VOLTERRA ROUGH EQUATIONS

by

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ABSTRACT

This article is devoted to extend the recently developed rough path theory for Volterra equations from [1] to the case of more rough noise and/or more singular Volterra kernels. It was already observed in [1] that the Volterra rough path introduced there did not satisfy any geometric relation, similar to that observed in classical rough path theory. Thus, an extension of the theory to more irregular driving signals requires a deeper understanding of the specific algebraic structure arising in the Volterra rough path. Inspired by the elements of "non-geometric rough paths" developed in [2] and [3] tree rooted index is a simple description of the Volterra rough path and the controlled Volterra process, and with this description the existence and uniqueness of the solution to rough volterra equations driven by more irregular signals are held.

1. INTRODUCTION

1.1 Introduction to rough paths

Rough paths analysis is a theory introduced by Terry Lyons in the pioneering paper [4] which aims to solve differential equations driven by functions with finite $p-$variation with $p > 1$, or by $\alpha-$Hölder continuous functions of order $\alpha \in (0, 1)$. Considering a (rough) differential equations of the form

$$dY_t = f(Y_t)dX_t, \qquad Y_0 = \zeta \in W. \tag{1.1}$$

Here, $X : [0, T] \to V$ is the driving or input signal, while $Y : [0, T] \to W$ is the output signal. As usual V and W are Banach spaces, and $f : W \to \mathcal{L}(V, W)$. When $dimV = d < \infty$, one may think of f as a collection of vector fields $(f_1, \cdots, f_d)$ on W. It is okay to think $V = \mathbb{R}^d$ and $W = \mathbb{R}^n$ but there is really no difference in the argument. Such equations are familiar from the theory of ODEs, and more specifically, control theory, where X is typically assumed to be absolutely continuous so that $dX_t = \dot{X}_t dt$. The case of SDEs, stochastic differential equations, with dX interpreted as Itô or Stratonovich differential of Brownian motion, is also well know. Both cases will be seen as special examples of RDEs, rough differential equations. For convenience, one may discuss (1.1) on the unit time interval. Indeed, equation (1.1) is invariant under time-reparametrization so that any (finite) time horizon may be rescaled to $[0, 1]$. Alternatively, global solutions on a larger time horizon are constructed successively, i.e. by concatenating $Y|_{[0,1]}$ (started at Y_0) with $Y|_{[1,2]}$ (started at Y_1) and so on. As a matter of fact, one shall construct solutions by a variation of the classical Picard iteration on intervals $[0, T]$, where $T \in (0, 1]$ will be chosen sufficiently small to guarantee invariance of suitable balls and the contraction property.

In order to solve equation (1.1), the first thing is to make sense of the expression

$$\int_0^t f(Y_s)dX_s, \tag{1.2}$$

where Y is itself the as yet unknown solution. Actually, since the function Y solves (1.1), one would expect the small-scale fluctuations of Y to look exactly like the small-scale fluctuation of X in the sense. The simplest way for Y to look like X is when $Y = G(X)$ for some sufficiently regular function G. Despite what one might guess, it turns out that this particular class of functions Y is already sufficiently rich so that knowing how to define integrals of the form $\int_0^t G(X_s)dX_s$ for (non-gradient) functions G allows to give a meaning to equations of the type (1.1), which is the approach originally developed on [4]. The value of such an integral does not depend on the parametrisation of X, which dovetails nicely with the fact that the α-Hölder of a function is also independent of its parametrisation.

To give a meaning of (1.2), it is necessary to define the space of rough path, more precisely the space of Hölder continuous rough path.

1.1.1 Rough path

A rough path on an interval $[0, T]$ with values in a Banach space V then consists of a continuous function $X : [0, T] \to V$, as well as a continuous second order process $\mathbb{X} : [0, T]^2 \to V \otimes V$, subject to certain algebraic and analytic conditions. For any $(s, u, t) \in \Delta_3$, $\mathbb{X}$ satisfy

$$\mathbb{X}_{ts} - \mathbb{X}_{us} - \mathbb{X}_{tu} = X_{us} \otimes X_{tu}, \quad \text{chen's relation.} \tag{1.3}$$

And $\mathbb{X}$ is self similar, that is

$$\mathbb{X}_{(\lambda t)(\lambda s)} \sim \lambda^{2\alpha} \mathbb{X}_{ts}. \tag{1.4}$$

The following definition is about a α-Hölder continuous function.

Definition 1.1.1. *For $\alpha \in (0, 1)$, define the space of $\alpha-$Hölder rough paths (over V), in symbols $C^\alpha([0, T], V)$ such that*

$$\|X\|_\alpha := \sup_{s \neq t \in [0,T]} \frac{|X_{ts}|}{|t - s|^\alpha} < \infty. \tag{1.5}$$

With Definition 1.1.1 in mind, it is ready to discuss rough path lift of a $\alpha-$Hölder continuous function. For a $\alpha-$Hölder continuous function $X : [0,T] \to V$, define the iterated integral as

$$\mathbb{X}_{ts} := \int_s^t X_{rs} \otimes dX_r. \tag{1.6}$$

Regarding the definition (1.6), the behaviour of iterated integrals $\mathbb{X} : [0,T]^2 \to V \otimes V$ suggests to impose the algebraic relation (1.3) and the analytic relation (1.4). Note that $t \mapsto (X_{t0}, \mathbb{X}_{t0})$ determines the entire second order process $\mathbb{X}$. In this sense, the pair $(X, \mathbb{X})$ is indeed a path, and not some two-parameter object, although it is often more convenient to consider it as one. This discussion motivates the following definition of our basic spaces of rough paths

Definition 1.1.2. *For $\alpha \in (0,1)$, define the space of α-Hölder rough path (over V), in symbols $\mathcal{C}^\alpha([0,T],V)$, as those pairs $(X,\mathbb{X}) =: \mathbf{X}$ such that*

$$\|X\|_\alpha := \sup_{s \neq t \in [0,T]} \frac{|X_{ts}|}{|t-s|^\alpha} < \infty, \qquad \|X\|_{2\alpha} := \sup_{s \neq t \in [0,T]} \frac{|\mathbb{X}_{ts}|}{|t-s|^{2\alpha}} < \infty, \tag{1.7}$$

and such that the algebraic constraint (1.3) is satisfied.

Notice that for an arbitrary path $X \in C^\alpha$ with values in some Banach space V it is far from obvious that this path can indeed be lifted to a rough path $(X,\mathbb{X}) \in \mathcal{C}^\alpha$. Anyway, assuming that we are provided with the data $\mathbf{X} = (X,\mathbb{X})$, then we know how to give meaning to the integral of components of X against other components of X ($\int_0^t X_s dX_s$): this is precisely what X encodes. Thanks to the Definition 1.1.1 of α-Hölder continuous, the integral $\int_0^t Y_s dX_s$ may be defined as a Riemann-Stieljes sums.

1.1.2 Young integral and Sewing lemma

Considering the Young integral $\int_s^t Y_r dX_r$ (as in [5]). Define the integral $\int_s^t Y_r dX_r$ as a limit of Riemann-Stieltjes sums, that is

$$\int_0^1 Y_t dX_t = \lim_{|\mathcal{P}| \to 0} \sum_{[s,t] \in \mathcal{P}} Y_s X_{ts}, \tag{1.8}$$

where $\mathcal{P}$ denotes a partition of $[0,1]$ (interpreted as a finite collection of essentially disjoint intervals such that $\cup\mathcal{P} = [0,1]$) and $|\mathcal{P}|$ denotes the length of the largest element of $\mathcal{P}$. The sum (1.8) converges if $X \in C^\alpha$ and $Y \in C^\beta$, provided $\alpha + \beta > 1$. And as a consequence of Young's inequality in [5], one has bound

$$\left| \int_s^t Y_r dX_r - Y_s X_{ts} \right| \le C \|Y\|_\beta \|X\|_\alpha |t-s|^{\alpha+\beta}. \tag{1.9}$$

According to the definition (1.6) of the iterated integral. The definition of Yong integral (1.8) can be extended to define $1-$form integral $\int_0^1 Y_t d\mathbf{X}_t$. The main insight of the theory of rough paths is that this seemingly unsurmountable barrier of $\alpha + \beta > 1$ (which reduces to $\alpha > \frac{1}{2}$ in the case $\alpha = \beta$) can be broken by adding additional structure to the problem. Indeed, for a rough path $\mathbf{X}$, we postulate the values $\mathbb{X}_{ts}$ of the integral of X against itself as given in (1.6). It is then intuitively clear that one should be able to define $\int Y d\mathbf{X}$ in a consistent way, consider $Y_t = F(X_t)$ for some sufficiently smooth $F : V \to \mathcal{L}(V, W)$. Similarly to (1.8), one have

$$\int_0^1 F(X_s) d\mathbf{X}_t = \lim_{|\mathcal{P}| \to 0} \sum_{[s,t] \in \mathcal{P}} F(X_s) X_{ts} + DF(X_s) \mathbb{X}_{ts}, \tag{1.10}$$

In the following, it is necessary to discuss when does the limit on the right hand side of (1.10) exist. Following [6], Introducing the space $C_2^{\alpha,\beta}([0,T], W)$ of functions Ξ from the simplex $0 \le s \le t \le T$ into W such that $\Xi_{tt} = 0$ and such that

$$\|\Xi\|_{\alpha,\beta} := \|\Xi\|_\alpha + \|\delta\Xi\|_\beta < \infty. \tag{1.11}$$

where $\|\Xi\|_\alpha = \sup_{s<t} \frac{|\Xi|}{|t-s|^\alpha}$ as usual, and also

$$\delta\Xi_{tus} = \Xi_{ts} - \Xi_{us} - \Xi_{tu}, \qquad \|\delta\Xi\|_\beta = \sup_{s<u<t} \frac{|\delta\Xi_{tus}|}{|t-s|^\beta}. \tag{1.12}$$

Provided that $\beta > 1$, it turns out that such functions are almost of the form $\Xi_{ts} = F_t - F_s$, for some α-Hölder continuous functions F (they would be if and only if $\delta\Xi = 0$). Indeed, it is possible to construct in a canonical way a function $\hat{\Xi}$ with $\delta\hat{\Xi} = 0$ and such that $\hat{\Xi}_{ts} \approx \Xi_{ts}$ for $|t-s| \ll 1$. It is now the time to introduce Sewing lemma.

Lemma 1.1.3. ***(Sewing lemma)*** *Let α and β be such that $0 < \alpha \leq 1 < \beta$. Then there exists a (unique) continuous map $\mathcal{I} : C_2^{\alpha,\beta}([0,T], W) \to C^\alpha([0,T], W)$ such that $(\mathcal{I}\Xi)_0 = 0$ and*

$$|(\mathcal{I}\Xi)_{ts} - \Xi_{ts}| \leq C\, |t-s|^\beta\,, \tag{1.13}$$

where C only depends on β and $\|\delta\Xi\|_\beta$. (The α-Hölder norm of $\mathcal{I}\Xi$ also depends on $\|\Xi\|_\alpha$ and hence on $\|\Xi\|_{\alpha,\beta}$.)

Applying the Sewing lemma 1.1.3 to the construction of (1.10), one can get

Theorem 1.1.4. ***(Lyons)****. Let $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^\alpha([0,T], V)$ for some $T > 0$ and $\alpha > \frac{1}{3}$, and let $F : V \to \mathcal{L}(V, W)$ be a C_b^2 function. Then, the rough integral defined in (1.10) exists and one has the bound*

$$\left|\int_0^t F(X_r)d\mathbf{X}_r - F(X_s)X_{ts} - DF(X_s)\mathbb{X}_{ts}\right| \\ \lesssim \|F\|_{C_b^2}\left(\|X\|_\alpha^3 + \|X\|_\alpha\|\mathbb{X}\|_{2\alpha}\right)|t-s|^{3\alpha}, \tag{1.14}$$

where the proportionality constant depends only on α. Furthermore, the indefinite rough integral is α-Hölder continuous on $[0,T]$ and we have the following quantitative estimate,

$$\left\|\int_0^\cdot F(X)d\mathbf{X}\right\|_\alpha \leq C\|F\|_{C_b^2}\left[\left(\|X\|_\alpha + \sqrt{\|\mathbb{X}\|_{2\alpha}}\right) \vee \left(\|X\|_\alpha + \sqrt{\|\mathbb{X}\|_{2\alpha}}\right)^{1/\alpha}\right], \tag{1.15}$$

where the constant C only depends on T and α and can be chosen uniformly in $T \leq 1$.

Another important notion that is useful in our Paper [7] is controlled path Y, relative to some reference path X. For the sake of the following definition, assuming that Y takes values in some Banach space, say $\bar{W}$. When it comes to the definition of a rough integral we typically take $\bar{W} = \mathcal{L}(V, W)$; although other choices can be useful.

Definition 1.1.5. *Given a path $X \in C^\alpha([0,T], V)$, we say that $Y \in C^\alpha([0,T], \bar{W})$ is controlled by X if there exists $Y' \in C^\alpha([0,T], \mathcal{L}(V, \bar{W}))$ so that the remainder term R^Y given implicitly through the relation*

$$Y_{ts} = Y'_s X_{ts} + R^Y_{ts}, \tag{1.16}$$

satisfies $\|R^Y\|_{2\alpha} < \infty$*. This defines the space of controlled rough paths,*

$$(Y, Y') \in \mathcal{D}_X^{2\alpha}([0,T], \bar{W}).$$

Although Y' *is not, in general, uniquely determined from* Y *(Gubinelli derivative of* Y*).*

With these notions at hand, it is now straight-forward to prove the following result, which is a slight reformulation of [6, Prop1]:

Theorem 1.1.6. ***(Gubinelli)****. Let* $T > 0$*, let* $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{L}^\alpha([0,T], V)$ *for some* $\alpha > \frac{1}{3}$*, and let* $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0,T], \mathcal{L}(V,W))$*. Then there exists a constant* C *depending only on* T *and* α *(and* C *can be chosen uniformly over* $T \in (0,1]$*) such that*
a) The integral defined in (1.10) exists and, for every pair s, t*, one has the bound*

$$\left| \int_s^t Y_r d\mathbf{X}_r - Y_s X_{ts} - Y_s' \mathbb{X}_{ts} \right| \leq C \left(\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha} \|Y'\|_\alpha \right) |t-s|^{3\alpha}. \tag{1.17}$$

b)The map from $\mathcal{D}_X^{2\alpha}([0,T], \mathcal{L}(V,W))$ *to* $\mathcal{D}_X^{2\alpha}([0,T], W)$ *given by*

$$(Y, Y') \mapsto (\int_0^\cdot Y_t d\mathbf{X}_t, Y), \tag{1.18}$$

is a continuous linear linear map between Banach spaces and one has the bound

$$\| \int_0^\cdot Y_t d\mathbf{X}_t, Y \|_{X,2\alpha} \leq \|Y\|_\alpha + \|Y'\|_{L^\infty} \|\mathbb{X}\|_{2\alpha} + C \left(\|X\|_\alpha \|R^Y\|_{2\alpha} + \|\mathbb{X}\|_{2\alpha} \|Y'\|_\alpha \right). \tag{1.19}$$

And for the integral of composition with regular functions. Let W and $\tilde{W}$ be two Banach spaces and let $\varphi : W \to \tilde{W}$ be a function in C_b^2. Let furthermore $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0,T], W)$ for some $X \in C^\alpha$. Then one can define a (candidate) controlled rough path $(\varphi(Y), \varphi(Y)') \in \mathcal{D}_X^{2\alpha}([0,T], \tilde{W})$ by

$$\varphi(Y)_t = \varphi(Y_t), \qquad \varphi(Y)_t' = D\varphi(Y_t) Y_t'. \tag{1.20}$$

It is straightforward to check that the corresponding remainder term does indeed satisfy the required bound. It is also straightforward to check that, as a consequence of the chain rule, this definition is consistent in the sense that $(\varphi \circ \psi)(Y, Y') = \varphi(\psi(Y, Y'))$. And $\|\varphi(Y), \varphi(Y)'\|_{X,2\alpha}$ satisfies

Lemma 1.1.7. *Let $\varphi \in C_b^2$, $(Y, Y') \in \mathcal{D}_X^{2\alpha}([0, T], W)$ for some $X \in C^\alpha$ with $|Y_0'| + \|Y, Y'\|_{X,2\alpha} \leq M \in [0, \infty)$. Let $(\varphi(Y), \varphi(Y)') \in \mathcal{D}_X^{2\alpha}([0, T], \tilde{W})$ be given by (1.20). Then, there exists a constant C depending only on $T > 0$ and $\alpha > \frac{1}{3}$ such that one has the bound*

$$\|\varphi(Y), \varphi(Y)'\|_{X,2\alpha} \leq C_{\alpha,T} M \|\varphi\|_{C_b^2} \left(1 + \|X\|_\alpha\right)^2 \left(|Y_0'| + \|Y, Y'\|_{X,2\alpha}\right).$$

At last, C can be chosen uniformly over $T \in (0, 1]$.

1.1.3 Solutions to rough differential equations

The aim of this section is to show that if f is regular enough and $(X, \mathbb{X}) \in \mathcal{C}^\beta$ with $\beta > \frac{1}{3}$. Consider a differential equations driven by the rough path $\mathbf{X} = (X, \mathbb{X})$ of the type

$$dY = f(Y) d\mathbf{X}.$$

Such an equation will yield solutions in $\mathcal{D}_X^{2\alpha}$ and will be interpreted in the corresponding integral formulation, where the integral of $f(Y)$ against X is defined using (1.10). More precisely, one has the following local existence and uniqueness result, which is from [8, Prop 8.3]

Theorem 1.1.8. *Given $\zeta \in W$, $f \in C^3(W, \mathcal{L}(V, W))$ and $\mathbf{X} = (X, \mathbb{X}) \in \mathcal{C}^\beta(\mathbb{R}_+, V)$ with $\beta \in (\frac{1}{3}, \frac{1}{2})$, there exists a unique element $(Y, Y') \in \mathcal{D}_X^{2\beta}([0, 1], W)$ such that*

$$Y_t = \zeta + \int_0^t f(Y_s) d\mathbf{X}_s, \qquad t < \tau, \tag{1.21}$$

for some $\tau > 0$. Here, the integral is interpreted in the sense of Theorem 1.1.6 and $f(Y) \in \mathcal{D}_X^{2\beta}$ is built from Y by Lemma 1.1.7. Furthermore, one has $Y' = f(Y)$ and, if $f \in C_b^3$, solutions are global in time.

1.2 Introduction to Malliavin Calculus

Malliavin calculus is one of the main tools of modern stochastic analysis. In a nutshell, this is theory providing a way of differentiating random variables defined on a Gaussian probability space (typically Wiener space) with respect to the underlying noise. This allows to develop an analysis on Wiener space, and infinite-dimensional generalisation of the usual analytical concepts we are familiar with on $\mathbb{R}^n$. In this introduction, we will recall some basic notions in Malliavin Calculus (mostly taken from [9]).

1.2.1 The Wiener Chaos Decomposition

The general setting for Malliavin calculus is a Gaussian probability space, i.e. a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ along with a closed subspace $\mathcal{H}$ of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ consisting of centered Gaussian random variables. It is often convenient to assume that $\mathcal{H}$ is isometric to another Hilbert space H, typically an L^2-space over a parameter set T. Recalling that a real-valued random variable X, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called Gaussian if its characteristic function $\varphi_X = \mathrm{e}^{\frac{q}{2}t^2}$ for some $q \geq 0$. And a family $(X_{\mathrm{i}})_{\mathrm{i} \in I}$ of real-valued random variables is called Gaussian family or jointly Gaussian, if for any $n \in \mathbb{N}$ and any choice $\mathrm{i}_1, \cdots, \mathrm{i}_n$ of distinct indices in I, the vector $(X_{\mathrm{i}_1}, \cdots, X_{\mathrm{i}_n})$ is a Gaussian vector. The definition of isonormal Gaussian processes is as follows.

Definition 1.2.1. *We say that a stochastic process $W = \{W(h), h \in H\}$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$ is an isonormal Gaussian process (or a Gaussian process on H) if W is a centered Gaussian family of random variables such that $E[W(h)W(g)] = \langle h , g \rangle_H$ for all $h, g \in H$.*

Remark 1.2.2. For any real, separable Hilbert space H we have that:

(1) there exists an H-isonormal Gaussian process W.

(2) The map $h \mapsto W(h)$ is an isometry, in particular, it is linear.

(3) W is a Gaussian family.

The range of the isonormal process W is the subspace $\mathcal{H}$ that was mentioned above. In order to get Wiener chaos decomposition, it is worthy studing the structure of the range of W.

For $n \in \mathbb{N}_0$, the n_{th} Hermite polynomial H_n is defined by $H_0 = 1$ and

$$H_n(x) = \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}), \tag{2.1}$$

for $n \geq 1$. It is easy to check that the Hermite polynomials have following basic properties: (1) $H_n'(x) = H_{n-1}(x)$; (2) $(n+1)H_{n+1}(x) = xH_n(x) - H_{n-1}(x)$; (3) $H_n(-x) = (-1)^n H_n(x)$. Inserting Gaussian random variables into Hermite polynomial, we get the following Lemma.

Lemma 1.2.3. *Let X, Y be standard Gaussian random variables which are disjointly Gaussian. Then for $n, m \geq 0$, we have*

$$E(H_n(X)H_m(Y)) = \begin{cases} 0, & \text{if} \quad n \neq m \\ \frac{1}{n!}(E(XY))^n, & \text{if} \quad n = m \end{cases}$$

With Lemma 1.2.3 at hand, Wiener chaos is defined by

Definition 1.2.4. *For each $n \geq 0$, we write $\mathcal{H}_n$ to denote the closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ generated by the random variables of the type $H_n(W(h)), h \in H$, $\|h\| = 1$. These space $\mathcal{H}_n$ is called the nth Wiener chaos of W.*

Note that $H_0 = 1$, the 0-th Wiener chaos $\mathcal{H}_0$ is the set of all constant functions, whereas $\mathcal{H}_1 = \{W(h) : h \in H\}$, since $H_1(x) = x$ and W is linear. The next result shows that $\oplus_{n=0}^{\infty} \mathcal{H}_n$ coincides with $L^2(\Omega, \mathcal{F}, \mathbb{P})$: this result is known as the Wiener-Itô chaotic decomposition of $L^2(\Omega, ,\mathbb{P})$.

Theorem 1.2.5. *The space $L^2(\Omega, \mathcal{G}, P)$ can be decomposed into the infinite orthogonal sum of the subspaces $\mathcal{H}_n$:*

$$L^2(\Omega, \mathcal{G}, P) = \oplus_{n=0}^{\infty} \mathcal{H}_n.$$

1.2.2 The derivative operator

Let S denote the set of all random variables of the form

$$f(W(h_1),\cdots,W(h_n)), \tag{2.2}$$

where $n \geq 1$, $f:\mathbb{R}^n \to \mathbb{R}$ is a C^∞-function such that f and its partial derivative have at most polynomial growth, and $h_i \in H, i = 1,\dots,n$. A random variable belonging to S is said to be smooth. The definition of Malliavin derivative is given by

Definition 1.2.6. *The derivative of a smooth random variable F of the form (2.2), is the H-valued random variable given by*

$$DF = \sum_{i=1}^{n} \partial_i f(W(h_1),\cdots,W(h_n))\, h_i. \tag{2.3}$$

Thanks to the Definition 1.2.6, Malliavin derivative D satisfies the following properties.

Remark 1.2.7. 1. For $h \in H$ and for F as in (2.2), observe that, almost surely,

$$\langle DF, h\rangle_H = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[f(W(h_1)+\epsilon\langle h_1,h\rangle_H+,\cdots,W(h_n)+\epsilon\langle h_n,h\rangle_H)-F].$$

This shows that DF may be seen as a directional derivative.

2. The operator D is closable from $L^P(\Omega)$ to $L^P(\Omega;H)$.

Also, for a smooth random variable F, the iteration of the operator D, say D^kF, is a random variable with value in $H^{\otimes k}$. Then for a fix $p \geq 1$ and $k \geq 0$, let $\mathbb{D}^{k,p}$ denote the closure of S with respect to the norm $\|\cdot\|_{k,p}$

$$\|F\|_{k,p} = \left[E(|F|^p) + \sum_{j=1}^{k} E(\|D^jF\|^p_{H^{\otimes j}})\right]^{\frac{1}{p}}. \tag{2.4}$$

Note that this family of spaces verifies properties: (i) $\mathbb{D}^{k+1,p} \subset \mathbb{D}^{k,p}$ if $k \geq 0$ and $p > q$. (ii) $\mathbb{D}^{0,p} = L^p(\Omega)$.

1.2.3 The divergence operator

This section is devoted to recall the divergence operator, defined as the adjoint of the derivative operator. Thanks to the property of the derivative operator D in Remark 1.2.7, divergence operator is defined by

Definition 1.2.8. *We denote by δ the adjoint of the operator D. That is, δ is an unbounded operator on $L^2(\Omega; H)$ with values in $L^2(\Omega)$ such that:*
(i) The domain of δ, denoted by $Dom\,\delta$, is the set of H-valued square integrable random variables $u \in L^2(\Omega; H)$ such that

$$|E(\langle DF, u\rangle_H)| \leq c\|F\|_2. \tag{2.5}$$

for all $F \in \mathbb{D}^{1,2}$, where c is some constant depending on u.

(ii) If u belongs to $Dom\,\delta$, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by

$$E(F\delta(u)) = E(\langle DF, u\rangle_H), \tag{2.6}$$

for any $F \in \mathbb{D}^{1,2}$.

The operator δ is called the divergence operator and is closed as the adjoint of an unbounded and densely defined operator. Divergence operator enjoys following property.

Proposition 1.2.9. *The space $\mathbb{D}^{1,2}(H)$ is included in the domain of δ. If $u, v \in \mathbb{D}^{1,2}(H)$, then*

$$E(\delta(u)\delta(v))) = E(\langle u, v\rangle_H) + E(Tr(Du \circ Dv)) \tag{2.7}$$

As a consequence of relation (2.7), we obtain the estimate

$$E(\delta(u)^2) \leq E(\|u\|_H^2) + E(\|Du\|_{H\otimes H}^2) = \|u\|_{1,2}^2. \tag{2.8}$$

This implies that the space $\mathbb{D}^{1,2}(H)$ is included in the domain of δ. In fact, if $u \in \mathbb{D}^{1,2}(H)$, there exists a sequence $u^n \in S_n$ converges to u in $L^2(\Omega)$ and Du^n converges to Du in

$L^2(\Omega; H \otimes H)$. Therefore, $\delta(u^n)$ converges in $L^2(\Omega)$ and its limit is $\delta(u)$. Moreover, (2.7) holds for any $u, v \in \mathbb{D}^{1,2}(H)$. With Section 1.2.3 at hand, it is now the time to review the definition of Skorohod and Stratonovich integral, and its relation.

1.2.4 The Skorohod integral and Stratonovich integral

Consider the separable Hilbert space $H = L^2(T, \mathcal{B}, \mu)$, where μ is a σ-finite atomless measure on a measurable space $(T, \mathcal{B})$. In this case the elements of $Dom\, \delta \subset L^2(T \times \Omega)$ are square integrable processes, and the divergence $\delta(u)$ is called the Skorohod stochastic integral of the process u. Say:

$$\delta(u) = \int_T u_t \, dW_t. \tag{2.9}$$

The space $\mathbb{D}^{1,2}(L^2(T))$, denoted by $\mathbb{L}^{1,2}$, coincides with the class of processes $u \in L^2(T \times \Omega)$ such that $u(t) \in \mathbb{D}^{1,2}$ for almost all t, and there exists a measurable version of the two-parameter process $D_s u_t$ verifying $E[\int_T \int_T (D_s u_t)\mu(ds)\mu(dt)] < \infty$. If u and v are two processes in the space $\mathbb{L}^{1,2}$, then Equation (2.7) can be written as

$$E(\delta(u)\delta(v)) = E(u_t v_t)\mu(d_t) + E(D_s u_t D_t v_s)\mu(ds)\mu(dt). \tag{2.10}$$

Owing to Definition 1.2.3 of divergence operator in hand, one may find that Skorohod integral is an extension of the Itô stochastic integral.

Proposition 1.2.10. *Let $W = \{W_t, t \in [0,1]\}$ be a one dimensional Brownian motion and consider an adapted process u such that $\int_0^1 u_t^2 dt < \infty$ a. s. Then $\delta(u)$ coincides with the Itô stochastic integral $\int_0^1 u_t dW_t$.*

Next, to introduce the definition of Stratonovich integral. Let π be an arbitrary partition of the interval $[0,1]$ of the form $\pi = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ and a family of random variables S^π by

$$S^\pi = \int_0^1 u_t W_t^\pi dt, \tag{2.11}$$

where

$$W_t^\pi = \sum_{i=0}^{n-1} \frac{W(t_{i+1}) - W(t_i)}{t_{i+1} - t_i} \mathbb{1}_{(t_i, t_{i+1}]}(t). \tag{2.12}$$

Definition 1.2.11. *We say that a measurable process $u = \{u_t, 0 \le t \le 1\}$ such that $\int_0^1 |u_t| dt < \infty$ a.s. is Stratonovich integral if the family S^π converges in probability as $|\pi| \to 0$, and in this case the limit will be denoted by $\int_0^1 u_t \circ dW_t$.*

In addition, assume that $u \in \mathbb{D}^{1,2}(H)$ and the derivative $D_s u_t$ exists and satisfies almost surely

$$\int_0^T \int_0^T |D_s u_t| |t-s|^{2H-2} ds\, dt < \infty, \qquad \text{and} \qquad \mathbb{E}\left[\|Du\|^2_{H^{\otimes l}}\right] < \infty.$$

Then the Stratonovich integral $\int_0^T u_t \circ dW_t$ exists, and Skorohod and Stratonovich stochastic integrals have following relation:

$$\int_0^T u_t \circ dW_t = \int_0^T u_t \, dW_t + a_H \int_0^T \int_0^T D_s\, u_t |t-s|^{2H-2} ds\, dt. \tag{2.13}$$

Furthermore, recalling that Meyer's inequality for the Skorohod integral: given $p > 1$ and an integer $k \ge 1$, there is a constant $c_{k,p}$ such that the k-th iterated Skorohod integral satisfies

$$\|(\delta^k(u)\|_p \le c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathcal{H}^{\otimes k})} \qquad \text{for all} \quad u \in \mathbb{D}^{k,p}(H^{\otimes k}). \tag{2.14}$$

1.2.5 Multiple Wiener-Itô integrals

Thanks to Section 1.2.1 and 1.2.3, the elements of the nth Wiener chaos $\mathcal{H}_n$ can be expressed as multiple stochastic integrals with respect to W.

Definition 1.2.12. *Let $m \ge 1$ and $f \in H^{\otimes m}$. The mth multiple integral of f is defined by $I_m(f) = \delta^m(f)$.*

It is easy to check that mth multiple integral I_m satisfies the following properties:

(i) I_m is linear,

(ii) $I_m(f) = I_m(\tilde{f})$, where $\tilde{f}$ denotes the symmetrization of f.

(iii)

$$E(I_m(f)I_q(g)) = \begin{cases} 0, & \text{if} \quad m \neq q \\ m!\langle \tilde{f}, \tilde{g} \rangle_{L^2(T^m)}, & \text{if} \quad m = q \end{cases}$$

Recalling the definition of the Hermite polynomials $\{H_m : m \geq 0\}$ given in (2.1) and the definition of δ in (2.9), one may get

Theorem 1.2.13. *Let f be such that $\|f\|_H = 1$. Then for any integer $m \geq 1$, we have*

$$H_m(W(f)) = I_m(f^{\otimes m}).$$

As a consequence, the liner operator I_m provides an isometry from H^p onto the mth Wiener chaos $\mathcal{H}_m$ of W.

The next statement provides a useful reformulation of the Wiener-Itô decomposition:

Theorem 1.2.14. *Every $F \in L^2(\Omega, \mathcal{F}, P)$ (recall that $\mathcal{F}$ denotes the σ-field generated by W) can be expanded as:*

$$F = \sum_{m=0}^{\infty} I_m(f_m).$$

Here $f_0 = E(F)$, and I_0 is the identity mapping on the constants. Furthermore, we can assume that the functions $f_n \in L^2(T^n)$ are symmetric and, in this case, uniquely determined by F.

This section is closed by a fundamental hypercontractivity property for multiple integrals, which shows that inside a fixed chaos, all the $L^m(\Omega)$-norm are equivalent.

Theorem 1.2.15. *For every $q > 0$ and every $p \geq 1$, there exists a constant $0 < k(q,p) < \infty$ such that*

$$E[|Y|^q]^{1/q} \leq k(q,p)E[Y^2]^{1/2},$$

for every random variable Y with the form of a pth multiple integral.

1.3 Summary of Volterra equations driven by rough signals

This section is a summary of the paper [1], which is devoted to give a meaning of the theory of rough paths in a Volterra setting with singular kernels. The main idea in order to achieve this goal is to extend the concept of a path $t \mapsto z_t$ to a two variable object $(t, \tau) \mapsto z_t^\tau$ for $(t, \tau) \in \Delta_2$, where Δ_2 is a simplex of two variables. This extension of the notion of path is motivated from the generic form of a Volterra integral

$$z_t^\tau = \int_0^t k(\tau, r) dx_r, \tag{3.1}$$

for some (possibly singular) kernel k and a Hölder continuous function x. And then giving a proper definition of the convolution product $*$, and arguing that the solution to a V-valued Volterra equation

$$y_t = \xi + \int_0^t k(t, r)\sigma(y_r) dx_r, \qquad \xi \in V. \tag{3.2}$$

That is, there are two goals in this paper:

(1) The path-wise construction of the Volterra paths in (3.1) as well as the algebraic and analytical properties of the associated Volterra-signature (as generalized from the concept of signatures in the theory of rough paths),

(2) Construction of solutions to (3.2).

In order to achieve these two goals, introducing the definition of Volterra rough paths is the first step.

1.3.1 Volterra Rough Paths

To define the Volterra rough paths, There are some notations that are used in this introduction (also in paper [1]) .

Notation 1.3.1. *Let $C > 0$ be a constant, the relation $a \leq Cb$ is defined by*

$$a \lesssim b$$

.

Another notation is about simplex Δ_n^T.

Notation 1.3.2. *Let $T > 0$ be a time horizon, and $n \geq 2$. Then the simplex Δ_n^T is defined by*

$$\Delta_n^T \Big\{ (s_1, \ldots, s_n) \in [0,T]^n; 0 \leq s_1 < \cdots < s_n \leq T \Big\}.$$

When this causes no ambiguity, Δ_n represents Δ_n^T.

Furthermore, operator δ is well known in the theory rough paths, and given by

Notation 1.3.3. *Let g be a path from Δ_2 to $\mathbb{R}^m$, and consider $(s,u,t) \in \Delta_3$. Then the quantity $\delta_u g_{ts}$ is defined by*

$$\delta_u g_{ts} = g_{ts} - g_{tu} - g_{us}. \tag{3.3}$$

With these notations at hand, it is ready to construct the Volterra signature over a smooth path. In this way the Volterra type integrals will be trivially defined and their algebraic and analytic properties can be checked. This constructions will rely on specific assumptions about the power type singularity of the kernel k appearing in (3.2). The main hypothesis are summarized as follows.

H Let k be a kernel $k : \Delta_2 \to \mathbb{R}$. Assume that there exists $\gamma \in (0,1)$ such that for all $(s,r,q,\tau) \in \Delta_4([0,T])$ and $\eta, \beta \in [0,1]$ we have

$$|k(\tau,r)| \lesssim |\tau - r|^{-\gamma} \tag{3.4}$$

$$|k(\tau,r) - k(q,r)| \lesssim |q-r|^{-\gamma-\eta} |\tau - q|^{\eta} \tag{3.5}$$

$$|k(\tau,r) - k(\tau,s)| \lesssim |\tau - r|^{-\gamma-\eta} |r-s|^{\eta} \tag{3.6}$$

$$|k(\tau,r) - k(q,r) - k(\tau,s) + k(q,s)| \lesssim |q-r|^{-\gamma-\beta} |r-s|^{\beta} \tag{3.7}$$

$$|k(\tau,r) - k(q,r) - k(\tau,s) + k(q,s)| \lesssim |q-r|^{-\gamma-\eta} |\tau - q|^{\eta}. \tag{3.8}$$

In the sequel a kernel fulfilling condition **(H)** will be called Volterra kernel of order $-\gamma$.

Thanks to these assumptions, it is now to state the definition of iterated Volterra integral.

Definition 1.3.4. *Consider a path* $x \in C^1([0,T];E)$ *and a Volterra kernel* $k : \Delta_2 \to \mathbb{R}$ *satisfying* (**H**). *The* iterated Volterra integral *of order* n *is a mapping* $\mathbf{z}^n : \Delta_3 \to E^{\otimes n}$ *given by*

$$(s,t,\tau) \mapsto \mathbf{z}_{ts}^{n,\tau} = \int_{t>r_n>\cdots>r_1>s} k(\tau, r_n) \bigotimes_{\mathrm{j}=1}^{n-1} k\left(r_{\mathrm{j}+1}, r_{\mathrm{j}}\right) dx_{r_{\mathrm{j}}}. \tag{3.9}$$

.

For $(s,u,t,\tau) \in \Delta_4$, it is not hard to check that $\mathbf{z}_{ts}^{n,\tau}$ as given in (3.9) satisfies

$$|\mathbf{z}_{ts}^{n,\tau}| \leq \frac{(\|x\|_1 \Gamma(1-\gamma))^n}{\Gamma(n(1-\gamma))} (\tau - s)^{-\gamma} (t-s)^{(n-1)(1-\gamma)+1},$$

Here Γ is the Gamma function.

Note that the Volterra signature does not have a multiplicative property ($\mathbf{X}_{tu} \otimes \mathbf{X}_{us} = \mathbf{X}_{ts}$) similar to the classical signature. So it is necessary to introduce an integral product behaving like a convolution extending the classical tensor product. It is not hard to check that this convolution product is well defined for a large class of Volterra paths, and provide an analogue of the extension theorem from the theory of rough paths (which guarantees in particular the existence of a Volterra signature). For all $0 \leq \mathrm{i} \leq n$, defining the convolution product $*$ as

$$\begin{aligned} &\mathbf{z}_{tu}^{n-\mathrm{i},\tau} * \mathbf{z}_{us}^{\mathrm{i},\cdot} \\ &:= \int_{t>r_n>\cdots>r_{\mathrm{i}+1}>u} \bigotimes_{\mathrm{j}=n}^{\mathrm{i}+1} k\left(r_{\mathrm{j}+1}, r_{\mathrm{j}}\right) dy_{r_{\mathrm{j}}} \otimes \int_{u>r_{\mathrm{i}}>\cdots>r_1>s} k\left(r_{\mathrm{i}+1}, r_{\mathrm{i}}\right) \bigotimes_{\mathrm{j}=\mathrm{i}-1}^{1} k\left(r_{\mathrm{j}+1}, r_{\mathrm{j}}\right) dx_{r_{\mathrm{j}}}, \end{aligned} \tag{3.10}$$

then $\mathbf{z}_{ts}^{n,\tau}$ as given in (3.9) satisfies

$$\mathbf{z}_{ts}^{n,\tau} = \sum_{\mathrm{i}=0}^{n} \mathbf{z}_{tu}^{n-\mathrm{i},\tau} * \mathbf{z}_{us}^{\mathrm{i},\cdot}, \tag{3.11}$$

Here the convention $\mathbf{z}^0 \equiv 1$ and $\mathbf{z}^n * 1 = 1 * \mathbf{z}^n = \mathbf{z}^n$.

It is the time to construct the Voterra rough paths. This means to generalize the processes of the form

$$z_{ts}^{\tau} = \int_s^t k(\tau, r) dx_r, \tag{3.12}$$

where x is an α-Hölder path and k a possibly singular kernel of order $-\gamma$. Consider the Volterra spaces as follows.

Definition 1.3.5. *Let E be a Banach space, and consider $(\alpha, \gamma) \in (0,1)^2$ with $\alpha - \gamma > 0$. Define the space of Volterra paths of index (α, γ), denoted by $\mathcal{V}^{(\alpha,\gamma)}(\Delta_2; \mathbb{R}^d)$, as the set of functions $z : \Delta_2 \to E$, given by $(t, \tau) \mapsto z_t^\tau$, with the condition $z_0^\tau = z_0 \in E$ for all $\tau \in (0, T]$, and satisfying*

$$\|z\|_{(\alpha,\gamma)} = \|z\|_{(\alpha,\gamma),1} + \|z\|_{(\alpha,\gamma),1,2} < \infty. \tag{3.13}$$

In (3.13), the 1-norm and (1,2)-norm are respectively defined as follows:

$$\|z\|_{(\alpha,\gamma),1} := \sup_{(s,t,\tau)\in\Delta_3} \frac{|z_{ts}^\tau|_E}{[|\tau - t|^{-\gamma}|t-s|^\alpha] \wedge |\tau - s|^{\alpha-\gamma}}, \tag{3.14}$$

$$\|z\|_{(\alpha,\gamma),1,2} := \sup_{\substack{(s,t,\tau',\tau)\in\Delta_4 \\ \eta\in[0,1],\zeta\in[0,\alpha-\gamma)}} \frac{|z_{ts}^{\tau\tau'}|_E}{|\tau-\tau'|^\eta|\tau'-t|^{-\eta+\zeta}\left([|\tau'-t|^{-\gamma-\zeta}|t-s|^\alpha] \wedge |\tau'-s|^{\alpha-\gamma-\zeta}\right)}, \tag{3.15}$$

with the convention $z_{ts}^\tau = z_t^\tau - z_s^\tau$ and $z_s^{\tau\tau'} = z_s^\tau - z_s^{\tau'}$. In addition, under the mapping

$$z \mapsto |z_0| + \|z\|_{(\alpha,\gamma)},$$

the space $\mathcal{V}^{(\alpha,\gamma)}(\Delta_2; E)$ is a Banach space.

The second goal is to construct a solution to rough Volterra equations like (3.2). Similar to classical rough path, Volterra version of the Sewing Lemma is necessary. The Volterra spaces $\mathcal{V}^{(\alpha,\gamma)}$ plays a similar role in Volterra Sewing Lemma as $\mathcal{C}^\alpha(V)$ in classical Sewing Lemma.

Definition 1.3.6. *Let $\alpha \in (0,1)$, $\gamma \in (0,1)$ with $\alpha - \gamma > 0$, $\kappa \in (0, \infty)$ and $\beta \in (1, \infty)$. Denote by $\mathcal{V}^{(\alpha,\gamma)(\beta,\kappa)}\left(\Delta_3\left[0;T\right];E\right)$, the space of all functions $\Xi : \Delta_3\left([0,T]\right) \to E$ such that*

$$\|\Xi\|_{\mathcal{V}^{(\alpha,\gamma)(\beta,\kappa)}} = \|\Xi\|_{(\alpha,\gamma)} + \|\delta\Xi\|_{(\beta,\kappa)} < \infty, \tag{3.16}$$

where δ is the operator defined by (3.3) *In (3.16), we also use the following convention: the norm $\|\varXi\|_{(\alpha,\gamma)}$ is given by* (3.16)*, while we have*

$$\|\delta\varXi\|_{(\alpha,\gamma)} = \|\delta\varXi\|_{(\alpha,\gamma),1} + \|\delta\varXi\|_{(\alpha,\gamma),1,2},$$

where the quantities $\|\delta\varXi\|_{(\beta,\gamma),1}$ and $\|\delta\varXi\|_{(\beta,\gamma),1,2}$ are slight modifications of (1.3.5) respectively defined by

$$\|\delta\varXi\|_{(\beta,\kappa),1} := \sup_{(s,m,t,\tau)\in\Delta_4} \frac{|\delta_m \varXi^\tau_{ts}|}{|\tau-t|^{-\kappa}|t-s|^\beta \wedge |\tau-s|^{\beta-\kappa}} \tag{3.17}$$

$$\|\delta\varXi\|_{(\beta,\kappa),1,2} := \sup_{\substack{(s,m,t,\tau',\tau)\in\Delta_5\\ \eta\in[0,1]}} \frac{|\delta_m \varXi^{\tau,\tau'}_{ts}|}{|\tau-\tau'|^\eta|\tau'-t|^{-\eta}\left(|\tau'-t|^{-\kappa}|t-s|^\beta \wedge |\tau'-s|^{\beta-\kappa}\right)}. \tag{3.18}$$

In the sequel the space $\mathcal{V}^{(\alpha,\gamma)(\beta,\kappa)}$ will be our space of abstract Volterra integrands.

It is now ready to state Sewing Lemma adapted to Volterra integrands.

Lemma 1.3.7. (Volterra sewing lemma) *Consider four exponents $\beta \in (1,\infty)$, $\kappa \in (0,1)$, $\alpha \in (0,1)$ and $\gamma \in (0,1)$ such that $\beta-\kappa \geq \alpha-\gamma > 0$. Let $\mathcal{V}^{(\alpha,\gamma)(\beta,\kappa)}$ and $\mathcal{V}^{(\alpha,\gamma)}$ be the spaces defined in Definition 1.3.6 and 1.3.5 respectively. Then there exists a linear continuous map $\mathcal{I} : \mathcal{V}^{(\alpha,\gamma)(\beta,\kappa)}(\Delta_3; E) \to \mathcal{V}^{(\alpha,\gamma)}(\Delta_3; E)$ such that the following holds true*

(i) *The quantity $\mathcal{I}(\varXi^\tau)_{ts} := \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} \varXi^\tau_{vu}$ exists for all $(s,t,\tau)\in\Delta_3$, where $\mathcal{P}$ is a generic partition of $[s,t]$ and $|\mathcal{P}|$ denotes the mesh size of the partition.*

(ii) *For all $(s,t,\tau)\in\Delta_3$ we have that*

$$|\mathcal{I}(\varXi^\tau)_{ts} - \varXi^\tau_{ts}| \lesssim \|\delta\varXi\|_{(\beta,\kappa),1}\left(|\tau-t|^{-\kappa}|t-s|^\beta \wedge |\tau-s|^{\beta-\kappa}\right), \tag{3.19}$$

while for $(s,t,\tau',\tau)\in\Delta_4$ we get

$$\begin{aligned}&|\mathcal{I}(\varXi^{\tau\tau'})_{ts} - \varXi^{\tau\tau'}_{ts}| \\ &\qquad\lesssim \|\delta\varXi\|_{(\beta,\kappa),1,2}\left[|\tau-\tau'|^\eta|\tau'-t|^{-\eta}\left(|\tau'-t|^{-\kappa}|t-s|^\beta \wedge |\tau'-s|^{\beta-\kappa}\right)\right].\end{aligned} \tag{3.20}$$

1.3.2 Convolution product in rough case: $\alpha - \gamma > \frac{1}{3}$

This section is devoted to define the first level and second level convolution product, which is equivalent to Chen's relation in Volterra context involves convolution type integrals. The first level convolution product is defined as follows:

Theorem 1.3.8. *Consider two Volterra paths $z \in \mathcal{V}^{(\alpha,\gamma)}$ and $y \in \mathcal{V}^{(\alpha',\gamma')}$ as given in Definition 2.2.4, where we recall that $\alpha, \gamma, \alpha', \gamma' \in (0,1)$, and define $\rho \equiv \alpha - \gamma > 0$ and $\rho' \equiv \alpha' - \gamma' > 0$. Then the convolution product is a bilinear operation on $\mathcal{V}^{(\alpha,\gamma)}$ given by*

$$z_{tu}^{\tau} * y_{us}^{\cdot} = \int_{t>r>u} dz_r^{\tau} \otimes y_{us}^{r} := \lim_{|\mathcal{P}|\to 0} \sum_{[u',v']\in\mathcal{P}} z_{v'u'}^{\tau} \otimes y_{us}^{u'}. \tag{3.21}$$

The integral is understood as a Volterra-Young integral for all $(s,u,t,\tau) \in \Delta_4$. Moreover, the following inequality holds true,

$$|z_{tu}^{\tau} * y_{us}^{\cdot}| \lesssim \|z\|_{(\alpha,\gamma),1} \|y\|_{(\alpha',\gamma'),1,2} \left[(\tau - t)^{-\gamma} (t-s)^{2\rho+\gamma} \wedge (\tau - s)^{2\rho} \right]. \tag{3.22}$$

In addition to Theorem 1.3.8, the rough Volterra formalism relies on a stack of iterated integrals verifying convolutional type algebraic identities. The other main assumption about this stack of integrals which should be seen as the equivalent of Chen's relation in our Volterra context is stated as:

Hypothesis 1.3.9. *Let $z \in \mathcal{V}^{(\alpha,\gamma)}$ be a Volterra path as given in Definition 1.3.5. For n such that $(n+1)\rho + \gamma > 1$, assume that there exists a family $\{\mathbf{z}^{\mathrm{j},\tau}; \mathrm{j} \le n\}$ such that $\mathbf{z}_{ts}^{\mathrm{j},\tau} \in (\mathbb{R}^m)^{\otimes \mathrm{j}}$, $\mathbf{z}^1 = z$ and verifying*

$$\delta_u \mathbf{z}_{ts}^{\mathrm{j},\tau} = \sum_{\mathrm{i}=1}^{\mathrm{j}-1} \mathbf{z}_{tu}^{\mathrm{j-i},\tau} * \mathbf{z}_{us}^{\mathrm{i},\cdot} = \int_s^t d\mathbf{z}_{tr}^{\mathrm{j-i},\tau} \otimes \mathbf{z}_{us}^{\mathrm{i},r}, \tag{3.23}$$

where the right hand side of (3.23) is defined in Theorem 1.3.8. In addition, for $\mathrm{j} = 1, \ldots, n$, $\mathbf{z}^{\mathrm{j}} \in \mathcal{V}^{(\mathrm{j}\rho+\gamma,\gamma)}$.

