Due to the concentration bound in Theorem 3.2.4, $\mathbb{E}_n[|U_1-n|]=\mathcal{O}(\sqrt{n})$, and so $\frac{1}{n}\mathbb{E}_n[|U_1-n|]=\mathcal{O}(n^{-1/2})$. Therefore, to prove our claim we only need to bound

$$\left|\frac{1}{n}\sum_{t=0}^{\infty} t\cdot P_n(|U_1-n|>t)-1\right|.$$

In fact, we will show that for any sequence $a=a_n$ such that $a_n \overset{n\to\infty}{\longrightarrow} \infty$ and $a_n \leq \sqrt{n}$ (for n sufficiently large) that

$$\left|\frac{1}{n}\sum_{t=0}^{\infty} t\cdot P_n(|U_1-n|>t)-1\right| = \mathcal{O}\left(a^4 n^{-1/2} + \left|A_n-\frac{1}{4}\right|a^2 + \mathrm{e}^{-Ca}\right),$$

then we will choose an appropriate sequence a_n to obtain the conclusion. To obtain the bound we need, we will rewrite the events involving U_1 in terms of the number of failures that occur in the coin tosses that determine the value of U_1. Let F_n be the number of failures in this sequence of coin tosses after the nth toss:

$$F_n := \sum_{\mathrm{j}=1}^{n}(1-\xi_{\mathrm{j}}).$$

Then we have that

$$\begin{aligned}
\left|\frac{1}{n}\sum_{t=0}^{\infty} t\cdot P_n(|U_1-n|>t)-1\right| &= \left|\frac{1}{n}\sum_{t=0}^{\infty} t\left(P_n(U_1>n+t)+P(U_1<n-t)\right)-1\right| \\
&= \left|\frac{1}{n}\sum_{t=0}^{\infty} t\cdot\left(P(F_{2n+t}<n)+P(F_{2n-t-1}\geq n)\right)-1\right|.
\end{aligned}$$

As in the periodic cookies case, we divide the sum into its "head" and "tail." Let

$$H_n(a) = \sum_{t=0}^{\lfloor a\sqrt{n}\rfloor} t \cdot \left(P(F_{2n+t} < n) + P(F_{2n-t-1} \geq n)\right)$$
$$T_n(a) = \sum_{t=\lfloor a\sqrt{n}\rfloor}^{\infty} t \cdot \left(P(F_{2n+t} < n) + P(F_{2n-t-1} \geq n)\right),$$

where $a = a_n$ can be any sequence which grows slowly with n (again, we will specify an appropriate choice of a_n at the end of the proof). We can handle the tail $T_n(a)$ by using the following result from the periodic cookies setting.

Lemma 3 (Claim 2.8 of [19])**.** *Let $a = a_n$ such that $\lim_{n\to\infty} a_n = \infty$. Then for all sufficiently large $n \in \mathbb{N}$,*

$$\frac{1}{n} T_n(a) \leq C\mathrm{e}^{-C'a}.$$

The proof of Lemma 3 is the same as the proof of Claim 2.8 in [19] in our setting, but we include it for completeness.

Proof of Lemma 3. By the concentration bound in Theorem 3.2.4, we have

$$T_n(a) = \sum_{t=\lfloor a\sqrt{n}\rfloor+1}^{\infty} t \cdot P_n[|U_1 - n| > t] \leq \sum_{\mathrm{i}=a}^{\infty} \sum_{t=\lfloor \mathrm{i}\sqrt{n}\rfloor+1}^{\lfloor(\mathrm{i}+1)\rfloor\sqrt{n}} t \cdot \exp\left(-\frac{Ct^2}{2n+t}\right)$$

Each of the terms in the inner sum is bounded by $4\mathrm{i}\sqrt{n}\exp\left(-\frac{C\mathrm{i}^2 n}{2n+\mathrm{i}\sqrt{n}}\right) \leq 4\mathrm{i}\sqrt{n}\exp\left(-C\mathrm{i}\right)$ for some constant C. Then

$$T_n(a) \leq \sum_{\mathrm{i}=a}^{\infty} \sqrt{n} \cdot 4\mathrm{i}\sqrt{n} \cdot \exp\left(-C\mathrm{i}\right) \leq 4n \sum_{\mathrm{i}=a}^{\infty} \mathrm{i} \cdot \exp\left(-C\mathrm{i}\right) \leq Cn\exp\left(-Ca\right),$$

completing our proof of the claim. □

We now turn our attention to the head of the sum $H_n(a)$. The main thrust of the argument is that $H_n(a)$ can be approximated by a sum over the standard normal cdf Φ.

Lemma 4 (Claim 2.6 of [19])**.** *Let $a > 0$ and let $n \in \mathbb{N}$ be such that $a \leq \sqrt{n}$. Then*

$$\left|\frac{1}{n}H_n(a) - \frac{1}{n}\sum_{t=0}^{\lfloor a\sqrt{n}\rfloor} 2t \cdot \Phi\left(\frac{-t}{\sqrt{8A_n}}\right)\right| \leq C\left(\frac{a^4}{\sqrt{n}} + \left|A_n - \frac{1}{4}\right| a^2\right)$$

Lemma 5 (Claim 2.7 of [19])**.** *Let $a = a_n$ such that $\lim_{n\to\infty} a_n = \infty$. Then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{t=0}^{\lfloor a\sqrt{n}\rfloor} 2t \cdot \Phi\left(\frac{-t}{\sqrt{8A_n}}\right) = 1,$$

where Φ is the standard normal cdf. Moreover,

$$\frac{1}{n}\sum_{t=0}^{\lfloor a\sqrt{n}\rfloor} 2t \cdot \Phi\left(\frac{-t}{\sqrt{8A_n}}\right) = 1 + \mathcal{O}\left(\frac{a}{\sqrt{n}} + \exp\left(-Ca\right)\right).$$

The proof of Lemma 5 is the same as the proof of Claim 2.7 in [19] because it does not depend on the specifics of the model, and therefore we refer the curious reader there.

Proof of Lemma 4. Let $\bar{q}_n = \frac{1}{n}\sum_{\mathrm{j}=1}^{n}(1 - p_{\mathrm{j}})$, let $\sigma_{\mathrm{j}}^2 = \mathbb{E}\left[\left((1-\xi_{\mathrm{j}}) - (1-p_{\mathrm{j}})\right)^2\right] = p_{\mathrm{j}}(1-p_{\mathrm{j}})$, and let $\rho_{\mathrm{j}} = \mathbb{E}\left[\left|\left((1-\xi_{\mathrm{j}}) - (1-p_{\mathrm{j}})\right)^3\right|\right]$ for all $\mathrm{j} \in \mathbb{N}$. By the Berry-Esseen theorem, there is a constant C such that for all $\alpha \in \mathbb{R}$,

$$\left|P\left(\frac{F_n - n\bar{q}_n}{\sqrt{nA_n}} \leq \alpha\right) - \Phi(\alpha)\right| \leq C \cdot \left(\sum_{\mathrm{i}=1}^{n}\sigma_{\mathrm{i}}^2\right)^{-3/2} \cdot \left(\sum_{\mathrm{i}=1}^{n}\rho_{\mathrm{i}}\right),$$

where Φ is the standard normal cdf. Since $\rho_{\mathrm{i}} \leq 1$ for each i and A_n is bounded, we have that

$$\left(\sum_{\mathrm{i}=1}^{n}\sigma_{\mathrm{i}}^2\right)^{-3/2} \cdot \left(\sum_{\mathrm{i}=1}^{n}\rho_{\mathrm{i}}\right) \leq (nA_n)^{-3/2} \cdot n \leq Cn^{-1/2}.$$

Therefore

$$P\left(\frac{F_n - n\bar{q}_n}{\sqrt{nA_n}} \leq \alpha\right) = \Phi(\alpha) + \mathcal{O}(n^{-1/2}). \tag{3.6}$$

We will now prove that we can replace A_n and $\bar{q}_n$ by their respective limits $\frac{1}{4}$ and $\frac{1}{2}$. The following lemma shows that we can do just that, and is analogous to Claim 2.9 in [19].

Lemma 6.

$$\left|P\left(\frac{F_n - n\bar{q}_n}{\sqrt{nA_n}} \le \alpha\right) - P\left(\frac{F_n - \frac{1}{2}n}{\sqrt{n(1/4)}} \le \alpha\right)\right| \le Cn^{-1/2} + C'\left|A_n - \frac{1}{4}\right|.$$

Proof of Lemma 6. The proof follows the same technique as the proof of Claim 2.9 in [19], but we must keep track of the error in terms of $b_n = |A_n - 1/4|$:

$$\begin{aligned}
&\left|P\left(\frac{F_n - n\bar{q}_n}{\sqrt{nA_n}} \le \alpha\right) - P\left(\frac{F_n - \frac{1}{2}n}{\sqrt{n(1/4)}} \le \alpha\right)\right| \\
&\qquad = \left|P\left(\frac{F_n - n\bar{q}_n}{\sqrt{nA_n}} \le \alpha\right) - P\left(\frac{F_n - n\bar{q}_n}{\sqrt{nA_n}} \le \sqrt{\frac{(1/4)}{A_n}}\alpha + \frac{n\left(\frac{1}{2} - \bar{q}_n\right)}{\sqrt{nA_n}}\right)\right| \\
&\qquad \le \left|\Phi(\alpha) - \Phi\left(\sqrt{\frac{(1/4)}{A_n}}\alpha + \frac{n\left(\frac{1}{2} - \bar{q}_n\right)}{\sqrt{nA_n}}\right)\right| + Cn^{-1/2},
\end{aligned}$$

where this last inequality follows from (3.6). Using the fact that Φ is $\frac{1}{\sqrt{2\pi}}$−Lipschitz, we can bound this expression by

$$\frac{1}{\sqrt{2\pi}}\left|\alpha - \left(\sqrt{\frac{(1/4)}{A_n}}\alpha + \frac{n\left(\frac{1}{2} - \bar{q}_n\right)}{\sqrt{nA_n}}\right)\right| + Cn^{-1/2} \le \frac{1}{\sqrt{2\pi}}\left(\alpha\left|1 - \sqrt{\frac{(1/4)}{A_n}}\right| + \frac{n\left|\frac{1}{2} - \bar{q}_n\right|}{\sqrt{nA_n}}\right) + Cn^{-1/2}.$$

By Lemma 1, $\bar{q}_n = \frac{1}{2} + \mathcal{O}(1/n)$, and so we can bound $n\left(\frac{1}{2} - \bar{q}_n\right)$ by a constant to obtain

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\left(\alpha\left|1 - \sqrt{\frac{(1/4)}{A_n}}\right| + \frac{n\left|\frac{1}{2} - \bar{q}_n\right|}{\sqrt{nA_n}}\right) + Cn^{-1/2} &\le \frac{1}{\sqrt{2\pi}}\left(\alpha\left|1 - \sqrt{\frac{(1/4)}{A_n}}\right| + \frac{C}{\sqrt{nA_n}}\right) + Cn^{-1/2} \\
&\overset{*}{=} \frac{1}{\sqrt{2\pi}}\left(\alpha\frac{\left|A_n - \frac{1}{4}\right|}{|\sqrt{A_n}\left(\sqrt{A_n} + \frac{1}{4}\right)|} + \frac{C}{\sqrt{nA_n}}\right) + Cn^{-1/2},
\end{aligned}$$

where the starred equality follows from multiplying the numerator and denominator of the first term by $\sqrt{A_n} + \sqrt{\frac{1}{4}}$. Recall that $A_n \to \frac{1}{4}$, and moreover $A_n = \frac{1}{n}\sum_{j=1}^{n} p_j(1 - p_j) \ge$

$(\min_j p_j(1-p_j)) > 0$ because we assume that $p_j \to \frac{1}{2}$ and that $p_j \notin \{0,1\}$ for all $j \in \mathbb{N}$. Therefore the term above is $\mathcal{O}\left(|A_n - \frac{1}{4}| + n^{-1/2}\right)$, and so we have established that

$$\frac{1}{\sqrt{2\pi}}\left|\alpha - \left(\sqrt{\frac{A}{A_n}}\alpha + \frac{n\left(\frac{1}{2}-\bar{q}_n\right)}{\sqrt{nA_n}}\right)\right| + Cn^{-1/2} \leq Cn^{-1/2} + C\left|A_n - \frac{1}{4}\right|,$$

which completes the proof of Lemma 6. □

We now have the tools that we need to finish the proof of Lemma 4. By Lemma 6, we have that

$$H_n(a) = \sum_{t=0}^{\lfloor a\sqrt{n}\rfloor} t\left[\Phi\left(\frac{-t}{2\sqrt{(2n+t)(1/4)}}\right) + 1 - \Phi\left(\frac{t+1}{2\sqrt{(2n-t-1)(1/4)}}\right) + \mathcal{O}\left(n^{-1/2} + \left|A_n - \frac{1}{4}\right|\right)\right].$$

Again using the fact that Φ is $\frac{1}{\sqrt{2\pi}}$−Lipschitz yields

$$\begin{aligned}\left|\Phi\left(\frac{-t}{2\sqrt{(2n+t)(1/4)}}\right) - \Phi\left(\frac{-t}{\sqrt{8n(1/4)}}\right)\right| &\leq \frac{1}{\sqrt{2\pi}}\left|\frac{-t}{\sqrt{(2n+t)}} - \frac{-t}{\sqrt{2n}}\right| \\ &= \frac{1}{\sqrt{2\pi}}\left|\frac{t^2}{\sqrt{2n(2n+t)}(\sqrt{2n+t}+\sqrt{2n})}\right| \\ &\leq \frac{Ct^2}{n^{3/2}}.\end{aligned}$$

The other term can be bounded in a similar way. Therefore,

$$\begin{aligned}\left|H_n(a) - \sum_{t=0}^{\lfloor a\sqrt{n}\rfloor} t\left[\Phi\left(\frac{-t}{\sqrt{2n}}\right) + 1 - \Phi\left(\frac{t}{\sqrt{2n}}\right)\right]\right| &\leq C\sum_{t=0}^{\lfloor a\sqrt{n}\rfloor}\left(\frac{t^3}{n^{3/2}} + tn^{-1/2} + \left|A_n - \frac{1}{4}\right|t\right) \\ &\leq C\left(a^4n^{1/2} + a^2n^{1/2} + \left|A_n - \frac{1}{4}\right|a^2n\right).\end{aligned}$$

