APPLIED TOPOLOGY AND ALGORITHMIC SEMI-ALGEBRAIC GEOMETRY

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ABSTRACT

Applied topology is a rapidly growing discipline aiming at using ideas coming from algebraic topology to solve problems in the real world, including analyzing point cloud data, shape analysis, etc. Semi-algebraic geometry deals with studying properties of semi-algebraic sets that are subsets of $\mathbb{R}^n$ and defined in terms of polynomial inequalities. Semi-algebraic sets are ubiquitous in applications in areas such as modeling, motion planning, etc. Developing efficient algorithms for computing topological invariants of semi-algebraic sets is a rich and well-developed field. However, applied topology has thrown up new invariants—such as persistent homology and barcodes—which give us new ways of looking at the topology of semi-algebraic sets. In this thesis, we investigate the interplay between these two areas. We aim to develop new efficient algorithms for computing topological invariants of semi-algebraic sets, such as persistent homology, and to develop new mathematical tools to make such algorithms possible.

1. INTRODUCTION

In this thesis we address two main problems:

- First, we explore a *weak* version of one of the long-lasting problems in algorithmic semi-algebraic geometry, namely designing an algorithm with a singly exponential complexity for computing semi-algebraic triangulations of a given semi-algebraic set. In particular, we present a singly exponential complexity simplicial replacement that is ℓ-equivalent (for any fixed $\ell \geqslant 0$) to a given semi-algebraic set. As a result, we obtain a reduction for the problem of computing the first ℓ homotopy groups of a semi-algebraic set to the combinatorial problem of computing the first ℓ homotopy groups of a finite simplicial complex.

- Second, we address a more recent notion from Computational Topology known as persistent homology. We present the first known singly exponential complexity algorithm to compute the *barcodes* up to dimension ℓ of the filtration of a given semi-algebraic set by the sub-level sets of a given polynomial. Our algorithm is a generalization of the corresponding results for computing the Betti numbers up to dimension ℓ of semi-algebraic sets where no filtration is present.

1.1 Real Algebraic Geometry

Algebraic Geometry is the study of geometric spaces that arise from solution sets of systems of polynomials over algebraically closed fields. A simple example is the zero set of a univariate polynomial of degree d with complex coefficients that always include d roots. However, if we consider a univariate polynomial of degree d over the real numbers, different situations concerning the zero set could arise. It is significant to notice that restriction to the real closed fields could give rise to problems that do not have any counterpart in Algebraic Geometry over the complex numbers. In practice, we are often interested in solution sets over the real numbers; examples of that include the algebra of constrained motion in molecules and robotic planning. *Real Algebraic Geometry* is a branch of Algebraic Geometry that focuses on the solution sets of a finite system of polynomial inequalities over the real numbers, or broadly a real closed field. In particular, Real Algebraic Geometry addresses algorithmic and quantitative problems concerning *semi-algebraic sets*. Let R be a real closed field, a basic semi-algebraic subset of R^k is defined as

$$S = \{\mathbf{x} \in \mathrm{R}^k \mid P_i(\mathbf{x}) > 0 \text{ or } P_i(\mathbf{x}) = 0,\ P_i \in \mathrm{R}[X_1, \dots X_k], 1 \leqslant i \leqslant n\},$$

where $\mathbf{x} = (x_1, \dots, x_k)$. A semi-algebraic set is a finite union of basic semi-algebraic sets.

Due to the fact that first-order theory of real closed fields admits quantifier-elimination, we often assume S is presented as input to our algorithms in terms of a *quantifier-free* formula $\phi(X_1, \dots, X_k)$, that is a formula with atoms of the form $P = 0, P > 0, P < 0$, $P \in \mathrm{D}[X_1, \dots, X_k]$, where D is an ordered domain contained in R. This assumption provides us with a framework to effectively analyze the algorithms. In particular, by *complexity of an algorithm* we mean the number of arithmetic operations and comparisons in the domain D. Therefore, the concept of complexity depends on the choice of the domain D. If we let $\mathrm{D} = \mathbb{R}$, then we can pick Blum-Shub-Smale machines to describe computations over the real numbers [1]. If we let $\mathrm{D} = \mathbb{Z}$, we can use Turing machines to explain the complexity of our algorithms by obtaining the bit-sizes of the coefficients of the input polynomials [2].

1.2 Computational Topology

Computational Topology provides mathematical tools from Algebraic Topology to determine the structure of the topological spaces. Topological properties encode connectivity and measure the complexity of the space. *Homology groups* are important topological invariants. Informally, in lower dimensions they are usually referred to as the connected components, the holes, and the void spaces, respectively in dimensions 0, 1, and 2; in the higher dimensions, they are known as p-dimensional holes. The rank of the p-th homology group, $\mathrm{H}_p(S)$, of a semi-algebraic set S defined over R, is known as the *p-th Betti number*, denoted by $b_p(S)$.

More recently, the emerging field of *Topological Data Analysis (TDA)* has provided new notions associated with a sequence of topological spaces, such as persistent homology and barcode. TDA has shown promising results in data analysis by effectively filtering out noisy data and capturing the underlying structure of the space (e.g., point cloud data) [3]. Extracted topological properties can be directly analyzed or used as features for downstream prediction tasks to classify and distinguish the global differences between spaces.

Persistent homology groups generalize ordinary homology groups of a space X where no filtration is present. Given a filtration $\mathcal{F} = (X_t)_{t \in T}$ of the topological space X, such that for any $s \leqslant t \Rightarrow X_s \subset X_t$, the *p-th persistent homology groups* of X is defined by $\mathrm{H}_p^{s,t}(\mathcal{F}) = \mathrm{Im}(i_p^{s,t})$, where $i_p^{s,t} : \mathrm{H}_p(X_s) \longrightarrow \mathrm{H}_p(X_t)$ is the homomorphism induced by the inclusion map $X_s \hookrightarrow X_t$. Persistent homology determines the lifetime of the homology classes in the filtration. Homology classes that persist for a longer period, through the filtration, best represent the properties of the underlying structure, and the rest are considered as noise. This characteristic is the basic principle of the persistent homology frameworks. The output of a persistent homology computation is usually expressed in the form of *barcodes* [4].

1.3 Algorithmic semi-algebraic geometry

One of the main problems in algorithmic semi-algebraic geometry is to design efficient (singly exponential complexity) algorithms to compute the topological invariants of semi-algebraic sets [5]. Given a closed and bounded semi-algebraic set $S \subset \mathrm{R}^k$ described by a quantifier-free formula involving s polynomials of degrees bounded by d, there exists an algorithm that generates a semi-algebraic triangulation of S. As a result, there exists a finite simplicial complex which is semi-algebraically homeomorphic to S, and hence has the same topological properties as S. However, the complexity of the algorithm is doubly exponential in k, i.e., $(sd)^{2^{O(k)}}$ [2, Chapter 5]. Note that if we have the semi-algebraic triangulation of S, then we can use linear algebra to compute Betti numbers in a polynomial time in the number of simplices.

Despite the tremendous effort to improve this upper bound, designing a singly exponential complexity algorithm to compute a semi-algebraic triangulation of S remains an open problem.

In this thesis we first improve the current state of the art algorithm for the weak version of the above problem, then we extend our result and explore a different direction, namely persistent homology. Below, we briefly present the main contributions and leave the precise statements of the results for their corresponding chapters.

1.3.1 Simplicial replacement of semi-algebraic sets

A semi-algebraic triangulation of the space encodes more information about the topology of the space compared to that of homology groups. In particular, a semi-algebraic triangulation determines the homeomorphism invariants, which are not easy to compute. More precisely, determining whether two simplicial complexes are homeomorphic is an undecidable problem [6]. Therefore, we relax the problem and consider a weaker equivalence—namely homotopies—instead of homeomorphism.

We will call a *simplicial replacement* of the semi-algebraic set S to be a simplicial complex K whose geometric realization $|K|$ is *homotopy equivalent* to S, and more importantly its complexity is bounded by $(sd)^{k^{O(1)}}$. Based on this notion, we will prove the following theorem, whose precise statements appear in Chapter 2.

In the statements below $\ell \in \mathbb{Z}_{\geqslant 0}$ is a fixed constant.

Theorem (cf. Theorems 2.2.1 and 2.2.1′ below)**.** *Given any closed semi-algebraic subset of $S \subset \mathrm{R}^k$, there exists a simplicial complex K homologically ℓ-equivalent to S whose size is bounded singly exponentially in k (as a function of the number and degrees of polynomials appearing in the description of S). If $\mathrm{R} = \mathbb{R}$, then K is ℓ-equivalent to S. Moreover, there exists an algorithm (Algorithm 3) which computes the complex K given S, and whose complexity is bounded singly exponentially in k.*

Previously, Basu et al. [7], [8] proposed algorithms to compute the first ℓ Betti numbers of the given semi-algebraic set with singly exponential complexity algorithms, $(sd)^{k^{O(1)}}$.

However, their algorithms only consider homology groups and disregard information about homotopy invariants; moreover, they do not produce a simplicial complex.

We also obtain the following corollary that gives an algorithmic reduction of the problem of computing the first ℓ homotopy groups of a given closed semi-algebraic set to a purely combinatorial problem.

Corollary (cf. Corollaries 1 and 2 below)**.** *Let* $\mathrm{R} = \mathbb{R}$*, there exists a reduction having singly exponential complexity, of the problem of computing the first* ℓ *homotopy groups of any given closed semi-algebraic subset* $S \subset \mathrm{R}^k$*, to the problem of computing the first* ℓ *homotopy groups of a finite simplicial complex. This implies that there exists an algorithm with singly exponential complexity which given as input a closed semi-algebraic set* $S \subset \mathbb{R}^k$ *guaranteed to be simply connected, outputs the description of the first* ℓ *homotopy groups of* S *(in terms of generators and relations).*

The algorithmic results mentioned above are consequences of a topological construction which can be interpreted as a generalization of the classical "nerve lemma" in topology. We state it here informally.

First, assume that there exists a "black-box" that given as input any closed semi-algebraic set $S \subset \mathrm{R}^k$, produces as output a cover of S by closed semi-algebraic subsets of S, which are homologically ℓ-connected.

Theorem (cf. Theorem 2.3.1 below)**.** *Given a black-box as above, there exists for every closed semi-algebraic set* S *a poset* $\mathbf{P}(S)$ *(see Definition 2.3.3 below) which depends on the given black-box, of controlled complexity (both in terms of the description of* S *and the complexity of the black-box), such that the geometric realization of the order-complex of* $\mathbf{P}(S)$ *is homologically* ℓ*-equivalent to* S*.*

1.3.2 Persistent homology of semi-algebraic sets

In Chapter 3, we build on our previous results and initiate the algorithmic problem of computing persistent homology groups of semi-algebraic sets equipped with a filtration of the sub-level sets of a polynomial. Previous literature has focused on finite filtrations of topological spaces, however, here we have filtrations of semi-algebraic sets by polynomials which are continuous functions. Therefore, one intermediate contribution of our work is that we reformulate the definition of barcode to extend to the continuous filtration, and then present a reduction from a barcode of a continuous filtration to the barcode of a finite filtration. This is possible since the topological type of the sub-level sets of a filtration of a semi-algebraic set by a semi-algebraic function changes at only finitely many values of the function.

We prove the following theorem stated below informally. The formal statement appears later in Chapter 3.

Theorem (cf. Theorem 3.2.1)**.** *There exists an algorithm (Algorithm 8) that takes as input a description of a closed and bounded semi-algebraic set $S \subset \mathrm{R}^k$, and a polynomial $P \in \mathrm{R}[X_1, \ldots, X_k]$, and outputs the "barcodes" (cf. Definition 3.2.5 below) in dimensions 0 to ℓ of the filtration of S by the sub-level sets of the polynomial P. The complexity of this algorithm is bounded singly exponentially in k (as a function of the number and degrees of polynomials appearing in the description of S).*

2. EFFICIENT SIMPLICIAL REPLACEMENT OF SEMI-ALGEBRAIC SETS

This chapter is organized as follows. In Section 2.1, we briefly review the previous works. In Section 2.2 we give precise statements of the main results summarized above after introducing the necessary definitions regarding the different notions of topological equivalence that we use. In Section 2.3 we define the key mathematical object (namely, a poset that we associate to any closed covering of a semi-algebraic set) and prove its main properties (Theorems 2.3.1 and 2.3.1′). In Section 2.4 we describe algorithms for computing efficient simplicial replacements of semi-algebraic sets thereby proving Theorems 2.2.1 and 2.2.1′. In Section 2.5, we explain the details of our implementation and report some of its results. Finally, in Section 2.6 we state some open questions and directions for future work in this area.

2.1 Background

Semi-algebraic subsets of R^k have tame topology. In particular, closed and bounded semi-algebraic subsets of R^k are semi-algebraically triangulable (see for example [2, Chapter 5]). This means that there exists a finite simplicial complex K, whose *geometric realization*, $|K|$, considered as a subset of R^N for some $N > 0$, is semi-algebraically homeomorphic to S. The semi-algebraic homeomorphism $|K| \to S$ is called a *semi-algebraic triangulation of* S. All topological properties of S are then encoded in the finite data of the simplicial complex K.

For instance, taking $\mathrm{R} = \mathbb{R}$, the (singular) homology groups, $\mathrm{H}_*(S)$[1], of S are isomorphic to the simplicial homology groups of the simplicial chain complex $\mathrm{C}_\bullet(K)$ of the simplicial complex K, and the latter is a complex of free $\mathbb{Z}$-modules having finite ranks.

[1] ↑In this chapter, we assume all homology and cohomology groups are with coefficients in $\mathbb{Z}$.

The problem of designing an efficient algorithm for obtaining semi-algebraic triangulations has attracted a lot of attention over the years. There exists a classical algorithm which takes as input a quantifier-free formula defining a semi-algebraic set S, and produces as output a semi-algebraic triangulation of S (see for instance [2, Chapter 5]). However, this algorithm is based on the technique of *cylindrical algebraic decomposition*, and hence the complexity of this algorithm is prohibitively expensive, being doubly exponential in k. More precisely, given a description by a quantifier-free formula involving s polynomials of degree at most d, of a closed and bounded semi-algebraic subset of $S \subset \mathrm{R}^k$, there exists an algorithm computing a semi-algebraic triangulation of $h : |K| \to S$, whose complexity is bounded by $(sd)^{2^{O(k)}}$. Moreover, the size of the simplicial complex K (measured by the number of simplices) is also bounded by $(sd)^{2^{O(k)}}$.

2.1.1 Doubly exponential vs singly exponential

One can ask whether the doubly exponential behavior for the semi-algebraic triangulation problem is intrinsic to the problem. One reason to think that it is not so comes from the fact that the ranks of the homology groups of S (following the same notation as in the previous paragraph), and so in particular those of the simplicial complex K, is bounded by $(O(sd))^k$ (see for instance [2, Chapter 7]), which is singly exponential in k. So it is natural to ask if this singly exponential upper bound on $\mathrm{rank}(\mathrm{H}_*(S))$ is "witnessed" by an efficient semi-algebraic triangulation of small (i.e. singly exponential) size. This is not known.

In fact, designing an algorithm with a *singly exponential* complexity for computing a semi-algebraic triangulation of a given semi-algebraic set has remained a holy grail in the field of algorithmic real algebraic geometry and little progress has been made over the last thirty years on this problem (at least for general semi-algebraic sets).

2.1.2 Comparison with prior and related results

As stated previously, there is no algorithm known for computing the Betti numbers of semi-algebraic sets having singly exponential complexity. However, algorithms with singly exponential complexity are known for computing certain (small) Betti numbers. The zero-th

Betti number of a semi-algebraic set is just the number of its semi-algebraically connected components. Counting the number of semi-algebraically connected components of a given semi-algebraic set is a well-studied problem and algorithms with singly exponential complexity are known for solving this problem [9]–[11]. In [7] a singly exponential complexity algorithm is given for computing the first Betti number of semi-algebraic sets, and this was extended to the first ℓ (for any fixed constant ℓ) Betti numbers in [8]. These algorithms do not produce a simplicial complex homotopy equivalent (or ℓ-equivalent) to the given semi-algebraic set.

In [12]–[14], the authors take a different approach. Working over $\mathbb{R}$, and given a well-conditioned semi-algebraic subset $S \subset \mathbb{R}^k$, they compute a witness complex whose geometric realization is k-equivalent to S. The size of this witness complex is bounded singly exponentially in k. However, the complexity depends on the condition number of the input (and so this bound is not uniform), and the algorithm will fail for ill-conditioned input when the condition number becomes infinite. This is unlike the kind of algorithms we consider here, which are supposed to work for all inputs and with uniform complexity upper bounds. So these approaches are not comparable.

While the approaches in [7], [8] and those in [12]–[14] are not comparable, since the meaning of what constitutes an algorithm and the notion of complexity are different, there is a common connection between the results of these papers and those in the current chapter which we elucidate below.

Separation of complexity into algebraic and combinatorial parts [2]

In the definition of complexity given earlier we are counting only arithmetic operations involving elements of the ring generated by the coefficients of the input formulas. Many algorithms in semi-algebraic geometry have the following feature. After a certain number of operations involving elements of the coefficient ring D, the problem is reduced to solving a combinatorial or a linear algebra problem defined over $\mathbb{Z}$.

[2] ↑Note that this notion of separation of complexity into algebraic and combinatorial parts is distinct from that used in [2], where "combinatorial part" refers to the part depending on the number of polynomials, and the"algebraic part" refers to the dependence on the degrees of the polynomials.

A typical example is an algorithm for computing the Betti numbers of a semi-algebraic set via computing a semi-algebraic triangulation. Once a simplicial complex whose geometric realization is semi-algebraically homeomorphic to the given semi-algebraic set has been computed, the problem of computing the Betti numbers of the given semi-algebraic set is reduced to linear algebra over $\mathbb{Z}$. Usually, this separation of the cost of an algorithm into a part that involves arithmetic operations over D, and a part that is independent of D, is not very important since often the complexity of the second part is subsumed by that of the first part. However, here the fact that we are only counting arithmetic operations in D is more significant. In one application that we discuss, namely that of computing the homotopy groups of a given semi-algebraic set (see Corollary 1), we give a reduction (having single exponential complexity) to a problem whose definition is independent of D, namely computing the homotopy groups of a simplicial complex. Note that the problem of deciding whether the first homotopy group of a simplicial complex is trivial or not is an undecidable problem (this fact follows from the undecidability of the word problem for groups [15]).

2.2 Precise statements of the main results

In this section we will describe in full detail the main results summarized in the Section 1.3.1. We first introduce certain preliminary definitions and notation.

2.2.1 Definitions of topological equivalence and complexity

We begin with the precise definitions of the two kinds of topological equivalence that here we are going to use.

Topological equivalences

Definition 2.2.1 (ℓ-equivalences)**.** *We say that a map $f : X \to Y$ between two topological spaces is an ℓ-equivalence, if the induced homomorphisms between the homotopy groups $f_* : \pi_i(X) \to \pi_i(Y)$ are isomorphisms for $0 \leqslant i \leqslant \ell$ [15, page 68].*

Remark 1. *Note that our definition of ℓ-equivalence deviates a little from the standard one which requires that homomorphisms between the homotopy groups $f_* : \pi_i(X) \to \pi_i(Y)$ are isomorphisms for $0 \leqslant i \leqslant \ell - 1$, and only an epimorphism for $i = \ell$. An ℓ-equivalence in our sense is an ℓ-equivalence in the traditional sense.*

The relation of ℓ-equivalence as defined above is not an equivalence relation since it is not symmetric. In order to make it symmetric one needs to "formally invert" ℓ-equivalences.

Definition 2.2.2 (ℓ-equivalent and homologically ℓ-equivalent)**.** *We will say that X is* ℓ-equivalent to *Y (denoted $X \sim_\ell Y$), if and only if there exists spaces, $X = X_0, X_1, \ldots, X_n = Y$ and ℓ-equivalences $f_1, \ldots, f_n$ as shown below:*

$$
\begin{tikzcd}
& X_1 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & X_3 \arrow[dl, "f_3"'] \arrow[dr, "f_4"] & \cdots & \cdots & X_{n-1} \arrow[dl, "f_{n-1}"'] \arrow[dr, "f_n"] & \\
X_0 & & X_2 & & \cdots & \cdots & & X_n
\end{tikzcd}
.
$$

It is clear that $\sim_\ell$ is an equivalence relation.

By replacing the homotopy groups, $\pi_i(\cdot)$ with homology groups $\mathrm{H}_i(\cdot)$ (resp. cohomology groups $\mathrm{H}^i(\cdot)$ with arrows reversed) in Definitions 2.2.1 and 2.2.2, we get the notion of two topological spaces X, Y being homologically ℓ-equivalent *(denoted $X \overset{h}{\sim}_\ell Y$) (resp.* cohomologically ℓ-equivalent *(denoted $X \overset{ch}{\sim}_\ell Y$)).*

This is a strictly weaker equivalence relation, since there are spaces X for which $\mathrm{H}_1(X) = 0$, but $\pi_1(X) \neq 0$.

We extend the above definitions to $\ell = -1$ by using the convention that $X \sim_{-1} Y$ (resp. $X \overset{h}{\sim}_{-1} Y$, $X \overset{ch}{\sim}_{-1} Y$), if and only if X, Y are both non-empty or both empty.

Definition 2.2.3 (ℓ-connected and homologically ℓ-connected)**.** *We say that a topological space X is* ℓ-connected, *for $\ell \geqslant 0$, if X is connected and $\pi_i(X) = 0$ for $0 < i \leqslant \ell$. We will say that X is (-1)-connected if X is non-empty. We say that X is* homologically ℓ-connected *if X is connected and $\mathrm{H}_i(X) = 0$ for $0 < i \leqslant \ell$.*

Definition 2.2.4 (Diagrams of topological spaces)**.** *A diagram of topological spaces is a functor, $X : J \to$* **Top***, from a small category J to* **Top***.*

We extend Definition 2.2.1 to diagrams of topological spaces. We denote by **Top** the category of topological spaces.

Definition 2.2.5 (ℓ-equivalence between diagrams of topological spaces)**.** *Let J be a small category, and $X, Y : J \to$ **Top** be two functors. We say a natural transformation $f : X \to Y$ is an ℓ equivalence, if the induced maps,*

$$f(j)_* : \pi_i(X(j)) \to \pi_i(Y(j))$$

are isomorphisms for all $j \in J$ and $0 \leqslant i \leqslant \ell$.

We will say that a diagram $X : J \to$ **Top** is ℓ-equivalent to the diagram $Y : J \to$ **Top** *(denoted as before by $X \sim_\ell Y$), if and only if there exists diagrams $X = X_0, X_1, \ldots, X_n = Y : J \to$ **Top** and ℓ-equivalences $f_1, \ldots, f_n$ as shown below:*

$$\begin{tikzcd}[column sep=small] & X_1 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & & X_3 \arrow[dl, "f_3"'] \arrow[dr, "f_4"] & \cdots & \cdots & X_{n-1} \arrow[dl, "f_{n-1}"'] \arrow[dr, "f_n"] & \\ X_0 & & X_2 & & \cdots & \cdots & & X_n \end{tikzcd}.$$

It is clear that $\sim_\ell$ is an equivalence relation.

In the above definition, by replacing the homotopy groups with homology (resp. cohomology) groups we obtain the notion of homological (resp. cohomological) ℓ-equivalence between diagrams, which we will denote as before by $\overset{h}{\sim}_\ell$ (resp. $\overset{ch}{\sim}_\ell$).

One particular diagram will be important in what follows.

Notation 1 (Diagram of various unions of a finite number of subspaces)**.** *Let J be a finite set, A a topological space, and $\mathcal{A} = (A_j)_{j \in J}$ a tuple of subspaces of A indexed by J.*

For any subset $J' \subset J$ [3]*, we denote*

$$\begin{aligned} \mathcal{A}^{J'} &= \bigcup_{j' \in J'} A_{j'}, \\ \mathcal{A}_{J'} &= \bigcap_{j' \in J'} A_{j'}, \end{aligned}$$

We consider 2^J as a category whose objects are elements of 2^J, and whose only morphisms are given by:

$$\begin{aligned} 2^J(J', J'') &= \emptyset \text{ if } J' \not\subset J'', \\ 2^J(J', J'') &= \{\iota_{J',J''}\} \text{ if } J' \subset J''. \end{aligned}$$

We denote by $\mathbf{Simp}^J(\mathcal{A}) : 2^J \to \mathbf{Top}$ the functor (or the diagram) defined by

$$\mathbf{Simp}^J(\mathcal{A})(J') = \mathcal{A}^{J'}, J' \in 2^J,$$

and $\mathbf{Simp}^J(\mathcal{A})(\iota_{J',J''})$ is the inclusion map $\mathcal{A}^{J'} \hookrightarrow \mathcal{A}^{J''}$.

$\mathcal{P}$-formulas and $\mathcal{P}$-semi-algebraic sets

Notation 2 (Realizations, $\mathcal{P}$-, $\mathcal{P}$-closed semi-algebraic sets)**.** *For any finite set of polynomials $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$, we call any quantifier-free first order formula ϕ with atoms, $P = 0, P < 0, P > 0, P \in \mathcal{P}$, to be a* $\mathcal{P}$-formula*. Given any semi-algebraic subset $Z \subset \mathrm{R}^k$, we call the realization of ϕ in Z, namely the semi-algebraic set*

$$\mathcal{R}(\phi, Z) \ := \ \{\mathbf{x} \in Z \mid \phi(\mathbf{x})\}$$

a $\mathcal{P}$-semi-algebraic subset of *Z.*

If $Z = \mathrm{R}^k$, we often denote the realization of ϕ in R^k by $\mathcal{R}(\phi)$.

[3]↑In this thesis $A \subset B$ will mean $A \cap B = A$ allowing the possibility that $A = B$. Also, when we denote $\alpha \prec \beta$ in a poset we allow the possibility $\alpha = \beta$, reserving $\alpha \precneqq \beta$ to denote $\alpha \prec \beta, \alpha \neq \beta$.

If $\Phi = (\phi_j)_{j \in J}$ is a tuple of formulas indexed by a finite set J, $Z \subset \mathrm{R}^k$ a semi-algebraic subset, we will denote by $\mathcal{R}(\Phi, Z)$ the tuple $(\mathcal{R}(\phi_j, Z))_{j \in J}$, and call it the realization of Φ in Z. For $J \subset J'$, we will denote by $\Phi|_{J'}$ the tuple $(\phi_j)_{j \in J'}$.

We say that a quantifier-free formula ϕ is closed *if it is a formula in disjunctive normal form with no negations, and with atoms of the form $P \geqslant 0, P \leqslant 0$ (resp. $P > 0, P < 0$), where $P \in \mathrm{D}[X_1, \ldots, X_k]$. If the set of polynomials appearing in a closed (resp. open) formula is contained in a finite set $\mathcal{P}$, we will call such a formula a $\mathcal{P}$-closed formula, and we call the realization, $\mathcal{R}(\phi)$, a $\mathcal{P}$-*closed semi-algebraic set.

We will also use the following notation.

Notation 3. *For $n \in \mathbb{Z}$ we denote by $[n] = \{0, \ldots, n\}$. In particular, $[-1] = \varnothing$.*

Finally, we are able to state the main results proved in this chapter.

2.2.2 Efficient simplicial replacements of semi-algebraic sets

Theorem 2.2.1. *There exists an algorithm that takes as input*

A. a $\mathcal{P}$-closed formula ϕ for some finite set $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$;

B. $\ell, 0 \leqslant \ell \leqslant k$;

and produces as output a simplicial complex $\Delta_\ell(\phi)$ such that $|\Delta_\ell(\phi)| \overset{h}{\sim}_\ell \mathcal{R}(\phi)$. The complexity of the algorithm is bounded by $(sd)^{k^{O(\ell)}}$, where $s = \mathrm{card}(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

More generally, there exists an algorithm that takes as input

A. a tuple $\Phi = (\phi_0, \ldots, \phi_N)$ of $\mathcal{P}$-closed formulas for some finite set $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$;

B. $\ell, 0 \leqslant \ell \leqslant k$;

and produces as output a simplicial complex $\Delta_\ell(\Phi)$, and for each $J \subset [N]$ a subcomplex $\Delta_\ell(\Phi|_J)$, such that

$$(J \to |\Delta_\ell(\Phi|_J)|)_{J \subset [N]} \overset{h}{\sim}_\ell \mathbf{Simp}^{[N]}(\mathcal{R}(\Phi)).$$

The complexity of the algorithm is bounded by $(Nsd)^{k^{O(\ell)}}$, where $s = \mathrm{card}(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

Theorem 2.2.1 is valid over arbitrary real closed fields. In the special case of $\mathrm{R} = \mathbb{R}$, we have the following stronger version of Theorem 2.2.1, where we are able to replace homological ℓ-equivalence by ℓ-equivalence.

Theorem 2.2.1′. *Let $\mathrm{R} = \mathbb{R}$. There exists an algorithm that takes as input*

A. a $\mathcal{P}$-closed formula ϕ for some finite set $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$;

B. $\ell, 0 \leqslant \ell \leqslant k$;

and produces as output a simplicial complex $\Delta_\ell(\phi)$ such that $|\Delta_\ell(\phi)| \sim_\ell \mathcal{R}(\phi)$. The complexity of the algorithm is bounded by $(sd)^{k^{O(\ell)}}$, where $s = \mathrm{card}(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

More generally, there exists an algorithm that takes as input

A. a tuple $\Phi = (\phi_0, \ldots, \phi_N)$ of $\mathcal{P}$-closed formulas for some finite set $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$;

B. $\ell, 0 \leqslant \ell \leqslant k$;

and produces as output a simplicial complex $\Delta_\ell(\Phi)$, and for each $J \subset [N]$ a subcomplex $\Delta_\ell(\Phi|_J)$ such that

$$(J \to |\Delta_\ell(\Phi|_J)|)_{J \subset [N]} \sim_\ell \mathbf{Simp}^{[N]}(\mathcal{R}(\Phi)).$$

The complexity of the algorithm is bounded by $(Nsd)^{k^{O(\ell)}}$, where $s = \mathrm{card}(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

Remark 2. *One main tool that we use is the Vietoris-Begle theorem (see proofs of Claims 2.3.1, 2.3.2). Since, there are many versions of the Vietoris-Begle theorem in the literature we make precise what we use below.*

It follows from [16, Main Theorem] that if $X \subset \mathbb{R}^m, Y \subset \mathbb{R}^n$ are compact semi-algebraic subsets (and so are locally contractible), and $f : X \to Y$ is a semi-algebraic continuous map such that for every $y \in Y$, $f^{-1}(y)$ is ℓ-connected, then f is an ℓ-equivalence. We will refer to this version of the Vietoris-Begle theorem as the homotopy version of the Vietoris-Begle theorem*. Since, ℓ-equivalence implies homological ℓ-equivalence (see for example [17, pp. 124, §4.1B]), f is a homological ℓ-equivalence as well.*

Alternatively, if we assume that $f^{-1}(y)$ is only homologically ℓ-connected for each $y \in Y$, then we can conclude that f is a homological ℓ-equivalence (see for example, the statement of the Vietoris-Begle theorem in [18]). This latter theorem is also valid for semi-algebraic maps between closed and bounded semi-algebraic sets over arbitrary real closed fields, once we know it for maps between compact semi-algebraic subsets over $\mathbb{R}$. This follows from a standard argument using the Tarski-Seidenberg transfer principle and the fact that homology groups of closed bounded semi-algebraic sets can be defined in terms of finite triangulations. We will refer to this version of the Vietoris-Begle theorem as the homological version of the Vietoris-Begle theorem.