In order to introduce the second level convolution product, the kind of topology for functions of the form $u^{1,2}$ is necessary.

Definition 1.3.10. *Let $\mathcal{W}_2^{(\alpha,\gamma)}$ denote the space of functions $u : \Delta_3 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 2}, \mathbb{R}^d)$ with a fixed initial condition $u_0^{p,q} = u_0$, endowed with the norm*

$$\left\|u^{1,2}\right\|_{(\alpha,\gamma)} := \left\|u^{1,2}\right\|_{(\alpha,\gamma),1} + \left\|u^{1,2}\right\|_{(\alpha,\gamma),1,2}. \tag{3.24}$$

The right hand side of (3.24) is defined as follows, recalling the convention $\rho = \alpha - \gamma$:

$$\left\|u^{1,2}\right\|_{(\alpha,\gamma),1} := \sup_{(s,t,\tau)\in\Delta_3} \frac{|u_{ts}^{\tau,\tau}|}{\left[|\tau - t|^{-\gamma} |t-s|^{\alpha}\right] \wedge |\tau - s|^{\rho}}, \tag{3.25}$$

and

$$\left\|u^{1,2}\right\|_{(\alpha,\gamma),1,2} := \left\|u^{1,2}\right\|_{(\alpha,\gamma),1,2,>} + \left\|u^{1,2}\right\|_{(\alpha,\gamma),1,2,<}, \tag{3.26}$$

where the norms $\|u^{1,2}\|_{(\alpha,\gamma),1,2,>}$ and $\|u^{1,2}\|_{(\alpha,\gamma),1,2,<}$ are respectively defined by

$$\left\|u^{1,2}\right\|_{(\alpha,\gamma),1,2,>} = \sup_{\substack{(s,t,r_1,r_2,r')\in\Delta_5 \\ \eta\in[0,1],\zeta\in[0,\alpha-\gamma)}} \frac{|u_{ts}^{r',r_2} - u_{ts}^{r',r_1}|}{h_{\eta,\zeta}(s,t,r_1,r_2,r')}, \tag{3.27}$$

$$\left\|u^{1,2}\right\|_{(\alpha,\gamma),1,2,<} = \sup_{\substack{(s,t,r',r_1,r_2)\in\Delta_5 \\ \eta\in[0,1],\zeta\in[0,\alpha-\gamma)}} \frac{|u_{ts}^{r_2,r'} - u_{ts}^{r_1,r'}|}{h_{\eta,\zeta}(s,t,r_1,r_2,r')}, \tag{3.28}$$

where the function h is defined by

$$\begin{aligned} h_{\eta,\zeta}(s,t,r_1,r_2,r') &= |r_2 - r_1|^{\eta} \left|\min(r_1,r_2,r') - t\right|^{-\eta+\zeta} \\ &\quad \times \left(\left[\left|\min(r_1,r_2,r') - t\right|^{-\gamma-\zeta} |t-s|^{\alpha}\right] \wedge \left|\min(r_1,r_2,r') - s\right|^{\alpha-\gamma-\zeta}\right). \end{aligned} \tag{3.29}$$

With the above definition at hand, it is now ready to recall the construction of second order convolution products in the rough case $\alpha - \gamma > \frac{1}{3}$.

Theorem 1.3.11. *Let $z \in \mathcal{V}^{(\alpha,\gamma)}$ be as given in Definition 1.3.5 with $\alpha, \gamma \in (0,1)$ satisfying $\rho = \alpha - \gamma > \frac{1}{3}$. We assume that $\mathbf{z}$ fulfills Hypothesis 1.3.9 with $n = 2$. Consider a function*

$y:\Delta_3\to\mathcal{L}((\mathbb{R}^d)^{\otimes 2},\mathbb{R}^d)$ with $\|y^{1,2}\|_{(\alpha,\gamma),1,2}<\infty$ and $y_0^{1,2}=y_0$, for a fixed initial condition $y_0\in\mathcal{L}((\mathbb{R}^d)^{\otimes 2},\mathbb{R}^d)$. For all fixed $(s,t,\tau)\in\Delta_3$ we have that

$$\mathbf{z}_{ts}^{2,\tau}*y_s^{1,2}:=\lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}}\mathbf{z}_{vu}^{2,\tau}y_s^{u,u}+(\delta_u\mathbf{z}_{vs}^{2,\tau})*y_s^{1,2} \tag{3.30}$$

is a well defined Volterra-Young integral. It follows that $*$ is a well defined bi-linear operation between the three parameters Volterra function $\mathbf{z}^2$ and a 3-parameter path y. Moreover, the following inequality holds

$$\begin{aligned}\left|\mathbf{z}_{ts}^{2,\tau}*y_s^{1,2}-\mathbf{z}_{ts}^{2,\tau}y_s^{s,s}\right|\lesssim\|y^{1,2}\|_{(\alpha,\gamma),1,2}\left(\|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma),1}+\|\mathbf{z}^1\|_{(\alpha,\gamma),1,2}\|\mathbf{z}^1\|_{(\alpha,\gamma),1}\right)\\ \times\left(\left[|\tau-t|^{-\gamma}|t-s|^{2\rho+\gamma}\right]\wedge|\tau-s|^{2\rho}\right).\end{aligned} \tag{3.31}$$

Similar to classical rough paths, controlled Volterra paths is crucial for a proper definition of rough Volterra equations.

Definition 1.3.12. *Let $z\in\mathcal{V}^{(\alpha,\gamma)}(E)$ for some $\rho=\alpha-\gamma>0$. Assume that there exists two functions $y:\Delta_2\to V$ and $y':\Delta_3\to\mathcal{L}(E,V)$, such that $y_0^\tau=y_0\in E$ for any $\tau\in[0,T]$ and $y_0'^{,p,q}=y_0'\in E$ for any $(q,p)\in\Delta_2$, and satisfying the relation*

$$y_{ts}^\tau=z_{ts}^\tau*y_s'^{,\tau,\cdot}+R_{ts}^\tau, \tag{3.32}$$

where $R\in\mathcal{V}_2^{(2\alpha,2\gamma)}(V)$ and $y'\in\mathcal{W}_2^{(\alpha,\gamma)}$. (Recall that the spaces $\mathcal{V}_2^{(2\alpha,2\gamma)}$ and $\mathcal{W}_2^{(\alpha,\gamma)}$ are respectively introduced in Definition 1.3.5 and Definition 1.3.10). Whenever (y,y') satisfies relation (3.32) we say that (y,y') is a Volterra path controlled by z (or controlled Volterra path in general) and we write $(y,y')\in\mathcal{D}_z^{(\alpha,\gamma)}(\Delta_2;V)$. We equip this space with a semi-norm $\|\cdot\|_{z,(\alpha,\gamma)}$ given by

$$\|y,y'\|_{z,(\alpha,\gamma)}=\|y'^{,\cdot_1,\cdot_2}\|_{(\alpha,\gamma)}+\|R\|_{(2\alpha,2\gamma)}. \tag{3.33}$$

Under the mapping $(y,y')\mapsto|y_0|+|y_0'|+\|y,y'\|_{z,(\alpha,\gamma)}$ the space $\mathcal{D}_z^{(\alpha,\gamma)}(\Delta_2;V)$ is a Banach space. The remainder term R in (3.32) with respect to a Volterra path $(y,y')\in\mathcal{D}_z^{(\alpha,\gamma)}$ will typically be denoted by R^y.

This leads to a rough integral given as a functional of the Volterra signature and the Volterra controlled paths, combined through the convolution product.

Theorem 1.3.13. *Let $x \in \mathcal{C}^\alpha$ and k be a Volterra kernel satisfying* (**H**) *with a parameter γ such that $\rho = \alpha - \gamma > \frac{1}{3}$. Thanks to Theorem 2.2.20, define $z_t^\tau = \int_0^t k(\tau, r)\, dx_r$ and assume there exists a second order Volterra rough path $\mathbf{z} \in \mathcal{V}^{\alpha,\gamma}(\Delta_2; E)$ built from z according to Definition 1.3.5. Additionally, suppose both components of $\mathbf{z}$ are uniformly bounded. Namely, we assume there exists an $M > 0$ such that*

$$\|\mathbf{z}\|_{(\alpha,\gamma)} := \|\mathbf{z}^1\|_{(\alpha,\gamma)} + \|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma)} \leq M, \tag{3.34}$$

where the two norm quantities corresponds to the norms given in Definition 1.3.5 and Definition 1.3.10. We now consider a controlled Volterra path $(y, y') \in \mathcal{D}_{\mathbf{z}^1}^{(\alpha,\gamma)}(\Delta_2; \mathcal{L}(E, V))$. Then the following holds true:

(i) *The following limit exists for all $(s,t,\tau) \in \Delta_3$,*

$$w_{ts}^\tau = \int_s^t k(\tau, r) y_r^r dx_r := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbf{z}_{vu}^{1,\tau} * y_u^{\cdot} + \mathbf{z}_{vu}^{2,\tau} * y_u^{\prime,\cdot_1,\cdot_2}. \tag{3.35}$$

(ii) *Let w be defined by (3.35). There exists a constant $C = C_{M,\alpha,\gamma}$ such that for all $(s,t) \in \Delta_2$ we have*

$$\begin{aligned}&\left|w_{ts}^\tau - \mathbf{z}_{ts}^{1,\tau} * y_s^{\cdot} - \mathbf{z}_{ts}^{2,\tau} * y_s^{\prime,\cdot_1,\cdot_2}\right| \\ &\qquad \leq C\|y, y'\|_{z,(\alpha,\gamma)} \|\mathbf{z}\|_{(\alpha,\gamma)} \left[|\tau - t|^{-\gamma}|t - s|^{3\rho+\gamma} \wedge |\tau - s|^{3\rho}\right].\end{aligned} \tag{3.36}$$

(iii) *For all $(s,t,p,q) \in \Delta_4$ and $\beta \in (0,1)$ we have*

$$\begin{aligned}&\left|w_{ts}^{qp} - \mathbf{z}_{ts}^{1,qp} * y_s^{\cdot} - \mathbf{z}_{ts}^{2,qp} * y_s^{\prime,\cdot_1,\cdot_2}\right| \\ &\qquad \leq C\|y, y'\|_{z,(\alpha,\gamma)} \|\mathbf{z}\|_{(\alpha,\gamma)} |p - q|^\beta \left[|q - t|^{-\gamma-\beta}|t - s|^{3\rho+\gamma} \wedge |q - s|^{3\rho-\beta}\right].\end{aligned} \tag{3.37}$$

(iv) *The couple* (w, w') *is a controlled Volterra path in* $\mathcal{D}_{\mathbf{z}^1}(\Delta_2, V)$*, where we recall that* w *is defined by (3.35) and* $w_t^{\prime,\tau,p} = y_t^p$.

The rough integral is then used in the construction of solutions to Volterra equations driven by Hölder noises with singular kernels.

Theorem 1.3.14. *Let* $\mathbf{z} \in \mathcal{V}^{(\alpha,\gamma)}(E)$ *with* $\alpha - \gamma > \frac{1}{3}$*. Assume that* $\mathbf{z}$ *satisfies the same hypothesis as in Theorem 1.3.13 and suppose* $f \in \mathcal{C}_b^4(V; \mathcal{L}(E, V))$*. Then there exists a unique Volterra solution in* $\mathcal{D}_{\mathbf{z}^1}^{(\alpha,\gamma)}(V)$ *to the equation*

$$y_t^\tau = y_0 + \int_0^t k(\tau, r) f(y_r^r)\, dx_r, \quad (t,\tau) \in \Delta^{(2)}([0,T]), \quad y_0 \in E, \tag{3.38}$$

where the integral is understood as a rough Volterra integral given in Theorem 1.3.13.

1.4 Summary of results in Chapter 1 and Chapter 2

1.4.1 The result of Chapter 1: Volterra equations driven by rough signals 2: higher order expansions

In this section, we will summarize the result in our first paper [7]: we extend rough path theory for Volterra equations from [1] in the rough case $\alpha - \gamma > \frac{1}{3}$ to the case $\alpha - \gamma > \frac{1}{4}$. It was already observed in [1] that the Volterra rough path introduced there did not satisfy any geometric relation, similar to that observed in classical rough path theory. Thus, an extension of the theory to more irregular driving signals requires a deeper understanding of the specific algebraic structure arising in the Volterra rough path. Inspired by the elements of "non-geometric rough paths" developed in [2] and [3] we provide a simple description of the Volterra rough path and the controlled Volterra process in terms of rooted trees, and with this description we are able to solve rough volterra equations in driven by more irregular signals.

Similar to the construction of the first and second level convolution product in [1], the third level Volterra convolution product is defined by

Theorem 1.4.1. *Let $z \in \mathcal{V}^{(\alpha,\gamma)}$ with $\alpha, \gamma \in (0,1)$ satisfying $\rho = \alpha - \gamma > \frac{1}{4}$, as given in Definition 2.2.4. We assume that $\mathbf{z}$ fulfills Hypothesis 2.2.16 with n=3. Consider a function $y : \Delta_4 \to \mathcal{L}((\mathbb{R}^m)^{\otimes 3}, \mathbb{R}^m)$ as given in Notation 2.3.3 such that $\|y^{1,2,3}\|_{(\alpha,\gamma),1,2,3} < \infty$ and $y_0^{1,2,3} = y_0$, where $\|y^{1,2,3}\|_{(\alpha,\gamma),1,2,3}$ is defined by (3.17). Then with Notation 2.2.14 in mind, we have for all fixed $(s,t,\tau) \in \Delta_3$ that*

$$\mathbf{z}_{ts}^{3,\tau} * y_s^{1,2,3} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbf{z}_{vu}^{3,\tau} y_s^{u,u,u} + \left(\delta_u \mathbf{z}_{vs}^{3,\tau}\right) * y_s^{1,2,3}, \tag{4.1}$$

$$\mathbf{z}_{ts}^{\overset{\bullet\,\bullet}{\vee},\tau} * y_s^{1,2,3} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbf{z}_{vu}^{\overset{\bullet\,\bullet}{\vee},\tau} y_s^{u,u,u} + \left(\delta_u \mathbf{z}_{vs}^{\overset{\bullet\,\bullet}{\vee},\tau}\right) * y_s^{1,2,3}. \tag{4.2}$$

is a well defined Volterra-Young integral. It follows that $*$ is a well defined bi-linear operation between the three parameters Volterra function $\mathbf{z}^3$ and a 4-parameter path y. Moreover, we have that

$$\begin{aligned}\left|\mathbf{z}_{ts}^{3,\tau} * y_s^{1,2,3} - \mathbf{z}_{ts}^{3,\tau} y_s^{s,s,s}\right| \lesssim \|y^{1,2,3}\|_{(\alpha,\gamma),1,2,3}\Big(&\|\mathbf{z}^3\|_{(3\rho+\gamma,\gamma),1} + \|\mathbf{z}^1\|_{(\alpha,\gamma),1,2}\|\mathbf{z}^2\|_{(\alpha,\gamma),1} \\ &+ \|\mathbf{z}^2\|_{(\alpha,\gamma),1,2}\|\mathbf{z}^1\|_{(\alpha,\gamma),1}\Big)\left(\left[|\tau-t|^{-\gamma}|t-s|^{3\rho+\gamma}\right] \wedge |\tau-s|^{3\rho}\right),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\left|\mathbf{z}_{ts}^{\overset{\bullet\,\bullet}{\vee},\tau} * y_s^{1,2,3} - \mathbf{z}_{ts}^{\overset{\bullet\,\bullet}{\vee},\tau} y_s^{s,s,s}\right| \lesssim \left\|y^{1,2,3}\right\|_{(\alpha,\gamma),1,2,3}\Big(&\|\mathbf{z}^{\overset{\bullet\,\bullet}{\vee}}\|_{(3\rho+\gamma,\gamma),1} + \|\mathbf{z}^{\bullet}\|_{(\alpha,\gamma),1,2}\|\mathbf{z}^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma),1} \\ &+ \|\mathbf{z}^{\bullet}\|^2_{(\alpha,\gamma),1,2}\|\mathbf{z}^{\bullet}\|_{(\alpha,\gamma),1}\Big)\left(\left[|\tau-t|^{-\gamma}|t-s|^{3\rho+\gamma}\right] \wedge |\tau-s|^{3\rho}\right).\end{aligned} \tag{4.4}$$

And then an integration of controlled processes $\int_s^t k(\tau,r)dx_r y_r^\tau$ enjoys following results:

Theorem 1.4.2. *Let $x \in \mathcal{C}^\alpha$ and k be a Volterra kernel satisfying Hypothesis 2.2.1 with a parameter γ such that $\rho = \alpha - \gamma > \frac{1}{4}$. Define $z_t^\tau = \int_0^t k(\tau,r)\,dx_r$ and assume there exists a tree indexed rough path $\mathbf{z} = \{\mathbf{z}^{\sigma,\tau}; \sigma \in \mathrm{T}_3\}$ above $\mathbf{z}^\tau$ satisfying Hypothesis 2.4.3. Additionally, suppose all components of $\mathbf{z}$ are uniformly bounded. Namely, we assume there exists an $M > 0$ such that*

$$\|\mathbf{z}\|_{(\alpha,\gamma)} := \|\mathbf{z}^{\bullet}\|_{(\alpha,\gamma)} + \|\mathbf{z}^{\overset{\bullet}{\bullet}}\|_{(2\rho+\gamma,\gamma)} + \|\mathbf{z}^{\overset{\bullet}{\overset{\bullet}{\bullet}}}\|_{(3\rho+\gamma,\gamma)} + \|\mathbf{z}^{\overset{\bullet\,\bullet}{\vee}}\|_{(3\rho+\gamma,\gamma)} \le M. \tag{4.5}$$

We now consider a controlled Volterra path $\mathbf{y} \in \mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}(\mathcal{L}(\mathbb{R}^m))$, as introduced in Definition 2.4.7. Then the following holds true:

(a) *The following limit exists for all $(s,t,\tau) \in \Delta_3$,*

$$w_{ts}^{\tau} = \int_s^t k(\tau,r)dx_r y_r^r := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \left(\mathbf{z}_{vu}^{\bullet,\tau} * y_u^{\cdot} + \mathbf{z}_{vu}^{\overset{\bullet}{\bullet},\tau} * y_u^{\bullet,\cdot,\cdot} + \mathbf{z}_{vu}^{\overset{\bullet}{\overset{\bullet}{\bullet}},\tau} * y_u^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot} + \mathbf{z}_{vu}^{\overset{\bullet\,\bullet}{\vee},\tau} * y_u^{\bullet\bullet,\cdot,\cdot,\cdot} \right) \tag{4.6}$$

(b) *Let w be defined by (4.13). There exists a constant $C = C_{M,\alpha,\gamma}$ such that for all $(s,t) \in \Delta_2$ we have*

$$\begin{aligned} &\left| w_{ts}^{\tau} - \left(\mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\cdot} + \mathbf{z}_{ts}^{\overset{\bullet}{\bullet},\tau} * y_s^{\bullet,\cdot,\cdot} + \mathbf{z}_{ts}^{\overset{\bullet}{\overset{\bullet}{\bullet}},\tau} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot} + \mathbf{z}_{ts}^{\overset{\bullet\,\bullet}{\vee},\tau} * y_s^{\bullet\bullet,\cdot,\cdot,\cdot} \right) \right| \\ &\leq C \left\| \left(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet} \right) \right\|_{\mathbf{z},(\alpha,\gamma)} \|\mathbf{z}\|_{(\alpha,\gamma)} \left(\left[|\tau - t|^{-\gamma} |t-s|^{4\rho+\gamma} \right] \wedge |\tau - s|^{4\rho} \right). \end{aligned} \tag{4.7}$$

(c) *For all $(s,t,p,q) \in \Delta_4$, $\eta \in [0,1]$ and $\zeta \in [0,4\rho)$ we have*

$$\begin{aligned} &\left| w_{ts}^{qp} - \left(\mathbf{z}_{ts}^{\bullet,qp} * y_s^{\cdot} + \mathbf{z}_{ts}^{\overset{\bullet}{\bullet},qp} * y_s^{\bullet,\cdot,\cdot} + \mathbf{z}_{ts}^{\overset{\bullet}{\overset{\bullet}{\bullet}},qp} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot} + \mathbf{z}_{ts}^{\overset{\bullet\,\bullet}{\vee},qp} * y_s^{\bullet\bullet,\cdot,\cdot,\cdot} \right) \right| \\ &\leq C \left\| \left(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet} \right) \right\|_{\mathbf{z},(\alpha,\gamma)} \|\mathbf{z}\|_{(\alpha,\gamma)} |p-q|^{\eta} |q-t|^{-\eta+\zeta} \left(\left[|q-t|^{-\gamma-\zeta} |t-s|^{4\rho+\gamma} \right] \wedge |q-s|^{4\rho-\zeta} \right). \end{aligned} \tag{4.8}$$

(d) *The triple $\mathbf{w} = (w, w^{\bullet}, w^{\overset{\bullet}{\bullet}}, 0)$ is a controlled Volterra path in $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}(\Delta_2, \mathbb{R}^m)$, where we recall that w is defined by (4.13), and where $w^{\bullet}$, $w^{\overset{\bullet}{\bullet}}$ are respectively given by*

$$w_t^{\bullet,\tau,p} = y_t^p, \quad \text{and} \quad w_t^{\overset{\bullet}{\bullet},\tau,q,p} = y_t^{\bullet,q,p}.$$

With Theorem 2.4.10 at hand, the composition of a Volterra controlled processes with a smooth function is still a controlled process.

Proposition 1.4.3. *Let $f \in C_b^4(\mathbb{R}^m)$ and assume $(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0) \in \hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma)}$ as given in Remark 2.4.12. Also recall our Notation 2.2.14 for matrix products. Then the composition*

$f(y)$ can be seen as a controlled path $f(y) = (\phi, \phi^{\bullet}, \phi^{\substack{\bullet\\ \bullet}}, \phi^{\bullet\bullet})$, where $\phi = f(y)$ and where in the decomposition (4.9) we have

$$\phi_t^{\bullet,q,p} = y_t^{\bullet,p} f'\left(y_t^q\right), \tag{4.9}$$

and where the second derivative $\phi^{\substack{\bullet\\ \bullet}}$ and $\phi^{\bullet\bullet}$ are respectively defined by

$$\phi_t^{\substack{\bullet\\ \bullet},r,q,p} = y_t^{\substack{\bullet\\ \bullet},q,p} f'\left(y_t^r\right), \quad \textit{and} \quad \phi_t^{\bullet\bullet,r,q,p} = \frac{1}{2}(y_t^{\bullet,q}) \otimes (y_t^{\bullet,p}) f''\left(y_t^r\right). \tag{4.10}$$

Moreover, there exists a constant $C = C_{M,\alpha,\gamma,\|f\|_{C_b^4}} > 0$ such that

$$\begin{aligned} \|(\phi, \phi^{\bullet}, \phi^{\substack{\bullet\\ \bullet}}, \phi^{\bullet\bullet})\|_{\mathbf{z};(\alpha,\gamma)} \leq C(1 + \|\mathbf{z}\|_{(\alpha,\gamma)})^3 \times \Bigg[\Big(|y_0^{\bullet}| + |y_0^{\substack{\bullet\\ \bullet}}| + \|(y, y^{\bullet}, y^{\substack{\bullet\\ \bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma)} \Big) \\ \vee \Big(|y_0^{\bullet}| + |y_0^{\substack{\bullet\\ \bullet}}| + \|(y, y^{\bullet}, y^{\substack{\bullet\\ \bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma)} \Big)^3 \Bigg]. \end{aligned} \tag{4.11}$$

Finally, the existence and uniqueness of Volterra type equations in the rough case $\rho = \alpha - \gamma > \frac{1}{4}$ is proved.

Theorem 1.4.4. *Let $z \in \mathcal{V}^{(\alpha,\gamma)}$ with $\alpha - \gamma > \frac{1}{4}$. Assume that z satisfies the same hypotheses as in Theorem 2.4.10, and suppose $f \in \mathcal{C}_b^5(\mathbb{R}^m; \mathcal{L}(\mathbb{R}^m))$. Recall that the space of controlled processes $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}(\mathbb{R}^m)$ is introduced in Definition 2.4.7. Then there exists a unique solution in $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}(\mathbb{R}^m)$ to the Volterra equation*

$$y_t^\tau = y_0 + \int_0^t k(\tau, r)\, dx_r f\left(y_r^r\right), \quad (t,\tau) \in \Delta_2([0,T]), \quad y_0 \in \mathbb{R}^m, \tag{4.12}$$

where the integral is understood as a rough Volterra integral according to Theorem 2.4.10.

1.4.2 The result of Chapter 2: Volterra rough path driven by fractional brownian motion

In this section, we are going to construct the Volterra rough path driven by a fractional Brownian motion with Hurst parameter $H > 1/2$ and $H = \frac{1}{2}$ so that it satisfies Defintion 3.2.8. It should be noticed that this regime leads to nontrivial rough paths development in the Volterra case, due to the singularity of the kernel k in (1.1). In order to show that the

Volterra rough paths $\{\mathbf{z}^1, \mathbf{z}^2\}$ satisfies the algebraic and analytic properties that are stated in Definition 3.2.8, the first step is to extend the classical Garsia-Rodemich-Rumsey inequality to suit our Volterra paths.

Definition 1.4.5. *Let $z : \Delta_3 \to \mathbb{R}^d$ be a continuous Volterra increment. Then for some parameters $p \geq 1$ and $\alpha, \gamma \in (0,1)$, $\zeta \in [0, \alpha - \gamma)$, $\eta \in [\zeta, 1]$ we define*

$$U^{\tau}_{(\alpha,\gamma),p,1}(z;\eta,\zeta) := \left(\int_{(v,w)\in\Delta_2^{\tau}} \frac{|z^{\tau}_{wv}|^{2p}}{|\tau - w|^{-2p(\eta-\zeta)} |\psi^1_{\alpha,\gamma+\zeta}(\tau, w, v)|^{2p} |w - v|^2} dvdw \right)^{\frac{1}{2p}} \tag{4.13}$$

$$U^{\tau}_{(\alpha,\gamma,\eta,\zeta),p,1,2}(z) := \left(\int_{(v,w,r',r)\in\Delta_4^{\tau}} \frac{|z^{rr'}_{wv}|^{2p}}{|\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(r, r', w, v)|^{2p} |w - v|^2 |r - r'|^2} dvdwdr'dr \right)^{\frac{1}{2p}}, \tag{4.14}$$

where recall that the functions $\psi^1, \psi^{1,2}$ are respectively defined in (2.1) and (2.2).

With the Definition 1.4.5, the Volterra GRR inequality is stated as follows:

Proposition 1.4.6. *Let $\mathbf{z} : \Delta_3 \to \mathbb{R}^d$. For $(\alpha, \gamma) \in (0,1)^2$ with $\alpha - \gamma > 0$, $\zeta \in [0, \alpha - \gamma)$, and $\eta \in [\zeta, 1]$, we assume that $\delta\mathbf{z} \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ where $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ is introduced in Definition 3.2.9. Suppose $\kappa \in (0, \alpha)$. Then for any $p > \frac{1}{\alpha-\kappa} \vee \frac{1}{\zeta}$, the following two bounds holds:*

$$\|\mathbf{z}\|_{(\kappa,\gamma),1} \lesssim U^{T}_{(\kappa,\gamma),1,p}(\mathbf{z}) + \|\delta\mathbf{z}\|_{(\kappa,\gamma),1}, \tag{4.15}$$

$$\|\mathbf{z}\|_{(\kappa,\gamma,\eta,\zeta),1,2} \lesssim U^{T}_{(\kappa,\gamma,\eta,\zeta),1,2,p}(\mathbf{z}) + \|\delta\mathbf{z}\|_{(\kappa,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p}),1,2} T^{2+\alpha-\kappa-\frac{1}{p}}. \tag{4.16}$$

It is now ready to construct the first level and second level of the Volterra rough path driven by a fBm as introduced in Notation 3.4.1.

Definition 1.4.7. *Consider a fractional Brownian motion $B : [0,T] \to \mathbb{R}^m$ as given in Notation 3.4.1 and a function h of the form $h^{\tau}_{ts}(r) = (\tau - r)^{-\gamma} \mathbb{1}_{[s,t]}(r)$ with $\gamma < 2H - 1$. Then for $(s,t,\tau) \in \Delta_3$ we define the increment $\mathbf{z}^{1,\tau,\mathrm{i}}_{ts} = \int_s^t (\tau - r)^{-\gamma} dB^{\mathrm{i}}_r$ as a Wiener integral of the form*

$$\mathbf{z}^{1,\tau,\mathrm{i}}_{ts} := B^{\mathrm{i}}(h^{\tau}_{ts}). \tag{4.17}$$

The second level of the Volterra rough path can be constructed in a similar to the first level of the Volterra rough path in Definition 1.4.7.

Definition 1.4.8. *We consider a fractional Brownian motion $B : [0, T] \to \mathbb{R}^m$ as given in Notation 3.4.1, as well as the first level of the Volterra rough path $\mathbf{z}^{1,\tau}$ defined by (4.13). As in Definition 3.4.2, we assume that $\gamma < 2H - 1$. Then for $(s, r, t, \tau) \in \Delta_4$, we set*

$$u_{ts}^{\tau,\mathrm{i}}(r) = (\tau - r)^{-\gamma} \mathbf{z}_{rs}^{1,r,\mathrm{i}} \mathbb{1}_{[s,t]}(r). \tag{4.18}$$

With this notation in hand, the increment $\mathbf{z}_{ts}^{2,\tau}$ is given as follows: if $\mathrm{i} \neq \mathrm{j}$ we define $\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}}$ as

$$\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}} = B^{\mathrm{j}}(u_{ts}^{\tau,\mathrm{i}}), \tag{4.19}$$

where (conditionally on B^{i}) the random variable $B^{\mathrm{j}}(u_{ts}^{\tau,\mathrm{i}})$ has to be interpreted as a Wiener integral. In the case $\mathrm{i} = \mathrm{j}$, we set

$$\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{i}} = \int_s^t u_{ts}^{\tau,\mathrm{i}}(r) dB_r^{\mathrm{i}}, \tag{4.20}$$

where the right hand side of (4.35) is defined as a Stratonovich integral like (4.11).

Once we get the Volterra rough path family $\{\mathbf{z}^{1,\tau}, \mathbf{z}^{2,\tau}\}$, it is now the time to verify that $\{\mathbf{z}^{1,\tau}, \mathbf{z}^{2,\tau}\}$ satisfies Definition 3.2.8.

Proposition 1.4.9. *The increment $\mathbf{z}^{1,\tau}$ introduced in Definition 3.4.2 is almost surely in the Volterra space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3; \mathbb{R}^m)$ for any $\alpha \in (\gamma, H)$, $\zeta \in [0, \alpha - \gamma)$ and $\eta \in [\zeta, 1]$, where $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3; \mathbb{R}^m)$ is introduced in Definition 2.2.4. In addition, for all $p \geq 1$ and $\alpha < H - \frac{3}{2p}$ we have that*

(i)

$$\mathbb{E}\left[\|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta)}^{2p}\right] < \infty, \tag{4.21}$$

(ii)

$$\delta_m \mathbf{z}_{ts}^{1,\tau,\mathrm{i}} = 0, \text{ for all } (s, m, t, \tau) \in \Delta_4 \text{ a.s.} \tag{4.22}$$

For $\mathbf{z}^{2,\tau}$, the analytic and algebraic properties is not hart to be checked.

Proposition 1.4.10. *Consider the second level $\mathbf{z}^{2,\tau}$ of the Volterra rough path, as defined in (4.34)-(4.35). Recall that $\delta\mathbf{z}^{2,\tau}$ is a path defined on Δ_4, and we refer to Definition 3.2.9 for the definition of $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_4;\mathbb{R}^m)$. We assume that $H > 1/2$, $\gamma < 2H-1$, $\alpha \in (\gamma, H)$, $\zeta \in [0, 2(\alpha-\gamma))$ and $\eta \in [\zeta, 1]$. Then almost surely we have*

(i)

$$\delta_m \mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}} = \mathbf{z}_{tm}^{1,\tau,\mathrm{j}} * \mathbf{z}_{ms}^{1,\cdot,\mathrm{i}}, \quad \textit{for all } (s,m,t,\tau) \in \Delta_4 \quad \textit{a.s.} \tag{4.23}$$

(ii)

$$\delta\mathbf{z}^{2,\tau} \in \mathcal{V}^{(2\alpha-\gamma,\gamma,\eta,\zeta)}(\Delta_4;\mathbb{R}^m). \tag{4.24}$$

(iii) *Moreover, for all $p \geq 1$ we have*

$$\mathbb{E}\left[\|\delta\mathbf{z}^{2,\tau}\|_{(2\alpha-\gamma,\gamma,\eta,\zeta)}^{2p}\right] < \infty, \tag{4.25}$$

where the norm above is understood as in (2.14).

2. CHAPTER1

2.1 Introduction

2.1.1 Background and description of the results

Volterra equations of the second kind are typically given on the form

$$y_t = y_0 + \int_0^t k_1(t,s)b(y_s)ds + \int_0^t k_2(t,s)\sigma(y_s)dx_s, \quad y_0 \in \mathbb{R}^d \tag{1.1}$$

where b and σ are sufficiently smooth functions, $x : [0,T] \to \mathbb{R}^d$ is a α-Hölder continuous path with $\alpha \in (0,1)$, and k_1 and k_2 are two possibly singular kernels, behaving like $|t-s|^{-\gamma}$ for some $\gamma \in [0,1)$ whenever $s \to t$. Such equations frequently appear in mathematical models for natural or social phenomena which exhibits some form of memory of its own past as it evolves in time (see e.g. [10] and the references therein). Most recently, Volterra equations of this form have become very popular in the modelling of stochastic volatility for financial asset prices. In this case the kernels $k_1(t,s)$ and $k_2(t,s)$ are typically assumed to be very singular when $s \to t$, and the path x is assumed to be a sample path of a Gaussian process (see e.g. [11]–[13]).

Whenever the driving noise x is sampled from a Brownian motion (or some other continuous semi-martingale), one may use traditional probabilistic techniques from stochastic analysis (see e.g. [14], [15]) in order to make sense of equations like (1.1). However, for more general driving noise x with rougher regularity than a Brownian motion, very little is known about solutions to Volterra equations. Inspired by the theory of rough paths [8], it is desirable to solve equation (1.1) in a purely pathwise sense relying only on the analytic behaviour of the sample paths of x. This would allow to remove the probabilistic restrictions imposed by classical stochastic analysis. However, due to the non-local nature of the equations induced by the kernels k_1 and k_2, the theory of rough paths can not directly be applied in order to solve singular Volterra equations of the form of (1.1). Indeed, the fundamental algebraic relations satisfied by a a classical rough path do not hold when the signal is influenced by a possibly singular kernel. Let us mention at this point a few contributions in the rough paths realm trying to overcome this obstacle:

(i) The articles [16], [17] handle some cases of rough Volterra equations thanks to an elaboration of traditional rough paths elements. However, the analysis was only valid for kernels with no singularities.

(ii) The paper [18] focuses on Volterra equations from a para-controlled calculus perspective. This elegant method is unfortunately restricted to first order rough paths type expansions, with inherent limits on both the irregularity of the driving process x and the singularity of the kernel k.

(iii) The contribution [13] investigates Volterra equations through the lens of regularity structures. Although only the strategy of the construction is outlined therein, we believe that a mere application of regularity structures techniques would only yield local existence and uniqueness results. It should also be mentioned that renormalization techniques are invoked in [13].

As the reader might see, the rough paths analysis of Volterra equations is thus far from being complete.

With those preliminary notions in mind, in the recent article [1] we initiated a rough path inspired study of singular Volterra equations, in a reduced form of (1.1) given by

$$u_t = u_0 + \int_0^t k(t,r) f(u_r) dx_r, \tag{1.2}$$

where f is a sufficiently regular function, x is a Hölder continuous path, and k is a singular kernel. To this end, we define

$$\Delta_n := \Delta_n\left([a,b]\right) = \{(x_1, \dots, x_n) \in [a,b]^n \mid a \le x_1 < \cdots < x_n \le b\}. \tag{1.3}$$

Next we introduce a class of two parameter paths $z : \Delta_2 \to \mathbb{R}^d$, needed to capture the possible singularity and regularity imposed by the kernels k_1 and k_2 and the driving noise x

in (1.1). These paths will then constitute the fundamental building blocks of the framework. The canonical example of such path is given by

$$z_t^\tau := \int_0^t k(\tau, s)dx_s, \quad \text{where} \quad t \leq \tau \in [0, T]. \tag{1.4}$$

For the moment, we may assume that x is a sufficiently regular path $x : [0, T] \to \mathbb{R}^d$, and $k(t, s)$ is an integrable (but possibly singular) kernel when $s \to t$, so that the above integral makes pathwise sense. We observe in particular that $t \mapsto z_t^t$ is just a standard Volterra integral (commonly referred to as a Volterra process in stochastic analysis). Heuristically one may think that the regularity arising from the mapping $\tau \mapsto z_t^\tau$ is induced by the behaviour of the kernel k while the regularity of the mapping $t \mapsto z_t^\tau$ is inherited by the regularity of x. By construction of a Volterra sewing lemma, we observed that this was indeed the case, even when x is only α-Hölder continuous for some $\alpha \in (0, 1)$. In general, we thus define a class of two variables paths in terms of the regularity in its upper and lower variable. This lead us in [1] to introduce two modifications of the classical Hölder semi-norms. The corresponding processes were then called Volterra paths.

Motivated by processes of the form (1.4), we constructed Volterra signatures as a collection of iterated integrals with respect to two-parameters Volterra paths. We also introduced a convolution product $*$, playing the role as the tensor product $\otimes$ in the classical rough path signature. The signature is then given as a family three-variable functions $\{(s, t, \tau) \mapsto \mathbf{z}_{ts}^{n,\tau}\}_{n\in\mathbb{N}}$, where, in the case of smooth x, each term is given by

$$\mathbf{z}_{ts}^{n,\tau} = \int_{\Delta_n([s,t])} k(\tau, r_n) \dots k(r_2, r_1)dx_{r_1} \otimes \cdots \otimes dx_{r_n}, \tag{1.5}$$

where we recall that $\Delta_n([s, t])$ is defined by (1.3). The algebraic structure associated with such iterated integrals resembles that of the tensor algebra of rough path theory, but where the tensor product is replaced by the convolution product. Together with Volterra signatures, we defined a class of controlled Volterra paths. Combining those two notions, it allowed to give a pathwise construction of solutions to Volterra equations of the form (1.1). Similarly to the theory of rough paths, the number of iterated integrals needed in order to give a pathwise definition of a rough Volterra integral is strongly dependent on the regularity of the path $x \in \mathsf{c}^\alpha([0,T];\mathbb{R}^d)$ and the singularity of the kernel k. Under the assumption that $|k(t,s)|$ behaves like $|t-s|^{-\gamma}$ when $s \to t$, the investigation in [1] was limited to the case when $\alpha - \gamma > \frac{1}{3}$, and thus only considers the first two components of the Volterra signature.

Our article [1] therefore left two important open questions, related to both the algebraic and probabilistic perspectives on rough paths theory:

(i) Algebraic aspects: Are there suitable algebraic relations describing the Volterra signature which are adaptable to prove existence and uniqueness of (1.1) in the case when $\alpha - \gamma < \frac{1}{3}$?

(ii) Probabilistic aspects: For what type of stochastic processes $\{x_t;\, t \in [0,T]\}$ and singular kernels k does there exist a collection of iterated integrals of the form of (1.5) almost surely satisfying the required algebraic and analytic relations?

The current article has to be seen as a step towards the answer of the algebraic problem mentioned above. Namely we investigate the case when $\alpha - \gamma < \frac{1}{3}$, and leave the probabilistic problem for a future work.

The rough Volterra picture gets significantly more involved when introducing a rougher signal x or a more singular kernel k. Indeed, the main challenge lies in the fact that the Volterra signature does not satisfy any geometric type property, in contrast with the classical rough paths situation. That is, classical integration by parts does not hold for Volterra iterated integrals, and therefore we do *not* have a relation of the form

$$\mathbf{z}^{2,\tau}_{ts} + (\mathbf{z}^{2,\tau}_{ts})^T = \mathbf{z}^{1,\tau}_{ts} * \mathbf{z}^{1,\cdot}_{ts},$$

where $(\cdot)^T$ denotes the transpose. Thus in order to consider $\alpha - \gamma$ lower than $\frac{1}{3}$, one needs to resort to different techniques than what is standard in the theory of rough paths.

Inspired by Martin Hairer's theory of regularity structures, we will in this article show that the Volterra signature is given with a Hopf algebraic type structure. Hence with the help of a description by rooted trees for the Volterra rough path, we are able to describe the necessary algebraic relations desired for the Volterra rough stochastic calculus. We will limit the scope of the current article to the case when $\alpha - \gamma > \frac{1}{4}$, and show that in order to prove existence and uniqueness of (1.1) in a "Volterra rough path" sense, one needs to introduce two more iterated integrals, as well as two more controlled Volterra derivatives than what is needed in the case $\alpha - \gamma > \frac{1}{3}$. We believe that the techniques developed here are an important stepping stone towards the goal of providing a rough paths framework for Volterra equations of the form of (1.1) in the general regime $\alpha - \gamma > 0$.

2.1.2 Organization of the paper

In section 2.2 we provide the necessary assumptions and preliminary results from [1]. In particular, we give the definition of Volterra paths, recall the Volterra sewing lemma and the convolution product between Volterra paths. Those results will play a central role for our subsequent analysis. Section 2.3 is devoted to the extension of the sewing lemma from the previous section to the case of two singularities, and we will apply this to create a third order convolution product between Volterra rough paths. In Section 2.4 we motivate the use of rooted trees to describe the Volterra rough path, and give a definition of controlled Volterra processes analogously. With this definition we prove both the convergence of a rough Volterra integral with respect to controlled Volterra paths, and that compositions of (sufficiently) smooth functions with a controlled Volterra path are again controlled Volterra paths. We conclude Section 2.4 with a proof of existence and uniqueness of Volterra equations driven by rough signals in the rougher regime.

2.1.3 Frequently used notation

We reserve the letter E to denote a Banach space, and we let the norm on E be denoted by $|\cdot|_E$. In subsequent sections, E will typically be given as $\mathbb{R}^d$ or $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^d)$ (The space of linear operators from $\mathbb{R}^m$ to $\mathbb{R}^d$). We will write $a \lesssim b$, whenever there exists a constant $C > 0$ (not depending on any parameters of significance) such that $a \leq Cb$. The space of continuous functions $f : X \to Y$ is denoted by $\mathsf{c}(X, Y)$. Whenever the codomain is not important, we use the shorter notation $\mathsf{c}(X)$. To denote that there exists a constant C which depends on a parameter p, we write $a \lesssim_p b$. For a one parameter path $f : [0, T] \to E$, we write $f_{ts} := f_t - f_s$, with a slight abuse of notation, we will later also use this notation for two variable functions of the form $f : [0,T]^2 \to \mathbb{R}^d$, where f_{ts} means evaluation in the point $(s,t) \in [0,T]^2$. We believe that it will always be clear from context what is meant. For $\alpha \in (0,1)$, we denote by $\mathsf{c}^\alpha([0,T];E)$ the standard space of α-Hölder continuous functions from $[0,T]$ into E, equipped with the norm $\|f\|_{\mathsf{c}^\alpha} := |f_0|_E + \|f\|_\alpha$, where $\|f\|_\alpha$ denotes the classical Hölder seminorm given by

$$\|f\|_\alpha := \sup_{(s,t)\in\Delta_2} \frac{|f_{ts}|}{|t-s|^\gamma}. \tag{1.6}$$

Whenever the domain and codomain is otherwise clear from the context, we will use the short hand notation c^α. We recall here that the n-simplex was already defined in (1.3). Throughout the article, we will frequently use the following simple bounds: for $(s,u,t) \in \Delta_3$ and $\gamma > 0$, then

$$|t-u|^\gamma \lesssim |t-s|^\gamma \quad \text{and} \quad |t-s|^{-\gamma} \lesssim |t-u|^{-\gamma}.$$

2.2 Assumptions and fundamentals of Volterra Rough Paths

We will start by presenting the necessary assumptions on the Volterra kernel k, as well as the driving noise x in (1.2). A full description (together with proofs) for the results recalled in this section can be found in [1].

Let us begin to present a working hypothesis for the type of kernels k, seen in (1.2), that we will consider in this article.

Hypothesis 2.2.1. *Let k be a kernel $k : \Delta_2 \to \mathbb{R}$, we assume that there exists $\gamma \in (0,1)$ such that for all $(s,r,q,\tau) \in \Delta_4([0,T])$ and $\eta, \beta \in [0,1]$ we have*

$$\begin{aligned}
|k(\tau,r)| &\lesssim |\tau - r|^{-\gamma} \\
|k(\tau,r) - k(q,r)| &\lesssim |q - r|^{-\gamma-\eta} |\tau - q|^{\eta} \\
|k(\tau,r) - k(\tau,s)| &\lesssim |\tau - r|^{-\gamma-\eta} |r - s|^{\eta} \\
|k(\tau,r) - k(q,r) - k(\tau,s) + k(q,s)| &\lesssim |q - r|^{-\gamma-\beta} |r - s|^{\beta} \\
|k(\tau,r) - k(q,r) - k(\tau,s) + k(q,s)| &\lesssim |q - r|^{-\gamma-\eta} |\tau - q|^{\eta}.
\end{aligned}$$

In the sequel a kernel fulfilling condition the Hypothesis 2.2.1 will be called Volterra kernel of order γ.

Remark 2.2.2. We limit our investigations in this article to the case of real valued Volterra kernels k for conciseness. The Volterra sewing lemma, and most results relating to Volterra rough paths are however easily extended to general Volterra kernels $k : \Delta_2 \to \mathcal{L}(E)$ for some Banach space E, by appropriate change of the bounds in 2.2.1, see e.g. [19], [20] where the Volterra sewing lemma from [1] is readily applied in an infinite dimensional setting.