Therefore, for some constant C independent of n, we have

$$\frac{1}{n}\left|H_n(a) - \sum_{t=0}^{\lfloor a\sqrt{n}\rfloor} t\left[2\Phi\left(\frac{-t}{\sqrt{2n}}\right)\right]\right| \leq C\left(a^4n^{-1/2} + \left|A_n - \frac{1}{4}\right|a^2\right),$$

thereby proving Lemma 4. □

We can now complete our proof by combining the error estimates for $H_n(a)$ and $T_n(a)$ given above. We see that

$$\begin{aligned}
\left|\frac{1}{n}\sum_{t=0}^{\infty} t\cdot P_n\left(|U_1-n|>t\right)-1\right| &= \left|\frac{1}{n}\left(H_n(a)+T_n(a)\right)-1\right| \\
&\leq \left|\frac{1}{n}H_n(a)-1\right|+\frac{1}{n}\left|T_n(a)\right| \\
&= \mathcal{O}\left(a^4 n^{-1/2}+\left|A_n-\frac{1}{4}\right|a^2+\mathrm{e}^{-Ca}\right).
\end{aligned}$$

To finish our proof that $\lim_{n\to\infty}\mathbb{E}_n[(U_1-n)^2]=2$, we only need to choose an appropriate sequence a_n. In the case that $b_n \leq n^{-1/2}$, we can take $a(n)=C\log n$ to obtain a rate of convergence of $\mathcal{O}\left(\frac{\log^4 n}{\sqrt{n}}\right)$, because we have that

$$b_n \log^4 b_n \leq \frac{(\log n^{-1/2})^4}{\sqrt{n}} = \mathcal{O}\left(\frac{\log^4 n}{\sqrt{n}}\right).$$

On the other hand, if $b_n \geq n^{-1/2}$, we will take $a(n) = -\frac{1}{C}\log\left(b_n\right)$ to obtain a rate of convergence of $\mathcal{O}\left(b_n\log^4 b_n\right)$. Finally, note that in this case $b_n\log^4 b_n$ is the dominant term, since

$$\frac{\log^4 n}{\sqrt{n}} = \frac{16(\log n^{-1/2})^4}{\sqrt{n}} \leq 16 b_n(\log b_n)^4 = \mathcal{O}\left(b_n\log^4 b_n\right).$$

Combining the two cases, we see that $|\nu(n)-2| = \mathcal{O}\left(n^{-1/2}\log^4 n + b_n\log^4 b_n\right)$. $\square$

Remark: As an immediate consequence of Lemma 2 and the calcuation of ρ in Section 3.3.1, we have that

$$\theta = \lim_{n\to\infty}\theta(n) = \lim_{n\to\infty}\frac{2\rho(n)}{\nu(n)} = \delta(\boldsymbol{p}).$$

Furthermore,

$$\begin{aligned}|\theta(n)-\delta| &= \left|\frac{2\rho(n)}{\nu(n)}-\delta\right| \\ &= \left|\frac{2\rho(n)-\rho(n)\nu(n)+\rho(n)\nu(n)-\nu(n)\delta}{\nu(n)}\right| \\ &\leq \frac{\rho(n)}{\nu(n)}\left|\nu(n)-2\right|+\left|\rho(n)-\delta\right|.\end{aligned}$$

Therefore, the rate at which $\theta(n)\to\theta$ depends on the rate at which $\rho(n)\to\rho$ and $\nu(n)\to\nu$. In Lemma 2, we showed that

$$|\nu(n)-n| = \mathcal{O}\left(\frac{\log^4 n}{\sqrt{n}}+b_n\log^4 b_n\right), \tag{3.7}$$

where $b_n=\frac{1}{4n}\sum_{\mathrm{j}=1}^{n}(2p_{\mathrm{j}}-1)^2$. Therefore, the rate of convergence for $\theta(n)$ depends on the rate at which $p_{\mathrm{j}}\to 1/2$ (recalling Example 2, this can be arbitrarily slowly). Also recall that

$$|\rho(n)-\rho| = \left|\sum_{\mathrm{j}=T_{n+1}}^{\infty}(2p_{\mathrm{j}}-1)\right| \leq \sup_{m\geq n}\left|\sum_{\mathrm{j}=m+1}^{\infty}(2p_{\mathrm{j}}-1)\right|,$$

and so the rate of convergence for $\theta(n)$ also relies on how quickly the tail of the δ series converges to 0 (recalling Example 3, this can also be arbitrarily slowly). If we wish to apply Theorem 3.2.3 to prove recurrence/transience of ERW, these two facts necessitate an assumption like the one in (3.2). We will now show that this assumption is sufficient to apply Theorem 3.2.3. We have already seen that under that assumption $|\rho(n)-\delta|=o\left(\frac{1}{\log n}\right)$. To bound $|\nu(n)-2|$, note that if $b_n\leq n^{-1/2}$ then the term $n^{-1/2}\log^4 n$ will dominate (3.7), and so $|\nu(n)-2|=\mathcal{O}\left(\frac{\log^4 n}{\sqrt{n}}\right)=o\left(\frac{1}{\log n}\right)$. If instead we have that $b_n\geq n^{-1/2}$, then $b_n\log^4 b_n$ will dominate (3.7). We claim that this bound is also $o\left(\frac{1}{\log n}\right)$. To see this, note that since we assume $\sum_{\mathrm{j}=n}^{\infty}(2p_{\mathrm{j}}-1)=o\left(\frac{1}{\log n}\right)$, then for all $\epsilon>0$ there exists n_ϵ such that

$$\left|\sum_{\mathrm{j}=n}^{\infty}(2p_{\mathrm{j}}-1)\right| < \frac{\epsilon}{\log n}, \quad \forall n\geq n_\epsilon.$$

Then for any $\epsilon > 0$ we will have that

$$|2p_n - 1| = \left|\sum_{\mathrm{j}=n}^{\infty}(2p_{\mathrm{j}} - 1) = \sum_{\mathrm{j}=n-1}^{\infty}(2p_{\mathrm{j}} - 1)\right| < \frac{2\epsilon}{\log n} \quad \forall n \geq n_\epsilon,$$

and so $|2p_{\mathrm{j}} - 1| = o\left(\frac{1}{\log n}\right)$. Because $b_n = \frac{1}{4}\sum_{\mathrm{j}=1}^{n}(2p_{\mathrm{j}} - 1)^2$, this implies that

$$b_n = \mathcal{O}\left(\frac{1}{\log^2 n}\right),$$

and plugging this in to (3.7) shows that

$$|\nu(n) - 2| = \mathcal{O}\left(\frac{(\log(\log n))^4}{(\log n)^2}\right) = o\left(\frac{1}{\log n}\right).$$

Consequently, $|\theta(n) - \delta| = o\left(\frac{1}{\log n}\right)$.

3.4 Example of Transient ERW with $\delta = 1$

In this section we will prove Theorem 3.2.2 by showing that ERW in the cookie environment in Example 3 is transient. We will give the same example that was given in [20]. Let $\boldsymbol{p} = (p_1, p_2, \ldots)$, where p_{j} is given by

$$p_{\mathrm{j}} = \begin{cases} \frac{5}{6} & k = 1, 2, 3 \\ \frac{1}{2} - \left(\frac{1}{2}\right)^{m+1} & k = 4^{4^m},\ m = 1, 2, \ldots \\ \frac{1}{2} & \text{otherwise} \end{cases}.$$

By Theorem 3.3.1, we have that $\nu = 2$ and $\rho = \theta = \delta(\boldsymbol{p})$. Recall that for this environment $\delta(\boldsymbol{p}) = 1$. Intuitively, after the walker visits a site in the environment $\boldsymbol{p}$ three times, the amount of drift they have consumed is already greater than 1. Subsequent visits to the site give either no drift or a slight negative drift, so that the total drift experienced by the walker slowly decreases to 1. Since the total drift the walker experiences at a site is only 1 “in the limit,” we may expect that ERW in $\boldsymbol{p}$ behaves like an ERW with $\delta > 1$.

We will show that

$$\theta(n) - 1 \geq \frac{2}{\log n}$$

for all n sufficiently large, and therefore by Theorem 3.2.3 ERW in $\boldsymbol{p}$ is transient. First note that

$$\theta(n) - 1 = \frac{\rho(n)}{\nu(n)}\left(2 - \nu(n)\right) + \left(\rho(n) - 1\right).$$

To handle the term with $\nu(n)$, note that

$$b_n = \left|A_n - \frac{1}{4}\right| = \frac{1}{4n}\sum_{\mathrm{j}=1}^{n}(2p_{\mathrm{j}} - 1)^2 \leq \frac{C}{n}$$

because $\sum(2p_{\mathrm{j}} - 1)^2$ converges, and so by Theorem 3.3.1 we have $|\nu(n) - 2| = \mathcal{O}\left(\frac{\log^4 n}{\sqrt{n}}\right)$. We will now find an expression for $\rho(n) - 1$. Recall that $\rho(n) = \mathbb{E}[\delta_{T_n}]$, and so for $n > 3$ we have

$$\rho(n) - 1 = \mathbb{E}\left[\delta_{T_n}\right] - 1 \overset{n \geq 3}{=} \mathbb{E}\left[2 + \sum_{k=4}^{T_n}\right] - 1 = 1 + \mathbb{E}\left[\sum_{k=4}^{T_n}(2p_k - 1)\right].$$

Now, let $C_{\boldsymbol{p}}(x) = \#\{\mathrm{j} \leq x : p_{\mathrm{j}} < 1/2\}$ be the number of negative drift cookies in the cookie stack $\boldsymbol{p}$ up to cookie x. We can then write

$$1 + \mathbb{E}\left[\sum_{k=4}^{T_n}(2p_k - 1)\right] = 1 + \mathbb{E}\left[\sum_{m=1}^{C_{\boldsymbol{p}}(T_n)} - \left(\frac{1}{2}\right)^m\right] = \mathbb{E}\left[\left(\frac{1}{2}\right)^{C_{\boldsymbol{p}}(T_n)}\right].$$

Then we have that

$$\mathbb{E}\left[\left(\frac{1}{2}\right)^{C_{\boldsymbol{p}}(T_n)}\right] \geq \left(\frac{1}{2}\right)^{\log_4(\log n)} P\left(C_{\boldsymbol{p}}(T_n) \leq \log_4(\log n)\right) = \frac{1}{\sqrt{\log n}} P\left(C_{\boldsymbol{p}}(T_n) \leq \log_4(\log n)\right).$$

Due to our choice of environment, the number of negative drift cookies in the first $3n$ cookies is no more than $\log_4(\log n)$. Therefore

$$\frac{1}{\sqrt{\log n}} P\left(C_{\boldsymbol{p}}(T_n) \leq \log_4(\log n)\right) \geq \frac{1}{\sqrt{\log n}} P(T_n \leq 3n) \geq \frac{1 - \mathrm{e}^{-Cn}}{\sqrt{\log n}},$$

where the last inequality follows from the concentration bound for T_n in Theorem 3.2.4. Combining this bound with the one for the $\nu(n)$ term, we see that for sufficiently large n

$$\theta(n) - 1 \geq \frac{1 - \mathrm{e}^{-Cn}}{\sqrt{\log n}} + \mathcal{O}\left(\frac{\log^4 n}{\sqrt{n}}\right) \geq \frac{2}{\log n}.$$

By Theorem 3.2.3, ERW in $\boldsymbol{p}$ is a.s. transient to ∞.

Remark: We close this section with a comparison between our model and previously studied ERW models with infinitely many non-placebo cookies per site. In the Markovian cookie stacks model, Kosygina and Peterson [16] showed that

$$|\rho(n) - \rho| \leq C\mathrm{e}^{-C'n}$$
$$|\nu(n) - \nu| \leq \frac{C''}{n}.$$

Note that the bound on $|\nu(n) - \nu|$ given for the periodic case (a special case of Markovian cookie stacks) in [19] is of the order $n^{-1/2}\log^4(n)$. These bounds help explain why critical ERW can be transient in finite-drift environments but not in Markovian cookie stack environments: environments with Markovian cookie stacks will always satisfy the first part of Theorem 2.3.4, essentially because under the assumptions in [16] Markov chains converge to their stationary distribution exponentially fast (and this is reflected in the rate at which $\rho(n) \to \rho$). However, we are able to construct finite-drift environments where $\rho(n) \to \rho$ as slowly as we like.

3.5 Additional Results Used

Lemma 7. *Let U_1 denote the number of offspring in the first generation of the forward branching-like process associated to excited random walk in $\boldsymbol{p}$, and let T_n be the trial in the sequence $\{\xi_\mathrm{j}\}_{\mathrm{j}\geq 1}$ on which the nth failure occurs. Then*

$$\mathbb{E}_n[U_1] = \mathbb{E}\left[\sum_{\mathrm{j}=1}^{T_n} \xi_\mathrm{j}\right] = \mathbb{E}\left[\sum_{\mathrm{j}=1}^{T_n} p_\mathrm{j}\right].$$

Proof. Our proof is taken directly from [20]. The statement is essentially a version of Wald's identity, and is proved in a similar way. First, we write

$$\mathbb{E}\left[\sum_{\mathrm{j}=1}^{T_n} \xi_k\right] = \sum_{\mathrm{j}=1}^{\infty} \mathbb{E}\left[\xi_{\mathrm{j}} \mathbf{1}_{\{T_n \geq \mathrm{j}\}}\right]. \tag{3.8}$$

Because the event $\{T_n \geq \mathrm{j}\} = \{T_n > \mathrm{j}-1\}$ depends only on $\xi_1, \ldots, \xi_{\mathrm{j}}$, we have $\mathbb{E}\left[\xi_{\mathrm{j}} \mathbf{1}_{\{T_n \geq \mathrm{j}\}}\right] = p_{\mathrm{j}} P(T_n \geq \mathrm{j})$. Using this fact, we can rewrite the right-hand side of (3.8) as

$$\sum_{\mathrm{j}=1}^{\infty} p_{\mathrm{j}} P(T_n \geq \mathrm{j}) = \sum_{\mathrm{j}=1}^{\infty} p_{\mathrm{j}} \mathbb{E}[\mathbf{1}_{\{T_n \geq \mathrm{j}\}}] = \mathbb{E}\left[\sum_{\mathrm{j}=1}^{\infty} p_{\mathrm{j}} \mathbf{1}_{\{T_n \geq \mathrm{j}\}}\right] = \mathbb{E}\left[\sum_{\mathrm{j}=1}^{T_n} p_{\mathrm{j}}\right],$$

thereby proving the claim. $\square$

4. LIMITING DISTRIBUTIONS OF TRANSIENT "HAVE YOUR COOKIE AND EAT IT" RANDOM WALK

In this chapter, we will study random walk in "have your cookie and eat it" environments (HYCRW) proposed by Pinsky [23]. We will begin with a description of the model. For each site $x \in \mathbb{Z}$, we place a single cookie of strength p at each site. Then, a walker is released at 0. Whenever the walker is at a site x with a cookie present, they choose to step to $x+1$ with probability p and to $x-1$ with probability $1-p$. If the walker decides to step left to $x-1$, they consume the cookie at x before taking the step, but the walker does not consume the cookie at x if they decide to step right. At sites with no cookie, the walker takes simple symmetric steps. This transition mechanism differs from ERW because after the walker arrives at a site, they first choose where to step next before (potentially) eating the cookie at that site. To keep track of this difference, it may be helpful to remember that in excited random walk, the "excitement" comes from eating cookies, but in the "have your cookie and eat it" random walk the excitement comes from *having* a cookie available.