2.2.3 Application to computing homotopy groups of semi-algebraic sets

One important new contribution of the current work compared to previous algorithms for computing topological invariants of semi-algebraic sets [7], [8] is that for any given semi-algebraic subset $S \subset \mathbb{R}^k$, our algorithms give information on not just the homology groups but the homotopy groups of S as well.

Computing homotopy groups of semi-algebraic sets is a considerably harder problem than computing homology groups. There is no algorithm to decide whether the fundamental group of a finite simplicial complex is trivial [15]. As such the problem of deciding whether the fundamental group of any semi-algebraic subset $S \subset \mathbb{R}^k$ is trivial or not is an undecidable problem.

On the other hand algorithms for computing topological invariants of a given semi-algebraic set $S \subset \mathrm{R}^k$, defined by a $\mathcal{P}$-formula where $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$, usually involve two kinds of operations.

(a) Arithmetic operations and comparisons amongst elements of the ring D;

(b) Operations that do not involve elements of D.

In our complexity bounds we only count the first kind of operations (i.e. those which involve elements of D).

From this point of view it makes sense to ask for any algorithmic problem involving formulas defined over D, if there is a reduction to another problem whose input is independent of D. Theorem 2.2.1′ gives precisely such a reduction for computing the first ℓ homotopy groups of any given semi-algebraic set defined by a formula involving coefficients from any fixed subring $\mathrm{D} \subset \mathbb{R}$.

Corollary 1. *For every fixed ℓ, and an ordered domain $\mathrm{D} \subset \mathbb{R}$, there exists a a reduction of the problem of computing the first ℓ homotopy groups of a semi-algebraic set defined by a quantifier-free formula with coefficients in D, to that of the problem of computing the first ℓ homotopy groups of a finite simplicial complex. The complexity of this reduction is bounded singly exponentially in the size of the input.*

While the problem of computing the fundamental group as well as the higher homotopy groups of a finite simplicial complex is clearly an extremely challenging problem, there has been recent breakthroughs. If a simplicial complex K is 1-connected then Čadek et al. [19] has given an algorithm for computing a description of the homotopy groups $\pi_i(|K|)$, $2 \leqslant i \leqslant \ell$, which has complexity polynomially bounded in the size of the simplicial complex K for every fixed ℓ. This result coupled with Theorem 2.2.1′ gives the following corollary.

Corollary 2. *Let $\mathrm{R} = \mathbb{R}, \mathrm{D} \subset \mathrm{R}$ and $\ell \geqslant 2$. There exists an algorithm that takes as input*

A. a $\mathcal{P}$-closed formula ϕ for some finite set $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$;

B. $\ell, 0 \leqslant \ell \leqslant k$;

such that $\mathcal{R}(\phi)$ is simply connected, and outputs descriptions of the abelian groups $\pi_i(\mathcal{R}(\phi))$, $2 \leqslant i \leqslant \ell$ in terms of generators and relations.

The complexity of the algorithm is bounded by $(sd)^{k^{O(\ell)}}$, where $s = \mathrm{card}(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

Remark 3. *Note that we do not have an effective algorithm for checking the hypothesis that the given semi-algebraic set is simply connected.*

Covers

A standard method in algebraic topology for computing homology/cohomology of a space X is by means of an appropriately chosen *cover*, $(V_\alpha \subset X)_{\alpha \in I}$, of X by open or closed subsets. Suppose that $X \subset \mathbb{R}^k$ is a closed or open semi-algebraic set. Let $\mathcal{V} = (V_\alpha \subset X)_{\alpha \in I}$ be a finite cover of X by open or closed semi-algebraic subsets, such that for each non-empty subset $J \subset I$, the intersection $V_J = \bigcap_{\alpha \in J} V_\alpha$ is either empty or contractible. We will say that such covers have the *Leray property* and refer to them as *Leray covers*. One can then associate to the cover $\mathcal{V}$, a simplicial complex, $\mathcal{N}(\mathcal{V})$, the nerve of $\mathcal{V}$ defined as follows.

The set of p-simplices of $\mathcal{N}(\mathcal{V})$ is defined by

$$\mathcal{N}(\mathcal{V})_p = \{\{\alpha_0, \ldots, \alpha_p\} \subset 2^I \mid V_{\alpha_0} \cap \cdots \cap V_{\alpha_p} \neq \emptyset\}.$$

It follows from a classical result of algebraic topology that the geometric realization $|\mathcal{N}(\mathcal{V})|$ is homotopy equivalent to X, and moreover for each $\ell \geqslant 0$, the geometric realization of the $(\ell+1)$-st skeleton of $\mathcal{N}(\mathcal{V})$,

$$\mathrm{sk}_{\ell+1}(\mathcal{N}(\mathcal{V})) = \{\sigma \in \mathcal{N}(\mathcal{V}) \mid \mathrm{card}(\sigma) \leqslant \ell + 2\}.$$

is homologically ℓ-equivalent (resp. ℓ-equivalent) to X (resp. when $\mathrm{R} = \mathbb{R}$).

The algorithms for computing the Betti numbers in [12]–[14] proceeds by computing the k-skeleton of the nerve of a cover having the Leray property whose size is bounded singly exponentially in k, and computing the simplicial homology groups of this complex. However, the bound on the size of the cover is not uniform but depends on a real valued parameter – namely the condition number of the input – and hence the size of the cover can become infinite. In fact, computing a singly exponential sized cover by semi-algebraic subsets having the Leray property of an arbitrary semi-algebraic sets is an open problem. If one solves this problem then one would also solve immediately the problem of designing an algorithm for computing all the Betti numbers of arbitrary semi-algebraic sets with singly exponential complexity in full generality.

The algorithms in [7], [8] which are able to compute some of the Betti numbers in dimensions > 0 also depends on the existence of small covers having size bounded singly exponentially, albeit satisfying a much weaker property than the Leray property. The weaker property is that only the sets $V_\alpha, \alpha \in I$ (i.e. the elements of the cover) are contractible. No assumption is made on the non-trivial finite intersections amongst the sets of the cover. Covers satisfying this weaker property can indeed be computed with singly exponential complexity (this is one of the main results of [7] but see Remark 5), and using this fact one is able to compute the first ℓ Betti numbers of semi-algebraic subsets of R^k for every fixed ℓ with singly exponential complexity. The algorithms in [7] and [8] do not construct a simplicial complex homotopy equivalent or ℓ-equivalent to the given semi-algebraic set S unlike the algorithm in [12].

Main technical contribution

The main technical result that underlies the algorithmic result of the current work is the following. Fix $0 \leqslant \ell \leqslant k$. Suppose for every closed and bounded semi-algebraic set S one has a covering of S by closed and bounded semi-algebraic subsets which are ℓ-connected (see Definition 2.2.3) and which has singly exponentially bounded complexity (meaning that the number of sets in the cover, the number of polynomials used in the quantifier-free formulas defining these sets and their degrees are all bounded singly exponentially in k). Moreover, since it is clear that contractible covers with singly exponential complexity exists, this is not a vacuous assumption. Using ℓ-connected covers repeatedly we build a simplicial complex of size bounded singly exponentially which is ℓ-equivalent to the given semi-algebraic set. The definition of this simplicial complex is a bit involved (much more involved than the nerve complex of a Leray cover) and appears in Section 2.3. Its main properties are encapsulated in Theorem 2.3.1.

Two remarks are in order.

Remark 4. *1. Firstly, the Leray property can be weakened to require that for every t-wise intersection, $V_J, \mathrm{card}(J) = t$ is either empty or $(\ell - t + 1)$-connected [20]. We call this the ℓ-*Leray property*. The nerve complex, $\mathcal{N}(\mathcal{V})$ is then ℓ-equivalent to X [20]. However,* *the property that we use is much weaker – namely that only the elements of the cover are ℓ-connected and we make no assumptions on the connectivity of the intersections of two or more sets of the cover. This is due to the fact that controlling the connectivity of the intersections is very difficult and we do not know of any algorithm with singly exponential complexity for computing covers having the ℓ-Leray property for $\ell \geqslant 1$.*

2. *Secondly, note that to be ℓ-connected is a weaker property than being contractible. Unfortunately, at present we do not know of algorithms for computing ℓ-connected covers, for $\ell > 0$ that has much better complexity asymptotically than the algorithm in [7] for computing covers by contractible semi-algebraic sets. However, it is still possible that there could be algorithms with much better complexity for computing ℓ-connected covers (at least for small ℓ) compared to computing contractible covers.*

2.3 Simplicial replacement in an abstract setting

We now arrive at the technical core of this chapter. Given a finite set J, a tuple, $\Phi = (\phi_j)_{j\in J}$, of closed formulas with k free variables, and numbers $i, m \geqslant 0$, we will describe the construction of a poset, that we denote by $\mathbf{P}_{m,i}(\Phi)$. We will assume that the realizations, $\mathcal{R}(\phi_j), j \in J$, of the formulas in the tuple are homologically ℓ-connected semi-algebraic subsets of R^k for some $\ell \geqslant 0$. In case $\mathrm{R} = \mathbb{R}$, substitute "ℓ-connected" for "homologically ℓ-connected". The poset $\mathbf{P}_{m,i}(\Phi)$ will have the property that the geometric realization of its order complex, $\Delta(\mathbf{P}_{m,i}(\Phi))$, is homologically $(m-1)$-equivalent ($(m-1)$-equivalent if $\mathrm{R} = \mathbb{R}$) to $\mathcal{R}(\Phi)^J$. More generally, for each $J' \subset J$, $\mathbf{P}_{m,i}(\Phi|_{J'})$ can be identified as a subposet of $\mathbf{P}_{m,i}(\Phi)$, and the diagram of inclusions of the corresponding geometric realizations is homologically $(m-1)$-equivalent to the diagram $\mathbf{Simp}^J(\mathcal{R}(\Phi))$ ($(m-1)$-equivalent if $\mathrm{R} = \mathbb{R}$) (cf. Theorems 2.3.1 and 2.3.1$'$). The poset $\mathbf{P}_{m,i}(\Phi)$ will then encode in a finite combinatorial way information which determines the first m homotopy groups of $\mathcal{R}(\Phi)^{J'}$ for all $J' \subset J$, and the morphisms $\pi_h(\mathcal{R}(\Phi)^{J'}) \to \pi_h(\mathcal{R}(\Phi)^{J''})$ induced by inclusions, for $0 \leqslant h \leqslant m-1$ and $J' \subset J'' \subset J$. (The significance of the subscript i in the notation $\mathbf{P}_{m,i}(\Phi)$ will become clear later.)

2.3.1 Outline of the main idea

We begin with an outline explaining the main ideas behind the construction. First observe that if the realizations of the sets in the given tuple, in addition to being ℓ-connected, satisfies the ℓ-Leray property (i.e. each t-wise intersections amongst them is $(\ell - t + 1)$-connected), then it follows from [20] that the poset of the non-empty intersections (with the poset relation being canonical inclusions) satisfies the property that the geometric realization of its order complex (see Definition 2.3.1) is ℓ-equivalent to $\mathcal{R}(\Phi)^J$. The same is true for all the subposets obtained by restricting the intersections to only amongst those indexed by some subset $J' \subset J$. However if the ℓ-Leray property fails to hold then the poset of canonical inclusions may fail to have the desired property.

Consider for example, the tuple

$$\Phi = (\phi_0, \phi_1),$$

where

$$\begin{aligned} \phi_0 &:= (X_1^2 + X_2^2 - 1 = 0) \wedge (X_2 \geqslant 0), \\ \phi_1 &:= (X_1^2 + X_2^2 - 1 = 0) \wedge (X_2 \leqslant 0). \end{aligned}$$

The realizations $\mathcal{R}(\phi_0), \mathcal{R}(\phi_1)$ are the upper and lower semi-circles covering the unit circle in the plane.

The intersection $\mathcal{R}(\phi_0) \cap \mathcal{R}(\phi_1) = \mathcal{R}(\phi_0 \wedge \phi_1)$ is the disjoint union of two points. The Hasse diagram of the poset of canonical inclusions of the sets defined by ϕ_0, ϕ_1, and $\phi_0 \wedge \phi_1$ is:

$$\begin{tikzcd} \phi_0 & & \phi_1 \\ & \phi_0 \wedge \phi_1 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

and the order complex of the poset is the simplicial complex shown in Figure 2.1. The geometric realization of the order complex is clearly not homotopy equivalent to the

$$\mathcal{R}(\Phi)^{\{0,1\}} = \mathcal{R}(\phi_0) \cup \mathcal{R}(\phi_1)$$

which is equal to the unit circle. This is not surprising since the cover of the circle by the two closed semi-circle is not a Leray cover (and in fact not ℓ-Leray for any $\ell \geqslant 0$).

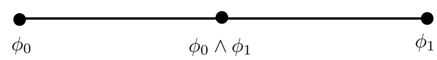


Figure 2.1. Order complex for non-Leray cover

One way of repairing this situation is to go one step further and choose a good (in this case ∞-connected) cover for the intersection $\mathcal{R}(\phi_0) \cap \mathcal{R}(\phi_1)$ defined by ψ_0, ψ_1, where

$$\begin{aligned} \psi_0 &:= (X_1 + 1 = 0) \wedge (X_2 = 0), \\ \psi_1 &:= (X_1 - 1 = 0) \wedge (X_2 = 0). \end{aligned}$$

The Hasse diagram of the poset of canonical inclusions of the sets defined by ϕ_0, ϕ_1, ψ_0, and ψ_1

$$\begin{tikzcd} \phi_0 & \phi_1 \\ \psi_0 \arrow[u] \arrow[ur] & \psi_1 \arrow[u] \arrow[ul] \end{tikzcd}$$

and the order complex of the poset is shown in Figure 2.2. It is easily seen to have the same homotopy type (homeomorphism type even in this case) to the circle.

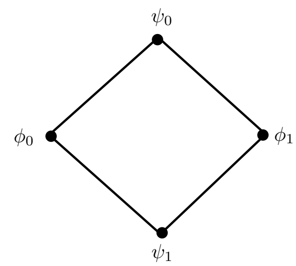


Figure 2.2. Order complex for modified poset

The very simple example given above motivates the definition of the poset $\mathbf{P}_{m,i}(\Phi)$ in general. We assume that we have available not just the given tuple of sets, and the non-empty intersections amongst them, but also that we can cover any given non-empty intersections that arise in our construction using ℓ-connected closed (resp. open) semi-algebraic sets (we do not assume that these covers satisfy the stronger ℓ-Leray property). The poset we define depends on the choice of these covers and not just on the formulas in the tuple Φ (unlike the standard nerve complex of the tuple $\mathcal{R}(\Phi)$). The choices that we make are encapsulated in the functions $\mathcal{I}_{k,i}$ and $\mathcal{C}_{k,i}$ below. In practice, they would correspond to some effective algorithm for computing well-connected covers of semi-algebraic sets.

Remark 5. *There is one technical detail that serves to obscure a little the clarity of the construction. It arises due to the fact that the only algorithm with single exponential complexity that exists in the literature for computing well connected (∞-connected or equivalently contractible) covers is the one in [7]. However, the algorithm requires that the polynomials describing the given set S be in strong general position (see Definition 2.4.1). In order to satisfy this requirement one needs to initially perturb the given polynomials and replace the given set by another one, say S', which is infinitesimally larger but has the same homotopy type as the given set S (see Lemma 2.3.1). The algorithm then computes closed formulas whose realizations cover S' and moreover are each semi-algebraically contractible. While there is a semi-algebraic retraction from S' to S, this retraction is not guaranteed to restrict to the elements of the cover. Our poset construction is designed to be compatible with the fact that the covers we assume to exist actually are covers of infinitesimally larger sets (i.e.*

that of S' instead of S following the notation of the previous sentence). This necessitates the use of iterated Puiseux extensions in what follows.

Of course, the introduction of infinitesimals could be avoided by choosing sufficiently small positive elements in the field R itself and thus avoid making extensions. This would be more difficult to visualize, and so we prefer to use the language of infinitesimal extensions. In the special case when $\mathrm{R} = \mathbb{R}$, we prefer not to make non-archimedean extensions, since we discuss homotopy groups, so we take the alternative approach. However, we believe that the infinitesimal language is conceptually easier to grasp and so we use it in the general case.

Before giving the definition of the poset we first need to introduce some mathematical preliminaries and notation.

2.3.2 Real closed extensions and Puiseux series

We will need some properties of Puiseux series with coefficients in a real closed field. We refer the reader to [2] for further details.

Notation 4. *For R a real closed field we denote by $\mathrm{R}\langle\varepsilon\rangle$ the real closed field of algebraic Puiseux series in ε with coefficients in R. We use the notation $\mathrm{R}\langle\varepsilon_1,\ldots,\varepsilon_m\rangle$ to denote the real closed field $\mathrm{R}\langle\varepsilon_1\rangle\langle\varepsilon_2\rangle\cdots\langle\varepsilon_m\rangle$. Note that in the unique ordering of the field $\mathrm{R}\langle\varepsilon_1,\ldots,\varepsilon_m\rangle$, $0 < \varepsilon_m \ll \varepsilon_{m-1} \ll \cdots \ll \varepsilon_1 \ll 1$.*

If $\bar\varepsilon$ denotes the (possibly infinite) sequence $(\varepsilon_1,\varepsilon_2,\ldots)$ we will denote by $\mathrm{R}\langle\bar\varepsilon\rangle$ the real closed field $\bigcup_{m\geqslant 0}\mathrm{R}\langle\varepsilon_1,\ldots,\varepsilon_m\rangle$.

Finally, given a finite sequence $(\bar\varepsilon_1,\ldots,\bar\varepsilon_m)$ we will denote by $\mathrm{R}\langle\bar\varepsilon_1,\ldots,\bar\varepsilon_m\rangle$ the real closed field $\mathrm{R}\langle\bar\varepsilon_1\rangle\langle\bar\varepsilon_2\rangle\cdots\langle\bar\varepsilon_m\rangle$.

Notation 5. *For elements $x\in\mathrm{R}\langle\varepsilon\rangle$ which are bounded over R we denote by $\lim_\varepsilon x$ to be the image in R under the usual map that sets ε to 0 in the Puiseux series x.*

Notation 6. *If R' is a real closed extension of a real closed field R, and $S\subset\mathrm{R}^k$ is a semi-algebraic set defined by a first-order formula with coefficients in R, then we will denote by $\mathrm{ext}(S,\mathrm{R}')\subset\mathrm{R}'^k$ the semi-algebraic subset of R'^k defined by the same formula.* [4] *It is well*

[4]↑Not to be confused with the homological functor $\mathrm{Ext}(\cdot,\cdot)$ which unfortunately also appears in this paper.

known that $\mathrm{ext}(S, \mathrm{R}')$ *does not depend on the choice of the formula defining* S *[2, Proposition 2.87].*

Notation 7. *Suppose* R *is a real closed field, and let* $X \subset \mathrm{R}^k$ *be a closed and bounded semi-algebraic subset, and* $X^+ \subset \mathrm{R}\langle\varepsilon\rangle^k$ *be a semi-algebraic subset bounded over* R*. Let for* $t \in \mathrm{R}, t > 0$, $\widetilde{X}_t^+ \subset \mathrm{R}^k$ *denote the semi-algebraic subset obtained by replacing* ε *in the formula defining* X^+ *by* t*, and it is clear that for* $0 < t \ll 1$, $\widetilde{X}_t^+$ *does not depend on the formula chosen. We say that* X^+ *is* monotonically decreasing to X*, and denote* $X^+ \searrow X$ *if the following conditions are satisfied.*

(a) for all $0 < t < t' \ll 1$, $\widetilde{X}_t^+ \subset \widetilde{X}_{t'}^+$*;*

(b)

$$\bigcap_{t>0} \widetilde{X}_t^+ = X;$$

or equivalently $\lim_\varepsilon X^+ = X$*.*

More generally, if $X \subset \mathrm{R}^k$ *be a closed and bounded semi-algebraic subset, and* $X^+ \subset \mathrm{R}\langle\varepsilon_1, \ldots, \varepsilon_m\rangle^k$ *a semi-algebraic subset bounded over* R*, we will say* $X^+ \searrow X$ *if and only if*

$$X_{m+1}^+ = X^+ \searrow X_m^+,\ X_m^+ \searrow X_{m-1}^+, \ldots, X_2^+ \searrow X_1^+ = X,$$

where for $i = 1, \ldots, m$, $X_i^+ = \lim_{\varepsilon_i} X_{i+1}^+$*.*

Note that if $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots)$ *is an infinite sequence, and* $X^+ \subset \mathrm{R}\langle\bar{\varepsilon}\rangle^k$ *is a semi-algebraic subset bounded over* R*, then there exists* $m \geqslant 1$*, and semi-algebraic subset* $X_m^+ \subset \mathrm{R}\langle\varepsilon_1, \ldots, \varepsilon_m\rangle^k$ *closed and bounded over* R*, such that* $X^+ = \mathrm{ext}(X_m^+, \mathrm{R}\langle\bar{\varepsilon}\rangle)$*.*

In this case, if $X \subset \mathrm{R}^k$ *be a closed and bounded semi-algebraic subset, we will say* $X^+ \searrow X$ *if and only if*

$$X_{m+1}^+ = X^+ \searrow X_m^+,\ X_m^+ \searrow X_{m-1}^+, \ldots, X_2^+ \searrow X_1^+ = X,$$

where for $i = 1, \ldots, m$, $X_i^+ = \lim_{\varepsilon_i} X_{i+1}^+$*.*

Finally, if $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_m$ *are sequences (possibly infinite),* $X \subset \mathrm{R}^k$ *be a closed and bounded semi-algebraic subset, and* $X^+ \subset \mathrm{R}\langle\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_m\rangle^k$ *a semi-algebraic subset bounded over* R*, we will say* $X^+ \searrow X$ *if and only if*

$$X^+_{m+1} = X^+ \searrow X^+_m,\ X^+_m \searrow X^+_{m-1}, \ldots, X^+_2 \searrow X^+_1 = X,$$

where for $i = 1, \ldots, m$*,* $X^+_i = \lim_{\bar{\varepsilon}_i} X^+_{i+1}$*.*

The following lemma will be useful later.

Lemma 2.3.1. *Let* $X \subset \mathrm{R}^k$ *be a closed and bounded semi-algebraic subset, and* $X^+ \subset \mathrm{R}\langle\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_m\rangle^k$ *a semi-algebraic subset bounded over* R*, such that* $X^+ \searrow X$*. Then,* $\mathrm{ext}(X, \mathrm{R}\langle\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_m\rangle)$ *is semi-algebraic deformation retract of* X^+*.*

Proof. See proof of Lemma 16.17 in [2]. □

Notation 8. *For* $x \in \mathrm{R}^k$ *and* $R \in \mathrm{R}$*,* $R > 0$*, we will denote by* $B_k(0, R)$ *the open Euclidean ball centered at* 0 *of radius* R*. We will denote by* $\overline{B_k(0, R)}$ *the closed Euclidean ball centered at* 0 *of radius* R*. If* R' *is a real closed extension of the real closed field* R *and when the context is clear, we will continue to denote by* $B_k(0, R)$ *the extension* $\mathrm{ext}(B_k(0, R), \mathrm{R}')$*, and similarly for* $\overline{B_k(0, R)}$*. This should not cause any confusion. Similarly, we will denote by* $\mathbf{S}^{k-1}(0, R)$ *the sphere of dimension* $k - 1$ *in* R^k *centered at* 0 *of radius* R*.*

We refer the reader to [2, Chapter 6] for the definitions of homology and cohomology groups of semi-algebraic sets over arbitrary real closed fields.

2.3.3 Definition of the poset $\mathbf{P}_{m,i}(\Phi)$

Simplified view of the definition of the poset $\mathbf{P}_{m,i}(\Phi)$

Before giving a precise definition of the poset $\mathbf{P}_{m,i}(\Phi)$, we first give a simplified version. We make the following two simplifications in order to illustrate the key idea.

(a) We ignore the role of the index i in what follows. The necessity of the extra parameter i is due to the fact that the hypothesis we assume (Hypothesis 1 in the following paragraph) is slightly stronger than we are able to assume for designing effective algorithms for computing the poset (see Remark 5). The actual hypothesis that we use is encapsulated in Property 2 below.

(b) Secondly, in order to keep a geometric view of the construction, we will talk about tuples $\mathcal{S} = (S_j)_{j \in J}$ of semi-algebraic sets, instead of tuples of formulas $\Phi = (\phi_j)_{j \in J}$ defining them. As above, in order to give an effective algorithms, and analyzing its complexity, we need to describe the poset in terms of formulas rather than sets, which we do in the precise definition that follows this simplified version.

We make the following hypothesis.

Hypothesis 1 (Black-box hypothesis)**.** *There exists a black-box (or algorithm) that given a closed and bounded semi-algebraic set $S \subset \mathbb{R}^k$ as input, produces a cover $(S_\alpha)_{\alpha \in \mathcal{C}(S)}$ of S by closed and bounded ℓ-connected semi-algebraic sets.*

Definition 2.3.1 (The order complex of a poset)**.** *Let $(\mathbf{P}, \preceq)$ be a poset. We denote by $\Delta(\mathbf{P})$ the simplicial complex whose simplices are chains of $\mathbf{P}$.*

Suppose $\mathcal{S} = (S_j)_{j \in J}$ is a finite tuple of ℓ-connected closed semi-algebraic subsets of $\mathbb{R}^k$. Our goal is to define a poset $\mathbf{P}_m(\mathcal{S})$ such that:

Property 1.

$$(|\Delta(\mathbf{P}_m(\mathcal{S}))| \overset{ch}{\sim}_m \mathcal{S}^J$$

(see Definition 2.3.1). We will say that the poset $\mathbf{P}_m(\mathcal{S})$ satisfies Property 1 for the pair $(m, \mathcal{S})$.

Remark 6. *We use cohomological m-equivalence in Property 1. In the final construction we will lose a dimension while passing from cohomological equivalence to (homological or homotopical) equivalence because of the use of the universal coefficients theorem (see the proof of Claim 2.3.5 inside the proof of Theorem 2.3.1), and we will end up with*

$$(|\Delta(\mathbf{P}_m(\mathcal{S}))| \sim_{m-1} \mathcal{S}^J.$$

The main idea is to approximate homotopically the diagram of sets

$$(\mathcal{S}_I)_{I \subset J, \mathrm{card}(I) \leqslant m+2}$$

(see Notation 1), and the inclusion maps

$$\mathcal{S}_{I'} \hookrightarrow \mathcal{S}_I, I \subset I',$$

by a corresponding diagram of (the geometric realizations of the order complexes of) posets

$$(\mathbf{P}_{m-\mathrm{card}(I)+1,I})_{I \subset J, \mathrm{card}(I) \leqslant m+2}$$

(where the poset $\mathbf{P}_{m-\mathrm{card}(I)+1,I}$ corresponds to $\mathcal{S}_I$), and poset inclusions

$$\mathbf{P}_{m-\mathrm{card}(I')+1,I'} \hookrightarrow \mathbf{P}_{m-\mathrm{card}(I)+1,I}, I \subset I'.$$

The construction is by induction on m (we call this the global induction below).

1. (Base case of the global induction, $m = -1$.) Suppose $\mathcal{S} = (S_j)_{j \in J}$ is a finite tuple of ℓ-connected closed and bounded semi-algebraic subsets of $\mathbb{R}^k$. We define the poset $\mathbf{P}_{-1}(\mathcal{S})$ to be just the index set J, with no non-trivial order relations. It is depicted in Figure 2.3a. It is clear that $\mathbf{P}_{-1}(\mathcal{S})$ satisfies Property 1 for the pair $(-1, \mathcal{S})$.

2. (Induction hypothesis of the global induction.) We assume that for each m', $-1 \leqslant m' < m$, and each finite tuple $\mathcal{S} = (S_j)_{j \in J}$ of ℓ-connected closed and bounded semi-algebraic subsets of $\mathbb{R}^k$, we have defined a poset $\mathbf{P}_{m'}(\mathcal{S})$ satisfying Property 1 for the pair $(m', \mathcal{S})$.

3. (Inductive step of the global induction, going from $< m$ to m.) Using the inductive hypothesis, we now define a poset $\mathbf{P}_m(\mathcal{S})$ satisfying Property 1 for the pair $(m, \mathcal{S})$, for any tuple $\mathcal{S}$ of ℓ-connected closed and bounded semi-algebraic subsets of $\mathbb{R}^k$.

   Fix a finite tuple $\mathcal{S} = (S_j)_{j \in J}$ of ℓ-connected closed and bounded semi-algebraic subsets of $\mathbb{R}^k$. We will define $\mathbf{P}_m(\mathcal{S})$ below in steps. The poset $\mathbf{P}_m(\mathcal{S})$ as a set will be a disjoint union of the index set J, and certain subposets $\mathbf{P}_{m-\mathrm{card}(I)+1,I}$, where I where $I \subset J, 2 \leqslant$

$\text{card}(I) \leqslant m+2$. We define the subposets $\mathbf{P}_{m-\text{card}(I)+1,I}$ by downward induction (we call this the local induction below) on $\text{card}(I)$, starting from the base case, $\text{card}(I) = m+2$.

(a) (Base case of the local induction, $\text{card}(I) = m+2$.) We first consider the semi-algebraic sets $\mathcal{S}_I$, $\text{card}(I) = m+2$. Associated to each such I, we define a poset, which we denoted by $\mathbf{P}_{-1,I}$ as follows: Using Hypothesis 1 applied to the semi-algebraic set $\mathcal{S}_I$ we obtain a cover $(\mathcal{S}_{I,\alpha})_{\alpha \in \mathcal{C}(\mathcal{S}_I)}$ of $\mathcal{S}_I$ by closed and bounded ℓ-connected semi-algebraic sets. We define

$$\mathbf{P}_{-1,I} = \mathbf{P}_{-1}((\mathcal{S}_{I,\alpha})_{\alpha \in \mathcal{C}(\mathcal{S}_I)}) = \mathcal{C}(\mathcal{S}_I)$$

with no non-trivial order relation. It is depicted in Figure 2.3a. It is clear that $\mathbf{P}_{-1,I}$ satisfies Property 1 for the pair $(-1, (\mathcal{S}_{I,\alpha})_{\alpha \in \mathcal{C}(\mathcal{S}_I)})$.

(b) (Going from $m+2$ to $m+1$.) Next we consider subsets I of cardinality $m+1$. For each such subset we construct a poset $\mathbf{P}_{0,I}$ satisfying two conditions:

i. For each set I', with $\text{card}(I') = \text{card}(I) + 1$, and $I \subset I'$, the poset $\mathbf{P}_{-1,I'}$ already defined is isomorphic to a sub-poset of $\mathbf{P}_{0,I}$;

ii. $|\Delta(\mathbf{P}_{0,I})|$ is cohomologically 0-equivalent to $\mathcal{S}_I$.