As mentioned in the introduction, one of the key ingredients in [1] is to consider processes $(t,\tau) \mapsto z_t^\tau$ indexed by Δ_2 (where we recall that the simplex Δ_n was defined in (1.3)). We begin this section with a recollection of the Hölder space containing such processes and introduce the Volterra sewing Lemma 2.2.11, we will then move on to introduce the convolution product and discuss its relation with the Volterra signature.

2.2.1 The space of Volterra paths

We begin this section by recalling the topology used to measure the regularity of processes like (1.4), and give a simple motivation for the introduction of this type of space. Before defining the proper spaces quantifying this type of regularity, let us introduce a notation:

Notation 2.2.3. *Let* $(\alpha, \gamma) \in (0,1)^2$ *be such that* $\alpha > \gamma$*. For* $(s,t,\tau) \in \Delta_3$*, we set*

$$\psi^1_{\alpha,\gamma}(\tau,t,s) = \left[|\tau - t|^{-\gamma}|t-s|^{\alpha}\right] \wedge |t-s|^{\alpha-\gamma}. \tag{2.1}$$

Considering two additional parameters $\zeta \in [0, \alpha-\gamma)$ *and* $\eta \in [\zeta, 1]$*, we also set*

$$\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(\tau,\tau',t,s) = |\tau-\tau'|^{\eta}|\tau'-t|^{-(\eta-\zeta)}\left(\left[|\tau'-t|^{-\gamma-\zeta}|t-s|^{\alpha}\right] \wedge |t-s|^{\alpha-\gamma-\zeta}\right) \tag{2.2}$$

We are now ready to introduce some functional spaces called $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$, which are also used in the definition of $\mathcal{V}^{(\alpha,\gamma)}$ in [1]. Those spaces are natural function sets when dealing with Volterra type regularities.

Definition 2.2.4. *Let* E *be a Banach space, and consider* $(\alpha,\gamma) \in (0,1)^2$ *with* $\alpha - \gamma > 0$*, and* $\zeta \in [0, \alpha-\gamma)$*,* $\eta \in [\zeta, 1]$*. We define the space of Volterra paths of index* $(\alpha,\gamma,\eta,\zeta)$*, denoted by* $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2; E)$*, as the set of functions* $z : \Delta_2 \to E$*, given by* $(t,\tau) \mapsto z^\tau_t$*, with the condition* $z^\tau_0 = z_0 \in E$ *for all* $\tau \in (0,T]$*, and satisfying*

$$\|z\|_{(\alpha,\gamma,\eta,\zeta)} = \|z\|_{(\alpha,\gamma),1} + \|z\|_{(\alpha,\gamma,\eta,\zeta),1,2} < \infty. \tag{2.3}$$

Recalling Notation 2.2.3, the 1-norms and (1,2)-norms in (2.3) are respectively defined as follows:

$$\|z\|_{(\alpha,\gamma),1} = \sup_{(s,t,\tau)\in\Delta_3} \frac{|z^\tau_{ts}|}{\psi^1_{\alpha,\gamma}(\tau,t,s)}, \tag{2.4}$$

$$\|z\|_{(\alpha,\gamma,\eta,\zeta),1,2} = \sup_{(s,t,\tau',\tau)\in\Delta_4} \frac{|z^{\tau\tau'}_{ts}|}{\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(\tau,\tau',t,s)}, \tag{2.5}$$

with the convention $z^\tau_{ts} = z^\tau_t - z^\tau_s$ *and* $z^{\tau\tau'}_s = z^\tau_s - z^{\tau'}_s$*. Notice that under the mapping*

$$z \mapsto |z_0| + \|z\|_{(\alpha,\gamma,\eta,\zeta)},$$

the space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2; E)$ *is a Banach space.*

Whenever the domain and codomain is otherwise clear from the context, we will simply write $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)} := \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_n; E)$. Throughout the article, we will typically let the Banach space E be given by $\mathbb{R}^d$ or $\mathcal{L}(\mathbb{R}^d)$.

Remark 2.2.5. As will be proved in Theorem 2.2.12 below, the typical example of path in $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ is given by z_t^τ defined as in (1.4), with suitable assumption on k and x. Note also that $\mathsf{c}^\alpha([0,1];\mathbb{R}^d) \subset \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2([0,1]);\mathbb{R}^d)$ for any $\gamma \in [0,1)$. Indeed, for a path $x \in \mathsf{c}^\alpha$, define $z_t^\tau = x_t$. Using that $|t-s|^\alpha \leq |\tau-s|^\alpha$, it is readily checked that $|z_{ts}^\tau| \lesssim |\tau-t|^{-\gamma}|t-s|^\alpha \wedge |\tau-s|^\alpha$. Furthermore, $z_{ts}^{\tau\tau'} = 0$, and thus $\|z\|_{(\alpha,\gamma,\eta,\zeta)} < \infty$ for any $\gamma \in (0,1)$.

Remark 2.2.6. We will also consider functions $u : \Delta_3 \to \mathbb{R}^d$, which, with a slight abuse of notation, will be denoted by the mapping $(s,t,\tau) \mapsto u_{ts}^\tau$. We then define the space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^d)$ analogously as in Definition 2.2.4, but where the increments of the path $(t,\tau) \mapsto z_t^\tau$ in the lower variable, appearing in (2.4) and (2.5), is simply replaced by the evaluation u_{ts}^τ and $u_{ts}^\tau - u_{ts}^{\tau'}$ respectively.

Remark 2.2.7. Similarly as for the classical Hölder spaces, we have the following elementary embedding: for $\beta < \alpha \in (0,1)$, with $\beta - \gamma > 0$, we have $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)} \hookrightarrow \mathcal{V}^{(\beta,\gamma,\eta,\zeta)}$. Indeed, suppose $y \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$, it is readily checked that

$$|y_{ts}^\tau| \lesssim |\tau-t|^{-\gamma}|t-s|^\alpha| \wedge |\tau-s|^{\alpha-\gamma} \leq T^{\alpha-\beta}(|\tau-t|^{-\gamma}|t-s|^\beta \wedge |\tau-s|^{\beta-\gamma}),$$

and thus $\|y\|_{(\beta,\gamma),1} \leq T^{\alpha-\beta}\|y\|_{(\alpha,\gamma),1}$. Similarly, one can also show that $\|y\|_{(\beta,\gamma,\eta,\zeta),1,2} \leq T^{\alpha-\beta}\|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}$, and thus $\|y\|_{(\beta,\gamma,\eta,\zeta)} \leq T^{\alpha-\beta}\|y\|_{(\alpha,\gamma,\eta,\zeta)}$.

The following lemma gives useful embedding results for $\mathcal{V}^{(\alpha,\gamma)}$ related to variations in the singularity parameter γ.

Lemma 2.2.8. *Let $\alpha,\gamma,\eta,\zeta \in (0,1)$ with $\alpha > \gamma$, $\zeta \in [0,\alpha-\gamma)$, $\eta \in [\zeta,1]$, and recall that $\rho = \alpha - \gamma$. Then for the spaces $\mathcal{V}^{(\alpha,\gamma)}$ given in Definition 2.2.4, the following inclusion holds true:*

$$\mathcal{V}^{(3\rho+\gamma,\gamma,\eta,\zeta)} \subset \mathcal{V}^{(3\rho+2\gamma,2\gamma,\eta,\zeta)} \subset \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}. \tag{2.6}$$

Proof. We will prove the second relation: $\mathcal{V}^{(3\rho+2\gamma,2\gamma,\eta,\zeta)} \subset \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$, the first relation being proved in a similar way. Moreover, in order to prove that $\mathcal{V}^{(3\rho+2\gamma,2\gamma,\eta,\zeta)} \subset \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$, we will show that $\|z\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)} \leq \|z\|_{(3\rho+2\gamma,2\gamma,\eta,\zeta)}$, for the $(\alpha,\gamma,\eta,\zeta)-$norms introduced in Definition 2.2.4. Also recall that the $(\alpha,\gamma,\eta,\zeta)-$norms are defined by (2.4) and (2.5). For sake of conciseness we will just prove that

$$\|z\|_{(3\rho+3\gamma,3\gamma),1} \leq \|z\|_{(3\rho+2\gamma,2\gamma),1}, \tag{2.7}$$

and leave the similar bound for the $(1,2)-$norm to the reader.

In order to prove (2.7), we refer again to (2.4). From this definition, it is readily checked that (2.7) can be reduced to prove the following relation:

$$|\tau-t|^{-3\gamma}|t-s|^{3\rho+3\gamma} \wedge |\tau-s|^{3\rho} \lesssim |\tau-t|^{-2\gamma}|t-s|^{3\rho+2\gamma} \wedge |\tau-s|^{3\rho}. \tag{2.8}$$

The proof of (2.8) will be split in 2 cases, according to the respective values of $|\tau-t|$ and $|t-s|$. In the sequel C_1 designates a strictly positive constant.

Case 1: $|\tau-t| \leq C_1|t-s|$. Let us write

$$|\tau-s|^{3\rho} = |\tau-s|^{3\rho+2\gamma}|\tau-s|^{-2\gamma}.$$

Then if $|\tau-t| \leq C_1|t-s|$, one has $|\tau-s|^{3\rho+2\gamma} = |\tau-t+t-s|^{3\rho+2\gamma} \lesssim |t-s|^{3\rho+2\gamma}$. Hence we get

$$|\tau-s|^{3\rho} \lesssim |t-s|^{3\rho+2\gamma}|\tau-s|^{-2\gamma} \lesssim |t-s|^{3\rho+2\gamma}|\tau-t|^{-2\gamma}. \tag{2.9}$$

Relation (2.8) is then immediately seem from (2.9).

Case 2: $|\tau-t| > C_1|t-s|$. In this case write

$$|t-s|^{3\rho+2\gamma}|\tau-t|^{-2\gamma} = |t-s|^{3\rho+3\gamma}|\tau-t|^{-3\gamma}\left(\frac{|\tau-t|}{|t-s|}\right)^{\gamma}.$$

Then resort to the fact that $|\tau-t| \geq C_1|t-s|$ in order to get $|\tau-t|^{\gamma}|t-s|^{-\gamma} \geq C_1^{\gamma}$. This yields

$$|t-s|^{3\rho+2\gamma}|t-s|^{-2\gamma} \gtrsim |t-s|^{3\rho+3\gamma}|\tau-t|^{-3\gamma},$$

from which (2.8) is readily checked.

Combining Case 1 and Case 2, we have thus finished the proof of (2.8). As mentioned above, this implies that (2.7) is true and achieves our claim (2.6). □

2.2.2 Volterra Sewing lemma

We begin with a recollection of the space of abstract Volterra integrands, to which the Volterra sewing Lemma 2.2.11 will apply. The typical path in this space exhibits different types of regularities/singularities in its arguments, similarly to Definition 2.2.4. As a necessary ingredient in the subsequent definition we introduce a particular notation, which will frequently be used throughout the article.

Notation 2.2.9. *Recall that the simplex Δ_n is defined by (1.3). For a path $g : \Delta_2 \to \mathbb{R}^d$ and $(s, u, t) \in \Delta_3$, we set*

$$\delta_u g_{ts} = g_{ts} - g_{tu} - g_{us} \tag{2.10}$$

We will consider δ as an operator from $\mathcal{C}(\Delta_2)$ to $\mathcal{C}(\Delta_3)$, where $\mathcal{C}(\Delta_n)$ denotes the spaces of continuous functions on Δ_n.

Definition 2.2.10. *Let $\alpha \in (0,1)$, $\gamma \in (0,1)$ with $\alpha - \gamma > 0$, and $\zeta \in [0, \alpha - \gamma)$, $\eta \in [\zeta, 1]$. We also consider two coefficients $\kappa \in (0,\infty)$ and $\beta \in (1,\infty)$. Denote by $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}\left(\Delta_3; \mathbb{R}^d\right)$, the space of all functions $\Xi : \Delta_3 \to \mathbb{R}^d$ such that*

$$\|\Xi\|_{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)} = \|\Xi\|_{(\alpha,\gamma,\eta,\zeta)} + \|\delta\Xi\|_{(\beta,\kappa,\eta,\zeta)} < \infty, \tag{2.11}$$

where δ is introduced in (2.10) and the norm $\|\Xi\|_{(\alpha,\gamma,\eta,\zeta)}$ is given by (2.3) (see also Remark 2.2.6). Similar to Definition 2.2.4, the quantity $\|\delta\Xi\|_{(\beta,\kappa,\eta,\zeta)}$ is defined by

$$\|\delta\Xi\|_{(\beta,\kappa,\eta,\zeta)} = \|\delta\Xi\|_{(\beta,\kappa),1} + \|\delta\Xi\|_{(\beta,\kappa,\eta,\zeta),1,2}, \tag{2.12}$$

and the 1-norms and (1,2)-norms in (2.12) are respectively defined as follows:

$$\|\delta \Xi\|_{(\beta,\kappa),1} := \sup_{(s,m,t,\tau)\in\Delta_4} \frac{|\delta_m \Xi^\tau_{ts}|}{\psi^1_{\beta,\kappa}(\tau,t,s)}, \tag{2.13}$$

$$\|\delta \Xi\|_{(\beta,\kappa,\eta,\zeta),1,2} := \sup_{(s,m,t,\tau',\tau)\in\Delta_5} \frac{|\delta_m \Xi^{\tau\tau'}_{ts}|}{\psi^{1,2}_{\beta,\kappa,\eta,\zeta}(\tau,\tau',t,s)}, \tag{2.14}$$

where ψ^1 and $\psi^{1,2}$ are given in Notation 2.2.3. In the sequel the space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}$ will be our space of abstract Volterra integrands.

With these two Volterra spaces in hand, we are ready to recall the Volterra sewing Lemma which can be found, together with a full proof, in [1, Lemma 21].

Lemma 2.2.11. *Consider six exponents $\beta \in (1,\infty)$, $\kappa \in (0,1)$, $\alpha \in (0,1)$, $\gamma \in (0,1)$, $\eta \in [0,1]$ and $\zeta \in [0,1]$ such that $\beta - \kappa \geq \alpha - \gamma > 0$, $0 \leq \zeta < \alpha - \gamma$, and $\zeta \leq \eta \leq 1$. Let $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}$ and $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ be the spaces given in Definition 2.2.10 and Definition 2.2.4 respectively. Then there exists a linear continuous map $\mathcal{I} : \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}\left(\Delta_3;\mathbb{R}^d\right) \to \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}\left(\Delta_2;\mathbb{R}^d\right)$ such that the following holds true.*

(i) The quantity $\mathcal{I}(\Xi^\tau)_{ts} := \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} \Xi^\tau_{vu}$ exists for all $(s,t,\tau)\in\Delta_3$, where $\mathcal{P}$ is a generic partition of $[s,t]$ and $|\mathcal{P}|$ denotes the mesh size of the partition. Furthermore, we define $\mathcal{I}(\Xi^\tau)_t := \mathcal{I}(\Xi^\tau)_{t0}$, and we have that $\mathcal{I}(\Xi^\tau)_{ts} = \mathcal{I}(\Xi^\tau)_t - \mathcal{I}(\Xi^\tau)_s$.

(ii) Recalling the Notation 2.2.3 of ψ^1 and $\psi^{1,2}$, for all $(s,t,\tau)\in\Delta_3$ we have

$$|\mathcal{I}\left(\Xi^\tau\right)_{ts} - \Xi^\tau_{ts}| \lesssim \|\delta\Xi\|_{(\beta,\kappa),1}\,\psi^1_{\beta,\kappa}(\tau,t,s), \tag{2.15}$$

while for $(s,t,\tau',\tau)\in\Delta_4$ we get

$$\left|\mathcal{I}(\Xi^{\tau\tau'})_{ts} - \Xi^{\tau\tau'}_{ts}\right| \lesssim \|\delta\Xi\|_{(\beta,\kappa,\eta,\zeta),1,2}\,\psi^{1,2}_{\beta,\kappa,\eta,\zeta}(\tau,\tau',t,s). \tag{2.16}$$

Lemma 2.2.11 is applied in [1] in order to get the construction of the path $(t,\tau)\mapsto z^\tau_t$ introduced in (1.4). We recall this result here, since z is at the heart of our future considerations.

Theorem 2.2.12. *Let $x \in \mathcal{C}^\alpha$ and k be a Volterra kernel of order $-\gamma$ satisfying Hypothesis 2.2.1, such that $\rho = \alpha - \gamma > 0$. We define an element $\Xi^\tau_{ts} = k(\tau, s)x_{ts}$. Then the following holds true:*

(i) There exists some coefficients $\beta > 1$, $\kappa > 0$, $0 \le \eta \le 1$ and $0 \le \zeta \le 1$ with $\beta - \kappa = \alpha - \gamma$, $0 \le \zeta < \alpha - \gamma$ and $\zeta \le \eta \le 1$such that $\Xi \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}$, where $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\eta,\zeta)}$ is given in Definition 2.2.10. It follows that the element $\mathcal{I}(\Xi^\tau)$ obtained in Lemma 2.2.11 is well defined as an element of $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ and we set $z^\tau_{ts} \equiv \mathcal{I}(\Xi^\tau)_{ts} = \int_s^t k(\tau, r)dx_r$.

(ii) According to the Notation 2.2.3 of ψ^1 and $\psi^{1,2}$, for $(s, t, \tau) \in \Delta_3$ z satisfies the bound

$$|z^\tau_{ts} - k(\tau, s)x_{ts}| \lesssim \psi^1_{\alpha,\gamma}(\tau, t, s),$$

and in particular it holds that $\|z\|_{(\alpha,\gamma),1} < \infty$.

(iii) For any $\eta \in [0, 1]$ and any $(s, t, q, p) \in \Delta_4$ we have

$$|z^{pq}_{ts}| \lesssim \psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(p, q, t, s),$$

where $z^{pq}_{ts} = z^p_t - z^q_t - z^p_s + z^q_s$. In particular it holds that $\|z\|_{(\alpha,\gamma,\eta,\zeta),1,2} < \infty$.

Remark 2.2.13. Thanks to Theorem 2.2.12, we know that a typical example of a Volterra path in $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ is given by the integral $\int_s^t k(\tau, r)dx_r$, as mentioned in Remark 2.2.5.

2.2.3 Convolution product in the rough case $\alpha - \gamma > \frac{1}{3}$

A second crucial ingredient in the Volterra formalism put forward in [1] is the notion of convolution product. In this section we show how this mechanism is introduced for first and second order convolutions, where we recall that second order convolutions were enough to handle the case $\rho = \alpha - \gamma > \frac{1}{3}$ in [1].
Let us first introduce a piece of notation which will prevail throughout the paper.

Notation 2.2.14. *In the sequel we will often consider products of the form* $y_s z_{ts}^{\tau}$*, where* y *and* z^{τ} *are increments lying respectively in* $\mathsf{c}([0,T])$ *and* $\mathsf{c}(\Delta_2)$*. For algebraic reasons due to our rough Volterra formalism, we will write this product as*

$$[(z_{ts}^{\tau})^{\mathsf{T}}\ y_s^{\mathsf{T}}]^{\mathsf{T}} \tag{2.17}$$

For obvious notational reason, we will simply abbreviate (2.17) into

$$z_{ts}^{\tau}\, y_s$$

In the same way, products of 3 (or more) elements of the form $f'(y_s)y_s z_{ts}^{\tau}$ *will be denoted as* $z_{ts}^{\tau} y_s f'(y_s)$ *without further notice.*

We now recall how the convolution with respect to z^{τ} is obtained, borrowing the following proposition from [1, Theorem 25].

Proposition 2.2.15. *We consider two Volterra paths* $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^d)$ *and* $y \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\mathcal{L}(\mathbb{R}^d))$ *as given in Definition 2.2.4, where we recall that* $\alpha,\gamma,\eta,\zeta \in (0,1)$ *with* $\rho = \alpha - \gamma > 0$, $0 \le \zeta < \rho$*, and* $\zeta \le \eta \le 1$*. Then the convolution product of the two Volterra paths* y *and* z *is a bilinear operation on* $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^d)$ *given by*

$$z_{tu}^{\tau} * y_{us}^{\cdot} = \int_{t>r>u} dz_r^{\tau} y_{us}^{r} := \lim_{|\mathcal{P}|\to 0} \sum_{[u',v']\in\mathcal{P}} z_{v'u'}^{\tau} y_{us}^{u'}. \tag{2.18}$$

The integral in (2.18) is understood as a Volterra-Young integral for all $(s,u,t,\tau) \in \Delta_4$*. Moreover, the following two inequalities holds for any* $(s,u,t,\tau,\tau') \in \Delta_5$*:*

$$|z_{tu}^{\tau} * y_{us}^{\cdot}| \lesssim \|z\|_{(\alpha,\gamma),1} \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}\, \psi^{1}_{(2\rho+\gamma),\gamma}(\tau,t,s), \tag{2.19}$$

$$\left|z_{tu}^{\tau'\tau} * y_{us}^{\cdot}\right| \lesssim \|z\|_{(\alpha,\gamma,\eta,\zeta),1,2} \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}\, \psi^{1,2}_{(2\rho+\gamma),\gamma,\eta,\zeta}(\tau,\tau',t,s), \tag{2.20}$$

where ψ^1 *and* $\psi^{1,2}$ *are given in Notation 2.2.3.*

In addition to Proposition 2.2.15, the rough Volterra formalism relies on a stack of iterated integrals verifying convolutional type algebraic identities. Thanks to Proposition 2.2.15 we can now state the main assumption about this stack of integrals, which should be seen as the equivalent of Chen's relation in our Volterra context.

Hypothesis 2.2.16. *Let $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ be a Volterra path as given in Definition 2.2.4. For n such that $(n+1)\rho + \gamma > 1$, we assume that there exists a family $\{\mathbf{z}^{\mathrm{j},\tau}; \mathrm{j} \leq n\}$ such that $\mathbf{z}^{\mathrm{j},\tau}_{ts} \in (\mathbb{R}^m)^{\otimes \mathrm{j}}$, $\mathbf{z}^1 = z$ and verifying*

$$\delta_u \mathbf{z}^{\mathrm{j},\tau}_{ts} = \sum_{\mathrm{i}=1}^{\mathrm{j}-1} \mathbf{z}^{\mathrm{j}-\mathrm{i},\tau}_{tu} * \mathbf{z}^{\mathrm{i},\cdot}_{us} = \int_s^t d\mathbf{z}^{\mathrm{j}-\mathrm{i},\tau}_{tr} \otimes \mathbf{z}^{\mathrm{i},r}_{us}, \tag{2.21}$$

where the right hand side of (2.21) is defined in Proposition 2.2.15. In addition, we suppose that for $\mathrm{j} = 1, \ldots, n$ we have $\mathbf{z}^{\mathrm{j}} \in \mathcal{V}^{(\mathrm{j}\rho+\gamma,\gamma,\eta,\zeta)}$.

The last notation we need to recall from [1] is the concept of second order convolution product. To this aim, we first introduce some basic notation about increments.

Notation 2.2.17. *We will denote by $u^{1,2}$ a function $u : \Delta_3 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 2}, \mathbb{R}^d)$ with two upper indices, namely,*

$$\Delta_3 \ni (s, \tau_1, \tau_2) \mapsto u^{\tau_2,\tau_1}_s \in \mathbb{R}^d.$$

The notation $u^{1,2}$ highlights the order of integration in future computations.

We now specify the kind of topology we will consider for functions of the form $u^{1,2}$.

Definition 2.2.18. *Let $\mathcal{W}^{(\alpha,\gamma,\eta,\zeta)}_2$ denote the space of functions $u : \Delta_3 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 2}, \mathbb{R}^d)$ with a fixed initial condition $u^{p,q}_0 = u_0$, endowed with the norm*

$$\left\| u^{1,2} \right\|_{(\alpha,\gamma,\eta,\zeta)} := \left\| u^{1,2} \right\|_{(\alpha,\gamma),1} + \left\| u^{1,2} \right\|_{(\alpha,\gamma,\eta,\zeta),1,2}. \tag{2.22}$$

The right hand side of (2.22) is defined as follows, recalling the convention $\rho = \alpha - \gamma$ and the definition (2.1) of ψ^1:

$$\left\| u^{1,2} \right\|_{(\alpha,\gamma),1} := \sup_{(s,t,\tau)\in\Delta_3} \frac{|u^{\tau,\tau}_{ts}|}{\psi^1_{\alpha,\gamma}(\tau,t,s)}, \tag{2.23}$$

and

$$\left\| u^{1,2} \right\|_{(\alpha,\gamma,\eta,\zeta),1,2} := \left\| u^{1,2} \right\|_{(\alpha,\gamma,\eta,\zeta),1,2,>} + \left\| u^{1,2} \right\|_{(\alpha,\gamma,\eta,\zeta),1,2,<}, \tag{2.24}$$

where the norms $\|u^{1,2}\|_{(\alpha,\gamma,\eta,\zeta),1,2,>}$ *and* $\|u^{1,2}\|_{(\alpha,\gamma,\eta,\zeta),1,2,<}$ *are respectively defined by*

$$\left\|u^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,>} = \sup_{(s,t,r_1,r_2,r')\in\Delta_5} \frac{|u_{ts}^{r',r_2} - u_{ts}^{r',r_1}|}{h_{\eta,\zeta}(s,t,r_1,r_2,r')}, \tag{2.25}$$

$$\left\|u^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,<} = \sup_{(s,t,r',r_1,r_2)\in\Delta_5} \frac{|u_{ts}^{r_2,r'} - u_{ts}^{r_1,r'}|}{h_{\eta,\zeta}(s,t,r_1,r_2,r')}, \tag{2.26}$$

where the function h *is defined by*

$$\begin{aligned} h_{\eta,\zeta}\left(s,t,r_1,r_2,r'\right) &= |r_2 - r_1|^{\eta} \left|\min(r_1,r_2,r') - t\right|^{-\eta+\zeta} \\ &\times \left(\left[\left|\min(r_1,r_2,r') - t\right|^{-\gamma-\zeta} |t-s|^{\alpha}\right] \wedge \left|\min(r_1,r_2,r') - s\right|^{\alpha-\gamma-\zeta}\right). \end{aligned} \tag{2.27}$$

Remark 2.2.19. In the sequel we will need to estimate differences of functions $u^{\cdot,\cdot} : \Delta_3 \to \mathcal{L}((\mathbb{R}^m)^{\otimes 2}, \mathbb{R}^m)$ of the form $|u_t^{\tau,q} - u_t^{\tau,p}|$. Those differences can be handled thanks to Definition 2.2.18 as follows:

$$\begin{aligned} |u_t^{\tau,q} - u_t^{\tau,p}| &\le |u_0^{\tau,q} - u_0^{\tau,p}| + |u_{t0}^{\tau,q} - u_{t0}^{\tau,p}| \\ &\le \|u\|_{(\alpha,\gamma),1,2} |q-p|^{\eta} |p-t|^{-\eta+\zeta} \left(\left[|p-t|^{-\gamma-\zeta} |t|^{\alpha}\right] \wedge |p|^{\rho-\zeta}\right). \end{aligned} \tag{2.28}$$

Since $\zeta \in [0,\rho)$ and $\eta \in [\zeta, 1]$, then we can set $\eta = \zeta$, that is

$$|u_t^{\tau,q} - u_t^{\tau,p}| \lesssim \|u\|_{(\alpha,\gamma),1,2} |q-p|^{\zeta} \lesssim \|u\|_{(\alpha,\gamma),1,2}.$$

we also have, for any $\tau \in [0,T]$,

$$|u_t^{\tau,\tau} - u_0^{\tau,\tau}| \le \|u\|_{(\alpha,\gamma),1} \left[|\tau - t|^{-\gamma} |t|^{\alpha} \wedge |\tau|^{\rho}\right] \lesssim \|u\|_{(\alpha,\gamma),1}. \tag{2.29}$$

With the above definition at hand, we are now ready to recall the construction of second order convolution products in the rough case $\alpha - \gamma > \frac{1}{3}$.

Theorem 2.2.20. *Let $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ be as given in Definition 2.2.4 with $\alpha, \gamma, \eta, \zeta \in (0,1)$ satisfying $\rho = \alpha - \gamma > \frac{1}{3}$, $\zeta \in [0, \rho)$ and $\eta \in [\zeta, 1]$. We assume that $\mathbf{z}$ fulfills Hypothesis 2.2.16 with $n = 2$. Consider a function $y : \Delta_3 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 2}, \mathbb{R}^d)$ with $\|y^{1,2}\|_{(\alpha,\gamma,\eta,\zeta),1,2} < \infty$ and $y_0^{1,2} = y_0$, for a fixed initial condition $y_0 \in \mathcal{L}((\mathbb{R}^d)^{\otimes 2}, \mathbb{R}^d)$. For all fixed $(s, t, \tau) \in \Delta_3$ we have that*

$$\mathbf{z}_{ts}^{2,\tau} * y_s^{1,2} := \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \mathbf{z}_{vu}^{2,\tau} y_s^{u,u} + (\delta_u \mathbf{z}_{vs}^{2,\tau}) * y_s^{1,2} \tag{2.30}$$

is a well defined Volterra-Young integral. It follows that $*$ is a well defined bi-linear operation between the three parameters Volterra function $\mathbf{z}^2$ and a 3-parameter path y. Moreover, the following inequality holds

$$\begin{aligned} \left| \mathbf{z}_{ts}^{2,\tau} * y_s^{1,2} - \mathbf{z}_{ts}^{2,\tau} y_s^{s,s} \right| \lesssim & \|y^{1,2}\|_{(\alpha,\gamma,\eta,\zeta),1,2} \\ & \times \left(\|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma),1} + \|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2} \|\mathbf{z}^1\|_{(\alpha,\gamma),1} \right) \psi^1_{(2\rho+\gamma),\gamma}(\tau, t, s), \end{aligned} \tag{2.31}$$

where ψ^1 is given in (2.1)

Remark 2.2.21. By Hypothesis 2.2.16, the term $(\delta_u \mathbf{z}_{vs}^{2,\tau}) * y_s^{1,2}$ in the right hand side of (2.30) can be rewritten as

$$\mathbf{z}_{vu}^{1,\tau} * \mathbf{z}_{us}^{1,\cdot} * y_s^{1,2},$$

where the convolution with $\mathbf{z}^{1,\tau}$ is defined through (2.18) and the inside integral concerns the second variable in $y^{1,2}$. As an example, if k, x are smooth functions and $\mathbf{z}_{vs}^{1,\tau} = \int_s^v k(\tau, r) dx_r$, then this convolution is understood in the following way

$$\mathbf{z}_{vu}^{1,\tau} * \mathbf{z}_{us}^{1,\cdot} * y_s^{1,2} = \int_u^v k(\tau, r_1) dx_{r_1} \otimes \int_s^u k(r_1, r_2) dx_{r_2} y_s^{r_1, r_2}.$$

Remark 2.2.22. Recalling that $\rho = \alpha - \gamma$, notice that Proposition 2.2.15 and Theorem 2.2.20 tell us how to define the n'th order convolution products under the condition $\rho > \frac{1}{3}$. We will follow a similar strategy to define third order convolution products and construct our solution to equation (1.2) with $\rho > \frac{1}{4}$ in the subsequent section.

2.3 Volterra rough paths for $\alpha - \gamma > \frac{1}{4}$

This section is devoted to the generalization of the concepts introduced in Section 2.2 to accommodate the case of Volterra rough paths with regularity $\rho = \alpha - \gamma > \frac{1}{4}$. One of the main issues encountered in this direction is to define third order convolution structures. To this end, we will state a version of our Volterra sewing Lemma 2.2.11 extended to the case of two types of Volterra singularities.

2.3.1 Volterra sewing lemma with two singularities

With the aim of extending the Volterra sewing Lemma 2.2.11 with one singularity to an increment exhibiting two singularities, we first introduce a new space of abstract integrands.

Definition 2.3.1. *Let $\alpha, \gamma, \eta, \zeta \in (0,1)$ with $\rho = \alpha - \gamma > \frac{1}{3}$, $0 \leq \zeta < \rho$ and $\zeta \leq \eta \leq 1$. We also consider three coefficients (β, κ, θ), with $(\kappa + \theta) \in (0,1)$, $\beta \in (1, \infty)$, and $\beta - \kappa - \theta \geq \alpha - \gamma > 0$. Denote by $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\theta,\eta,\zeta)}(\Delta_4; \mathbb{R}^d)$, the space of all functions of the form $\Delta_4 \ni (v,s,t,\tau) \mapsto (\varXi_v^\tau)_{ts} \in \mathbb{R}^d$ such that the following norm is finite:*

$$\|\varXi\|_{\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\theta,\eta,\zeta)}} = \|\varXi\|_{(\alpha,\gamma,\eta,\zeta)} + \|\delta\varXi\|_{(\beta,\kappa,\theta,\eta,\zeta)}. \tag{3.1}$$

In equation (3.1), the operator δ is introduced in (2.10), the quantity $\|\varXi\|_{(\alpha,\gamma,\eta,\zeta)}$ is given by (2.3) and the term $\|\delta\varXi\|_{(\beta,\kappa,\theta,\eta,\zeta)}$ takes the double singularity into account. Namely we have

$$\|\delta\varXi\|_{(\beta,\kappa,\theta,\eta,\zeta)} = \|\delta\varXi\|_{(\beta,\kappa,\theta),1} + \|\delta\varXi\|_{(\beta,\kappa,\theta,\eta,\zeta),1,2},$$

where

$$\|\delta\varXi\|_{(\beta,\kappa,\theta),1} := \sup_{(v,s,m,t,\tau)\in\Delta_5} \frac{|\delta_m(\varXi_v^\tau)_{ts}|}{\phi^1_{\beta,\kappa,\theta}(\tau,t,s,v)}, \tag{3.2}$$

and the term $\|\delta \Xi\|_{(\beta,\kappa,\theta,\eta,\zeta),1,2}$ *is defined by*

$$\|\delta \Xi\|_{(\beta,\kappa,\theta,\eta,\zeta),1,2} := \sup_{(v,s,m,t,\tau',\tau)\in\Delta_6} \frac{\left|\delta_m(\Xi_v^{\tau\tau'})_{ts}\right|}{\phi^{1,2}_{\beta,\kappa,\theta,\eta,\zeta}(\tau,\tau',t,s,v)}, \tag{3.3}$$

where the function $\psi^1_{\beta,\kappa,\theta}(\tau,t,s,v)$ *and* $\psi^{1,2}_{\beta,\kappa,\theta,\eta,\zeta}(\tau,\tau',t,s,v)$ *are respectively given by*

$$\phi^1_{\beta,\kappa,\theta}(\tau,t,s,v) = \left[|\tau-t|^{-\kappa}|t-s|^{\beta}|s-v|^{-\theta}\right] \wedge |\tau-v|^{\beta-\kappa-\theta} \tag{3.4}$$

$$\phi^{1,2}_{\beta,\kappa,\theta,\eta,\zeta}(\tau,\tau',t,s,v) = |\tau-\tau'|^{\eta}|\tau'-t|^{-\eta+\zeta}\left(\left[|\tau'-t|^{-\kappa-\zeta}|t-s|^{\beta}|s-v|^{-\theta}\right]\wedge|\tau'-v|^{\beta-\kappa-\theta-\zeta}\right). \tag{3.5}$$

Notice that we will use $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\theta,\eta,\zeta)}$ *as a space of abstract Volterra integrands with a double singularity.*

With this new space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\theta,\eta,\zeta)}$ at hand, we are ready to state the Volterra sewing Lemma with two singularities alluded to above.

Lemma 2.3.2. *Consider seven exponents* $(\alpha,\gamma,\eta,\zeta)$, *and* $(\beta,\kappa,\theta,\eta,\zeta)$, *with* $\beta\in(1,\infty)$, $(\kappa+\theta)\in(0,1)$, $\alpha\in(0,1)$ *and* $\gamma\in(0,1)$ *such that* $\beta-\kappa-\theta\geq\alpha-\gamma>0$, $\zeta\in[0,\alpha-\gamma)$ *and* $\eta\in[\zeta,1]$. *Let* $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\theta,\eta,\zeta)}$ *and* $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ *be the spaces given in Definition 2.3.1 and Definition 2.2.4 respectively. Then there exists a linear continuous map* $\mathcal{I}:\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)(\beta,\kappa,\theta,\eta,\zeta)}\left(\Delta_4;\mathbb{R}^d\right)\to\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}\left(\Delta_3;\mathbb{R}^d\right)$ *such that the following holds true.*

(i) *The quantity* $\mathcal{I}(\Xi^\tau_v)_{ts} := \lim_{|\mathcal{P}|\to0}\sum_{[u,w]\in\mathcal{P}}(\Xi^\tau_v)_{wu}$ *exists for all* $(v,s,t,\tau)\in\Delta_4$, *where* $\mathcal{P}$ *is a generic partition of* $[s,t]$ *and* $|\mathcal{P}|$ *denotes the mesh size of the partition. Furthermore, we define* $\mathcal{I}(\Xi^\tau_v)_t := \mathcal{I}(\Xi^\tau_v)_{t0}$, *and have* $\mathcal{I}(\Xi^\tau_v)_{ts} = \mathcal{I}(\Xi^\tau_v)_{t0}-\mathcal{I}(\Xi^\tau_v)_{s0}$.

(ii) *For all* $(v,s,t,\tau)\in\Delta_4$ *we have*

$$|\mathcal{I}(\Xi^\tau_v)_{ts}-(\Xi^\tau_v)_{ts}| \lesssim \|\delta\Xi\|_{(\beta,\kappa,\theta),1}\,\phi^1_{\beta,\kappa,\theta}(\tau,t,s,v), \tag{3.6}$$

while for $(v,s,t,\tau',\tau)\in\Delta_5$ *we get*

$$\left|\mathcal{I}(\Xi_v^{\tau\tau'})_{ts}-(\Xi_v^{\tau\tau'})_{ts}\right|\lesssim\|\delta\Xi\|_{(\beta,\kappa,\theta),1,2}\,\phi^{1,2}_{\beta,\kappa,\theta,\eta,\zeta}(\tau,\tau',t,s,v),\tag{3.7}$$

where ϕ^1 *and* $\phi^{1,2}$ *are the functions given by* (3.4) *and* (3.5).

Proof. This is an extension of [1, Lemma 21]. Let us consider the n-th order dyadic partition $\mathcal{P}^n$ of $[s,t]$ where each set $[u,w]\in\mathcal{P}^n$ has length $2^{-n}|t-s|$. We define the n-th order Riemann sum of Ξ_v^τ, denoted $\mathcal{I}^n(\Xi_v^\tau)_{ts}$, as follows

$$\mathcal{I}^n(\Xi_v^\tau)_{ts}=\sum_{[u,w]\in\mathcal{P}^n}(\Xi_v^\tau)_{wu}\,.$$

Our aim is to show that the sequence $\{\mathcal{I}^n(\Xi_v^\tau);n\geq1\}$ converges to an element $\mathcal{I}(\Xi_v^\tau)$ which fulfills relation (3.6). To this aim we begin to consider the difference $\mathcal{I}^{n+1}(\Xi_v^\tau)-\mathcal{I}^n(\Xi_v^\tau)$. A series of elementary computations reveals that

$$\mathcal{I}^{n+1}(\Xi_v^\tau)_{ts}-\mathcal{I}^n(\Xi_v^\tau)_{ts}=-\sum_{[u,w]\in\mathcal{P}^n}\delta_m(\Xi_v^\tau)_{wu},\tag{3.8}$$

where $m=\frac{w+u}{2}$ and where we recall that δ is given by relation (2.10). Plugging relation (3.2) into (3.8), it is easy to check that

$$\sum_{[u,w]\in\mathcal{P}^n}|\delta_m(\Xi_v^\tau)_{wu}|\lesssim\|\delta\Xi\|_{(\beta,\kappa,\theta),1}\sum_{[u,w]\in\mathcal{P}^n}|\tau-w|^{-\kappa}|u-v|^{-\theta}|w-u|^{\beta}.\tag{3.9}$$

We will upper bound the right hand side above. Invoking the fact that $\beta>1$ and $|w-u|=2^{-n}|t-s|$, for $u,w\in\mathcal{P}^n$ we write

$$\sum_{[u,w]\in\mathcal{P}^n}|\tau-w|^{-\kappa}|u-v|^{-\theta}|w-u|^{\beta}\leq2^{-n(\beta-1)}|t-s|^{\beta-1}\sum_{[u,w]\in\mathcal{P}^n}|\tau-w|^{-\kappa}|u-v|^{-\theta}|w-u|.\tag{3.10}$$

With the definition of Riemann sums in mind, the term

$$\sum_{[u,w]\in\mathcal{P}^n}|\tau-w|^{-\kappa}|u-v|^{-\theta}|w-u|$$

in the right hand side of (3.10) can be dominated by the following finite integral (recall that $\kappa + \theta < 1$):

$$\int_s^t |\tau - x|^{-\kappa} |x - v|^{-\theta} dx.$$

In addition, some elementary calculations show that the above integral can be upper bounded as follows,

$$\int_s^t |\tau - x|^{-\kappa} |x - v|^{-\theta} dx \lesssim |\tau - t|^{-\kappa} |s - v|^{-\theta} |t - s| \wedge |t - v|^{1-\kappa-\theta}. \tag{3.11}$$

Plugging the inequality (3.11) into (3.10), we thus get

$$\sum_{[u,w]\in\mathcal{P}^n} |\tau - w|^{-\kappa} |u - v|^{-\theta} |w - u|^{\beta} \\ \lesssim 2^{-n(\beta-1)} \left(\left[|\tau - t|^{-\kappa} |s - v|^{-\theta} |t - s|^{\beta} \right] \wedge |\tau - v|^{\beta-\kappa-\theta} \right).$$

Then taking (3.9) into account, relation (3.8) can be recast as

$$|\mathcal{I}^{n+1}(\Xi_v^\tau)_{ts} - \mathcal{I}^n(\Xi_v^\tau)_{ts}| \\ \lesssim 2^{-n(\beta-1)} \|\delta\Xi\|_{(\beta,\kappa,\theta),1} \left(\left[|\tau - t|^{-\kappa} |s - v|^{-\theta} |t - s|^{\beta} \right] \wedge |\tau - v|^{\beta-\kappa-\theta} \right). \tag{3.12}$$

Since $\beta > 1$, then (3.12) implies that the sequence $\{\mathcal{I}^n(\Xi_v^\tau); n \geq 1\}$ is Cauchy. It thus converges to a quantity $\mathcal{I}(\Xi_v^\tau)_{ts}$ which satisfies (3.6). The rest of this proof is the same as [8, Lemma 4.2], which means that the element $\mathcal{I}(\Xi_v^\tau)$ has finite $\|\cdot\|_{(\beta,\kappa,\theta),1}$ norm. The proof of relation (3.7) is very similar to (3.6), and left to the reader for sake of conciseness. We just define an increment $\Xi_v^{\tau,\tau'}$ instead of Ξ_v^τ and then proceed as in (3.8)-(3.12). The proof is now complete. □

2.3.2 Third order convolution products in the rough case $\alpha - \gamma > \frac{1}{4}$

In this section we establish a proper definition of third order convolution products. Let us first introduce the class of integrands we shall consider for those products.

Notation 2.3.3. *Similarly to Notation 2.2.17, we denote by $u^{1,2,3}$ a function $u : \Delta_4 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 3}, \mathbb{R}^d)$ given by*

$$(s, \tau_1, \tau_2, \tau_3) \mapsto u_s^{\tau_3,\tau_2,\tau_1}.$$

To motivate the upcoming analysis and in order to get a better intuition of what is meant by third order convolution products, let us first give a definition of the third order convolution product for smooth functions, and prove a useful relation for the construction of this convolution.

Definition 2.3.4. *Let x be a continuously differentiable function and consider a Volterra kernel k which fulfills Hypothesis 2.2.1 with $\gamma < 1$. Let also $f : \Delta_4 \to \mathcal{L}((\mathbb{R}^m)^{\otimes 3}, \mathbb{R}^m)$ be a smooth function given in Notation 2.3.3. Then recalling our Notation 2.2.14 for $\tau \geq t > s \geq v$ the convolution $\mathbf{z}_{ts}^{3,\tau} * f_v^{1,2,3}$ is defined by*

$$\mathbf{z}_{ts}^{3,\tau} * f_v^{1,2,3} = \int_{t>r_1>s} k(\tau, r_1) dx_{r_1} \otimes \int_{r_1>r_2>s} k(r_1, r_2) dx_{r_2} \otimes \int_{r_2>r_3>s} k(r_2, r_3) dx_{r_3} f_v^{r_1,r_2,r_3}. \tag{3.13}$$

Lemma 2.3.5. *Under the same conditions as in Definition 2.3.4, let $\mathbf{z}_{ts}^{3,\tau} * f_s^{1,2,3}$ be the increment given by (3.13). Consider $(s,t) \in \Delta_2$ and a generic partition $\mathcal{P}$ of $[s,t]$. Then we have*

$$\mathbf{z}_{ts}^{3,\tau} * f_s^{1,2,3} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbf{z}_{vu}^{3,\tau} * f_s^{1,2,3} + \left(\delta_u \mathbf{z}_{vs}^{3,\tau}\right) * f_s^{1,2,3}. \tag{3.14}$$

Proof. Starting from expression (3.13), it is readily seen that

$$\mathbf{z}_{ts}^{3,\tau} * f_s^{1,2,3} = \sum_{[u,v]\in\mathcal{P}} \int_{v>r_1>u} k(\tau, r_1) dx_{r_1} \otimes \int_{r_1>r_2>s} k(r_1, r_2) dx_{r_2} \otimes \int_{r_2>r_3>s} k(r_2, r_3) dx_{r_3} f_s^{r_1,r_2,r_3}.$$

Then for each $[u,v] \in \mathcal{P}$, divide the region $\{v > r_1 > u\} \cap \{r_1 > r_2 > r_3 > s\}$ into

$$\{v > r_1 > r_2 > r_3 > u\} \cup \{v > r_1 > r_2 > u > r_3 > s\} \cup \{v > r_1 > u > r_2 > r_3 > s\}.$$

This yields a decomposition of $\mathbf{z}_{ts}^{3,\tau} * f_s^{1,2,3}$ of the form

$$\mathbf{z}_{ts}^{2,\tau} * f_s^{1,2,3} = \sum_{[u,v]\in\mathcal{P}} A_{vu}^{\tau} + B_{vu}^{\tau} + C_{vu}^{\tau},$$

where A^{τ}_{vu} , B^{τ}_{vu} , and C^{τ}_{vu} are respectively given by

$$
\begin{aligned}
A^{\tau}_{vu} &= \int_{v>r_1>u} k(\tau, r_1) dx_{r_1} \otimes \int_{r_1>r_2>u} k(r_1, r_2) dx_{r_2} \otimes \int_{r_2>r_3>u} k(r_2, r_3) dx_{r_3} f^{r_1,r_2,r_3}_s \\
B^{\tau}_{vu} &= \int_{v>r_1>u} k(\tau, r_1) dx_{r_1} \otimes \int_{r_1>r_2>u} k(r_1, r_2) dx_{r_2} \otimes \int_{u>r_3>s} k(r_2, r_3) dx_{r_3} f^{r_1,r_2,r_3}_s \\
C^{\tau}_{vu} &= \int_{v>r_1>u} k(\tau, r_1) dx_{r_1} \otimes \int_{u>r_2>s} k(r_1, r_2) dx_{r_2} \otimes \int_{r_2>r_3>s} k(r_2, r_3) dx_{r_3} f^{r_1,r_2,r_3}_s .
\end{aligned}
$$

We recognize the term A^{τ}_{vu} as the expression $\mathbf{z}^{3,\tau}_{vu} * f^{1,2,3}_s$ given by Definition 3.13. Moreover, we can check that $B^{\tau}_{vu} = \mathbf{z}^{2,\tau}_{vu} * \mathbf{z}^{1,\cdot}_{us} * f^{1,2,3}_s$, and $C^{\tau}_{vu} = \mathbf{z}^{1,\tau}_{vu} * \mathbf{z}^{2,\cdot}_{us} * f^{1,2,3}_s$. Then since $\mathbf{z}^{3,\tau}$ satisfies (2.21), we have $B^{\tau}_{vu} + C^{\tau}_{vu} = (\delta_u \mathbf{z}^{3,\tau}_{vs}) * f^{1,2,3}_s$. This finishes the proof of our claim (3.14). □

In order to generalize the notion of convolution product beyond the scope of Definition 2.3.4 to accommodate rough signals x, let us introduce the kind of norm we shall consider for processes with 3 upper variables of the form $u^{1,2,3}$, and in that connection introduce another Volterra-Hölder space equipped with this new norm.