Remark: At first blush, it may seem that HYCRW is simply ERW with Geo$(1-p)$ cookies of strength p at each site. However, the HYCRW model has an additional level of self-interaction: the effective number of cookies at each site is not fixed ahead of time, but depends on the path that the walker takes. In spite of this, our main result can be explained at a heuristic level by comparing these two models (see remark below Theorem 4.2.1).

Our main task for this chapter will be to identify the limiting distributions for transient HYCRW. We will begin by reviewing some of the necessary background results that are due to Pinsky [23].

4.1 Review of Pinsky's HYCRW Results

In the following results, we use $P_1(\cdot)$ to denote the law of HYCRW started from 1, i.e. with $P(X_0 = 1) = 1$. We note that many of Pinsky's results hold in more generality, for instance, with random cookie strengths at each site or for deterministic, spatially periodic "have your cookie and eat it" environments. We will state these results only for the HYCRW with a cookie of deterministic strength p at each site, since that is the model that our main result pertains to.

Theorem 4.1.1 (Lemma 1 of [23]). *Let $\{X_n\}_{n\geq 0}$ be a random walk in a "have your cookie and eat it" environment with $p \in [1/2, 1)$. Then*

1. *if $P_1(T_0 = \infty) = 0$, the walk is recurrent, i.e. $P(X_n$ visits each site i.o.$) = 1$.*

2. *if $P_1(T_0 = \infty) > 0$, the walk is transient, i.e. $P(\lim_{n\to\infty} X_n = \infty) = 1$.*

Theorem 4.1.2 (Recurrence/Transience of HYCRW, Theorem 1 of [23]). *Let $\{X_n\}_{n\geq 0}$ be a random walk in a "have your cookie and eat it" environment with $p \in [1/2, 1]$. Then*

1. *if $p \leq 2/3$, $P_1(T_0 = \infty) = 0$, and so the walk is recurrent.*

2. *if $p > 2/3$, then*

$$P_1(T_0 = \infty) \in \left(\frac{3p-2}{p}, \frac{3p-2}{p(2p-1)}\right),$$

and so the walk is transient.

Theorem 4.1.3 (Ballisticity of HYCRW, Theorem 3 of [23]). *Let $\{X_n\}_{n\geq 0}$ be a random walk in a "have your cookie and eat it" environment. Then*

1. *if $p < 3/4$, then $\frac{X_n}{n} \to 0$ almost surely.*

2. *if $p > 3/4$, then $\lim_{n\to\infty} \frac{X_n}{n} \geq 4p - 3$, with equality if $p \in (4/5, 1]$.*

Remark: In [23], Pinsky states that the second part of Theorem 4.1.3 holds for $p \in (10/11, 1]$. This seems to be due to a slight error in arithmetic. If one carefully carries out the calculations in the proof of Theorem 4.1.3 from [23], one obtains the version of the theorem listed above. Pinsky conjectures in [23] that equality holds all the way down to $p = 3/4$, which would demonstrate non-ballisticity in the borderline case $p = 3/4$. Theorem 4.2.1 below implies that when $p = 3/4$, $\frac{X_n}{n} \to 0$ in probability as $n \to \infty$.

4.2 Main Result

Theorem 4.2.1 (Limiting Distributions for Transient HYCRW)**.** *Let $\{X_n\}_{n\geq 0}$ be a random walk in a "have your cookie and eat it" environment, suppose $p \in (2/3, 1)$ so that the walk is transient, and let $T_n = \inf\{t : X_t = n\}$ be the corresponding hitting times. Finally, let $\alpha = \frac{2p-1}{1-p}$. If $p \in \left(\frac{2}{3}, \frac{3}{4}\right)$, then for some constant b and for all $x \in \mathbb{R}$,*

$$P\left(\frac{T_n}{n^{2/\alpha}} \leq x\right) \to L_{\alpha/2,b}(x), \qquad P\left(\frac{X_n}{n^{\alpha/2}} \leq x\right) \to 1 - L_{\alpha/2,b}\left(x^{-2/\alpha}\right).$$

If $p = 3/4$, then for some b and for all $x \in \mathbb{R}$

$$P\left(\frac{T_n - nD(n)/c}{n} \leq x\right) \to L_{1,b}(x), \qquad P\left(\frac{X_n - cn\Gamma(n)}{c^2 n \log^{-2} n} \leq x\right) \to 1 - L(-x^2/a),$$

where $D(n) \sim \log n$ and $\Gamma(n) \sim \log^{-1} n$. Furthermore, there exists a constant c such that

$$\frac{T_n}{\log n} \to \frac{1}{c} \quad \text{and} \quad \frac{X_n}{n/\log n} \to c \quad \text{as } n \to \infty.$$

If $p \in \left(\frac{3}{4}, \frac{5}{6}\right)$, then for some b and for all $x \in \mathbb{R}$ we have

$$P\left(\frac{T_n - n/v}{n^{2/\alpha}} \leq x\right) \to L_{\alpha/2,b}(x), \qquad P\left(\frac{X_n - nv}{v^{1+2/\alpha} n^{2/\alpha}} \leq x\right) \to 1 - L_{\alpha/2,b}(x).$$

If $p = 5/6$, then

$$P\left(\frac{T_n - n/v}{b\sqrt{n \log n}} \leq x\right) \to \Phi(x), \qquad P\left(\frac{X_n - nv}{bv^{3/2}\sqrt{n \log n}} \leq x\right) \to 1 - \Phi(x) = \Phi(x).$$

Finally, if $p > 5/6$, then

$$P\left(\frac{T_n - n/v}{b\sqrt{n}} \leq x\right) \to \Phi(x), \qquad P\left(\frac{X_n - nv}{bv^{3/2}\sqrt{n}} \leq x\right) \to 1 - \Phi(x) = \Phi(x).$$

Remark: The limiting distributions for transient HYCRW are very similar to those of the M-cookie ERW listed in Table 2.2, only with δ replaced by $\alpha = \frac{2p-1}{1-p}$. At a heuristic level, the walker in a HYCRW experiences a drift of size $2p - 1$ whenever they visit a site with a cookie, and since the number of right steps before the first left step at a site follows a Geo$(1-p)$ distribution the walker will on average visit a site $\frac{1}{1-p}$ times before consuming the cookie at that site. Therefore, the amount of drift the walker experiences at each site should be on average $\frac{2p-1}{1-p}$.

Our proof of Theorem 4.2.1 will follow the technique discussed in Section 2.3.3. In particular, we will prove the following tail asymptotics for σ_1 and for W_1. Recall that we define

$$\sigma_0 = 0, \quad \sigma_k = \inf\{t > \sigma_{k-1} : V_t = 0\}, \qquad W_k = \sum_{j=\sigma_{k-1}}^{\sigma_k - 1} V_j,$$

and that the $(\sigma_k - \sigma_{k-1}, W_k)$ are i.i.d. random variables. We must then prove the following two statements.

Theorem 4.2.2. *Let $p > 2/3$ and let $\alpha = \frac{2p-1}{1-p}$. Then*

$$\lim_{n\to\infty} n^{\alpha} P_0^V(\sigma_1 > n) = C_1 \in (0, \infty). \tag{4.1}$$

Theorem 4.2.3. *Let $p > 2/3$ and let $\alpha = \frac{2p-1}{1-p}$. Then*

$$\lim_{n\to\infty} n^{\alpha/2} P_0^V(W_1 > n) = C_2 \in (0, \infty). \tag{4.2}$$

These two statements will allow us to identify the limiting distributions for the hitting times T_n in the manner described in Section 2.3.3. Then, we will need to do some technical work to translate them into limiting distributions for the walk X_n. Our proof will follow the same strategy used by Kosygina and Mountford [15] and Kosygina and Peterson [16], but some of the technical results must be redone. In fact, to prove Theorems 4.2.2 and 4.2.3 we can repeat verbatim the proof of Theorems 2.1 and 2.2 in [15], provided that we prove the following four technical results.

Before stating them, we provide some additional notation. Let $\{Z_t\}_{t\geq 0}$ be a stochastic process. The probability measure associated to $\{Z_t\}_{t\geq 0}$ with $P(Z_0 = z) = 1$ will be written $P^Z_z(\cdot)$. We will also need to define the the lower and upper exit times of Z:

$$\sigma^Z_x = \inf\{\mathrm{j} > 0 : Z_\mathrm{j} \geq x\},$$
$$\tau^Z_x = \inf\{\mathrm{j} > 0 : Z_\mathrm{j} \leq x\}.$$

With this notation in hand, we can now state the four technical results to be proved.

Lemma 8 (Diffusion Approximation). *Fix arbitrary $\epsilon > 0$, $y > \epsilon$, and a sequence $y_n \to y$ as $n \to \infty$. Define $Y^{\epsilon,n}(t) = \frac{V_{\lfloor nt\rfloor \wedge \sigma_{\epsilon n}}}{n}$, $t \geq 0$. Then the process $Y^{\epsilon,n}$ converges in the Skorokhod (J_1) topology to $Y\left(\cdot \wedge \sigma^Y_\epsilon\right)$, where Y is the solution of the following stochastic differential equation:*

$$dY(t) = \left(1 - \frac{2p-1}{1-p}\right)\, dt + \sqrt{2Y(t)^+} dB_t, \quad Y(0) = y.$$

This lemma shows that the BBLP, when appropriately scaled, can be approximated by a squared Bessel process of generalized dimension $2(1-\alpha)$ on $[\epsilon n, \infty)$. To handle the case where the BBLP drops below ϵn, we will use the fact that Y^α (or $\log Y$ when $p = 2/3$) is a local martingale, and as such for all $a > 1$ and $\mathrm{j} \in \mathbb{N}$ we have that

$$P\left(\tau_{a^{\mathrm{j}-1}} < \tau_{a^{\mathrm{j}+1}} | Y(0) = a^\mathrm{j}\right) = \frac{a^\alpha}{1 + a^\alpha}.$$

The rescaled version of the BBLP is close to being a martingale, and so we can put a bound on the probability that this process exits the interval $(a^{\mathrm{j}-1}, a^{\mathrm{j}+1})$ from below.

Lemma 9 (Exit Distributions). *Let $a > 1$, $|x - a^\mathrm{j}| \leq a^{2\mathrm{j}/3}$, and let γ be the exit time of the interval $(a^{\mathrm{j}-1}, a^{\mathrm{j}+1})$. Then for all sufficiently large* $\mathrm{j} \in \mathbb{N}$

$$\left| P^V_x\left(V_\gamma \leq a^{\mathrm{j}-1}\right) - \frac{a^\alpha}{1 + a^\alpha} \right| \leq a^{-\mathrm{j}/4}.$$

Since the rescaled BBLP is still a discrete process, whenever it leaves the interval $(a^{\mathrm{j}-1}, a^{\mathrm{j}+1})$ it is possible that it is far away from the two endpoints (this cannot happen for the limiting process since it is continuous). Therefore, part of proving Lemma 9 is to control the probability of the rescaled BBLP "overshooting" the boundary of the interval by a large amount.

Lemma 10 (Overshoot Lemma)**.** *There are positive constants C_3 and C_4 and $N \in \mathbb{N}$ such that for all $x \geq N$ and $y \geq 0$, we have that*

$$\max_{0 \leq z \leq x} P\left(V_{\tau_x} > x + y | \tau_x < \sigma_0\right) \leq C_3 \left(\mathrm{e}^{-C_4 y^2/x} + \mathrm{e}^{-C_5 y}\right),$$

and

$$\max_{x < z < 4x} P\left(V_{\sigma_x \wedge \tau_{4x}} < x - y\right) \leq C_3 \mathrm{e}^{-C_4 y^2/x}.$$

In short, when the rescaled process exits an interval, it is overwhelmingly likely that it is near the boundary. Once we have proven Lemma 10, the proof of Lemma 9 is identical to the proof of the corresponding result in [15] (with Lemma 10 replacing a similar result there), and so we will not repeat it here.