We apply Hypothesis 1, to the semi-algebraic set $\mathcal{S}_I$ as input and obtain a cover $(\mathcal{S}_{I,\alpha})_{\alpha \in \mathcal{C}(\mathcal{S}_I)}$ of $\mathcal{S}_I$ by closed and bounded ℓ-connected semi-algebraic sets. We let

$$\mathbf{P}_{-1,I} = \mathbf{P}_{-1}((\mathcal{S}_{I,\alpha})_{\alpha \in \mathcal{C}(\mathcal{S}_I)}).$$

Let J_I be the union of the indexing set $\mathcal{C}(\mathcal{S}_I)$, with the posets $\mathbf{P}_{-1,I'}$ for each I' with $I \subset I', \text{card}(I') = \text{card}(I) + 1$. Notice that for each $\alpha \in J_I$, there is an ℓ-connected closed and bounded semi-algebraic set associated to it. Denote this set by $D(\alpha)$.

For every pair $\alpha, \beta \in J_I$, we again apply Hypothesis 1 to obtain a cover of $D(\alpha) \cap D(\beta)$ by ℓ-connected closed and bounded semi-algebraic sets, $(S_{I,\alpha,\beta,\gamma})_{\gamma \in I_{\alpha,\beta}}$ where $I_{\alpha,\beta} = \mathcal{C}(D(\alpha) \cap D(\beta))$. The poset $\mathbf{P}_{0,I}$ is defined to be the set $J_I \cup \bigcup_{\alpha,\beta \in J_I} I_{\alpha,\beta}$, and the non-trivial order relations are $\gamma \lneq \alpha, \beta$ for each $\gamma \in I_{\alpha,\beta}$. It is depicted in Figure 2.3b.

(c) (Local induction hypothesis.) We assume that we have already defined the posets $\mathbf{P}_{m-\mathrm{card}(I')+1,I'}$, with $\mathrm{card}(I') > \mathrm{card}(I)$.

(d) (Inductive step in general for the local induction.) We construct the poset $\mathbf{P}_{m-\mathrm{card}(I)+1,I}$ as follows. We apply Hypothesis 1 with the semi-algebraic set $\mathcal{S}_I$ as input and obtain a cover $(\mathcal{S}_{I,\alpha})_{\alpha\in\mathcal{C}(\mathcal{S}_I)}$ of $\mathcal{S}_I$ by closed and bounded ℓ-connected semi-algebraic sets. Let J_I be the union of the indexing set $\mathcal{C}(\mathcal{S}_I)$, with the posets $\mathbf{P}_{m-\mathrm{card}(I')+1,I'}$ for each I' with $I \subset I', \mathrm{card}(I') = \mathrm{card}(I) + 1$. Notice that for each $\alpha \in J_I$, there is an ℓ-connected closed and bounded semi-algebraic set associated to it. Denote this set by $D(\alpha)$.

We define the poset $\mathbf{P}_{m-\mathrm{card}(I)+1,I}$ using the global induction hypothesis. The global inductive hypothesis gives us that for any finite tuple of ℓ-connected closed and bounded semi-algebraic set (in particular, the tuple of sets $(D(\alpha))_{\alpha\in J_I}$) we have defined a poset $\mathbf{P}_{m-\mathrm{card}(I)+1}((D(\alpha))_{\alpha\in J_I})$, which satisfies Property 1 for the pair $(m - \mathrm{card}(I) + 1, (D(\alpha))_{\alpha\in J_I})$ (since $m - \mathrm{card}(I) + 1 < m$).

We define

$$\mathbf{P}_{m-\mathrm{card}(I)+1,I} = \mathbf{P}_{m-\mathrm{card}(I)+1}((D(\alpha))_{\alpha\in J_I}).$$

This finishes the local induction and we have defined $\mathbf{P}_{m-\mathrm{card}(I)+1,I}$, for each $I \subset J, 2 \leqslant \mathrm{card}(I) \leqslant m + 2$.

Finally, we define

$$\mathbf{P}_m(\mathcal{S}) = J \cup \bigcup_{I\subset J, 2\leqslant \mathrm{card}(I)\leqslant m+2} \mathbf{P}_{m-\mathrm{card}(I)+1,I}. \tag{2.1}$$

The partial order in the poset $\mathbf{P}_m(\mathcal{S})$ is specified as follows. By the local induction, each of the poset $\mathbf{P}_{m-\mathrm{card}(I)+1,I}$ comes with a partial order. We extend these orders as follows:

(a) For each $I \subset I' \subset J$, with $2 \leqslant \mathrm{card}(I) \leqslant \mathrm{card}(I') \leqslant m + 2$, there is a subposet of $\mathbf{P}_{m-\mathrm{card}(I)+1,I}$ canonically isomorphic to the poset $\mathbf{P}_{m-\mathrm{card}(I')+1,I'}$. For each element α of the former and the corresponding element α' of the latter we set $\alpha' \precneqq \alpha$.

(b) For each $j \in J$, and $\alpha \in \mathbf{P}_{m-\mathrm{card}(I)+1,I}, j \in I$, we set the element $\alpha \precneqq j$.

This ends the definition of the poset $\mathbf{P}_m(\mathcal{S})$ completing the global induction. Figure 2.3c depicts $\mathbf{P}_m(\mathcal{S})$ in terms of subposets $\mathbf{P}_{m-\mathrm{card}(I)+1,I}$. In Claim 2.3.11 we will show that the height of the poset $\mathbf{P}_m(\mathcal{S})$ is bounded by $2m+2$.

Notice that for any chain $\alpha_k \precnsim \alpha_{k-1} \precnsim \ldots \precnsim \alpha_0$ of elements in $\mathbf{P}_m(\mathcal{S})$, we have a sequence of inclusion maps of semi-algebraic sets $D(\alpha_k) \hookrightarrow D(\alpha_{k-1}) \hookrightarrow \ldots \hookrightarrow D(\alpha_0)$. It is depicted in Figure 2.4 for a hypothetical space with four elements in the initial covering.

The following two examples are illustrative.

Example 1. *Let $\ell = \infty, m \geqslant 2$, $\mathcal{S} = (S_1, S_2)$, where S_1, S_2 are the closed upper and lower hemispheres of the unit sphere in $\mathbb{R}^3$ (see Figure 2.5a).*

Using (2.1) we get

$$\mathbf{P}_m(\mathcal{S}) = \{1,2\} \cup \mathbf{P}_{m-2+1,\{1,2\}}. \tag{2.2}$$

Let $\mathcal{C}(\mathcal{S}_{\{1,2\}})$ be the cover of $\mathcal{S}_{\{1,2\}}$ by two closed semi-circles T_3, T_4, and let $\mathcal{T} = (T_3, T_4)$.

Note that $T_3 \cap T_4$ is a set containing two points W_5, W_6 (say), and the only possibility for $\mathcal{C}(T_3 \cap T_4)$, is the tuple $\mathcal{W} = (W_5, W_6)$. Then,

$$\mathbf{P}_{m-1}(\mathcal{T}) = \{3,4\} \cup \mathbf{P}_{m-2,\{3,4\}} \tag{2.3}$$

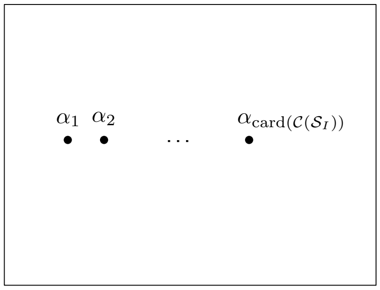
and the subposet $\mathbf{P}_{m-2,\{3,4\}}$ is isomorphic to the poset

$$\mathbf{P}_{m-2}(\mathcal{W}) = \{5,6\}. \tag{2.4}$$

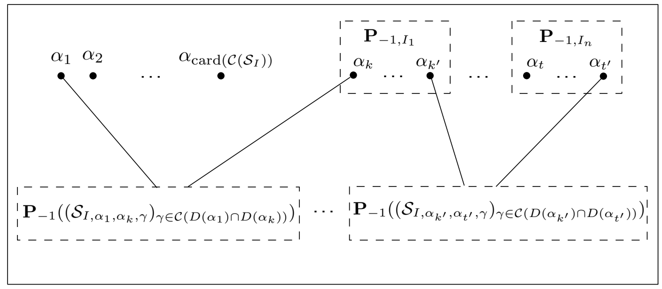
Substituting (2.4) into (2.3) and (2.3) into (2.2) we finally obtain that the Hasse diagram of the poset $\mathbf{P}_m(\mathcal{S})$ is

$$\begin{tikzcd}
1 & 2 \\
3 \arrow[u] \arrow[ur] & 4 \arrow[u] \arrow[ul] \\
5 \arrow[u] \arrow[ur] & 6 \arrow[u] \arrow[ul]
\end{tikzcd}$$

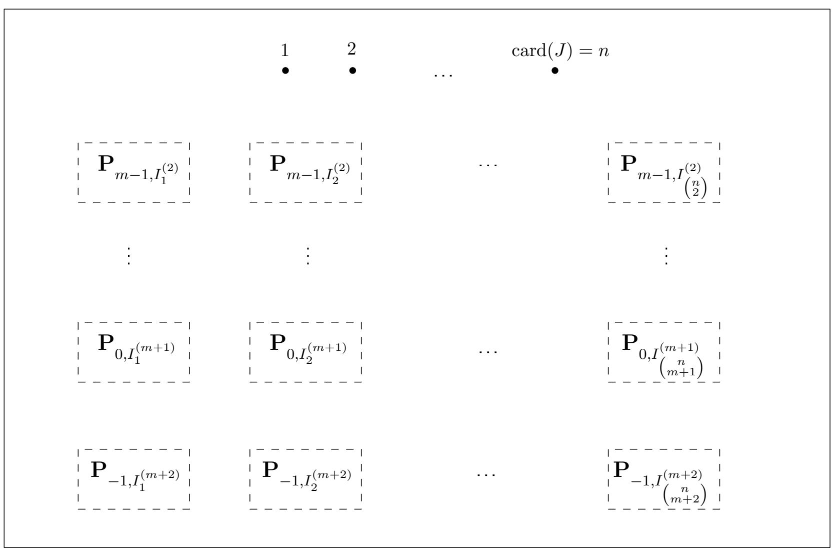
The order complex of this poset is homotopy equivalent (in fact, in this case is homeomorphic) to the sphere.



(a) $\mathbf{P}_{-1,I} = \mathbf{P}_{-1}((\mathcal{S}_{I,\alpha})_{\alpha \in \mathcal{C}(\mathcal{S}_I)})$: The elements of the poset, i.e. J_I, correspond to the elements of the cover $\mathcal{C}(\mathcal{S}_I)$, with no non-trivial order relation.



(b) $\mathbf{P}_{0,I}$: At the top level, the elements of $\mathbf{P}_{0,I}$ correspond to the cover $\mathcal{C}(\mathcal{S}_I)$ and elements of the posets $\mathbf{P}_{-1,I_i}$, where $\mathrm{card}(I_i) = \mathrm{card}(I) + 1$ and $I \subset I_i$. At the bottom level we have elements of the posets $\mathbf{P}_{-1}((\mathcal{S}_{I,\alpha_i,\alpha_j,\gamma})_{\gamma \in \mathcal{C}(D(\alpha_i) \cap D(\alpha_j))})$—shown as a box—for every pair α_i and α_j at the top level. The order relations are between the pairs and the elements of their corresponding posets at the bottom level.



(c) $\mathbf{P}_m(\mathcal{S}) = J \cup \bigcup_{I \subset J, 2 \leqslant \mathrm{card}(I) \leqslant m+2} \mathbf{P}_{m-\mathrm{card}(I)+1,I}$: The top level of the poset corresponds to the elements of J. Next, we have elements of the posets $\mathbf{P}_{m-1,I_i^{(2)}}$ where $I_i^{(2)} \subset J$ and $\mathrm{card}(I_i^{(2)}) = 2$—denoted by the superscript (2). Similarly at the lower levels, we have elements of the posets corresponded to subsets $I_i^{(m')} \subset J$ with $\mathrm{card}(I_i^{(m')}) = m'$ and $m' \leqslant m+2$. The partial order relations are defined between $j \in \{1, \ldots, n\}$ at the top level and the elements of $\mathbf{P}_{m-1,I_i^{(2)}}$, if $j \in I_i^{(2)}$. Furthermore, in addition to the order relations within each poset, if $I_j^{(m'-1)} \subset I_i^{(m')}$ then $\mathbf{P}_{m-m'+1,I_i^{(m')}} \hookrightarrow \mathbf{P}_{m-m'+2,I_j^{(m'-1)}}$, hence for each element α' of the $\mathbf{P}_{m-m'+1,I_i^{(m')}}$ and the corresponding element α of the $\mathbf{P}_{m-m'+2,I_j^{(m'-1)}}$ we set $\alpha' \lnsim \alpha$.

Figure 2.3. A simple illustration of the simplified view of the poset.

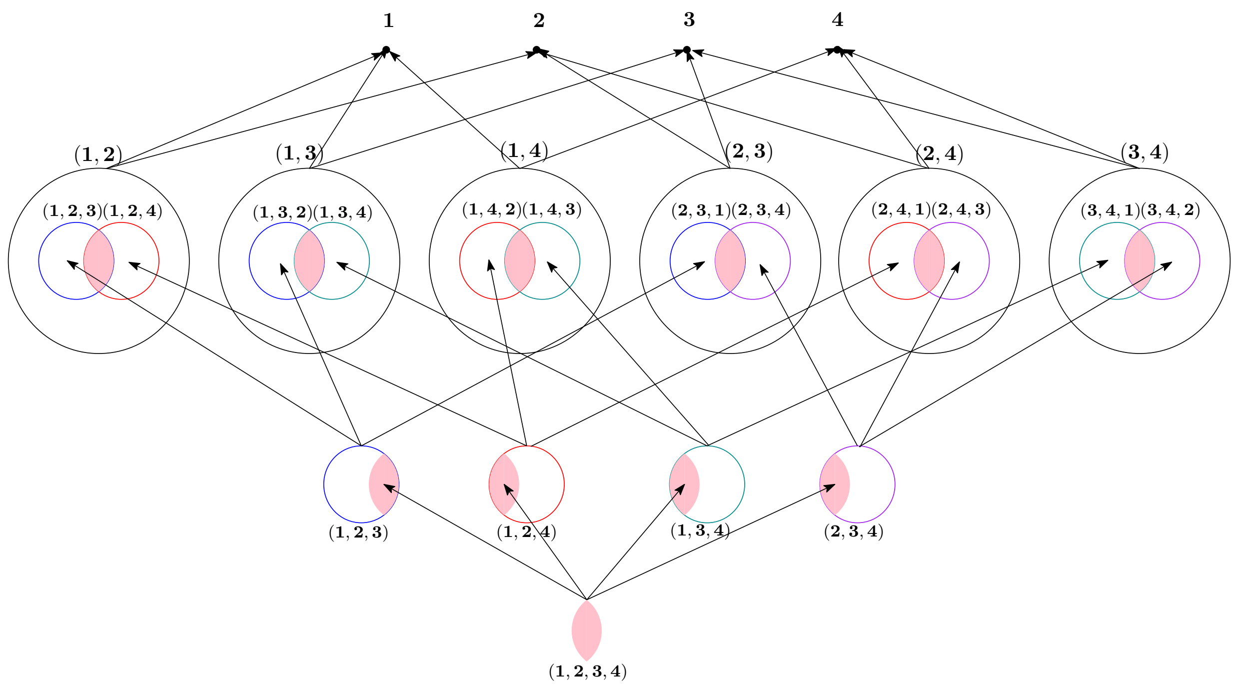


Figure 2.4. Poset $\mathbf{P}_m(\mathcal{S})$ such that $|\Delta(\mathbf{P}_m(\mathcal{S}))|$ is m-equivalent to $\bigcup_{j\in J} S_j$ with $m = 2,\ J = \{1,2,3,4\}$.

Example 2. *Now let $\ell = m = 2$, $\mathcal{S} = (S_1, S_2)$, where S_1, S_2 are the closed upper and lower hemispheres of the unit sphere in $\mathbb{R}^k$, $k > 5$. That is S_1 (resp. S_2) is the intersection of the unit sphere in R^k, with the set defined by $X_k \geqslant 0$ (resp. $X_k \leqslant 0$).*

Using (2.1) we get

$$\mathbf{P}_m(\mathcal{S}) = \{1,2\} \cup \mathbf{P}_{m-2+1,\{1,2\}}.$$

Let $\mathcal{C}(\mathcal{S}_{\{1,2\}})$ be the cover of $\mathcal{S}_{\{1,2\}}$ by two closed semi-spheres T_3, T_4, (i.e. T_3 (resp. T_4) is the intersection of $\mathcal{S}_{\{1,2\}}$ with $X_{k-1} \geqslant 0$ (resp. $X_{k-1} \leqslant 0$), and let $\mathcal{T} = (T_3, T_4)$.

Note that $W_5 = T_3 \cap T_4$ is a $(k-3)$-dimensional sphere, and since $k > 5$, W_5 is 2-connected and we can take $\mathcal{C}(W_5) = (W_5)$.

$$\mathbf{P}_1(\mathcal{T}) = \{3,4\} \cup \{5\}$$

with Hasse diagram

$$\begin{tikzcd}
3 & & 4 \\
& 5 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Finally we obtain that the Hasse diagram of the poset $\mathbf{P}_2(\mathcal{S})$ *is*

$$\begin{tikzcd}
1 & & 2 \\
3 \arrow[u] \arrow[urr] & & 4 \arrow[u] \arrow[ull] \\
& 5 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The order complex of this poset is contractible and is 2*-equivalent (but in this case not homotopy equivalent) to* $\mathbf{S}^{k-1}$ *for* $k > 5$*.*

With the definition of the poset $\mathbf{P}_m(\mathcal{S})$ it is possible to prove the following theorem. We do not include a proof of this theorem since it is subsumed by Theorem 2.3.1′.

Theorem. *With the same notation as in the Definition of* $\mathbf{P}_m(\mathcal{S})$ *defined above:*

$$|\Delta(\mathbf{P}_m(\mathcal{S}))| \sim_{m-1} \bigcup_{j\in J} S_j.$$

More generally, we have the diagrammatic homological $(m-1)$*-equivalence*

$$(J' \mapsto |\Delta(\mathbf{P}_m(\mathcal{S}|_{J'})|)_{J'\in 2^J} \overset{h}{\sim}_{m-1} \mathbf{Simp}^J(\mathcal{S}),$$

where $\mathcal{S}|_{J'} = (S_j)_{j\in J'}$*.*

We now return to the precise definition of the poset $\mathbf{P}_{m,i}(\Phi)$ that we are going to the use.

Precise definition of $\mathbf{P}_{m,i}(\Phi)$

We begin with a few useful notation that we will use in the construction.

Notation 9. *We will denote by* $\mathcal{F}_{\mathrm{R},k}$ *the set of quantifier-free formulas with coefficients in* R *and* k *variables, whose realizations are closed in* R^k*.*

We also use the following convenient notation.

Notation 10 (The relation $\subset_{\leqslant n}$)**.** *For any* $n \in \mathbb{Z}_{\geqslant 0}$*, and sets* A, B*, we will write* $A \subset_{\leqslant n} B$ *to mean* $A \subset B$ *and* $0 < \mathrm{card}(A) \leqslant n$*.*

We are now in a position to define a poset associated to a given finite tuple of formulas that will play the key technical role in the rest of the paper.

We first fix the following.

A. Let $\mathrm{R} = \mathrm{R}_0 \subset \mathrm{R}_1 \subset \mathrm{R}_2 \subset \cdots$ be a sequence of extensions of real closed fields.

B. We also fix two sequences of functions,

$$\mathcal{I}_{i,k} : \mathcal{F}_{\mathrm{R}_i,k} \rightarrow \mathbb{Z}_{\geqslant -1},$$

and

$$\mathcal{C}_{i,k} : \mathcal{F}_{\mathrm{R}_i,k} \rightarrow \bigcup_{p \geqslant 0} (\mathcal{F}_{\mathrm{R}_{i+1},k})^{[p]},$$

Remark 7. *The definition of the poset* $\mathbf{P}_{m,i}(\cdot)$ *given below does not depend on any specific properties of the functions* $\mathcal{I}_{i,k}(\cdot)$ *and* $\mathcal{C}_{i,k}(\cdot)$*. Later we will prove key topological properties of* $\mathbf{P}_{m,i}(\cdot)$ *(see Theorems 2.3.1 and 2.3.1′ below) under certain assumptions on* $\mathcal{I}_{i,k}(\cdot)$ *and* $\mathcal{C}_{i,k}(\cdot)$ *(see Properties 2 and 2′ below).*

For each $i \geqslant 0$, and $-1 \leqslant m \leqslant k$, a non-empty finite set J, and $\Phi \in (\mathcal{F}_{\mathrm{R}_i,k})^J$, we define a poset $(\mathbf{P}_{m,i}(\Phi), \prec)$.

We first need an auxilliary definition which will be used in the definition of the underlying set, $\mathbf{P}_{m,i}(\Phi)$, of the poset $(\mathbf{P}_{m,i}(\Phi), \prec)$.

Definition 2.3.2. *Let* J *be a non-empty finite set, and* $\Phi \in (\mathcal{F}_{\mathrm{R}_i,k})^J$*. We first define for each subset* $I \subset_{\leqslant m+2} J$*, a set* $J_{m,i,I,\Phi}$*, and an element* $\Phi_{m,i,I,J} \in (\mathcal{F}_{\mathrm{R}_{i+1},k})^{J_{m,i,I,\Phi}}$ *(using downward induction on* $\mathrm{card}(I)$*).*

Base case (card$(I) = m+2$)*: In this case we define,*

$$J_{m,i,I,\Phi} = \{I\} \times [\mathcal{I}_{i,k}(\bigwedge_{j\in I} \Phi(j))], \tag{2.5}$$

and for $(I,p) \in J_{m,i,I,\Phi}$,

$$\Phi_{m,i,I,J}((I,p)) = \mathcal{C}_{i,k}(\bigwedge_{j\in I} \Phi(j))(p).$$

Inductive step: Suppose we have defined $J_{m,i,I',\Phi}$ *and* $\Phi_{m,i,I',J}$ *for all* I' *with* $\mathrm{card}(I') = \mathrm{card}(I)+1$. *We define*

$$J_{m,i,I,\Phi} = \left(\{I\} \times [\mathcal{I}_{i,k}(\bigwedge_{j\in I} \Phi(j)]\right) \cup \bigcup_{I\subset I'\subset J, \mathrm{card}(I')=\mathrm{card}(I)+1} J_{m,i,I',\Phi}, \tag{2.6}$$

and

$$\begin{aligned} \Phi_{m,i,I,J}(\alpha) &= \mathcal{C}_{i,k}(\bigwedge_{j\in I} \Phi(j))(p), \text{ if } \alpha = (I,p) \in \{I\} \times [\mathcal{I}_{i,k}(\textstyle\bigwedge_{j\in I} \Phi(j))], \\ &= \Phi_{m,i,I',J}(\alpha), \text{ if } \alpha \in J_{m,i,I',\Phi} \text{ for some } I' \supset I, \text{ with} \\ &\qquad \mathrm{card}(I') = \mathrm{card}(I)+1. \end{aligned}$$

The following properties of $J_{m,i,I,\Phi}$ and $\Phi_{m,i,I,J}$ are obvious from the above definition. Using the same notation as in Definition 2.3.2:

Lemma 2.3.2. *(a)* $\mathrm{card}(J_{m,i,I,\Phi}) < \infty$ *for each* $I \subset_{\leqslant m+2} J$;

(b) For $I, I' \subset J$ *with* $\mathrm{card}(I \cup I') \leqslant m+2$,

$$J_{m,i,I\cup I',\Phi} \subset J_{m,i,I,\Phi} \cap J_{m,i,I',\Phi}.$$

(c) If $I' \subset I \subset_{\leqslant m+2} J \subset J'$, *then* $J_{m,i,I,\Phi} \subset J'_{m,i,I',\Phi}$, *and for* $\alpha \in J_{m,i,I,\Phi}$, $\Phi_{m,i,I,J}(\alpha) = \Phi_{m,i,I',J'}(\alpha)$.

Proof. Follows directly from Definition 2.3.2. ∎

We now define the set $\mathbf{P}_{m,i}(\Phi)$.

Definition 2.3.3 (The underlying set of the poset $(\mathbf{P}_{m,i}(\Phi), \prec)$)**.** *We define the set $\mathbf{P}_{m,i}(\Phi)$ using induction on m.*

Base case ($m = -1$): For each finite set J, and $\Phi \in (\mathcal{F}_{\mathrm{R}_i,k})^J$ we define

$$\mathbf{P}_{-1,i}(\Phi) = \bigcup_{j \in J} \{\{j\}\} \times \{\varnothing\}.$$

Inductive step: Suppose we have defined the sets $(\mathbf{P}_{m',i'}(\Phi'), \prec)$ for all m' with $-1 \leqslant m' < m$, $i' \geqslant 0$, for all non-empty finite sets J' and all $\Phi' \in (\mathcal{F}_{\mathrm{R}_{i'},k})^{J'}$.

We complete the inductive step by defining:

$$\mathbf{P}_{m,i}(\Phi) = \bigcup_{j \in J} \{\{j\}\} \times \{\varnothing\} \cup \bigcup_{I \subset J, 1 < \mathrm{card}(I) \leqslant m+2} \{I\} \times \mathbf{P}_{m-\mathrm{card}(I)+1,i+1}(\Phi_{m,i,I,J}). \tag{2.7}$$

We now specify the partial order on $\mathbf{P}_{m,i}(\Phi)$. For this it will be useful to have the following alternative characterization of the elements of the poset $\mathbf{P}_{m,i}(\Phi)$ as *tuples of sets*. This characterization follows simply by unravelling the inductive definition of the set $\mathbf{P}_{m,i}(\Phi)$ given above.

Characterization of the elements of the poset $\mathbf{P}_{m,i}(\Phi)$ as tuples of sets

The elements of $\mathbf{P}_{m,i}(\Phi)$ are all finite tuples of sets (of varying lengths)

$$(I_0, \ldots, I_r, \varnothing),$$

satisfying the following conditions.

1. I_0 is a subset of $J_0 = J$, $\mathrm{card}(I_0) = 1$ if $r = 0$, and $2 \leqslant \mathrm{card}(I_0) \leqslant m + 2$ otherwise.

2. I_1 is a subset of $J_1 = (J_0)_{m_0,i_0,I_0,\Phi_0}$ (see Eqn. (2.6), Definition 2.3.3) with

$$
\begin{aligned}
m_0 &= m, \\
i_0 &= i, \\
\Phi_0 &= \Phi,
\end{aligned}
$$

and

$$2 \leqslant \mathrm{card}(I_1) \leqslant (m_0 - \mathrm{card}(I_0) + 1) + 2.$$

3. I_2 is a subset of $J_2 = (J_1)_{m_1,i_1,I_1,\Phi_1}$, where

$$
\begin{aligned}
m_1 &= m_0 - \mathrm{card}(I_0) + 1, \\
i_1 &= i_0 + 1, \\
\Phi_1 &= (\Phi_0)_{m_0,i_0,I_0,J_0},
\end{aligned}
$$

and

$$2 \leqslant \mathrm{card}(I_2) \leqslant (m_1 - \mathrm{card}(I_1) + 1) + 2.$$

4. Continuing in the above fashion,

$$I_{r-1} \subset J_{r-1} = (J_{r-2})_{m_{r-2},i_{r-2},I_{r-2},\Phi_{r-2}}, \tag{2.8}$$

where

$$
\begin{aligned}
m_{r-2} &= m_{r-3} - \mathrm{card}(I_{r-3}) + 1, \\
i_{r-2} &= i_{r-3} + 1, \\
\Phi_{r-2} &= (\Phi_{r-3})_{m_{r-3},i_{r-3},I_{r-3},J_{r-3}},
\end{aligned}
$$

and

$$2 \leqslant \mathrm{card}(I_{r-1}) \leqslant m_{r-2} + 2 = (m + r - 1 - \sum_{j=0}^{r-2} \mathrm{card}(I_j)) + 2. \tag{2.9}$$

5. Finally,

$$I_r \subset J_r = (J_{r-1})_{m_{r-1},i_{r-1},I_{r-1},\Phi_{r-1}},$$

where

$$\Phi_{r-1} = (\Phi_{r-2})_{m_{r-2},i_{r-2},I_{r-2},J_{r-2}},$$

and

$$\mathrm{card}(I_r) = 1.$$

(We show later (see Claim 2.3.8) that for tuples $(I_0, \ldots, I_r, \emptyset)$ satisfying the above conditions, $r \leqslant m+1$.)

Definition 2.3.4 (Partial order on $\mathbf{P}_{m,i}(\Phi)$)**.** *The partial order $\prec$ on $\mathbf{P}_{m,i}(\Phi)$ is defined as follows.*

For $\alpha = (I_0^\alpha, \ldots, I_{r_\alpha}^\alpha, \emptyset), \beta = (I_0^\beta, \ldots, I_{r_\beta}^\beta, \emptyset) \in \mathbf{P}_{m,i}(\Phi)$,

$$\beta \prec \alpha \Leftrightarrow (r_\alpha \leqslant r_\beta) \text{ and } I_j^\alpha \subset I_j^\beta, 0 \leqslant j \leqslant r_\alpha. \tag{2.10}$$

2.3.4 Main properties of the poset $\mathbf{P}_{m,i}(\Phi)$

We will now state and prove the important properties of the poset $\mathbf{P}_{m,i}(\Phi)$ that motivates its definition.

Lemma 2.3.3. *For each $J' \subset J'' \subset J$, and $-1 \leqslant m' \leqslant m'' \leqslant m$, we have a poset inclusion,*

$$\mathbf{P}_{m',i}(\Phi|_{J'}) \hookrightarrow \mathbf{P}_{m'',i}(\Phi|_{J''}).$$

Proof. Follows from Definition 2.3.3 and Part (c) of Lemma 2.3.2. □

We now state a lemma which will be useful later, that states a key property of the partial order relation in $\mathbf{P}_{m,i}(\Phi)$. Using the same notation as in Definition 2.3.3:

Lemma 2.3.4. *Suppose that $I' \subset I \subset J$.*

(a) The poset $\mathbf{P}_{m-\mathrm{card}(I)+1,i+1}(\Phi_{m,i,I,J})$ is a subposet of $\mathbf{P}_{m-\mathrm{card}(I')+1,i+1}(\Phi_{m,i,I',J})$.

(b) For each $\alpha, \alpha' \in \mathbf{P}_{m-\mathrm{card}(I)+1,i+1}(\Phi_{m,i,I,J})$,

$$\alpha \prec_{\mathbf{P}_{m-\mathrm{card}(I)+1,i+1}(\Phi_{m,i,I,J})} \alpha' \Leftrightarrow (I, \alpha) \prec_{\mathbf{P}_{m,i}(\Phi)} (I', \alpha').$$

Proof. Part (a) follows from the fact that $J_{m,i,I,\Phi} \subset J_{m,i,I',\Phi}$, $m-\mathrm{card}(I)+1 \leqslant m-\mathrm{card}(I')+1$, and Lemma 2.3.3.

Part (b) follows immediately from the definition of the partial order on $\mathbf{P}_{m,i}(\Phi)$ (see Definition 2.3.4). □

Let R be a real closed field and $R \in \mathrm{R}, R > 0$. We say that the tuple

$$((\mathrm{R}_i)_{i \geqslant 0}, R, k, (\mathcal{I}_{i,k})_{i \geqslant 0}, (\mathcal{C}_{i,k})_{i \geqslant 0})$$

satisfies the *homological ℓ-connectivity property over* R if it satisfies the following conditions.