Definition 2.3.6. *Let $\mathcal{W}^{(\alpha,\gamma,\eta,\zeta)}_3$ denote the space of functions $u : \Delta_4 \to \mathcal{L}((\mathbb{R}^d)^{\otimes 3}, \mathbb{R}^d)$ as given in Notation 2.3.3 with $u^{\tau_1,\tau_2,\tau_3}_0 = u_0 \in \mathcal{L}((\mathbb{R}^d)^{\otimes 3}, \mathbb{R}^d)$ and such that $\|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta)} < \infty$, where the norm $\|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta)}$ is defined by*

$$
\left\| u^{1,2,3} \right\|_{(\alpha,\gamma,\eta,\zeta)} := \left\| u^{1,2,3} \right\|_{(\alpha,\gamma),1} + \left\| u^{1,2,3} \right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} . \tag{3.15}
$$

More specifically, recalling the definition (2.1) for ψ^1,the $\|\cdot\|_{(\alpha,\gamma),1}$ and $\|\cdot\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$ norms in (3.15) are respectively defined by

$$
\left\| u^{1,2,3} \right\|_{(\alpha,\gamma),1} := \sup_{(s,t,\tau)\in\Delta_3} \frac{|u^{\tau,\tau,\tau}_{ts}|}{\psi^1_{\alpha,\gamma}(\tau,t,s)}, \tag{3.16}
$$

and

$$
\left\| u^{1,2,3} \right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} := \left\| u^{1,2,3} \right\|_{(\alpha,\gamma,\eta,\zeta),1,2} + \left\| u^{1,2,3} \right\|_{(\alpha,\gamma,\eta,\zeta),1,3} + \left\| u^{1,2,3} \right\|_{(\alpha,\gamma,\eta,\zeta),2,3} . \tag{3.17}
$$

In the right hand side of (3.17), similarly to (2.25)-(2.26), we have set $\|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2}$ *as the sum* $\|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,>} + \|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,<}$*, with*

$$\left\|u^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,>} = \sup_{(s,t,r,r_1,r_2,r')\in\Delta_6} \frac{|u_{ts}^{r',r_2,r} - u_{ts}^{r',r_1,r}|}{h_{\eta,\zeta}(s,t,r_1,r_2,r,r')}, \tag{3.18}$$

$$\left\|u^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,<} = \sup_{(s,t,r,r',r_1,r_2)\in\Delta_6} \frac{|u_{ts}^{r_2,r',r} - u_{ts}^{r_1,r',r}|}{h_{\eta,\zeta}(s,t,r_1,r_2,r,r')}. \tag{3.19}$$

Here we define h as follows:

$$\begin{aligned} h_{\eta,\zeta}(s,t,r_1,r_2,r,r') &= |r_2 - r_1|^{\eta} \left|\min(r_1,r_2,r,r') - t\right|^{-\eta+\zeta} \\ &\times \left(\left[\left|\min(r_1,r_2,r,r') - t\right|^{-\gamma-\zeta} |t-s|^{\alpha}\right] \wedge \left|\min(r_1,r_2,r,r') - s\right|^{\alpha-\gamma-\zeta}\right). \end{aligned} \tag{3.20}$$

Moreover, the norms $\|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),2,3}$ *and* $\|u^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,3}$ *in (3.17) are defined similarly to relations (3.18)-(3.19).*

Remark 2.3.7. Notice that Definition 2.3.6 has been introduced so that the increments $y^{u,u,u} - y^{r,r,r}$ can be controlled by (3.17). Indeed, we have for any $\eta \in [0,1]$ and $\zeta \in [0,\rho)$

$$\begin{aligned} &|y_{ts}^{u,u,u} - y_{ts}^{r,r,r}| = |y_{ts}^{u,u,u} - y_{ts}^{u,r,r} + y_{ts}^{u,r,r} - y_{ts}^{r,r,r}| \le |y_{ts}^{u,u,u} - y_{ts}^{u,r,r}| + |y_{ts}^{u,r,r} - y_{ts}^{r,r,r}| \\ &\le \left(\|y\|_{(\alpha,\gamma,\eta,\zeta),2,3} + \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}\right) |u-r|^{\eta} |r-t|^{-\eta+\zeta} \left(\left[|r-t|^{-\gamma-\zeta} |t-s|^{\alpha}\right] \wedge |r-s|^{\rho-\zeta}\right) \\ &\lesssim \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} |u-r|^{\eta} |r-t|^{-\eta+\zeta} \left(\left[|r-t|^{-\gamma-\zeta} |t-s|^{\alpha}\right] \wedge |r-s|^{\rho-\zeta}\right) \\ &\le \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} |u-r|^{\eta} |r-t|^{-\eta+\zeta} |r-s|^{\rho-\zeta}. \end{aligned} \tag{3.21}$$

Hence similarly to (2.29), we let $\eta = \zeta$ and we obtain

$$|y_{ts}^{u,u,u} - y_{ts}^{r,r,r}| \lesssim \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}. \tag{3.22}$$

Thanks to Hypothesis 2.2.16 and Definition 2.3.6, we can now state a general convolution product for functions defined on Δ_4. As mentioned above, it has to be seen as a generalization of Definition 2.3.4 to a rough context.

Theorem 2.3.8. *Let $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ with $\alpha, \gamma, \eta, \zeta \in (0,1)$ satisfying $\rho = \alpha - \gamma > \frac{1}{4}$, $\zeta \in [0, \alpha-\gamma)$ and $\eta \in [\zeta, 1]$ as given in Definition 2.2.4. We assume that $\mathbf{z}$ fulfills Hypothesis 2.2.16 with n=3. Consider a function $y : \Delta_4 \to \mathcal{L}((\mathbb{R}^m)^{\otimes 3}, \mathbb{R}^m)$ as given in Notation 2.3.3 such that $\|y^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} < \infty$ and $y_0^{1,2,3} = y_0$, where $\|y^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$ is defined by (3.17). Then with Notation 2.2.14 in mind, we have for all fixed $(s,t,\tau) \in \Delta_3$ that*

$$\mathbf{z}_{ts}^{3,\tau} * y_s^{1,2,3} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbf{z}_{vu}^{3,\tau} y_s^{u,u,u} + \left(\delta_u \mathbf{z}_{vs}^{3,\tau}\right) * y_s^{1,2,3}. \tag{3.23}$$

is a well defined Volterra-Young integral. It follows that $*$ is a well defined bi-linear operation between the three parameters Volterra function $\mathbf{z}^3$ and a 4-parameter path y. Moreover, we have that

$$\begin{aligned}\left|\mathbf{z}_{ts}^{3,\tau} * y_s^{1,2,3} - \mathbf{z}_{ts}^{3,\tau} y_s^{s,s,s}\right| \lesssim \|y^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}\Big(&\|\mathbf{z}^3\|_{(3\rho+\gamma,\gamma),1} + \|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}\|\mathbf{z}^2\|_{(\alpha,\gamma),1} \\ &+ \|\mathbf{z}^2\|_{(\alpha,\gamma,\eta,\zeta),1,2}\|\mathbf{z}^1\|_{(\alpha,\gamma),1}\Big)\, \psi^1_{(3\rho+\gamma),\gamma}(\tau,t,s),\end{aligned} \tag{3.24}$$

where ψ^1 is given in (2.1).

Remark 2.3.9. Similarly to Remark 2.2.21, the term $(\delta_u \mathbf{z}_{vs}^{3,\tau}) * y_s^{1,2,3}$ is defined thanks to the fact that (according to relation (2.21))

$$\delta_u \mathbf{z}_{vs}^{3,\tau} * y_s^{1,2,3} = \mathbf{z}_{vu}^{2,\tau} * \mathbf{z}_{us}^{1,\cdot} * y^{1,2,3} + \mathbf{z}_{vu}^{1,\tau} * \mathbf{z}_{us}^{2,\cdot} * y^{1,2,3}, \tag{3.25}$$

and the convolutions with respect to $\mathbf{z}^{1,\tau}$, $\mathbf{z}^{2,\tau}$ in (3.25) are respectively defined by Theorem 2.2.15 and Theorem 2.2.20.

Proof of Theorem 2.3.8. We first prove (3.23). To this aim, for a generic partition $\mathcal{P}$ of $[s,t]$ let us denote by $\mathcal{I}_{\mathcal{P}}$ the approximation of the right hand side of (3.23). Specifically we set $\mathcal{I}_{\mathcal{P}} := \sum_{[u,v]\in\mathcal{P}} (\Xi_s^\tau)_{vu}$, where

$$(\Xi_s^\tau)_{vu} = \mathbf{z}_{vu}^{3,\tau} y_s^{u,u,u} + \left(\delta_u \mathbf{z}_{vs}^{3,\tau}\right) * y_s^{1,2,3}. \tag{3.26}$$

We now compute $\delta_r(\Xi_s^\tau)_{vu}$ in order to check that the extended Volterra sewing Lemma 2.3.2 can be applied in our context. Recall that

$$\delta_r\left(\Xi_s^\tau\right)_{vu} = \left(\Xi_s^\tau\right)_{vu} - \left(\Xi_s^\tau\right)_{vr} - \left(\Xi_s^\tau\right)_{ru}, \quad \text{for all} \quad \tau > v > r > u > s.$$

Moreover, we know from Hypothesis 2.2.16 that

$$\delta_r \mathbf{z}_{vu}^{3,\tau} = \mathbf{z}_{vr}^{2,\tau} * \mathbf{z}_{ru}^{1,\cdot} + \mathbf{z}_{vr}^{1,\tau} * \mathbf{z}_{ru}^{2,\cdot}.$$

Therefore, a few elementary computations reveal that

$$\delta_r\left(\mathbf{z}_{vu}^{3,\tau} y_s^{u,u,u}\right) = -\mathbf{z}_{vr}^{3,\tau}\left(y_s^{r,r,r} - y_s^{u,u,u}\right) + \left(\mathbf{z}_{vr}^{2,\tau} * \mathbf{z}_{ru}^{1,\cdot} + \mathbf{z}_{vr}^{1,\tau} * \mathbf{z}_{ru}^{2,\cdot}\right) y_s^{u,u,u} \tag{3.27}$$

$$\delta_r\left(\left(\delta_u \mathbf{z}_{vs}^{3,\tau}\right) * y_s^{1,2,3}\right) = -\left(\mathbf{z}_{vr}^{2,\tau} * \mathbf{z}_{ru}^{1,\cdot} + \mathbf{z}_{vr}^{1,\tau} * \mathbf{z}_{ru}^{2,\cdot}\right) * y_s^{1,2,3}, \tag{3.28}$$

Combining (3.27) and (3.28), we thus get

$$\delta_r\left(\Xi_s^\tau\right)_{vu} = -\left(Q_{vru}^1 + Q_{vru}^2 + Q_{vru}^3\right), \tag{3.29}$$

where the quantities Q_{vru}^1, Q_{vru}^2, Q_{vru}^3 are defined by

$$\begin{aligned}
Q_{vru}^1 &= \mathbf{z}_{vr}^{3,\tau}\left(y_s^{r,r,r} - y_s^{u,u,u}\right) \\
Q_{vru}^2 &= \mathbf{z}_{vr}^{2,\tau} * \mathbf{z}_{ru}^{1,\cdot} * \left(y_s^{1,2,3} - y_s^{u,u,u}\right) \\
Q_{vru}^3 &= \mathbf{z}_{vr}^{1,\tau} * \mathbf{z}_{ru}^{2,\cdot} * \left(y_s^{1,2,3} - y_s^{u,u,u}\right)
\end{aligned}$$

We will bound each of the above terms separately.

Applying (3.21) with $\zeta = 0$, and invoking the definition of $\|\mathbf{z}^3\|_{(3\rho+\gamma,\gamma),1}$ in (2.4), and using that $r \in [u, v]$ we have for any $\eta \in [0, 1]$

$$\left|Q_{vru}^1\right| \lesssim \left\|y^{1,2,3}\right\|_{(\alpha,\gamma,\eta,0),1,2,3} \left\|\mathbf{z}^3\right\|_{(3\rho+\gamma,\gamma),1} |u-s|^{-\eta}|\tau - v|^{-\gamma}|v-u|^{3\rho+\gamma+\eta}, \tag{3.30}$$

We then choose η such that $3\rho+\gamma+\eta>1$ and at the same time $\eta+\gamma<1$, which is always possible since $\rho>0$, to obtain the desired regularity. For the term Q^2_{vru}, we invoke the bound in (2.31), and observe that

$$\begin{aligned}\left|Q^2_{vru}\right| &\leq \left|\mathbf{z}^{2,\tau}_{vr}\right| \left|\mathbf{z}^{1,r}_{ru} * (y^{r,r,3}_s - y^{u,u,u}_s)\right| \\ &\quad + \|\hat{y}\|_{(\alpha,\gamma,\eta,\zeta),1,2} \left(\|\mathbf{z}^1\|^2_{(\alpha,\gamma,\eta,\zeta)} + \|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma,\eta,\zeta)}\right) |\tau-v|^{-\gamma}|v-u|^{3\rho+\gamma} \wedge |\tau-u|^{3\rho} \end{aligned} \tag{3.31}$$

where $\hat{y}^{l,w}_{ru} = \mathbf{z}^{1,\cdot}_{ru} * (y^{l,w,3}_s - y^{u,u,u}_s)$, and we will need to find a bound for $\|\hat{y}\|_{(\alpha,\gamma,\eta,\zeta),1,2}$. Note that convolution only happens in the first term of $y^{l,w,3}_s - y^{u,u,u}_s$. By (2.20) it follows that

$$\|\hat{y}\|_{(\alpha,\gamma,\eta,\zeta),1,2} \lesssim \|\mathbf{z}\|_{(\alpha,\gamma),1}\|y^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),2,3}|v-u|^{\eta}|u-s|^{-\eta}.$$

Furthermore, from (2.19) it is readily checked that

$$\left|\mathbf{z}^{1,r}_{ru} * (y^{r,2,3}_s - y^{u,u,u}_s)\right| \lesssim \|\mathbf{z}^1\|_{(\alpha,\gamma),1}\|y^{r,2,3} - y^{u,u,u}_s\|_{(\alpha,\gamma,\eta,\zeta),1,2}|r-u|^{\rho}$$

We continue to investigate the first terms in (3.31). From the above regularity estimate it follows that

$$\left|\mathbf{z}^{2,\tau}_{vr}\right| \left|\mathbf{z}^{1,\cdot}_{ru} * \left(y^{1,2,3}_s - y^{u,u,u}_s\right)\right| \lesssim \left\|\mathbf{z}^2\right\|_{(2\rho+\gamma,\gamma,\eta,\zeta)} \left\|\mathbf{z}^1\right\|_{(\alpha,\gamma,\eta,\zeta)} \|y\|_{(\alpha,\gamma)} |\tau-v|^{-\gamma}|v-u|^{3\rho+\gamma+\eta}|u-s|^{-\eta}.$$

Combining our estimates for the different terms on the right hand side of (3.31), we have that

$$\left|Q^2_{vru}\right| \lesssim \left\|y^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \left\|\mathbf{z}^2\right\|_{(2\rho+\gamma,\gamma,\eta,\zeta)} \left\|\mathbf{z}^1\right\|_{(\alpha,\gamma,\eta,\zeta)} |\tau-v|^{-\gamma}|v-u|^{3\rho+\gamma+\eta}|u-s|^{-\eta}, \tag{3.32}$$

By similar computations as for the bound for Q^2, we obtain a bound for Q^3 given by

$$\left|Q^3_{vru}\right| \lesssim \left\|y^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \left\|\mathbf{z}^1\right\|_{(\alpha,\gamma,\eta,\zeta)} \left\|\mathbf{z}^2\right\|_{(2\rho+\gamma,\gamma,\eta,\zeta)} |\tau-v|^{-\gamma}|v-u|^{3\rho+\gamma+\eta}|u-s|^{-\eta}. \tag{3.33}$$

Plugging (3.30)-(3.33) into (3.29), we have thus obtained

$$|\delta_r (\Xi_s^\tau)_{vu}| \lesssim C_{y,\mathbf{z}} |\tau - v|^{-\gamma} |u-s|^{-\eta} |v-u|^{3\rho+\gamma+\eta}, \tag{3.34}$$

where the constant $C_{y,\mathbf{z}}$ used above is given explicitly as

$$c_{y,\mathbf{z}} = \left\| y^{1,2,3} \right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \left(\|\mathbf{z}^3\|_{(3\rho+\gamma,\gamma,\eta,\zeta)} + 2 \left\|\mathbf{z}^2\right\|_{(2\rho+\gamma,\gamma,\eta,\zeta)} \left\|\mathbf{z}^1\right\|_{(\alpha,\gamma,\eta,\zeta)} \right).$$

Starting from (3.34), one can now check that

$$\|\delta \Xi\|_{(3\rho+\gamma+\eta,\gamma,\eta),1} < \infty, \tag{3.35}$$

where the norm in the left hand side of (3.35) is defined by (3.2). In the same way, we let the patient reader check that $\|\Xi\|_{(3\rho+\gamma+\eta,\gamma,\eta),1,2} < \infty$, where the $\|\cdot\|_{(3\rho+\gamma+\eta,\gamma,\eta),1,2}$ norm is introduced in (3.3). Since we have chosen η such that $3\rho+\gamma+\eta > 1$ and $\gamma+\eta < 1$, we can apply Lemma 2.3.2 to the increment Ξ and recall the Notation 2.2.3 of ψ^1, $\psi^{1,2}$, which directly yields our claims (3.6) and (3.7). □

Remark 2.3.10. The general convolution $\mathbf{z}^{3,\tau} * y_s^{1,2,3}$ is given in (3.23), for a path y defined on Δ_4. If we wish to consider the convolution restricted to a path $y_s^{1,2}$ defined on Δ_3, a natural way to proceed is to define

$$\mathbf{z}_{ts}^{3,\tau} * y_s^{1,2} := \mathbf{z}_{ts}^{3,\tau} * \hat{y}^{1,2,3}, \quad \text{with} \quad \hat{y}^{r_1,r_2,r_3} = y^{r_2,r_3}.$$

This means that the path $\hat{y}$ has no dependence in r_1. Therefore resorting to the notations (2.23)-(2.24), and (3.16)-(3.17), it is not difficult to check that

$$\begin{aligned}
\left\|\hat{y}^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,>} &= \left\|y^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta)1,2,<}, & \left\|\hat{y}^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,2,<} &= 0,\\
\left\|\hat{y}^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,3,>} &= \left\|y^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta)1,2,>}, & \left\|\hat{y}^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),1,3,<} &= 0,\\
\left\|\hat{y}^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),2,3,>} &= \left\|y^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta)1,2,>}, & \left\|\hat{y}^{1,2,3}\right\|_{(\alpha,\gamma,\eta,\zeta),2,3,<} &= \left\|y^{1,2}\right\|_{(\alpha,\gamma,\eta,\zeta)1,2,<}.
\end{aligned}$$

Hence we have $\|\hat{y}^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \lesssim \|y^{1,2}\|_{(\alpha,\gamma,\eta,\zeta),1,2}$, where $\|\hat{y}^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$ is given in (3.17) and the norm $\|y^{1,2}\|_{(\alpha,\gamma,\eta,\zeta),1,2}$ is introduced in (2.24). This will be invoked for our rough path constructions in the upcoming section.

Remark 2.3.11. In our applications to rough Volterra equations we will consider the case $\rho = \alpha - \gamma \in (\frac{1}{4}, \frac{1}{3})$. Therefore it is sufficient to show that the convolution product $*$ can be performed on the third level of a Volterra rough path.

2.4 Stochastic calculus for Volterra rough paths

In this section we carry out some of the main steps leading to a proper differential calculus in a Volterra context. That is, we show how to integrate a Volterra controlled process in Section 2.4.1, and we solve Volterra type equations in Section 2.4.2.

2.4.1 Volterra controlled processes and rough Volterra integration

We begin with a proper definition of rough Volterra integration in rough case $\alpha - \gamma > \frac{1}{4}$. As usual in rough integration theory, one needs to specify a proper class of processes which can be integrated with respect to the driving noise. As we will see, a non-geometric rough path type theory based on tree type expansions are needed, in order to construct a well defined rough Volterra integral. We therefore begin with some motivation for tree type expansions for iterated integrals.

Tree expansions setting

In Hypothesis 2.2.16, we have introduced the notion of a convolutional rough path $\mathbf{z}$ above z. While $\mathbf{z}$ satisfies the Chen type relation (2.21), it cannot be considered as a geometric rough path (see e.g. [8]). The reader might check for instance that for a path $\mathbf{z}^{1,\tau}$ given by the mapping $(t,\tau) \mapsto \int_0^t k(\tau, r) dx_r$, the component $\mathbf{z}^{2,\tau}$ will *not* satisfy the component-wise relation

$$\left(\mathbf{z}_{ts}^{2,\tau}\right)^{\mathrm{ii}} = \frac{1}{2}\left((\mathbf{z}_{ts}^{1,\tau})^{\mathrm{i}}\right)^2, \qquad \mathrm{i} = 1, \cdots, d.$$

Hence in order to define a rough path type calculus of order 2 related to z^{τ}, we have to invoke techniques related to non geometric settings. The standard language in this kind of context is related to the Hopf algebra of trees. We give a brief account on those notions in the current section, referring to [3] for further details.

Let $\mathcal{T}_3$ be the set of rooted trees with at most 3 vertices, whose vertices are decorated by labels from the alphabet $\{1, \ldots, d\}$. A full description of the undecorated version of $\mathcal{T}_3$ is given by

$$\mathcal{T}_3 = \left\{ \bullet,\ \begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix},\ \begin{smallmatrix}\bullet\\ \bullet\\ \bullet\end{smallmatrix},\ \overset{\bullet\,\bullet}{\bullet} \right\}. \tag{4.1}$$

In the sequel we will use the operation $[\,\cdot\,]$ on trees. Namely for $\sigma_1, \ldots, \sigma_m \in \mathcal{T}_3$ and $a \in \{1, \ldots, d\}$. we define $\sigma = [\sigma_1 \cdots \sigma_k]_a$ as the tree for which $\sigma_1, \ldots, \sigma_k$ are attached to a new root with label a. For instance in the unlabeled case we have

$$[\,1\,] = \bullet \qquad [\bullet] = \begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix} \qquad [\begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix}] = \begin{smallmatrix}\bullet\\ \bullet\\ \bullet\end{smallmatrix} \qquad [\bullet\bullet] = \overset{\bullet\,\bullet}{\bullet}.$$

It is thus readily checked that any tree in $\mathcal{T}_3$ can be constructed iteratively from smaller trees thanks to the operation $[\,\cdot\,]$. Let us also mention that we always assume that the order of the branches in each tree does not matter, in the sense that $[\sigma_1 \cdots \sigma_m]_{\mathrm{i}} = [\sigma_{\boldsymbol{\pi}_1} \cdots \sigma_{\boldsymbol{\pi}_m}]_{\mathrm{i}}$ for all permutations $\boldsymbol{\pi}$ of $\{1, \ldots, m\}$.

The set $\mathcal{T}_3$ can be turned into a Hopf algebra when equipped with a suitable coproduct and antipode. This elegant structure is applied and discussed in detail in [3], but is not necessary in our context. However, we shall use some of the notation contained in [3] for our future computations.

Notation 2.4.1. *For any tree $\sigma \in \mathcal{T}_3$, the quantity $|\sigma|$ denotes the numbers of vertices in σ. We call the set $\mathcal{F}_2$ a forest consisting of elements with 2 vertices or less. Namely, $\mathcal{F}_2$ is defined by*

$$\mathcal{F}_2 = \{\bullet,\ \begin{smallmatrix}\bullet\\ \bullet\end{smallmatrix},\ \bullet\bullet\}.$$

Remark 2.4.2. Note that the operation $[\,\cdot\,]$ sends the set $\{1\} \cup \mathcal{F}_2$ into $\mathcal{T}_3$.

Tree indexed rough path and controlled processes.

We have already introduced the family $\{\mathbf{z}^{\mathrm{j},\tau}, \mathrm{j} = 1,2,3\}$ in Hypothesis 2.2.16. These objects will be identified with tree indexed objects below. On top of this family, our computations will also hinge on an additional function called $\mathbf{z}^{\overset{\bullet\,\bullet}{\vee},\tau}$. Similarly to (3.13), whenever x is a continuously differentiable function and k satisfies Hypothesis 2.2.1, the increment $\mathbf{z}^{\overset{\bullet\,\bullet}{\vee},\tau}$ is defined by

$$\mathbf{z}^{\overset{\bullet\,\bullet}{\vee},\tau}_{ts} = \int_s^t k(\tau,r)\left(\int_s^r k(r,l)dx_l\right)\otimes\left(\int_s^r k(r,w)dx_w\right)\otimes dx_r\,. \tag{4.2}$$

However, for a generic rough signal x we need some more abstract assumptions which are summarized below.

Hypothesis 2.4.3. *Let $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ be a Volterra path as given in Definition 2.2.4. Recall that α,γ,η,ζ satisfies $4\rho+\gamma>1$ where $\rho=\alpha-\gamma$, $\zeta\in[0,\rho)$ and $\eta\in[\zeta,1]$. We assume the existence of a family $\mathbf{z}=\{\mathbf{z}^{\sigma,\tau},\sigma\in\mathcal{T}_3\}$ such that $\mathbf{z}^{\sigma,\tau}_{ts}\in(\mathbb{R}^m)^{\otimes|\sigma|}$. This family is defined by*

$$\mathbf{z}^{\bullet,\tau}=\mathbf{z}^{1,\tau},\qquad \mathbf{z}^{\overset{\bullet}{\underset{\bullet}{|}},\tau}=\mathbf{z}^{2,\tau},\qquad \mathbf{z}^{\begin{smallmatrix}\bullet\\|\\\bullet\\|\\\bullet\end{smallmatrix},\tau}=\mathbf{z}^{3,\tau},$$

where $\mathbf{z}^{1,\tau},\mathbf{z}^{2,\tau},\mathbf{z}^{3,\tau}$ are introduced in Hypothesis 2.2.16. Moreover the increment $\mathbf{z}^{\overset{\bullet\,\bullet}{\vee},\tau}$ satisfies the algebraic relation

$$\delta_u\mathbf{z}^{\overset{\bullet\,\bullet}{\vee},\tau}_{ts} = 2\mathbf{z}^{\overset{\bullet}{\underset{\bullet}{|}},\tau}_{tu} * \mathbf{z}^{\bullet,\cdot}_{us} + \mathbf{z}^{\bullet,\tau}_{tu} * (\mathbf{z}^{\bullet,\cdot}_{us})^{\otimes 2}\ , \tag{4.3}$$

where the right hand side of (4.3) is defined in Proposition 2.2.15. Analytically, we require each $\mathbf{z}^{\sigma,\tau}$ to be an element of $\mathcal{V}^{(|\sigma|\rho+\gamma,\gamma)}$, and we define

$$|||\mathbf{z}|||_{(\alpha,\gamma,\eta,\zeta)} := \sum_{\sigma\in\mathcal{T}^3}\|\mathbf{z}^\sigma\|_{(|\sigma|\rho+\gamma,\gamma,\eta,\zeta)}. \tag{4.4}$$

Remark 2.4.4. Note that $|||\cdot|||$ does not define a seminorm of any sort, but is rather meant as a convenient way to collect the seminorm terms concerning $\mathbf{z}^\sigma$ for $\sigma\in\mathcal{T}^3$.

Together with the elements in Hypothesis 2.4.3, we will also make use of the family $\{\mathbf{z}^\delta;\delta\in\mathcal{F}_2\}$ in the sequel. To this aim, let us now introduce the element $\mathbf{z}^{\bullet\bullet}$.

Notation 2.4.5. *As stated in Hypothesis [2.4.3](#), we have set* $\mathbf{z}^{\bullet,\tau} = \mathbf{z}^{1,\tau}$*. In addition, we also define* $\mathbf{z}^{\bullet\bullet,\tau}$ *as*

$$\mathbf{z}^{\bullet\bullet,\tau}_{ts} = \int_s^t k(\tau, r) dx_r \int_s^t k(\tau, l) dx_l = (\mathbf{z}^{\bullet,\tau}_{ts})^{\otimes 2} . \tag{4.5}$$

Therefore we can recast (4.3) as

$$\delta_u \mathbf{z}^{\curlyvee,\tau}_{ts} = 2\mathbf{z}^{\dot{\mid},\tau}_{tu} * \mathbf{z}^{\bullet,\cdot}_{us} + \mathbf{z}^{\bullet,\tau}_{tu} * \mathbf{z}^{\bullet\bullet,\tau}_{ts} . \tag{4.6}$$

Assuming Hypothesis 2.4.3 holds, similarly to Theorem 2.3.8, we now give a convolution result for $\mathbf{z}^{\curlyvee,\tau}$.

Theorem 2.4.6. *Let* $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ *with* $\alpha, \gamma, \eta, \zeta \in (0,1)$ *satisfying* $\rho = \alpha - \gamma > \frac{1}{4}$, $\zeta \in [0, \alpha - \gamma)$ *and* $\eta \in [\zeta, 1]$*. We assume that the Volterra rough path* $\mathbf{z}$ *over* z *fulfills Hypothesis 2.4.3. Consider a function* $y : \Delta_4 \to \mathcal{L}((\mathbb{R}^m)^{\otimes 3}, \mathbb{R}^m)$ *as given in Notation 2.3.3 such that* $\|y^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} < \infty$ *and* $y_0^{1,2,3} = y_0$*, where* $\|y^{1,2,3}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$ *is defined by (3.17). Then with Notation 2.2.14 in mind, we have for all fixed* $(s,t,\tau) \in \Delta_3$ *that*

$$\mathbf{z}^{\curlyvee,\tau}_{ts} * y^{1,2,3}_s = \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \mathbf{z}^{\curlyvee,\tau}_{vu} y^{u,u,u}_s + \left(\delta_u \mathbf{z}^{\curlyvee,\tau}_{vs} \right) * y^{1,2,3}_s . \tag{4.7}$$

is a well defined Volterra-Young integral. It follows that $*$ *is a well defined bi-linear operation between the three parameters Volterra function* $\mathbf{z}^3$ *and a 4-parameter path* y*. Moreover, we have that*

$$\left| \mathbf{z}^{\curlyvee,\tau}_{ts} * y^{1,2,3}_s - \mathbf{z}^{\curlyvee,\tau}_{ts} y^{s,s,s}_s \right| \lesssim \left\| y^{1,2,3} \right\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \mathbf{z} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert^3_{(\alpha,\gamma)} \psi^1_{(3\rho+\gamma),\gamma}(\tau, t, s), \tag{4.8}$$

where ψ^1 *is defined by (2.1).*

Proof. The proof goes along the same lines as the proof of Theorem 2.3.8, and is omitted for sake of conciseness. □

We are now ready to introduce the natural class of processes one can integrate with respect to $\mathbf{z}$, called Volterra controlled processes.

Definition 2.4.7. *Let $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ for some $\rho = \alpha - \gamma > 0$, $\zeta \in [0, \rho)$, $\eta \in [\zeta, 1]$, and consider a Volterra path $y : \Delta_2 \to \mathbb{R}^d$. We assume that there exists a family $\{y^\sigma ; \sigma \in \mathcal{F}_2\}$, with $\mathcal{F}_2$ as in Notation 2.4.1, such that the following holds true.*

(i) *y^σ is a function from $\Delta_{|\sigma|+2}$ to $\mathcal{L}((\mathbb{R}^d)^{\otimes|\sigma|}, \mathbb{R}^d)$, and y^σ has $|\sigma| + 1$ upper arguments. The initial conditions are respectively given by*

$$y_0^{\bullet,p,q} = y_0^{\bullet}, \qquad y_0^{\overset{\bullet}{\bullet},p,q,r} = y_0^{\overset{\bullet}{\bullet}}, \qquad y_0^{\bullet\bullet,p,q,r} = y_0^{\bullet\bullet}, \qquad \textit{for all} \quad (r, q, p) \in \Delta_3.$$

(ii) *The family $\{y^\sigma ; \sigma \in \mathcal{F}_2\}$ is related to the increments of y^τ in the following way: for $(s, t, \tau) \in \Delta_3$ we have*

$$y_{ts}^\tau = \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\tau,\cdot} + \mathbf{z}_{ts}^{\overset{\bullet}{\bullet},\tau} * y_s^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot} + \mathbf{z}_{ts}^{\bullet\bullet,\tau} * y_s^{\bullet\bullet,\tau,\cdot,\cdot} + R_{ts}^{y,\tau}, \tag{4.9}$$

and

$$y_{ts}^{\bullet,\tau,p} = \mathbf{z}_{ts}^{\bullet,\tau} * (y_s^{\overset{\bullet}{\bullet},\tau,p,\cdot} + 2y_s^{\bullet\bullet,\tau,p,\cdot}) + R_{ts}^{\bullet,\tau,p}, \tag{4.10}$$

where $y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet} \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$, $R^\bullet \in \mathcal{W}_2^{(2\rho+2\gamma,2\gamma,\eta,\zeta)}(\mathcal{L}(\mathbb{R}^d))$ and $R^y \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}(\mathbb{R}^d)$ (recall that $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ and $\mathcal{W}_2^{(2\rho+2\gamma,2\gamma,\eta,\zeta)}$ are introduced in Definition 2.2.4 and Definition 2.2.18 respectively).

Whenever $\mathbf{y} \equiv (y, y^\bullet, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet})$ satisfies relation (4.9)-(4.10), we say that $\mathbf{y}$ is a Volterra path controlled by $\mathbf{z}$ (or simply controlled Volterra path) and we write $\mathbf{y} \in \mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2; \mathbb{R}^m)$. We equip this space with a semi-norm $\|\cdot\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}$ given by

$$\begin{aligned}
\|\mathbf{y}\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} &= \left\|\left(y, y^\bullet, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet}\right)\right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} \\
&= \|y^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)} + \|y^{\bullet\bullet}\|_{(\alpha,\gamma,\eta,\zeta)} + \|R^y\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)} + \|R^\bullet\|_{(2\rho+2\gamma,2\gamma,\eta,\zeta)}.
\end{aligned} \tag{4.11}$$

where the norms in (4.11) are respectively defined by (2.3) and (2.22). Equipped with the norm

$$\mathbf{y} = \left(y, y^\bullet, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet}\right) \mapsto |y_0| + |y_0^\bullet| + |y_0^{\overset{\bullet}{\bullet}}| + |y_0^{\bullet\bullet}| + \left\|\left(y, y^\bullet, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet}\right)\right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)},$$

the space $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}$ is a Banach space.

Remark 2.4.8. It is easily seen from (4.9) and (4.10) that if $\mathbf{y} \in \mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}$, then $y, y^{\bullet} \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$. Indeed, we observe directly from (4.10) that

$$\|y^{\bullet}\|_{(\alpha,\gamma,\eta,\zeta)} \lesssim \|\mathbf{z}^{\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}(|y_0^{\substack{\bullet\\\bullet}}| + |y_0^{\bullet,\bullet}| + \|y^{\substack{\bullet\\\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)} + \|y^{\bullet,\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}) + \|R^{\bullet}\|_{(2\rho+2\gamma,2\gamma,\eta,\zeta)},$$

where the quantities on the right hand side are finite by assumption. Furthermore, by relation (4.9) we then have that

$$\|y\|_{(\alpha,\gamma,\eta,\zeta)} \leq \|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)} \left(|y_0^{\bullet}| + |y_0^{\substack{\bullet\\\bullet}}| + |y_0^{\bullet,\bullet}| + \|y^{\bullet}\|_{(\alpha,\gamma,\eta,\zeta)} + \|y^{\substack{\bullet\\\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)} + \|y^{\bullet,\bullet}\|_{(\alpha,\gamma,\eta,\zeta)} + \|R^{y}\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)} \right). \tag{4.12}$$

Remark 2.4.9. According to Definition 2.4.7, the function $y^{\bullet}$ is defined on Δ_3 and has two upper variables, while $y^{\substack{\bullet\\\bullet}}$ and $y^{\bullet\bullet}$ are defined on Δ_4 and have three upper arguments. Therefore in (4.11) the norm $\|y^{\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}$ has to be understood as a norm in $\mathcal{W}_2^{(\alpha,\gamma,\eta,\zeta)}$, while the norms $\|y^{\substack{\bullet\\\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)}$ and $\|y^{\bullet\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}$ have to be considered as norms in $\mathcal{W}_3^{(\alpha,\gamma,\eta,\zeta)}$. The readers is referred to Definition 2.2.18 and 2.3.6 for the definition of $\mathcal{W}_2^{(\alpha,\gamma,\eta,\zeta)}$ and $\mathcal{W}_3^{(\alpha,\gamma,\eta,\zeta)}$ respectively. We stick to the notation $\|\cdot\|_{(\alpha,\gamma,\eta,\zeta)}$ for the norm on those different spaces, for notational ease.

Integration of controlled processes

Our next step is to show that we may construct a Volterra rough integral in the rough case $\alpha - \gamma > \frac{1}{4}$, and then prove that the Volterra rough integral of a controlled path with respect to a driving Hölder noise $x \in \mathcal{C}^{\alpha}$ is again a controlled Volterra path.

Theorem 2.4.10. *For $\alpha, \zeta, \eta \in (0,1)$, let $x \in \mathcal{C}^{\alpha}([0,T];\mathbb{R}^d)$ and k be a Volterra kernel satisfying Hypothesis 2.2.1 with a parameter γ such that $\rho = \alpha - \gamma > \frac{1}{4}$, $0 \leq \zeta < \rho$, and $\zeta \leq \eta \leq 1$. Define $z_t^{\tau} = \int_0^t k(\tau, r)\, dx_r$ and assume there exists a tree indexed rough path $\mathbf{z} = \{\mathbf{z}^{\sigma,\tau}; \sigma \in \mathcal{T}_3\}$ above z satisfying Hypothesis 2.4.3. Let $M > 0$ be a constant such that $|||\mathbf{z}|||_{(\alpha,\gamma,\eta,\zeta)} \leq M$. We now consider a controlled Volterra path $\mathbf{y} \in \mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}(\mathcal{L}(\mathbb{R}^d))$, as introduced in Definition 2.4.7. Then the following holds true:*

(i) *Define* $\Xi^{\tau}_{vu} := \mathbf{z}^{\bullet,\tau}_{vu} * y^{\cdot}_{u} + \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vu} * y^{\bullet,\cdot,\cdot}_{u} + \mathbf{z}^{\overset{\overset{\bullet}{\bullet}}{\bullet},\tau}_{vu} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{u} + \mathbf{z}^{\overset{\bullet\bullet}{\bullet},\tau}_{vu} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{u}$. *The following limit exists for all* $(s,t,\tau) \in \Delta_3$,

$$w^{\tau}_{ts} = \int_s^t k(\tau,r)dx_r y^r_r := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \Xi^{\tau}_{vu}. \tag{4.13}$$

(ii) *Let* w *be defined by* (4.13). *Recalling the Notation* 2.2.3 *of* ψ^1 *and* $\psi^{1,2}$, *there exists a positive constant* $C = C_{M,\alpha,\gamma}$ *such that for all* $(s,t,\tau) \in \Delta_3$ *we have*

$$|w^{\tau}_{ts} - \Xi^{\tau}_{ts}| \leq C \left\| \left(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet}\right) \right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} |||\mathbf{z}|||_{(\alpha,\gamma,\eta,\zeta)} \, \psi^1_{(4\rho+\gamma),\gamma}(\tau,t,s). \tag{4.14}$$

(iii) *There exists a positive constant* $C = C_{M,\alpha,\gamma}$ *such that for all* $(s,t,p,q) \in \Delta_4$, *we have*

$$|w^{qp}_{ts} - \Xi^{qp}_{vu}| \leq C \left\| \left(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet}\right) \right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} |||\mathbf{z}|||_{(\alpha,\gamma,\eta,\zeta)} \, \psi^{1,2}_{(4\rho+\gamma),\gamma,\eta,\zeta}(p,q,t,s) \tag{4.15}$$

(iv) *The triple* $\mathbf{w} = (w, w^{\bullet}, w^{\overset{\bullet}{\bullet}}, 0)$ *is a controlled Volterra path in* $\mathcal{D}^{(\alpha,\gamma,\eta,\zeta)}_{\mathbf{z}}(\Delta_2, \mathbb{R}^m)$, *where we recall that* w *is defined by* (4.13), *and where* $w^{\bullet}$, $w^{\overset{\bullet}{\bullet}}$ *are respectively given by*

$$w^{\bullet,\tau,p}_t = y^p_t, \quad \text{and} \quad w^{\overset{\bullet}{\bullet},\tau,q,p}_t = y^{\bullet,q,p}_t.$$

Remark 2.4.11. From Theorem 2.4.10, we also can find a bound for $\|R^w\|_{(3\alpha,3\gamma,\eta,\zeta)}$ and $\|R^{w^{\bullet}}\|_{(2\alpha,2\gamma,\eta,\zeta)}$.
Specifically, according to Theorem 2.4.10 *(ii)* we have

$$\|R^w\|_{(3\alpha,3\gamma,\eta,\zeta)} \lesssim \left\| \left(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet}\right) \right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} |||\mathbf{z}|||_{(\alpha,\gamma,\eta,\zeta)} \tag{4.16}$$

Moreover, thanks to Theorem 2.4.10 *(iv)* we have $w^{\bullet,\tau,p}_t = y^p_t$. Recalling (4.9) together with (4.16), we obtain

$$\left\|R^{w^{\bullet}}\right\|_{(2\alpha,2\gamma,\eta,\zeta)} \lesssim \left\| \left(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet}\right) \right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} |||\mathbf{z}|||_{(\alpha,\gamma,\eta,\zeta)} \tag{4.17}$$

Proof of Theorem 2.4.10. Let $\varXi$ be given as in *(i)*. Thanks to Proposition 2.2.15, Theorem 2.2.20, Theorem 2.3.8, and Theorem 2.4.6, $\varXi$ is well-defined. Our global strategy is to show that the Volterra sewing lemma can be applied to $\varXi$. In order to do so, let us compute $\delta_m \varXi^{\tau}_{vu}$ for $(u,m,v,\tau) \in \Delta_4$. Owing to (2.21), as well as some elementary properties of the operator δ, we get

$$\delta_m \left(\varXi^{\tau}_{vu}\right) := A^{\tau}_{vmu} + B^{\tau}_{vmu}. \tag{4.18}$$

where the quantities A^{τ}_{vmu} and B^{τ}_{vmu} are given by

$$A^{\tau}_{vmu} = -\left(\mathbf{z}^{\bullet,\tau}_{vm} * y^{\cdot}_{mu} + \mathbf{z}^{\substack{\bullet\\ \bullet},\tau}_{vm} * y^{\bullet,\cdot,\cdot}_{mu} + \mathbf{z}^{\substack{\bullet\\ \bullet\\ \bullet},\tau}_{vm} * y^{\substack{\bullet\\ \bullet},\cdot,\cdot,\cdot}_{mu} + \mathbf{z}^{\substack{\bullet\,\bullet\\ \bullet},\tau}_{vm} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{mu}\right) \tag{4.19}$$

and

$$B^{\tau}_{vmu} = \delta_m \mathbf{z}^{\substack{\bullet\\ \bullet},\tau}_{vu} * y^{\bullet,\cdot,\cdot}_{u} + \delta_m \mathbf{z}^{\substack{\bullet\\ \bullet\\ \bullet},\tau}_{vu} * y^{\substack{\bullet\\ \bullet},\cdot,\cdot,\cdot}_{u} + \delta_m \mathbf{z}^{\substack{\bullet\,\bullet\\ \bullet},\tau}_{vu} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{u}. \tag{4.20}$$

Due to the assumption that $(y, y^{\bullet}, y^{\substack{\bullet\\ \bullet}}, y^{\bullet\bullet}) \in \mathcal{D}^{(\alpha,\gamma,\eta,\zeta)}_{\mathbf{z}}$, we have that for any $(s,t,\tau) \in \Delta_3$

$$y^{\cdot}_{ts} = \mathbf{z}^{\bullet,\cdot}_{ts} * y^{\bullet,\cdot,\cdot}_{s} + \mathbf{z}^{\substack{\bullet\\ \bullet},\cdot}_{ts} * y^{\substack{\bullet\\ \bullet},\cdot,\cdot,\cdot}_{s} + \mathbf{z}^{\bullet\bullet,\cdot}_{ts} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{s} + R^{y,\cdot}_{ts},$$

and

$$y^{\bullet,\cdot,\cdot}_{ts} = \mathbf{z}^{\bullet,\cdot}_{ts} * \left(y^{\substack{\bullet\\ \bullet},\cdot,\cdot,\cdot}_{s} + 2y^{\bullet\bullet,\cdot,\cdot,\cdot}_{s}\right) + R^{\bullet,\cdot,\cdot}_{ts}.$$

Plugging the above two relations into (4.19), we obtain

$$\begin{aligned}
A^{\tau}_{vmu} = &- \mathbf{z}^{\bullet,\tau}_{vm} * \mathbf{z}^{\bullet,\cdot}_{mu} * y^{\bullet,\cdot,\cdot}_{u} - \mathbf{z}^{\bullet,\tau}_{vm} * \mathbf{z}^{\substack{\bullet\\ \bullet},\cdot}_{mu} * y^{\substack{\bullet\\ \bullet},\cdot,\cdot,\cdot}_{u} - \mathbf{z}^{\bullet,\tau}_{vm} * \mathbf{z}^{\bullet\bullet,\cdot}_{mu} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{u} - \mathbf{z}^{\bullet,\tau}_{vm} * R^{y,\cdot}_{mu} \\
&- \mathbf{z}^{\substack{\bullet\\ \bullet},\tau}_{vm} * \mathbf{z}^{\bullet,\cdot}_{mu} * y^{\substack{\bullet\\ \bullet},\cdot,\cdot,\cdot}_{u} - 2\mathbf{z}^{\substack{\bullet\\ \bullet},\tau}_{vm} * \mathbf{z}^{\bullet,\cdot}_{mu} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{u} - \mathbf{z}^{\substack{\bullet\\ \bullet},\tau}_{vm} * R^{\bullet,\cdot,\cdot}_{mu} \\
&- \mathbf{z}^{\substack{\bullet\\ \bullet\\ \bullet},\tau}_{vm} * y^{\substack{\bullet\\ \bullet},\cdot,\cdot,\cdot}_{mu} - \mathbf{z}^{\substack{\bullet\,\bullet\\ \bullet},\tau}_{vm} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{mu}.
\end{aligned} \tag{4.21}$$

Thanks to Hypothesis 2.2.16 and Hypothesis 2.4.3, plugging in the algebraic relations from (2.21) and (4.3) into (4.20), we have

$$
\begin{aligned}
B^{\tau}_{vum} = & \mathbf{z}^{\bullet,\tau}_{vm} * \mathbf{z}^{\bullet,\cdot}_{mu} * y^{\bullet,\cdot,\cdot}_{u} + \mathbf{z}^{\bullet,\tau}_{vm} * \mathbf{z}^{\overset{\bullet}{\bullet},\cdot}_{mu} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{u} + \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vm} * \mathbf{z}^{\bullet,\cdot}_{mu} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{u} \\
& + 2\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vm} * \mathbf{z}^{\bullet,\cdot}_{mu} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{u} + \mathbf{z}^{\bullet,\tau}_{vm} * (\mathbf{z}^{\bullet,\cdot}_{mu})^{\otimes 2} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{u}.
\end{aligned} \tag{4.22}
$$

We now insert (4.21) and (4.22) into (4.18). Let us also recall that $\mathbf{z}^{\bullet\bullet,\cdot}_{mu} = (\mathbf{z}^{\bullet,\cdot}_{mu})^{\otimes 2}$ according to (4.5). Then some elementary algebraic manipulations and cancellations show that

$$
\delta_m\left(\Xi^{\tau}_{vu}\right) = -\mathbf{z}^{\overset{\bullet}{\overset{\bullet}{\bullet}},\tau}_{vm} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{mu} - \mathbf{z}^{\overset{\bullet\bullet}{\bullet},\tau}_{vm} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{mu} - \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vm} * R^{\bullet,\cdot,\cdot}_{mu} - \mathbf{z}^{\bullet,\tau}_{vm} * R^{y,\cdot}_{mu}. \tag{4.23}
$$

We now bound successively the 4 terms in the right hand side of (4.23). First we apply a small variant of (3.24) and (4.8), which takes into account the fact that increments of the form $y^{\overset{\bullet}{\bullet}}_{mu}$ and $y^{\bullet\bullet}_{mu}$ are considered. We also bound the terms involving $y^{\overset{\bullet}{\bullet},u,u,u}_{mu}$, $y^{\bullet\bullet,u,u,u}_{mu}$ properly in (3.24) and (4.8). Resorting to (4.11) and the definition (2.1) of ψ^1, we get

$$
\begin{aligned}
& \left| \mathbf{z}^{\overset{\bullet}{\overset{\bullet}{\bullet}},\tau}_{vm} * y^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot}_{mu} + \mathbf{z}^{\overset{\bullet\bullet}{\bullet},\tau}_{vm} * y^{\bullet\bullet,\cdot,\cdot,\cdot}_{mu} \right| \\
& \lesssim \left(\|y^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} + \|y^{\bullet\bullet}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \right) |||\mathbf{z}|||_{(\alpha,\gamma,\eta,\zeta)} \, |u-m|^{\rho} \, |\tau - m|^{-\gamma} \, |v-m|^{3\rho+\gamma}.
\end{aligned} \tag{4.24}
$$

where we recall that $|||\mathbf{z}|||_{(\alpha,\gamma,\eta,\zeta)}$ was defined in (4.4). Next invoking the fact that $R^y \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$ and Proposition 2.2.15, together with ψ^1 as given in (2.1), we obtain

$$
|\mathbf{z}^{\bullet,\tau}_{vm} * R^{y,\cdot}_{mu}| \leq \|R^y\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)} \|\mathbf{z}^{\bullet}\|_{(\alpha,\gamma,\eta,\zeta)} |\tau - v|^{-\gamma} |u-m|^{4\rho+\gamma}. \tag{4.25}
$$

Eventually, resorting to Theorem 2.2.20 and owing to the fact that $R^{\bullet} \in \mathcal{W}_2^{(2\rho+2\gamma,2\gamma,\eta,\zeta)}$, we can check that

$$
\left| \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{vm} * R^{\bullet,\cdot,\cdot}_{mu} \right| \leq \|R^{\bullet}\|_{(2\rho+2\gamma,2\gamma,\eta,\zeta),1,2} |||\mathbf{z}|||^2_{(\alpha,\gamma,\eta,\zeta)} \, |\tau - v|^{-\gamma} \, |v-u|^{4\rho+\gamma}. \tag{4.26}
$$

Plugging (4.24), (4.25) and (4.26) into (4.23), we have thus obtained

$$|\delta_m \Xi^\tau_{vu}| \lesssim \left\| \left(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet} \right) \right\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} \|\!|\mathbf{z}|\!\|^2_{(\alpha,\gamma,\eta,\zeta)} |\tau - v|^{-\gamma} |v-u|^{4\rho+\gamma}. \tag{4.27}$$

Recall that by assumption, $\rho > \frac{1}{4}$, and therefore $\beta \equiv 4\rho + \gamma > 1$. We have thus obtained that $\|\delta\Xi\|_{(\beta,\gamma),1} < \infty$. Following along the same lines above, it is readily checked that also $\|\delta\Xi\|_{(\beta,\gamma,\eta,\zeta),1,2} < \infty$. Therefore we apply directly the Volterra sewing Lemma 2.2.11 in order to achieve the claims in (4.13), (4.14) and (4.15).