Lemma 11 (Main Lemma)**.** *For each $a \in (1, 2]$ there exists an $\ell_0 \in \mathbb{N}$ and a small positive number λ such that if ℓ, m, u, $x \in \mathbb{N}$ satisfy $\ell_0 \leq \ell < m < u$ and let $|x - a^m| < a^{2m/3}$ then*

$$\frac{h^-(m) - h^-(\ell)}{h^-(u) - h^-(\ell)} \leq P_x^V\left(\sigma_{a_\ell}^V > \tau_{a_u}^V\right) \leq \frac{h^+(m) - h^+(\ell)}{h^+(u) - h^+(\ell)},$$

where for $\mathrm{j} \geq 1$

$$h^{\pm}(\mathrm{j}) = \begin{cases} \prod_{\mathrm{i}=1}^{\mathrm{j}} \left(a^{\alpha} \mp a^{-\lambda \mathrm{i}}\right), & \text{for } \alpha > 0 \\ \prod_{\mathrm{i}=1}^{\mathrm{j}} \left(a^{-\alpha} \mp a^{-\lambda \mathrm{i}}\right)^{-1}, & \text{for } \alpha < 0 \\ \mathrm{j} \mp \frac{1}{\mathrm{j}}, & \text{for } \alpha = 0. \end{cases}$$

For $\alpha \neq 0$, we also have that there functions $K_1 : \mathbb{N} \to (0,\infty)$ and $K_2 : \mathbb{N} \to (0,\infty)$ such that $K_\mathrm{i}(\ell) \to 1$ as $\ell \to \infty$ and for all $\mathrm{j} > \ell$

$$K_1(\ell) a^{(\mathrm{j}-\ell)\alpha} \leq \frac{h^{\pm}(\mathrm{j})}{h^{\pm}(\ell)} \leq K_2(\ell) a^{(\mathrm{j}-\ell)\alpha}.$$

The proof of Lemma 11 is identical to the proof of Lemma 5.3 of [15]. The only difference in the proof is that we must use the results from Lemmas 9 and 10 instead of the corresponding results in that paper. Therefore, we will omit the proof here and refer the reader to [15] for complete details.

Before discussing the proofs of Theorems 4.2.2 and 4.2.3 given these results, it will be helpful to compute the necessary parameters of the BBLP. In particular, we will compute

$$\hat{\mu} = \lim_{n\to\infty} \frac{\mathbb{E}_n[V_1]}{n} \qquad \hat{\rho} = \mathbb{E}_n[V_1 - \hat{\mu} n]$$
$$\hat{\nu} = \lim_{n\to\infty} \frac{\mathbb{E}_n[(V_1 - \hat{\mu} n)^2]}{n} \qquad \hat{\theta} = \frac{2\hat{\rho}}{\hat{\nu}},$$

where in this section $\mathbb{E}_n[\,\cdot\,] = \mathbb{E}[\,\cdot\,|V_0 = n]$. Note that these parameters are defined in the same way as those of the FBLP as defined in Section 2.2.1. We collect the values of these limits and note the asymptotics of $\hat{\rho}$ and $\hat{\nu}$ in the following proposition.

Proposition 4.2.1 (Parameters of BBLP associated to HYCRW)**.** *Let X_n be excited random walk in a "have your cookie and eat it" environment with $p \in (0,1)$, and let $\{V_k\}_{k\geq 0}$ be the associated backward branching-like process. Then*

$$\hat{\mu} = 1 \qquad \hat{\rho} = 1 - \frac{2p-1}{1-p} = 1 - \alpha$$
$$\hat{\nu} = 2 \qquad \hat{\theta} = 1 - \frac{2p-1}{1-p} = 1 - \alpha.$$

Furthermore there exist constants C_1 and C_2 so that for all $n \in \mathbb{N}$ we have

$$|\mathbb{E}_n[V_1 - n] - (1-\alpha)| \leq C_1 n p^{n+1}$$
$$\left| \frac{\mathbb{E}_n[(V_1 - n)^2]}{n} - 2 \right| \leq \frac{C_2}{n}.$$

Proof of Proposition 4.2.1. We will begin by computing $\hat{\mu}$. Recall how the BBLP is defined: given $V_0 = n$, the value of V_1 is determined by the number of failures that occur in the sequence of Bernoulli trials $\{\xi_{\mathrm{j}}\}_{\mathrm{j}\geq 1}$ before the $(n+1)$th success (the "+1" comes from the extra immigrant that is added to each generation of the BBLP before reproduction). For HYCRW, the $\xi_{\mathrm{j}} \sim \mathrm{Ber}(p)$ if $\xi_k = 1$ for all $k < \mathrm{j}$, but otherwise $\xi_{\mathrm{j}} \sim \mathrm{Ber}(1/2)$. For that reason, it will be advantageous to condition on how many successes occur in this sequence of trials before the first failure. Let S_k be the event that exactly k successes occur in $\{\xi_{\mathrm{j}}\}_{\mathrm{j}\geq 1}$ before the first failure in the sequence, or formally that $S_k = \{\xi_1 = \cdots = \xi_k = 1, \xi_{k+1} = 0\}$. Also, let $\{H_{\mathrm{i}}\}_{\mathrm{i}\geq 1}$ denote a family of independent geometric random variables $H_{\mathrm{i}} \sim \mathrm{Geo}(1/2)$ each with mean 1, let $G_x = H_1 + \cdots + H_x$, where we use the convention that $G_0 = 0$. Then G_x is a branching process where each organism has Geo(1/2) offspring. Then

$$\begin{aligned}
\mathbb{E}_n[V_1] &= \sum_{k=0}^{n+1} \mathbb{E}_n[V_1|S_k]P(S_k) \\
&= \sum_{k=0}^{n} \left(1 + \mathbb{E}[G_{n+1-k}]\right) p^k(1-p) \\
&= (n+2)\sum_{k=0}^{n} p^k(1-p) - \sum_{k=0}^{n} kp^k(1-p) \\
&= (n+2)\left(1 - p^{n+1}\right) - \sum_{k=0}^{n} kp^k(1-p).
\end{aligned}$$

Dividing through by n and taking limits shows that $\hat{\mu} = 1$. To compute $\hat{\rho}$, observe that

$$\begin{aligned}
\hat{\rho} = \lim_{n\to\infty} \mathbb{E}_n[V_1 - n] &= \lim_{n\to\infty}\left(2\left(1 - p^{n+1}\right) - \sum_{k=0}^{n} kp^k(1-p) - np^{n+1}\right) \\
&= \lim_{n\to\infty}\left(2\left(1 - p^{n+1}\right) - \frac{p}{1-p}\left(np^{n+1} - (n+1)p^n + 1\right) - np^{n+1}\right) \quad (4.3) \\
&= 2 - \frac{p}{1-p} \\
&= 1 - \frac{2p-1}{1-p}.
\end{aligned}$$

To prove the asymptotic statement about $\hat{\rho}$ above, note that by (4.3) we have that

$$\left|\mathbb{E}_n[V_1 - n] - \left(1 - \frac{2p-1}{1-p}\right)\right| = \frac{1}{1-p}\left((3-2p)p^{n+1} + np^{n+1}\right),$$

and so there exists a constant C such that $|\mathbb{E}_n[V_1 - n] - \hat{\rho}| \leq Cnp^{n+1}$ for all $n \in \mathbb{N}$. We now turn our attention to calculating $\hat{\nu}$. Since $n\hat{\nu} = \mathbb{E}_n[(V_1-n)^2] = \mathbb{E}_n[V_1^2] - 2n\mathbb{E}_n[V_1] + n^2$, the only ingredient we are missing is $\mathbb{E}_n[V_1^2]$, which we can compute by again conditioning on the event S_k :

$$\mathbb{E}_n[V_1^2] = \sum_{k=0}^{n+1} \mathbb{E}_n[V_k^2|S_k]P(S_k) = \sum_{k=0}^{n} \mathbb{E}[(1+G_{n+1-k})^2]p^k(1-p).$$

Now, $\mathbb{E}[(1+G_{n+1-k})^2] = \mathrm{Var}(G_{n+1-k}) + \mathbb{E}[1+G_{n+1-k}]^2 = (n+1-k)\mathrm{Var}(H_1) + (n+2-k)^2$, and since $H_1 \sim \mathrm{Geo}(1/2)$ we have $\mathrm{Var}(H_1) = 2$. Therefore, we can write

$$\begin{aligned}
\mathbb{E}_n[V_1^2] &= \sum_{k=0}^{n} \Big(2(n+1-k) + (n+2-k)^2\Big)\, p^k(1-p) \\
&= \sum_{k=0}^{n} \Big(k^2 - 2nk - 6k + n^2 + 6n + 6\Big)\, p^k(1-p) \\
&= \sum_{k=0}^{n} k^2p^k(1-p) - (2n+6)\sum_{k=0}^{n} kp^k(1-p) + (n^2+6n+6)\sum_{k=0}^{n} p^k(1-p).
\end{aligned}$$

If we set

$$\begin{aligned}
A_n &= \sum_{k=0}^{n} k^2p^k(1-p) = \frac{p}{(1-p)^2}\Big((2n^2+2n+1)p^{n+1} - n^2p^{n+2} - (n+1)^2p^n + p + 1\Big), \\
B_n &= \sum_{k=0}^{n} kp^k(1-p) = \frac{p}{1-p}\Big(np^{n+1} - (n+1)p^n + 1\Big), \\
C_n &= \sum_{k=0}^{n} p^k(1-p) = 1 - p^{n+1},
\end{aligned}$$

the above expression becomes

$$\mathbb{E}_n[V_1^2] = A_n - (2n+6)B_n + (n^2+6n+6)C_n.$$

Note also that $A_n = \mathcal{O}\left(n^2p^n\right)$ and $B_n = \mathcal{O}\left(np^n\right)$. Together with the fact that $\mathbb{E}_n[V_1] = (n+2)C_n - B_n$, we can see that

$$\mathbb{E}_n\left[(V_1-n)^2\right] = \mathbb{E}_n[V_1^2] - 2n\mathbb{E}_n[V_1] + n^2$$

$$
\begin{aligned}
&= A_n - (2n+6)B_n + (n^2+6n+6)C_n - 2n\left((n+2)C_n - B_n\right) + n^2 \\
&= A_n - 6B_n + n^2 p^{n+1} + (2n+6)C_n \\
&= (2n+6)\left(1 - p^{n+1}\right) + \mathcal{O}(n^2 p^n).
\end{aligned}
$$

Dividing through by n and taking $n \to \infty$ shows that $\nu = 2$. Furthermore, we have that

$$
\left| \frac{\mathbb{E}_n\left[(V_1 - n)^2\right]}{n} - 2 \right| \leq C \cdot np^n + \frac{6}{n}(1 - p^{n+1}) \leq \frac{C_2}{n}
$$

for some constant C_2. To finish the proof, note that $\hat{\theta} = \frac{2\hat{\rho}}{\hat{\nu}} = \hat{\rho} = 1 - \alpha$. □

We close this section with a discussion of the proofs of Theorems 4.2.2 and 4.2.3. Again, once we have proven the technical results in Lemmas 8 - 11, we can simply repeat the proofs of Theorems 2.1 and 2.2 in [15], replacing the technical results in that argument with Lemmas 8 - 11. For that reason, we will only give a summary of the proofs, and refer the reader to [15] for complete details.

The main idea of the proof is to approximate the BBLP, stopped upon its first time reaching 0, by a squared Bessel process of dimension $2(1-\alpha)$. Lemma 8 shows that this approximation is good until the BBLP drops below ϵn for the first time. Furthermore, the approximating process $Y(t)$ has the property that the area under its path is of the correct order, so that

$$
\lim_{y \to \infty} y^{\alpha} P_1^Y \left(\int_0^{\tau_0} Y(t)\, dt > y^2 \right) = C \in (0, \infty).
$$

The vast majority of the proof is technical work with the purpose of verifying that this property is shared by the BBLP. This can be done because V is unlikely to overshoot the boundary of an interval by too much due to Lemma 10, and because it is close to being a martingale we can estimate the probability that it exits a given interval . Finally, a few technical results must be proven in order to show $P\left(\sum_{\mathrm{j}=1}^{\sigma_1 - 1} V_{\mathrm{j}} > n\right)$ decays at a commensurate rate with $P_1^Y\left(\int_0^{\tau_0} Y_t\, dt > y^2\right)$, including among others that

$$
\lim_{n \to \infty} P_{\epsilon n}^V(W_1 > n^2) = P_1^Y \left(\int_0^{\tau_0} Y(t)\, dt > \epsilon^{-2} \right).
$$

In sum, the approximation of the BBLP by this squared Bessel process is good enough that we can transfer the tail decay for the area under its path to the total progeny between two extinction times of the BBLP.

We will now show how to use Theorems 4.2.2 and 4.2.3 to deduce the limiting distributions of X_n.

4.3 Proof of Theorem 4.2.1

In this section we will prove our main result, assuming Theorems 4.2.2 and 4.2.3 in the preceding section. First, we define some additional notation. Throughout this section, we will use $Z_{a,b}$ to represent the stable random variable with distribution function $L_{a,b}(x)$ (refer to Section 2.3.3 for the characteristic function of these random variables).

Recall that the hitting times T_n of the HYCRW can be expressed in terms of D_n^k, the number of left steps from site k by the time the walk reaches n for the first time:

$$T_n = n + 2\sum_{k=0}^{n} D_n^k + 2\sum_{k<0} D_n^k \overset{\text{law}}{=} n + 2\sum_{\mathrm{j}=0}^{n} V_{\mathrm{j}} + 2\sum_{\mathrm{j}<0} V_{\mathrm{j}}. \tag{4.4}$$

We will use this connection to establish limit laws for the hitting times T_n, which by a relatively simple calculation can be converted into a limit law for the position of the walk X_n. We will address the cases in increasing order of complexity.