Property 2. *1. For each $i \geqslant 0$, $\mathrm{R}_i = \mathrm{R}\langle \bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_i \rangle$ where for $j = 1, \ldots, i$, $\bar{\varepsilon}_j$ denotes the sequence $\varepsilon_{j,1}, \varepsilon_{j,2}, \ldots$.*

2. For each $\phi \in \mathcal{F}_{\mathrm{R}_i,k}$:

(a) If $\mathcal{R}(\phi, \overline{B_k(0,R)})$ is empty then, $\mathcal{I}_{i,k}(\phi) = -1$.

(b)

$$\left(\bigcup_{j \in [\mathcal{I}_{i,k}(\phi)]} \mathcal{R}(\mathcal{C}_{i,k}(\phi)(j), \overline{B_k(0,R)})) \right) \searrow \left(\mathcal{R}(\phi, \overline{B_k(0,R)}) \right)$$

(see Notation 7). Notice that in the case $\mathcal{R}(\phi, \overline{B_k(0,R)})$ is empty, $\mathcal{I}_{i,k}(\phi) = -1$, hence $[\mathcal{I}_{i,k}(\phi)] = \emptyset$, and so $\bigcup_{j \in [\mathcal{I}_{i,k}(\phi)]} \mathcal{R}(\mathcal{C}_{i,k}(\phi)(j), \overline{B_k(0,R)})$ is an empty union, and is thus empty as well.

(c) For $j \in [\mathcal{I}_{i,k}(\phi)]$, $\mathcal{R}(\mathcal{C}_{i,k}(\phi)(j), \overline{B_k(0,R)}))$ is homologically ℓ-connected.

Notation 11. *Let ϕ be a quantifier-free formula with coefficients in $\mathrm{R}[\bar{\varepsilon}]$. Then ϕ is defined over $\mathrm{R}[\bar{\varepsilon}'_1, \bar{\varepsilon}'_2, \ldots, \bar{\varepsilon}'_i]$ where $\bar{\varepsilon}'_j$ is a finite sub-sequence of the sequence $\bar{\varepsilon}_j$. For $\bar{t} = (\bar{t}_1, \ldots, \bar{t}_i)$, where for $1 \leqslant j \leqslant i$, $\bar{t}_j$ is a tuple of elements of $\mathbb{R}$ of the same length as $\bar{\varepsilon}'_j$, we will denote by $\phi_{\bar{t}}$ the formula defined over $\mathbb{R}$ obtained by replacing $\bar{\varepsilon}'_j$ by $\bar{t}_j$ in the formula ϕ.*

For any finite sequence $\bar{t} = (t_1, \ldots, t_N)$, by the phrase "for all sufficiently small and positive $\bar{t}$" we will mean " for all sufficiently small $t_1 \in \mathbb{R}_{>0}$, and having chosen t_1, for all sufficiently small $t_2 \in \mathbb{R}_{>0}$, ... ".

We will say that

$$((\mathrm{R}_i)_{i\geqslant 0}, R, k, (\mathcal{I}_{i,k})_{i\geqslant 0}, (\mathcal{C}_{i,k})_{\geqslant 0})$$

satisfies the *ℓ-connectivity property over* $\mathrm{R} = \mathbb{R}$ if it satisfies the following conditions.

Property 2′. *1.* $\mathrm{R}_0 = \mathbb{R}$ *and for each* , $i > 0$, $\mathrm{R}_i = \mathrm{R}\langle\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_i\rangle$.

2. For each $\phi \in \mathcal{F}_{\mathrm{R}_i,k}$*:*

(a) If $\mathcal{R}(\phi, \overline{B_k(0,R)})$ *is empty then,* $\mathcal{I}_{i,k}(\phi) = -1$.

(b)

$$\left(\bigcup_{j\in[\mathcal{I}_{i,k}(\phi)]} \mathcal{R}(\mathcal{C}_{i,k}(\phi)(j), \overline{B_k(0,R)}))\right) \searrow \left(\mathcal{R}(\phi, \overline{B_k(0,R)})\right)$$

(c) For $j \in [\mathcal{I}_{i,k}(\phi)]$, *and all sufficiently small and positive* $\bar{t}$,

$$\mathcal{R}(\mathcal{C}_{i,k}(\phi)(j)_{\bar{t}}, \overline{B_k(0,R)}))$$

is ℓ-connected.

The following two theorems give the important topological properties of the posets defined above that will be useful for us.

Theorem 2.3.1. *Suppose that the tuple*

$$((\mathrm{R}_i)_{i\geqslant 0}, R, k, (\mathcal{I}_{i,k})_{i\geqslant 0}, (\mathcal{C}_{i,k})_{i\geqslant 0})$$

satisfies the homological ℓ-connectivity property over R *(see Property 2). Then, for* $-1 \leqslant m \leqslant \ell$, *every finite set* J, *and* $\Phi \in (\mathcal{F}_{k,\mathrm{R}_i})^J$, *such that for each* $j \in J$, $\mathcal{R}(\Phi(j), \overline{B_k(0,R)})$ *is homologically ℓ-connected,*

$$|\Delta(\mathbf{P}_{m,i}(\Phi))| \overset{h}{\sim}_{m-1} \mathcal{R}(\Phi, \overline{B_k(0,R)})^J. \tag{2.11}$$

More generally, we have the diagrammatic homological $(m-1)$*-equivalence*

$$(J' \mapsto |\Delta(\mathbf{P}_{m,i}(\Phi|_{J'})|)_{J' \in 2^J} \overset{h}{\sim}_{m-1} \mathbf{Simp}^J(\mathcal{R}(\Phi, \overline{B_k(0,R)})). \tag{2.12}$$

In the case $\mathrm{R} = \mathbb{R}$ we can derive a stronger conclusion from a stronger assumption.

Theorem 2.3.1′. *Suppose that*

$$((\mathrm{R}_i)_{i \geqslant 0}, R, k, (\mathcal{I}_{i,k})_{i \geqslant 0}, (\mathcal{C}_{i,k})_{i \geqslant 0})$$

satisfies the ℓ*-connectivity property over* $\mathrm{R} = \mathbb{R}$ *(cf. Property 2′).*

Then, for $-1 \leqslant m \leqslant \ell$*, each finite set* J*, and* $\Phi \in (\mathcal{F}_{\mathrm{R},k})^J$*, such that for each* $j \in J$*,* $\mathcal{R}(\Phi(j), \overline{B_k(0,R)})$ *is* ℓ*-connected,*

$$|\Delta(\mathbf{P}_{m,i}(\Phi))| \sim_{m-1} \mathcal{R}(\Phi, \overline{B_k(0,R)})^J.$$

More generally, we have the diagrammatic $(m-1)$*-equivalence:*

$$(J' \mapsto |\Delta(\mathbf{P}_{m,i}(\Phi|_{J'})|)_{J' \in 2^J} \sim_{m-1} \mathbf{Simp}^J(\mathcal{R}(\Phi, \overline{B_k(0,R)})). \tag{2.13}$$

Before proving Theorems 2.3.1 and 2.3.1′ we discuss an example.

2.3.5 Example of the sphere $\mathbf{S}^2$ in R^3

In order to illustrate the main ideas behind the definition of the poset, $\mathbf{P}_{m,i}(\Phi)$, defined above we discuss a very simple example. Starting from a cover of the two dimensional unit sphere in $\mathbb{R}^3$ by two closed hemispheres, we show how we construct the associated poset. We will assume that there is an algorithm available as a black-box which given any closed formula ϕ such that $\mathcal{R}(\phi)$ is bounded, produces a tuple of quantifier-free closed formulas as output, such that

(a) the realization of each formula in the tuple is contractible;

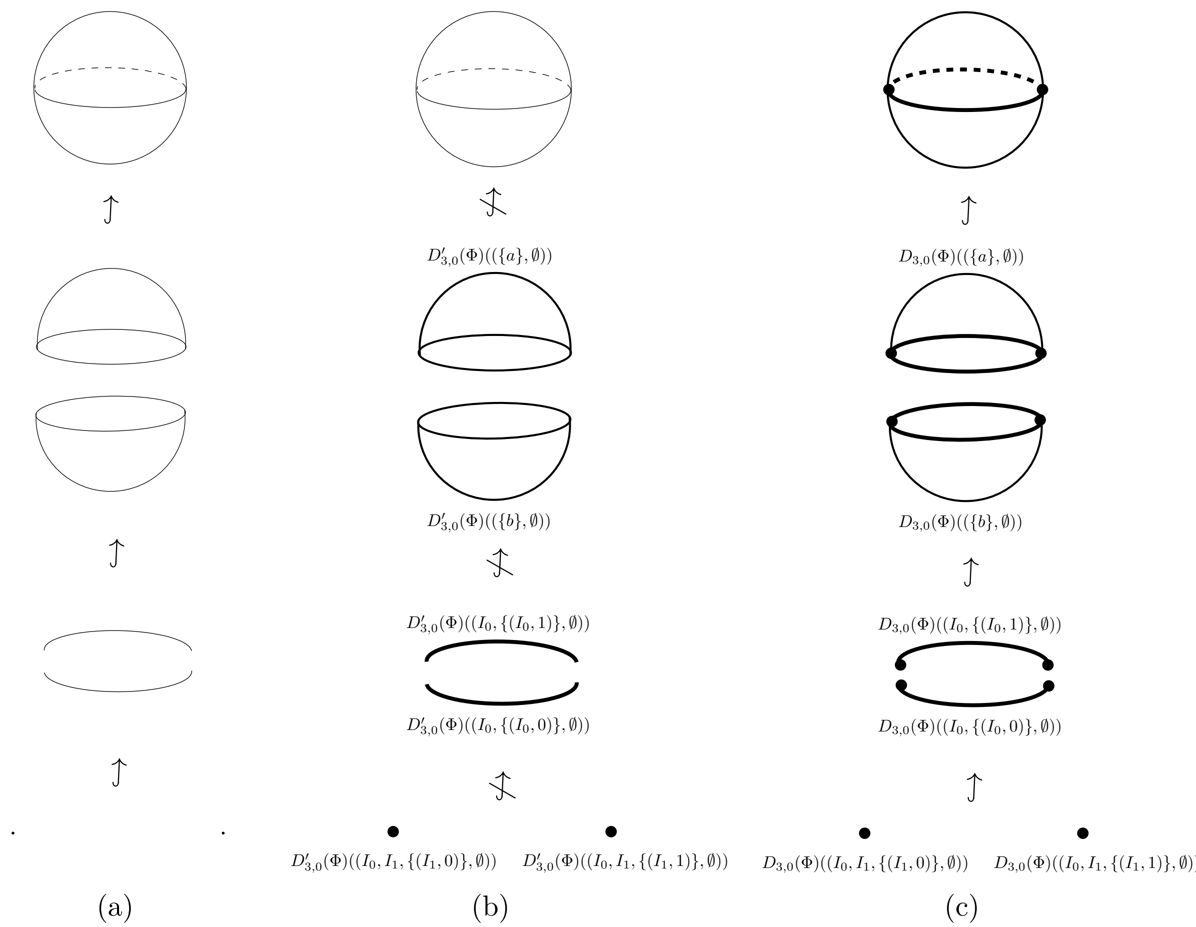


Figure 2.5. (a) The ideal situation, (b) $D'_{m,i}(\Phi)(.)$, and (c) $D_{m,i}(\Phi)(.)$

(b) the union of the realizations is a semi-algebraic set infinitesimally larger than $\mathcal{R}(\phi)$, and such that $\mathcal{R}(\phi)$ is a semi-algebraic deformation retract of the union.

Therefore, at each step of our construction the cover by contractible sets that we consider, is actually a cover of a semi-algebraic set which is infinitesimally larger than that but with the same homotopy type as the original set. As a result, the *inclusion property* – namely, that each element of the cover is included in the set that it is part of a cover of – which is expected from the elements of a cover will not hold.

We first describe the situation in the case when Part (b) above is replaced with:

(b′) the union of the realizations is equal to $\mathcal{R}(\phi)$.

We call this the *ideal situation*. Figure 2.5a displays three levels of the construction in the ideal situation for the sphere. In the first step, we have two closed contractible hemispheres that cover the whole sphere. The intersection of the two hemispheres is a circle, and the next level shows the two closed semi-circles as its cover. The bottom level consists of two points which is the intersection of these semi-circles. Clearly, the inclusion property holds in this case.

Unfortunately, as mentioned before we cannot assume that we are in the ideal situation. This is because the only algorithm with a singly exponential complexity that is currently known for computing covers by contractible sets, satisfies Property (b) rather than the ideal Property (b′). In the non-ideal situation we will obtain in the first step a cover of an infinitesimally thickened sphere by two thickened hemispheres where the thickening is in terms of some infinitesimal $\varepsilon_0, 0 < \varepsilon_0 \ll 1$. The intersection of these two thickened hemispheres is a thickened circle, and which is covered by two thickened semi-circles whose union is infinitesimally larger than the thickened circle. The new infinitesimal is ε_1 and $0 < \varepsilon_1 \ll \varepsilon_0 \ll 1$. Finally, in the next level, the intersection of the two thickened semi-circles is covered by two thickened points involving a third infinitesimal ε_2, such that $0 < \varepsilon_2 \ll \varepsilon_1 \ll \varepsilon_0 \ll 1$.

We associate to each element $\alpha \in \mathbf{P}_{m,i}(\Phi)$ two semi-algebraic sets $D_{m,i}(\Phi)(\alpha)$, $D'_{m,i}(\Phi)(\alpha)$. The association $D_{m,i}(\Phi)(\cdot)$ is functorial in the sense that if $\alpha, \beta \in \mathbf{P}_{m,i}(\Phi)$, then $\alpha \prec \beta \Leftrightarrow D_{m,i}(\Phi)(\alpha) \subset D_{m,i}(\Phi)(\beta)$. This functoriality is important since it allows us to define the homotopy colimit of the functor $D_{m,i}(\Phi)$. The association $\alpha \mapsto D'_{m,i}(\Phi)(\alpha)$ does not have the functorial property. However, it follows directly from its definition that $D'_{m,i}(\Phi)$ is contractible (or ℓ-connected in the more general setting). Finally, we are able to show that $D'_{m,i}(\Phi)(\alpha)$ is homotopy equivalent to $D_{m,i}(\Phi)(\alpha)$ for each $\alpha \in \mathbf{P}_{m,i}(\Phi)$, and thus the functor $D_{m,i}(\Phi)$ has the advantage of being functorial as well as satisfying the connectivity property.

In this example, we display $D'_{m,i}(\Phi)(\alpha)$ and $D_{m,i}(\Phi)(\alpha)$ for all different $\alpha \in \mathbf{P}_{m,i}(\Phi)$ in Figures 2.5b and 2.5c.

For the rest of this example we assume the covers of sphere are in the ideal situation. This assumption will not change the poset $\mathbf{P}_{m,i}(\Phi)$ that we construct.

In order to reconcile with the notation used in the definition of the poset $\mathbf{P}_{m,i}(\Phi)$, we will assume that the different covers described above (which are not Leray but ∞-connected) correspond to the values of the maps $\mathcal{I}_{i,3}$ and $\mathcal{C}_{i,3}$ evaluated at the corresponding formulas which we describe more precisely below.

Step 1. Let a, b denote the closed upper and lower hemispheres of the sphere $\mathbf{S}^2(0,1) \subset \mathbb{R}^3$, defined by formulas

$$\begin{aligned}\phi_a &:= (X_1^2 + X_2^2 + X_3^2 - 1 = 0) \wedge (X_3 \geqslant 0),\\ \phi_b &:= (X_1^2 + X_2^2 + X_3^2 - 1 = 0) \wedge (X_3 \leqslant 0).\end{aligned}$$

Let $J = J_0 = \{a, b\}$, and $\Phi \in \mathcal{F}^J_{\mathbb{R},3}$ be defined by $\Phi(a) = \phi_a, \Phi(b) = \phi_b$. Moreover, since $\mathrm{card}(J) = 2$,

$$\mathbf{P}_{3,0}(\Phi) = \{(\{a\}, \varnothing), (\{b\}, \varnothing)\} \cup \bigcup_{I_0 \subset J, \mathrm{card}(I_0)=2} \{I_0\} \times \mathbf{P}_{2,1}(\Phi_{3,0,I_0,J_0}).$$

Following the notation used in Definition 2.3.3, let $I_0 = J_0 = J = \{a, b\}$.

Step 2. We suppose that $\mathcal{I}_{0,3}(\phi_a \wedge \phi_b) = 1$, and $\mathcal{C}_{0,3}(\phi_a \wedge \phi_b)(0) = \phi_c$, $\mathcal{C}_{0,3}(\phi_a \wedge \phi_b)(1) = \phi_d$, where

$$\begin{aligned}\phi_c &:= (X_1^2 + X_2^2 + X_3^2 - 1 = 0) \wedge (X_3 = 0) \wedge (X_2 \geqslant 0),\\ \phi_d &:= (X_1^2 + X_2^2 + X_3^2 - 1 = 0) \wedge (X_3 = 0) \wedge (X_2 \leqslant 0),\end{aligned}$$

denote the two semi-circles.

$$J_1 = J_{3,0,I_0,\Phi} = \{I_0\} \times [1] = \{(I_0, 0), (I_0, 1)\},$$

$$\Phi_1 = \Phi_{3,0,I_0,J_0},$$

$$\Phi_1((I_0, 0)) = \phi_c,$$

$$\Phi_1((I_0, 1)) = \phi_d.$$

$$\mathbf{P}_{2,1}(\Phi_1) = \{(\{(I_0, 0)\}, \emptyset), (\{(I_0, 1)\}, \emptyset)\} \cup \bigcup_{I_1 \subset J_1, \mathrm{card}(I_1)=2} \{I_1\} \times \mathbf{P}_{1,2}((\Phi_1)_{2,1,I_1,J_1}).$$

Now let $I_1 = J_1$.

Step 3. Suppose that $\mathcal{I}_{1,3}(\phi_c \wedge \phi_d) = 1$, and $\mathcal{C}_{1,3}(\phi_c \wedge \phi_d)(0) = \phi_e$, $\mathcal{C}_{1,3}(\phi_c \wedge \phi_d)(1) = \phi_f$, where

$$\begin{aligned}
\phi_e &:= (X_1^2 + X_2^2 + X_3^2 - 1 = 0) \wedge (X_3 = 0) \wedge (X_2 = 0) \wedge (X_1 + 1 = 0),\\
\phi_f &:= (X_1^2 + X_2^2 + X_3^2 - 1 = 0) \wedge (X_3 = 0) \wedge (X_2 = 0) \wedge (X_1 - 1 = 0).
\end{aligned}$$

$$J_2 = (J_1)_{2,1,I_1,\Phi_1} = \{I_1\} \times [1] = \{(I_1, 0), (I_1, 1)\},$$

$$\Phi_2 = (\Phi_1)_{2,1,I_1,J_1}$$

$$\Phi_2((I_1, 0)) = \phi_e,$$

$$\Phi_2((I_1, 1)) = \phi_f.$$

$$\mathbf{P}_{1,2}(\Phi_2) = \{(\{(I_1, 0)\}, \varnothing), (\{(I_1, 1)\}, \varnothing)\} \cup \bigcup_{I_2 \subset J_2, \mathrm{card}(I_2)=2} \{I_2\} \times \mathbf{P}_{0,3}((\Phi_2)_{1,2,I_2,J_2}).$$

Let $I_2 = J_2$.

Step 4. Since $\mathcal{I}_{2,3}(\phi_e \wedge \phi_f) = -1$, hence $\mathbf{P}_{0,3}((\Phi_2)_{1,2,I_2,J_2}) = \varnothing$, and from Step 3

$$\mathbf{P}_{1,2}(\Phi_2) = \{(\{(I_1, 0)\}, \varnothing), (\{(I_1, 1)\}, \varnothing)\}.$$

Step 5. With these choices of the values of $\mathcal{I}_{\cdot,3}$ and $\mathcal{C}_{\cdot,3}$ for the specific formulas described above, and $\ell = \infty$, from Step 2 and Step 4, the Hasse diagram of the poset $\mathbf{P}_{2,1}(\Phi_1)$ is as follows.

$$\begin{tikzcd}
(\{(I_0, 0)\}, \varnothing) & (\{(I_0, 1)\}, \varnothing) \\
(I_1, \{(I_1, 0)\}, \varnothing) \arrow[u] \arrow[ur] & (I_1, \{(I_1, 1)\}, \varnothing) \arrow[u] \arrow[ul]
\end{tikzcd}$$

Step 6. Finally, from Step 1 and Step 5, the Hasse diagram of the poset $\mathbf{P}_{3,0}(\Phi)$ is shown below.

$$\begin{tikzcd}
(\{a\}, \varnothing) & (\{b\}, \varnothing) \\
(I_0, \{(I_0, 0)\}, \varnothing) \arrow[u] \arrow[ur] & (I_0, \{(I_0, 1)\}, \varnothing) \arrow[u] \arrow[ul] \\
(I_0, I_1, \{(I_1, 0)\}, \varnothing) \arrow[u] \arrow[ur] & (I_0, I_1, \{(I_1, 1)\}, \varnothing) \arrow[u] \arrow[ul]
\end{tikzcd}$$

The order complex, $\Delta(\mathbf{P}_{3,0}(\Phi))$ is displayed below and clearly $|\Delta(\mathbf{P}_{3,0}(\Phi))|$ is homeomorphic to $\mathbf{S}^2(0,1)$.

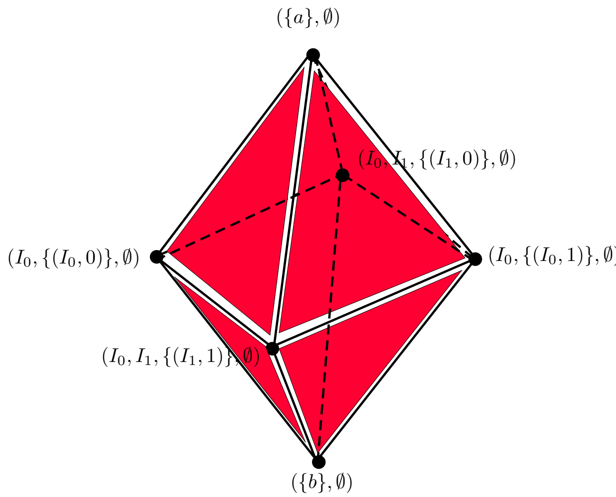


Figure 2.6. The order complex, $\Delta(\mathbf{P}_{3,0}(\Phi))$

2.3.6 Proofs of Theorems 2.3.1 and 2.3.1′

In this section we prove Theorem 2.3.1 as well as Theorem 2.3.1′. We first give an outline of the proof of Theorem 2.3.1.

Outline of the proof of Theorem 2.3.1

In order to prove that $|\Delta(\mathbf{P}_{m,i}(\Phi))|$ is homologically $(m-1)$-equivalent to $\mathcal{R}(\Phi)^J$, we give two homological $(m-1)$-equivalences. The source of both these maps is a semi-algebraic set which is defined as the homotopy colimit of a certain functor $D_{m,i}$ from the poset category $\mathbf{P}_{m,i}(\Phi)$ to **Top** taking its values in semi-algebraic subsets of R^k_{i+m+1}. The targets are $|\Delta(\mathbf{P}_{m,i}(\Phi))|$ and $\mathcal{R}(\Phi)^J$. Taken together these two homological $(m-1)$-equivalences imply that $|\Delta(\mathbf{P}_{m,i}(\Phi))|$ and $\mathcal{R}(\Phi)^J$ are homologically $(m-1)$-equivalent.

In what follows, we first define the functor $D_{m,i}$ as well as an associated map $D'_{m,i}$, also taking values in semi-algebraic sets, and prove the main properties of these objects that we are going to need in the proof of Theorem 2.3.1.

Definition of $D_{m,i}, D'_{m,i}$

We now define for each $\alpha = (I_0, \ldots, I_r, \emptyset) \in \mathbf{P}_{m,i}(\Phi)$, a closed semi-algebraic subset $D_{m,i}(\alpha) \subset \overline{B_k(0,R)} \subset \mathrm{R}^k_{i+m+1}$, and also a semi-algebraic set $D'_{m,i}(\alpha) \subset \mathrm{R}^k_{i+r}$.

We define $D_{m,i}, D'_{m,i}$ by induction on m. For $m = -1$, we define for $j \in J$,

$$D_{-1,i}(\Phi)((\{j\}, \emptyset)) = D'_{-1,i}(\Phi)((\{j\}, \emptyset)) = \mathcal{R}(\Phi(j), \overline{B_k(0,R)}) \subset \mathrm{R}^k_i.$$

We now define $D_{m,i}(\Phi), D'_{m,i}(\Phi) : \mathbf{P}_{m,i}(\Phi) \to \mathbf{Top}$, using the fact that they are already defined for all $-1 \leqslant m' < m$. We define:

$$\begin{aligned}
D_{m,i}(\Phi)((\{j\}, \emptyset)) &= \mathrm{ext}(\mathcal{R}(\Phi(j), \overline{B_k(0,R)}), \mathrm{R}_{i+m+1}) \cup \\
&\quad \bigcup_{(I,\alpha) \in \mathbf{P}_{m,i}(\Phi), j \in I} \mathrm{ext}(D_{m-\mathrm{card}(I)+1,i+1}(\Phi_{m,i,I,J})(\alpha), \mathrm{R}_{i+m+1}), \\
D_{m,i}(\Phi)((I,\alpha)) &= \mathrm{ext}(D_{m-\mathrm{card}(I)+1,i+1}(\Phi_{m,i,I,J})(\alpha), \mathrm{R}_{i+m+1}), \\
&\quad I \subset_{\leqslant m+2} J, \mathrm{card}(I) > 1,\ \alpha \in \mathbf{P}_{m-\mathrm{card}(I)+1,i+1}(\Phi_{m,i,I,J}),
\end{aligned}$$

$$D'_{m,i}(\Phi)((\{j\}, \emptyset)) = \mathcal{R}(\Phi(j), \overline{B_k(0,R)}), \tag{2.14}$$

and

$$D'_{m,i}(\Phi)((I,\alpha)) = D'_{m-\mathrm{card}(I)+1,i+1}(\Phi_{m,i,I,J})(\alpha),$$

for $I \subset_{\leqslant m+2} J, \mathrm{card}(I) > 1$, $\alpha \in \mathbf{P}_{m-\mathrm{card}(I)+1,i+1}(\Phi_{m,i,I,J})$.

The following lemma is obvious from the definition of $D_{m,i}(\alpha)$ given above.

Lemma 2.3.5. *For each $\alpha, \beta \in \mathbf{P}_{m,i}(\Phi)$ with $\alpha \prec \beta$, the morphism $D_{m,i}(\Phi)(\alpha \prec \beta)$: $D_{m,i}(\Phi)(\alpha) \to D_{m,i}(\Phi)(\beta)$ is an inclusion. So, $D_{m,i}(\Phi)$ is a functor from the poset category $(\mathbf{P}_{m,i}(\Phi), \prec)$ to* **Top**.

Remark 8. *Unlike $D_{m,i}$, $D'_{m,i}$ is not necessarily a functor.*

Lemma 2.3.6. *For each $\alpha \in \mathbf{P}_{m,i}(\Phi)$,*

$$D_{m,i}(\Phi)(\alpha) \searrow D'_{m,i}(\Phi)(\alpha).$$

Proof. Let

$$\alpha = (I_0^\alpha, \ldots, I_{r_\alpha}^\alpha = \{j_\alpha\}, \varnothing)$$

with $I_h^\alpha \subset J_h^\alpha, 0 \leqslant h \leqslant r_\alpha$ following the same notation as in Section 2.3.3 (with an added superscript $^\alpha$).

First observe that

$$D_{m,i}(\Phi)(\alpha) = \mathrm{ext}(D'_{m,i}(\Phi)(\alpha), \mathrm{R}_{i+m+1}) \cup \bigcup_{\beta \precneqq \alpha} D_{m,i}(\Phi)(\beta). \tag{2.15}$$

We now prove that for each $\alpha \in \mathbf{P}_{m,i}(\Phi)$:

$$D_{m,i}(\Phi)(\alpha) \searrow D'_{m,i}(\Phi)(\alpha), \tag{2.16}$$

and

$$\bigcup_{\beta \precneqq \alpha} D_{m,i}(\Phi)(\beta) \searrow \bigcup_{\beta \precneqq \alpha} \lim_{\bar{\varepsilon}_{i+r_\alpha+1}} D'_{m,i}(\Phi)(\beta) \subset D'_{m,i}(\Phi)(\alpha). \tag{2.17}$$

The proof is by induction on the maximum length, length(α), of any chain with α as the maximal element.

We first note that if $\mathrm{R}' = \mathrm{R}\langle\bar{\varepsilon}\rangle$, and $X \subset \mathrm{R}^k$ is a semi-algebraic subset, then

$$\lim_{\bar{\varepsilon}} \mathrm{ext}(X, \mathrm{R}') = \overline{X}.$$

This follows easily from the definition of $\mathrm{ext}(X, \mathrm{R}')$ and standard properties of $\lim_{\bar{\varepsilon}}$. In particular, if X is a closed semi-algebraic set, then

$$\lim_{\bar{\varepsilon}} \mathrm{ext}(X, \mathrm{R}') = X.$$

Base case of the induction, length$(\alpha) = 1$: It follows from (2.15) and the fact that that $D'_{m,i}(\Phi)(\alpha)$ is a closed semi-algebraic set, that (2.16) holds if α is a minimal element of the poset $\mathbf{P}_{m,i}(\Phi)$ (and so length$(\alpha) = 1$). In this case (2.17) is trivially true.

Induction hypothesis: We assume now that (2.16) and (2.17) is true for all $\alpha \in \mathbf{P}_{m,i}(\Phi)$, with length$(\alpha) < t$.

Inductive step: Suppose that $\alpha \in \mathbf{P}_{m,i}(\Phi)$, with length$(\alpha) = t$. The inductive hypothesis implies that (2.16) and (2.17) both hold with α replaced by α' for all $\alpha' \precneqq \alpha$.

Using the fact that $D'_{m,i}(\Phi)(\alpha)$ is closed, it is easy to check that (2.17) implies (2.16). So we need to prove only (2.17). Using the induction hypothesis we have for each $\beta \precneqq \alpha$

$$\bigcup_{\beta \precneqq \alpha} D_{m,i}(\Phi)(\beta) \searrow \bigcup_{\beta \precneqq \alpha} D'_{m,i}(\Phi)(\beta). \tag{2.18}$$

Now observe that for any $\beta \in \mathbf{P}_{m,i}(\Phi)$, $\beta \precneqq \alpha$ if and only if there exist $j'_\alpha \in I^\alpha_{r_\alpha - 1}, j'_\alpha \neq j_\alpha$ and $j''_\alpha \in (J^\alpha_{r_\alpha})_{m^\alpha_{r_\alpha}, i^\alpha_{r_\alpha}, \{j_\alpha, j'_\alpha\}, \Phi_{r_\alpha}}$, such that

$$\beta \prec \gamma(j''_\alpha) = (I^\alpha_0, \ldots, I^\alpha_{r_\alpha - 1}, \{j_\alpha, j'_\alpha\}, \{j''_\alpha\}, \emptyset),$$

where we assume that $I^\alpha_{-1} = J$.