We now proceed to prove the last claim, *(iv)* . To this aim, observe that the bound in (4.14) together with the fact that $\mathbf{z}^{\overset{\bullet}{\bullet}}, \mathbf{z}^{\overset{\bullet\,\bullet}{\vee}} \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}(\mathbb{R}^d)$, implies the existence of a function $R \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}(\mathbb{R}^m)$ such that

$$w^\tau_{ts} = \mathbf{z}^{\bullet,\tau}_{ts} * y^{\cdot}_s + \mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\bullet,\cdot,\cdot}_s + R^\tau_{ts}. \tag{4.28}$$

From (4.28) it is readily seen that w^τ can be decomposed as a controlled Volterra path in $\mathcal{D}^{(\alpha,\gamma,\eta,\zeta)}_{\mathbf{z}}(\Delta_2, \mathbb{R}^m)$ with $w^{\bullet,\tau,p}_t = y^p_t$, $w^{\overset{\bullet}{\bullet},\tau,q,p}_t = y^{\bullet,q,p}_t$, $w^{\bullet\bullet,\tau,q,p}_t = 0$. This finishes our proof. $\square$

Remark 2.4.12. From Theorem 2.4.10 (d), we know that the process w defined by (4.13) satisfies

$$w^{\bullet,\tau,p}_t = y^p_t = w^{\bullet,p}_t.$$

Therefore $w^{\bullet}$ depends on two variables instead of 3 variables in the general definition (4.9). In the same way, we have

$$w^{\overset{\bullet}{\bullet},\tau,q,p}_t = y^{\bullet,q,p}_t = w^{\overset{\bullet}{\bullet},q,p}_t,$$

that is, $w^{\overset{\bullet}{\bullet}}$ depends on three variables (vs 4 variables in the general definition (4.9)). Therefore we can refine Theorem 2.4.10 and state that the Volterra rough integration sends

$(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, y^{\bullet\bullet}) \in \mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^d)$ to a controlled process $(w, w^{\bullet}, w^{\overset{\bullet}{\bullet}}, 0) \in \hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^d)$, where the space $\hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^d)$ is defined by

$$
\begin{aligned}
\hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^d) := \Big\{ \Big(w, w^{\bullet}, w^{\overset{\bullet}{\bullet}}, 0\Big) \in \mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2; \mathbb{R}^d) \,\Big|\, & w_s^{\bullet,\tau,p} = w_s^{\bullet,p}, \\
& w_s^{\overset{\bullet}{\bullet},\tau,q,p} = w_s^{\overset{\bullet}{\bullet},q,p}, \text{and } w_s^{\bullet\bullet,\tau,q,p} = 0 \Big\}.
\end{aligned}
\tag{4.29}
$$

The composition of a Volterra controlled processes with a smooth function

With Remark 2.4.12 in mind, we will now prove that one can compose processes in $\hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}$ and still get a controlled process.

Proposition 2.4.13. *Let $f \in C_b^4(\mathbb{R}^d; \mathbb{R}^m)$ and assume $(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0) \in \hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^d)$ as given in Remark 2.4.12. Also recall our Notation 2.2.14 for matrix products. Then the composition $f(y)$ can be seen as a controlled path $(\phi, \phi^{\bullet}, \phi^{\overset{\bullet}{\bullet}}, \phi^{\bullet\bullet})$, where $\phi = f(y)$ and where in the decomposition (4.9) we have*

$$
\phi_t^{\bullet,q,p} = y_t^{\bullet,p} f'\left(y_t^q\right), \tag{4.30}
$$

$$
\phi_t^{\overset{\bullet}{\bullet},r,q,p} = y_t^{\overset{\bullet}{\bullet},q,p} f'\left(y_t^r\right), \quad \text{and} \quad \phi_t^{\bullet\bullet,r,q,p} = \frac{1}{2}(y_t^{\bullet,q}) \otimes (y_t^{\bullet,p}) f''\left(y_t^r\right). \tag{4.31}
$$

Moreover, there exists a constant $C = C_{\alpha,\gamma,\|f\|_{C_b^4}} > 0$ such that

$$
\begin{aligned}
\|(\phi, \phi^{\bullet}, \phi^{\overset{\bullet}{\bullet}}, \phi^{\bullet\bullet})\|_{\mathbf{z};(\alpha,\gamma,\eta,\zeta)} \leq C(1 + \|\!|\mathbf{z}|\!\|_{(\alpha,\gamma,\eta,\zeta)})^3 \Big[& \Big(|y_0^{\bullet}| + |y_0^{\overset{\bullet}{\bullet}}| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\Big) \\
& \vee \Big(|y_0^{\bullet}| + |y_0^{\overset{\bullet}{\bullet}}| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\Big)^3 \Big].
\end{aligned}
\tag{4.32}
$$

Proof. We separate this proof into two parts: in the first step we will find the appropriate expression for $\phi^{\bullet}$, $\phi^{\overset{\bullet}{\bullet}}$ and $\phi^{\bullet\bullet}$ (namely (4.30) and (4.31)), as well as proving that $(\phi, \phi^{\bullet}, \phi^{\overset{\bullet}{\bullet}}, \phi^{\bullet\bullet}) \in \mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}$. In the second step, we will prove relation (4.32). Without loss of generality, we do the below analysis component-wise for $f(y) = (f_1(y), \ldots, f_m(y))$, where each $f_{\mathrm{i}} : \mathbb{R}^d \to \mathbb{R}$ for $\mathrm{i} = 1, \ldots, m$. With a slight abuse of notation, we drop the subscript notation, and still just write $f(y)$ representing each component.

Step 1: Expression for $\phi^{\bullet}$, $\phi^{\text{\textbullet\!\textbullet}}$ and $\phi^{\bullet\bullet}$. An elementary application of Taylor's formula enables us to decompose the increment $f(y^\tau)_{ts}$ into

$$f(y^\tau)_{ts} = y^\tau_{ts} f'(y^\tau_s) + \frac{1}{2}(y^\tau_{ts})^{\otimes 2} f''(y^\tau_s) + r^\tau_{ts}, \tag{4.33}$$

where $r^\tau_{ts} = \frac{1}{6}(y^\tau_{ts})^{\otimes 3} \int_0^1 f^{(3)}(c^\tau_{ts}(\theta))d\theta$, where $c^\tau_{ts}(\theta) = \theta y^\tau_s + (1-\theta)y^\tau_t$. It is readily checked from (4.33) that $r \in \mathcal{V}^{(3\alpha, 3\gamma, \eta, \zeta)}$. Indeed, it follows directly that

$$\|r\|_{(3\alpha,3\gamma),1} \lesssim \|y\|^3_{(\alpha,\gamma),1} \|f\|_{C^3_b}.$$

Furthermore, for $(s, t, \tau, \tau') \in \Delta_4$, we have

$$\begin{aligned}|r^{\tau'\tau}_{ts}| \leq 3|y^{\tau'\tau}_{ts}||(y^{\tau'}_{ts})^{\otimes 2} + (y^\tau_{ts})^{\otimes 2}|\|f\|_{C^3_b} \\ + \|y\|^3_{(\alpha,\gamma),1}\|f\|_{C^4_b}(|y^{\tau'\tau}_s| + |y^{\tau'\tau}_t|)(|\tau - t|^{-\gamma}|t-s|^\alpha \wedge |\tau - s|^\rho)^3.\end{aligned}$$

It is simply checked that the following inequality hold:

$$|y^{\tau'\tau}_s| \lesssim |y_0| + \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}|\tau' - \tau|^\eta[|\tau - s|^{\eta-\zeta}|s^\alpha| \wedge |s|^{\rho-\zeta}],$$

and thus it follows that

$$\|r\|_{(3\alpha,3\gamma,\eta,\zeta),1,2} \lesssim (\|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}\|y\|^2_{(\alpha,\gamma),1} + \|y\|^3_{(\alpha,\gamma),1}(|y_0| + \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}))\|f\|_{C^3_b}.$$

Combining the above estimates, we get

$$\|r\|_{(3\alpha,3\gamma,\eta,\zeta)} \lesssim (|y_0| + \|y\|_{(\alpha,\gamma,\eta,\zeta)})^3 \|f\|_{C^4_b}. \tag{4.34}$$

Now observe that $(y, y^{\bullet}, y^{\substack{\bullet\\ \vert\\ \bullet}}, 0) \in \hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}$ where $\hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}$ is the subset of the space $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}$ as defined in Remark 2.4.12. In particular, y, $y^{\bullet}$, $y^{\substack{\bullet\\ \vert\\ \bullet}}$ satisfy relation (4.9). Then taking squares in the relation (4.9), we end up with

$$(y_{ts}^{\tau})^{\otimes 2} = (\mathbf{z}_{ts}^{\bullet,\tau} * y_{s}^{\bullet,\tau,\cdot})^{\otimes 2} + \tilde{r}_{ts}^{\tau}, \tag{4.35}$$

where the reminder term $\tilde{r}_{ts}^{\tau}$ is defined by

$$\begin{aligned} \tilde{r}_{ts}^{\tau} = {} & (\mathbf{z}_{ts}^{\substack{\bullet\\ \vert\\ \bullet},\tau} * y_{s}^{\substack{\bullet\\ \vert\\ \bullet},\tau,\cdot,\cdot})^{\otimes 2} + (R_{ts}^{y,\tau})^{\otimes 2} + (\mathbf{z}_{ts}^{\substack{\bullet\\ \vert\\ \bullet},\tau} * y_{s}^{\substack{\bullet\\ \vert\\ \bullet},\tau,\cdot,\cdot}) \otimes (\mathbf{z}_{ts}^{\bullet,\tau} * y_{s}^{\bullet,\tau,\cdot}) + (\mathbf{z}_{ts}^{\bullet,\tau} * y_{s}^{\bullet,\tau,\cdot}) \otimes (\mathbf{z}_{ts}^{\substack{\bullet\\ \vert\\ \bullet},\tau} * y_{s}^{\substack{\bullet\\ \vert\\ \bullet},\tau,\cdot,\cdot}) \\ & + (\mathbf{z}_{ts}^{\bullet,\tau} * y_{s}^{\bullet,\tau,\cdot}) \otimes R_{ts}^{y,\tau} + R_{ts}^{y,\tau} \otimes (\mathbf{z}_{ts}^{\bullet,\tau} * y_{s}^{\bullet,\tau,\cdot}) + (\mathbf{z}_{ts}^{\substack{\bullet\\ \vert\\ \bullet},\tau} * y_{s}^{\substack{\bullet\\ \vert\\ \bullet},\tau,\cdot,\cdot}) \otimes R_{ts}^{y,\tau} + R_{ts}^{y,\tau} \otimes (\mathbf{z}_{ts}^{\substack{\bullet\\ \vert\\ \bullet},\tau} * y_{s}^{\substack{\bullet\\ \vert\\ \bullet},\tau,\cdot,\cdot}). \end{aligned} \tag{4.36}$$

Plugging (4.35) into (4.33), we get

$$f(y^{\tau})_{ts} = y_{ts}^{\tau} f'(y_s^{\tau}) + \frac{1}{2}(\mathbf{z}_{ts}^{\bullet,\tau} * y_{s}^{\bullet,\cdot})^{\otimes 2} f''(y_s^{\tau}) + \frac{1}{2}\tilde{r}_{ts}^{\tau} f''(y_s^{\tau}) + r_{ts}^{\tau},$$

We now invoke (4.9) in order to further decompose y_{ts}^{τ} above. We end up with

$$f(y^{\tau})_{ts} = \mathbf{z}_{ts}^{\bullet,\tau} * y_{s}^{\bullet,\cdot} f'(y_s^{\tau}) + \mathbf{z}_{ts}^{\substack{\bullet\\ \vert\\ \bullet},\tau} * y_{s}^{\substack{\bullet\\ \vert\\ \bullet},\cdot,\cdot} f'(y_s^{\tau}) + \frac{1}{2}(\mathbf{z}_{ts}^{\bullet,\tau} * y_{s}^{\bullet,\cdot})^{\otimes 2} f''(y_s^{\tau}) + R_{ts}^{\phi,\tau}, \tag{4.37}$$

where the reminder R^{ϕ} is defined by

$$R_{ts}^{\phi,\tau} = R_{ts}^{y,\tau} f'(y_s^{\tau}) + \frac{1}{2}\tilde{r}_{ts}^{\tau} f''(y_s^{\tau}) + r_{ts}^{\tau}. \tag{4.38}$$

Thanks to (4.37) and the definition of the relations in (4.30)-(4.31), the proof of $(\phi, \phi^{\bullet}, \phi^{\substack{\bullet\\ \vert\\ \bullet}}, \phi^{\bullet\bullet}) \in \mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}$ are now reduced to proving the following two claims:

- Claim 1: The remainder term $R_{ts}^{\phi,\tau}$ in (4.38) is of order 3. Specifically, due to (4.33) and the fact that $(y, y^{\bullet}, y^{\substack{\bullet\\ \vert\\ \bullet}}, 0) \in \hat{\mathcal{D}}^{(\alpha,\gamma)}$, relation (4.38) this is reduced to the following claim:

$$\tilde{r} \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}. \tag{4.39}$$

- Claim 2: $\phi^{\bullet}$ fulfills relation (4.10), which can be written as

$$\phi^{\bullet,\cdot,\cdot} - \mathbf{z}^{\bullet,\cdot} * (\phi^{\overset{\bullet}{\bullet},\cdot,\cdot,\cdot} + 2\phi^{\bullet\bullet,\cdot,\cdot,\cdot}) \in \mathcal{W}_2^{(2\rho+2\gamma,2\gamma,\eta,\zeta)}, \tag{4.40}$$

where we recall that $\phi^{\bullet}$, $\phi^{\overset{\bullet}{\bullet}}$, $\phi^{\bullet\bullet}$ are defined by (4.30)-(4.31).

In the following, we will prove those two Claims separately.

Proof of Claim 1. According to relation (4.36), there are eight terms to evaluate in $\tilde{r}$. For conciseness, we can consider one of these terms, say the increment I_{ts}^{τ} defined by $I_{ts}^{\tau} = (\mathbf{z}_{ts}^{\overset{\bullet}{\bullet},\tau} * y_s^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot}) \otimes (\mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\tau,\cdot})$, and the remaining terms will follow directly from similar considerations. To this aim, a first observation is that since $(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0) \in \hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}$ as given in (4.29), then both $y^{\bullet}$ and $y^{\overset{\bullet}{\bullet}}$ don't dependent on τ and we have $I_{ts}^{\tau} = (\mathbf{z}_{ts}^{\overset{\bullet}{\bullet},\tau} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot}) \otimes (\mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot})$. Moreover, due to the fact that I is part of the reminder $\tilde{r}$, we have to evaluate $\|I\|_{(3\rho+2\gamma,2\gamma,\eta,\zeta)}$. Owing to Definition 2.2.4, this is equivalent to evaluate

$$\|I\|_{(3\rho+2\gamma,2\gamma,\eta,\zeta)} = \|I\|_{(3\rho+2\gamma,2\gamma),1} + \|I\|_{(3\rho+2\gamma,2\gamma,\eta,\zeta),1,2}. \tag{4.41}$$

In order to upper bound the right hand side of (4.41), it suffices to estimate $|I_{ts}^{\tau}|$ and $|I_{ts}^{qp}|$. Some elementary computations reveal that

$$|I_{ts}^{\tau}| = \left| \left(\mathbf{z}_{ts}^{\overset{\bullet}{\bullet},\tau} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot} \right) \otimes (\mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot}) \right| \lesssim \left| \mathbf{z}_{ts}^{\overset{\bullet}{\bullet},\tau} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot} \right| \left| \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \right|, \tag{4.42}$$

and

$$\begin{aligned} |I_{ts}^{qp}| = |I_{ts}^{q} - I_{ts}^{p}| &= \left| \left(z_{ts}^{\overset{\bullet}{\bullet},q} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot} \right) \otimes (\mathbf{z}_{ts}^{\bullet,q} * y_s^{\bullet,\cdot}) - \left(\mathbf{z}_{ts}^{\overset{\bullet}{\bullet},p} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot} \right) \otimes (\mathbf{z}_{ts}^{\bullet,p} * y_s^{\bullet,\cdot}) \right| \\ &\lesssim |\mathbf{z}_{ts}^{\bullet,qp} * y_s^{\bullet,\cdot}| \left| \mathbf{z}_{ts}^{\overset{\bullet}{\bullet},q} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot} \right| + \left| \mathbf{z}_{ts}^{\overset{\bullet}{\bullet},qp} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot} \right| |\mathbf{z}_{ts}^{\bullet,q} * y_s^{\bullet,\cdot}|. \end{aligned} \tag{4.43}$$

To bound the right hand side of (4.42), thanks to a slight variation of Proposition 2.2.15 and Theorem 2.2.20, we have

$$|I_{ts}^{\tau}| \lesssim (|y_0^{\overset{\bullet}{\bullet}}| + |y_0^{\bullet}| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)})^2 \|\!|\mathbf{z}|\!\|_{(\alpha,\gamma,\eta,\zeta)}^2 \, \psi^1_{(3\rho+2\gamma),2\gamma}(\tau, t, s).$$

Similarly, we can bound $|I^{qp}_{ts}|$ is the following way:

$$|I^{qp}_{ts}| \lesssim (|y^{\overset{\bullet}{\bullet}}_0| + |y^{\bullet}_0| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)})^2 |||\mathbf{z}|||^2_{(\alpha,\gamma,\eta,\zeta)}\, \psi^{1,2}_{(3\rho+2\gamma),2\gamma,\eta,\zeta}(p, q, t, s) \tag{4.44}$$

It follows by definition of the quantities $\|I\|_{(3\rho+2\gamma,2\gamma),1}$ and $\|I\|_{(3\rho+2\gamma,2\gamma,\eta,\zeta),1,2}$ as given in Definition 2.2.4, that

$$\|I\|_{(3\rho+2\gamma,2\gamma),1} \vee \|I\|_{(3\rho+2\gamma,2\gamma,\eta,\zeta),1,2} \lesssim (|y^{\overset{\bullet}{\bullet}}_0| + |y^{\bullet}_0| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{(\alpha,\gamma,\eta,\zeta)})^2 |||\mathbf{z}|||^2_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} \tag{4.45}$$

which implies $I \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$ according to Lemma 2.2.8. Similarly, we let the patient reader check that

$$(\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\tau,\cdot}_s) \otimes (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot}_s) \in \mathcal{V}^{(3\rho+2\gamma,2\gamma,\eta,\zeta)}, \quad (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}_s)^{\otimes 2} \in \mathcal{V}^{(4\rho+2\gamma,2\gamma,\eta,\zeta)}, \quad (R^y_{ts})^{\otimes 2} \in \mathcal{V}^{(6\rho+6\gamma,6\gamma,\eta,\zeta)}, \tag{4.46}$$

as well as

$$(\mathbf{z}^{\bullet,\tau}_{ts} * y^{\bullet,\cdot}_s) \otimes R^y_{ts} \in \mathcal{V}^{(4\rho+4\gamma,4\gamma,\eta,\zeta)}, \quad (\mathbf{z}^{\overset{\bullet}{\bullet},\tau}_{ts} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}_s) \otimes R^y_{ts} \in \mathcal{V}^{(5\rho+4\gamma,4\gamma,\eta,\zeta)}. \tag{4.47}$$

In fact the appropriate norm for each of these terms is easily seen to be bounded by the product $(|||\mathbf{z}|||_{(\alpha,\gamma,\eta,\zeta)} + |||\mathbf{z}|||^2_{(\alpha,\gamma,\eta,\zeta)})(|y^{\overset{\bullet}{\bullet}}_0| + |y^{\bullet}_0| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{(\alpha,\gamma,\eta,\zeta)})^2$. Combining (4.45), (4.46) and (4.47), we have thus obtained that $\tilde{r} \in \mathcal{V}^{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$, and it follows that

$$\|\tilde{r}\|_{(3\rho+3\gamma,\gamma,\eta,\zeta)} \lesssim (|||\mathbf{z}|||_{(\alpha,\gamma,\eta,\zeta)} + |||\mathbf{z}|||^2_{(\alpha,\gamma,\eta,\zeta)})(|y^{\overset{\bullet}{\bullet}}_0| + |y^{\bullet}_0| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)})^2 \tag{4.48}$$

Proof of Claim 2. Before proving relation (4.40), we will give some algebraic insight on the terms of ϕ^{σ} for $\sigma \in \mathcal{F}_2$. Indeed, resorting to (4.37), we can safely set

$$\phi^{\bullet,q,p}_t = y^{\bullet,p}_t f'(y^q_t), \tag{4.49}$$

as stated in (4.30). According to (4.37), we also let

$$\phi_t^{\overset{\bullet}{\bullet},r,q,p} = y_t^{\overset{\bullet}{\bullet},q,p} f'\left(y_t^r\right), \qquad \phi_t^{\bullet\bullet,r,q,p} = \frac{1}{2}(y_t^{\bullet,q}) \otimes (y_t^{\bullet,p}) f''\left(y_t^r\right). \tag{4.50}$$

With relation (4.9) in mind, we can rewrite (4.37) as

$$f(y_t^\tau) - f(y_s^\tau) = \mathbf{z}_{ts}^{\bullet,\tau} * \phi_s^{\bullet,\tau,\cdot} + \mathbf{z}_{ts}^{\overset{\bullet}{\bullet},\tau} * \phi_s^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot} + (\mathbf{z}_{ts}^{\bullet,\tau})^{\otimes 2} * \phi_s^{\bullet\bullet,\tau,\cdot,\cdot} + R_{ts}^{\phi,\tau} \tag{4.51}$$

Let us briefly give a few details regarding the expressions on the right hand side of (4.51). Specifically, we will explain how to compute $\mathbf{z}_{ts}^{\bullet,\tau} * \phi_s^{\bullet,\tau,\cdot} = \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} f'(y_s^\tau)$. Referring to Notation 2.2.14, the expression $\mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} f'(y_s^\tau)$ can be rewritten as $[(\mathbf{z}_{ts}^{\bullet,\tau})^\intercal * (y_s^{\bullet,\cdot} f'(y_s^\tau))^\intercal]^\intercal = f'(y_s^\tau) y_s^{\bullet,\cdot} * \mathbf{z}_{ts}^{\bullet,\tau}$, where we have used also that $y_s^{\bullet,\cdot} f'(y_s^\tau)$ can be rewritten as $[(y_s^{\bullet,\cdot})^\intercal (f'(y_s^\tau))^\intercal]^\intercal = f'(y_s^\tau) y_s^{\bullet,\cdot}$. In addition, notice that $f'(y_s) \in \mathbb{R}^d$, $y_s^{\bullet} \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ and $\mathbf{z}_{ts}^{\bullet,\tau} \in \mathbb{R}^d$. Therefore the quantity $\mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} f'(y_s^\tau)$ has to be interpreted as an inner product, and we let the patient reader perform the same kind of manipulation for the term $z_{ts}^{\overset{\bullet}{\bullet},\tau} * y^{\overset{\bullet}{\bullet}} f'(y_s^\tau)$. In the end we get that both the left hand side and the right hand side of (4.51) are real-valued.

Now we are ready to prove (4.40). To this aim we set

$$J_{ts}^{\tau,\cdot} := \phi_{ts}^{\bullet,\tau,\cdot} - \mathbf{z}_{ts}^{\bullet,\tau} * (\phi_s^{\overset{\bullet}{\bullet},\tau,\cdot,\cdot} + 2\phi_s^{\bullet\bullet,\tau,\cdot,\cdot}). \tag{4.52}$$

Our claim (4.40) amounts to show that $J \in \mathcal{W}_2^{(2\rho+2\gamma,\gamma,\eta,\zeta)}$, with $\mathcal{W}_2^{(2\rho+2\gamma,2\gamma,\eta,\zeta)}$ given in Definition 2.2.18. Thanks to (4.49) and (4.50), we first write

$$J_{ts}^{\tau,\cdot} = y_t^{\bullet,\cdot}\left(f'(y_t^\tau) - f'(y_s^\tau)\right) + y_{ts}^{\bullet,\cdot} f'(y_s^\tau) - \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot} f'(y_s^\tau) - \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\cdot} f''\left(y_s^\tau\right). \tag{4.53}$$

We now invoke (4.10), recalling that $y^{\bullet\bullet} = 0$ since we have assumed that $y \in \hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}$. Plugging this information in (4.53), we end up with

$$J_{ts}^{\tau,\tau} = y_t^{\bullet,\tau}\left(f'(y_t^\tau) - f'(y_s^\tau)\right) + R_{ts}^{\bullet,\tau} f'(y_s^\tau) - \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\tau} f''(y_s^\tau). \tag{4.54}$$

Let us apply a Taylor expansion to the first term of right hand side of (4.54). Specifically we write

$$f'(y_t^\tau) - f'(y_s^\tau) - y_{ts}^\tau f''(y_s^\tau) = F_{ts}^{(2),\tau},$$

where the term $F_{ts}^{(2),\tau} = (F_{ts}^{(2),\tau,1}, \dots, F_{ts}^{(2),\tau,d})$ is defined as a reminder in a Taylor expansion. Namely consider multi-indices $\beta = (\beta_1, \dots, \beta_d)$ with $\beta_{\rm i} \in \{0,1,2\}$. We set $|\beta| = \sum_{\rm j=1}^d \beta_{\rm j}$ and $|\beta|! = \Pi_{\rm j=1}^d \beta_{\rm j}!$. Then for $\rm i = 1, \dots, d$, $F_{ts}^{(2),\tau,\rm i}$ is given by

$$F_{ts}^{(2),\tau,\rm i} = 2 \sum_{|\beta|=2} \frac{(y_{ts}^\tau)^{\otimes|\beta|}}{\beta!} \int_0^1 (1-r)\partial^\beta \left(\partial_{\rm i} f(y_s^\tau + r y_{st}^\tau)\right) dr. \tag{4.55}$$

With expression (4.55) in hand and recalling (4.54), we thus get

$$\begin{aligned} J_{ts}^{\tau,\tau} &= y_t^{\bullet,\tau}\left(f'(y_t^\tau) - f'(y_s^\tau) - y_{ts}^\tau f''(y_s^\tau)\right) + y_t^{\bullet,\tau} y_{ts}^\tau f''(y_s^\tau) - \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\tau} f''(y_s^\tau) + R_{ts}^{\bullet,\tau} f'(y_s^\tau) \\ &= y_t^{\bullet,\tau} F_{ts}^{(2),\tau} + R_{ts}^{\bullet,\tau} f'(y_s^\tau) + y_t^{\bullet,\tau} y_{ts}^\tau f''(y_s^\tau) - \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\tau} f''(y_s^\tau), \end{aligned} \tag{4.56}$$

Furthermore, we plug in identity (4.9) in the above expansion in order to expand the term $y_t^{\bullet,\tau} y_{ts}^\tau f''(y_s^\tau)$ in (4.56). This yields

$$\begin{aligned} J_{ts}^{\tau,\tau} &= y_t^{\bullet,\tau} F_{ts}^{(2),\tau} + R_{ts}^{\bullet,\tau} f'(y_s^\tau) + y_t^{\bullet,\tau} \mathbf{z}_{ts}^{\genfrac{}{}{0pt}{}{\bullet}{\bullet},\tau} * y_s^{\genfrac{}{}{0pt}{}{\bullet}{\bullet},\cdot,\cdot} f''(y_s^\tau) + y_t^{\bullet,\tau} R_{ts}^\tau f''(y_s^\tau) \\ &\quad + y_t^{\bullet,\tau} \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} f''(y_s^\tau) - \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\tau} f''(y_s^\tau). \end{aligned} \tag{4.57}$$

Next we resort to the forthcoming identity (4.70) in order to handle the term $\mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} \otimes y_s^{\bullet,\tau} f''(y_s^\tau)$ above. One obtains that the last two terms in (4.57) combine into one term $y_{ts}^{\bullet,\tau} \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} f''(y_s^\tau)$. We end up with

$$\begin{aligned} J_{ts}^{\tau,\tau} =& y_t^{\bullet,\tau} F_{ts}^{(2),\tau} + R_{ts}^{\bullet,\tau} f'(y_s^\tau) + y_t^{\bullet,\tau} \mathbf{z}_{ts}^{\genfrac{}{}{0pt}{}{\bullet}{\bullet},\tau} * y_s^{\genfrac{}{}{0pt}{}{\bullet}{\bullet},\cdot,\cdot} f''(y_s^\tau) \\ &+ y_t^{\bullet,\tau} R_{ts}^\tau f''(y_s^\tau) + y_{ts}^{\bullet,\tau} \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} f''(y_s^\tau). \end{aligned} \tag{4.58}$$

In the same way, we let the patient reader check that we can rewrite $J_{ts}^{q,q} - J_{ts}^{p,p}$ as

$$J_{ts}^{q,q} - J_{ts}^{p,p} = J_1^{qp} + J_2^{qp} + J_3^{qp} + J_4^{qp} + J_5^{qp}, \tag{4.59}$$

where the terms J_1^{qp}, J_2^{qp}, J_3^{qp}, J_4^{qp}, and J_5^{qp} are defined respectively by

$$
\begin{aligned}
J_1^{qp} &= y_t^{\bullet,qp} F_{ts}^{(2),q} + y_t^{\bullet,p}\left(F_{ts}^{(2),q} - F_{ts}^{(2),p}\right)\\
J_2^{qp} &= \left(y_t^{\bullet,p}\mathbf{z}_{ts}^{\overset{\bullet}{\bullet},qp} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot} + y_t^{\bullet,qp}\mathbf{z}_{ts}^{\overset{\bullet}{\bullet},q} * y^{\overset{\bullet}{\bullet},\cdot,\cdot}\right) f''(y_s^q) + \left(y_t^{\bullet,p}\mathbf{z}_{ts}^{\overset{\bullet}{\bullet},p} * y_s^{\overset{\bullet}{\bullet},\cdot,\cdot}\right)\left(f''(y_s^q) - f''(y_s^p)\right)\\
J_3^{qp} &= \left(y_t^{\bullet,p} R_{ts}^{qp} + y_t^{\bullet,qp} R_{ts}^{q}\right) f''(y_s^q) + y_t^{\bullet,p} R_{ts}^{p}\left(f''(y_s^q) - f''(y_s^p)\right), \qquad (4.60)\\
J_4^{qp} &= \left(y_{ts}^{\bullet,qp}\mathbf{z}_{ts}^{\bullet,q} * y_s^{\bullet,\cdot} + y_{ts}^{\bullet,p}\mathbf{z}_{ts}^{\bullet,qp} * y_s^{\bullet,\cdot}\right) f''(y_s^q) + y_{ts}^{\bullet,p}\mathbf{z}_{ts}^{\bullet,p} * y_s^{\bullet,\cdot}\left(f''(y_s^q) - f''(y_s^p)\right)\\
J_5^{qp} &= R_{ts}^{\bullet,qp} f'(y_s^q) + R_{ts}^{\bullet,p}\left(f'(y_s^q) - f'(y_s^p)\right).
\end{aligned}
$$

With (4.58)-(4.60) at hand, and recalling Definition 2.2.18 for the spaces $\mathcal{W}$, it is readily checked, using the information of the regularities in the different terms of J_{i} for $\mathrm{i} = 1, \dots, 5$ that $J \in \mathcal{W}_2^{(2\rho+2\gamma,2\gamma,\eta,\zeta)}$. We omit further details, as the arguments follows directly along the same lines as in previous computations in the proof of claim 1.

Summarizing our analysis so far, we have now proved both Claim 1 and Claim 2 above. Therefore we obtain that $(\phi, \phi^{\bullet}, \phi^{\overset{\bullet}{\bullet}}, \phi^{\bullet\bullet})$ is an element of $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}$.

Step 2: Proof of relation (4.32). According to the definition (4.11) for the norm in $\mathcal{D}_{\mathbf{z}}^{(\alpha,\gamma)}$, we have

$$
\|(\phi, \phi^{\bullet}, \phi^{\overset{\bullet}{\bullet}}, \phi^{\bullet\bullet})\|_{\mathbf{z};(\alpha,\gamma,\eta,\zeta)} = \|\phi^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)} + \|\phi^{\bullet\bullet}\|_{(\alpha,\gamma,\eta,\zeta)} + \|R^{\phi}\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)} + \|R^{\phi^{\bullet}}\|_{(2\rho+2\gamma,2\gamma,\eta,\zeta)}. \qquad (4.61)
$$

In the following, we will bound four terms in the right hand side of (4.61) separately.

We begin to handle the term $\|\phi^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta)}$ in (4.61). We recall that $\phi^{\overset{\bullet}{\bullet}}$ is given by (4.31), and its (α,γ)-norm is introduced in Definition 2.3.6. According to this definition, it is thus enough to bound $\|\phi^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma),1}$ and $\|\phi^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$. Towards this aim, we write

$$
\begin{aligned}
\left|\phi_{ts}^{\overset{\bullet}{\bullet},\tau,\tau,\tau}\right| &= \left|y_t^{\overset{\bullet}{\bullet},\tau,\tau} f'(y_t^{\tau}) - y_s^{\overset{\bullet}{\bullet},\tau,\tau} f'(y_s^{\tau})\right| = \left|y_t^{\overset{\bullet}{\bullet},\tau,\tau}\left(f'(y_t^{\tau}) - f'(y_s^{\tau})\right) + y_{ts}^{\overset{\bullet}{\bullet},\tau,\tau} f'(y_s^{\tau})\right|\\
&\lesssim \|f\|_{C_b^2}\left(|y_0^{\overset{\bullet}{\bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2}\right)\psi_{\alpha,\gamma}^{1}(\tau,t,s), \qquad (4.62)
\end{aligned}
$$

where psi^1 as given in (2.1). This yields

$$\|\phi^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma),1} \lesssim \|f\|_{C_b^2}(|y_0^{\substack{\bullet\\ \bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2}). \tag{4.63}$$

We now wish to handle the norm $\|\phi^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$ in (3.15). Otherwise stated, we wish to bound the terms in the right hand side of (3.17) for $\phi^{\substack{\bullet\\ \bullet}}$. For the term $\|\phi^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2,>}$, we thus write

$$\begin{aligned}
\left|\phi_{ts}^{\substack{\bullet\\ \bullet},p',p_2,p} - \phi_{ts}^{\substack{\bullet\\ \bullet},p',p_1,p}\right| &= \left|y_t^{\substack{\bullet\\ \bullet},p_2,p} f'(y_t^{p'}) - y_s^{\substack{\bullet\\ \bullet},p_2,p} f'(y_s^{p'}) - y_t^{\substack{\bullet\\ \bullet},p_1,p} f'(y_t^{p'}) + y_s^{\substack{\bullet\\ \bullet},p_1,p} f'(y_s^{p'})\right| \\
&\leq \left|\left(y_{ts}^{\substack{\bullet\\ \bullet},p_2,p} - y_{ts}^{\substack{\bullet\\ \bullet},p_1,p}\right) f'(y_t^{p'})\right| + \left|\left(y_s^{\substack{\bullet\\ \bullet},p_2,p} - y_s^{\substack{\bullet\\ \bullet},p_1,p}\right)\left(f'(y_t^{p'}) - f'(y_s^{p'})\right)\right|.
\end{aligned}$$

In addition, owing to Remark 2.4.9 and (3.20) and since $\mathbf{y} \in \hat{D}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}$, we have $y^{\substack{\bullet\\ \bullet}} \in \mathcal{W}_2^{(\alpha,\gamma,\eta,\zeta)}$. Due to the fact that y is also an element of $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ according to Remark 2.4.8, we get

$$\left|\phi_{ts}^{\substack{\bullet\\ \bullet},p',p_2,p} - \phi_{ts}^{\substack{\bullet\\ \bullet},p',p_1,p}\right| \lesssim \|f\|_{C_b^2}(|y_0^{\substack{\bullet\\ \bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2})\, \psi_{\alpha,\gamma,\eta,\zeta}^{1,2}(p_2,p_1,t,s), \tag{4.64}$$

where $\psi^{1,2}$ as given in (2.2). We thus have

$$\|\phi^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2} \lesssim \|f\|_{C_b^2}(|y_0^{\substack{\bullet\\ \bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2}). \tag{4.65}$$

Moreover, it is easily seen that $\|\phi^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,3}$ and $\|\phi^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta),2,3}$ are bounded exactly in the same way as (4.65). Hence we get the following bound for $\|\phi^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3}$:

$$\|\phi^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2,3} \lesssim \|f\|_{C_b^2}(|y_0^{\substack{\bullet\\ \bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2}). \tag{4.66}$$

Eventually, plugging (4.63) and (4.66) into (3.15), we obtain the desired bound for $\|\phi^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta)}$:

$$\|\phi^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta)} \lesssim \|f\|_{C_b^2}(|y_0^{\substack{\bullet\\ \bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\substack{\bullet\\ \bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2}). \tag{4.67}$$

We let the reader check that the term $\|\phi^{\bullet\bullet}\|_{(\alpha,\gamma,\eta,\zeta)}$ in (4.61) can be treated in a similar way. Indeed, $\phi^{\bullet\bullet}$ has to be considered as a process in $\mathcal{W}_3$, exactly like $\phi^{\substack{\bullet\\ \bullet}}$. Therefore owing to the definition (4.31) of $\phi^{\bullet\bullet}$ and to the definition (3.17) of the $(1,2,3)$-norm in $\mathcal{W}_3$, we get the following bound along the same lines as (4.62)-(4.67):

$$\|\phi^{\bullet\bullet}\|_{(\alpha,\gamma,\eta,\zeta)} \lesssim \|f\|_{C_b^2}(|y_0^{\overset{\bullet}{\bullet}}| + \|y\|_{(\alpha,\gamma),1} + \|y^{\overset{\bullet}{\bullet}}\|_{(\alpha,\gamma,\eta,\zeta),1,2}). \tag{4.68}$$

We are now ready to bound the fourth term $\|R^{\phi^{\bullet}}\|_{(2\rho+2\gamma,2\gamma,\eta,\zeta)}$ in the right hand side of (4.61). To this aim, recall that according to (4.10) we have

$$R_{ts}^{\bullet,\tau,p} = y_{ts}^{\bullet,\tau,p} - \mathbf{z}_{ts}^{\bullet,\tau} * \left(y_s^{\overset{\bullet}{\bullet},\tau,p,\cdot} + 2y_s^{\bullet\bullet,\tau,p,\cdot} \right).$$

Comparing this expression to (4.52), we get $R^{\phi^{\bullet}} = J$. Now recall that J has been analyzed through a decomposition in (4.58)-(4.60). Note that all the terms appearing in the decomposition are directly bounded due to the fact that $(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0) \in \hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma)}$. It is therefore readily checked that

$$\begin{aligned}\|R^{\phi^{\bullet}}\|_{(2\rho+2\gamma,2\gamma,\eta,\zeta)} \leq C(1 + \|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)})^3 \Big[&\left(|y_0^{\bullet}| + |y_0^{\overset{\bullet}{\bullet}}| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} \right) \\ &\vee \left(|y_0^{\bullet}| + |y_0^{\overset{\bullet}{\bullet}}| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} \right)^3 \Big].\end{aligned}$$

Eventually, we handle the term $\|R^{\phi}\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)}$ in (4.61). Recall that R^{ϕ} is given by (4.38), and that we have already bounded the term r and $\tilde{r}$ in (4.34) and (4.48) respectively. Furthermore, it follows directly that $\|R^{y}\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)} \leq \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma.\eta,\zeta)}$. Combining the above considerations, we see that

$$\begin{aligned}\|R^{\phi}\|_{(3\rho+3\gamma,3\gamma,\eta,\zeta)} \lesssim (1 + \|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)})^3 \Big[&\left(|y_0| + |y_0^{\bullet}| + |y_0^{\overset{\bullet}{\bullet}}| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} \right) \\ &\vee \left(|y_0| + |y_0^{\bullet}| + |y_0^{\overset{\bullet}{\bullet}}| + \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)} \right)^3 \Big].\end{aligned}$$

Gathering the bounds found above, it is now evident that

$$
\begin{aligned}
\|(\phi,\phi^{\bullet},\phi^{\overset{\bullet}{\bullet}},\phi^{\bullet\bullet})\|_{\mathbf{z};(\alpha,\gamma,\eta,\zeta)} \lesssim \left(1+\|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)}\right)^{3} \Bigg[& \left(|y_0|+|y_0^{\bullet}|+|y_0^{\overset{\bullet}{\bullet}}|+\|(y,y^{\bullet},y^{\overset{\bullet}{\bullet}},0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\right) \\
& \vee \left(|y_0|+|y_0^{\bullet}|+|y_0^{\overset{\bullet}{\bullet}}|+\|(y,y^{\bullet},y^{\overset{\bullet}{\bullet}},0)\|_{\mathbf{z},(\alpha,\gamma,\eta,\zeta)}\right)^{3}\Bigg],
\end{aligned}
$$

where the hidden constant depends on $\|f\|_{C_b^4}$, α, and γ. The above relation is exactly (4.32), which concludes our proof. $\square$

Remark 2.4.14. In Proposition 2.4.13 we have obtained useful bounds on the composition map from $\hat{D}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2([0,T]);\mathbb{R}^d)$ to $D_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2([0,T]);\mathbb{R}^m)$. Let us now choose a parameter β such that $\beta<\alpha$ and we still have $\beta-\gamma>\frac{1}{4}$. We will in the next section onsider the composition map from $\hat{\mathcal{D}}_{\mathbf{z}}^{(\beta,\gamma,\eta,\zeta)}(\Delta_2([0,T]);\mathbb{R}^d)$ to $\mathcal{D}_{\mathbf{z}}^{(\beta,\gamma,\eta,\zeta)}(\Delta_2([0,T]);\mathbb{R}^m)$. Due to Remark 2.2.7, it is readily checked that there exists a constant $C=C_{M,\alpha,\beta,\gamma,\eta,\zeta,\|f\|_{C_b^5}}$ such that,

$$
\begin{aligned}
\|(\phi,\phi^{\bullet},\phi^{\overset{\bullet}{\bullet}},\phi^{\bullet\bullet})\|_{\mathbf{z};(\beta,\gamma,\eta,\zeta)} \le C\left(1+\|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)}\right)^{3} & \left(\left[|y_0^{\bullet}|+|y_0^{\overset{\bullet}{\bullet}}|+\|(y,y^{\bullet},y^{\overset{\bullet}{\bullet}},0)\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}\right]\right) \\
& \vee \left(\left[|y_0^{\bullet}|+|y_0^{\overset{\bullet}{\bullet}}|+\|(y,y^{\bullet},y^{\overset{\bullet}{\bullet}},0)\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}\right]^{3}\right) T^{\alpha-\beta}. \quad (4.69)
\end{aligned}
$$

We close this section by presenting a technical result which leads to some useful cancellations in the rough path expansion (4.57).