4.3.1 Zero Speed, $p \in (2/3, 3/4)$

Let $p \in (\frac{2}{3}, \frac{3}{4})$, so that the HYCRW is transient to $+\infty$ but has speed $v = 0$. Then observe that

$$\frac{T_n}{n^{2/\alpha}} \overset{\text{law}}{=} n^{1-2/\alpha} + \frac{2\sum_{\mathrm{j}=0}^{n} V_{\mathrm{j}}}{n^{2/\alpha}} + \frac{2\sum_{\mathrm{j}<0} V_{\mathrm{j}}}{n^{2/\alpha}}. \tag{4.5}$$

Note that for $p < 3/4$, we have $1 - 2/\alpha < 0$. Also, since the walk is transient to ∞, $\sum_{\mathrm{j}<0} V_{\mathrm{j}}$ is almost surely finite, so the last term in the above sum will converge to 0. Therefore, if we wish to prove a scaling limit for T_n, it suffices to prove one for $\sum_{\mathrm{j}=0}^{n} V_{\mathrm{j}}$. To this end, we write

$$\frac{\sum_{\mathrm{j}=0}^{n} V_{\mathrm{j}}}{n^{2/\alpha}} = \frac{\sum_{\mathrm{i}=1}^{N_n} W_{\mathrm{i}}}{n^{2/\alpha}} + \frac{\sum_{\mathrm{j}=\sigma_n+1}^{n} V_{\mathrm{j}}}{n^{2/\alpha}},$$

where $N_n = \max\{\mathrm{i} \geq 0 : \sigma_{\mathrm{i}} \leq n\}$ is the number of times the BBLP reaches 0 by time n. The rightmost term is bounded above by $W_{N_n+1}/n^{2/\alpha}$, which converges to 0 in probability (see, for instance, section XI.5 of [10]). Moreover, due to the tail asymptotics for W_1 in Theorem 4.2.3 there exists $\tilde{b} > 0$ such that the first term converges to the stable random variable $Z_{\alpha/2,\tilde{b}}$ by Theorem I.3.2 of [13] (note the lack of centering term since $\alpha/2 < 1$). That is, we have proven that there exists $b > 0$ such that

$$\lim_{n\to\infty} P\left(\frac{T_n}{n^{2/\alpha}} \leq x\right) = P(Z_{\alpha/2,b} \leq x) \quad \forall x \in \mathbb{R}.$$

We must now translate this statement into a statement about the walk X_n. Let $\overline{X}_n = \sup_{\mathrm{i}\leq n} X_{\mathrm{i}}$, and note that $P(T_m > n) = P(\overline{X}_n < m)$. Then

$$\begin{aligned}
P\left(\frac{\overline{X}_n}{n^{\alpha/2}} < x\right) &= P\left(\overline{X}_n < xn^{\alpha/2}\right) \\
&= P\left(\overline{X}_n < \lceil xn^{\alpha/2}\rceil\right) \\
&= P\left(T_{\lceil xn^{\alpha/2}\rceil} > n\right) \\
&= P\left(\frac{T_{\lceil xn^{\alpha/2}\rceil}}{(\lceil xn^{\alpha/2}\rceil)^{2/\alpha}} > \frac{n}{(\lceil xn^{2/\alpha}\rceil)^{2/\alpha}}\right) \\
&= P\left(\frac{T_{\lceil xn^{\alpha/2}\rceil}}{(\lceil xn^{\alpha/2}\rceil)^{2/\alpha}} > \frac{1}{x^{2/\alpha}} + o(1)\right) \overset{n\to\infty}{\longrightarrow} P\left(Z_{\alpha/2,b} > \frac{1}{x^{2/\alpha}}\right) = 1 - L_{\alpha/2,b}(x^{-2/\alpha}).
\end{aligned}$$

This proves the limiting distribution for $\overline{X}_n$. To prove the same statement with $\tilde{X}_n = \inf_{\mathrm{i}\geq n} X_{\mathrm{i}}$, note that for all $m, n, p \in \mathbb{N}$

$$\left\{\overline{X}_n > m\right\} \subset \left\{\tilde{X}_n < m\right\} \subset \left\{\overline{X}_n < m + p\right\} \cup \left\{\tilde{X}_{T_{m+p}} < m\right\}. \tag{4.6}$$

We can then complete the proof by using the following lemma from [15].

Lemma 12 (Lemma 9.1 of [15])**.**

$$\lim_{n\to\infty}\sup_{n\geq 1} P\left(\tilde{X}_{T_n} < n-k\right) = 0.$$

Lemma 12 shows that the probability of the rightmost event will tend to 0. Then, we can simply choose $p := p(n)$ to be any sequence which grows more slowly than $n^{\alpha/2}$, then use 4.6 to prove that $\tilde{X}_n$ has the same limiting distribution as $\bar{X}_n$. Since $\tilde{X}_n \leq X_n \leq \bar{X}_n$, we see that X_n follows the same limiting distribution as $\bar{X}_n$ and $\tilde{X}_n$.

4.3.2 Positive Speed

We divide the cases where $p > 3/4$ according to whether we obtain Gaussian or non-Gaussian limits.

Non-Gaussian Limits, $p \in (3/4, 5/6)$

Now, let $p \in (\frac{3}{4}, \frac{5}{6})$. In this regime, the HYCRW is transient and has speed $v \geq 4p-3 > 0$. Since $\alpha/2 > 1$ in this case, we will require a centering term. Recall that the random variables $(\sigma_n - \sigma_{n-1}, W_n)_{n\geq 1}$ are independent, and that $\sigma_n - \sigma_{n-1} \overset{\text{law}}{=} \sigma_1$ and $W_n \overset{\text{law}}{=} W_1$ for all $n \in \mathbb{N}$. By the tail asymptotics of W_1 given in (4.2), W_1 is in the domain of attraction for $Z_{\alpha/2,b}$. By a standard fact from renewal theory, we have that

$$\lim_{n\to\infty} \frac{N_n}{n} = \frac{1}{\mathbb{E}_0^V[\sigma_1]} =: \lambda.$$

Note that the expectation in the denominator is finite due to the asymptotics for σ_1 given in 4.2.2. Additionally, there is a constant C_4 such that, for any $\epsilon > 0$, it is true for sufficiently large n that

$$P_0^V\left(|N_n - \lambda n| > C_4\sqrt{n}\right) < \epsilon.$$

We expect the centering term for T_n to be $\frac{n}{v}$ because $\lim_{n\to\infty}\frac{T_n}{n}=\frac{1}{v}$ almost surely. This will help us to calculate the proper centering term for $\sum W_{\mathrm{i}}$. As before, we use (4.4) and (4.5) to express T_n in terms of W_{i}:

$$\lim_{n\to\infty}\frac{T_n}{n}=\lim_{n\to\infty}\frac{n+2\sum_{\mathrm{j}=0}^{n}V_{\mathrm{j}}}{n}=\lim_{n\to\infty}1+\frac{2\sum_{\mathrm{i}=1}^{N_n}W_{\mathrm{i}}+2\sum_{\mathrm{j}=\sigma_n}^{n}V_{\mathrm{j}}}{n}=1+2\lambda\mathbb{E}_0^V[W_1].$$

A little algebra reveals that $\mathbb{E}_0^V[W_1]=\frac{1}{2}(v^{-1}-1)\lambda$. With this information in hand, we can identify the limiting distribution for T_n in this case.

$$\begin{aligned}\frac{\sum_{\mathrm{j}=0}^{n}V_{\mathrm{j}}-(v^{-1}-1)n/2}{n^{2/\alpha}}&=\frac{\sum_{\mathrm{i}=1}^{N_n}W_{\mathrm{i}}+\sum_{\mathrm{j}=\sigma_{N_n+1}}^{n}V_{\mathrm{j}}-(v^{-1}-1)n/2}{n^{2/\alpha}}\\&=\frac{\sum_{\mathrm{i}=1}^{N_n}(W_{\mathrm{i}}-\mathbb{E}_0^V[W_1])+N_n\mathbb{E}_0^V[W_1]+\sum_{\mathrm{j}=\sigma_{N_n+1}}^{n}V_{\mathrm{j}}-(v^{-1}-1)n/2}{n^{2/\alpha}}\\&=\frac{\sum_{\mathrm{i}=1}^{N_n}(W_{\mathrm{i}}-\mathbb{E}_0^V[W_1])}{n^{2/\alpha}}+\mathbb{E}_0^V[W_1]\frac{N_n-\lambda n}{n^{2/\alpha}}+\frac{\sum_{\mathrm{j}=\sigma_{N_n+1}}^{n}V_{\mathrm{j}}}{n^{2/\alpha}}.\end{aligned}$$

When we take $n\to\infty$, the first term will converge to $Z_{\alpha/2,\tilde{b}}$ for some $\tilde{b}>0$ by Theorem I.3.2 of [13]. For the second term, we use the fact $N_n-\lambda n=\mathcal{O}(\sqrt{n})$ and $2/\alpha>1/2$ in this regime, and again we bound the third term by $W_{N_n+1}/n^{2/\alpha}$, which converges in probability to 0. The upshot is that, for some $b>0$, we have that

$$\lim_{n\to\infty}P\left(\frac{T_n-v^{-1}n}{n^{2/\alpha}}<x\right)=P(Z_{\alpha/2,b}<x)\quad\forall x\in\mathbb{R}.$$

As before, we can translate this into a statement about X_n by using the relationship $\{T_m>n\}=\{\overline{X}_n<m\}$:

$$\begin{aligned}P\left(\frac{\overline{X}_n-nv}{\beta n^{2/\alpha}}\right)&=P\left(\overline{X}_n<\beta xn^{2/\alpha}+vn\right)\\&=P\left(\overline{X}_n<\lceil\beta xn^{2/\alpha}+vn\rceil\right)\\&=P\left(T_{\lceil\beta xn^{2/\alpha}+vn\rceil}>n\right)\\&=P\left(\frac{T_{\lceil\beta xn^{2/\alpha}+vn\rceil}-v^{-1}\lceil\beta xn^{2/\alpha}+vn\rceil}{\lceil\beta xn^{2/\alpha}+vn\rceil^{2/\alpha}}>\frac{n-v^{-1}\lceil\beta xn^{2/\alpha}+vn\rceil}{\lceil\beta xn^{2/\alpha}+vn\rceil^{2/\alpha}}\right).\\&=P\left(\frac{T_{\lceil\beta xn^{2/\alpha}+vn\rceil}-v^{-1}\lceil\beta xn^{2/\alpha}+vn\rceil}{(\beta xn^{2/\alpha}+vn)^{2/\alpha}}>\frac{n-v^{-1}(\beta xn^{2/\alpha}+vn)}{(\beta xn^{2/\alpha}+vn)^{2/\alpha}}+o(1)\right).\end{aligned}$$

$$= P\left(\frac{T_{\lceil \beta x n^{2/\alpha}+vn\rceil} - v^{-1}\lceil \beta x n^{2/\alpha}+vn\rceil}{(\beta x n^{2/\alpha}+vn)^{2/\alpha}} > \frac{-\beta x n^{2/\alpha}}{v(\beta x n^{2/\alpha}+vn)^{2/\alpha}} + o(1)\right).$$

$$= P\left(\frac{T_{\lceil \beta x n^{2/\alpha}+vn\rceil} - v^{-1}\lceil \beta x n^{2/\alpha}+vn\rceil}{(\beta x n^{2/\alpha}+vn)^{2/\alpha}} > \frac{-\beta x n^{2/\alpha}}{v^{1+\frac{2}{\alpha}} n^{2/\alpha}(\frac{\beta}{v} x n^{2/\alpha-1}+1)^{2/\alpha}} + o(1)\right).$$

$$= P\left(\frac{T_{\lceil \beta x n^{2/\alpha}+vn\rceil} - v^{-1}\lceil \beta x n^{2/\alpha}+vn\rceil}{(\beta x n^{2/\alpha}+vn)^{2/\alpha}} > \frac{-\beta x}{v^{1+\frac{2}{\alpha}} (\frac{\beta}{v} x n^{2/\alpha-1}+1)^{2/\alpha}} + o(1)\right).$$

Now, if we were to take $n \to \infty$, the right-hand side of the inequality would converge to $-\beta x/v^{1+2/\alpha}$ (note that $2/\alpha < 1$ since $p > 3/4$). Therefore, if we take $\beta = v^{1+2/\alpha}$ we have shown that

$$\lim_{n\to\infty} P\left(\frac{\overline{X}_n - nv}{v^{1+2/\alpha} n^{2/\alpha}}\right) = 1 - P(Z_{\alpha/2,b} > -x) = P(Z_{\alpha/2,b} > -x) = 1 - P(Z_{\alpha/2,b} \le x) \quad \forall x \in \mathbb{R}.$$

We can then translate the limiting distribution for $\overline{X}_n$ into a limiting distribution for X_n as we did in the previous section.

Gaussian Limits, $p \in (5/6, 1)$

Let $p \in (\frac{5}{6}, 1)$. Since $\alpha > 4$ when $p > 5/6$, the tail asymptotics in (4.2) imply that $\mathbb{E}_0^V[W_1^2] < \infty$. This case can therefore be handled by applying the central limit theorem for Markov chains to V, for instance Theorem I.16.1 of [6]. Using that theorem and the connection between T_n and V, there is a constant $b > 0$ such that

$$\lim_{n\to\infty} P\left(\frac{T_n - v^{-1}n}{b\sqrt{n}} < x\right) = \Phi(x).$$

We can then write the following regarding the maximum position of the walk:

$$\begin{aligned} P\left(\frac{\overline{X}_n - vn}{\beta\sqrt{n}} < x\right) &= P(\overline{X}_n < \lceil \beta\sqrt{n}x + vn\rceil \\ &= P(T_{\lceil \beta\sqrt{n}x+vn\rceil} > n) \\ &= P\left(\frac{T_{\lceil \beta\sqrt{n}x+vn\rceil} - v^{-1}\lceil \beta\sqrt{n}x+vn\rceil}{b\lceil \beta\sqrt{n}x+vn\rceil^{1/2}} > \frac{n - v^{-1}\lceil \beta\sqrt{n}x+vn\rceil}{b\lceil \beta\sqrt{n}x+vn\rceil^{1/2}}\right) \end{aligned}$$

$$= P\left(\frac{T_{\lceil \beta\sqrt{n}x+vn\rceil} - v^{-1}\lceil \beta\sqrt{n}x + vn\rceil}{b\lceil \beta\sqrt{n}x + vn\rceil^{1/2}} > \frac{n - v^{-1}(\beta\sqrt{n}x + vn)}{b(\beta\sqrt{n}x + vn)^{1/2}} + o(1)\right)$$
$$= P\left(\frac{T_{\lceil \beta\sqrt{n}x+vn\rceil} - v^{-1}\lceil \beta\sqrt{n}x + vn\rceil}{b\lceil \beta\sqrt{n}x + vn\rceil^{1/2}} > \frac{-\beta\sqrt{n}x}{bv^{3/2}(\frac{\beta}{v}\sqrt{n}x + n)^{1/2}} + o(1)\right)$$

If we select $\beta = bv^{3/2}$ and take $n \to \infty$, we obtain

$$\lim_{n\to\infty} P\left(\frac{\overline{X}_n - vn}{bv^{3/2}\sqrt{n}} < x\right) = 1 - \Phi(-x) = \Phi(x) \quad \forall x \in \mathbb{R}.$$

4.3.3 Boundary Cases

We will now handle the boundary cases $p = 3/4$ and $p = 5/6$. We will begin with the more mild of the two.