Using the above observation we have that

$$\bigcup_{\beta \precneqq \alpha} D'_{m,i}(\Phi)(\beta) = \bigcup_{j'_\alpha \in I^\alpha_{r_\alpha - 1}, j'_\alpha \neq j_\alpha} \ \bigcup_{j''_\alpha \in (J^\alpha_{r_\alpha})_{m^\alpha_{r_\alpha}, i^\alpha_{r_\alpha}, \{j_\alpha, j'_\alpha\}, \Phi_{r_\alpha}}} \left(\bigcup_{\beta \prec \gamma(j''_\alpha)} D'_{m,i}(\Phi)(\beta) \right), \tag{2.19}$$

where

$$\gamma(j''_\alpha) = (I^\alpha_0, \ldots, I^\alpha_{r_\alpha - 1}, \{j_\alpha, j'_\alpha\}, \{j''_\alpha\}, \emptyset).$$

Applying hypothesis (2.17) we have that

$$\left(\bigcup_{\beta \precneqq \gamma(j''_\alpha)} D'_{m,i}(\Phi)(\beta) \right) \searrow \lim_{\bar{\varepsilon}_{i+r+2}} \bigcup_{\beta \precneqq \gamma(j''_\alpha)} D'_{m,i}(\Phi)(\beta) \subset D'_{m,i}(\Phi)(\gamma(j''_\alpha)). \tag{2.20}$$

Also observe that,

$$\left(\bigcup_{j''_\alpha \in J_{m,i,\{j_\alpha,j'_\alpha\},\Phi}} D'_{m,i}(\Phi)(\gamma(j''_\alpha))\right) \searrow \left(D'_{m,i}(\Phi)(\alpha) \cap D'_{m,i}(\Phi)(\alpha')\right) \subset D'_{m,i}(\Phi)(\alpha), \tag{2.21}$$

where

$$\alpha' = (I_0^\alpha, \ldots, I_{r_\alpha - 1}^\alpha, \{j'_\alpha\}, \emptyset).$$

Finally, (2.17) now follows from (2.18), (2.19), (2.20), and (2.21).

□

Lemma 2.3.7.

$$\left(\bigcup_{\alpha \in \mathbf{P}_{m,i}(\Phi)} D_{m,i}(\Phi)(\alpha)\right) \searrow \mathcal{R}(\Phi, \overline{B_k(0,R)})^J.$$

In particular, $\mathrm{ext}(\mathcal{R}(\Phi, \overline{B_k(0,R)})^J, \mathrm{R}_i)$ *is a semi-algebraic deformation retract of*

$$\bigcup_{\alpha \in \mathbf{P}_{m,i}(\Phi)} D_{m,i}(\Phi)(\alpha).$$

Proof. First note that for each $j \in J$, $(\{j\}, \emptyset)$ is a maximal element of the poset $\mathbf{P}_{m,i}(\Phi)$. It now follows from Lemma 2.3.5 that

$$\bigcup_{\alpha \in \mathbf{P}_{m,i}(\Phi)} D_{m,i}(\Phi)(\alpha) = \bigcup_{j \in J} D_{m,i}(\Phi)((\{j\}, \emptyset)).$$

The lemma now follows from Lemma 2.3.6 and Eqn.(2.14). □

Notation 12. *We will denote the deformation retraction*

$$\bigcup_{\alpha \in \mathbf{P}_{m,i}(\Phi)} D_{m,i}(\Phi)(\alpha) \to \mathrm{ext}(\mathcal{R}(\Phi, \overline{B_k(0,R)})^J, \mathrm{R}_i)$$

in Lemma 2.3.7 by $r_{m,i}(\Phi)$*.*

In the proof of Theorem 2.3.1 we need the notion of the *homotopy colimit* of a functor which we define below.

We fix a real closed field R in the following definition.

Definition 2.3.5 (The topological standard n-simplex)**.** *We denote by*

$$|\Delta^n| = \{(t_0, \ldots, t_n) \in \mathrm{R}^{n+1}_{\geqslant 0} | \sum_{i=0}^{n} t_i = 1\}$$

the standard n-simplex defined over R*. For $0 \leqslant i \leqslant n$, we define the face operators,*

$$d_n^i : |\Delta^{n-1}| \to |\Delta^n|,$$

by

$$d_n^i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

Definition 2.3.6 (Homotopy colimit)**.** *Let $(\mathbf{P}, \prec)$ be a poset category and*

$$D : (\mathbf{P}, \prec) \to \mathbf{Top}$$

a functor taking its values in closed and bounded semi-algebraic subsets of R^k, and such that the morphisms $D(\alpha \prec \beta)$ are inclusion maps. The homotopy colimit of D is the quotient space [5]

$$\mathrm{hocolim}(D) = \left(\coprod_{\alpha_0 \precneqq \cdots \precneqq \alpha_p} |\Delta^p| \times D(\alpha_0) \right) \Big/ \sim \ ,$$

where the equivalence relation $\sim$ is defined as follows.

For a chain $\sigma = (\alpha_0 \precneqq \cdots \precneqq \alpha_p)$, $t \in |\Delta^p|$, and $x \in D(\alpha_0)$, we denote by $(t, x)_\sigma$, the image of (t, x) under the canonical inclusion of $|\Delta^p| \times D(\alpha_0)$ (corresponding to the chain σ) in the disjoint union $\coprod_{\alpha_0 \precneqq \cdots \precneqq \alpha_p} |\Delta^p| \times D(\alpha_0)$.

Using the above notation the equivalence relation $\sim$ is defined by:

$$(d_p^i(t), x)_\sigma \sim (t, x)_{\sigma'}, \tag{2.22}$$

[5]↑which is a semi-algebraic set defined over R, being a quotient space of a proper semi-algebraic equivalence relation, (see for example [21, page 166])

for $x \in D(\alpha_0)$ *and* $t \in |\Delta^{p-1}|$*, where* $\sigma = (\alpha_0 \lneq \cdots \lneq \alpha_p)$ *and*

$$\sigma' = \begin{cases} (\alpha_1 \lneq \cdots \lneq \alpha_p) & \textit{if } i = 0, \\ (\alpha_0 \lneq \cdots \alpha_{i-1} \lneq \alpha_{i+1} \lneq \cdots \lneq \alpha_p) & \textit{if } 0 < i < p, \\ (\alpha_0 \lneq \cdots \lneq \alpha_{p-1}) & \textit{if } i = p. \end{cases}$$

We denote by $\pi_1^D : \mathrm{hocolim}(D) \to |\Delta(\mathbf{P})|$*,* $\pi_2^D : \mathrm{hocolim}(D) \to \mathrm{colim}(D)$ *the canonical maps, where* $|\Delta(\mathbf{P})|$ *is the geometric realization of the order complex of* $\mathbf{P}$ *(see Definition 2.3.1). More precisely,* π_1^D *is the map induced from the projection map*

$$\coprod_{\alpha_0 \lneq \cdots \lneq \alpha_p} |\Delta^p| \times D(\alpha_0) \to \coprod_{\alpha_0 \lneq \cdots \lneq \alpha_p} |\Delta^p|$$

after taking the quotient by $\sim$*, and* π_2^D *is the composition of the map induced by the projection*

$$\coprod_{\alpha_0 \lneq \cdots \lneq \alpha_p} |\Delta^p| \times D(\alpha_0) \to \coprod_{\alpha_0 \lneq \cdots \lneq \alpha_p} D(\alpha_0),$$

and the canonical map to the colimit of the functor D*, which in this case is simply the union* $\bigcup_{\alpha \in \mathbf{P}} D(\alpha)$*.*

The following example illustrates the definition given above.

Example 3. *Consider the poset* $\mathbf{P} = \{a, b, c\}$*, with three elements with* $c \lneq a, c \lneq b$ *as the only non-trivial ordering relation (Hasse diagram shown below).*

$$\begin{tikzcd} a & & b \\ & c \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Let $D : \mathbf{P} \to \mathbf{Top}$ be the functor, with

$$\begin{aligned} D(a) &= \mathcal{R}((X_1^2 + X_2^2 - 4 = 0) \wedge (X_2 \geqslant 0)), \\ D(b) &= \mathcal{R}((X_1^2 + X_2^2 - 4 = 0) \wedge (X_2 \leqslant 0)), \\ D(c) &= D(a) \cap D(b) \\ &= \{(-2,0),(2,0)\}. \end{aligned}$$

The homotopy colimit of the functor D is then the quotient of the disjoint union of the spaces

$$\Delta^0 \times D(a), \Delta^0 \times D(b), \Delta^0 \times D(c),$$
$$\Delta^1 \times D(c), \Delta^1 \times D(c)$$

corresponding to the chains $(a),(b),(c),(c \lneqq a),(c \lneqq b)$ by the equivalence relation defined in Eqn. (2.22). The non-trivial identifications induced by the quotienting are given by (following the notation introduced in Definition 2.3.6)

$$((0,1),(-2,0))_{(c \lneqq a)} \sim ((1),(-2,0))_{(c)},$$
$$((0,1),(2,0))_{(c \lneqq a)} \sim ((1),(2,0))_{(c)},$$
$$((1,0),(-2,0))_{(c \lneqq a)} \sim ((1),(-2,0))_{(a)},$$
$$((1,0),(2,0))_{(c \lneqq a)} \sim ((1),(2,0))_{(a)},$$
$$((0,1),(-2,0))_{(c \lneqq b)} \sim ((1),(-2,0))_{(c)},$$
$$((0,1),(2,0))_{(c \lneqq b)} \sim ((1),(2,0))_{(c)},$$
$$((1,0),(-2,0))_{(c \lneqq b)} \sim ((1),(-2,0))_{(b)},$$
$$((1,0),(2,0))_{(c \lneqq b)} \sim ((1),(2,0))_{(b)}.$$

The quotient space (as a semi-algebraic set) is shown below in Figure 2.7.

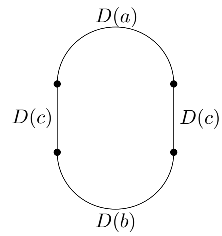


Figure 2.7. Homotopy colimit of the functor D in Example 3.

Proof of Theorem 2.3.1. The theorem will follow from the following two claims.

Claim 2.3.1. *The map* $\pi_1^{D_{m,i}(\Phi)} : \mathrm{hocolim}(D_{m,i}(\Phi)) \to |\Delta(\mathbf{P}_{m,i}(\Phi))|$ *is a homological* ℓ*-equivalence (and so a homological* $(m-1)$*-equivalence).*

Claim 2.3.2. *The map*

$$F_{m,i}(\Phi) = r_{m,i}(\Phi) \circ \pi_2^{D_{m,i}(\Phi)} : \mathrm{hocolim}(D_{m,i}(\Phi)) \to \mathrm{ext}(\mathcal{R}(\Phi, \overline{B_k(0,R)})^J, \mathrm{R}_i)$$

is a homological $(m-1)$*-equivalence.*

We first deduce the proof of the theorem from these two claims. The homological $(m-1)$-equivalence in (2.11) now follows from Claims 2.3.1, 2.3.2 and Lemma 2.3.7.

The diagrammatic homological $(m-1)$-equivalence in (2.12) follows from the commutativity of the following diagrams of maps.

For each pair $J', J'' \subset J$, with $J' \subset J''$ we have the following commutative diagram, where the vertical arrows are inclusions, and the slanted arrows induce isomorphisms in the homology groups up to dimension $m-1$.

$$\begin{tikzcd}[column sep=small]
& \mathrm{hocolim}(D_{m,i}(\Phi|_{J'})) \arrow[dl, "\pi_1^{D_{m,i}(\Phi|_{J'})}"'] \arrow[d] \arrow[dr, "F_{m,i}(\Phi|_{J'})"] & \\
|\Delta(\mathbf{P}_{m,i}(\Phi|_{J'}))| \arrow[dd] & & \mathrm{ext}(\mathcal{R}(\Phi, \overline{B_k(0,R)})^{J'}, \mathrm{R}_i) \arrow[dd] \\
& \mathrm{hocolim}(D_{m,i}(\Phi|_{J''})) \arrow[dl, "\pi_1^{D_{m,i}(\Phi|_{J''})}"'] \arrow[dr, "F_{m,i}(\Phi|_{J''})"] & \\
|\Delta(\mathbf{P}_{m,i}(\Phi|_{J''}))| & & \mathrm{ext}(\mathcal{R}(\Phi, \overline{B_k(0,R)})^{J''}, \mathrm{R}_i)
\end{tikzcd}.$$

This implies that we have the following diagram of morphisms where both arrows are homological $(m-1)$-equivalences:

$$\begin{tikzcd}
& (J' \mapsto \mathrm{hocolim}(D_{m,i}(\Phi|_{J'})))_{J' \in 2^J} \arrow[dl] \arrow[dr] & \\
(J' \mapsto |\Delta(\mathbf{P}_{m,i}(\Phi|_{J'}))|)_{J' \in 2^J} & & \mathbf{Simp}^J(\mathcal{R}(\Phi, \overline{B_k(0,R)})).
\end{tikzcd}$$

This proves that the diagrams

$$(J' \mapsto |\Delta(\mathbf{P}_{m,i}(\Phi|_{J'}))|)_{J' \in 2^J}$$

and

$$\mathbf{Simp}^J(\mathcal{R}(\Phi, \overline{B_k(0,R)}))$$

are homologically $(m-1)$-equivalent.

We now proceed to prove Claims 2.3.1 and 2.3.2.

Proof of Claim 2.3.1. Let $t \in |\Delta(\mathbf{P}_{m,i}(\Phi))|$. Then there exists a unique simplex σ of the simplicial complex $\Delta(\mathbf{P}_{m,i}(\Phi))$ of the smallest possible dimension such that $t \in |\sigma|$. Let $\alpha_0 \precnsim \cdots \precnsim \alpha_p$ be the chain in $\mathbf{P}_{m,i}(\Phi)$ corresponding to σ. Then,

$$(\pi_1^{D_{m,i}(\Phi)})^{-1}(t) = \{t\} \times D_{m,i}(\Phi)(\alpha_0).$$

It is clear from its definition that $D'_{m,i}(\Phi)(\alpha)$ is homologically ℓ-connected. From Lemma 2.3.6 it follows that so is $D_{m,i}(\Phi)(\alpha)$. It now follows from the homological version of the Vietoris-Begle theorem (see Remark 2) that $\pi_1^{D_{m,i}(\Phi)}$ is a homological ℓ-equivalence. $\square$

Proof of Claim 2.3.2. The claim will follow from the following claims. Let

$$x \in \mathrm{ext}(\mathcal{R}(\Phi, \overline{B_k(0,R)})^J, \mathrm{R}_i).$$

We will prove that the fiber $(F_{m,i}(\Phi))^{-1}(x)$ is homologically $(m-1)$-connected which will suffice to prove that $F_{m,i}(\Phi)$ is a homological $(m-1)$-equivalence by the homological version of Vietoris-Begle theorem (see Remark 2).

In order to study the fiber $(F_{m,i}(\Phi))^{-1}(x)$ we define for each $I \subset_{\leqslant m+2} J$ the following posets of $\mathbf{P}_{m,i}(\Phi)$.

We define

$$\begin{aligned} \mathbf{P}_I(x) \quad &= \quad \{(I,\alpha) \in \{I\} \times \mathbf{P}_{m-\mathrm{card}(I)+1,i+1}(\Phi_{m,i,I,J}) \mid \\ &\qquad x \in \lim_{\bar{\varepsilon}} D_{m-\mathrm{card}(I)+1,i+1}(\Phi_{m,i,I,J})(\alpha)\} \subset \mathbf{P}_{m,i}(\Phi), \end{aligned}$$

and

$$\mathbf{Q}_I(x) = \bigcup_{I \subset I' \subset_{\leqslant m+2} J} \mathbf{P}_{I'}(x).$$

The motivation behind the definition of the posets $\mathbf{P}_I(x), \mathbf{Q}_I(x)$ is as follows. First observe that

$$(F_{m,i}(\Phi))^{-1}(x) = \left| \bigcup_{j \in J} \Delta(\mathbf{Q}_{\{j\}}(x)) \right|, \tag{2.23}$$

and

$$\bigcap_{j \in I} \mathbf{Q}_{\{j\}}(x) = \mathbf{Q}_I(x). \tag{2.24}$$

Our strategy for proving the homological $(m-1)$-connectedness of $(F_{m,i}(\Phi))^{-1}(x)$ is to use the closed covering provided by (2.23) and then use the cohomological Mayer-Vietoris spectral sequence to reduce the problem to studying the connectivity of the various $|\Delta(\mathbf{Q}_I(x))|$ using (2.24). Finally, we prove (see Claim 2.3.5) that for each I, $|\Delta(\mathbf{P}_I(x))|$ is homologically equivalent to $|\Delta(\mathbf{Q}_I(x))|$. This last fact allows us to use induction on the cardinality of I to prove the required connectivity statement for the corresponding $|\Delta(\mathbf{Q}_I(x))|$.

We now return to the proof of Claim 2.3.2. Since, for each I', with $I \subset I' \subset_{\leqslant m+2} J$,

$$\mathbf{P}_{m-\mathrm{card}(I')+1,i+1}(\Phi_{m,i,I',J}) \subset \mathbf{P}_{m-\mathrm{card}(I)+1,i+1}(\Phi_{m,i,I,J}),$$

there is an injective map,

$$\mathbf{P}_{I'}(x) \to \mathbf{P}_I(x), (I', \alpha) \mapsto (I, \alpha).$$

Thus there is a map

$$\theta_I(x) : \mathbf{Q}_I(x) \to \mathbf{P}_I(x),$$

defined by

$$\theta_I(x)((I', \alpha)) = (I, \alpha),$$

for each $(I', \alpha) \in \mathbf{Q}_I(x)$, where $I \subset I' \subset_{\leqslant m+2} J$.

It is obvious from the above definition and the definition of the partial order in $\mathbf{P}_{m,i}(\Phi)$, that the map $\theta_I(x)$ is a map of posets (i.e. a map respecting the partial orders of the two posets).

Claim 2.3.3. *The map $\theta_I(x)$ induces a simplicial map $\Theta_I(x) : \Delta(\mathbf{Q}_I(x)) \to \Delta(\mathbf{P}_I(x))$. Moreover, the corresponding map $|\Theta_I(x)| : |\Delta(\mathbf{Q}_I(x))| \to |\Delta(\mathbf{P}_I(x))|$, between the geometric realizations, is a homological equivalence.*

Proof. Since the map $\theta_i(x)$ is a poset map, it carries a chain of $\mathbf{Q}_I(x)$ to a chain of $\mathbf{P}_I(x)$. This implies that $\theta_I(x)$ induces a simplicial map $\Theta_I(x) : \Delta(\mathbf{Q}_I(x)) \to \Delta(\mathbf{P}_I(x))$.

We now prove the second half of the claim. We are going to use the poset fiber theorem proved in [22, Lemma 3.2] (also [23, Corollary 3.4]).

For $n \geqslant 0$, we denote by $\mathcal{B}_n$ the complete Boolean lattice on a set with n elements. It is a well known fact (see for example [24]) that $|\Delta(\mathcal{B}_n)|$ is homeomorphic to $[0,1]^n$, and is thus contractible.

Let $(I, \alpha) \in \mathbf{P}_I(x)$, and $I' \subset_{\leqslant m+2} J$ be the unique maximal subset of J such that $(I', \alpha) \in \mathbf{P}_{I'}(x)$ (see the schematic diagram in Figure 2.8 which depicts subposet of the poset shown in Figure 2.4).

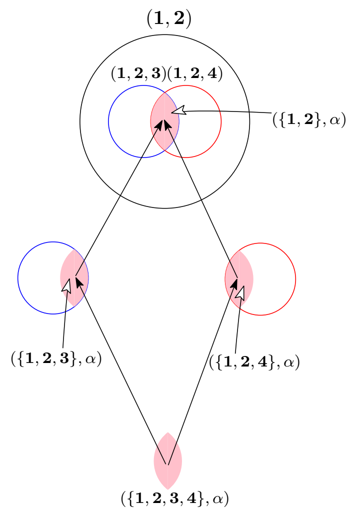


Figure 2.8. $\theta_I(x)^{-1}((I, \alpha))$ with $I = \{1, 2\}$, and $I' = \{1, 2, 3, 4\}$

Then,

$$\theta_I(x)^{-1}((I, \alpha)) = \{(I'', \alpha) \mid I \subset I'' \subset I'\}.$$

Hence, the poset $\theta_I(x)^{-1}((I, \alpha))$ is isomorphic as a poset to $\mathcal{B}_{\mathrm{card}(I')-\mathrm{card}(I)}$. Thus, $|\Delta(\theta_I(x)^{-1}((I, \alpha)))|$ is contractible.

Moreover, for each $(I'', \alpha) \in \theta_I(x)^{-1}((I, \alpha))$,

$$\theta_I(x)^{-1}((I, \alpha))_{\succ(I'', \alpha)} = \{(I''', \alpha) \mid I \subset I''' \subset I''\},$$

and hence $\theta_I(x)^{-1}((I, \alpha))_{\succ(I'', \alpha)}$ is isomorphic to $\mathcal{B}_{\mathrm{card}(I'')-\mathrm{card}(I)}$. This proves that $|\Delta(\theta_I(x)^{-1}((I, \alpha))_{\succ(I'', \alpha)})|$ is contractible for each $(I'', \alpha) \in \theta_I(x)^{-1}((I, \alpha))$.

It now follows from the poset fiber theorem [22, Lemma 3.2] (also [23, Corollary 3.4]) that the poset map $\theta_I(x)$ induces a homological equivalence $|\Theta_I(x)| : |\Delta(\mathbf{Q}_I(x))| \to |\Delta(\mathbf{P}_I(x))|$.

$\square$

Observe that Claim 2.3.3 implies in particular that if $\mathrm{card}(I) = 1$, then $|\mathbf{Q}_I(x)|$ is contractible if non-empty.

Claim 2.3.4. *For* $x \in \mathrm{ext}(\mathcal{R}(\Phi, \overline{B_k(0,R)})^{J''}, \mathrm{R}_i) = \lim_{\bar{\varepsilon}} \bigcup_{\alpha \in \mathbf{P}_{m,i}(\Phi)} D_{m,i}(\Phi)(\alpha)$,

$$\begin{aligned} \mathrm{H}^j((F_{m,i}(\Phi))^{-1}(x)) &\cong \mathbb{Z}, \ \textit{for } j = 0, \\ &= 0, \ \textit{for } 0 < j \leqslant m. \end{aligned} \tag{2.25}$$

Proof. The proof is by induction on m starting with the case $m = 0$. The case $m = -1$ is trivial.

Base case ($m = 0$). We need to show that for

$$x \in \mathrm{ext}(\mathcal{R}(\Phi, \overline{B_k(0,R)})^{J}, \mathrm{R}_i) = \lim_{\bar{\varepsilon}} \bigcup_{\alpha \in \mathbf{P}_{0,i}(\Phi)} D_{0,i}(\Phi)(\alpha),$$

$(F_{0,i}(\Phi))^{-1}(x)$ is connected.

First note that

$$F_{0,i}(\Phi) = r_{0,i}(\Phi) \circ \pi_2^{D_{0,i}(\Phi)},$$

and $r_{0,i}(\Phi)$ is a semi-algebraic deformation retraction. Hence, $r_{0,i}(\Phi)^{-1}(x)$ is closed and semi-algebraically connected (in fact contractible).

Let $J(x) = \{j \in J \mid D_{0,i}(\Phi)((\{j\}, \emptyset)) \cap r_{0,i}(\Phi)^{-1}(x) \neq \emptyset\}$. Since, the sets $D_{0,i}(\Phi)((\{j\}, \emptyset))$, $j \in J(x)$ is a covering of the closed and semi-algebraically connected set $r_{0,i}(\Phi)^{-1}(x)$ by closed sets, it follows that the union

$$\bigcup_{j \in J(x)} D_{0,i}(\Phi)((\{j\}, \emptyset))$$

is semi-algebraically connected as well. It follows that given any $j, j' \in J(x)$, there exists a sequence $j = j_0, j_1, \ldots, j_N = j'$ such that for each $h, 0 \leqslant h \leqslant N-1$,

$$D_{0,i}(\Phi)((\{j_h\}, \emptyset)) \cap D_{0,i}(\Phi)((\{j_{h+1}\}, \emptyset)) \cap r_{0,i}(\Phi)^{-1}(x) \neq \emptyset.$$

So there exists for each $h, 0 \leqslant h \leqslant N-1$ $j'' = (\{j_h, j_{h+1}\}, p) \in J_{0,i,\{j_h,j_{h+1}\},\Phi}$ such that

$$\mathcal{R}(\Phi_{\{j_h,j_{h+1}\}}(p)) \cap r_{m,i}(\Phi)^{-1}(x) \neq \varnothing.$$

So there exists $\alpha = (\{j''\}, \varnothing) \in \mathbf{P}_{-1,i+1}(\Phi_{\{j_h,_{h+1}\}})$, such that

$$D_{0,i}(\Phi)((\{j_h, j_{h+1}\}, \alpha)) \cap r_{0,i}(\Phi)^{-1}(x) \neq \varnothing,$$

and so

$$(\{j_h, j_{h+1}\}, \alpha) \in (F_{0,i}(\Phi))^{-1}(x).$$

Moreover,

$$(\{j_h, j_{h+1}\}, \alpha) \precneqq (\{j_h\}, \varnothing), (\{j_{h+1}\}, \varnothing)$$

(using Lemma 2.3.4). This implies that $(\{j_h\}, \varnothing), (\{j_{h+1}\}, \varnothing)$, and thus every pair of the form $(\{j\}, \varnothing), (\{j'\}, \varnothing)$ in $(F_{0,i}(\Phi))^{-1}(x)$ belongs to the same connected component of $(F_{0,i}(\Phi))^{-1}(x)$. Since, for every element of the form $(\{j_h, j_{h+1}\}, \alpha) \in (F_{0,i}(\Phi))^{-1}(x)$ we have

$$(\{j_h, j_{h+1}\}, \alpha) \precneqq (\{j_h\}, \varnothing), (\{j_{h+1}\}, \varnothing) \in (F_{0,i}(\Phi))^{-1}(x),$$

$(\{j_h, j_{h+1}\}, \alpha)$ belong to the same connected component of $(F_{0,i}(\Phi))^{-1}(x)$ as

$$(\{j_h\}, \varnothing), (\{j_{h+1}\}, \varnothing)$$

as well. Together, these facts imply that $(F_{0,i}(\Phi))^{-1}(x)$ is connected. This proves the claim in the base case.

Inductive step. Suppose we have proved the theorem for all $m', 0 \leqslant m' < m$, $i \geqslant 0$, all finite J', and $\Phi' \in (\mathcal{F}_{k,\mathrm{R}_i})^{J'}$. We now prove it for m, i, J, Φ.

$$x \in \operatorname{ext}(\mathcal{R}(\Phi, \overline{B_k(0,R)})^J, R_i) = \lim_{\bar{\varepsilon}} \bigcup_{\alpha \in \mathbf{P}_{m,i}(\Phi)} D_{m,i}(\Phi)(\alpha),$$

Recall from (2.23) that

$$(F_{m,i}(\Phi))^{-1}(x) = \left|\bigcup_{j\in J} \Delta(\mathbf{Q}_{\{j\}}(x))\right|.$$

Let

$$J' = \{j \in J \ \mid \ \mathbf{Q}_{\{j\}}(x) \neq \varnothing\}.$$

So

$$(F_{m,i}(\Phi))^{-1}(x) = |\bigcup_{j\in J'} \Delta(\mathbf{Q}_{\{j\}}(x))|.$$

It follows from the Mayer-Vietoris exact sequence in cohomology for closed subspaces (see for example, [25, page 148]) that there exists a spectral sequence

$$E_r^{p,q} \Rightarrow \mathrm{H}^{p+q}\left(\left|\bigcup_{j\in J'} \Delta(\mathbf{Q}_{\{j\}}(x))\right|\right)$$

whose E_1 term is given by

$$E_1^{p,q} = \bigoplus_{I\subset J',\mathrm{card}(I)=p+1} \mathrm{H}^q\left(\left|\bigcap_{j\in I} \Delta(\mathbf{Q}_{\{j\}}(x))\right|\right).$$

Notice that

$$\bigcap_{j\in I} \mathbf{Q}_{\{j\}}(x) = \mathbf{Q}_I(x),$$

and it follows from Claim 2.3.3 that $|\Delta(\mathbf{Q}_I(x))|$ is homotopy equivalent to $|\Delta(\mathbf{P}_I(x))|$.

So we get,

$$E_1^{p,q} = \bigoplus_{I\subset J',\mathrm{card}(I)=p+1} \mathrm{H}^q(|\Delta(\mathbf{P}_I(x))|).$$

Now for I, with $\mathrm{card}(I) > 1$, we can apply the induction hypothesis to deduce that

$$\begin{aligned}
\mathrm{H}^j(|\Delta(\mathbf{P}_I(x))|) &\cong \mathbb{Z}, \text{ for } j = 0, \\
&= 0, \text{ for } 0 < j \leqslant m - \mathrm{card}(I) + 1.
\end{aligned}$$

We can deduce from this that

$$
\begin{aligned}
E_1^{p,0} &\cong \bigoplus_{I \subset J', \mathrm{card}(I)=p+1} \mathbb{Z}, \\
E_1^{p,q} &\cong 0, \text{ for } 0 < q \leqslant m-p.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
E_2^{0,0} &\cong \mathbb{Z}, \\
E_2^{p,0} &\cong 0, p > 0 \\
E_2^{p,q} &\cong 0, \text{ for } p \geqslant 0, 0 < q \leqslant m-p.
\end{aligned}
$$

$\square$

Note that it follows from Claim 2.3.5 and the Mayer-Vietoris spectral sequence argument used in its proof that ror any

$$J' \subset \{ j \in J \mid \mathbf{Q}_{\{j\}}(x) \neq \emptyset \},$$

$$
\begin{aligned}
\mathrm{H}^j \left(\left| \bigcup_{j \in J'} \Delta(\mathbf{Q}_{\{j\}}(x)) \right| \right) &\cong \mathbb{Z}, \text{ for } j = 0, \\
&= 0, \text{ for } 0 < j \leqslant m.
\end{aligned} \tag{2.26}
$$

Claim 2.3.5. *For*

$$x \in \mathrm{ext}(\mathcal{R}(\Phi, \overline{B_k(0,R)})^J, \mathrm{R}_i) = \lim_{\bar{\varepsilon}} \bigcup_{\alpha \in \mathbf{P}_{m,i}(\Phi)} D_{m,i}(\Phi)(\alpha),$$

$(F_{m,i}(\Phi))^{-1}(x)$ *is homologically* $(m-1)$*-connected.*

Proof. Let $X = (F_{m,i}(\Phi))^{-1}(x)$. It follows from [26, Theorem 12, page 248] that there exists a short exact sequence:

$$0 \to \mathrm{Ext}(\mathrm{H}^{q+1}(X), \mathbb{Z}) \to \mathrm{H}_q(X) \to \mathrm{Hom}(\mathrm{H}^q(X), \mathbb{Z}) \to 0.$$

Thus, for each $q > 0$

$$\mathrm{H}^{q+1}((F_{m,i}(\Phi))^{-1}(x)) = \mathrm{H}^q((F_{m,i}(\Phi))^{-1}(x)) = 0$$

implies that $\mathrm{H}_q((F_{m,i}(\Phi))^{-1}(x)) = 0$.