Lemma 2.4.15. *Let $f\in C_b^4(\mathbb{R}^d)$ and assume $(y,y^{\bullet},y^{\overset{\bullet}{\bullet}},0)\in\hat{\mathcal{D}}_{\mathbf{z}}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^d)$ as given in Remark 2.4.12. Also recall our Notation 2.2.14 for matrix products. Then for any $(s,t,\tau)\in\Delta_3$, we have*

$$
y_s^{\bullet,\tau}\mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot} f''(y_s^{\tau}) = \mathbf{z}_{ts}^{\bullet,\tau} * y_s^{\bullet,\cdot}\otimes y_s^{\bullet,\tau} f''(y_s^{\tau}) \tag{4.70}
$$

Proof. Let L (respectively M) be the left hand side (respectively right hand side) of (4.70) . Recalling the dimension considerations after equation (4.51), notice that both L and M are elements of $\mathbb{R}^{d\times d}$. For $a\in\mathbb{R}^d$, we consider the matrix products aL and aM in the sense of

Notation 2.2.14. In particular, our Notation 2.2.14 implies that aL has to be interpreted as $f''(y_s^\tau)y_s^{\bullet,\cdot} * \mathbf{z}_{ts}^{\bullet,\tau} y_s^{\bullet,\tau} a$. Expressing this in coordinates we get

$$\begin{aligned} aL &= \sum_{\mathrm{i,j}=1}^{m} f''(y_s^\tau)^{\mathrm{ij}} \sum_{\mathrm{i}_1=1}^{m} y_s^{\bullet,\cdot,\mathrm{ii}_1} \mathbf{z}_{ts}^{\bullet,\tau,\mathrm{i}_1} \sum_{\mathrm{j}_1=1}^{m} y_s^{\bullet,\tau,\mathrm{jj}_1} a^{\mathrm{j}_1} \\ &= \sum_{\mathrm{i,j,i_1,j_1}=1}^{m} f''(y_s^\tau)^{\mathrm{ij}} y_s^{\bullet,\cdot,\mathrm{ii}_1} \mathbf{z}_{ts}^{\bullet,\tau,\mathrm{i}_1} y_s^{\bullet,\tau,\mathrm{jj}_1} a^{\mathrm{j}_1}. \end{aligned} \tag{4.71}$$

Similarly, the product aM can be expressed as

$$aM = f''(y_s^\tau)^{\mathrm{ij}} y_s^{\bullet,\tau} \otimes y_s^{\bullet,\tau} * \mathbf{z}_{ts}^{\bullet,\cdot} \cdot a = \sum_{\mathrm{i,j,i_1,j_1}=1}^{m} f''(y_s^\tau)^{\mathrm{ij}} y_s^{\bullet,\cdot,\mathrm{ii}_1} y_s^{\bullet,\tau,\mathrm{jj}_1} \mathbf{z}_{ts}^{\bullet,\tau,\mathrm{i}_1} a^{\mathrm{j}_1}. \tag{4.72}$$

Comparing (4.71) and (4.72), it is clear that $aL = aM$ for any $a \in R^d$. Thus $L = M$, which finishes the proof. □

2.4.2 Rough Volterra Equations

In this section we gather all the element of stochastic calculus put forward in Sections 2.3.2-2.4.1, in order to achieve one of main goals in this paper. Namely we will solve Volterra type equations in a very rough setting.

We start by introducing a new piece of notation.

Notation 2.4.16. *Let us define a new space* $\mathcal{D}_{\mathbf{z};\mathbf{y}_0}^{(\beta,\gamma,\eta,\zeta)}\left(\Delta_2^T\left(\left[0,\bar{T}\right]\right);\mathbb{R}^d\right)$, *where* $\mathbf{y}_0$ *is of the form* $(y_0, y_0^{\bullet}, y_0^{\ddagger}, y_0^{\bullet\bullet})$. *For* $0 \le a < b \le T$ *we define a simplex type set* $\Delta_2^T([a,b])$ *as follows,*

$$\Delta_2^T([a,b]) = \left\{(s,\tau) \in [a,b] \times [0,T] \,\middle|\, a \le s \le \tau \le T\right\}. \tag{4.73}$$

Note that the first component of $(s,\tau) \in \Delta_2^T([a,b])$ *is restricted to* $[a,b]$ *while the second component is allowed to vary in the whole interval* $[0,T]$. *Without loss of generality, we assume that* $\|\mathbf{z}\|_{(\alpha,\gamma)} \le M \in \mathbb{R}_+$. *As in Remark 2.4.14, we choose a parameter* $\beta < \alpha$ *but still satisfying* $\beta - \gamma > 1/4$. *Let us also consider a time horizon* $\bar{T} \le T$ *(this* $\bar{T}$ *will be* *made small enough to perform a contraction argument later on). We will work on a space* $\mathcal{D}^{(\beta,\gamma)}_{\mathbf{z},\mathbf{y}_0}(\Delta_2^T([0,\bar{T}]);\mathbb{R}^m)$ *defined by*

$$\mathcal{D}^{(\beta,\gamma,\eta,\zeta)}_{\mathbf{z},\mathbf{y}_0}(\Delta_2^T([0,\bar{T}]);\mathbb{R}^m) = \Big\{ \left(y, y^{\bullet}, y^{\substack{\bullet\\ \bullet}}, y^{\bullet\bullet}\right) \in \mathcal{D}^{(\beta,\gamma,\eta,\zeta)}_{\mathbf{z}}\left(\Delta_2^T\left(\left[0,\bar{T}\right]\right);\mathbb{R}^m\right) \Big| \\ \mathbf{y}_0 = \{y_0^{\tau}, y_0^{\bullet,\tau}, y_0^{\substack{\bullet\\ \bullet},\tau,\tau}, y_0^{\bullet\bullet,\tau,\tau}\} = \{y_0, y_0^{\bullet}, y_0^{\substack{\bullet\\ \bullet}}, y_0^{\bullet\bullet}\}\Big\}. \quad (4.74)$$

Notice that the norm on $D^{(\beta,\gamma,\eta,\zeta)}_{\mathbf{z},\mathbf{y}_0}$ *is still defined by* (4.11)*. The only difference between* $D^{(\beta,\gamma,\eta,\zeta)}_{\mathbf{z},\mathbf{y}_0}$ *and* $D^{(\beta,\gamma,\eta,\zeta)}_{\mathbf{z}}$ *in Definition 2.4.7 is that* $D^{(\beta,\gamma,\eta,\zeta)}_{\mathbf{z},\mathbf{y}_0}$ *has an affine space structure, in contrast with the Banach space nature of* $D^{(\beta,\gamma,\eta,\zeta)}_{\mathbf{z}}$*.*

We are now ready to solve Volterra type equations in the rough case $\alpha - \gamma > \frac{1}{4}$.

Theorem 2.4.17. *Consider a path* $x \in \mathsf{c}^{\alpha}([0,T];\mathbb{R}^d)$*, and let* $k : \Delta_2 \to \mathbb{R}$ *be a Volterra kernel of order* γ*, with* $\alpha - \gamma > \frac{1}{4}$*. Define* $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2;\mathbb{R}^d)$ *by* $z_t^{\tau} = \int_0^t k(\tau,r)dx_r$ *and assume there exists a tree indexed Volterra rough path* $\mathbf{z} = \{\mathbf{z}^{\sigma,\tau};\sigma\in\mathcal{T}_3\}$ *above* z *satisfying Hypothesis 2.4.3. Additionally, suppose* $f \in \mathcal{C}_b^5(\mathbb{R}^m;\mathcal{L}(\mathbb{R}^d;\mathbb{R}^m))$*. Then there exists a unique solution in* $\mathcal{D}^{(\alpha,\gamma,\eta,\zeta)}_{\mathbf{z}}(\mathbb{R}^m)$ *to the Volterra equation*

$$y_t^{\tau} = y_0 + \int_0^t k(\tau,r)\,dx_r f(y_r^r), \quad (t,\tau)\in\Delta_2([0,T]), \quad y_0\in\mathbb{R}^m, \tag{4.75}$$

where the integral is understood as a rough Volterra integral according to Theorem 2.4.10.

Proof. We will proceed in a classical way by (i) Establishing a fixed point argument on a small interval. (ii) Patching the solutions obtained on the small intervals. Since this procedure is standard, we will skip some details.

We wish to solve (4.75) in a class of controlled processes. This means that the right hand side of (4.75) has to be understood according to Theorem 2.4.10. In particular referring to Theorem 2.4.10 *(iv)*, the controlled process $\mathbf{y}$ will be of the form $\mathbf{y} = \{y, y^{\bullet}, y^{\substack{\bullet\\ \bullet}}, 0\}$. In the remainder of the proof, we will consider a controlled path $\mathbf{y} \in \mathcal{D}^{(\beta,\gamma,\eta,\zeta)}_{\mathbf{z};\mathbf{y}_0}\left(\Delta_2^T\left(\left[0,\bar{T}\right]\right);\mathbb{R}^m\right)$

as given in (4.74), that is a controlled processes $\mathbf{y}$ starting from an initial value $\mathbf{y}_0 = (y_0, f(y_0), f(y_0)f'(y_0), 0)$. As in Remark 2.4.14, we consider a parameter β such that

$$\beta < \alpha, \text{ and } \beta - \gamma > \frac{1}{4}. \tag{4.76}$$

In addition, we introduce a mapping

$$\mathcal{M}_{\bar{T}} : \mathcal{D}_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma,\eta,\zeta)}\left(\Delta_2^T\left(\left[0,\bar{T}\right]\right);\mathbb{R}^m\right) \to \mathcal{D}_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma,\eta,\zeta)}\left(\Delta_2^T\left(\left[0,\bar{T}\right]\right);\mathbb{R}^m\right), \tag{4.77}$$

such that for all $\left(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0\right) \in \mathcal{D}_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma,\eta,\zeta)}(\mathbb{R}^m)$, we have

$$\begin{aligned}\mathcal{M}_{\bar{T}}&\left(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0\right)_t^{\tau} \\ &= \left\{\left(y_0 + \int_0^t k(\tau, r)\, dx_r f(y_r^r),\ f(y_t^\tau), f(y_t^\tau))\, f'(y_t^\tau), 0\right) \middle| (t,\tau) \in \Delta_2^T\left(\left[0,\bar{T}\right]\right)\right\}.\end{aligned} \tag{4.78}$$

We are now ready to implement the first piece (i) of the general strategy described above.

Step 1: Invariant ball on a small interval. In this step, our goal is to show that there exists a ball of radius 1 in $\mathcal{D}_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma)}(\Delta_2^T([0,\bar{T}]);\mathbb{R}^m)$ which is left invariant by $\mathcal{M}_{\bar{T}}$ provided that $\bar{T}$ is small enough. To this aim, we introduce some additional notation. Namely for $\mathbf{y}$ as in (4.78) we define a controlled process $\mathbf{w}$ in the following way:

$$(s,t,\tau) \mapsto \mathbf{w}_{ts}^{\tau} = \left(w_{ts}^{\tau}, w_{ts}^{\bullet,\tau}, w_{ts}^{\overset{\bullet}{\bullet},\tau}, 0\right) = \mathcal{M}_{\bar{T}}\left(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0\right)_{ts}^{\tau}, \tag{4.79}$$

where we recall that $\mathcal{M}_{\bar{T}}$ is defined by (4.78). Next consider the unit ball $\mathcal{B}_{\bar{T}}$ within the space $\mathcal{D}_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma,\eta,\zeta)}(\Delta_2^T([0,\bar{T}]);\mathbb{R}^m)$, defined by

$$\mathcal{B}_{\bar{T}} = \left\{\left(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0\right) \in \mathcal{D}_{\mathbf{z},\mathbf{y}_0}^{(\beta,\gamma,\eta,\zeta)}\left(\Delta_2^T\left(\left[0,\bar{T}\right]\right);\mathbb{R}^m\right) \middle| \|(y, y^{\bullet}, y^{\overset{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq 1\right\}. \tag{4.80}$$

In order to bound the process defined by (4.79), notice that $\mathcal{M}_{\bar{T}}$ is given as the Volterra type integral of $\phi = f(y)$. Hence according to (4.69) there exists a constant C such that

$$\|(\phi, \phi^{\bullet}, \phi^{\overset{\bullet}{\bullet}}, \phi^{\bullet\bullet})\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \lesssim \left(1 + \|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)}\right)^3 \left(1 + Q^3\right) \bar{T}^{\alpha-\beta}, \tag{4.81}$$

where we have set

$$Q = |f(y_0)| + |f(y_0)f'(y_0)| + \|(y, y^{\bullet}, y^{\substack{\bullet\\ \bullet}}, 0)\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}. \tag{4.82}$$

In addition, our process $\mathbf{w}$ is defined in (4.79) as

$$w^{\tau}_{ts} = \int_s^t k(\tau, r) dx_r \phi^r_r.$$

Thus an easy extension of (4.14)-(4.17) to a process $\phi \in \mathcal{D}^{(\beta,\gamma,\eta,\zeta)}_{\mathbf{z}}$ with β satisfying (4.76) yields

$$\|\mathbf{w}\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq C\|(\phi, \phi^{\bullet}, \phi^{\substack{\bullet\\ \bullet}}, \phi^{\bullet\bullet})\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}\|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)} \leq C\left(1 + \|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)}\right)^4 \left(1 + Q^3\right) \bar{T}^{\alpha-\beta}, \tag{4.83}$$

for a universal constant which can change from line to line. Furthermore, since we have assumed that $\|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)} \leq M$, one can recast (4.83) as

$$\|\mathbf{w}\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq C\left(1 + M^4\right)\left(1 + Q^3\right) \bar{T}^{\alpha-\beta}. \tag{4.84}$$

Considering $\bar{T} \leq \left(C\left(1 + M^4\right)\left(1 + Q^3\right)\right)^{\frac{1}{\alpha-\beta}}$ and back to our definition (4.79), it is now easily seen that $\mathcal{B}_{\bar{T}}$ in (4.80) is left invariant by the map $\mathcal{M}_{\bar{T}}$. This completes the proof of step 1.

Next, we handle the second piece (ii) of the general strategy described above.

Step 2: $\mathcal{M}_{\bar{T}}$ is contractive. The aim of this step is to prove that $\mathcal{M}_{\bar{T}}$ is a contraction mapping on $\mathcal{D}^{(\alpha,\gamma,\eta,\zeta)}_{\mathbf{z},\mathbf{y}_0}(\Delta^T_2([0, \bar{T}]); \mathbb{R}^m)$. That is, we will show that there exists a small $\hat{T} \leq \bar{T}$ and a constant $0 < q < 1$ such that for two paths $\mathbf{y} = (y, y^{\bullet}, y^{\substack{\bullet\\ \bullet}}, 0)$ and $\tilde{\mathbf{y}} = (\tilde{y}, \tilde{y}^{\bullet}, \tilde{y}^{\substack{\bullet\\ \bullet}}, 0)$ in $\mathcal{D}^{(\beta,\gamma)}_{\mathbf{z},\mathbf{y}_0}(\Delta^T_2([0, \hat{T}]); \mathbb{R}^m)$ we have

$$\left\|\mathcal{M}_{\bar{T}}\left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\substack{\bullet\\ \bullet}} - \tilde{y}^{\substack{\bullet\\ \bullet}}, 0\right)\right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq q\left\|\left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\substack{\bullet\\ \bullet}} - \tilde{y}^{\substack{\bullet\\ \bullet}}, 0\right)\right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}. \tag{4.85}$$

To this aim, we set $F = f(y) - f(\tilde{y})$, and consider the controlled path $\mathbf{F} = (F, F^{\bullet}, F^{\stackrel{\bullet}{\bullet}}, F^{\bullet\bullet}) \in \mathcal{D}_{\mathbf{z}}^{(\beta,\gamma,\eta,\zeta)}(\Delta_2^T([0,\hat{T}]); \mathbb{R}^m)$ defined through Proposition 2.4.13. According to expression (4.78), we have

$$\mathcal{M}_{\hat{T}}\left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\stackrel{\bullet}{\bullet}} - \tilde{y}^{\stackrel{\bullet}{\bullet}}, 0\right)^{\tau}_{ts} = \left\{\left(\int_s^t k\left(\tau, r\right) dx_r F^r_r,\ F^{\tau}_{st},\ F^{\bullet,\tau,\tau}_{ts}, 0\right) \Big| \left(s,t,\tau\right) \in \Delta_3^T\left([0,\hat{T}]\right)\right\}. \quad (4.86)$$

Hence in order to prove (4.85), it is sufficient to bound the right hand side of (4.86). Now similarly to Step 1, thanks to Remark 2.4.11 and upper bounds (4.14)-(4.15), we obtain

$$\left\|\mathcal{M}_{\hat{T}}\left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\stackrel{\bullet}{\bullet}} - \tilde{y}^{\stackrel{\bullet}{\bullet}}, 0\right)\right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq C\|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta)}\left\|\left(F, F^{\bullet}, F^{\stackrel{\bullet}{\bullet}}, F^{\bullet\bullet}\right)\right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}\hat{T}^{\alpha-\beta}. \quad (4.87)$$

In the following, we will bound $\|(F, F^{\bullet}, F^{\stackrel{\bullet}{\bullet}}, F^{\bullet\bullet})\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}$, that is, we need to find a bound for $\|(F, F^{\bullet}, F^{\stackrel{\bullet}{\bullet}}, F^{\bullet\bullet})\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}$ with respect to $\|(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\stackrel{\bullet}{\bullet}} - \tilde{y}^{\stackrel{\bullet}{\bullet}}, 0)\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}$. Recalling that $F = f(y) - f(\tilde{y})$ and the definition (4.30)-(4.31), we can rewrite $\mathbf{F}$ as

$$\mathbf{F} = \Big(f(y) - f(\tilde{y}), f(y)f'(y) - f(\tilde{y})f'(\tilde{y}), f(y)f'(y)f'(y) - f(\tilde{y})f'(\tilde{y})f'(\tilde{y}), \frac{1}{2}f(y)f(y)f''(y) - \frac{1}{2}f(\tilde{y})f(\tilde{y})f''(\tilde{y})\Big). \quad (4.88)$$

The strategy to bound $\|\mathbf{F}\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} = \|(F, F^{\bullet}, F^{\stackrel{\bullet}{\bullet}}, F^{\bullet\bullet})\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}$ as given in (4.88) is very similar to the classical rough path case as explained in [8]. Due to the fact that both $\mathbf{y}$ and $\tilde{\mathbf{y}}$ sit in the ball $\mathcal{B}_T$ defined by (4.80), we let the patient reader to check that there exists a constant $\tilde{C} = \tilde{C}_{M,\alpha,\gamma,\|f\|_{C_b^5}}$ such that

$$\left\|\left(F, F^{\bullet}, F^{\stackrel{\bullet}{\bullet}}, F^{\bullet\bullet}\right)\right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq \tilde{C}\left\|\left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\stackrel{\bullet}{\bullet}} - \tilde{y}^{\stackrel{\bullet}{\bullet}}, 0\right)\right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}. \quad (4.89)$$

Reparting (4.89) into (4.87), we thus get the existence of a constant C such that

$$\left\|\mathcal{M}_{\hat{T}}\left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\stackrel{\bullet}{\bullet}} - \tilde{y}^{\stackrel{\bullet}{\bullet}}, 0\right)\right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq CM\left\|\left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\stackrel{\bullet}{\bullet}} - \tilde{y}^{\stackrel{\bullet}{\bullet}}, 0\right)\right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)}\hat{T}^{\alpha-\beta}. \quad (4.90)$$

By choosing $\hat{T}$ small enough such that $q \equiv CM\hat{T}^{\alpha-\beta} < 1$, we can recast (4.90) as

$$\left\| \mathcal{M}_{\hat{T}} \left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\substack{\bullet\\ \bullet}} - \tilde{y}^{\substack{\bullet\\ \bullet}}, 0 \right) \right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} \leq q \left\| \left(y - \tilde{y}, y^{\bullet} - \tilde{y}^{\bullet}, y^{\substack{\bullet\\ \bullet}} - \tilde{y}^{\substack{\bullet\\ \bullet}}, 0 \right) \right\|_{\mathbf{z},(\beta,\gamma,\eta,\zeta)} .$$

It follows that $\mathcal{M}_{\bar{T}}$ is contractive on $\mathcal{D}_{\mathbf{z}}^{(\beta,\gamma,\eta,\zeta)}(\Delta_2^T([0,\bar{T}]);\mathbb{R}^m)$, which completes the proof of Step 2.

Combining Step 1 and Step 2, we have proved that if a small enough $\hat{T}$ is chosen then $\mathcal{M}_{\hat{T}}$ admits a unique fixed point $\mathbf{y} = (y, y^{\bullet}, y^{\substack{\bullet\\ \bullet}}, 0)$ in the ball $\mathcal{B}_{\hat{T}}$ defined by (4.80). This fixed point is the unique solution to (4.75) in $\mathcal{B}_{\hat{T}}$. In addition, owing to (4.82) plus the fact that f, f' are uniformly bounded, it is easily proved that the choice of $\hat{T}$ can again be done uniformly in the starting point y_0. Hence the solution on $[0,T]$ is constructed iteratively on intervals $[k\hat{T},\, (k+1)\hat{T}]$. The proof of Theorem 2.4.17 is now finished. $\square$

3. CHAPTER2

3.1 Introduction

In [1] we proposed a new methodology based on the theory of rough paths to treat Banach-valued Volterra equations of the form

$$y_t = y_0 + \int_0^t k(t,s)f(y_s)dx, \tag{1.1}$$

where k, defined on $[0,T]^2$ is possibly with a singularity on the diagonal of the form $|t-s|^{-\gamma}$ for some $\gamma \geq 0$, and the driving signal x is only assumed to be Hölder continuous (and with Hölder regularity possibly lower than $\frac{1}{2}$). The Volterra rough path framework is developed around a splitting of the arguments in a Volterra process, in the sense that one lifts the classical form of Volterra process $z_t := \int_0^t k(t,s)dx_s$ defined on $[0,T]$, to a two parameter object defined on the simplex $\Delta_2[0,T] := \{(s,t) \in [0,T]^2 | s \leq t\}$ given formally by

$$z_t^\tau := \int_0^t k(\tau,s)dx_s, \quad t \leq \tau.$$

Clearly, when the two parameter object is restricted to the diagonal in $[0,T]^2$, we have $z_t^t = z_t$, obtaining the classical type of Volterra process. The advantage of viewing the Volterra process as this two parameter object is that one can easily distinguish between the regularity contributed by the driving signal versus the possible singularity obtained from the kernel k, thus making pathwise regularity analysis easier, and sewing based arguments more straightforward.

In a similar spirit as for classical rough paths, the idea is to lift the Volterra signal $(t,\tau) \mapsto z_t^\tau$ to a signature type object, satisfying certain algebraic relations, which is called the Volterra signature. In contrast to classical rough path theory, the Volterra signature does not satisfy Chen's relation with the tensor product, but a convolution type product is required in order to obtain an equivalent algebraic relation. The Volterra signature, in combination with certain "controlled Volterra paths" is then used to construct solutions to (1.1) in a purely pathwise manner.

Although [1] provides the basic framework for Volterra rough paths as mentioned above, several important questions relating to this theory was left open. On the analytic side, [1] only deals with the case when $\alpha - \gamma \geq 1/3$ (where we recall that α is the regularity of the signal, while γ is the possible order of singularity from the kernel k). The problem of extending this regime was dealt with in the article [7], where the algebraic framework was described for $\alpha - \gamma \geq 1/4$. In a very recent article [21], Bruned and Kastetsiadis extends this even further to all $\alpha - \gamma > 0$ by invoking algebraic theories similar to that used for non-geometric rough paths [2] and regularity structures [22].

Another important step for completeness of the framework for Volterra rough paths, is to provide a complete probabilistic picture of how to lift a Volterra stochastic process into a Volterra rough path, analogues to the rough path lift for stochastic processes. As the framework for Volterra rough paths relies on spaces for Hölder volterra paths with two parameters (one corresponding to regularity and one to singularity), a direct application of the classical Kolmogorov continuity theorem will not provide sufficient answers, and so new arguments needs to be developed, specifically suited for the type of Hölder spaces used. This is what we will deal with in this article.

We begin with a recollection of the basic framework of Volterra rough paths, including a construction of the so called convolution product, and describe the exact type of Hölder spaces we will work with. We then extend the classical Garsia-Rodemich-Rumsey inequality to suit our Volterra paths. To this end, ...

3.2 Preliminary results

In [1] and [7], our Volterra rough formalism was based on certain spaces of functions having specific regularity/singularity features. Before defining the proper spaces quantifying this type of regularity, let us introduce some notation:

Notation 3.2.1. *Let $T > 0$ be a time horizon, and $n \geq 2$. Then the simplex Δ_n^T is defined by*

$$\Delta_n^T \Big\{ (s_1, \dots, s_n) \in [0, T]^n; 0 \leq s_1 < \cdots < s_n \leq T \Big\}.$$

When this causes no ambiguity, we will abbreviate Δ_n^T as Δ_n. For $(s,t) \in \Delta_2$, we designate by $\mathcal{P}$ a generic partition of $[s,t]$. Two successive points in this partition are written as $[u,v] \in \mathcal{P}$.

The functions quantifying our regularities are also labeled in the following notation.

Notation 3.2.2. *Let $(\alpha,\gamma) \in (0,1)^2$ be such that $\alpha > \gamma$. For $(s,t,\tau) \in \Delta_3$, we set*

$$\psi^1_{\alpha,\gamma}(\tau,t,s) = \left[|\tau - t|^{-\gamma}|t-s|^{\alpha}\right] \wedge |t-s|^{\alpha-\gamma}. \tag{2.1}$$

Considering two additional parameters $\zeta \in [0,\alpha-\gamma)$ and $\eta \in [\zeta,1]$, we also set

$$\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(\tau,\tau',t,s) = |\tau-\tau'|^{\eta}|\tau'-t|^{-(\eta-\zeta)}\left(\left[|\tau'-t|^{-\gamma-\zeta}|t-s|^{\alpha}\right] \wedge |t-s|^{\alpha-\gamma-\zeta}\right) \tag{2.2}$$

We are now ready to introduce some functional spaces called $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$, which are also used in the definition of $\mathcal{V}^{(\alpha,\gamma)}$ in [1], [7]. Those spaces are natural function sets when dealing with Volterra type regularities.

Definition 3.2.3. *Let $m \geq 1$, $(\alpha,\gamma) \in (0,1)^2$ with $\alpha-\gamma > 0$, and $\zeta \in [0,\alpha-\gamma)$, $\eta \in [\zeta,1]$. Throughout the article we consider functions $z : \Delta_3 \to \mathbb{R}^m$ of the form $(s,t,\tau) \mapsto z^{\tau}_{ts}$, such that $z^{\tau}_0 = z_0$ for all $\tau \in (0,T]$. We define the space of Volterra paths of index $(\alpha,\gamma,\eta,\zeta)$, denoted by $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^m)$, as the set of such functions satisfying*

$$\|z\|_{(\alpha,\gamma,\eta,\zeta)} = \|z\|_{(\alpha,\gamma),1} + \|z\|_{(\alpha,\gamma,\eta,\zeta),1,2} < \infty. \tag{2.3}$$

Recalling Notation 3.2.1 and 3.2.2, the 1-norms and (1,2)-norms in (2.3) are respectively defined as follows:

$$\|z\|_{(\alpha,\gamma),1} = \sup_{(s,t,\tau)\in\Delta_3} \frac{|z^{\tau}_{ts}|}{\psi^1_{\alpha,\gamma}(\tau,t,s)}, \tag{2.4}$$

$$\|z\|_{(\alpha,\gamma,\eta,\zeta),1,2} = \sup_{(s,t,\tau',\tau)\in\Delta_4} \frac{|z^{\tau\tau'}_{ts}|}{\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(\tau,\tau',t,s)}, \tag{2.5}$$

with the convention $z^\tau_{ts} = z^\tau_t - z^\tau_s$ and $z^{\tau\tau'}_s = z^\tau_s - z^{\tau'}_s$. Notice that under the mapping

$$z \mapsto |z_0| + \|z\|_{(\alpha,\gamma,\eta,\zeta)},$$

the space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ is a Banach space.

Remark 3.2.4. This remark has to be changed according to our new version of [7]. The spaces $\mathcal{V}^{(\alpha,\gamma)}$ defined in [1], [7] are based on a different norm than (2.3). Namely the norm in $\mathcal{V}^{(\alpha,\gamma)}$ introduced therein could be spelled out as

$$\|\mathbf{z}\|_{(\alpha,\gamma)} = \|\mathbf{z}\|_{(\alpha,\gamma),1} + \sup_{\zeta\in[0,\alpha-\gamma),\,\eta\in[\zeta,1]} \|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta),1,2}, \tag{2.6}$$

where the norms in the right hand side above are still defined by (2.4)-(2.5). However, the sup in ζ, η in (2.6) is delicate to handle for the stochastic processes we shall consider in this paper. We thus let the patient reader check the following assertion: all the theoretical considerations in [1], [7] are still correct if we replace $\mathcal{V}^{(\alpha,\gamma)}$ by a space $\mathcal{W}$ of the form

$$\mathcal{W}(\Delta_3;\mathbb{R}^m) = \bigcap_{\mathrm{i}=1}^{M} \mathcal{V}^{(\alpha_\mathrm{i},\gamma_\mathrm{i},\eta_\mathrm{i},\zeta_\mathrm{i})}(\Delta_3;\mathbb{R}^m), \quad \text{with} \quad M<\infty, \tag{2.7}$$

for some specific values of $(\alpha_\mathrm{i},\gamma_\mathrm{i},\eta_\mathrm{i},\zeta_\mathrm{i})$. The norm on $\mathcal{W}$ is given by

$$\|\mathbf{z}\|_{\mathcal{W}} = \sum_{\mathrm{i}}^{M} \|\mathbf{z}\|_{(\alpha_\mathrm{i},\gamma_\mathrm{i},\eta_\mathrm{i},\zeta_\mathrm{i})}. \tag{2.8}$$

More specifically, in [1] we consider $M=2$ and we use $(\alpha_1,\gamma_1,\eta_1,\zeta_1) = (\alpha,\gamma,\eta,\zeta)$ in Lemma 22, and $(\alpha_1,\gamma_1,\eta_1,\zeta_1) = (\alpha,\gamma,\eta,0)$ in Theorem 32. In the current paper, in for example Proposition 3.4.9, we consider $M=3$, let $(\alpha_1,\gamma_1,\eta_1,\zeta_1) = (\alpha,\gamma,0,0)$, $(\alpha_2,\gamma_2,\eta_2,\zeta_2) = (\alpha,\gamma,\eta,\zeta)$ and $(\alpha_3,\gamma_3,\eta_3,\zeta_3) = (\alpha,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p})$ for some fixed p determined in Proposition 3.4.9 and 3.4.13.

Remark 3.2.5. As mentioned in [7, Remark 2.6], the spaces $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ enjoy embedding properties of the form $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)} \subset \mathcal{V}^{(\beta,\gamma,\eta,\zeta)}$ for $0 < \alpha < \beta < 1$. In addition, the norms defined by (2.3)-(2.5) verify the following relation on $[0,T]$:

$$\|y\|_{(\beta,\gamma),1} \leq T^{\alpha-\beta}\|y\|_{(\alpha,\gamma),1}, \quad \|y\|_{(\beta,\gamma,\eta,\zeta),1,2} \leq T^{\alpha-\beta}\|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}, \quad \|y\|_{(\beta,\gamma,\eta,\zeta)} \leq T^{\alpha-\beta}\|y\|_{(\alpha,\gamma,\eta,\zeta)}.$$

Convolution products also played a crucial role for the considerations in [7]. Let us recall a proposition establishing the existence of such convolution products in a general setting.

Proposition 3.2.6. *We consider two Volterra paths $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^d)$ and $y \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\mathcal{L}(\mathbb{R}^d))$ as given in Definition 3.2.3, where we recall that $\alpha, \gamma \in (0,1)$. Define $\rho = \alpha - \gamma$, and assume that $\rho > 0$, $\zeta \in [0,\rho)$ and $\eta \in [\zeta, 1]$. Then the convolution product of the two Volterra paths y and z is a bilinear operation on $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^d)$ given by*

$$z_{tu}^{\tau} * y_{us}^{\cdot} = \int_{t>r>u} dz_r^{\tau} y_{us}^{r} := \lim_{|\mathcal{P}|\to 0} \sum_{[u',v']\in\mathcal{P}} z_{v'u'}^{\tau} y_{us}^{u'}, \tag{2.9}$$

where $\mathcal{P}$ is a generic partition of $[u,t]$ for which we recall Notation 3.2.1. The integral in (2.9) is understood as a Volterra-Young integral for all $(s,u,t,\tau) \in \Delta_4$. Moreover, the following two inequalities hold for any $\eta \in [0,1]$, $\zeta \in [0, 2\rho)$ and any tuple (s,u,t,τ,τ') lying in Δ_5:

$$|z_{tu}^{\tau} * y_{us}^{\cdot}| \lesssim \|z\|_{(\alpha,\gamma),1}\|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}\, \psi^{1}_{2\alpha-\gamma,\gamma}(\tau,t,s), \tag{2.10}$$

$$\left|z_{tu}^{\tau'\tau} * y_{us}^{\cdot}\right| \lesssim \|z\|_{(\alpha,\gamma,\eta,\zeta),1,2}\|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}\, \psi^{1,2}_{2\alpha-\gamma,\gamma,\eta,\zeta}(\tau,\tau',t,s). \tag{2.11}$$

Let us also recall the definition of an operator δ acting on increments, which is useful for rough paths constructions.

Notation 3.2.7. *Let g be a path from Δ_2 to $\mathbb{R}^m$, and consider $(s,u,t) \in \Delta_3$. Then the quantity $\delta_u g_{ts}$ is defined by*

$$\delta_u g_{ts} = g_{ts} - g_{tu} - g_{us}. \tag{2.12}$$

With Definition 3.2.3 and Proposition 3.2.6 in hand, we are now ready to state the main assumption we have used in [7]. Namely our Volterra rough paths analysis relies on the ability to define a stack $\{z^{\mathrm{j},\tau}; \mathrm{j} \leq n\}$ of Volterra iterated integrals according to the following definition.

Definition 3.2.8. *A Volterra rough path above z is a family $\{\mathbf{z}^{\mathrm{j},\tau}; \mathrm{j} \leq n\}$, where n satisfies $n = \lfloor \rho^{-1} \rfloor$ for $\rho = \alpha - \gamma > 0$. This family is assumed to enjoy the following properties:*

(i) $\mathbf{z}^1 = z$ *and* $\mathbf{z}^{\mathrm{j},\tau}_{ts} \in (\mathbb{R}^m)^{\otimes \mathrm{j}}$.

(ii) For all $\mathrm{j} \leq n$ *and* $(s,t,\tau) \in \Delta_3$ *we have*

$$\delta_u \mathbf{z}^{\mathrm{j},\tau}_{ts} = \sum_{\mathrm{i}=1}^{\mathrm{j}-1} \mathbf{z}^{\mathrm{j}-\mathrm{i},\tau}_{tu} * \mathbf{z}^{\mathrm{i},\cdot}_{us} = \int_s^t d\mathbf{z}^{\mathrm{j}-\mathrm{i},\tau}_{tr}\, \mathbf{z}^{\mathrm{i},r}_{us}, \tag{2.13}$$

where the right hand side of (2.13) is given by Proposition 3.2.6.

(iii) For all $\mathrm{j} = 1, \ldots, n$*, we have* $\mathbf{z}^{\mathrm{j}} \in \mathcal{V}^{(\mathrm{j}\rho+\gamma,\gamma,\eta,\zeta)}$.

As the reader might have seen, our Definition 3.2.8 is a natural extension of the more classical definition of rough path [23], adapted to our context with singularities at $t = \tau$ and prominent role of convolution products.In the decomposition (2.13), we would like to quantify the regularity of some paths depending on a variable $(s,u,t,\tau) \in \Delta_4$. We label a small variation of Definition 3.2.3 in this sense (see also in [7, Definition 2.9]).

Definition 3.2.9. *As in Definition 2.2.4, consider $m \geq 1$, $(\alpha,\gamma) \in (0,1)^2$ with $\alpha - \gamma > 0$, and $\zeta \in [0, \alpha-\gamma)$, $\eta \in [\zeta, 1]$. Let $\mathbf{z} : \Delta_4 \to \mathbb{R}^m$ be of the form $(s,u,t,\tau) \mapsto \mathbf{z}^{\tau}_{tus}$. The definition of $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^m)$ can be extended in order to define a space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_4;\mathbb{R}^m)$, by using the same definition as (2.3). That is we have $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_4;\mathbb{R}^m)$ if*

$$\|z\|_{(\alpha,\gamma,\eta,\zeta)} = \|z\|_{(\alpha,\gamma),1} + \|z\|_{(\alpha,\gamma,\eta,\zeta),1,2} < \infty. \tag{2.14}$$

The quantities $\|z\|_{(\alpha,\gamma),1}$ *and* $\|z\|_{(\alpha,\gamma,\eta,\zeta),1,2}$ *in* (2.14) *are slight modifications of* (2.4) *and* (2.5)*, respectively defined by*

$$\|\mathbf{z}\|_{(\alpha,\gamma),1} = \sup_{(s,u,t,\tau)\in\Delta_4} \frac{|\mathbf{z}^{\tau}_{tus}|}{\psi^1_{\alpha,\gamma}(\tau,t,s)}, \tag{2.15}$$

and

$$\|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta),1,2} = \sup_{(s,u,t,\tau',\tau)\in\Delta_5} \frac{|\mathbf{z}^{\tau\tau'}_{tus}|}{\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(\tau,\tau',t,s)}. \tag{2.16}$$

3.3 An extension of Garsia-Rodemich-Rumsey's inequality

This section is devoted to extend Garsia-Rodemich-Rumsey's celebrated result [24] to the Volterra space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ introduced in Definition 3.2.3. To this aim, let us introduce an integral norm which will quantify the regularity of our processes.

Definition 3.3.1. *Let* $z:\Delta_3\to\mathbb{R}^d$ *be a continuous Volterra increment. Then for some parameters* $p\geq 1$ *and* $\alpha,\gamma\in(0,1)$*,* $\zeta\in[0,\alpha-\gamma)$*,* $\eta\in[\zeta,1]$ *we define*

$$U^{\tau}_{(\alpha,\gamma),p,1}(z;\eta,\zeta) := \left(\int_{(v,w)\in\Delta_2^\tau} \frac{|z^{\tau}_{wv}|^{2p}}{|\tau-w|^{-2p(\eta-\zeta)}|\psi^1_{\alpha,\gamma+\zeta}(\tau,w,v)|^{2p}|w-v|^2} dvdw\right)^{\frac{1}{2p}} \tag{3.1}$$

$$U^{\tau}_{(\alpha,\gamma,\eta,\zeta),p,1,2}(z) := \left(\int_{(v,w,r',r)\in\Delta_4^\tau} \frac{|z^{rr'}_{wv}|^{2p}}{|\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(r,r',w,v)|^{2p}|w-v|^2|r-r'|^2} dvdwdr'dr\right)^{\frac{1}{2p}}, \tag{3.2}$$

where recall that the functions $\psi^1,\psi^{1,2}$ *are respectively defined in* (2.1) *and* (2.2).

Remark 3.3.2. Notice that if we set

$$D^{\tau}(w,v) = \frac{|z^{\tau}_{wv}|^{2p}}{|\tau-w|^{-2p(\eta-\zeta)}|\psi^1_{\alpha,\gamma}(\tau,w,v)|^{2p}|w-v|^2},$$

then we trivially have $D^{\tau}(w,v)\geq 0$. Plugging this information in relation (3.1), we get that $\tau\mapsto U^{\tau}_{\alpha,\gamma,p,1}(z;\eta,\zeta)$ is a non-decreasing function. Thus for $\tau\leq T$ we have $U^{\tau}_{(\alpha,\gamma),p,1}(z;\eta,\zeta)\leq U^{T}_{(\alpha,\gamma),p,1}(z;\eta,\zeta)$.

Remark 3.3.3. The quantity $U^{\tau}_{(\alpha,\gamma),p,1}(z;\eta,\zeta)$ evaluated at $\eta=\zeta=0$, will be denoted by $U^{\tau}_{(\alpha,\gamma),p,1}(z)$ for notational sake.

We now state and prove our extension of Garsia-Rodemich-Rumsey's inequality, that is a theorem relating the functional U and the regularity of processes in $\mathcal{V}^{(\alpha,\gamma)}$.

Lemma 3.3.4. *Let $\mathbf{z}:\Delta_3\to\mathbb{R}^d$ be a continuous increment. Then for any $\kappa\in(0,1)$, $\gamma\in[0,\kappa)$, $\zeta\in[0,\kappa-\gamma)$ and $\eta\in[\zeta,1]$ there exists a universal constant $C>0$ such that for all $(s,t,\tau)\in\Delta_3$*

$$|\mathbf{z}^{\tau}_{ts}|\le C|\tau-t|^{-(\eta-\zeta)}\psi^{1}_{\kappa,\gamma+\zeta}(\tau,t,s)\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)+\|\delta\mathbf{z}\|^{[s,t]}_{(\kappa,\gamma,\eta,\zeta),1}\right), \tag{3.3}$$

where the quantity $\|\delta\mathbf{z}\|^{[s,t]}_{(\kappa,\gamma,\eta,\zeta),1}$ is defined as

$$\|\delta\mathbf{z}\|^{[s,t]}_{(\kappa,\gamma,\eta,\zeta),1}=\sup_{s\le u<v\le t}\frac{|\delta_u\mathbf{z}^{\tau}_{vs}|}{|\tau-v|^{-(\eta-\zeta)}\psi^{1}_{\kappa,\gamma+\zeta}(\tau,v,s)}. \tag{3.4}$$

In particular, for $\eta=\zeta=0$, we have

$$|\mathbf{z}^{\tau}_{ts}|\lesssim\psi^{1}_{\kappa,\gamma}(\tau,t,s)\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z})+\|\delta\mathbf{z}\|_{(\alpha,\gamma),1}\right), \tag{3.5}$$

where $\|\delta\mathbf{z}\|_{(\alpha,\gamma),1}$ is given by (2.15).

Proof. Consider a tuple $(s,t,\tau)\in\Delta_3$, with $t-s<\frac{T}{2}$. Let us first construct a sequence of points $(s_k)_{k\ge0}$, such that $s_k\in[0,T]$ and converging to s by induction. Namely we set $s_0=t$, we suppose that $s_0,s_1,\dots,s_k$ have been constructed, and let $D_k=(s,\frac{s_k+s}{2})$. We also introduce a function I as follows:

$$I(v):=\int_s^v\frac{|\mathbf{z}^{\tau}_{vu}|^{2p}}{|\tau-v|^{-2p(\eta-\zeta)}|\psi^{1}_{\kappa,\gamma+\zeta}(\tau,v,u)|^{2p}|v-u|^{2}}du. \tag{3.6}$$

According to the value of I, we define two subsets of the interval D_k:

$$A_k \ := \ \left\{ v \in D_k \middle| \quad I(v) > \frac{4(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta))^{2p}}{|s_k - s|} \right\}, \tag{3.7}$$

$$B_k \ := \ \left\{ v \in D_k \middle| \quad \frac{|\mathbf{z}^{\tau}_{s_k v}|^{2p}}{|\tau - s_k|^{-2p(\eta-\zeta)}|\psi^1_{\kappa,\gamma+\zeta}(\tau, s_k, v)|^{2p}|s_k - v|^2} > \frac{4I(s_k)}{|s_k - s|} \right\}, \tag{3.8}$$

where we recall again that $\psi^1_{\kappa,\gamma+\zeta}(\tau, v, u)$ is given by (2.1). We claim that $A_k \cup B_k \subset D_k$, where the inclusion is strict. Toward this aim, observe that the set of (u, v) such that

$$s < u < v < \frac{s_k + s}{2}$$

is included in $[0, T)^2$. Hence due to the definition (3.1) of $U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)$ we get

$$\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p} \geq \int_{A_k} dv\, I(v). \tag{3.9}$$

Therefore thanks to relation (3.7) defining A_k, we get

$$\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p} > \frac{4\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p}}{|s_k - s|}\mu(A_k), \tag{3.10}$$

where $\mu(A_k)$ denotes the Lebesgue measure of set A_k. It is thus readily checked from (3.10) that

$$\mu(A_k) < \frac{|s_k - s|}{4} = \frac{\mu(D_k)}{2}. \tag{3.11}$$

Let us argue similarly for the set B_k. Namely note that since the set B_k defined by (3.8) is a subset of (s, s_k), we have

$$I(s_k) \geq \int_{B_k} \frac{|\mathbf{z}^{\tau}_{s_k v}|^{2p}}{|\tau - s_k|^{-2p(\eta-\zeta)}|\psi^1_{\kappa,\gamma+\zeta}(\tau, s_k, v)|^{2p}|s_k - v|^2} dv. \tag{3.12}$$

Thus plugging the definition (3.8) of B_k in the right hand side of (3.12), we get

$$I(s_k) > \frac{4\mu(B_k)}{|s_k - s|} I(s_k),$$

from which we obtain again that

$$\mu(B_k) < \frac{|s_k - s|}{4} = \frac{\mu(D_k)}{2} \tag{3.13}$$

Combining (3.11) and (3.13), we have thus obtained

$$\mu(A_k) < \frac{\mu(D_k)}{2}, \quad \text{and} \quad \mu(B_k) < \frac{\mu(D_k)}{2},$$

Thus we get

$$\mu(A_k) + \mu(B_k) < \mu(D_k), \tag{3.14}$$

from which we easily deduce that $A_k \cup B_k$ is a strict subset of D_k. Then we can choose s_{k+1} arbitrarily in $D_k \setminus (A_k \cup B_k)$. Summarizing our considerations so far, for all n we have been able to construct a family $\{s_0, \dots, s_n\}$ such that for all $0 \le k \le n$, we have $0 \le s_k - s \le \frac{t-s}{2^k}$ and the following 2 conditions are met:

$$\begin{aligned} \frac{|\mathbf{z}^\tau_{s_k s_{k+1}}|^{2p}}{|\tau - s_k|^{-2p(\eta-\zeta)}|\psi^1_{\kappa,\gamma+\zeta}(\tau, s_k, s_{k+1})|^{2p}|s_k - s_{k+1}|^2} &\le \frac{4I(s_k)}{|s_k - s|}, \\ I(s_{k+1}) &\le \frac{4\left(U^\tau_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p}}{|s_k - s|}. \end{aligned} \tag{3.15}$$

With (3.15) in hand, let us decompose $\mathbf{z}^\tau_{ts}$ into

$$\mathbf{z}^\tau_{ts} = \mathbf{z}^\tau_{s_{n+1}s} + \sum_{k=0}^{n} \left(\mathbf{z}^\tau_{s_k s_{k+1}} + \delta_{s_{k+1}} \mathbf{z}^\tau_{s_k s}\right). \tag{3.16}$$

Let us now bound the term $\mathbf{z}^\tau_{s_k s_{k+1}}$ in (3.16). To this aim, notice that since $s_{k+1} \notin B_k$, we have

$$\frac{|\mathbf{z}^\tau_{s_k s_{k+1}}|^{2p}}{|\tau - s_k|^{-2p(\eta-\zeta)}|\psi^1_{\kappa,\gamma}(\tau, s_k, s_{k+1})|^{2p}|s_k - s_{k+1}|^2} \le 4\frac{I(s_k)}{|s_k - s|}. \tag{3.17}$$

Moreover, we also have $s_k \notin A_{k-1}$. Hence we obtain

$$I(s_k) < \frac{4\left(U^\tau_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p}}{|s_{k-1} - s|}. \tag{3.18}$$

Gathering (3.17) and (3.18), we thus get

$$\frac{|\mathbf{z}^{\tau}_{s_k s_{k+1}}|^{2p}}{|\tau - s_k|^{-2p(\eta-\zeta)}|\psi^1_{\kappa,\gamma+\zeta}(\tau, s_k, s_{k+1})|^{2p}|s_k - s_{k+1}|^2} < \frac{16\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p}}{|s_k - s||s_{k-1} - s|} \lesssim \frac{\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p}}{|s_k - s|^2}, \quad (3.19)$$

where we have used the fact that $|s_k - s| \le |s_{k-1} - s|$ for the second inequality. In addition, thanks to $|s_k - s_{k+1}| \le |s_k - s|$, it is easily seen that we can recast (3.19) as

$$|\mathbf{z}^{\tau}_{s_k s_{k+1}}| \lesssim U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\,\psi^1_{\kappa,\gamma+\zeta}(\tau, s_k, s)|\tau - s_k|^{-(\eta-\zeta)}. \quad (3.20)$$

Next recall that η is assumed to be larger than ζ, and are we also supposing that $0 \le \zeta < \kappa - \gamma$. Thus owing to the fact that $|\tau - s_k| \ge |\tau - t|$, $|s_k - s| \lesssim 2^{-k}(t-s)$, and recalling the expression (2.1) for ψ^1, we end up with

$$\left|\mathbf{z}^{\tau}_{s_k s_{k+1}}\right| \lesssim \frac{U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)}{2^{k(\kappa-\gamma-\zeta)}}\psi^1_{\kappa,\gamma+\zeta}(\tau,t,s)|\tau - t|^{-(\eta-\zeta)}$$

Summing this inequality over k (and using that $\kappa - \gamma - \zeta > 0$), we get the following bound for the right hand side of (3.16):

$$\left|\sum_{k=0}^{n}\mathbf{z}^{\tau}_{s_k s_{k+1}}\right| \lesssim U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\,\psi^1_{\kappa,\gamma+\zeta}(\tau,t,s)|\tau - t|^{-(\eta-\zeta)}. \quad (3.21)$$

Now we turn to bound the second term $\delta_{s_{k+1}}\mathbf{z}^{\tau}_{s_k s}$ in the right hand side of (3.16). It is clear that

$$|\delta_{s_{k+1}}\mathbf{z}^{\tau}_{s_k s}| \lesssim \|\delta\mathbf{z}\|^{[s,t]}_{(\kappa,\gamma,\eta,\zeta),1}\,|\tau - t|^{-(\eta-\zeta)}\psi^1_{\kappa,\gamma+\zeta}(\tau, s_k, s),$$

recalling that $\|\delta\mathbf{z}\|^{[s,t]}_{(\kappa,\gamma,\eta,\zeta),1}$ as given in (3.4). Hence similarly to (3.21), we obtain

$$|\sum_{k=0}^{n}\delta_{s_{k+1}}\mathbf{z}^{\tau}_{s_k s}| \lesssim \|\delta\mathbf{z}\|^{[s,t]}_{(\kappa,\gamma,\eta,\zeta),1}\,|\tau - t|^{-(\eta-\zeta)}\psi^1_{\kappa,\gamma+\zeta}(\tau,t,s). \quad (3.22)$$

Plugging (3.21) and (3.22) into (3.16), and letting $n \to \infty$, we get relation (3.3) thanks to the continuity of $\mathbf{z}$. This completes the proof. $\square$

In preparation for the next proposition, we recall here the classical Sobolev embedding inequality. The particular form of the inequality stated here is as a consequence of the classical Garsia-Rodemich-Rumsey inequality [25], and can be found stated in the below form in [26, pp. 2].