When $p = 5/6$ we have $\alpha = 4$, and the tail asymptotics for σ and W are

$$P_0^V(\sigma_1 > n) \sim n^{-4} \quad \text{and} \quad P_0^V(W_1 > n) \sim n^{-2}.$$

We can then directly apply chapter XVII.5 of [10] to see that the distribution of W is in the domain of attraction for a normal distribution, requiring normalization by $C\sqrt{n \log n}$.

The case $p = 3/4$ is by far the most delicate. We will closely follow the approach taken in Appendix B of [16]. Let $p = 3/4$, so that $\alpha = 2$. Then Theorems 4.2.2 and 4.2.3 give that

$$P_0^V(\sigma_1 > n) \sim n^{-2} \quad \text{and} \quad P_0^V(W_1 > n) \sim n^{-1}. \tag{4.7}$$

Let $m(t) = \mathbb{E}_0^V[W_1 \mathbf{1}_{\{W_1 > t\}}]$ be the mean of W_1 truncated at t, and note that by (4.7) we have that $m(t) \sim C_2 \log t$. By Theorem 3.7.2 of [9], there exist constants $b' > 0$ and $\xi' \in \mathbb{R}$ such that

$$\lim_{n\to\infty} P_0^V\left(\frac{\sum_{k=1}^n W_k - nm(n)}{n} \le x\right) = L_{1,b',\xi'}(x), \quad \forall x \in \mathbb{R}, \tag{4.8}$$

where $L_{1,b',\xi'}$ is the distribution of the 1-stable random variable with characteristic exponent

$$\log \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ux} L_{1,b',\xi'}(dx) = \mathrm{i}u\xi' - b'|u| \left(1 + \frac{2\mathrm{i}}{\pi} \log|u| \operatorname{sign}(u)\right).$$

The same theorem also implies the existence of $A > 0$ such that

$$\lim_{n\to\infty} P_0^V \left(\frac{\sigma_n - n\bar{\sigma}}{A\sqrt{n \log n}}\right) = \Phi(x), \quad \forall x \in \mathbb{R}, \tag{4.9}$$

where $\bar{\sigma} = \mathbb{E}[\sigma_1]$. We now claim that the difference between $n^{-1}\sum_{\mathrm{i}=1}^{n} V_{\mathrm{i}}$ and $n^{-1}\sum_{k=1}^{\lfloor n/\bar{\sigma}\rfloor} W_k$ converges to 0 in probability. Let $R = \{|r_{n/\bar{\sigma}} - n| > n^{3/4}\}$, then indeed for any $\epsilon > 0$

$$\begin{aligned}
&P_0^V \left(\left|\sum_{\mathrm{i}=1}^{n} V_{\mathrm{i}} - \sum_{k=1}^{\lfloor n/\bar{\sigma}\rfloor} W_k\right| > \epsilon n\right)\\
&= P_0^V \left(\left|\sum_{\mathrm{i}=1}^{n} V_{\mathrm{i}} - \sum_{k=1}^{\lfloor n/\bar{\sigma}\rfloor} W_k\right| > \epsilon n \mid R\right) P(R) + P_0^V \left(\left|\sum_{\mathrm{i}=1}^{n} V_{\mathrm{i}} - \sum_{k=1}^{\lfloor n/\bar{\sigma}\rfloor} W_k\right| > \epsilon n \mid R^c\right) P(R^c)\\
&\le P_0^V(R) + P_0^V \left(\left|\sum_{\mathrm{i}=1}^{n} V_{\mathrm{i}} - \sum_{k=1}^{\lfloor n/\bar{\sigma}\rfloor} W_k\right| > \epsilon n \mid R^c\right)\\
&= P_0^V(R) + P_0^V \left(\sum_{k:|k-n/\bar{\sigma}|\le n^{3/4}+1} V_k > \epsilon n\right)\\
&\le P_0^V(R) + P_0^V \left(\sum_{k:|k-n/\bar{\sigma}|\le n^{3/4}+1} W_k > \epsilon n\right)\\
&= P_0^V(R) + P_0^V \left(\left|\sum_{k=1}^{2\lfloor n^{3/4}\rfloor+3} W_k\right| > \epsilon n\right).
\end{aligned}$$

As $n \to \infty$, both of these terms go to 0, the first due to (4.9) and the second due to (4.8), proving our claim. Because of the relationship noted in (4.5), T_n and $n + 2\sum_{\mathrm{i}=1}^{n} V_{\mathrm{i}}$ have the same limiting distributions. In particular, we have

$$\begin{aligned}
\lim_{n\to\infty} P_0^V \left(\frac{T_n - 2(\frac{n}{\bar{\sigma}})m(\frac{n}{\bar{\sigma}})}{n} \le x\right) &= \lim_{n\to\infty} P_0^V \left(\frac{n + 2\sum_{\mathrm{i}=1}^{n} V_{\mathrm{i}} - 2(\frac{n}{\bar{\sigma}})m(\frac{n}{\bar{\sigma}})}{n} \le x\right)\\
&= \lim_{n\to\infty} P \left(\frac{\sum_{k=1}^{\lfloor n/\bar{\sigma}\rfloor} W_k - \frac{n}{\bar{\sigma}}m(\frac{n}{\bar{\sigma}})}{(\frac{n}{\bar{\sigma}})} \ge \frac{(x-1)\bar{\sigma}}{2}\right)
\end{aligned}$$

$$= L_{1,b',\xi'}\left(\frac{(x-1)\bar{\sigma}}{2}\right) = L_{1,b,\xi}(x),$$

where $b = \frac{2b'}{\bar{\sigma}}$ and $\xi = 1 + \frac{2\xi'}{\bar{\sigma}} - \frac{4b'}{\pi\bar{\sigma}}\log(\frac{2}{\bar{\sigma}})$. That is, we have shown

$$\lim_{n\to\infty} P\left(\frac{T_n - nD(n)}{n} \le x\right) = L_{1,b,\xi}(x) \quad \forall x \in \mathbb{R},$$

where $D(n) = \xi + \frac{2}{\bar{\sigma}}m(\frac{n}{\bar{\sigma}})$, so that $D(t) \sim a^{-1}\log t$, where $a = \frac{\bar{\sigma}}{2C_2}$. It remains to translate this result into a statement about the walk X_n. Toward this end, we define a function $\Gamma(t)$ that will be similar to an inverse of $sD(s)$:

$$\Gamma(t) = \inf\{s > 0 \;:\; sD(s) \ge t\}.$$

Note that $\Gamma(t)$ has the property that $\Gamma(t) \sim \frac{at}{\log t}$. We will now justify our claim that $\Gamma(t)$ is "close" to being an inverse of $sD(s)$. First, $sD(s) = s(\xi + \frac{2}{\bar{\sigma}}\mathbb{E}_0^V[W_1\mathbf{1}_{\{W_1\le s\}}])$ is a strictly increasing function once s is sufficiently large since $\lim_{s\to\infty}\mathbb{E}_0^V[W_1\mathbf{1}_{\{W_1\le s\}}] = \infty$, so eventually $\xi + \mathbb{E}_0^V[W_1\mathbf{1}_{\{W_1\le s\}}] > 0$. Additionally, $sD(s)$ is right continuous. If $sD(s)$ has a discontinuity at s_0, it will be a jump discontinuity of size no larger than $2(\frac{s_0}{\bar{\sigma}})^2P(W_1 = \frac{s_0}{\bar{\sigma}})$. Therefore,

$$|\Gamma(t)D(\Gamma(t)) - t| \le 2\left(\frac{\Gamma(t)}{\bar{\sigma}}\right)^2 P\left(W_1 = \frac{\Gamma(t)}{\bar{\sigma}}\right) \le o(\Gamma(t))$$

because (4.7) implies that $tP(W_1 = t) = o(1)$. In sum, $\Gamma(t)$ has the property that

$$\Gamma(t)D(\Gamma(t)) = t + o(\Gamma(t)).$$

Now let $m_{n,x} = \lceil \Gamma(n) + \frac{xn}{(\log n)^2}\rceil \vee 0$. Since $m_{n,x} \sim \Gamma(n)$ as $n \to \infty$, we have that

$$\begin{aligned}\lim_{n\to\infty}\frac{n - m_{n,x}D(m_{n,x})}{m_{n,x}} &= \lim_{n\to\infty}\frac{n - m_{n,x}D(\Gamma(n))}{m_{n,x}} \\ &= \lim_{n\to\infty}\frac{n - (\Gamma(n) + \frac{xn}{(\log n)^2}D(\Gamma(n))}{\Gamma(n) + \frac{xn}{(\log n)^2}}\end{aligned}$$

$$= \lim_{n\to\infty} \frac{-\frac{xn}{(\log n)^2} D(\Gamma(n))}{\Gamma(n) + \frac{xn}{(\log n)^2}} = -\frac{x}{a^2}.$$

Letting $M_{n,x} = m_{n,x} + \lceil \sqrt{n} \rceil$, we can also see that

$$\lim_{n\to\infty} \frac{n - M_{n,x} D(M_{n,x})}{M_{n,x}} = -\frac{x}{a^2}. \tag{4.10}$$

To complete the proof, we use the fact that $\{T_{m+r} > n\} = \{\overline{X}_n < m + r\}$ to rewrite (4.6) in the form:

$$\{T_m > n\} = \{\overline{X}_n > m\} \subset \{X_n < m\} \subset \{T_{m+r} > n\} \cup \left\{ \inf_{k \geq T_{m+r}} X_k < m \right\}. \tag{4.11}$$

Having done so, we can use (4.11) to see that

$$\begin{aligned}
P\left(\frac{T_{m_{n,x}} - m_{n,x} D(m_{n,x})}{m_{n,x}} > \frac{n - m_{n,x} D(m_{n,x})}{m_{n,x}} \right) &= P(T_{m_{n,x}} > n) \\
&= P(\overline{X}_n < m_{n,x}) \\
&\leq P(X_n < m_{n,x}) \\
&= P\left(\frac{X_n - \Gamma(n)}{n/(\log n)^2} < x \right) \\
&\leq P\left(T_{M_{n,x}} > n \right) + \mathcal{O}(n^{-1/2}).
\end{aligned}$$

Taking $n \to \infty$ shows that

$$\lim_{n\to\infty} P\left(\frac{X_n - \Gamma(n)}{n(\log n)^2} < x \right) = 1 - L_{1,b,\xi}(-x^2/a),$$

and this final limit completes the proof of the main result.

4.4 Proofs of Lemmas Supporting Tail Asymptotics

In this section we will provide proofs for Lemmas 8 – 11. In order to prove Lemma 8, we will appeal to a technical lemma due to Kosygina and Peterson [16].

Lemma 13 (Lemma 7.1 of [16]). *Let $b \in \mathbb{R}$, $D > 0$, and $Y(t)$, $t \geq 0$ be a solution of*

$$dY(t) = b\ dt + \sqrt{DY(t)^+}dB(t), \quad Y(0) = x > 0, \tag{4.12}$$

where $B(t)$, $t \geq 0$ is a standard Brownian motion. Let $Z_n := \{Z_{n,k}\}_{k\geq 0}$ be integer-valued Markov chains such that:

(i) there is a sequence $N_n \in \mathbb{N}$, $N_n \to \infty$, $N_n = o(n)$ as $n \to \infty$, function $f : \mathbb{N} \to [0, \infty)$ such that $f(x) \to 0$ as $x \to \infty$, and function $g : \mathbb{N} \to [0, \infty)$, $g(x) \searrow 0$ as $x \to \infty$ such that

(E) $|\mathbb{E}[Z_{n,1} - Z_{n,0}|Z_{n,0} = m]| \leq f(m \vee N_n)$

(V) $\left|\dfrac{\mathrm{Var}(Z_{n,1}|Z_{n,0=m}}{(m \vee N_n)} - D\right| \leq g(m \vee N_n)$

(ii) for each T, $r > 0$

$$\mathbb{E}\left[\max_{1\leq k\leq (Tn)\wedge\tau_{rn}^{Z_n}} (Z_{n,k} - Z_{n,k-1})^2\right] = o(n^2) \quad \textit{as } n \to \infty,$$

where $\tau_x^{Z_n} = \inf\{k \geq 0 : Z_{n,k} \geq x\}$.

Set $Z_{n,0} = \lfloor nx_n \rfloor$, $x_n \to x$ as $n \to \infty$ and $Y_n(t) = Z_{n,\lfloor nt\rfloor}/n$, $t \geq 0$. Then as $n \to \infty$, Y_n converges in distribution to Y with respect to the Skorokhod (J_1) topology.

We can derive Lemma 8 from Lemma 13 in the same way it was done in [16]. Namely, we will construct a modified version of the BBLP, denoted by $\bar{V}$, for which it will be easier to check the conditions of Lemma 13. Then, we can couple V and $\bar{V}$ together, so that the two processes match until the first time that they fall below N_n, and since $N_n = o(n)$ this yields a diffusion approximation until falling below $n\epsilon$ for the first time for every $\epsilon > 0$.