The claim now follows from (2.25). $\square$

Claim 2.3.2 now follows from Claim 2.3.5 and the homological version of the Vietoris-Begle theorem (see Remark 2). $\square$

This completes the proof of Theorem 2.3.1. $\square$

Proof of Theorem 2.3.1′. Since the proof closely mirrors that of the proof of Theorem 2.3.1 we only point out the places where it differs. For each $\alpha \in \mathbf{P}_{m,i}(\Phi)$, we replace the infinitesimals $\bar{\varepsilon}_0, \ldots, \bar{\varepsilon}_m$, by sequences of appropriately small enough positive elements $\bar{t}_0, \ldots, \bar{t}_m$ of $\mathbb{R}$, in the formula defining the set $D_{m,i}(\Phi)(\alpha)$, and denote the set defined by the new formula (which are semi-algebraic subset of $\mathbb{R}^k$) by $\widetilde{D}_{m,i}(\Phi)(\alpha)$. Similarly, we will denote the retraction

$$\bigcup_{\alpha \in \mathbf{P}_{m,i}(\Phi)} \widetilde{D}_{m,i}(\Phi)(\alpha) \to \mathcal{R}(\Phi, \overline{B_k(0,R)})^J$$

by $\widetilde{r}_{m,i}(\Phi)$, and the composition

$$\widetilde{r}_{m,i}(\Phi) \circ \pi_2^{\widetilde{D}_{m,i}(\Phi)} : \mathrm{hocolim}(\widetilde{D}_{m,i}(\Phi)) \to \mathcal{R}(\Phi, \overline{B_k(0,R)})^J$$

by $\widetilde{F}_{m,i}(\Phi)$.

Claims 2.3.1 and 2.3.2 are replaced by:

Claim 2.3.1′. *The map* $\pi_1^{\widetilde{D}_{m,i}(\Phi)} : \mathrm{hocolim}(\widetilde{D}_{m,i}(\Phi)) \to |\Delta(\mathbf{P}_{m,i}(\Phi))|$ *is an* ℓ*-equivalence (and so an* $(m-1)$*-equivalence).*

Claim 2.3.2′. *The map*

$$\widetilde{F}_{m,i}(\Phi) = \widetilde{r}_{m,i}(\Phi) \circ \pi_2^{\widetilde{D}_{m,i}(\Phi)} : \operatorname{hocolim}(\widetilde{D}_{m,i}(\Phi)) \to \mathcal{R}(\Phi, \overline{B_k(0,R)})^J$$

is an $(m-1)$*-equivalence.*

The proof of Claim 2.3.1′ is the same as the proof of Claim 2.3.1 replacing homologically ℓ-connected by just ℓ-connected, and using the homotopy version of the Vietoris-Begle theorem (see Remark 2).

For the proof of Claim 2.3.2′ we need an extra argument to deduce the $(m-1)$-connectivity of the fibers of the map $\widetilde{F}_{m,i}(\Phi)$ from the fact that they are homologically $(m-1)$-connected which is already proved in Claim 2.3.5. In order to do this we apply Hurewicz's isomorphism theorem which requires simple connectivity of the fibers $(\widetilde{F}_{m,i}(\Phi))^{-1}(x)$, which is the content of the following claim.

Claim 2.3.6. *For* $x \in \mathcal{R}(\Phi, \overline{B_k(0,R)})^J$*, and* $m \geqslant 1$*,* $(\widetilde{F}_{m,i}(\Phi))^{-1}(x)$ *is simply connected. In other words,* $(\widetilde{F}_{m,i}(\Phi))^{-1}(x)$ *is connected, and*

$$\pi_1((\widetilde{F}_{m,i}(\Phi))^{-1}(x)) \cong 0.$$

Proof. Let

$$J' = \{j \in J \mid \mathbf{Q}_{\{j\}}(x) \neq \varnothing\}.$$

So

$$(\widetilde{F}_{m,i}(\Phi))^{-1}(x) = \left| \bigcup_{j \in J'} \Delta(\mathbf{Q}_{\{j\}}(x)) \right|.$$

We prove the stronger statement that for all non-empty subsets $J'' \subset J'$,

$$\left| \bigcup_{j \in J''} \Delta(\mathbf{Q}_{\{j\}}(x)) \right|$$

is simply connected.

We argue using induction on $\operatorname{card}(J'')$. If $\operatorname{card}(J'') = 1$, then $\Delta(\mathbf{Q}_{\{j\}}(x))$, where $J'' = \{j\}$, is a cone and so $|\Delta(\mathbf{Q}_{\{j\}}(x))|$ is contractible and hence simply connected.

Suppose, we have already proved that the claim holds for all subsets of J' of cardinality strictly smaller than that of J''. Let $j'' \in J''$. Then, by the induction hypothesis, we have that $\left|\bigcup_{j' \in J'' - \{j''\}} \Delta(\mathbf{Q}_{\{j'\}}(x))\right|$ is simply connected.

We first show that

$$|\Delta(\mathbf{Q}_{\{j''\}}(x))| \cap \left|\bigcup_{j' \in J'' - \{j''\}} \Delta(\mathbf{Q}_{\{j'\}}(x))\right|$$

is connected, which is equivalent to proving that

$$\mathrm{H}^0(|\Delta(\mathbf{Q}_{\{j''\}}(x))| \cap \bigcup_{j' \in J'' - \{j''\}} |\Delta(\mathbf{Q}_{\{j'\}}(x))|) \cong \mathbb{Z}.$$

The Mayer-Vietoris exact sequence in cohomology gives the following exact sequence:

$$\begin{gathered}\mathrm{H}^0(\bigcup_{j' \in J''} |\Delta(\mathbf{Q}_{\{j'\}}(x))|) \to \mathrm{H}^0(|\Delta(\mathbf{Q}_{\{j''\}}(x))|) \oplus \mathrm{H}^0(\bigcup_{j' \in J'' - \{j''\}} |\Delta(\mathbf{Q}_{\{j'\}}(x))|) \to \\ \mathrm{H}^0(|\Delta(\mathbf{Q}_{\{j''\}}(x))| \cap \bigcup_{j' \in J'' - \{j''\}} |\Delta(\mathbf{Q}_{\{j'\}}(x))|) \to \mathrm{H}^1(\bigcup_{j' \in J''} |\Delta(\mathbf{Q}_{\{j'\}}(x))|).\end{gathered}$$

Applying (2.26) we have an exact sequence

$$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{H}^0\left(|\Delta(\mathbf{Q}_{\{j''\}}(x))| \cap \bigcup_{j' \in J'' - \{j''\}} |\Delta(\mathbf{Q}_{\{j'\}}(x))|\right) \to 0,$$

where the first map is the diagonal embedding. This implies that

$$\mathrm{H}^0\left(|\Delta(\mathbf{Q}_{\{j''\}}(x))| \cap \bigcup_{j' \in J'' - \{j''\}} |\Delta(\mathbf{Q}_{\{j'\}}(x))|\right) \cong \mathbb{Z}.$$

Finally, using the fact that $|\Delta(\mathbf{Q}_{\{j''\}}(x))|$ is simply connected, it follows from the Seifert-van Kampen's theorem [26, page 151] that $\left|\bigcup_{j \in J''} \Delta(\mathbf{Q}_{\{j\}}(x))\right|$ is simply connected. $\square$

We also have the obvious analog of Lemma 2.3.7.

Lemma 2.3.7′. *The semi-algebraic set* $\mathcal{R}(\Phi, \overline{B_k(0,R)})^J$ *is a semi-algebraic deformation retract of*

$$\bigcup_{\alpha \in \mathbf{P}_{m,i}(\Phi)} \widetilde{D}_{m,i}(\Phi)(\alpha),$$

and hence $\mathcal{R}(\Phi, \overline{B_k(0,R)})^J$ *and* $\bigcup_{\alpha \in \mathbf{P}_{m,i}(\Phi)} \widetilde{D}_{m,i}(\Phi)(\alpha)$ *are semi-algebraically homotopy equivalent.*

Proof. Similar to proof of Lemma 2.3.7 and omitted. □

Proof of Claim 2.3.2′. It follows from Claim 2.3.5, Claim 2.3.6, and Hurewicz isomorphism theorem [26, Theorem 5, page 398], that for

$$x \in \mathcal{R}(\Phi, \overline{B_k(0,R)})^J$$

and $m \geqslant 1$, $(\widetilde{F}_{m,i}(\Phi))^{-1}(x)$ is $(m-1)$-connected. Claim 2.3.2′ now follows from the previous statement and the homotopy version of the Vietoris-Begle theorem (see Remark 2). □

Finally, Theorem 2.3.1′ follows from Claims 2.3.1′, 2.3.2′ and Lemma 2.3.7′. □

2.3.7 Upper bound on the size of the simplicial complex $\Delta(\mathbf{P}_{m,i}(\Phi))$

We now prove an upper bound on the size of the simplicial complex $\Delta(\mathbf{P}_{m,i}(\Phi))$ assuming a "singly exponential" upper bound on the function $\mathcal{I}_{i,k}(\cdot)$ and $\mathcal{C}_{i,k}(\cdot)$.

Definition 2.3.7. *For any closed formula* ϕ *with coefficients in a real closed field* R*, let the* size *of* ϕ, $\mathbf{Comp}(\phi)$ *be the product of the number of polynomials appearing in the formula* ϕ *and the maximum amongst the degrees of these polynomials. Similarly, if* J *is any finite set, and* $\Phi \in (\mathcal{F}_{\mathrm{R},k})^J$*, we denote by* $\mathbf{Comp}(\Phi)$ *the product of the total number of polynomials appearing in the formulas* $\Phi(j), j \in J$*, and the maximum amongst the degrees of these polynomials.*

Theorem 2.3.2. *Suppose that there exists $c > 0$ such that for each $\phi \in \mathcal{F}_{\mathrm{R}_i,k}$,*

$$\begin{aligned} \mathcal{I}_{i,k}(\phi) &\leqslant (\mathbf{Comp}(\phi))^{k^c}, \\ \max_{j \in [\mathcal{I}_{i,k}(\phi)]} \mathbf{Comp}(\mathcal{C}_{i,k}(\phi)(j)) &\leqslant (\mathbf{Comp}(\phi))^{k^c}. \end{aligned} \tag{2.27}$$

Let J be a finite set and $\Phi \in (\mathcal{F}_{\mathrm{R}_i,k})^J$. Then the number of simplices in $\Delta(\mathbf{P}_{m,i}(\Phi))$ is bounded by

$$(\mathrm{card}(J) D)^{k^{O(m)}},$$

where

$$D = \mathbf{Comp}(\Phi).$$

Proof. Recall that the elements of $\mathbf{P}_{m,i}(\Phi)$ are finite tuples

$$(I_0, \ldots, I_r, \emptyset),$$

where for each, $h, 0 \leqslant h \leqslant r$, I_h is a subset of a certain set J_h defined in Section 2.3.3.

We first bound the cardinalities of the various J_h's occurring in the sequence above.

Claim 2.3.7. *For any $i' \geqslant 0$, $m' \geqslant -1$, finite set J', $I' \subset_{m'+2} J'$, and $\Phi' \in (\mathcal{F}_{\mathrm{R}_{i'},k})^{J'}$,*

$$\mathrm{card}(J'_{m',i',I',\Phi'}) \leqslant (\mathrm{card}(J'))^{m'+1} (\mathbf{Comp}(\Phi'))^{k^c}.$$

Proof of Claim 2.3.7. Let for each fixed i, k,

$$F(M', N', m', D') = \max_{\substack{J', \mathrm{card}(J') = N', \\ I' \subset_{m'+2} J', \mathrm{card}(I') = M', \\ \Phi' \in \mathcal{F}_{\mathrm{R}_i,k}, \mathbf{Comp}(\Phi') = D'}} \mathrm{card}(J'_{m',i,I',\Phi'}).$$

Using Eqns. (2.5) and (2.6) and Eqn. (2.27), we obtain:

$$\begin{aligned} F(m'+2, N', D') &\leqslant D'^{k^c}, \\ F(M', N, D') &\leqslant D'^{k^c} + (N' - M') F(M'+1, N', D'), \text{ for } 1 < M' < m'+2. \end{aligned}$$

It follows that

$$\begin{aligned} F(M', N', D') &\leqslant D'^{k^c}(1 + N' + N'^2 + \cdots + N'^{m'+2-M'}) \\ &\leqslant D'^{k^c} N'^{m'+1} \text{ for } 1 < M' \leqslant m' + 2. \end{aligned}$$

The claim follows from the above inequality. $\square$

Claim 2.3.8. *For* $(I_0, \ldots, I_r, \phi) \in \mathbf{P}_{m,i}(\Phi)$, $r \leqslant m + 1$.

Proof of Claim 2.3.8. The claim follows from the fact that $\mathrm{card}(I_0), \ldots, \mathrm{card}(I_{r-1}) \geqslant 2$, and hence it follows from Eqn. (2.9) that

$$2r \leqslant \sum_{0 \leqslant j < r} \mathrm{card}(I_j) \leqslant m + (r - 1) + 2.$$

It follows that

$$r \leqslant m + 1.$$

$\square$

Claim 2.3.9. *For every tuple* $(I_0, \ldots, I_r, \varnothing) \in \mathbf{P}_{m,i}(\Phi)$, $0 \leqslant h \leqslant r$,

$$\begin{aligned} \mathbf{Comp}(\Phi_h(\alpha)) &\leqslant D^{k^{ch}}, \text{ for } \alpha \in J_h, \\ \mathrm{card}(J_h) &\leqslant N^{(m+1)^h} D^{(k(m+1))^{ch}}, \end{aligned}$$

where $J_h, \Phi_h, 0 \leqslant j \leqslant r$ *are defined in Eqn.* (2.8)*, and* $N = \mathrm{card}(J)$.

Proof of Claim 2.3.9. The claim is obviously true for $h = 0$. Also, note that for each $h, 0 \leqslant h \leqslant r$,

$$m_h \leqslant m.$$

The claim now follows by induction on h, using the inductive definitions of J_h, Φ_h (see Eqn. (2.8)), Eqn. (2.27), and Claim 2.3.7. $\square$

Claim 2.3.10.

$$\mathrm{card}(\mathbf{P}_{m,i}(\Phi)) \leqslant (\mathrm{card}(J) D)^{k^{O(m)}}.$$

Proof of Claim 2.3.10. In order to bound the cardinality of $\mathbf{P}_{m,i}(\Phi)$, we bound the number of possible choices of $I_0, \ldots, I_r$ for $(I_0, \ldots, I_r, \varnothing) \in \mathbf{P}_{m,i}(\Phi)$.

It follows from Eqn. (2.9), that for each $h, 0 \leqslant h \leqslant r$,

$$\begin{aligned}
\mathrm{card}(I_h) &\leqslant m - \sum_{t=0}^{h-1} \mathrm{card}(I_t) + h + 2 \\
&\leqslant m - 2h + h + 2 \text{ (since } \mathrm{card}(I_t) \geqslant 2, 0 \leqslant t < r) \\
&\leqslant m - h + 2 \\
&\leqslant m + 2.
\end{aligned}$$

Since by Claim 2.3.9 for $0 \leqslant h \leqslant r$,

$$\mathrm{card}(J_h) \leqslant N^{(m+1)^h} D^{(k(m+1))^{ch}},$$

the number of choices for I_h is clearly bounded by

$$\sum_{t=2}^{m+2} \binom{N^{(m+1)^h} D^{(k(m+1))^{ch}}}{h} \leqslant N^{m^{O(h)}} D^{k^{O(h)}},$$

noting that $m \leqslant k$. The above inequality, together with the fact that $r \leqslant m+1$ (by Claim 2.3.8), proves the claim. $\square$

Claim 2.3.11. *The length of any chain in $\mathbf{P}_{m,i}(\Phi)$ is bounded by $2m + 2$.*

Proof of Claim 2.3.11. Suppose that $\alpha = (I_0^\alpha, \ldots, I_{r_\alpha}^\alpha, \varnothing), \beta = (I_0^\beta, \ldots, I_{r_\beta}^\beta, \varnothing) \in \mathbf{P}_{m,i}(\Phi)$, $\beta \precnsim \alpha$ and $\alpha \neq \beta$.

It follows from Eqn. (2.10) that

$$(r_\alpha \leqslant r_\beta) \text{ and } I_h^\alpha \subset I_h^\beta, 0 \leqslant h \leqslant r_\alpha.$$

In particular, this implies that $0 < \sum_{h=0}^{r_\alpha} \mathrm{card}(I_h^\alpha) < \sum_{h=0}^{r_\beta} \mathrm{card}(I_h^\beta)$. Since for any $(I_0, \ldots, I_r, \emptyset) \in \mathbf{P}_{m,i}(\Phi)$, we have that

$$\sum_{0 \leqslant h < r} \mathrm{card}(I_h) \leqslant m + r + 2,$$

$$\mathrm{card}(I_r) = 1,$$

and

$$r \leqslant m + 1,$$

it follows immediately that the length of a chain in $\mathbf{P}_{m,i}(\Phi)$ is bounded by $2m + 2$. □

The theorem follows from Claims 2.3.8, 2.3.9, 2.3.10 and 2.3.11. □

2.4 Simplicial replacement: algorithm

We begin with describing some preliminary algorithms that we will need.

2.4.1 Algorithmic Preliminaries

The following algorithm is described in [2]. We briefly recall the input, output and complexity. We made a small and harmless modification to the input by requiring that the closed semi-algebraic of which the covering is being computed is contained in the closed ball of radius R centered at the origin, rather than in the sphere of radius R. This is done to avoid complicating notation down the road and is not significant since the algorithm can be easily modified to accommodate this change without any change in the complexity estimates.

Algorithm 1 (Covering by Contractible Sets)

Input:

A. a finite set of s polynomials $\mathcal{P} \subset \mathrm{D}[\bar{\varepsilon}][X_1, \ldots, X_k]$ in strong k-general position on R^k, with $\deg(P_i) \leqslant d$ for $1 \leqslant i \leqslant s$,

B. a $\mathcal{P}$-closed formula ϕ such that semi-algebraic set $\mathcal{R}(\phi) \subset \overline{B_k(0,R)}$, for some $R > 0$, $R \in \mathrm{R}$.

Output:

(a) a finite set of polynomials $\mathcal{H} \subset \mathrm{D}[\bar{\varepsilon}, \bar{\zeta}][X_1, \ldots, X_k]$, where $\bar{\zeta} = (\zeta_1, \ldots, \zeta_{2\mathrm{card}(\mathcal{H})})$;

(b) a tuple of $\mathcal{H}$-formulas $(\theta_\alpha)_{\alpha \in I}$ such that each $\mathcal{R}(\theta_\alpha, \mathrm{R}\langle\bar{\varepsilon}, \bar{\zeta}\rangle^k), \alpha \in I$ is a closed semi-algebraically contractible set, and

(c)
$$\bigcup_{\alpha \in I} \mathcal{R}(\theta_\alpha, \mathrm{R}\langle\bar{\varepsilon}, \bar{\zeta}\rangle^k) = \mathcal{R}(\psi, \mathrm{R}\langle\bar{\varepsilon}, \bar{\zeta}\rangle^k).$$

Complexity: The complexity of the algorithm is bounded by $(\mathrm{card}(\mathcal{P})^{(k+1)^2} D^{k^{O(1)}}$, where $D = \max_{P \in \mathcal{P}} \deg_{\bar{X}, \bar{\varepsilon}}(P)$. Moreover,

$$\begin{aligned} \mathrm{card}(I), \mathrm{card}(\mathcal{H}) &\leqslant (\mathrm{card}(\mathcal{P}) D)^{k^{O(1)}}, \\ \deg_{\bar{Y}}(H), \deg_{\bar{\varepsilon}}(H), \deg_{\bar{\zeta}}(H) &\leqslant D^{k^{O(1)}}. \end{aligned}$$

Suppose that $\bar{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_t)$, and that each polynomial in $\mathcal{P}$ depends on at most m of the ε_i's. Then, each polynomial appearing in $\mathcal{H}$ depends on at most $m(k+1)^2$ of ε_i's, and on at most one of the ζ_i's.

Remark 9. *Note that the last claim in the complexity of Algorithm 1, namely that each polynomial appearing in any of the formulas θ_α depends on at most $m(k+1)^2$ of ε_i's, and on at most one of the ζ_i's, does not appear explicitly in [2], but is evident on a close examination of the algorithm. It is also reflected in the fact that the combinatorial part (i.e. the part depending on* $\mathrm{card}(\mathcal{P})$*) of the complexity of Algorithm 16.14 in [2] is bounded by* $\mathrm{card}(\mathcal{P})^{(k+1)^2}$*. This is because the Algorithm 16.14 in [2] has a "local property", namely that* *all computations involve at most a small number (in this case $(k+1)^2$) polynomials in the input at a time.*

Notation 13. *Let $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{D}[X_1, \ldots, X_k]$. For $1 \leqslant i \leqslant s$, let*

$$H_i = 1 + \sum_{1 \leqslant j \leqslant k} i^j X_j^{d'}.$$

where d' is the smallest number strictly bigger than the degree of all the polynomials in $\mathcal{P}$.

For ϕ a $\mathcal{P}$-closed formula, we will denote by $\phi^\star(\zeta)$ the formula obtained from ϕ by replacing any occurrence of $P_i \geqslant 0$ with $P_i \geqslant -\zeta H_i$, and any occurrence of $P_i \leqslant 0$ with $P_i \leqslant \zeta H_i$, for each $i, 1 \leqslant i \leqslant s$.

Definition 2.4.1. *Let $\mathcal{P} \subset \mathrm{R}[X_1, \ldots, X_k]$ be a finite set. We say that $\mathcal{P}$ is in* ℓ-general position, *if no more than ℓ polynomials belonging to $\mathcal{P}$ have a common zero in R^k.*

The set $\mathcal{P}$ is in strong ℓ-general position *if moreover any ℓ polynomials belonging to $\mathcal{P}$ have at most a finite number of common zeros in R^k.*

Lemma 2.4.1. *The set*

$$\mathrm{Def}(\mathcal{P}, \zeta) = \{P_i \pm \zeta H_i \mid 1 \leqslant i \leqslant s\}$$

is in strong k-general position.

Proof. See proof of Proposition 13.6 in [2]. □

Lemma 2.4.2. *Let $R \in \mathrm{R}, R > 0$. The semi-algebraic set $\mathrm{ext}(\mathcal{R}(\phi, \overline{B_k(0,R)}), \mathrm{R}\langle\zeta\rangle)$ is semi-algebraically homotopy equivalent to $\mathcal{R}(\phi^\star(\zeta), \cap\overline{B_k(0,R)})$.*

Proof. Follows from Lemma 2.3.1. □

2.4.2 Algorithm for computing simplicial replacement

We now describe an algorithm that given a tuple of formula Φ and $m, i \geqslant 0$, computes the corresponding poset $\mathbf{P}_{m,i}(\Phi)$, using Algorithm 1 to compute $\mathcal{I}_{j,k}(\phi)$ and $\mathcal{C}_{j,k}(\phi)$ for different j and ϕ which arise in the course of the execution of the algorithm.

Algorithm 2 (Computing the poset $\mathbf{P}_{m,i}(\Phi)$)

Input:

(a) $\ell, 0 \leqslant \ell \leqslant k$, $m, -1 \leqslant m \leqslant \ell$, $i, 0 \leqslant i \leqslant m+2$.

(b) A finite set of polynomials $\mathcal{P} \subset \mathrm{D}[\bar{\varepsilon}_0, \ldots, \bar{\varepsilon}_i][X_1, \ldots, X_k]$, where D is an ordered domain contained in a real closed field R.

(c) An element $r \in \mathrm{D}$, $r > 0$.

(d) For each $j, 0 \leqslant j \leqslant N$, a $\mathcal{P}$-closed formula ϕ_j, such that $\mathcal{R}(\phi_j, \overline{B_k(0, 1/r)})$ is homologically ℓ-connected (and ℓ-connected if $\mathrm{R} = \mathbb{R}$).

Output:

The poset $\mathbf{P}_{m,i}(\Phi)$ (see Definition 2.3.3), where Φ is defined by $\Phi(j) = \phi_j, j \in [N]$, and the various $\mathcal{I}_{\cdot,k}(\cdot)$ $\mathcal{C}_{\cdot,k}(\cdot)$ are obtained by calls to Algorithm 1.

Procedure:

1: $J \leftarrow [N]$.

2: **if** $m = -1$ **then**

3: Output

$$\mathbf{P}_{-1,i}(\Phi) = \{(\{j\}, \phi) \mid j \in J\},$$

and the order relation to be the trivial one – namely for $j, j' \in J$,

$$(\{j\}, \varnothing) \prec (\{j'\}, \varnothing) \Leftrightarrow j = j'.$$

4: **else**

5:

$$\mathcal{P} \leftarrow \mathcal{P} \cup \left\{ r^2 \sum_{i=1}^{k} X_i^2 - 1 \right\}.$$

6: **for** $j \in J$ **do**

7:

$$\Phi(j) \leftarrow \Phi(j) \wedge \left(r^2 \sum_{i=0}^{k} X_i^2 - 1 \leqslant 0 \right).$$

8: **end for**

for each subset $I \subset_{\leqslant m+2} J$ **do**

Use Definition 2.3.2 to compute $J_{m,i,I,\Phi}$ and $\Phi_{m,i,I,J}$, using Algorithm 1 with input $\mathrm{Def}(\mathcal{P}, \varepsilon)$, and the formula $\bigwedge_{j \in I} \Phi(j)^{\star}(\varepsilon)$, (noting that $\mathcal{R}(\bigwedge_{j \in I} \Phi(j)^{\star}(\varepsilon))$ is contained in $\overline{B_k(0, 2/r)}$), to compute $\mathcal{I}_{i,k}(\bigwedge_{j \in I} \Phi(j))$ and $\mathcal{C}_{i,k}((\bigwedge_{j \in I} \Phi(j)))$. The polynomials appearing in the formulas in $\mathcal{C}_{i,k}((\bigwedge_{j \in I} \Phi(j)))$ have coefficients in $\mathrm{D}[\bar{\varepsilon}_0, \ldots, \bar{\varepsilon}_i, \bar{\varepsilon}_{i+1}]$, where $\bar{\varepsilon}_{i+1} = (\varepsilon, \bar{\zeta})$, and $\bar{\zeta}$ is a new tuple of infinitesimals.

end for

for $I \subset J, 1 < \mathrm{card}(I) \leqslant m+2$ **do**

Use Algorithm 2 recursively with input $\ell, m - \mathrm{card}(I) + 1, i + 1, \mathcal{P}_I, \Phi_{m,i,I,J}, r$, where $\mathcal{P}_I \subset \mathrm{D}[\bar{\varepsilon}_0, \ldots, \bar{\varepsilon}_{i+1}]$ is the set of polynomials occurring in $\Phi_{m,i,I,J}$.

$$\mathbf{P}_{m,i}(\Phi) \leftarrow \{(\{j\}, \phi) \mid j \in J\} \cup \bigcup_{I \subset J, 1 < \mathrm{card}(I) \leqslant m+2} \{I\} \times \mathbf{P}_{m-\mathrm{card}(I)+1, i+1}(\Phi_{m,i,I,J}).$$

Define partial order $\prec$ on $\mathbf{P}_{m,i}(\Phi)$ as in Definition 2.3.3.

end for

end if

Complexity: The complexity of the algorithm, as well as $\mathrm{card}(\mathbf{P}_{m,i}(\Phi))$, are bounded by

$$(Nsd)^{k^{O(m)}},$$

where $s = \mathrm{card}(\mathcal{P})$, and $d = \max_{P \in \mathcal{P}} \deg(P)$.

Proof of correctness. The algorithm follows Definition 2.3.3. □

Complexity analysis. The bound on $\mathrm{card}(\mathbf{P}_{m,i}(\Phi))$ is a consequence of Theorem 2.3.2. The complexity of the algorithm follows from the complexity of the Algorithm 1 and an argument as in the proof of Theorem 2.3.2.

There is one additional point to note that in the recursive calls algorithm the arithmetic operations take place in a larger ring, namely - $\mathrm{D}[\bar{\varepsilon}_0, \ldots, \bar{\varepsilon}_{m+2}]$.

It follows from the complexity of Algorithm 1 that the number of different infinitesimals occurring in each polynomial that is computed in the course of Algorithm 2 is bounded by $k^{O(m)}$, and these infinitesimals occur with degrees bounded by $d^{k^{O(m)}}$. Hence each arithmetic operation involving the coefficients with these polynomials costs $\left(d^{k^{O(m)}}\right)^{k^{O(m)}} = d^{k^{O(m)}}$ arithmetic operations in the ring D. This does not affect the asymptotics of the complexity, where we measure arithmetic operations in the ring D. □

Remark 10. *Suppose we define (following the same notation as in Properties 2 and 2′ and Algorithm 2) for* $\phi \in \mathcal{F}_{\mathrm{R}_i,k}$*,*

$$\begin{aligned}\mathcal{I}_{i,k}(\Phi) &= \mathrm{card}(I) - 1,\\ \mathcal{C}_{i,k}(\Phi) &= (\theta_\alpha)_{\alpha\in I},\end{aligned}$$

where $(\theta_\alpha)_{\alpha\in I}$ *is the output of Algorithm 1 with input the set of polynomials appearing in the definition of* $\phi^*(\varepsilon)$*, the closed formula* $\phi^*(\varepsilon)$*, and* R *set to* $1/r$ *(as in Line 10 of Algorithm 2).*

Then it follows from the correctness of Algorithm 1, that (denoting by $\mathrm{R}_i = \mathrm{R}\langle\bar{\varepsilon}_0,\ldots,\bar{\varepsilon}_i\rangle$ *as in Algorithm 2) the tuple*

$$((\mathrm{R}_i)_{i\geqslant 0}, 1/r, k, (\mathcal{I}_{i,k})_{i\geqslant 0}, (\mathcal{C}_{i,k})_{i\geqslant 0})$$

satisfies the homological ℓ*-connectivity property over* R *(resp.* ℓ*-connectivity property if* $\mathrm{R} = \mathbb{R}$*) for every* $\ell \geqslant 0$ *(see Property 2 and Property 2′).*

Algorithm 3 (Simplicial replacement)

Input:

(a) A finite set of polynomials $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$ where D is an ordered domain contained in a real closed field R.

(b) An integer $N \geqslant 0$, and for each $i \in [N]$, a $\mathcal{P}$-closed formula ϕ_i.

(c) $\ell, 0 \leqslant \ell \leqslant k$.

Output:

A simplicial complex Δ and for each $I \subset [N]$ a subcomplex $\Delta_I \subset \Delta$ such that there is a diagrammatic homological ℓ-equivalence

$$(I \mapsto \Delta_I)_{I \subset [N]} \overset{h}{\sim}_\ell \mathbf{Simp}^{[N]}(\mathcal{R}(\Phi)),$$

where $\Phi(i) = \phi_i, i \in [N]$. In case $\mathrm{R} = \mathbb{R}$, then the simplicial complex Δ and the subcomplexes Δ_I satisfy the stronger property, namely:

$$(I \mapsto \Delta_I)_{I \subset [N]} \sim_\ell \mathbf{Simp}^{[N]}(\mathcal{R}(\Phi)),$$

where $\Phi(i) = \phi_i, i \in [N]$.