Proposition 3.3.5. *Let $h : [a,b] \to \mathbb{R}^d$ be continuous. Then for any $p > \frac{1}{\alpha}$ the following inequality holds*

$$|h_{ts}| \lesssim_{\alpha,p} |t-s|^{\alpha} \left(\int_a^b \int_a^u \frac{|h_{uv}|^p}{|u-v|^{2+p\alpha}} dv du \right)^{\frac{1}{p}}, \tag{3.23}$$

where we have set $h_{ts} = h_t - h_s$ for $(s,t) \in \Delta_2$.

We follow up with a technical lemma, combining Proposition 3.3.5 with Lemma 3.3.4.

Lemma 3.3.6. *Let $\mathbf{z} : \Delta_3 \to \mathbb{R}^d$ be continuous. Consider some parameters γ, $\alpha \in (0,1)$ with $\gamma < \alpha$, $\zeta \in [0, \alpha-\gamma)$ and $\eta \in [\zeta, 1]$ as in Lemma 3.3.4. Recall that $\psi^{1,2}$ is defined by (2.2) and the quantities U are introduced in Definition 3.2. Then for any $p > \frac{1}{\alpha-\gamma}$, the following inequality holds for any $(s,t,\tau',r',\tau) \in \Delta_5^T$,*

$$\begin{aligned}\left(\frac{|\mathbf{z}_{ts}^{\tau\tau'}|}{\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(\tau,\tau',t,s)} \right)^{2p} &\lesssim U^T_{(\alpha,\gamma,\eta,\zeta),p,1,2}(\mathbf{z}) \\ &\quad + \int_{\tau'}^{\tau} \int_{\tau'}^{r} \sup_{0 \le s < u < v \le r'} \frac{|\delta_u \mathbf{z}_{vs}^{rr'}|^{2p}}{\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(r,r',v,s)^{2p} |r-r'|^2} \, dr' \, dr. \end{aligned} \tag{3.24}$$

Proof. First, since $\mathbf{z}$ is continuous, we apply Proposition 3.3.5 to the increment $\mathbf{z}_{ts}^{\tau} - \mathbf{z}_{ts}^{\tau'}$, and we get

$$\frac{\left|\mathbf{z}_{ts}^{\tau\tau'}\right|}{|\tau'-\tau|^{\eta}} \lesssim \left(\int_{\tau'}^{\tau} \int_{\tau'}^{r} \frac{\left|\mathbf{z}_{ts}^{rr'}\right|^{2p}}{|r-r'|^{2+2p\eta}} dr' dr \right)^{1/2p}. \tag{3.25}$$

Moreover, comparing (2.1) and (2.2) it is easily seen that

$$\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(\tau,\tau',t,s) = |\tau-\tau'|^{\eta} \, |\tau'-t|^{-(\eta-\zeta)} \, \psi^{1}_{\alpha,\gamma+\zeta}(\tau',t,s). \tag{3.26}$$

Plugging (3.26) in (3.25), we end up with

$$\left(\frac{|\mathbf{z}_{ts}^{\tau\tau'}|}{\psi_{\alpha,\gamma,\eta,\zeta}^{1,2}(\tau,\tau',t,s)}\right)^{2p} \lesssim I(\tau,\tau',t,s), \tag{3.27}$$

where we have set

$$I(\tau,\tau',t,s) = \int_{\tau'}^{\tau}\int_{\tau'}^{r} \frac{\left|\mathbf{z}_{ts}^{rr'}\right|^{2p}}{|\tau'-t|^{-2p(\eta-\zeta)}\left|\psi_{\alpha,\gamma+\zeta}^{1}(\tau',t,s)\right|^{2p}|r-r'|^{2+2p\eta}} dr'\, dr.$$

Invoking the fact that $t \le \tau' \le r' \le \tau$ and $\eta - \zeta \ge 0$ we have $|\tau'-t|^{\eta-\zeta} \le |r'-t|^{\eta-\zeta}$. Hence it immediately follows that

$$I(\tau,\tau',t,s) \lesssim \int_{\tau'}^{\tau}\int_{\tau'}^{r} \frac{|\mathbf{z}_{ts}^{rr'}|^{2p}}{|r'-t|^{-2p(\eta-\zeta)}\left|\psi_{\alpha,\gamma+\zeta}^{1}(r',t,s)\right|^{2p}|r-r'|^{2+2p\eta}}\, dr'\, dr. \tag{3.28}$$

We now fix r and apply inequality (3.3) to the Volterra path $(r',t,s) \mapsto \mathbf{z}_{ts}^{rr'}$. We get

$$\left|\mathbf{z}_{ts}^{rr'}\right| \lesssim |r'-t|^{-(\eta-\zeta)}\, \psi_{\alpha,\gamma+\zeta}^{1}(r',t,s)\left(U_{(\alpha,\gamma),p,1}^{r'}(\mathbf{z}^{r,\cdot};\eta,\zeta) + \|\delta\mathbf{z}^{r,\cdot}\|_{(\alpha,\gamma,\eta,\zeta),1}^{[s,t]}\|\right). \tag{3.29}$$

We now plug (3.29) into (3.28), recall the definition (3.1) of $U^{r'}$, resort to (3.26) again and use the expression of (3.4) for $\|\delta\mathbf{z}^{r,\cdot}\|_{(\alpha,\gamma,\eta,\zeta),1}^{[s,t]}$. We end up with

$$I(\tau,\tau',t,s) \lesssim I_1(\tau,\tau') + I_2(\tau,\tau',t,s), \tag{3.30}$$

where I_1 and I_2 are respectively given by

$$I_1(\tau,\tau') = \int_{\tau'}^{\tau}\int_{\tau'}^{r}\int_0^{r'}\int_0^{v} \frac{|\mathbf{z}_{vu}^{rr'}|^{2p}}{\left|\psi_{\alpha,\gamma,\eta,\zeta}^{1,2}(r,r',v,u)\right|^{2p}|v-u|^2|r-r'|^2}\, du\, dv\, dr'\, dr,$$

$$I_2(\tau,\tau',t,s) = \int_{\tau'}^{\tau}\int_{\tau'}^{r}\sup_{0\le s<u<v\le t} \frac{|\delta_u\mathbf{z}_{vs}^{rr'}|^{2p}}{\left|\psi_{\alpha,\gamma+\zeta}^{1}(r',v,s)\right|^{2p}|r'-v|^{-2p(\eta-\zeta)}|r-r'|^{2+2p\eta}}\, dr'\, dr.$$

Going back to (3.2), it is now readily checked that

$$I_1(\tau,\tau') \le \left(U^{\tau}_{(\alpha,\gamma,\eta,\zeta),p,1,2}(\mathbf{z})\right)^{2p} \le \left(U^{T}_{(\alpha,\gamma,\eta,\zeta),p,1,2}(\mathbf{z})\right)^{2p}. \tag{3.31}$$

Furthermore, another application of (3.26) reveals that

$$I_2(\tau,\tau',t,s) = \int_{\tau'}^{\tau}\int_{\tau'}^{r} \sup_{0\le s<u<v\le t} \frac{|\delta_u \mathbf{z}^{rr'}_{vs}|^{2p}}{\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(r,r',v,u)^{2p}|r-r'|^2}\, dr'\, dr. \tag{3.32}$$

Plugging (3.31)-(3.32) into (3.30) and then back to in (3.27), this achieves the proof of our claim (3.24). □

Now we will combine Lemma 3.3.4 and 3.3.6 to obtain a modified Garsia-Rodemich-Rumsey inequality tailored to Volterra rough paths.

Proposition 3.3.7. *Let $\mathbf{z}:\Delta_3\to\mathbb{R}^d$. For $(\alpha,\gamma)\in(0,1)^2$ with $\alpha-\gamma>0$, $\zeta\in[0,\alpha-\gamma)$, and $\eta\in[\zeta,1]$, we assume that $\delta\mathbf{z}\in\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ where $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ is introduced in Definition 3.2.9. Suppose $\kappa\in(0,\alpha)$. Then for any $p>\frac{1}{\alpha-\kappa}\vee\frac{1}{\zeta}$, the following two bounds holds:*

$$\|\mathbf{z}\|_{(\kappa,\gamma),1} \lesssim U^{T}_{(\kappa,\gamma),1,p}(\mathbf{z}) + \|\delta\mathbf{z}\|_{(\kappa,\gamma),1}, \tag{3.33}$$

$$\|\mathbf{z}\|_{(\kappa,\gamma,\eta,\zeta),1,2} \lesssim U^{T}_{(\kappa,\gamma,\eta,\zeta),1,2,p}(\mathbf{z}) + \|\delta\mathbf{z}\|_{(\kappa,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p}),1,2}\, T^{2+\alpha-\kappa-\frac{1}{p}}. \tag{3.34}$$

Proof. We begin by proving (3.33). It follows directly from (3.5) that for any $0<\kappa<\alpha$

$$|\mathbf{z}^{\tau}_{ts}| \lesssim \psi^{1}_{\kappa,\gamma}(\tau,t,s)\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z}) + \|\delta\mathbf{z}\|_{(\alpha,\gamma),1}\right),$$

Using that $\tau\mapsto U^{\tau}$ is increasing (see Remark 3.3.2) and taking supremum over τ on the right hand side above, it is easily seen that (3.33) holds. We now move on to prove (3.34). To this aim, we shall spell out the right hand side of (3.24) in a slightly different way. Namely note that for $\delta\mathbf{z} \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ and $\eta < \eta'$, we have

$$\int_{\tau'}^{\tau}\int_{\tau'}^{r} \sup_{0\le s<u<v\le t} \frac{|\delta_u \mathbf{z}_{vs}^{rr'}|^{2p}}{\psi^{1,2}_{\kappa,\gamma,\eta,\zeta}(r,r',v,s)^{2p}|r-r'|^2}\, dr'\, dr$$
$$\lesssim \|\delta\mathbf{z}\|^{2p}_{(\alpha,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p}),1,2} \int_{\tau'}^{\tau}\int_{\tau'}^{r} \sup_{0\le s<u<v\le t} \frac{\psi^{1,2}_{\alpha,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p}}(r,r',v,s)^{2p}}{\psi^{1,2}_{\kappa,\gamma,\eta,\zeta}(r,r',v,s)^{2p}|r-r'|^2}\, dr'\, dr, \quad (3.35)$$

where we have used the Definition 2.5 of the $(1,2)$-norm. Furthermore, since $p > \frac{1}{\alpha-\kappa}$ and $s, v \in [0,T]$ it is readily checked that

$$\frac{\psi^{1,2}_{\alpha,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p}}(r,r',v,s)^{2p}}{\psi^{1,2}_{\kappa,\gamma,\eta,\zeta}(r,r',v,s)^{2p}|r-r'|^2} \lesssim |v-s|^{2p(\alpha-\kappa)-2} \le T^{2p(\alpha-\kappa)-2}. \quad (3.36)$$

Hence the right hand side of (3.24) can be upper bounded by

$$C_{T,p,\alpha,\kappa}\|\delta\mathbf{z}\|^{2p}_{(\alpha,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p}),1,2}.$$

Plugging this information into (3.24), the proof of (3.34) is now easily achieved.

□

3.4 Volterra rough path driven by fractional Brownian motion

In this section, we are going to construct the Volterra rough path driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. In this paper we will focus on the case $H > 1/2$ (see also the Brownian case $H = \frac{1}{2}$ in the next section). It should be noticed that this regime leads to nontrivial rough paths development in the Volterra case, due to the singularity of the kernel k in (1.1). Let us first recall some basic facts about the stochastic calculus of variations with respect to fractional Brownian motion.

3.4.1 Malliavin calculus preliminaries

This section is devoted to review some elementary information of Malliavin calculus (mostly borrowed from [9]) that we will use in Section 3.4.2 and Section 3.4.3. We first introduce the notation for our main process of interest.

Notation 3.4.1. *In the sequel we denote by* $B = \{(B_t^1, \ldots, B_t^m),\ t \in [0,T]\}$ *a standard* m*-dimensional fractional Brownian motion with Hurst parameter* $H \in (\frac{1}{2}, 1)$*. Recall that* B *is a centered Gaussian process with independent coordinates. For each component* B^{i}*, the covariance function* R *is defined by*

$$R(s,t) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{4.1}$$

We now say a few words about Cameron-Martin type spaces related to each component B^{i} in Notation 3.4.1. Namely let $\mathcal{H}$ be the Hilbert space defined as the closure of the set of step functions on the interval $[0,T]$ with respect to the scalar product

$$\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]} \rangle_{\mathcal{H}} = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Under our assumption $H > 1/2$, it is easy to see that the covariance of fBm (4.1) can be written as

$$R(s,t) = a_H \int_0^t \int_0^s |u-v|^{2H-2} du dv,$$

where the constant a_H is defined by $a_H = H(2H-1)$. This implies that

$$\langle f, g \rangle_{\mathcal{H}} = a_H \int_0^T \int_0^T f_u g_v |u-v|^{2H-2} du dv, \tag{4.2}$$

for any pair of step functions f and g on $[0,T]$. Therefore $\mathcal{H}$ can also be seen as the completion of step functions with respect to the inner product (4.2). We now introduce a family of additional spaces $|\mathcal{H}|^{\otimes l}$ which will be useful for our computations. Namely for $l \geq 1$ we define $|\mathcal{H}|^{\otimes l}$ as the linear space of measurable functions f on $[0,T]^l \subset \mathbb{R}^l$ such that

$$\|f\|^2_{|\mathcal{H}|^{\otimes l}} := a_H^l \int_{[0,T]^{2l}} |f_{\mathbf{u}}||f_{\mathbf{v}}||u_1 - v_1|^{2H-2} \cdots |u_l - v_l|^{2H-2} d\mathbf{u} d\mathbf{v} < \infty, \tag{4.3}$$

where we write $\mathbf{u} = (u_1, \cdots, u_l)$, $\mathbf{v} = (v_1, \ldots, v_l) \in [0,T]^l$. Notice that $|\mathcal{H}|^{\otimes l}$ is a subset of $\mathcal{H}^{\otimes l}$. The main interest of the spaces $|\mathcal{H}|$ is due to the fact that while $\mathcal{H}^{\otimes l}$ contains distributions, the space $|\mathcal{H}|^{\otimes l}$ is a space of functions.

For each component B^{i}, the mapping $\mathbb{1}_{[0,t]} \mapsto B^{\mathrm{i}}_t$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by B^{i}. We denote this isometry by $h \mapsto B^{\mathrm{i}}(h)$. In this way, $\{B^{\mathrm{i}}(h), h \in \mathcal{H}\}$ is an isonormal Gaussian process indexed by the Hilbert space $\mathcal{H}$. Namely, we have

$$\mathbb{E}\left[B^{\mathrm{i}}(f)\, B^{\mathrm{i}}(g)\right] = \langle f, g \rangle_{\mathcal{H}}. \tag{4.4}$$

It is also worth mentioning that the Wiener integral can be approximated by Riemann type sums. Namely for $h \in \mathcal{H}$ the following limit holds true in $L^2(\Omega)$:

$$B^{\mathrm{i}}(h) = \lim_{|\mathcal{P}| \to 0} \sum_{[r,v] \in \mathcal{P}} B^{\mathrm{i}}_{vr}\, h(r), \tag{4.5}$$

where the Riemann sum is written similarly to (2.9) and we recall that $B^{\mathrm{i}}_{vr} = B^{\mathrm{i}}_v - B^{\mathrm{i}}_r$.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B_{s_1}, \ldots, B_{s_N}),$$

where $N \geq 1$ and $f \in C_b^\infty(\mathbb{R}^{m \times N})$. For each $\mathrm{j} = 1, \ldots, m$ and $t \in [0,T]$, the partial Malliavin derivative of F with respect to the component B^{j} is defined for $F \in \mathcal{S}$ as the $\mathcal{H}$-valued random variable

$$D^{\mathrm{j}}_t F = \sum_{\mathrm{i}}^{N} \frac{\partial f}{\partial x^{\mathrm{j}}_{\mathrm{i}}}(B_{s_1}, \ldots, B_{s_N}) \mathbb{1}_{[0,s_{\mathrm{i}}]}(t), \qquad t \in [0,T], \tag{4.6}$$

where $x_{\mathrm{i}}^{\mathrm{j}}$ stands for the $\mathrm{j}-th$ component of x. We can iterate this procedure to define higher order derivatives $D^{\mathrm{j}_1,\dots,\mathrm{j}_l}F$, which take values in $\mathcal{H}^{\otimes l}$. For any $p \geq 1$ and integer $k \geq 1$, we define the Sobolev space $\mathbb{D}^{k,p}$ as the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p = \mathbb{E}[|F|^p] + \mathbb{E}\left[\sum_{\mathrm{i}=1}^{k}\left(\sum_{\mathrm{j}_1,\dots,\mathrm{j}_l}^{m} \|D^{\mathrm{j}_\mathrm{i},\dots,\mathrm{j}_l}F\|_{\mathcal{H}^{\otimes l}}^2\right)^{p/2}\right]. \tag{4.7}$$

If V is Hilbert space, $\mathbb{D}^{k,p}(V)$ denotes the corresponding Sobolev space of V-valued random variables.

For any $\mathrm{j} = 1, \dots, m$, we denote by $\delta^{\diamond,\mathrm{j}}$ the adjoint of the derivative operator D^{j}. For a process $\{u_t;\, t \in [0,T]\}$, we say $u \in \mathrm{Dom}\,\delta^{\diamond,\mathrm{j}}$ if there is a $\delta^{\diamond,\mathrm{j}}(u) \in L^2(\mathbb{R}^m)$ such that for any $F \in \mathbb{D}^{k,p}$ the following duality relation holds

$$\mathbb{E}\left[\langle u, D^{\mathrm{j}}F\rangle_{\mathcal{H}}\right] = \mathbb{E}\left[\delta^{\diamond,\mathrm{j}}(u)F\right]. \tag{4.8}$$

The random variable $\delta^{\diamond,\mathrm{j}}(u)$ is also called the Skorohod integral of u with respect to the fBm B^{j}, and we use the notation $\delta^{\diamond,\mathrm{j}}(u) = \int_0^T u_t \delta^{\diamond} B_t^{\mathrm{j}}$. It is well known that $\mathbb{D}^{1,2}(\mathcal{H}) \subset \mathrm{Dom}\,(\delta^{\diamond,\mathrm{j}})$ for all $\mathrm{j} = 1, \dots, m$.

We now introduce a pathwise type integral defined on the Wiener space, called Stratonovich integral. Namely let $u = \{u_t, t \in [0,T]\}$ be a continuous stochastic process, and let $\mathcal{P}$ be a generic partition of $[s,t]$. Following [9, Section 3.1], we define

$$B_t^{\mathrm{i},\mathcal{P}} = \sum_{[r,v]\in\mathcal{P}} \frac{B_{vr}^{\mathrm{i}}}{v-r} \mathbb{1}_{[r,v]}(t), \qquad \text{and} \qquad S_{ts}^{\mathrm{i},\mathcal{P}} = \int_s^t u_r\, B_r^{\mathrm{i},\mathcal{P}} dr. \tag{4.9}$$

Then the Stratonovich integral of u with respect to B^{i} is defined as

$$\int_s^t u_r\, dB_r^{\mathrm{i}} = \lim_{|\mathcal{P}|\to 0} S_{ts}^{\mathrm{i},\mathcal{P}}, \tag{4.10}$$

where the limit is understood in probability. On the other hand, assume that u is C^κ-Hölder with $\kappa + H > 1$. Moreover we suppose that $u \in \mathbb{D}^{1,2}(\mathcal{H})$ and the derivative $D_s^{\mathrm{j}} u_t$ exists and satisfies almost surely

$$\int_0^T \int_0^T |D_s^{\mathrm{j}} u_t||t-s|^{2H-2} ds\, dt < \infty, \qquad \text{and} \qquad \mathbb{E}\left[\|D^{\mathrm{j}} u\|^2_{|\mathcal{H}|^{\otimes l}}\right] < \infty.$$

Then the Stratonovich integral $\int_0^T u_t dB_t^{\mathrm{j}}$ exists, and we have the following relation between Skorohod and Stratonovich stochastic integrals:

$$\int_0^T u_t dB_t^{\mathrm{j}} = \int_0^T u_t \delta^\diamond B_t^{\mathrm{j}} + a_H \int_0^T \int_0^T D_s^{\mathrm{j}} u_t |t-s|^{2H-2} ds\, dt. \tag{4.11}$$

We close this section by spelling out Meyer's inequality for the Skorohod integral: given $p > 1$ and an integer $k \geq 1$, there is a constant $c_{k,p}$ such that the k-th iterated Skorohod integral satisfies

$$\|(\delta^\diamond)^k(u)\|_p \leq c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathcal{H}^{\otimes k})} \qquad \text{for all,} \quad u \in \mathbb{D}^{k,p}(\mathcal{H}^{\otimes k}). \tag{4.12}$$

3.4.2 First level of the Volterra rough path

In this section, we will construct the first level of the Volterra rough path driven by a fBm as introduced in Notation 3.4.1. We start by defining our main object of study.

Definition 3.4.2. *Consider a fractional Brownian motion $B : [0,T] \to \mathbb{R}^m$ as given in Notation 3.4.1 and a function h of the form $h_{ts}^\tau(r) = (\tau - r)^{-\gamma} \mathbb{1}_{[s,t]}(r)$ with $\gamma < 2H - 1$. Then for $(s,t,\tau) \in \Delta_3$ we define the increment $\mathbf{z}_{ts}^{1,\tau,\mathrm{i}} = \int_s^t (\tau - r)^{-\gamma}\, dB_r^{\mathrm{i}}$ as a Wiener integral of the form*

$$\mathbf{z}_{ts}^{1,\tau,\mathrm{i}} := B^{\mathrm{i}}(h_{ts}^\tau). \tag{4.13}$$

Remark 3.4.3. Note that for the particular type of integrand h considered in Definition 3.4.2, the process $B^{\mathrm{i}}(h_{ts}^\tau)$ is additive in its lower variables, in the sense that

$$B^{\mathrm{i}}(h_{ts}^\tau) = B^{\mathrm{i}}(h_{t0}^\tau) - B^{\mathrm{i}}(h_{s0}^\tau). \tag{4.14}$$

Thus defining $\mathbf{z}^{1,\tau}_t := \mathbf{z}^{1,\tau}_{t0}$ we have that $\mathbf{z}^1$ is defined on the simplex Δ_2.

With Definition 3.4.2 in hand, we now estimate the second moment of $\mathbf{z}^{1,\tau,\mathrm{i}}_{ts}$ and $\mathbf{z}^{1,\tau\tau',\mathrm{i}}_{ts}$.

Lemma 3.4.4. *Consider the Volterra rough path $\mathbf{z}^1$ as given in (4.13). Then for $(s,t,\tau) \in \Delta_3$, we have*

$$\mathbb{E}[(\mathbf{z}^{1,\tau,\mathrm{i}}_{ts})^2] \lesssim \left|\psi^1_{(H,\gamma)}(\tau,t,s)\right|^2. \tag{4.15}$$

While for $(s,t,\tau',\tau) \in \Delta_4$, $\zeta \in [0, H-\gamma)$, and $\eta \in [\zeta, 1]$, we get

$$\mathbb{E}[(\mathbf{z}^{1,\tau\tau',\mathrm{i}}_{ts})^2] \lesssim \left|\psi^{1,2}_{(H,\gamma,\eta,\zeta)}(\tau,,\tau',t,s)\right|^2, \tag{4.16}$$

where ψ^1 and $\psi^{1,2}$ are given in Notation 3.2.2.

Proof. We first prove relation (4.15). According to (4.13) and (4.4), we can compute $\mathbb{E}[(\mathbf{z}^{1,\tau,\mathrm{i}}_{ts})^2]$ as

$$\mathbb{E}[(\mathbf{z}^{1,\tau,\mathrm{i}}_{ts})^2] = \mathbb{E}\left[B^{\mathrm{i}}(h^\tau_{ts})\, B^{\mathrm{i}}(h^\tau_{ts})\right] = \langle h^\tau_{ts}, h^\tau_{ts}\rangle_{\mathcal{H}} \tag{4.17}$$

Owing to relation (4.2) for the inner product in $\mathcal{H}$, we thus obtain

$$\mathbb{E}[(\mathbf{z}^{1,\tau,\mathrm{i}}_{ts})^2] = H(2H-1)\iint_{[s,t]\times[s,t]} (\tau-r)^{-\gamma}(\tau-l)^{-\gamma}\,|r-l|^{2H-2}\,drdl. \tag{4.18}$$

Notice that the function $(\tau-r)^{-\gamma}(\tau-l)^{-\gamma}\,|r-l|^{2H-2}$ is symmetric. Hence we can recast (4.18) as

$$\mathbb{E}[(\mathbf{z}^{1,\tau,\mathrm{i}}_{ts})^2] = 2H(2H-1)\int_s^t (\tau-r)^{-\gamma}dr \int_r^t (\tau-l)^{-\gamma}(l-r)^{2H-2}dl. \tag{4.19}$$

In the right hand side of (4.19), we first estimate the integral

$$\int_r^t (\tau-l)^{-\gamma}(l-r)^{2H-2}dl := J. \tag{4.20}$$

Since $l \in (r,t)$ in (4.20), we proceed to a change of variable $l = r + \theta(t-r)$. We obtain

$$J = (t-r)^{2H-1}\int_0^1 (\tau - r - \theta(t-r))^{-\gamma}\,\theta^{2H-2}d\theta \leq (t-r)^{2H-1}(\tau-r)^{-\gamma}\int_0^1 (1-\theta)^{-\gamma}\theta^{2H-2}d\theta. \tag{4.21}$$

Recall that we have assumed that $\gamma < 2H-1 < 1$. Moreover $H > 1/2$ and thus $2H-2 > -1$. Hence the right hand side of (4.21) can be expressed in terms of Beta functions in the following way:

$$\int_0^1 (1-\theta)^{-\gamma}\theta^{2H-2}d\theta = \mathrm{Beta}(1-\gamma, 2H-1) < \infty.$$

Therefore the integral J as given in (4.20) can be bounded by

$$J = \int_r^t (\tau - l)^{-\gamma}(l-r)^{2H-2}dl \lesssim (t-r)^{2H-1}(\tau-r)^{-\gamma}. \tag{4.22}$$

Plugging (4.22) into (4.18), we thus get

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,\mathrm{i}})^2] \lesssim \int_s^t (\tau-r)^{-2\gamma}(t-r)^{2H-1}dr. \tag{4.23}$$

We now bound the right hand side of (4.23) in two different ways. First since $(\tau-r) > (t-r)$, we have

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,\mathrm{i}})^2] \lesssim \int_s^t (t-r)^{2H-2\gamma-1}dr \lesssim (t-s)^{2H-2\gamma}, \tag{4.24}$$

where we have resorted to the fact that $\gamma < 2H-1 < H$ for the second inequality. Next we also use the fact that $(\tau - r) > (\tau - t)$ in the right hand side of (4.23), which allows to write

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,\mathrm{i}})^2] \lesssim (\tau-t)^{-2\gamma}\int_s^t (t-r)^{2H-1}dr \lesssim (\tau-t)^{-2\gamma}(t-s)^{2H}. \tag{4.25}$$

Combining (4.24) and (4.25), we end up with the following estimate for the second moment of $\mathbf{z}_{ts}^{1,\tau,\mathrm{i}}$:

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,\mathrm{i}})^2] \lesssim \left[(\tau-t)^{-2\gamma}(t-s)^{2H}\right] \wedge (t-s)^{2H-2\gamma} = \left(\psi^1_{(H,\gamma)}(\tau,t,s)\right)^2, \tag{4.26}$$

where we have appealed to the definition (2.1) of ψ^1 for the second identity. Relation (4.26) is the desired result (4.15).

Next, we will prove the inequality (4.16). To this aim, we first note that owing to (4.13), we have the following expression for $\mathbf{z}_{ts}^{1,\tau\tau',\mathrm{i}}$,

$$\mathbf{z}_{ts}^{1,\tau\tau',\mathrm{i}} = \mathbf{z}_{ts}^{1,\tau,\mathrm{i}} - \mathbf{z}_{ts}^{1,\tau',\mathrm{i}} = B^{\mathrm{i}}(h_{ts}^{\tau} - h_{ts}^{\tau'}). \tag{4.27}$$

Similarly to (4.19), we can thus rewrite $\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',\mathrm{i}})^2]$ as

$$\begin{aligned}\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',\mathrm{i}})^2] &= \langle h_{ts}^{\tau} - h_{ts}^{\tau'}, h_{ts}^{\tau} - h_{ts}^{\tau'}\rangle_{\mathcal{H}} \\ &= 2H(2H-1)\int_s^t \left[(\tau'-r)^{-\gamma} - (\tau-r)^{-\gamma}\right] dr \int_r^t \left[(\tau'-l)^{-\gamma} - (\tau-l)^{-\gamma}\right](l-r)^{2H-2} dl.\end{aligned} \tag{4.28}$$

We now recall an elementary inequality on increments of negative power functions. Namely for $\tau > \tau' > r$ and $\eta \in [0,1]$ we have

$$(\tau-r)^{-\gamma} - (\tau'-r)^{-\gamma} \lesssim (\tau-\tau')^{\eta}(\tau'-r)^{-\eta-\gamma}.$$

Plugging this upper bound into the right hand side of (4.28), we obtain

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',\mathrm{i}})^2] \lesssim |\tau-\tau'|^{2\eta}\int_s^t (\tau'-r)^{-\eta-\gamma}\int_r^t (\tau'-l)^{-\eta-\gamma}(l-r)^{2H-2} dl. \tag{4.29}$$

The expression (4.29) is now very similar to (4.19). Therefore with the same steps as for (4.20)-(4.25), for some $\zeta \in [0, H-\gamma)$ and $\eta \in [\zeta, 1]$, we get

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',\mathrm{i}})^2] \lesssim |\tau-\tau'|^{2\eta}|\tau'-t|^{-2(\eta-\zeta)}\left(\left[(\tau-t)^{-2\gamma-2\zeta}(t-s)^{2H}\right] \wedge (t-s)^{2H-2\gamma-2\zeta}\right). \tag{4.30}$$

According to the definition (2.2) of $\psi^{1,2}$, (4.30) is equivalent to

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',\mathrm{i}})^2] \lesssim \left|\psi^{1,2}_{(H,\gamma,\eta,\zeta)}(\tau,\tau',t,s)\right|^2.$$

This finishes the proof of (4.16). $\square$

Remark 3.4.5. One can easily extend the computation of Lemma 4.15 in order to get more general bounds for covariance functions. Namely for any $(s,u,v,\tau) \in \Delta_4$, and recalling the expression (2.1) for ψ^1 we have

$$\mathbb{E}[\mathbf{z}_{us}^{1,\tau,\mathrm{i}}\mathbf{z}_{vs}^{1,\tau,\mathrm{i}}] \lesssim \left|\psi^1_{(H,\gamma)}(\tau,v,s)\right|^2. \tag{4.31}$$

Similarly for any $(s,u,v,\tau',\tau) \in \Delta_5$ and recalling our definition (2.2) for $\psi^{1,2}$, we obtain

$$\mathbb{E}[\mathbf{z}_{us}^{1,\tau\tau',\mathrm{i}}\mathbf{z}_{vs}^{1,\tau\tau',\mathrm{i}}] \lesssim \left|\psi^{1,2}_{(H,\gamma,\eta,\zeta)}(\tau,\tau',v,s)\right|^2, \tag{4.32}$$

where $0 \le \zeta < H-\gamma$ and $\zeta \le \eta \le 1$.

3.4.3 Second level of the Volterra rough path

In this section we turn our attention to the construction of a nontrivial Volterra rough path above a fBm. More specifically our aim is to construct a family $\{\mathbf{z}^{1,\tau},\mathbf{z}^{2,\tau}\}$ verifying Definition 3.2.8. Let us start with the definition of $\mathbf{z}^{2,\tau}$.

Definition 3.4.6. *We consider a fractional Brownian motion $B:[0,T]\to\mathbb{R}^m$ as given in Notation 3.4.1, as well as the first level of the Volterra rough path $\mathbf{z}^{1,\tau}$ defined by (4.13). As in Definition 3.4.2, we assume that $\gamma < 2H-1$. Then for $(s,r,t,\tau)\in\Delta_4$, we set*

$$u_{ts}^{\tau,\mathrm{i}}(r) = (\tau-r)^{-\gamma}\mathbf{z}_{rs}^{1,r,\mathrm{i}}\mathbb{1}_{[s,t]}(r). \tag{4.33}$$

With this notation in hand, the increment $\mathbf{z}_{ts}^{2,\tau}$ is given as follows: if $\mathrm{i}\neq\mathrm{j}$ we define $\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}}$ as

$$\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}} = B^{\mathrm{j}}(u_{ts}^{\tau,\mathrm{i}}), \tag{4.34}$$

where (conditionally on B^{i}*) the random variable* $B^{\mathrm{j}}(u_{ts}^{\tau,\mathrm{i}})$ *has to be interpreted as a Wiener integral. In the case* $\mathrm{i} = \mathrm{j}$*, we set*

$$\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{i}} = \int_s^t u_{ts}^{\tau,\mathrm{i}}(r) dB_r^{\mathrm{i}}, \tag{4.35}$$

where the right hand side of (4.35) *is defined as a Stratonovich integral like* (4.11)*.*

Remark 3.4.7. Having the Definition 3.4.2 of $\mathbf{z}^{1,\tau}$ in mind when considering the process $u^{\tau,\mathrm{i}}$ in (4.33), we get that $\mathbf{z}^{2,\tau}$ in (4.34)-(4.35) is formally interpreted as

$$\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}} = \int_s^t (\tau - r)^{-\gamma} \int_s^r (r-l)^{-\gamma} \, dB_l^{\mathrm{i}} dB_r^{\mathrm{j}}. \tag{4.36}$$

Below we will show that $\mathbf{z}^{2,\tau}$ can indeed be considered as the double iterated integral in (4.36).

Similarly to what we did for $\mathbf{z}^1$, we will now estimate the second moment of $\mathbf{z}^{2,\tau}$.

Proposition 3.4.8. *Consider the second level* $\mathbf{z}^{2,\tau}$ *of the Volterra rough path, as defined in* (4.34)-(4.35)*. Recall that* H, γ *satisfy* $H > 1/2$ *and* $\gamma < 2H - 1$*. Then for* $(s,t,\tau) \in \Delta_3$ *and any* $\mathrm{i}, \mathrm{j} = 1, \ldots, d$*, we have*

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}}\right)^2\right] \lesssim \left|\psi^1_{(2H-\gamma,\gamma)}(\tau,t,s)\right|^2. \tag{4.37}$$

As far as the $(1,2)$*-type increments are considered, for* $(s,t,\tau',\tau) \in \Delta_4$*,* $\zeta \in [0, 2(H-\gamma))$*, and* $\eta \in [\zeta, 1]$*, we get*

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau\tau',\mathrm{i},\mathrm{j}}\right)^2\right] \lesssim \left|\psi^{1,2}_{(2H-\gamma,\gamma,\eta,\zeta)}(\tau,\tau',t,s)\right|^2, \tag{4.38}$$

where ψ^1 *and* $\psi^{1,2}$ *are given in Notation 3.2.2.*

Proof. We will prove relation (4.37) in the following, (4.38) can be treated in a similar way and is left to the reader, for sake of conciseness. According to Remark 3.4.7, we consider $\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}}$ and $\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{i}}$ as different integrals. Therefore we will split the proof of (4.37) into two parts: $\mathrm{i} \neq \mathrm{j}$ and $\mathrm{i} = \mathrm{j}$.

Step 1: Relation (4.37) *for* $\mathrm{i} \neq \mathrm{j}$. In this step, we will show that (4.37) holds for $\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}}$ as given in (4.34). According to Definition 3.4.6, we consider the integral (4.34) as a conditional Wiener integral. Namely due to the independence of B^{i} and B^{j} we can write

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}}\right)^2\right] = \mathbb{E}\left\{\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}}\right)^2 \,\middle|\, B^{\mathrm{i}}\right]\right\} = \mathbb{E}\left\{\mathbb{E}\left[\left(B^{\mathrm{j}}(u_{ts}^{\tau,\mathrm{i}})\right)^2 \,\middle|\, B^{\mathrm{i}}\right]\right\}, \tag{4.39}$$

where we recall that $u_{ts}^{\tau,\mathrm{i}}$ is defined by (4.33). Furthermore, relation (4.39) for Wiener integral reads

$$\mathbb{E}\left[\left(B^{\mathrm{j}}(u_{ts}^{\tau,\mathrm{i}})\right)^2 \,\middle|\, B^{\mathrm{i}}\right] = \|u_{ts}^{\tau,\mathrm{i}}\|_{\mathcal{H}}^2, \tag{4.40}$$

and thus

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}}\right)^2\right] = \mathbb{E}\left[\|u_{ts}^{\tau,\mathrm{i}}\|_{\mathcal{H}}^2\right]. \tag{4.41}$$

In order to bound the right hand side of (4.41), we resort to the expression (4.2) for the inner product in $\mathcal{H}$. This yields

$$\begin{aligned}\mathbb{E}\left[\|u_{ts}^{\tau,\mathrm{i}}\|_{\mathcal{H}}^2\right] = \mathbb{E}[\langle u_{ts}^{\tau,\mathrm{i}}, u_{ts}^{\tau,\mathrm{i}}\rangle_{\mathcal{H}}] = H(2H-1)\mathbb{E}\Big[\int_s^t\int_s^t &\left((\tau-r_1)^{-\gamma}\int_s^{r_1}(r_1-l_1)^{-\gamma}dB_{l_1}^{\mathrm{i}}\right)\\ &\times\left((\tau-r_2)^{-\gamma}\int_s^{r_2}(r_2-l_2)^{-\gamma}dB_{l_2}^{\mathrm{i}}\right)|r_1-r_2|^{2H-2}dr_1dr_2\Big].\end{aligned}$$

Thanks to an easy application of Fubini's theorem, and invoking the symmetry of the integrand like in (4.19) we get

$$\begin{aligned}\mathbb{E}\left[\|u_{ts}^{\tau,\mathrm{i}}\|_{\mathcal{H}}^2\right] = 2H(2H-1)\int_s^t\int_s^{r_1}&(\tau-r_1)^{-\gamma}(\tau-r_2)^{-\gamma}|r_1-r_2|^{2H-2}\\ &\times\mathbb{E}\left[\int_s^{r_1}(r_1-l_1)^{-\gamma}dB_{l_1}^{\mathrm{i}}\int_s^{r_2}(r_2-l_2)^{-\gamma}dB_{l_2}^{\mathrm{i}}\right]dr_2dr_1.\end{aligned} \tag{4.42}$$

Moreover, owing to (4.31), and recalling the definition (2.1) of ψ^1, we have

$$\mathbb{E}\left[\int_s^{r_1}(r_1-l_1)^{-\gamma}dB_{l_1}^{\mathrm{i}}\int_s^{r_2}(r_2-l_2)^{-\gamma}dB_{l_2}^{\mathrm{i}}\right] = \mathbb{E}\left[\mathbf{z}_{r_1s}^{1,r_1,\mathrm{i}}\mathbf{z}_{r_2s}^{1,r_2,\mathrm{i}}\right] \lesssim |r_1-s|^{2H-2\gamma}. \tag{4.43}$$

Plugging (4.43) into (4.42), we thus get

$$\mathbb{E}\left[\|u_{ts}^{\tau,\mathrm{i}}\|_{\mathcal{H}}^2\right] \lesssim \int_s^t \int_s^{r_1} (\tau - r_1)^{-\gamma}(\tau - r_2)^{-\gamma}|r_1 - r_2|^{2H-2}|r_1 - s|^{2H-2\gamma} dr_1 dr_2. \tag{4.44}$$

Similarly to what we did for (4.18)-(4.25) in the proof of Lemma 3.4.4, we evaluate the right hand side of (4.44) thanks to elementary integral bounds and the use of Beta functions. We let the patient reader check that we get

$$\mathbb{E}\left[\|u_{ts}^{\tau,\mathrm{i}}\|_{\mathcal{H}}^2\right] \lesssim \left[|\tau - t|^{-2\gamma}|t - s|^{4H-2\gamma}\right] \wedge |t - s|^{4H-4\gamma}. \tag{4.45}$$

Plugging (4.45) into (4.40), we thus obtain

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}}\right)^2\right] \lesssim \left[|\tau - t|^{-2\gamma}|t - s|^{4H-2\gamma}\right] \wedge |t - s|^{4H-4\gamma} = \left|\psi^1_{(2H-\gamma,\gamma)}(\tau, t, s)\right|^2, \tag{4.46}$$

where we have invoked the definition (2.1) of ψ^1. This is the desired result (4.37).

Step 2: Relation (4.37) *for* $\mathrm{i} = \mathrm{j}$*.* In this step, we will show that relation (4.37) holds for $\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{i}}$ defined by (4.35). According to Definition 3.4.6, we consider (4.35) as a Stratonovich integral like (4.11). We thus recast (4.35) as

$$\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{i}} = \int_s^t u_{ts}^{\tau,\mathrm{i}}(r) dB_r^{\mathrm{i}} = \int_s^t u_{ts}^{\tau,\mathrm{i}}(r) \delta B_r^{\mathrm{i}} + H(2H-1) \int_s^t \int_s^t D_l^{\mathrm{i}} \left(u_{ts}^{\tau,\mathrm{i}}(r)\right) |r - l|^{2H-2} dr\, dl, \tag{4.47}$$

where $u_{ts}^{\tau,\mathrm{i}}(r)$ as given in (4.33). Taking square and expectation on both sides of (4.47), we obtain

$$\mathbb{E}\left[\left(\int_s^t u_{ts}^{\tau,\mathrm{i}}(r) dB_r^{\mathrm{i}}\right)^2\right] \lesssim J_1 + J_2, \tag{4.48}$$

where the terms J_1 and J_2 are respectively defined by

$$J_1 = \mathbb{E}\left[\left(\int_s^t u_{ts}^{\tau,\mathrm{i}}(r) \delta B_r^{\mathrm{i}}\right)^2\right], \tag{4.49}$$

$$J_2 = (H(2H-1))^2\, \mathbb{E}\left[\left(\int_s^t \int_s^t D_l^{\mathrm{i}} \left(u_{ts}^{\tau,\mathrm{i}}(r)\right) |r - l|^{2H-2} dr\, dl\right)^2\right]. \tag{4.50}$$

In the following, we will estimate J_1 and J_2 separately.

In order to upper bound J_1, we recall that the integral $\int_s^t u_{ts}^{\tau,\mathrm{i}}(r)\delta^\diamond B_r^{\mathrm{i}}$ in the right hand side of (4.49) is interpreted as a Skorohod integral of the form $\delta^\diamond(u_{ts}^{\tau,\mathrm{i}})$. Resorting to (4.12), we thus have

$$J_1 = \|\delta^\diamond\left(u_{ts}^{\tau,\mathrm{i}}\right)\|_2^2 \lesssim \|u_{ts}^{\tau,\mathrm{i}}\|_{\mathbb{D}^{1,2}(\mathcal{H}^{\otimes 2})}^2. \tag{4.51}$$

Let us now handle the right hand side of (4.51). Owing to (4.7), we get

$$J_1 \lesssim \mathbb{E}\left[\|u_{ts}^{\tau,\mathrm{i}}\|_{\mathcal{H}}^2\right] + \mathbb{E}\left[\|D^{\mathrm{i}}\left(u_{ts}^{\tau,\mathrm{i}}\right)\|_{\mathcal{H}^{\otimes 2}}^2\right]. \tag{4.52}$$

Notice that the first term of the right hand side of (4.52) is what we upper bounded in Step 1. Thanks to (4.45), we obtain

$$\mathbb{E}\left[\|u_{ts}^{\tau,\mathrm{i}}\|_{\mathcal{H}}^2\right] \lesssim \left[|\tau - t|^{-2\gamma}|t-s|^{4H-2\gamma}\right] \wedge |t-s|^{4H-4\gamma}. \tag{4.53}$$

In order to estimate the second term in the right hand side of (4.52), let us first compute the partial Malliavin derivative $D_l^{\mathrm{i}}(u_{ts}^{\tau,\mathrm{i}}(r))$ of $u_{ts}^{\tau,\mathrm{i}}(r)$ with respect to B^{i}. Specifically, we gather (4.13) and (4.33) in order to get

$$u_{ts}^{\tau,\mathrm{i}}(r) = (\tau - r)^{-\gamma} B^{\mathrm{i}}(h_{rs}^r)\mathbb{1}_{[s,t]}(r), \qquad \text{with} \qquad h_{rs}^r(l) = (r-l)^{-\gamma}\mathbb{1}_{[s,r]}(l).$$

Thanks to (4.6), we thus get

$$D_l^{\mathrm{i}}\left(u_{ts}^{\tau,\mathrm{i}}(r)\right) = (\tau - r)^{-\gamma} h_{rs}^r(l)\mathbb{1}_{[s,t]}(r) = (\tau - r)^{-\gamma}(r-l)^{-\gamma}\mathbb{1}_{[s,r]}(l)\mathbb{1}_{[s,t]}(r). \tag{4.54}$$

Plugging (4.54) into the second term of the right hand side of (4.52), and having the definition (4.3) of $\mathcal{H}^{\otimes 2}$-norms in mind, we obtain

$$\begin{aligned}\|D^{\mathrm{i}} u_{ts}^{\tau,\mathrm{i}}\|_{\mathcal{H}^{\otimes 2}}^2 = (H(2H-1))^2 \int_s^t\int_s^t\int_s^{r_1}\int_s^{r_2} (\tau - r_1)^{-\gamma}(\tau - r_2)^{-\gamma} \\ \times (r_1 - l_1)^{-\gamma}(r_2 - l_2)^{-\gamma}|l_1 - l_2|^{2H-2}|r_1 - r_2|^{2H-2} dl_1 dl_2 dr_1 dr_2.\end{aligned} \tag{4.55}$$

The right hand side of (4.55) can be estimated by elementary calculus similarly to (4.18)-(4.26). We let the patient reader check that

$$\mathbb{E}\left[\|D^{\mathrm{i}}u_{ts}^{\tau,\mathrm{i}}\|_{\mathcal{H}^{\otimes 2}}^{2}\right] \lesssim \left[|\tau-t|^{-2\gamma}|t-s|^{4H-2\gamma}\right] \wedge |t-s|^{4H-4\gamma}. \tag{4.56}$$

Eventually plugging (4.56) and (4.53) into (4.52), we end up with

$$J_1 \lesssim \left[|\tau-t|^{-2\gamma}|t-s|^{4H-2\gamma}\right] \wedge |t-s|^{4H-4\gamma}. \tag{4.57}$$

Next we upper bound J_2 as given in (4.50). Recalling that we have computed $D_l^{\mathrm{i}}(u_{ts}^{\tau,\mathrm{i}}(r))$ in (4.54) and plugging this identity into (4.50), we obtain

$$J_2 = (H(2H-1))^2 \left(\int_s^t \int_s^r (\tau-r)^{-\gamma}(r-l)^{-\gamma}|r-l|^{2H-2} dr\, dl\right)^2.$$

Along the same lines as for the computations from (4.19) to (4.25), and recalling the fact that $\gamma < 2H-1 < 1$, we get the following upper bound for J_2,

$$J_2 \lesssim \left[|\tau-t|^{-2\gamma}|t-s|^{4H-2\gamma}\right] \wedge |t-s|^{4H-4\gamma}. \tag{4.58}$$

Eventually plugging (4.58) and (4.57) into (4.48) and recalling again the definition (2.1) of ψ^1, we get

$$\mathbb{E}\left[\left(\int_s^t u_r dB_r^{\mathrm{i}}\right)^2\right] \lesssim \left[|\tau-t|^{-2\gamma}|t-s|^{4H-2\gamma}\right] \wedge |t-s|^{4H-4\gamma} = \left|\psi^1_{(2H-\gamma,\gamma)(\tau,t,s)}\right|^2.$$

This completes the proof of Step 2.