Proof of Lemma 8. We will begin by constructing the modified branching process that we will apply Lemma 13 to. Recall that the BBLP can be written as

$$V_k = \sum_{m=1}^{V_{k-1}+1} G_m^k,$$

where G_m^x is the number of failures between the $m-1$th and mth success in the coin tosses at site x $\{\xi_{\mathrm{j}}^x\}_{\mathrm{j}\geq 1}$. We can then write

$$V_k = V_{k-1} + 1 + \sum_{m=1}^{V_{k-1}+1} \left(G_m^k - 1\right), \quad k \geq 1.$$

Now, for any sequence $N_n = o(n)$ and for n large enough that $N_n \ll ny_n$, we define

$$\bar{V}_{n,0} = \lfloor ny_n \rfloor, \quad \bar{V}_{n,k} := \bar{V}_{n,k-1} + 1 + \sum_{\mathrm{j}=1}^{(\bar{V}_{n,k-1}+1)\vee N_n} \left(G_{\mathrm{j}}^k - 1\right), \quad k \geq 1.$$

The only difference between the two processes is that if $\bar{V}_{n,k}$ ever falls below N_n, we "top up" the process by increasing the number of organisms to N_n before reproduction. Since we assume that $N_n \ll \lfloor ny_n \rfloor$, the modified process $\bar{V}_{n,k}$ agrees with the BBLP V_k for as long as the two processes stay above level $N_n < \epsilon n$, and so the two agree at least until exiting $[n\epsilon, \infty)$. We must now verify the conditions of Lemma 13 for $\bar{V}$. Note that by Proposition 4.2.1, condition (i) is satisfied with $f(x) = C_1 x p^{x+1}$, $g(x) = C_2/x$, $D = \nu = 2$ and $b = 1-\alpha$, and so the bulk of the work will be to show that condition (ii) is met. To that end, fix $T, r > 0$. Then, we need to show that

$$\lim_{n\to\infty} \frac{1}{n^2} \mathbb{E}\left[\max_{1\leq k\leq (Tn)\wedge\tau_{rn}^{\bar{V}_n}} \left|1 + \sum_{\mathrm{j}=1}^{(\bar{V}_{n,k-1})\vee N_n} \left(G_{\mathrm{j}}^k - 1\right)\right|^2\right] = 0,$$

where $\tau_{rn}^{\bar{V}_n} = \inf\{k : \bar{V}_{n,k} \geq rn\}$. In order to do so, observe that if we take n large enough that $rn > N_n$ we have

$$\begin{aligned}
\frac{1}{n^2} \mathbb{E}\left[\max_{1\leq k\leq (Tn)\wedge\tau_{rn}^{\bar{V}_n}} \left|1 + \sum_{\mathrm{j}=1}^{(\bar{V}_{n,k-1})\vee N_n} \left(G_{\mathrm{j}}^k - 1)\right)\right|^2\right] &\leq \frac{1}{n^2} \mathbb{E}\left[\max_{1\leq k\leq (Tn)} \max_{N_n\leq m\leq rn+1} \left|\sum_{\mathrm{j}=1}^{m} (G_{\mathrm{j}}^k - 1)\right|^2\right] \\
&\leq \frac{1}{n^2} \sum_{y=0}^{\infty} P\left(\max_{1\leq k\leq (Tn)} \max_{N_n\leq m\leq rn+1} \left|\sum_{\mathrm{j}=1}^{m} (G_{\mathrm{j}}^k - 1)\right|^2 > y\right) \\
&\leq \frac{r^{3/2}}{\sqrt{n}} + rT \max_{N_n\leq m\leq rn+1} \sum_{y\geq (rn)^{3/2}} P\left(\left|\sum_{\mathrm{j}=1}^{m} (G_{\mathrm{j}}^k - 1)\right| > \sqrt{y}\right),
\end{aligned}$$

where the last line comes from bounding the probabilities for $y < (rn)^{3/2}$ by 1. Therefore, we have reduced the problem of verifying condition (i) to finding appropriate bounds on $P\left(\left|\sum(G_j^k - 1)\right| > \sqrt{y}\right)$. To do so, we will need Lemma A.2 from [16].

Lemma 14 (Lemma A.2 of [16], Theorem III.15 of [22])**.** *Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. non-negative random variables with $\mathbb{E}[Y_1] = \mu$ and $\mathbb{E}[e^{\lambda_0 Y_1}] < \infty$ for some $\lambda_0 > 0$. Then there exists a constant $C > 0$ such that*

$$P\left(\left|\sum_{k=1}^{n} Y_k - \mu n\right| \geq y\right) \leq \exp\left\{-C\frac{y^2}{y \vee n}\right\}.$$

The calculations that follow will not depend on k, and so we will suppress the k superscript, writing G_j for G_j^k. We will start by considering the right-tail probabilities. Let $\{\xi_j\}_{j\geq 1}$ be the sequence of Bernoulli random variables that occur at a fixed site k, and recall that $\xi_j \sim \text{Ber}(p)$ until a failure has occurred at that site, after which $\xi_j \sim \text{Ber}(1/2)$. Also let $\{\gamma_j\}_{j\geq 1}$ be a collection of i.i.d. Ber(1/2) random variables and let $H_j \sim \text{Geo}(1/2)$ be a geometric random variable with mean 1. Then

$$P\left(\sum_{j=1}^{m}(G_j - 1) > y\right) = P\left(\sum_{j=1}^{m} G_j > m + y\right) = P\left(\sum_{j=1}^{2m+y} \xi_j < m\right)$$

because $y+n$ failures occur before n successes in $\{\xi_j\}_{j\geq 1}$ exactly when the number of successes in the first $2n+y$ Bernoulli trials is less than n. Now, we condition on when the first failure occurs. Let $E_{y/2}$ be the event that the first failure occurs before the $\lfloor y/2 \rfloor$ trial:

$$E_{y/2} = \left\{\xi_1 + \cdots + \xi_{\lfloor y/2 \rfloor} < \lfloor y/2 \rfloor\right\},$$

Then we have that

$$\begin{aligned} P\left(\sum_{j=1}^{2m+y} \xi_j < m\right) &\leq P\left(\sum_{j=1}^{2m+y} \xi_j < m \middle| E_{\lfloor y/2 \rfloor}\right) P(E_{\lfloor y/2 \rfloor}) + P\left(E^C_{\lfloor y/2 \rfloor}\right) \\ &\leq P\left(\sum_{j=\lfloor y/2 \rfloor}^{2m+y} \gamma_j < m\right) + p^{y/2} \end{aligned}$$

$$
\begin{aligned}
&\le P\left(\sum_{\mathrm{j}=1}^{2m+\lceil y/2\rceil} \gamma_{\mathrm{j}} < m\right) + p^{y/2} \\
&= P\left(\sum_{\mathrm{j}=1}^{m} H_{\mathrm{j}} > m + y/2\right) + p^{y/2} \\
&= P\left(\sum_{\mathrm{j}=1}^{m} (H_{\mathrm{j}} - 1) > y/2\right) + p^{y/2} \\
&\le \exp\left\{-C\frac{y^2}{y \vee m}\right\} + p^{y/2},
\end{aligned}
$$

where in the last line we have used Lemma 14. We can handle the left tail probabilities in a similar way. To that end, we can write

$$
P\left(\sum_{\mathrm{j}=1}^{m} G_{\mathrm{j}} < m - y\right) = P\left(\sum_{\mathrm{j}=1}^{2m-y-1} (1-\xi_{\mathrm{j}}) < m - y\right),
$$

since fewer than $m-y$ failures occur before m successes if and only if the number of failures in the first $2m-y-1$ trials does not exceed $m-y$. Note that $(1-\xi_{\mathrm{j}})$ are $\mathrm{Ber}(1-p)$ random variables before the first failure occurs in $\{\xi_{\mathrm{j}}\}_{\mathrm{j}\ge 1}$, and are $\mathrm{Ber}(1/2)$ afterward. If we set $m' = m - y$ (we assume $m > y$, otherwise the probability in question is 0) we can rewrite the above expression:

$$
P\left(\sum_{\mathrm{j}=1}^{2m-y-1} (1-\xi_{\mathrm{j}}) < m - y\right) = P\left(\sum_{\mathrm{j}=1}^{2m'+y-1} (1-\xi_{\mathrm{j}}) < m'\right),
$$

and this places us in the same situation we had for the right tail probability. Repeating that argument yields

$$
P\left(\sum_{\mathrm{j}=1}^{m} G_{\mathrm{j}} < m - y\right) \le \exp\left\{-C\frac{y^2}{y \vee m'}\right\} + p^{y/2} \le \exp\left\{-C\frac{y^2}{y \vee m}\right\} + p^{y/2}.
$$

We can now apply these estimates to complete our proof of Lemma 8:

$$
\max_{N_n \le m \le rn+1} \sum_{y \ge (rn)^{3/2}} P\left(\left|\sum_{\mathrm{j}=1}^{m} (G_{\mathrm{j}} - 1)\right| > \sqrt{y}\right) \le \max_{N_n \le m \le rn+1} \sum_{y \ge (rn)^{3/2}} \exp\left\{-C\frac{y}{\sqrt{y} \vee m}\right\} + p^{\sqrt{y}/2}
$$

$$\leq \sum_{y \geq (rn)^{3/2}} \exp\left\{-C\frac{y}{\sqrt{y} \vee (rn+1)}\right\} + p^{\sqrt{y}/2}$$
$$\leq \left(\sum_{y \geq (rn)^{3/2}} \exp\left\{-C\frac{y}{\sqrt{y} \vee (y^{2/3}+1)}\right\} + p^{\sqrt{y}/2}\right) \xrightarrow{n\to\infty} 0,$$

which shows that $\bar{V}$ satisfies the stated diffusion approximation. Since V and $\bar{V}$ agree with each other until the two fall below $N_n = o(n)$, V satisfies the same diffusion approximation before exiting the interval $[n\epsilon, \infty)$, which completes our proof of Lemma 8. □

Next, We will prove the Overshoot Lemma (Lemma 10) since we will need it to prove Lemma 9.

Proof of Lemma 10. Our proof will closely follow the proof of Lemma 6.3 in [16]. To prove the first part of Lemma 10, we will use the following inequality from pp. 595-596 of [15]:

$$\max_{0\leq z<x} P_z^V\left(V_{\tau_x} > x+y|\tau_x < \sigma_0\right) \leq \max_{0\leq z<x} \frac{P_z^V\left(V_1 > x+y\right)}{P_z^V\left(V_1 \geq x\right)}. \tag{4.13}$$

Rewriting V_1 above as a sum of G_{j} random variables as above, we can see that

$$\begin{aligned}\max_{0\leq z<x} P_z^V(V_{\tau_x} > x+y \mid \tau_x < \sigma_0) &\leq \max_{0\leq z<x} \frac{P\left(z+1+\sum_{\text{j}=1}^{z+1}(G_{\text{j}}-1) > x+y\right)}{P\left(z+1+\sum_{\text{j}=1}^{z+1}(G_{\text{j}}-1) \geq x\right)} \\ &= \max_{0\leq m<x} \frac{P\left(\sum_{\text{j}=1}^{x-m}(G_{\text{j}}-1) > y+m\right)}{P\left(\sum_{\text{j}=1}^{x-m}(G_{\text{j}}-1) \geq m\right)},\end{aligned} \tag{4.14}$$

where in (4.14) we have made the substitution $m = x - z - 1$. To bound the probability of interest, we will need to place an upper bound on the numerator and lower bound on the denominator of (4.14). For the lower bound, we will need the following lemma.

Lemma 15.

$$\inf_{n\geq 1} P\left(\sum_{\text{j}=1}^{n}(G_{\text{j}}-1) \geq 0\right) > 0.$$

We will leave the proof of Lemma 15 for Section 4.5. Now, let $\rho_m = \min\left\{n \geq 1 : \sum_{\mathrm{j}=1}^{n}(G_{\mathrm{j}} - 1) \geq m\right\}$. Then

$$
\begin{aligned}
P\left(\sum_{\mathrm{j}=1}^{x-m}(G_{\mathrm{j}} - 1) \geq m\right) &= \sum_{n=1}^{x-m} P\left(\sum_{\mathrm{j}=1}^{x-m}(G_{\mathrm{j}} - 1) \geq m,\ \rho_m = n\right) \\
&\geq \sum_{n=1}^{x-m} P\left(\sum_{\mathrm{j}=n+1}^{x-m}(G_{\mathrm{j}} - 1) \geq 0,\ \rho_m = n\right) \\
&= \sum_{n=1}^{x-m} P\left(\sum_{\mathrm{j}=n+1}^{x-m}(G_{\mathrm{j}} - 1) \geq 0\right) P\left(\rho_m = n\right) \\
&\geq \sum_{n=1}^{x-m} \inf_{k\geq 1} P\left(\sum_{\mathrm{j}=1}^{k}(G_{\mathrm{j}} - 1) \geq 0\right) P(\rho_m = n) \geq CP(\rho_m \leq x - m),
\end{aligned}
$$

where the last inequality follows from Lemma 15. Now, we will place an upper bound on the numerator of (4.13). To do so, observe that

$$
\begin{aligned}
&P\left(\sum_{\mathrm{j}=1}^{x-m}(G_{\mathrm{j}} - 1) > m + y\right) \\
&\leq P\left(\sum_{\mathrm{j}=1}^{x-m}(G_{\mathrm{j}} - 1) > m + y/2\right) \\
&\leq \sum_{n=1}^{x-m} P\left(\sum_{\mathrm{j}=1}^{n}(G_{\mathrm{j}} - 1) > m + y/2,\ \rho_m = n\right) + \sum_{n=1}^{x-m} P\left(\sum_{\mathrm{j}=n+1}^{x-m}(G_{\mathrm{j}} - 1) > y/2,\ \rho_m = n\right) \\
&\leq \sum_{n=1}^{x-m} \max_{l<m} P(G_1 > m - l + y/2 \mid G_1 - 1 \geq m - l)P(\rho_m = n) \\
&\qquad\qquad\qquad + \sum_{n=1}^{x-m} \max_{k<x} P\left(\sum_{\mathrm{j}=1}^{k}(G_{\mathrm{j}} - 1) > y/2\right) P(\rho_m = n) \\
&= \left(\max_{l<m} P(G_1 - 1 > m - l + y/2 \mid G_1 - 1 \geq m - l) + \max_{k<x} P\left(\sum_{\mathrm{j}=1}^{k}(G_{\mathrm{j}} - 1) > y/2\right)\right) P(\rho_m \leq x - m) \\
&\leq \left(\left(\frac{1}{2}\right)^{\lfloor y/2 \rfloor} + p^{y/2} \max_{k<x}\ \exp\left\{-C\frac{(y/2)^2}{(y/2) \vee k}\right\}\right) P(\rho_m \leq x - m) \\
&= \left(\left(\frac{1}{2}\right)^{\lfloor y/2 \rfloor} + p^{y/2} + \exp\left\{-C'\frac{y^2}{y \vee x}\right\}\right) P(\rho_m \leq x - m),
\end{aligned}
$$

where the last inequality comes from the right-tail proven above. Combined with the lower bound on the denominator, we have that

$$P_z\left(V_{\tau_x} > x + y | \tau_x < \sigma_0\right) \leq \max_{0 \leq m < x} \frac{\left(\left(\frac{1}{2}\right)^{\lfloor y/2 \rfloor} + p^{y/2} + \exp\left\{-C' \frac{y^2}{y \vee x}\right\}\right) P(\rho_m \leq x - m)}{C P\left(\rho_m \leq x - m\right)}$$
$$\leq C_3 \exp\left\{-C_4 \frac{y^2}{y \vee x}\right\},$$

where C_3 and C_4 come from adjusting constants appropriately. This concludes the proof of the first part of Lemma 10. The proof of the second part is similar, but we will present it here for completeness since it is not contained in either of [15], [16]. We are considering

$$\max_{x < z < 4x} P_z(V_{\sigma_x \wedge \tau_{4x}} < x - y).$$

We will begin by establishing an inequality similar to the first step in the proof of the first part of Lemma 10, which we formulate as a lemma:

Lemma 16.

$$P_z(V_{\sigma_x \wedge \tau_{4x}} < x - y) \leq \max_{x < r < 4x} \frac{P_r(V_1 < x - y)}{P_r(V_1 \leq x)}.$$

The proof of Lemma 16 is a somewhat tedious calculation, and so we banish it to Section 4.5 below. Once it is proven, though, we can rewrite V_1 as we did in the proof of the first part of Lemma 10, and then we can quickly obtain

$$\max_{x < z < 4x} \frac{P_z(V_1 < x - y)}{P_z(V_1 \leq x)} = \max_{-3x < m < 0} \frac{P_z\left(\sum_{\mathrm{j}=1}^{x-m}(G_{\mathrm{j}} - 1) < m - y\right)}{P_z\left(\sum_{\mathrm{j}=1}^{x-m}(G_{\mathrm{j}} - 1) \leq m\right)}, \tag{4.15}$$

where we have again made the substitution $m = x - z - 1$. As before, we will put a lower bound on the denominator and an upper bound on the numerator of (4.15). To put a lower bound on the denominator, we need to use the fact that

$$\inf_{n \geq 1} P\left(\sum_{\mathrm{j}=1}^{n}(G_{\mathrm{j}} - 1) \leq 0\right) > 0. \tag{4.16}$$

Since the proof of (4.16) is essentially the same as the proof of Lemma 15, we omit it. To place a lower bound on the denominator, let $\rho'_m = \min\{n \geq 1 : \sum_{\mathrm{j}=1}^{n}(G_\mathrm{j} - 1) \leq m\}$, then

$$P\left(\sum_{\mathrm{j}=1}^{x-m}(G_\mathrm{j}-1) \leq m\right) = \sum_{n=1}^{x-m} P\left(\sum_{\mathrm{j}=1}^{x-m}(G_\mathrm{j}-1) \leq m,\ \rho'_m = n\right) \geq \sum_{n=1}^{x-m} P\left(\sum_{\mathrm{j}=n+1}^{x-m}(G_\mathrm{j}-1) \leq 0\right) P(\rho'_m = n)$$
$$\geq C' P(\rho'_m \leq x - m),$$

where in the last line we have used (4.16). For the upper bound on the numerator, in the spirit of the proof of the first part of Lemma 10 we write

$$\begin{aligned}
&P\left(\sum_{\mathrm{j}=1}^{x-m}(G_\mathrm{j}-1) \leq m\right) \\
&\leq \sum_{n=1}^{x-m} P\left(\sum_{\mathrm{j}=1}^{n}(G_\mathrm{j}-1) < m - y/2, \rho'_m = n\right) + \sum_{n=1}^{x-m} P\left(\sum_{\mathrm{j}=n+1}^{x-m}(G_\mathrm{j}-1) < -y/2,\ \rho'_m = n\right) \\
&\leq \left[\max_{l<m} P(G_1 - 1 < m - y/2 + l \mid G_1 - 1 < m + l) + \max_{x<k<4x} P\left(\sum_{\mathrm{j}=1}^{k}(G_\mathrm{j}-1) < -y/2\right)\right] \sum_{n=1}^{x-m} P(\rho'_m = n) \\
&\leq P(\rho'_m \leq x - m)\left[\left(\frac{1}{2}\right)^{\lfloor y/2 \rfloor} + \max_{x<k<4x} P\left(\sum_{\mathrm{j}=1}^{k}(G_\mathrm{j}-1) < -y/2\right)\right].
\end{aligned}$$

Applying the left-tail estimates proven above and adjusting constants as necessary gives the desired bound on the numerator, and we can then finish the proof of the second part of Lemma 10 in the same way as the first part. $\square$

Having proved Lemma 10, we now have all the tools we need to prove Lemma 9. The proof is essentially the same as the proof of Lemma 5.2 in [15], and so we will only summarize it here. Let $s(x) \in C^\infty([0,\infty))$ be a function with compact support such that

$$s(x) = \begin{cases} x^\alpha, & \alpha \neq 0 \\ \log x, & \alpha = 0 \end{cases},$$

for all $x \in \left(\frac{2}{3a}, \frac{3a}{2}\right)$. We then show that $s\left(V_k\right)$ is close to being a martingale before it exits the interval $(a^{\mathrm{j}-1}, a^{\mathrm{j}})$. To that end, let $\mathcal{H}_k = \sigma\left(V_{\mathrm{i}} : \mathrm{i} \leq k\right)$ and consider

$$\mathbb{E}\left[s\left(\frac{V_{k+1}}{a^{\mathrm{j}}}\right)\middle|\mathcal{H}_k\right].$$

By Taylor expansion of $s(x)$ on the event $\{\gamma > k\}$, we have that

$$\mathbb{E}\left[s\left(\frac{V_{k+1}}{a^{\mathrm{j}}}\right)\middle|\mathcal{H}_k\right] = s\left(\frac{V_k}{a^{\mathrm{j}}}\right) + \frac{s'\left(\frac{V_k}{a_{\mathrm{j}}}\right)}{a^{\mathrm{j}}}\mathbb{E}\left[V_{k+1} - V_k|\mathcal{H}_k\right] + \frac{s''\left(\frac{V_k}{a^{\mathrm{j}}}\right)}{2a^{2\mathrm{j}}}\mathbb{E}\left[(V_{k+1} - V_k)^2\middle|\mathcal{H}_k\right] + r_{n,k},$$

where $r_{n,k}$ is the Taylor remainder term. Now, we can use Proposition 4.2.1 to note that:

$$\begin{aligned} &\left|\mathbb{E}\left[V_{k+1} - V_k|\mathcal{H}_k\right] - (1-\alpha)\right| \leq C_1 \mathrm{e}^{-C'V_k} \leq C_1 \mathrm{e}^{-C'a^{\mathrm{j}-1}}, \\ &\left|\mathbb{E}\left[(V_{k+1} - V_k)^2\middle|\mathcal{H}_k\right] - 2V_k\right| \leq C_2. \end{aligned}$$

We can also bound $r_{n,k}$:

$$r_{n,k} \leq \frac{1}{6}\|s'''\|_\infty \left(\mathbb{E}\left[\left(\frac{V_{k+1} - V_k}{a^{\mathrm{j}}}\right)^4\middle|\mathcal{H}_k\right]\right)^{3/4},$$

and we can then use a fourth moment bound like the one found in Lemma A.3 of [16] and the fact that $V_k \leq a^{\mathrm{j}+1}$ to show $r_{n,k} \leq Ca^{-3\mathrm{j}/2}$. Taking each of these bounds together, we have that

$$\mathbb{E}\left[s\left(\frac{V_{k+1}}{a^{\mathrm{j}}}\right)\middle|\mathcal{H}_k\right] = s\left(\frac{V_{k+1}}{a^{\mathrm{j}}}\middle|\mathcal{H}_k\right) + \frac{1}{a^{\mathrm{j}}}\left[(1-\alpha)s'\left(\frac{V_k}{a^{\mathrm{j}}}\right) + \frac{V_k}{a^{\mathrm{j}}}s''\left(\frac{V_k}{a^{\mathrm{j}}}\right)\right] + R_{\mathrm{j},k}, \tag{4.17}$$

where $|R_{\mathrm{j},k}| \leq Ca^{-3\mathrm{j}/2}$ on $\{\gamma > k\}$. Since $s(x) = x^\alpha$ in $[1/a, a]$, the middle term in the right-hand side of (4.17) is 0 on $\{\gamma > k\}$, and therefore $s\left(\frac{V_{n\wedge\gamma}}{a_{\mathrm{j}}}\right) - \sum_{k=0}^{(n\wedge\gamma)-1} R_{\mathrm{j},n}$ is a martingale with respect to $\mathcal{H}_n$. We can then use Lemmas 8 and 10 to complete the proof in the same way it is done in the proof of Lemma 5.3 of [15], showing that both

$$\mathbb{E}\left[s\left(\frac{V_\gamma}{a^{\mathrm{j}}}\right)\right] \approx s\left(\frac{x}{a^{\mathrm{j}}}\right),$$

$$\mathbb{E}\left[s\left(\frac{V_\gamma}{a^{\mathrm{j}}}\right)\right] \approx P(V_\gamma \leq a^{\mathrm{j}-1})a^{-\alpha} + P(V_\gamma \geq a^{\mathrm{j}+1})a^{\alpha},$$

and Lemma 9 follows.

4.5 Proofs of Needed Technical Results

Here we collect proofs of the technical results necessary to prove of Lemmas 8 – 11.

Proof of Lemma 15. We want to show that

$$\inf_{n\geq 1} P\left(\sum_{\mathrm{j}=1}^{n}(G_{\mathrm{j}}-1) \geq 0\right) > 0.$$

Now, the event $\{\sum_{\mathrm{j}=1}^{n}(G_{\mathrm{j}}-1) \geq 0\}$ occurs if and only if there are less than n successes in the first $2n-1$ Bernoulli trials. Therefore, we have

$$P\left(\sum_{\mathrm{j}=1}^{n}(G_{\mathrm{j}}-1) \geq 0\right) = P\left(\sum_{\mathrm{j}=1}^{2n-1}\xi_{\mathrm{j}} < n\right),$$

where ξ_{j} represent the j^{th} Bernoulli trial and let γ_{j} be i.i.d. Bernoulli$(1/2)$ random variables. Then we can write

$$\begin{aligned} P\left(\sum_{\mathrm{j}=1}^{2n-1}\xi_{\mathrm{j}} < n\right) \geq (1-p)P\left(\sum_{\mathrm{i}=1}^{2n-2}\gamma_{\mathrm{j}} < n\right) &\geq (1-p)P\left(\sum_{\mathrm{i}=1}^{2n}\gamma_{\mathrm{j}} < n\right) \\ &= (1-p)P\left(\sum_{\mathrm{i}=1}^{2n}(\gamma_{\mathrm{j}}-1/2) < 0\right). \end{aligned} \tag{4.18}$$

The probability in (4.18) converges to $1/2$ by the Central Limit Theorem. This, combined with the fact that $P\left(\sum_{\mathrm{j}=1}^{n}(G_{\mathrm{j}}-1) \geq 0\right)$ is positive for each n completes the proof. $\square$

Proof of Lemma 16. We want to show that

$$P_z(V_{\sigma_x\wedge\tau_{4x}} < x-y) \leq \max_{x<r<4x}\frac{P_r(V_1 < x-y)}{P_r(V_1 \leq x)}. \tag{4.19}$$

First, we split the probability in (4.19) according to which of τ_{4x} and σ_x is smaller:

$$\begin{aligned}P_z(V_{\sigma_x\wedge\tau_{4x}} < x-y) &= P_z(V_{\sigma_x\wedge\tau_{4x}} < x-y,\ \sigma_x < \tau_{4x}) + P_z(V_{\sigma_x\wedge\tau_{4x}} < x-y,\ \tau_{4x} < \sigma_x) \\ &= P_z(V_{\sigma_x} < x-y,\ \sigma_x < \tau_{4x}) + P_z(V_{\tau_{4x}} < x-y,\ \sigma_x < \tau_{4x}).\end{aligned}$$

Then, we condition on the value of σ_x and do some arithmetic:

$$\begin{aligned}&P_z(V_{\sigma_x} < x-y,\ \sigma_x < \tau_{4x}) \\ &= \sum_{n=1}^{\infty} P_z(V_{\sigma_x} < x-y,\ \sigma_x = n,\ \sigma_x < \tau_{4x}) \\ &= \sum_{n=1}^{\infty} P_z(V_n < x-y,\ V_n \le x,\ \sigma_x = n,\ \sigma_x < \tau_{4x}) \\ &= \sum_{n=1}^{\infty}\sum_{r=x+1}^{4x-1} P_z(V_n < x-y,\ V_n \le x, V_{n-1} = r,\ x < V_{\mathrm{j}} < 4x\ \forall \mathrm{j} \in \{1,2,\ldots,n-2\}) \\ &= \sum_{n=1}^{\infty}\sum_{r=x+1}^{4x-1} P_z(V_n < x-y \mid V_n \le x, V_{n-1} = r) \\ &\qquad\qquad \times P(V_n < x-y,\ V_n \le x,\ V_{n-1} = r,\ x < V_{\mathrm{j}} < 4x\ \forall \mathrm{j} \in \{1,2,\ldots,n-2\}) \\ &= \sum_{n=1}^{\infty}\sum_{r=x+1}^{4x-1} \frac{P_z(V_n < x-y,\ V_n \le x,\ V_{n-1} = r)}{P_z(V_n \le x,\ V_{n-1} = r)} P_z(\sigma_x < \tau_{4x},\ \sigma_x = n,\ V_{n-1} = r) \\ &= \sum_{n=1}^{\infty}\sum_{r=x+1}^{4x-1} \frac{P_z(V_n < x-y \mid V_{n-1} = r)\cancel{P_z(V_{n-1} = r)}}{P_z(V_n \le x \mid V_{n-1} = r)\cancel{P_z(V_{n-1} = r)}} P_z(\sigma_x < \tau_{4x},\ \sigma_x = n,\ V_{n-1} = r) \\ &\le \max_{x<r<4x} \frac{P_r(V_1 < x-y)}{P_r(V_1 \le x)} \sum_{n=1}^{\infty}\sum_{r=x+1}^{4x-1} P_z(\sigma_x < \tau_{4x},\ \sigma_x = n,\ V_{n-1} = r) \\ &= \max_{x<r<4x} \frac{P_r(V_1 < x-y)}{P_r(V_1 \le x)} P_z(\sigma_x < \tau_{4x}).\end{aligned}$$

Repeating the same calculation with $P_z\left(V_{\tau_{4x}} < x-y, \tau_{4x} < \sigma_x\right)$ allows us to extract the desired inequality. $\square$

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VITA

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