Procedure:

1: Let $0 < \varepsilon_0 \ll \delta < 1$ be infinitesimals.

2: **for** $0 \leqslant i \leqslant N$ **do**

3: Call Algorithm 1 with input $\mathrm{Def}(\mathcal{P}, \varepsilon_0)$, the formula $\phi_i^\star(\varepsilon_0) \wedge 4\delta^2(X_1^2 + \cdots + X_k^2) - 1 \leqslant 0$ as input and let $\Phi_i = (\phi_{i,1}, \ldots, \phi_{i,N_i})$ be the output.

4: $\mathcal{P}_i \leftarrow$ the set of polynomials appearing in the formula Φ_i.

5: **end for**

6: $\mathcal{P}' \leftarrow \bigcup_{i \in [N]} \mathcal{P}_i$.

7: **for** $0 \leqslant i \leqslant n$ **do**

8: $J_i \leftarrow \{(i,j) \mid 1 \leqslant j \leqslant N_i\}$.

9: **end for**

$J \leftarrow \bigcup_{i \in [N]} J_i$.

Let $\Psi \in (\mathcal{F}_{\mathrm{R}\langle \delta, \varepsilon_0 \rangle, k})^J$ be defined by $\Psi((i,j)) = \phi_{i,j}$.

Call Algorithm 2 with input

$$(\ell + 1, m + 1, 0, \mathcal{P}', J, \delta, \Psi),$$

and let $\mathbf{P}_{m,0}(\Psi)$ denote the output.

Output the simplicial complex $\Delta(\mathbf{P}_{m,0}(\Psi))$, and for each subset $I \subset [N]$, the subcomplex $\Delta(\mathbf{P}_{m,0}(\Psi|_{\bigcup_{i \in I} J_i}))$.

Complexity: The complexity of the algorithm is bounded by $(sd)^{k^{O(\ell)}}$, where $s = \mathrm{card}(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.

Proof of correctness. The correctness of the algorithm follows from the correctness of Algorithm 2, Remark 10, and Theorems 2.3.1 and 2.3.1′. □

Complexity analysis. The complexity bound follows from the complexity bounds of Algorithms 1 and 2. □

Proofs of Theorems 2.2.1 and 2.2.1′. Both theorems now follows from the correctness and the complexity analysis Algorithm 3. □

2.5 Implementation

We implemented the construction corresponding to the simplified view of the poset, $\mathbf{P}_m(\mathcal{S})$, and will report some of its results. Our implementation closely follows the construction; however, to facilitate implementation and not to work directly with semi-algebraic sets, we made the following assumptions:

1. The input of the procedure is an index set J corresponding to the elements of a contractible cover of $\mathcal{S}$, and an integer m.

2. There exists a black-box that takes an index set I ($I \subset J$), and returns an index set J_I corresponded to the elements of $\mathcal{C}(\mathcal{S}_I)$.

One technical detail regarding the implementation (and the output) is the naming convention of the elements of the poset, which is slightly different from what we described in Definition 2.3.3. More precisely, in our implementation, the elements of the poset are tuples of length two (tuples with length > 2 are divided into two parts using parenthesis, recursively). That is because when the procedure begins with a given index set I, it first uses the black-box to obtain a set of indices J_I corresponding to the elements of $\mathcal{C}(\mathcal{S}_I)$. Then the elements of the new cover are labeled by (I, i) for $i \in J_I$, and the procedure continues recursively.

We incorporated SageMath [6] Python modules to construct the order complex from the poset and to compute the Betti numbers. The following simple examples demonstrate the output of the program.

Example 4. ***(Sphere with a cover of size two)***

In this example, similar to the example given in Section 2.3.5, we have a two-dimensional unit sphere in $\mathbb{R}^3$ covered by two closed hemispheres labeled by 0 and 1. Below is the output of the program with input $J = \{0, 1\}$ and $m = 3$.

```
================Sphere==================
Betti numbers:{0: 1, 1: 0, 2: 1}
height of the poset: 3
width of the poset: 2
number of elements: 6
number of relations: 12
------------------------
Elements:
([0], [])
([1], [])
([0, 1], ([([0, 1], 0)], []))
([0, 1], ([([0, 1], 1)], []))
([0, 1], ([([0, 1], 0), ([0, 1], 1)], ([([([0, 1], 0), ([0, 1], 1)], 0)], [])))
([0, 1], ([([0, 1], 0), ([0, 1], 1)], ([([([0, 1], 0), ([0, 1], 1)], 1)], [])))

------------------------
Order relations:
('([0, 1], ([([0, 1], 0)], []))', '([0], [])')
('([0, 1], ([([0, 1], 1)], []))', '([0], [])')
('([0, 1], ([([0, 1], 0), ([0, 1], 1)], ([([([0, 1], 0), ([0, 1], 1)], 0)], [])))',
                                                                  '([0], [])')
('([0, 1], ([([0, 1], 0), ([0, 1], 1)], ([([([0, 1], 0), ([0, 1], 1)], 1)], [])))',
                                                                  '([0], [])')
('([0, 1], ([([0, 1], 0)], []))',
                          '([1], [])')
('([0, 1], ([([0, 1], 1)], []))',
                          '([1], [])')
```

[6] https://github.com/sagemath/sage

```
('([0, 1], ([([0, 1], 0), ([0, 1], 1)], ([(([([0, 1], 0), ([0, 1], 1)], 0)], []))))',
                                                                                  '([1], [])')
('([0, 1], ([([0, 1], 0), ([0, 1], 1)], ([(([([0, 1], 0), ([0, 1], 1)], 1)], []))))',
                                                                                  '([1], [])')
('([0, 1], ([([0, 1], 0), ([0, 1], 1)], ([(([([0, 1], 0), ([0, 1], 1)], 0)], []))))',
                                                   '([0, 1], ([([0, 1], 0)], []))')
('([0, 1], ([([0, 1], 0), ([0, 1], 1)], ([(([([0, 1], 0), ([0, 1], 1)], 1)], []))))',
                                                   '([0, 1], ([([0, 1], 0)], []))')
('([0, 1], ([([0, 1], 0), ([0, 1], 1)], ([(([([0, 1], 0), ([0, 1], 1)], 0)], []))))',
                                                   '([0, 1], ([([0, 1], 1)], []))')
('([0, 1], ([([0, 1], 0), ([0, 1], 1)], ([(([([0, 1], 0), ([0, 1], 1)], 1)], []))))',
                                                   '([0, 1], ([([0, 1], 1)], []))')
```

Example 5. ***(Sphere with a cover of size four)***

In this example again, we have a two-dimensional unit sphere in $\mathbb{R}^3$*, however, covered by four elements as shown in Figure 2.9. Below is the output of the program with input* $J = \{0, 1, 2, 3\}$ *and* $m = 3$ *(order relations are omitted).*

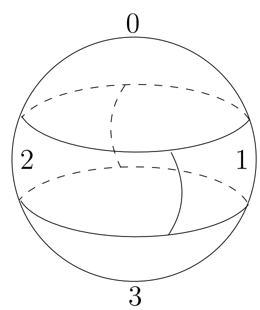


Figure 2.9. A contractible cover of Sphere by four elements.

```
================Sphere===================
Betti numbers:{0: 1, 1: 0, 2: 1}
height of the poset: 3
width of the poset: 18
number of elements: 38
number of relations: 108
————————————————————
Elements:
([0], [])
([1], [])
([2], [])
([3], [])
([0, 1, 2], ([([0, 1, 2], 0)], []))
([0, 1, 2], ([([0, 1, 2], 1)], []))
([1, 2, 3], ([([1, 2, 3], 0)], []))
```

```
([1, 2, 3], ([([1, 2, 3], 1)], []))
([0, 1], ([([0, 1], 0)], []))
([0, 1], ([([0, 1, 2], 0)], []))
([0, 1], ([([0, 1, 2], 1)], []))
([0, 1], ([([0, 1], 0), ([0, 1, 2], 0)], ([([([0, 1], 0), ([0, 1, 2], 0)], 0)], [])))
([0, 1], ([([0, 1], 0), ([0, 1, 2], 1)], ([([([0, 1], 0), ([0, 1, 2], 1)], 0)], [])))
([0, 2], ([([0, 2], 0)], []))
([0, 2], ([([0, 1, 2], 0)], []))
([0, 2], ([([0, 1, 2], 1)], []))
([0, 2], ([([0, 2], 0), ([0, 1, 2], 0)], ([([([0, 2], 0), ([0, 1, 2], 0)], 0)], [])))
([0, 2], ([([0, 2], 0), ([0, 1, 2], 1)], ([([([0, 2], 0), ([0, 1, 2], 1)], 0)], [])))
([1, 2], ([([1, 2], 0)], []))
([1, 2], ([([1, 2], 1)], []))
([1, 2], ([([0, 1, 2], 0)], []))
([1, 2], ([([0, 1, 2], 1)], []))
([1, 2], ([([1, 2, 3], 0)], []))
([1, 2], ([([1, 2, 3], 1)], []))
([1, 2], ([([1, 2], 0), ([0, 1, 2], 0)], ([([([1, 2], 0), ([0, 1, 2], 0)], 0)], [])))
([1, 2], ([([1, 2], 0), ([1, 2, 3], 0)], ([([([1, 2], 0), ([1, 2, 3], 0)], 0)], [])))
([1, 2], ([([1, 2], 1), ([0, 1, 2], 1)], ([([([1, 2], 1), ([0, 1, 2], 1)], 0)], [])))
([1, 2], ([([1, 2], 1), ([1, 2, 3], 1)], ([([([1, 2], 1), ([1, 2, 3], 1)], 0)], [])))
([1, 3], ([([1, 3], 0)], []))
([1, 3], ([([1, 2, 3], 0)], []))
([1, 3], ([([1, 2, 3], 1)], []))
([1, 3], ([([1, 3], 0), ([1, 2, 3], 0)], ([([([1, 3], 0), ([1, 2, 3], 0)], 0)], [])))
([1, 3], ([([1, 3], 0), ([1, 2, 3], 1)], ([([([1, 3], 0), ([1, 2, 3], 1)], 0)], [])))
([2, 3], ([([2, 3], 0)], []))
([2, 3], ([([1, 2, 3], 0)], []))
([2, 3], ([([1, 2, 3], 1)], []))
([2, 3], ([([2, 3], 0), ([1, 2, 3], 0)], ([([([2, 3], 0), ([1, 2, 3], 0)], 0)], [])))
([2, 3], ([([2, 3], 0), ([1, 2, 3], 1)], ([([([2, 3], 0), ([1, 2, 3], 1)], 0)], [])))
```

Example 6. ***(Torus with a cover of size four)***

In this example, let $\mathcal{S}$ *be the two-dimensional torus embedded in* $\mathbb{R}^3$*, covered by four elements in the following way. A horizontal cut slices torus into two pieces, followed by two vertical cuts to form four contractible elements of the covering. Below is the output of the program with input* $J = \{0, 1, 2, 3\}$ *and* $m = 2$ *(elements and the order relations are omitted).*

```
================Torus====================
Betti numbers:{0: 1, 1: 2, 2: 1, 3: 648, 4: 0}
height of the poset: 5
width of the poset: 1520
number of elements: 2808
number of relations: 17400
```

Example 6 demonstrates that the singly exponential size of the poset could be formidably expensive for practical applications.

2.6 Future work and open problems

We conclude this chapter by stating some open problems and possible future directions of research in this area.

1. It is an interesting problem to try to make the poset $\mathbf{P}_{m,i}(\Phi)$ in Theorem 2.3.1 smaller in size and more efficiently computable. For instance, in Theorem 2.3.2 one should be able to improve the dependence on $\mathrm{card}(J)$.

2. There are some recent work in algorithmic semi-algebraic geometry where algorithms have been developed for computing the first few Betti numbers of semi-algebraic subsets of R^k having special properties. For example, in [27] the authors give an algorithm to compute the first ℓ Betti numbers of semi-algebraic subsets of R^k defined by *symmetric* polynomials of degrees bounded by some constant d. The complexity of the algorithm is doubly exponential in both d and ℓ (though polynomial in k for fixed d and ℓ). This algorithm uses semi-algebraic triangulations which leads to the doubly exponential complexity. It is an interesting problem to investigate whether by applying the efficient simplicial replacement of the current work the dependence on d and ℓ can be improved.

3. PERSISTENT HOMOLOGY OF SEMI-ALGEBRAIC SETS

In this chapter, we consider the problem of computing persistent homology groups of semi-algebraic sets by a filtration of the sub-level sets of a polynomial. We believe the previous literature has not addressed this problem. First, in Section 3.1, we provide an overview of the main contributions. Then in Section 3.2, we give the precise statements of the main result after introducing the necessary definitions. In Section 3.3, we prove the key proposition (Proposition 3.3.3) which allows us to efficiently reduce the filtration of the sub-level sets of a polynomial to a finite filtrations. In Section 3.4, after introducing certain necessary preliminaries, we describe our algorithm for computing barcodes of semi-algebraic filtrations and analyze its complexity (thereby proving Theorem 3.2.1). Finally, in Section 3.5 we state some open questions and directions for future work in this area.

3.1 Introduction

Persistent homology groups are associated to a filtration of topological spaces; hence, they generalize ordinary homology groups where the filtration is trivial (i.e., constant). The current state of the problem of computing homology groups of semi-algebraic sets was discussed thoroughly in the previous chapter (see Section 2.1). We begin here with a motivational example in Topological Data Analysis.

Given a subset $X = \{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$ of R^k and a distance function (e.g., Euclidean distance)—X represents as the data points sampled from a sub-manifold M of R^k—the goal is to infer the topological properties of the underlying structure of X (i.e., space M). At a high level, a common approach to accomplish this is to construct a filtration of simplicial complexes over X. One of the well-known filtration, is the filtration of *Čech complexes* [3] which can be described as follows.

Let $S \subset \mathrm{R}^{k+1}$, be the semi-algebraic set defined by the formula,

$$\phi(X_1, \ldots, X_k, T) := \bigvee_{i=1}^{n} \left(|\mathbf{X} - \mathbf{x}^{(i)}|^2 - T \leqslant 0 \right),$$

where $\mathbf{X} = (X_1, \ldots, X_k)$. We take the polynomial $P = T$, to construct the filtration. For $t \geqslant 0$, we denote by S_t the union of closed Euclidean balls, $\overline{B_k(\mathbf{x}, t)}$, of radius t centered at the points $\mathbf{x}$. Notice that each S_t is a semi-algebraic set indexed by t. In particular, $S_0 = X$. Also, for $0 \leqslant t \leqslant t'$, we have that $S_t \subset S_{t'}$. Thus, $\mathcal{F} = (S_t)_{t \geqslant 0}$ is an increasing family of semi-algebraic sets indexed by $t \geqslant 0$. This is an example of a semi-algebraic filtration. The main rationale for considering this filtration is that nerve complex of the family of convex sets $\overline{B_k(\mathbf{x}, t)}$, $\mathbf{x} \in \mathbf{X}$ approximates homotopically the underlying manifold M, and each homology class of M would show up in the homology of S_t for some values of t. Hence, different homology groups would appear by varying the parameter t. A diagram known as *barcode* is used to illustrate the lifetime of homology classes. In the barcode, each homology class is represented as a horizontal line segment. The line segments span the period that the corresponding topological properties exist along the parameter axis (i.e., t). The barcode of the filtration $(S_t)_{t \geqslant 0}$ is a tool for filtering out spurious homology (noise) from that which genuinely reflects the topology of M (see [3], [4], [28]). In the semi-algebraic world, they play a similar role—for example, as a measure of topological similarity of two given semi-algebraic sets, which is much finer (because of the presence of the continuous parameters) to just the sequence of Betti numbers.

As stated earlier, the main goal in this chapter is to design an efficient (singly exponential complexity algorithm) that takes as input a quantifier-free formula describing a closed semi-algebraic set $S \subset \mathrm{R}^k$ as well as a polynomial $P \in \mathrm{R}[X_1, \ldots, X_k]$, and outputs the barcodes up to dimension ℓ for some fixed $\ell \geqslant 0$ of the filtration of S by the sub-level sets of the function P on S, thereby generalizing the algorithm in [8] for computing the first ℓ Betti numbers with a similar complexity. There are several intermediate steps needed to achieve this goal. These intermediate steps have been used recently in other applications (that we mention in Section 3.1.1 below) and hence could be of independent interest. We outline them below.

3.1.1 Main contributions

The main contributions of our work are as follows.

1. We reformulate the definition of barcodes in order to treat continuous as well as finite filtrations in a uniform manner. This is important in the current application since we consider filtrations of semi-algebraic sets by polynomial functions which are by nature examples of continuous filtrations (since they are indexed by R). However, we show that the barcode of this continuous filtration is equal to another finite one (see Propositions 3.3.1 and 3.3.3). In order for such an equality to make sense it is important that persistent homology of a filtration should be defined in a uniform way for arbitrary ordered index set. It is possible to have a completely categorical description of persistent homology which applies to very general filtration [29]. We avoid categorical language and give an *elementary definition of barcodes* directly in terms of sub-quotients of homology groups (see Definition 3.2.5) which we believe could be useful in other applications as well. For example, the definition given in this chapter is used in a crucial way in [30] to define *harmonic barcodes* (which carry more information than the classical barcodes). In this application it is very important to identify the persistent homology spaces with certain subspaces of the chain spaces of the ambient simplicial complex. This is possible using our definitions.

2. We give a definition of barcodes for semi-algebraic maps which are not necessarily proper (Definition 3.2.8) generalizing the one for proper maps—and we believe that this could form the basis of generalizing the results of the current papers to arbitrary semi-algebraic sets and maps.

3. By an application of a standard theorem in real algebraic geometry (Hardt triviality theorem [31]) we can deduce that the topological type of the sub-level sets of a filtration of a semi-algebraic set by a semi-algebraic function changes at only finitely many values of the function. This implies that the barcode of the original filtration is equal to that of a finite filtration (after proper definition of barcodes encompassing both the finite and the continuous case as mentioned earlier). However, an algorithm based on Hardt

triviality theorem would inevitably lead to a doubly exponential sized filtration—since the proof of this theorem (see for example proof of [2, Theorem 5.46]) depends on taking semi-algebraic triangulations for which only a doubly exponential complexity algorithm is known to exist. A second important contribution of the current work is an algorithm with singly exponential complexity (see Algorithm 6 below) for reducing a given continuous filtration of a semi-algebraic set by a polynomial to a filtration of simplicial complexes indexed by a *finite* subset of R, such that the barcode of this finite filtration is equal to that of the continuous filtration in dimensions up to ℓ. The two main ingredients for this algorithms are:

(a) mathematical techniques introduced in [32] for bounding the number of homotopy types of fibers of a semi-algebraic map;

(b) the algorithm for efficiently computing simplicial replacements of semi-algebraic sets—Theorem 2.2.1 in Chaper 2.

We note that Algorithm 6 has other applications as well. For example, it plays a key role in a recent work on computing a homology basis of the first homology group of a given semi-algebraic set with singly exponential complexity [33].

4. The last (and perhaps the most important) contribution is an algorithm with a singly exponential complexity that computes the barcodes of a semi-algebraic filtration up to dimension ℓ for any fixed $\ell \geqslant 0$. After having reduced to the case of finite semi-algebraic filtration using Algorithm 6, we then compute the barcode of this finite filtration of finite simplicial complexes (cf. Algorithms 7 and 8) using Definition 3.2.5 and standard algorithms from linear algebra.

We remark that it is plausible that after ensuring the finiteness of the filtration, the last step of computing the barcode could be achieved by an appropriate extension of the algorithm for computing the first few Betti numbers of semi-algebraic sets described in [8]. However, this extension would be non-trivial and we prefer to use directly Algorithm 2 in Chapter 2 for which no extension is needed.

The importance of the assumption that the input semi-algebraic subset be closed and bounded is discussed in Section 3.2.2.

3.2 Precise definitions and statements of the main results

In this section, we define precisely persistent homology and barcodes of filtrations in Section 3.2.1. Then in Section 3.2.2 we define semi-algebraic filtrations and state the main algorithmic result of this section (Theorem 3.2.1).

3.2.1 Persistent homology and barcodes

Let T be an ordered set, and $\mathcal{F} = (X_t)_{t \in T}$, a tuple of subspaces of X, such that $s \leqslant t \Rightarrow X_s \subset X_t$. We call $\mathcal{F}$ a filtration of the topological space X.

We now recall the definition of the *persistent homology groups* associated to a filtration [28], [34]. Since we only consider homology groups with rational coefficients, all homology groups in what follows are finite dimensional $\mathbb{Q}$-vector spaces.

Notation 14. *For $s, t \in T, s \leqslant t$, and $p \geqslant 0$, we let $i_p^{s,t} : \mathrm{H}_p(X_s) \longrightarrow \mathrm{H}_p(X_t)$, denote the homomorphism induced by the inclusion $X_s \hookrightarrow X_t$.*

Definition 3.2.1. *[28] For each triple $(p, s, t) \in \mathbb{Z}_{\geqslant 0} \times T \times T$ with $s \leqslant t$ the* persistent homology group, *$\mathrm{H}_p^{s,t}(\mathcal{F})$ is defined by*

$$\mathrm{H}_p^{s,t}(\mathcal{F}) = \mathrm{Im}(i_p^{s,t}).$$

Note that $\mathrm{H}_p^{s,t}(\mathcal{F}) \subset \mathrm{H}_p(X_t)$, and $\mathrm{H}_p^{s,s}(\mathcal{F}) = H_p(X_s)$.

Notation 15. *We denote by $b_p^{s,t}(\mathcal{F}) = \dim_{\mathbb{Q}}(\mathrm{H}_p^{s,t}(\mathcal{F}))$.*

Persistent homology measures how long a homology class persists in the filtration, in other words considering the homology classes as topological features, it gives an insight about the time (thinking of the indexing set T of the filtration as time) that a topological feature appears (or is born) and the time it disappears (or dies). This is made precise as follows.

Definition 3.2.2. *For $s \leqslant t \in T$, and $p \geqslant 0$,*

- *we say that a homology class $\gamma \in \mathrm{H}_p(X_s)$ is* born *at time s, if $\gamma \notin \mathrm{H}_p^{s',s}(\mathcal{F})$, for any $s' < s$;*
- *for a class $\gamma \in \mathrm{H}_p(X_s)$ born at time s, we say that γ* dies *at time t,*
  - *if $i_p^{s,t'}(\gamma) \notin \mathrm{H}_p^{s',t'}(\mathcal{F})$ for all s', t' such that $s' < s \leqslant t' < t$,*
  - *but $i_p^{s,t}(\gamma) \in \mathrm{H}_p^{s'',t}(\mathcal{F})$, for some $s'' < s$.*

Remark 11. *Note that the homology classes that are born at time s, and those that are born at time s and dies at time t, as defined above are not subspaces of $\mathrm{H}_p(X_s)$. In order to be able to associate a "multiplicity" to the set of homology classes which are born at time s and dies at time t we interpret them as classes in certain* subquotients *of $\mathrm{H}_*(X_s)$ in what follows.*

First observe that it follows from Definition 3.2.1 that for all $s' \leqslant s \leqslant t$ and $p \geqslant 0$, $\mathrm{H}_p^{s',t}(\mathcal{F})$ is a subspace of $\mathrm{H}_p^{s,t}(\mathcal{F})$, and both are subspaces of $\mathrm{H}_p(X_t)$. This is because the homomorphism $i_p^{s',t} = i_p^{s,t} \circ i_p^{s',s}$, and so the image of $i_p^{s',t}$ is contained in the image of $i_p^{s,t}$. It follows that, for $s \leqslant t$, the union of $\bigcup_{s'<s} \mathrm{H}_p^{s',t}(\mathcal{F})$ is an increasing union of subspaces, and is itself a subspace of $\mathrm{H}_p(X_t)$. In particular, setting $t = s$, $\bigcup_{s'<s} \mathrm{H}^{s',s}(\mathcal{F})$ is a subspace of $\mathrm{H}_p(X_s)$.

With the same notation as above:

Definition 3.2.3 (Subspaces of $\mathrm{H}_p(X_s)$)**.** *For $s \leqslant t$, and $p \geqslant 0$, we define*

$$
\begin{aligned}
M_p^{s,t}(\mathcal{F}) &= \bigcup_{s'<s} (i_p^{s,t})^{-1}(\mathrm{H}_p^{s',t}(\mathcal{F})), \\
N_p^{s,t}(\mathcal{F}) &= \bigcup_{s'<s\leqslant t'<t} (i_p^{s,t'})^{-1}(\mathrm{H}_p^{s',t'}(\mathcal{F})),
\end{aligned}
$$

Remark 12. *The "meaning" of these subspaces are as follows.*

(a) For every fixed $s \in T$, $M_p^{s,t}(\mathcal{F})$ is a subspace of $\mathrm{H}_p(X_s)$ consisting of homology classes in $\mathrm{H}_p(X_s)$ which are

"born before time s, or born at time s and dies at t or earlier"

(b) Similarly, for every fixed $s \in T$, $N_p^{s,t}(\mathcal{F})$ is a subspace of $\mathrm{H}_p(X_s)$ consisting of homology classes in $\mathrm{H}_p(X_s)$ which are

"born before time s, or born at time s and dies strictly earlier than t"

The dimensions of $M_p^{s,t}(\mathcal{F})$ and $N_p^{s,t}(\mathcal{F})$ are given in Eqn. (3.10) and (3.11) in Proposition 3.3.4 below.

We now define certain *subquotients* of the homology groups of $\mathrm{H}_p(X_s), s \in T, p \geqslant 0$, in terms of the subspaces defined above in Definition 3.2.3.

Definition 3.2.4 (Subquotients associated to a filtration)**.** *For $s \leqslant t$, and $p \geqslant 0$, we define*

$$\begin{aligned} P_p^{s,t}(\mathcal{F}) &= M_p^{s,t}(\mathcal{F})/N_p^{s,t}(\mathcal{F}), \\ P_p^{s,\infty}(\mathcal{F}) &= \mathrm{H}_p(X_s)/\bigcup_{s \leqslant t} M_p^{s,t}(\mathcal{F}). \end{aligned}$$

We will call

(a) $P_p^{s,t}(\mathcal{F})$ the space of p-dimensional cycles born at time s and which dies at time t*; and*

(b) $P_p^{s,\infty}(\mathcal{F})$ the space of p-dimensional cycles born at time s and which never die*.*

Remark 13. *Notice that $M_p^{s,t}(\mathcal{F}) \subset M_p^{s,t'}(\mathcal{F})$ for $t \leqslant t'$, and hence $\bigcup_{s \leqslant t} M_p^{s,t}(\mathcal{F})$ is a subspace of $\mathrm{H}_p(X_s)$, and $N_p^{s,t}(\mathcal{F})$ is a subspace of $M_p^{s,t}(\mathcal{F})$. Therefore, these subquotients are vector spaces and have well defined dimensions.*

Finally, we are able to achieve our goal of defining the multiplicity of a bar as the dimension of an associated vector space and define the barcode of a filtration.

Definition 3.2.5 (Persistent multiplicity, barcode)**.** *We will denote for $s \in T, t \in T \cup \{\infty\}$,*

$$\mu_p^{s,t}(\mathcal{F}) = \dim P_p^{s,t}(\mathcal{F}), \tag{3.1}$$

and call $\mu_p^{s,t}(\mathcal{F})$ the persistent multiplicity of p-dimensional cycles born at time s and dying at time t if $t \neq \infty$, or never dying in case $t = \infty$*.*

Finally, we will call the set

$$\mathcal{B}_p(\mathcal{F}) = \{(s, t, \mu_p^{s,t}(\mathcal{F})) \mid \mu_p^{s,t}(\mathcal{F}) > 0\} \tag{3.2}$$

the p-dimensional barcode associated to the filtration $\mathcal{F}$*.*

We will call an element $b = (s, t, \mu_p^{s,t}(\mathcal{F})) \in \mathcal{B}_p(\mathcal{F})$ a bar of $\mathcal{F}$ of multiplicity $\mu_p^{s,t}(\mathcal{F})$*.*

Remark 14. *Note that the notion of* persistent multiplicity *has been defined previously in the context of finite filtrations (see [35]). The definition of $\mu_p^{s,t}(\mathcal{F})$ given in Eqn. (3.1) generalizes that given in* loc.cit. *in the case of finite filtrations, who defined it using Eqn. (3.9) in Proposition 3.3.4 stated below. Our definition gives a geometric meaning to this number as a dimension of a certain vector space (a subquotient of $\mathrm{H}_p(X_s)$), and we prove that it agrees with that given in* loc.cit. *in Proposition 3.3.4. Also, it is important to note for what follows that our definition of a barcode applies uniformly to all filtrations with index coming from an ordered set, and we make no additional assumption on the indexing set.*

Remark 15 (Continuous vs finite filtrations)**.** *In most applications the filtration $\mathcal{F}$ is assumed to be finite (i.e. the ordered set T is finite). Since we are considering filtration of semi-algebraic sets by the sub-level sets of a polynomial function, our filtration is indexed by* R *and is an example of a continuous (infinite) filtration. Nevertheless, we will reduce to the finite filtration case by proving that the barcode of the given filtration is equal to that of a finite filtration. A general theory encompassing both finite and infinite filtrations using a categorical view-point has been developed (see [29], [36]). We avoid using the categorical definitions and the module-theoretic language used in [36]. We will prove directly the equality of the barcodes of the infinite and the corresponding finite filtration(cf. Proposition 3.3.3) that is* *important in designing our algorithm, starting from the definition of persistent multiplicities given above.*

We now give a concrete example of a barcode associated to a (infinite) filtration.

Example 7. *Let S be the two-dimensional torus (topologically $\mathbf{S}^1 \times \mathbf{S}^1$) embedded in $\mathbb{R}^3$, and $\mathcal{F}$ be the filtration of the torus by the sub-level sets of the height function (depicted in Figure 3.1a). We denote by $S_{\leqslant t}$ the subset of the torus having "height" $\leqslant t$.*

We consider homology in dimensions 0, 1 and 2.

Informally, one observes that a 0-dimensional homology class is born at time t_0 which never dies. There are two 1-dimensional homology classes, the horizontal loop born at time t_2 and the vertical loop born at time t_4, which also never die. Lastly, there is a 2-dimensional homology class born at time t_5 which never dies. Since there are no homology classes of the same dimension being born and dying at the same time, multiplicities in all the cases are 1.