Eventually, combining Step 1 and Step 2, relation (4.37) holds for the increment $\mathbf{z}_{ts}^{2,\tau}$ as given in (4.34)-(4.35). This concludes the proof of (4.37). □

3.4.4 Properties of Volterra rough path family $\{\mathbf{z}^{1,\tau}, \mathbf{z}^{2,\tau}\}$

We have constructed a Volterra rough path family $\{\mathbf{z}^{1,\tau}, \mathbf{z}^{2,\tau}\}$ and we have also upper bounded their moment in Section 3.4.2 and Section 3.4.3. In this section, we will verify that $\{\mathbf{z}^{1,\tau}, \mathbf{z}^{2,\tau}\}$ satisfies Definition 3.2.8. We start by checking the analytic part of Definition 3.2.8 for $\mathbf{z}^{1,\tau}$.

Proposition 3.4.9. *The increment $\mathbf{z}^{1,\tau}$ introduced in Definition 3.4.2 is almost surely in the Volterra space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^m)$ for any $\alpha \in (\gamma, H)$, $\zeta \in [0, \alpha-\gamma)$ and $\eta \in [\zeta, 1]$, where $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^m)$ is introduced in Definition 3.2.3. In addition, for all $p \geq 1$ and $\alpha < H - \frac{3}{2p}$ we have that*

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta)}\right] < \infty. \tag{4.59}$$

Proof. In this proof, we will turn to Proposition 3.3.7 in order to prove (4.59). According to the definition (2.3) of Volterra norms, it suffices to show that $\mathbb{E}[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}]$ and $\mathbb{E}[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta),1,2}]$ are finite. We will separate the study of those two moments.

Step 1: Estimate for the 2p moment of 1-norm. Let us first upper bound $\mathbb{E}[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}]$. Towards this aim, consider a fixed Volterra exponent $\gamma < \alpha < H$ and a parameter $p > 1$ to be determined later on. Relation (3.33) is then equivalent to

$$\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1} \lesssim \left(U^T_{(\alpha,\gamma),1,p}(\mathbf{z}^1)\right)^{2p} + \|\delta\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}. \tag{4.60}$$

Let us handle the term $\|\delta\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}$ in the right hand side of (4.60). Gathering the definitions in (4.13) and (2.12), for $(s,m,t) \in \Delta_3$ we have

$$\delta_m \mathbf{z}^{1,\tau,\mathrm{i}}_{ts} = B^{\mathrm{i}}(h^\tau_{ts}) - B^{\mathrm{i}}(h^\tau_{tm}) - B^{\mathrm{i}}(h^\tau_{ms}). \tag{4.61}$$

Moreover recalling that $h^\tau_{ts}(r) = (\tau - r)^{-\gamma}\mathbb{1}_{[s,t]}(r)$, it is readily checked that $h^\tau_{ts} - h^\tau_{tm} - h^\tau_{ms} = 0$. We thus get $\delta\mathbf{z}^{1,\tau,\mathrm{i}} = 0$ and (4.60) is reduced to

$$\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1} \lesssim \left(U^T_{(\alpha,\gamma),1,p}(\mathbf{z}^1)\right)^{2p}. \tag{4.62}$$

Taking expectations on both sides of (4.62) and recalling the definition (3.1) of $U^T_{(\alpha,\gamma),1,p}$, we obtain

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}\right] \lesssim \mathbb{E}\left[\int_{(v,w)\in\Delta_2^\tau} \frac{|z^{1,\tau}_{wv}|^{2p}}{|\psi^1_{\alpha,\gamma}(\tau,w,v)|^{2p}|w-v|^2}\, dv\, dw\right]. \tag{4.63}$$

Invoking Fubini's theorem and the fact that $\mathbf{z}^{1;\tau}_{wv}$ is a Gaussian random variable, we thus get

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}\right] \lesssim \int_{(v,w)\in\Delta_2^\tau} \frac{\mathbb{E}\left[|z^{1,\tau}_{wv}|^2\right]^p}{|\psi^1_{\alpha,\gamma}(\tau,w,v)|^{2p}|w-v|^2} dvdw. \tag{4.64}$$

We can now apply (4.15), and hence relation (4.64) reads

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}\right] \lesssim \int_{(v,w)\in\Delta_2^\tau} \frac{|\psi^1_{H,\gamma}(\tau,w,v)|^{2p}}{|\psi^1_{\alpha,\gamma}(\tau,w,v)|^{2p}|w-v|^2} dvdw. \tag{4.65}$$

Recalling the definition (2.1) of ψ^1, we obtain

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}\right] \lesssim \int_{(v,w)\in\Delta_2^\tau} \frac{\left[|\tau-w|^{-2p\gamma}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma)}}{\left(\left[|\tau-w|^{-2p\gamma}|w-v|^{2p\alpha}\right]\wedge|w-v|^{2p(\alpha-\gamma)}\right)|w-v|^2} dvdw. \tag{4.66}$$

In order to upper bound the right hand side of (4.66), we split set Δ_2^τ into two subsets

$$E_1 = \left\{(v,w)\in\Delta_2^\tau \Big|\, |\tau-w| \le |w-v|\right\}, \quad \text{and} \quad E_2 = \left\{(v,w)\in\Delta_2^\tau \Big|\, |\tau-w| > |w-v|\right\}.$$

Then relation (4.66) is equivalent to

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}\right] \lesssim I_1 + I_2, \tag{4.67}$$

where I_1 and I_2 are respectively given by

$$I_1 = \int_{E_1} \frac{\left[|\tau-w|^{-2p\gamma}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma)}}{\left[|\tau-w|^{-2p\gamma}|w-v|^{2p\alpha+2}\right]\wedge|w-v|^{2p(\alpha-\gamma)+2}} dvdw, \tag{4.68}$$

$$I_2 = \int_{E_2} \frac{\left[|\tau-w|^{-2p\gamma}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma)}}{\left[|\tau-w|^{-2p\gamma}|w-v|^{2p\alpha+2}\right]\wedge|w-v|^{2p(\alpha-\gamma)+2}} dvdw. \tag{4.69}$$

In the following, we will estimate I_1 and I_2 separately.

In order to upper bound I_1, we first note that for any $(v, w) \in E_1$, we have $|\tau - w| \leq |w - v|$. Thus

$$|w - v|^{2p(H-\gamma)} = |w - v|^{2pH} |w - v|^{-2p\gamma} \leq |\tau - w|^{-2p\gamma} |w - v|^{2pH}, \tag{4.70}$$

and we trivially get

$$\left[|\tau - w|^{-2p\gamma} |w - v|^{2pH} \right] \wedge |w - v|^{2p(H-\gamma)} = |w - v|^{2p(H-\gamma)}. \tag{4.71}$$

In the same way, on E_1 we can write

$$\left[|\tau - w|^{-2p\gamma} |w - v|^{2pH+2} \right] \wedge |w - v|^{2p(H-\gamma)+2} = |w - v|^{2p(H-\gamma)+2}. \tag{4.72}$$

Plugging (4.71) and (4.72) into (4.68), we get

$$I_1 = \int_{E_1} \frac{|w - v|^{2p(H-\gamma)}}{|w - v|^{2p(\alpha-\gamma)+2}} = \int_{E_1} |w - v|^{2p(H-\alpha)-2} dv dw. \tag{4.73}$$

Similarly, reverting the inequality in (4.70) we get that

$$I_2 = \int_{E_2} \frac{|\tau - w|^{-2p\gamma} |w - v|^{2pH}}{|\tau - w|^{-2p\gamma} |w - v|^{2p\alpha+2}} dv dw = \int_{E_2} |w - v|^{2p(H-\alpha)-2} dv dw. \tag{4.74}$$

Now gathering (4.73) and (4.74) into (4.67), we end up with

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1} \right] \lesssim \int_{(v,w)\in\Delta_2^\tau} |w - v|^{2p(H-\alpha)-2} dv dw. \tag{4.75}$$

The right hand side above is easily checked to be finite as long as $\alpha < H - \frac{1}{2p}$.

Step 2: Estimate for the $2p$ *moment of* $(1,2)$*-norm.* Next, we will show that $\mathbb{E}[\|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}]$ is finite. Similarly to the proof for the 1-norm in Step 1, considering again $p \geq 1$. Then resorting to (3.34), we get

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta),1,2} \right] \lesssim \mathbb{E}\left[\left(U^T_{(\alpha,\gamma,\eta,\zeta),1,2,p}(\mathbf{z}^1) \right)^{2p} \right].$$

As in Step 1, recalling the definition (3.2) of $U^T_{(\alpha,\gamma,\eta,\zeta),1,2,p}(\mathbf{z})$, invoking Fubini's theorem and thanks to the fact that $\mathbf{z}^{1,\tau\tau'}$ is a Gaussian random variable, we obtain

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta),1,2}\right] \lesssim \int_{(v,w,r',r)\in\Delta_4^\tau} \frac{\mathbb{E}^p\left[|z^{1,rr'}_{wv}|^2\right]}{|\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(r,r',w,v)|^{2p}|w-v|^2|r-r'|^2}\, dvdwdr'dr. \tag{4.76}$$

In addition, owing to (4.16), relation (4.76) yields

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta),1,2}\right] \lesssim \int_{(v,w,r',r)\in\Delta_4^\tau} \frac{|\psi^{1,2}_{H,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p}}(r,r',w,v)|^{2p}}{|\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(r,r',w,v)|^{2p}|w-v|^2|r-r'|^2} dvdwdr'dr. \tag{4.77}$$

We now recall the definition (2.2) of $\psi^{1,2}$ and plug this identity into (4.77). We get

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta),1,2}\right] \lesssim \int_{(v,w,r',r)\in\Delta_4^\tau} g_{(H,\alpha,\gamma,\eta,\zeta,p)}(r,r',w,v)\, dvdwdr'dr, \tag{4.78}$$

where $g_{(H,\alpha,\gamma,\eta,\zeta,p)}(r,r',w,v)$ is given by

$$\begin{aligned} &g_{(H,\alpha,\gamma,\eta,\zeta,p)}(r,r',w,v) = \\ &\frac{|r-r'|^{2p(\eta+\frac{1}{p})}|r'-w|^{-2p(\eta-\zeta)}\left(\left[|r'-w|^{-2p(\gamma+\zeta+\frac{1}{p})}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma-\zeta-\frac{1}{p})}\right)}{|r-r'|^{2p\eta}|r'-w|^{-2p(\eta-\zeta)}\left(\left[|r'-w|^{-2p(\gamma+\zeta)}|w-v|^{2p\alpha}\right]\wedge|w-v|^{2p(\alpha-\gamma-\zeta)}\right)|w-v|^2|r-r'|^2}. \end{aligned} \tag{4.79}$$

Thanks to cancellations, we can simplify the right hand side of (4.79) as

$$g_{(H,\alpha,\gamma,\eta,\zeta,p)}(r,r',w,v) = \frac{\left[|r'-w|^{-2p(\gamma+\zeta)-2}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma-\zeta)-2}}{\left(\left[|r'-w|^{-2p(\gamma+\zeta)}|w-v|^{2p\alpha}\right]\wedge|w-v|^{2p(\alpha-\gamma-\zeta)}\right)|w-v|^2}. \tag{4.80}$$

Plugging (4.80) into (4.78), we thus get

$$\begin{aligned} &\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta),1,2}\right] \\ &\lesssim \int_{(v,w,r',r)\in\Delta_4^\tau} \frac{\left[|r'-w|^{-2p(\gamma+\zeta)-2}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma-\zeta)-2}}{\left(\left[|r'-w|^{-2p(\gamma+\zeta)}|w-v|^{2p\alpha}\right]\wedge|w-v|^{2p(\alpha-\gamma-\zeta)}\right)|w-v|^2}\, dvdwdr'dr. \end{aligned} \tag{4.81}$$

Notice that the right hand side of (4.81) is now very similar to the right hand side of (4.66). Therefore with the same steps as for (4.66)-(4.75), we obtain that

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta),1,2}\right] \lesssim \int_{(v,w,r',r)\in\Delta_4^\tau} |w-v|^{2p(H-\alpha)-4} dv dw dr' dr < \infty. \tag{4.82}$$

The right hand side of (4.82) is finite as long as $p > \frac{3}{2}(H-\alpha)^{-1}$, or equivalently $\alpha < H - \frac{3}{2p}$.

Step 3: (4.59) *holds for any* $p \geq 1$. Invoking (4.60) we immediately have

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta)}\right] \lesssim \mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}\right] + \mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta),1,2}\right]. \tag{4.83}$$

Furthermore, combining (4.75) and (4.82) in the right hand side of (4.83). We end up with

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta)}\right] < \infty, \quad \text{for any} \quad p \geq 1. \tag{4.84}$$

This is the desired result (4.59). And it is easy to check that (4.84) yields

$$\|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta)} < \infty \quad \text{a.s.}. \tag{4.85}$$

This means that $\mathbf{z}^1$ is almost surely in the Volterra space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^m)$. □

With Proposition 3.4.9 in hand, we finish the study of $\mathbf{z}^1$ by proving the algebraic relation (2.13) for $\mathbf{z}^{1,\tau}$ in more detail.

Proposition 3.4.10. *The increment* $\mathbf{z}^{1,\tau,\mathrm{i}}_{ts}$ *as given in* (4.13) *satisfies relation* (2.13)*, that is*

$$\delta_m \mathbf{z}^{1,\tau,\mathrm{i}}_{ts} = 0, \ \textit{for all } (s,m,t,\tau) \in \Delta_4 \textit{ a.s.} \tag{4.86}$$

Proof. For fixed $(s,m,t,\tau) \in \Delta_4$, we have obtained in (4.61) that $\delta_m \mathbf{z}^{1,\tau,\mathrm{i}}_{ts} = 0$ almost surely. We will now prove that

$$(t,\tau) \in \Delta_2 \mapsto \mathbf{z}^{1,\tau}_t \in \mathbb{R}^m \text{ is a continuous function.} \tag{4.87}$$

By a standard argument, which consists in taking limits on rational points, this will achieve our claim (4.86).

The proof of (4.87) relies on Lemma 3.4.4. Indeed, according to (4.15) for (s, t, τ) in Δ_3, we have

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{1,\tau}\right)^2\right] \lesssim |t-s|^{2(H-\gamma)} . \tag{4.88}$$

In the same way thanks to (4.16) applied with $\zeta = \eta = H - \gamma - \epsilon$ with a small $\epsilon > 0$, we get

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{1,\tau\tau'}\right)^2\right] \lesssim |\tau - \tau'|^{H-\gamma-\epsilon} |t-s|^{\epsilon} . \tag{4.89}$$

Gathering (4.88) and (4.89), we end up with the following inequality, valid for $(s, t, \tau', \tau) \in \Delta_4$:

$$\|\mathbf{z}_{ts}^{1,\tau\tau'}\|_{L^2(\Omega)} \lesssim |\tau - \tau'|^{H-\gamma-\epsilon} + |t-s|^{H-\gamma} . \tag{4.90}$$

Moreover $\mathbf{z}_{ts}^{1,\tau\tau'}$ is a Gaussian random variable. Hence the upper bound (4.90) can be extended to arbitrary norms in $L^p(\Omega)$. Therefore a standard application of Kolmogorov's criterion yields the continuity property (4.88) for $\mathbf{z}^{1,\tau}$. This finishes our proof. $\square$

We now turn to the analysis $\mathbf{z}^{2,\tau}$. We start this study by verifying the algebraic relation (2.13) for $\mathbf{z}^{2,\tau}$.

Proposition 3.4.11. *The increment $\mathbf{z}_{ts}^{2,\tau}$ as given in (4.34)-(4.35) satisfies relation (2.13), that is*

$$\delta_m \mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}} = \mathbf{z}_{tm}^{1,\tau,\mathrm{j}} * \mathbf{z}_{ms}^{1,\cdot,\mathrm{i}}, \quad \textit{for all } (s,m,t,\tau) \in \Delta_4 \quad \textit{a.s.} \tag{4.91}$$

Proof. In order to show (4.91), we first prove that (4.91) holds for fixed $(s, m, t) \in \Delta_3^\tau$. According to Definition 3.4.6, we will separate the proof into two cases $\mathrm{i} \neq \mathrm{j}$ and $\mathrm{i} = \mathrm{j}$.

Step 1: (4.91) holds for fixed $(s, m, t) \in \Delta_3^\tau$ when $\mathrm{i} \neq \mathrm{j}$. In this step, let us handle the case $\mathrm{i} \neq \mathrm{j}$. For any $(s, m, t) \in \Delta_3^\tau$, gathering (4.34) and (2.12), we have

$$\delta_m \mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}} = B^{\mathrm{j}}\left(u_{ts}^{\tau,\mathrm{i}}\right) - B^{\mathrm{j}}\left(u_{tm}^{\tau,\mathrm{i}}\right) - B^{\mathrm{j}}\left(u_{ms}^{\tau,\mathrm{i}}\right) , \tag{4.92}$$

where we recall that the process u is defined by (4.33). In order to calculate the right hand side of (4.92), it is thus sufficient to compute $\delta_m u_{ts}^{\tau,\mathrm{i}} = u_{ts}^{\tau,\mathrm{i}} - u_{tm}^{\tau,\mathrm{i}} - u_{ms}^{\tau,\mathrm{i}}$. To this aim, according to the definition (4.33) of $u^{\tau,\mathrm{i}}$, we obtain

$$\begin{aligned}\delta_m u_{ts}^{\tau,\mathrm{i}}(r) &= u_{ts}^{\tau,\mathrm{i}}(r) - u_{tm}^{\tau,\mathrm{i}}(r) - u_{ms}^{\tau,\mathrm{i}}(r) \\ &= (\tau - r)^{-\gamma}\, \mathbf{z}_{rs}^{1,r,\mathrm{i}} \mathbb{1}_{[s,t]}(r) - (\tau - r)^{-\gamma}\, \mathbf{z}_{rm}^{1,r,\mathrm{i}} \mathbb{1}_{[m,t]}(r) - (\tau - r)^{-\gamma}\, \mathbf{z}_{rs}^{1,r,\mathrm{i}} \mathbb{1}_{[s,m]}(r).\end{aligned}$$

Resorting to the definition (4.13) of $\mathbf{z}_{rs}^{1,r,\mathrm{i}}$, we thus get

$$\delta_m u_{ts}^{\tau,\mathrm{i}}(r) = (\tau - r)^{-\gamma} \left(B^{\mathrm{i}}(h_{rs}^r) \mathbb{1}_{[s,t]}(r) - B^{\mathrm{i}}(h_{rm}^r) \mathbb{1}_{[m,t]}(r) - B^{\mathrm{i}}(h_{rs}^r) \mathbb{1}_{[s,m]}(r) \right), \tag{4.93}$$

where the expression for h is given in Definition 3.4.2. The right hand side of (4.93) can be simplified by elementary calculus, we thus let the patient reader check that we have

$$\delta_m u_{ts}^{\tau,\mathrm{i}}(r) = (\tau - r)^{-\gamma}\, B^{\mathrm{i}}\left(h_{ms}^r\right) \mathbb{1}_{[m,t]}(r). \tag{4.94}$$

Furthermore, according to the definition of h in Definition 3.4.2, we have that $(\tau - r)^{-\gamma}\, \mathbb{1}_{[m,t]}(r) = h_{tm}^{\tau}(r)$. Hence (4.94) can be recast as

$$\delta_m u_{ts}^{\tau,\mathrm{i}}(r) = h_{tm}^{\tau}(r) B^{\mathrm{i}}\left(h_{ms}^r\right). \tag{4.95}$$

Plugging (4.95) into (4.92), we thus have

$$\delta_m \mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}} = B^{\mathrm{j}}\left(h_{tm}^{\tau} B^{\mathrm{i}}\left(h_{ms}^r\right)\right). \tag{4.96}$$

Resorting to the definition (4.5) of $B^{\mathrm{j}}(h)$, the right hand side of (4.96) can be written as

$$\delta_m \mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}} = \lim_{|\mathcal{P}|\to 0} \sum_{[r,v]} B_{vr}^{\mathrm{j}}\, h_{tm}^{\tau}(r)\, B^{\mathrm{i}}(h_{ms}^r), \tag{4.97}$$

where we recall that $\mathcal{P}$ is a generic partition of $[m,t]$ whose mesh $|\mathcal{P}|$ is converging to 0, and where the limit holds in $L^2(\Omega)$. We now consider a subsequence of partitions in order to get an almost sure convergence in (4.97). According to the definition (2.9) of convolution product we end up with

$$\delta_m \mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{j}} = \mathbf{z}_{tm}^{1,\tau,\mathrm{j}} * \mathbf{z}_{ms}^{1,\cdot,\mathrm{i}},$$

where we have used the definition (4.13) of $\mathbf{z}^{1,\tau}$.

Step 2: (4.91) *holds for fixed* $(s,m,t) \in \Delta_3^\tau$ *when* $\mathrm{i} = \mathrm{j}$. In this step, we will deal with the case $\mathrm{i} = \mathrm{j}$ for the the second level of the Volterra rough path. For any $(s,m,t) \in \Delta_3^\tau$, according to the definition (4.35) of $\mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{i}}$, we obtain

$$\delta_m \mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{i}} = \delta_m \left(\int_s^t u_{ts}^{\tau,\mathrm{i}}(r) dB_r^{\mathrm{i}} \right), \tag{4.98}$$

where the integral above is understood in the Stratonovich sense. According to (4.10), we have

$$\int_s^t u_{ts}^{\tau,\mathrm{i}}(r) dB_r^{\mathrm{i}} = \lim_{|\mathcal{P}|\to 0} S_{ts}^{\mathrm{i},\mathcal{P}}, \quad \text{where} \quad S_{ts}^{\mathrm{i},\mathcal{P}} = \int_s^t u_{ts}^{\mathrm{i}}(r) B_r^{\mathrm{i},\mathcal{P}} dr.$$

Now for a fixed $\mathcal{P}$, elementary algebraic manipulations show that

$$\delta_m S_{ts}^{\mathrm{i},\mathcal{P}} = \mathbf{z}_{tm}^{1,\tau,\mathrm{i},\mathcal{P}} * \mathbf{z}_{ms}^{1,\cdot,\mathrm{i}}, \tag{4.99}$$

where $\mathbf{z}_{tm}^{1,\tau,\mathrm{j},\mathcal{P}}$ is defined by (4.10). Taking limits on both sides of (4.99) as $\mathcal{P} \to 0$, we get

$$\delta_m \mathbf{z}_{ts}^{2,\tau,\mathrm{i},\mathrm{i}} = B^{\mathrm{i}} \left(h_{tm}^\tau B^{\mathrm{i}}(h_{ms}^r) \right) = \mathbf{z}_{tm}^{1,\tau,\mathrm{i}} * \mathbf{z}_{ms}^{1,\cdot,\mathrm{i}},$$

which proves (4.91) for $\mathrm{i} = \mathrm{j}$.

Step 3: (4.91) *holds for all* $(s,m,t) \in \Delta_3^\tau$. The proof of this fact, based on Kolmogorov's criterion for continuity of stochastic processes, is very similar to the considerations in Proposition 3.4.10. For sake of conciseness, it is omitted here. The proof of (4.91) is now complete. □

With Proposition 3.4.11 in hand, we are now ready to check the regularity of $\delta\mathbf{z}^{2,\tau}$.

Proposition 3.4.12. *Consider the second level* $\mathbf{z}^{2,\tau}$ *of the Volterra rough path, as defined in* (4.34)-(4.35). *Recall that* $\delta\mathbf{z}^{2,\tau}$ *is a path defined on* Δ_4, *and we refer to Definition 3.2.9* *for the definition of* $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_4;\mathbb{R}^m)$*. We assume that* $H > 1/2$*,* $\gamma < 2H-1$*,* $\alpha \in (\gamma, H)$*,* $\zeta \in [0, 2(\alpha-\gamma))$ *and* $\eta \in [\zeta, 1]$*. Then almost surely we have*

$$\delta \mathbf{z}^{2,\tau} \in \mathcal{V}^{(2\alpha-\gamma,\gamma,\eta,\zeta)}(\Delta_4;\mathbb{R}^m). \tag{4.100}$$

Moreover, for all $p \geq 1$ *we have*

$$\mathbb{E}\left[\|\delta \mathbf{z}^{2,\tau}\|^{2p}_{(2\alpha-\gamma,\gamma,\eta,\zeta)}\right] < \infty, \tag{4.101}$$

where the norm above is understood as in (2.14)*.*

Proof. In this proof, we will show that (4.101) holds for any $p \geq 1$, and it is easy to check that (4.100) is a direct result of (4.101). Toward this aim, according to the definition (2.14), it is necessary to prove that $\mathbb{E}[\|\delta \mathbf{z}^{2,\tau}\|^{2p}_{(2\alpha-\gamma,\gamma),1}]$ and $\mathbb{E}[\|\delta \mathbf{z}^{2,\tau}\|^{2p}_{(2\alpha-\gamma,\gamma),1,2}]$ are finite. Thanks to (4.91), for any $(s,u,t,\tau) \in \Delta_4$ we have

$$\delta_u \mathbf{z}^{2,\tau}_{ts} = \mathbf{z}^{1,\tau}_{tu} * \mathbf{z}^{1,\cdot}_{us}. \tag{4.102}$$

Hence resorting to (2.10), we get

$$\left|\delta_u \mathbf{z}^{2,\tau}_{ts}\right| = \left|\mathbf{z}^{1,\tau}_{tu} * \mathbf{z}^{1,\cdot}_{us}\right| \lesssim \|\mathbf{z}^1\|_{(\alpha,\gamma),1} \|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}\, \psi^1_{2\alpha-\gamma,\gamma}(\tau,t,s). \tag{4.103}$$

Dividing by $\psi^1_{2\alpha-\gamma,\gamma}(\tau,t,s)$ on both sides of (4.103), and then taking supremum over $(s,u,t,\tau) \in \Delta_4$, we obtain

$$\|\delta \mathbf{z}^2\|_{(2\alpha-\gamma,\gamma),1} \leq \|\mathbf{z}^1\|_{(\alpha,\gamma),1} \|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}\ , \tag{4.104}$$

where we have used the definition (2.15) of 1−norm for the Volterra space $\mathcal{V}^{(2\alpha-\gamma,\gamma,\eta,\zeta)}(\Delta_4;\mathbb{R}^m)$. Similarly, resorting to (2.11) and (2.16), for any $(s,u,t,\tau',\tau) \in \Delta_5$, $\zeta \in [0, 2\alpha-2\gamma)$ and $\eta \in [\zeta, 1]$, we let the patient reader check that we have

$$\|\delta \mathbf{z}^2\|_{(2\alpha-\gamma,\gamma,\eta,\zeta),1,2} \leq \|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2} \|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}. \tag{4.105}$$

Combining (4.104) and (4.105), and recalling the definition (2.14) again, we thus obtain

$$\|\delta \mathbf{z}^2\|_{(2\alpha-\gamma,\gamma,\eta,\zeta)} = \|\delta \mathbf{z}^2\|_{(2\alpha-\gamma,\gamma),1} + \|\delta \mathbf{z}^2\|_{(2\alpha-\gamma,\gamma,\eta,\zeta),1,2} \lesssim \|\mathbf{z}^1\|^2_{(\alpha,\gamma,\eta,\zeta)}. \tag{4.106}$$

Taking $2p$ moments on both sides of (4.106), we thus get

$$\mathbb{E}\left[\|\delta \mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma,\eta,\zeta)}\right] \lesssim \mathbb{E}\left[\|\mathbf{z}^1\|^{4p}_{(\alpha,\gamma,\eta,\zeta)}\right]. \tag{4.107}$$

According to (4.59), the right hand side of (4.107) is finite. This means that we have

$$\mathbb{E}\left[\|\delta \mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma,\eta,\zeta)}\right] < \infty, \qquad \text{for any} \qquad p \geq 1. \tag{4.108}$$

This is the desired result. $\square$

Finally, let us close this section by giving the proof of the regularity result for $\mathbf{z}^{2,\tau}$.

Proposition 3.4.13. *Under the same assumption as for Proposition 3.4.12, the second level of the Volterra rough path $\mathbf{z}^{2,\tau}$ introduced in (4.34)-(4.35) is almost surely an element of $\mathcal{V}^{(2\alpha-\gamma,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^m)$ for any $\alpha \in (\gamma, H)$, $\zeta \in [0, 2(\alpha-\gamma))$ and $\eta \in [\zeta, 1]$. Furthermore, for all $p \geq 1$ we have that*

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma,\eta,\zeta)}\right] < \infty. \tag{4.109}$$

Proof. Our strategy to prove this Proposition is the same as for the proof of Proposition 3.4.9, that is we will appeal to the Volterra GRR Lemma 3.3.7 to show that $\mathbb{E}[\|\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma),1}]$ and $\mathbb{E}[\|\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma,\eta,\zeta),1,2}]$ are both finite. Let us first show that $\mathbb{E}[\|\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma),1}]$ is finite. Consider a fixed Volterra exponent $\alpha \in (\gamma, H)$ and a parameter $p \geq 1$ to be determined later, relation (3.33) reads

$$\|\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma),1} \lesssim \left(U^T_{(2\alpha-\gamma,\gamma),1,p}(\mathbf{z}^2)\right)^{2p} + \|\delta \mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma),1}. \tag{4.110}$$

Taking expectation on both sides of (4.110), we obtain

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma),1}\right] \lesssim \mathbb{E}\left[\left(U^T_{(2\alpha-\gamma,\gamma),1,p}(\mathbf{z}^2)\right)^{2p}\right] + \mathbb{E}\left[\|\delta \mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma),1}\right]. \tag{4.111}$$

Recalling (4.101), the second term of the right hand side of (4.111) is finite. In order to upper bound the left hand side of (4.111), it is thus sufficient to estimate the first term $\mathbb{E}[(U^T_{(2\alpha-\gamma,\gamma),1,p}(\mathbf{z}^2))^{2p}]$. Toward this aim, we set

$$A = \mathbb{E}\left[\left(U^T_{(2\alpha-\gamma,\gamma),1,p}(\mathbf{z}^2)\right)^{2p}\right].$$

Recalling the definition (3.2) of $U^T_{(2\alpha-\gamma,\gamma),1,p}$, we have

$$A = \mathbb{E}\left[\int_{(v,w)\in\Delta_2^\tau} \frac{|z^{2,\tau}_{wv}|^{2p}}{|\psi^1_{2\alpha-\gamma,\gamma}(\tau,w,v)|^{2p}|w-v|^2}\,dv\,dw\right]. \tag{4.112}$$

Observe that $\mathbf{z}^{2,\tau}_{wv}$ is an element of the second chaos of the fBm B, on which all L^p norms are equivalent. Hence invoking Fubini's theorem, we get

$$A \lesssim \int_{(v,w)\in\Delta_2^\tau} \frac{\mathbb{E}^p\left[|z^{2,\tau}_{wv}|^2\right]}{|\psi^1_{2\alpha-\gamma,\gamma}(\tau,w,v)|^{2p}|w-v|^2}\,dv\,dw. \tag{4.113}$$

We now apply (4.37) to the right hand side of (4.113), we obtain

$$A \lesssim \int_{(v,w)\in\Delta_2^\tau} \frac{|\psi^1_{2H-\gamma,\gamma}(\tau,w,v)|^{2p}}{|\psi^1_{2\alpha-\gamma,\gamma}(\tau,w,v)|^{2p}|w-v|^2}\,dvdw. \tag{4.114}$$

Notice that relation (4.114) is almost the same as (4.65). Hence we can carry out the same procedure going from (4.65) to (4.75) in the proof of Proposition 3.4.9. We end up with

$$A \lesssim \int_{(v,w)\in\Delta_2^\tau} |w-v|^{4p(H-\alpha)-2}\,dvdw. \tag{4.115}$$

Eventually plugging (4.115) into (4.111), we get

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma),1}\right] \lesssim \int_{(v,w)\in\Delta_2^\tau} |w-v|^{4p(H-\alpha)-2}\,dvdw + \mathbb{E}\left[\|\delta\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma),1}\right]. \tag{4.116}$$

Recalling (4.101) again, the right hand side above is easily checked to be finite as long as $p > \frac{1}{4}(H-\alpha)^{-1}$. Considering such a p (which is allowed since $z^{2,\tau}_{wv}$ admits moments of all orders), we thus obtain

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma),1}\right] < \infty. \tag{4.117}$$

Next we will show that $\mathbb{E}[\|\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma,\eta,\zeta),1,2}]$ is finite. Similarly to the steps going from (4.110) to (4.116), we resort to (3.34) in order to get

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma,\eta,\zeta),1,2}\right] \lesssim \int_{(v,w,r',r)\in\Delta_4^\tau} |w-v|^{4p(H-\alpha)-4}\, dvdwdr'dr \\ + \mathbb{E}\left[\|\delta\mathbf{z}^{2,\tau}\|^{2p}_{(2\alpha-\gamma,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p}),1,2}\right]. \tag{4.118}$$

Owing to (4.101), the right hand side of (4.118) is finite as long as $p > \frac{3}{4(H-\alpha)}$. Eventually combining (4.117) and (4.118), and recalling our definition (2.3) of $(\alpha,\gamma,\eta,\zeta)$-norm, we trivially get that

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma,\eta,\zeta)}\right] < \infty. \tag{4.119}$$

This completes the proof. □

3.5 Volterra rough path driven by Brownian motion

3.5.1 Analysis on wiener space

The Malliavin calculus preliminaries for a Brownian motion are similar to what we wrote in Section 3.4.1 for a fBm. Keeping most of our previous notation, let us just highlight the main differences between the two situations.

(i) Our notation for the Brownian driving process is $W = (W^1, \ldots, W^m)$. The covariance function for each independent component is

$$R(s,t) = s \wedge t.$$

(ii) The space $\mathcal{H}$ is $L^2([0,T])$, with inner product

$$\langle f, g \rangle_{\mathcal{H}} = \int_0^T f_u g_u du.$$

(iii) Malliavin derivatives, Sobolev spaces and Skorohod integrals are defined similarly to (4.6)-(4.7)-(4.8). However, for an adapted process u in $\mathbb{D}^{1,2}(\mathcal{H})$, the Skorohod integral $\delta^{\diamond,\mathrm{j}}(u)$ coincides with Itô's integral.

(iv) The Stratonovich integral can be defined as in (4.10). If Du enjoys suitable continuity properties and is an adapted process, the conection formula (4.11) reads

$$\int_0^T u_t dB_t^{\mathrm{j}} = \int_0^T u_t \delta^{\diamond} B_t^{\mathrm{j}} + \int_0^T D_t^{\mathrm{j}} u_t dt.$$

Notice that in a rough path setting the stochastic integrals with respect to W are naturally understood in the Stratonovich sense.

3.5.2 Definition of the Volterra rough path

In this section we will deal with the case of a driving noise given by a m-dimensional Brownian motion W. This case is rougher than in Section 3.4, although arguably already addressed in the classical reference. Nevertheless, it should be noticed that a rough path point of view on equation (1.1) driven by a Brownian motion is still useful, due to convenient continuity properties with respect to the Volterra signature.

Definition 3.5.1. *Consider a Brownian motion $W : [0,T] \to \mathbb{R}^m$ and a function h of the form $h_{ts}^{\tau}(r) = (\tau - r)^{-\gamma} \mathbb{1}_{[s,t]}(r)$ with $\gamma < \frac{1}{2}$. Then for $(s,t,\tau) \in \Delta_3$ we define the increment $\mathbf{z}_{ts}^{1,\tau,\mathrm{i}} = \int_s^t (\tau - r)^{-\gamma} \, dW_r^{\mathrm{i}}$ as a Wiener integral of the form*

$$\mathbf{z}_{ts}^{1,\tau,\mathrm{i}} := W^{\mathrm{i}}(h_{ts}^{\tau}). \tag{5.1}$$

With Definition 3.5.1 in hand, let us find a bound for second moment of $\mathbf{z}^1$.

Lemma 3.5.2. *Consider the Volterra rough path* $\mathbf{z}^1$ *as given in* (4.13)*. Then for* $(s,t,\tau) \in \Delta_3$*, we have*

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,\mathrm{i}})^2] \lesssim \left|\psi^1_{(\frac{1}{2},\gamma)}(\tau,t,s)\right|^2. \tag{5.2}$$

While for $(s,t,\tau',\tau) \in \Delta_4$*,* $\zeta \in [0, \frac{1}{2} - \gamma)$*, and* $\eta \in [\zeta, 1]$*, we get*

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',\mathrm{i}})^2] \lesssim \left|\psi^{1,2}_{(\frac{1}{2},\gamma,\eta,\zeta)}(\tau,,\tau',t,s)\right|^2, \tag{5.3}$$

where ψ^1 *and* $\psi^{1,2}$ *are given in Notation 3.2.2.*

Proof. In this proof, we will show that (5.2) is held for any $(s,t,\tau) \in \Delta_3$. (5.3) can be proved in a similar way. Toward to this aim, according to (3.5.1), we have

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,\mathrm{i}})^2] = \mathbb{E}[(\int_s^t (\tau - r)^{-\gamma}\, dW_r^{\mathrm{i}})^2]. \tag{5.4}$$

Recalling that B is a Brownian motion, resorting to isometry, (5.4) reads

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,\mathrm{i}})^2] = \int_s^t (\tau - r)^{-2\gamma}\, dr. \tag{5.5}$$

Thanks to some elementary calculus, we obtain

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,\mathrm{i}})^2] \lesssim \left[|\tau - t|^{-2\gamma}|t-s|\right] \wedge |t-s|^{1-2\gamma} = \left|\psi^1_{\frac{1}{2},\gamma}(\tau,t,s)\right|^2. \tag{5.6}$$

Notice that we have used the definition (2.1) of $\psi^1_{\frac{1}{2},\gamma}(\tau,t,s)$. This is the desired result (5.2). $\square$

Next we turn our attention to construct the second level Volterra rough path over a Brownian motion.

Definition 3.5.3. *We consider a Brownian motion $B : [0,T] \to \mathbb{R}^m$, and the first level of the Volterra rough path $\mathbf{z}^{1,\tau}$ defined by (5.1). As in Definition 3.5.1, we assume that $\gamma < \frac{1}{2}$. Then for $(s,r,t,\tau) \in \Delta_4$, we set*

$$u_{ts}^{\tau,\mathrm{i}}(r) = (\tau - r)^{-\gamma} \mathbf{z}_{rs}^{1,r,\mathrm{i}} \mathbb{1}_{[s,t]}(r). \tag{5.7}$$

With this notation in hand, we define the increment $\mathbf{z}_{ts}^{2,\tau}$ as a Itô-integral of the form

$$\mathbf{z}_{ts}^{2,\tau,\mathrm{i,j}} = B^{\mathrm{j}}(u_{ts}^{\tau,\mathrm{i}}), \qquad \textit{for any} \quad \mathrm{i,j} \in \{1,2,\cdots,m\}^2. \tag{5.8}$$

Similarly to what we did for $\mathbf{z}^1$, we will now estimate the second moment of $\mathbf{z}^2$.

Proposition 3.5.4. *Consider the second level $\mathbf{z}^{2,\tau}$ of the Volterra rough path, as defined in (5.8). Recall that γ satisfy $\gamma < \frac{1}{2}$. Then for $(s,t,\tau) \in \Delta_3$, we have*

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau}\right)^2\right] \lesssim \left|\psi^1_{(1-\gamma,\gamma)}(\tau,t,s)\right|^2. \tag{5.9}$$

As far as the $(1,2)$-type increments are considered, for $(s,t,\tau',\tau) \in \Delta_4$, $\zeta \in [0, 2(\frac{1}{2} - \gamma))$, and $\eta \in [\zeta, 1]$, we get

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau\tau'}\right)^2\right] \lesssim \left|\psi^{1,2}_{(1-\gamma,\gamma,\eta,\zeta)}(\tau,\tau',t,s)\right|^2, \tag{5.10}$$

where ψ^1 and $\psi^{1,2}$ are given in Notation 3.2.2.

Proof. This proof is very similar to the proof of (3.5.2). We will prove (5.9), and let the patient read show that (5.10) holds for $(s,t,\tau',\tau) \in \Delta_4$.According to (5.8) , we have

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau}\right)^2\right] = \mathbb{E}\left[\left(W^{\mathrm{j}}(u_{ts}^{\tau,\mathrm{i}})\right)^2\right]. \tag{5.11}$$

Thanks to isometry for Brownian motion, we obtain

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau}\right)^2\right] = \int_s^t \mathbb{E}\left[\left(u_{ts}^{\tau,\mathrm{i}}\right)^2\right] dr. \tag{5.12}$$

Recalling the definition (5.7) of u, we get

$$\mathbb{E}\left[\left(u_{ts}^{\tau,\mathrm{i}}\right)^2\right]=\mathbb{E}\left[(\tau-r)^{-2\gamma}\left(\mathbf{z}_{rs}^{1,r,\mathrm{i}}\right)^2\mathbb{1}_{[s,t]}(r)\right]=(\tau-r)^{-2\gamma}\mathbb{E}\left[\left(\mathbf{z}_{rs}^{1,r,\mathrm{i}}\right)^2\right]\mathbb{1}_{[s,t]}(r). \tag{5.13}$$

Similarly to proof of Proposition 3.5.2, we let patient reader check that we have

$$\mathbb{E}\left[\left(\mathbf{z}_{rs}^{1,r,\mathrm{i}}\right)^2\right]\lesssim(r-s)^{1-2\gamma}. \tag{5.14}$$

Plugging (5.14) into (5.13), we find a bound for $\mathbb{E}\left[\left(u_{ts}^{\tau,\mathrm{i}}\right)^2\right]$ as follows

$$\mathbb{E}\left[\left(u_{ts}^{\tau,\mathrm{i}}\right)^2\right]\lesssim(\tau-r)^{-2\gamma}(r-s)^{1-2\gamma}\mathbb{1}_{[s,t]}(r). \tag{5.15}$$

Eventually plugging (5.15) into (5.12), we thus get

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau}\right)^2\right]\lesssim\int_s^t(\tau-r)^{-2\gamma}(r-s)^{1-2\gamma}dr. \tag{5.16}$$

In order to get a bound for right hand side of (5.16), the process is very similar to (4.23)-(4.25) as for Proposition 3.4.4. We finally obtain

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau}\right)^2\right]\lesssim\left[|\tau-t|^{-2\gamma}|t-s|^{2-2\gamma}\right]\wedge|t-s|^{2-4\gamma}=\left|\psi^1_{(1-\gamma,\gamma)}(\tau,t,s)\right|^2. \tag{5.17}$$

We have appealed the definition (2.1) of ψ^1 at the second identity of above equation. We completes the proof. $\square$

We have constructed a Volterra rough path family $\{\mathbf{z}^{1,\tau},\mathbf{z}^{2,\tau}\}$ over a Brownian motion and we have also upper bounded their moment. We will close this paper with verifying that $\{\mathbf{z}^{1,\tau},\mathbf{z}^{2,\tau}\}$ satisfies Definition 3.2.8. Let us first state that $\mathbf{z}^{1,\tau}$ satisfies all properties that mentioned in Definition 3.2.8.

Proposition 3.5.5. *Consider the increment $\mathbf{z}^{1,\tau}$ introduced in Definition 3.5.1. Then for any $\alpha\in(\gamma,\frac{1}{2})$, $\zeta\in[0,\alpha-\gamma)$ and $\eta\in[\zeta,1]$, we have* *(i)* $\mathbf{z}^{1,\tau}$ *is almost surely in the Volterra space* $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^m)$ *where* $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^m)$ *is introduced in Definition 3.2.3.*

(ii) For all $p \geq 1$ *we have that*

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta)}\right] < \infty. \tag{5.18}$$

(iii)

$$\delta_m \mathbf{z}^{1,\tau,\mathrm{i}}_{ts} = 0, \text{ for all } (s,m,t,\tau) \in \Delta_4 \text{ a.s.} \tag{5.19}$$

The proof is very similar to the proof as for Proposition 3.4.9-3.4.10, we let the patient reader check them. Similarly, we have following Proposition for $\mathbf{z}^2$.

Proposition 3.5.6. *Consider the second level* $\mathbf{z}^{2,\tau}$ *of the Volterra rough path as defined in* (5.8)*. Then the following properties hold for any* $\gamma < \frac{1}{2}$, $\alpha \in (\gamma, \frac{1}{2})$, $\zeta \in [0, 2(\alpha-\gamma))$ *and* $\eta \in [\zeta, 1]$.

(i) Recalling the definition (2.12) *of* δ*, then* $\delta_m \mathbf{z}^{2,\tau}_{ts}$ *satisfies relation* (2.13)*, that is*

$$\delta_m \mathbf{z}^{2,\tau,\mathrm{i},\mathrm{j}}_{ts} = \mathbf{z}^{1,\tau,\mathrm{j}}_{tm} * \mathbf{z}^{1,\cdot,\mathrm{i}}_{ms}, \quad \text{for all } (s,m,t,\tau) \in \Delta_4 \quad \text{a.s..} \tag{5.20}$$

With (5.20) *in hand, we get*

$$\delta \mathbf{z}^{2,\tau} \in \mathcal{V}^{(2\alpha-\gamma,\gamma,\eta,\zeta)}(\Delta_4;\mathbb{R}^m) \qquad \text{a.s..} \tag{5.21}$$

In addition, for all $p \geq 1$ *we have that*

$$\mathbb{E}\left[\|\delta \mathbf{z}^{2,\tau}\|^{2p}_{(2\alpha-\gamma,\gamma,\eta,\zeta)}\right] < \infty. \tag{5.22}$$

(ii) $\mathbf{z}^{2,\tau}$ *is almost surely an element of* $\mathcal{V}^{(2\alpha-\gamma,\gamma,\eta,\zeta)}$*. Furthermore, for all* $p \geq 1$ *we have that*

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\alpha-\gamma,\gamma,\eta,\zeta)}\right] < \infty. \tag{5.23}$$

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