More formally, following Definitions 3.2.3, 3.2.4 and 3.2.5, we obtain:

(Case p = 0) *If $t_0 \leqslant t < \infty$ then (using Definition 3.2.3)*

$$M_0^{t_0,t}(\mathcal{F}) = 0,$$

and hence (using Definitions 3.2.4 and 3.2.5)

$$P_0^{t_0,t}(\mathcal{F}) = 0, \text{ and } \mu_0^{t_0,t}(\mathcal{F}) = 0.$$

On the other hand,

$$P_0^{t_0,\infty}(\mathcal{F}) = \mathrm{H}_0(S_{\leqslant t_0}),$$

implying

$$\mu_0^{t_0,\infty}(\mathcal{F}) = 1.$$

(Case p = 1) *For $t_2 \leqslant t < \infty$,*

$$M_1^{t_2,t}(\mathcal{F}) = 0,$$

and hence

$$P_1^{t_2,t}(\mathcal{F}) = 0, \text{ \textit{and} } \mu_1^{t_2,t}(\mathcal{F}) = 0.$$

Moreover,

$$P_1^{t_2,\infty}(\mathcal{F}) = \mathrm{H}_1(S_{\leqslant t_2}),$$

and therefore,

$$\mu_1^{t_2,\infty}(\mathcal{F}) = 1.$$

For $t_4 \leqslant t < \infty$*,*

$$M_1^{t_4,t}(\mathcal{F}) = N_1^{t_4,t}(\mathcal{F}) = \mathrm{H}_1(S_{<t_4}),$$

and hence

$$P_1^{t_4,t}(\mathcal{F}) = 0, \text{ \textit{and} } \mu_1^{t_4,t}(\mathcal{F}) = 0.$$

Moreover,

$$P_1^{t_4,\infty}(\mathcal{F}) = \mathrm{H}_1(S_{\leqslant t_4})/\mathrm{H}_1(S_{<t_4}),$$

and therefore

$$\mu_1^{t_4,\infty}(\mathcal{F}) = 1.$$

(Case $p = 2$*)* *For* $t_5 \leqslant t < \infty$*,*

$$M_2^{t_5,t}(\mathcal{F}) = 0,$$

and hence

$$P_2^{t_5,t}(\mathcal{F}) = 0, \text{ \textit{and} } \mu_2^{t_5,t}(\mathcal{F}) = 0.$$

Moreover,

$$P_2^{t_5,\infty}(\mathcal{F}) = \mathrm{H}_2(S),$$

and therefore

$$\mu_2^{t_5,\infty}(\mathcal{F}) = 1.$$

Therefore the barcodes are as follows (using Eqn. (3.2)).

$$
\begin{aligned}
\mathcal{B}_0(\mathcal{F}) &= \{(t_0, +\infty, 1)\}, \\
\mathcal{B}_1(\mathcal{F}) &= \{(t_2, +\infty, 1), (t_4, +\infty, 1)\}, \\
\mathcal{B}_2(\mathcal{F}) &= \{(t_5, +\infty, 1)\}.
\end{aligned}
$$

Figure 3.1b illustrates the corresponding bars. Notice that even though the filtration $\mathcal{F}$ is an infinite filtration indexed by $\mathbb{R}$, the barcodes, $\mathcal{B}_p(\mathcal{F})$, are finite.

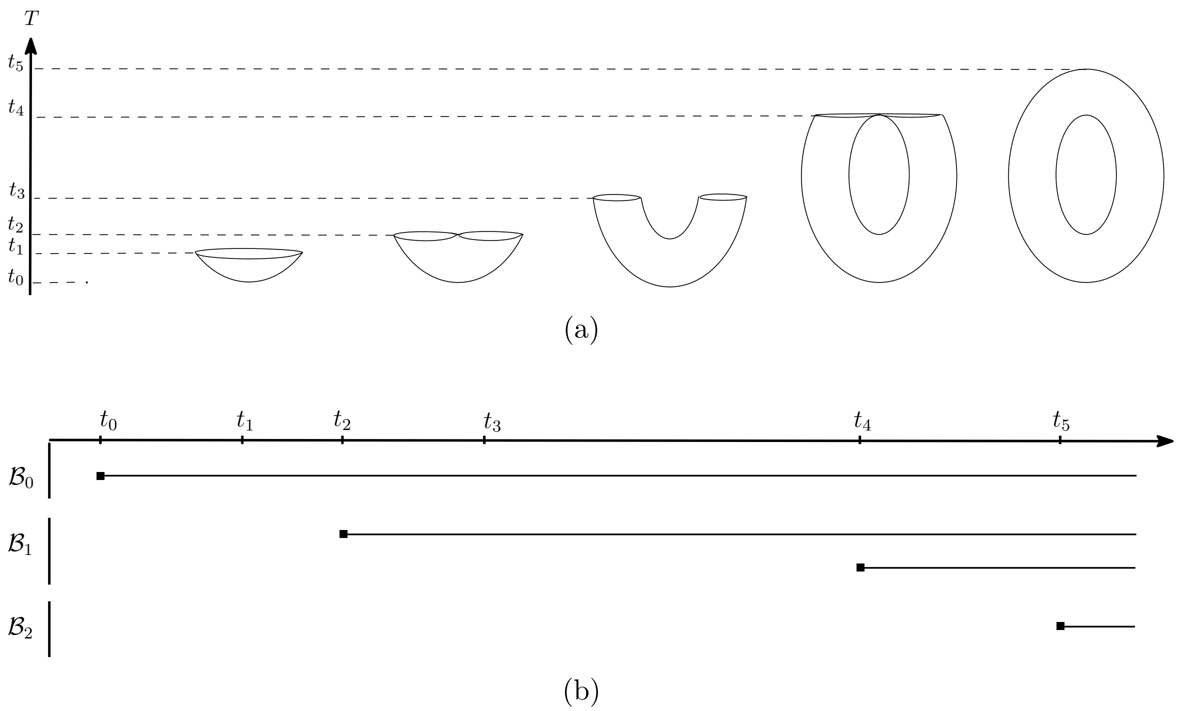


Figure 3.1. (a) Torus filtered by the sub-level sets of the height function, (b) corresponding barcodes for homology classes of dimension 0, 1 and 2.

The main type of filtration that we consider in this work is filtration of semi-algebraic sets by the sub-level sets of continuous semi-algebraic functions—which we define below.

3.2.2 Semi-algebraic filtrations

We consider the algorithmic problem of computing the dimensions of persistent homology groups and barcodes of the filtration induced on a given semi-algebraic set by a polynomial function.

Definition 3.2.6. *Let $S \subset \mathrm{R}^k$ be a semi-algebraic set and $P : S \to \mathrm{R}$ a continuous semi-algebraic map.*

For $t \in \mathrm{R} \cup \{\pm\infty\}$, let

$$S_{P \leqslant t} = \{x \in S \mid P(x) \leqslant t\}.$$

Then, $(S_{P \leqslant t})_{t \in \mathrm{R} \cup \{\pm\infty\}}$ is a filtration of the semi-algebraic set S indexed by $\mathrm{R} \cup \{\pm\infty\}$, and we will denote this filtration by $\mathcal{F}(S, P)$.

Notation 16. *For $p \geqslant 0$, we will denote*

$$\mathcal{B}_p(S, P) = \mathcal{B}_p(\mathcal{F}(S, P)).$$

Remark 16. *In the definition of $\mathcal{B}_p(\mathcal{F}(S, P))$ we need to specify the homology theory we are using. For a semi-algebraic set X defined over an arbitrary real closed field R we take homology groups $\mathrm{H}_*(X) = \mathrm{H}_*(X, \mathbb{Q})$ as defined in [37, (3.6), page 141]. It agrees with singular homology in case $\mathrm{R} = \mathbb{R}$.*

Remark 17. *Note also that the barcode of a polynomial function restricted to a semi-algebraic set S gives important topological information about the function P on S. It allows one to define a p-dimensional distance between two such polynomial functions restricted to S, by defining a notion of distance between two barcodes. Various distances have been proposed but the most commonly used one is the so called "bottle-neck distance" [35]. An algorithm with singly exponential complexity for computing the barcode of a polynomial also gives an algorithm with singly exponential complexity for computing such distances as well. To our knowledge the algorithmic problem of computing barcodes of polynomial functions on semi-algebraic sets have not been considered prior to our work.*

We prove the following theorem.

Theorem 3.2.1. *There exists an algorithm that takes as input:*

1. *a finite set of polynomials, $\mathcal{P} \subset \mathrm{D}[X_1, \ldots, X_k]$;*

2. *a $\mathcal{P}$-closed formula ϕ such that $\mathcal{R}(\phi)$ is bounded;*

3. a polynomial $P \subset \mathrm{D}[X_1, \ldots, X_k]$*;*

4. $\ell \geqslant 0$*;*

and computes $\mathcal{B}_p(\mathcal{R}(\phi), P)$*, for* $0 \leqslant p \leqslant \ell$*. The complexity of the algorithm is bounded by* $(sd)^{k^{O(\ell)}}$*, where* $s = \mathrm{card}(\mathcal{P})$*, and* d *is the maximum amongst the degrees of* P *and the polynomials in* $\mathcal{P}$*.*

Barcodes of non-proper maps

Notice that in Theorem 3.2.1 we only consider semi-algebraic sets S which are closed and bounded. In particular, this implies that any continuous semi-algebraic function on S is a proper map $S \to \mathrm{R}$ (i.e. the inverse image of a closed and bounded semi-algebraic set is closed and bounded).

One reason to assume the properness is that for non-proper semi-algebraic maps $P : S \to \mathrm{R}$, the barcode $\mathcal{B}_p(S, P)$ may not reflect the topology of S as illustrated in the following example.

Example 8. *Let* $S \subset \mathrm{R}^2$ *be the (unbounded) semi-algebraic set defined by the formula*

$$\phi := (0 < X_1 < 1) \wedge (X_1(X_1 - 1)X_2 - 1 = 0)$$

(depicted in Figure 3.2), and let $P = X_1$*. Consider the semi-algebraic filtration* $\mathcal{F}(S, P)$*. Note that* P *restricted to* S *is not a proper semi-algebraic map (*$P^{-1}([0, 1])$ *is not bounded).*

It is clear that for $p > 0$*,*

$$\mathcal{B}_p(S, P) = \emptyset.$$

We claim that even for $p = 0$ *(contrary to the expectation)*

$$\mathcal{B}_p(S, P) = \emptyset.$$

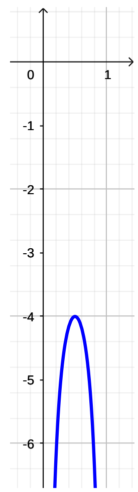


Figure 3.2. $S = \{(x_1, x_2) \mid 0 < x_1 < 1, x_1(x_1 - 1)x_2 - 1 = 0\}$

To see this observe that for all $s \leqslant 0$, $t \geqslant s$, we have that

$$\begin{aligned} M_0^{s,t}(\mathcal{F}(S,P)) &= \bigcup_{s'<s} (i_p^{s,t})^{-1}(\mathrm{H}_0^{s',t}(\mathcal{S})) \\ &= 0, \\ \mathrm{H}_0(S_{P\leqslant s}) &= 0, \end{aligned}$$

since $S_{P\leqslant s} = \emptyset$ for $s \leqslant 0$. This shows that

$$\mu_0^{s,t}(\mathcal{F}(S,P)) = 0, s \leqslant 0, t \geqslant s. \tag{3.3}$$

For $s > 0$, and $t \geqslant s$, it follows from Definition 3.2.5 that

$$M_0^{s,t}(\mathcal{F}(S,P)) = N_0^{s,t}(\mathcal{F}(S,P)) = \mathrm{H}_0(S_{P\leqslant s}),$$

proving that

$$\mu_0^{s,t}(\mathcal{F}(S,P)) = 0, s > 0, t \geqslant s. \tag{3.4}$$

Together Eqns. (3.3) and (3.4) imply that

$$\mathcal{B}_0(S,P) = \emptyset.$$

In order to have a more reasonable definition of barcodes (and allow "bars" which have open endpoints) we propose the following definition. We use two notions from real algebraic geometry – that of the real spectrum and the real closed extension of R by the field of Puiseux series.

Let $S \subset \mathrm{R}^k$ be an arbitrary semi-algebraic set and $P : S \to \mathrm{R}$ a continuous semi-algebraic function. We define a new filtration $\widetilde{\mathcal{F}}(S, P)$ as follows.

The indexing set of the new filtration will the set

$$\widetilde{\mathrm{R}} = \{-\infty, +\infty\} \cup \bigcup_{x \in \mathrm{R}} \{x_-, x, x_+\},$$

on which a total order is specified by

$$-\infty < x_- < x < x_+ < y_- < y < y_+ < \infty,$$

for all $x < y$ in R. (The ordered set $\widetilde{\mathrm{R}}$ is the *real spectrum* of the ring $\mathrm{R}[X]$ – see for example [38, page 134]).

We now define the filtration $\widetilde{\mathcal{F}}(S, P)$.

Definition 3.2.7 (Filtration for semi-algebraic maps not necessarily proper)**.** *For* $\tilde{t} \in \widetilde{\mathrm{R}}$ *define*

$$\begin{aligned}
\widetilde{S}_{\tilde{t}} &= \mathrm{ext}(S, \mathrm{R}\langle\varepsilon\rangle)_{P=-1/\varepsilon}, \text{ if } \tilde{t} = -\infty, \\
&= \mathrm{ext}(S, \mathrm{R}\langle\varepsilon\rangle)_{P\leqslant t-\varepsilon}, \text{ if } \tilde{t} = t_-, t \in \mathrm{R}, \\
&= \mathrm{ext}(S, \mathrm{R}\langle\varepsilon\rangle)_{P\leqslant t}, \text{ if } \tilde{t} = t \in \mathrm{R}, \\
&= \mathrm{ext}(S, \mathrm{R}\langle\varepsilon\rangle)_{P\leqslant t+\varepsilon}, \text{ if } \tilde{t} = t_+, t \in \mathrm{R}, \\
&= \mathrm{ext}(S, \mathrm{R}\langle\varepsilon\rangle)_{P=1/\varepsilon}, \text{ if } \tilde{t} = +\infty.
\end{aligned}$$

(where $\mathrm{R}\langle\varepsilon\rangle$ *is the field of algebraic Puisuex series in* ε*, and* $\mathrm{ext}(\cdot, \mathrm{R}\langle\varepsilon\rangle)$ *denotes the extension of a semi-algebraic subset of* R^k *to* $\mathrm{R}\langle\varepsilon\rangle^k$ *– see Notation 4 and Notation 6).*

Definition 3.2.8 (Barcode for filtration induced by a semi-algebraic map not necessarily proper)**.** *For $S \subset \mathrm{R}^k$ an arbitrary semi-algebraic set and $P : S \to \mathrm{R}$ a continuous semi-algebraic function, we define*

$$\widetilde{\mathcal{B}}_p(S, P) = \mathcal{B}_p(\widetilde{\mathcal{F}}(S, P)).$$

It is easy now to verify that for the pair S, P in Example 8

$$\widetilde{\mathcal{B}}_0(S, P) = \{(0_+, +\infty, 1)\}.$$

Note that

$$\widetilde{\mathcal{B}}_p(S, P) \subset \widetilde{\mathrm{R}} \times \widetilde{\mathrm{R}} \times \mathbb{Z}_{>0}.$$

Using Hardt triviality theorem, one can deduce that $\widetilde{\mathcal{B}}_p(S, P)$ is a finite set. We will formally prove this statement later for proper semi-algebraic maps (see Proposition 3.3.1).

The barcode for a proper semi-algebraic map takes its value in $\mathrm{R} \times \mathrm{R} \times \mathbb{Z}_{>0}$ which is properly contained in the $\widetilde{\mathrm{R}} \times \widetilde{\mathrm{R}} \times \mathbb{Z}_{>0}$. It is not difficult to prove that in case $P : S \to \mathrm{R}$ is a proper semi-algebraic map, the new definition of barcode agrees with the previous one.

We record the above mentioned facts in the following proposition for future reference and omit the proofs. We will not use it in here since we restrict ourselves to the proper case.

Proposition 3.2.1. *For any continuous semi-algebraic map $P : S \to \mathrm{R}$ and for all $p \geqslant 0$, $\widetilde{B}(S, P)$ is a finite set. Moreover, if P is a proper semi-algebraic map, then for all $p \geqslant 0$,*

$$\widetilde{\mathcal{B}}_p(S, P) = \mathcal{B}_p(S, P).$$

Proof. Omitted. ☐

3.3 Continuous to finite filtration

In this section we describe how to efficiently reduce the problem of computing the barcode of a continuous semi-algebraic filtration to that of a finite filtration of semi-algebraic sets. The mathematical results are encapsulated in Propositions 3.3.1 and 3.3.3 stated and proved in Section 3.3.1. Then in Section 3.3.2 we prove a formula used to compute the barcode of a finite filtration (Proposition 3.3.4). This formula is not new (see [35, page 152][35]), however, it is important to deduce that from our new definition of barcodes.

Recall that we are interested in the persistent homology of filtrations of semi-algebraic sets by the sub-level sets of a polynomial. Recall also (cf. Definition 3.2.6) that for a closed and bounded semi-algebraic set $S \subset \mathrm{R}^k$, $P \in \mathrm{R}[X_1, \ldots, X_k]$, and $t \in \mathrm{R} \cup \{\pm\infty\}$, we denote the filtration

$$(S_{P \leqslant t} = \{x \in S \mid P(x) \leqslant t\})_{t \in \mathrm{R} \cup \{\pm\infty\}}$$

by $\mathcal{F}(S, P)$.

Our first observation is that, even though the indexing set $\mathrm{R} \cup \{\pm\infty\}$ is infinite, for each $p \geqslant 0$, the barcode $\mathcal{B}_p(\mathcal{F}(S, P))$ is a finite set (cf. Example 7).

Proposition 3.3.1. *For each $p \geqslant 0$, the cardinality of $\mathcal{B}_p(\mathcal{F}(S, P))$ is finite.*

3.3.1 Reduction to the case of a finite filtration

We will now prove a result (cf. Proposition 3.3.3 below) from which Proposition 3.3.1 will follow. Our strategy is to identify a finite set of values $\{s_0, \ldots, s_M\} \subset \mathrm{R}$, such that the semi-algebraic homotopy type of the increasing family $S_{P \leqslant t}$ (as t goes from $-\infty$ to ∞), can change only when t crosses one of the s_i's. This would imply that the barcode, $\mathcal{B}_p(\mathcal{F}(S, P))$, of the infinite filtration $\mathcal{F}(S, P)$, is *equal* to the barcode of the *finite filtration* $\varnothing \subset S_{P \leqslant s_0} \subset \cdots \subset S_{P \leqslant s_M} \subset S$ (cf. Proposition 3.3.3 below). In addition, we will obtain a bound on the number M in terms of the number of polynomials appearing in the definition of S and their degrees, as well as the degree of the polynomial P. The technique used in the proofs of these results are adaptations of the technique used in the proof of the main result (Theorem 2.1) in [32], which gives a singly exponential bound on the number of distinct homotopy types amongst the fibers of a semi-algebraic map in [32]. We need a slightly different statement than that of Theorem 2.1 in [32]. However, our situation is simpler since we only need the result for maps to R (rather than to R^n as is the case in [32, Theorem 2.1]).

Outline of the reduction

Before delving into the detail we first give an outline of the main idea behind the reduction to the finite filtration case. The key mathematical result that we need is the following. Given a semi-algebraic subset $X \subset \mathrm{R}^{k+1}$, obtain a semi-algebraic partition of $\mathrm{R} \cup \{\pm\infty\}$ into points $-\infty = s_{-1} < s_0 < s_1 < \cdots < s_M < s_{M+1} = \infty$, and open intervals $(s_i, s_{i+1}), -1 \leqslant i \leqslant M$, such that the homotopy type of $X_t = X \cap \pi_{k+1}^{-1}$ stays constant over each open interval (s_i, s_{i+1}) (here π_{k+1} denotes the projection on the last coordinate). In our application the fibers X_t will be a non-decreasing in t (in fact, X_t will be equal to $S_{P \leqslant t}$) but we do not need this property to hold for obtaining the partition mentioned above.

The following example is illustrative.

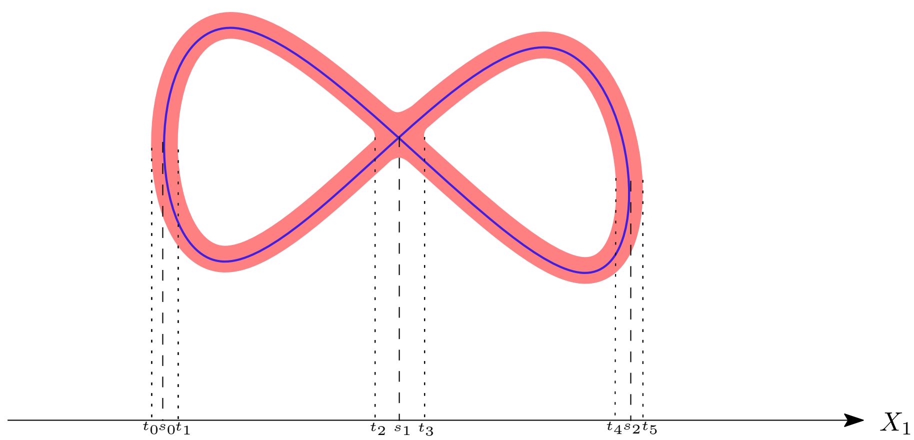


Figure 3.3. Homotopy types of fibers

Suppose that $X \subset \mathrm{R}^2$ is a singular curve shown in blue in Figure 3.3. We define a semi-algebraic tubular neighborhood $X^\star(\varepsilon)$ of X using an infinitesimal ε (shown in red), whose boundary has good algebraic properties – namely, in this case a finite number of critical values $t_0 < t_1 < \cdots < t_5$ for the projection map onto the chosen coordinate X_{k+1} which is shown as X_1 in the figure. The t_i's give a partition of $\mathrm{R}\langle\varepsilon\rangle$ rather than that of R, and over each interval (t_i, t_{i+1}) the semi-algebraic homeomorphism type of $X^\star(\varepsilon)$ (but not necessarily the semi-algebraic homotopy type of $\mathrm{ext}(X, \mathrm{R}\langle\varepsilon\rangle)$) stay constant. Clearly this partition does not have the homotopy invariance property with respect to the set $\mathrm{ext}(X, \mathrm{R}\langle\varepsilon\rangle)$. However, the intervals $(t_1, t_2) \cap \mathrm{R} = (s_0, s_1)$ and $(t_3, t_4) \cap \mathrm{R} = (s_1, s_2)$ does have the require property with respect to X, and the points s_0, s_1, s_2 gives us the require partition.

In the general case the definition of the tube $X^\star(\varepsilon)$ is more involved and uses more than one infinitesimal (cf. Notation 18). The set of points corresponding to the t_i's in the above example is defined precisely in Proposition 3.3.2 where the important property of the partition of $\mathrm{R}\langle\bar{\varepsilon}\rangle$ they induce is also proved. The passage from the t_i's to the s_i's and the important property satisfied by the s_i's is described in Lemma 3.3.4. The finite set of values $\{s_0, \ldots, s_M\} \subset \mathrm{R}$ is then used to define a finite filtration of the given semi-algebraic set, and the fact that this finite filtration has the same barcode as the infinite filtration we started with is proved in Proposition 3.3.3. Proposition 3.3.3 immediately implies Proposition 3.3.1.

There are several further technicalities involved in converting the above construction into an efficient algorithm. These are explained in Section 3.4. The complexity of the whole procedure is bounded singly exponentially.

Proof of Proposition 3.3.1

We begin by fixing some notation.

Notation 17. *For $\mathcal{Q} \subset \mathrm{R}[X_1, \ldots, X_k]$ we will denote by*

$$\mathrm{Z}(\mathcal{Q}, \mathrm{R}^k) = \{x \in \mathrm{R}^k | \bigwedge_{Q \in \mathcal{Q}} Q(x) = 0\}.$$

For $Q \in \mathrm{R}[X_1, \ldots, X_k]$, we will denote by $\mathrm{Z}(Q, \mathrm{R}^k) = \{x \in \mathrm{R}^k \mid Q(x) = 0\}$.

Definition 3.3.1. *Let $\mathcal{Q}$ be a finite subset of $\mathrm{R}[X_1, \ldots, X_k]$. A* sign condition *on $\mathcal{Q}$ is an element of $\{0, 1, -1\}^{\mathcal{Q}}$. We say that $\mathcal{Q}$* realizes *the sign condition σ at $x \in \mathrm{R}^k$ if*

$$\bigwedge_{Q \in \mathcal{Q}} \mathbf{sign}(Q(x)) = \sigma(Q).$$

The realization of the sign condition σ *is*

$$\mathcal{R}(\sigma) = \{x \in \mathrm{R}^k \mid \bigwedge_{Q \in \mathcal{Q}} \mathbf{sign}(Q(x)) = \sigma(Q)\}.$$

The sign condition σ is realizable *if $\mathcal{R}(\sigma)$ is non-empty. We denote by* $\mathrm{Sign}(\mathcal{Q})$ *the set of realizable sign conditions of $\mathcal{Q}$.*

Let $R \in \mathrm{R}$ with $R > 0$, and let

$$\mathcal{P} = \{P_0, P_1, \ldots, P_s\} \subset \mathrm{R}[X_1, \ldots, X_k],$$

with $P_0 = X_1^2 + \cdots + X_k^2 - R$. Let $P \in \mathrm{R}[X_1, \ldots, X_k]$, and also let ϕ be a closed $(\mathcal{P} - \{P_0\})$-formula, and $\widetilde{\phi}$ be $\phi \wedge (P_0 \leqslant 0) \wedge (P - Y \leqslant 0)$, where Y is a new variable. So ϕ is a $(\mathcal{P} \cup \{P - Y\})$-closed formula. Let $P_{s+1} = P - Y$.

Notation 18. *For $\bar{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_{s+1})$, we denote by $\phi^\star(\bar{\varepsilon})$, the $\mathcal{P}^\star(\bar{\varepsilon})$-closed formula obtained by replacing each occurrence of $P_i \geqslant 0$ in ϕ by $P_i + \varepsilon_i \geqslant 0$ (resp. $P_i \leqslant 0$ in $\widetilde{\phi}$ by $P_i - \varepsilon_i \leqslant 0$) for $0 \leqslant i \leqslant s+1$, where*

$$\mathcal{P}^\star(\bar{\varepsilon}) = \bigcup_{0 \leqslant i \leqslant s+1} \{P_i + \varepsilon_i, P_i - \varepsilon_i\}.$$

Observe that

$$S^\star(\bar{\varepsilon}) := \mathcal{R}(\phi^\star(\bar{\varepsilon})) \subset \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+1}$$

is a $\mathcal{P}^\star(\bar{\varepsilon})$-closed semi-algebraic set, and we define $\Sigma_\phi \subset \{-1, 0, 1\}^{\mathcal{P}^\star(\bar{\varepsilon})}$ by

$$S^\star(\bar{\varepsilon}) = \bigcup_{\sigma \in \Sigma_\phi, \mathcal{R}(\sigma) \neq \varnothing} \mathcal{R}(\sigma). \tag{3.5}$$

Lemma 3.3.1. *For each $\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon})$, $\mathrm{Z}(\mathcal{Q}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+1})$ is either empty or is a non-singular $(k+1-\mathrm{card}(\mathcal{Q}))$-dimensional real variety such that at every point $(x_1, \ldots, x_k, y) \in \mathrm{Z}(\mathcal{Q}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+1})$, the $(\mathrm{card}(\mathcal{Q}) \times (k+1))$-Jacobi matrix,*

$$\left(\frac{\partial P}{\partial X_i}, \frac{\partial P}{\partial Y}\right)_{P \in \mathcal{Q}\ 1 \leqslant i \leqslant k}$$

has the maximal rank $\mathrm{card}(\mathcal{Q})$.

Proof. See [32]. □

Now let $\pi_{k+1} : \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+1} \to \mathrm{R}\langle\bar{\varepsilon}\rangle$ denote the projection to the last (i.e. the Y) coordinate, and $\pi_{[1,k]} : \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+1} \to \mathrm{R}\langle\bar{\varepsilon}\rangle^{k}$ denote the projection to the first (i.e. $(X_1, \ldots, X_k)$) k coordinates.

For any semi-algebraic subset $S \subset \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+1}$, and $T \subset \mathrm{R}\langle\bar{\varepsilon}\rangle$, we denote by $S_T = \pi_{[1,k]}(S \cap \pi_{k+1}^{-1}(T))$. For $t \in \mathrm{R}\langle\varepsilon\rangle$, we will denote by $S_{\leqslant t} = S_{(-\infty,t]}$, and $S_t = S_{\{t\}}$.

Notation 19 (Critical points and critical values)**.** *For $\mathcal{Q} \subset \mathcal{P}^\star(\bar{\varepsilon})$, we denote by* $\mathrm{Crit}(\mathcal{Q})$ *the subset of* $\mathrm{Z}(\mathcal{Q}, \mathrm{R}\langle\bar{\varepsilon}\rangle^{k+1})$ *at which the the Jacobian matrix,*

$$\left(\frac{\partial P}{\partial X_i}\right)_{P\in\mathcal{Q}, 1\leqslant i\leqslant k}$$

is not of the maximal possible rank. We denote $\mathrm{crit}(\mathcal{Q}) = \pi(\mathrm{Crit}(\mathcal{Q}))$*.*

Lemma 3.3.2. *The set*

$$\bigcup_{\mathcal{Q}\subset\mathcal{P}^\star(\bar{\varepsilon})} \mathrm{crit}(\mathcal{Q})$$

is finite.

Proof. Follows from Lemma 3.3.1 and the semi-algebraic Sard's lemma (see for example [2, Theorem 5.56]). □

Lemma 3.3.3. *The partitions*

$$\begin{aligned} \mathrm{R}^{k+1} &= \bigcup_{\sigma\in\mathrm{Sign}(\mathcal{P}^\star(\bar{\varepsilon}))} \mathcal{R}(\sigma), \\ S^\star(\bar{\varepsilon}) &= \bigcup_{\sigma\in\Sigma_\phi} \mathcal{R}(\sigma), \end{aligned}$$

are compatible Whitney stratifications of R^{k+1} *and* $S^\star(\bar{\varepsilon})$ *respectively.*

Proof. Follows directly from the definition of Whitney stratification (see [39], [40]), and Lemma 3.3.1. □

We are now in a position to prove the key mathematical result that allows us to reduce the filtration of a semi-algebraic set by the sub-level sets of a polynomial to the case of a finite filtration.

Proposition 3.3.2. *Suppose*

$$\bigcup_{\mathcal{Q}\subset\mathcal{P}^\star(\bar{\varepsilon})} \mathrm{crit}(\mathcal{Q}) = \{t_0, \ldots, t_N\},$$

with $t_0 < t_1 < \cdots < t_N$ *(cf. Lemma 3.3.2). Then for* $0 \leqslant i < N$, $a, b \in \mathrm{R}$ *such that* $(a, b) \subset (t_i, t_{i+1}) \cap \mathrm{R}$, *and for any* $c \in (a, b)$, *the inclusion*

$$\mathcal{R}(\phi(\cdot, a)) \hookrightarrow \mathcal{R}(\phi(\cdot, c))$$

is a semi-algebraic homotopy equivalence.

Proof. The proof is an adaptation of a proof of a similar result in [32] (Lemma 3.8), though our situation is much simpler. It follows from Lemma 3.3.3 that the semi-algebraic set

$$\widehat{S^\star(\bar{\varepsilon})} := S^\star(\bar{\varepsilon}) \backslash \pi_{k+1}^{-1}(\{t_0, \ldots, t_N\})$$

is a Whitney-stratified set. Moreover, $\pi_{k+1}|_{\widehat{S^\star(\bar{\varepsilon})}}$ is a proper stratified submersion. By Thom's first isotopy lemma (in the semi-algebraic version, over real closed fields [40]) the map $\pi_{k+1}|_{\widehat{S^\star(\bar{\varepsilon})}}$ is a locally trivial fibration.

Now let $0 \leqslant i < N$. It follows that for $a', b' \in \mathrm{R}\langle\bar{\varepsilon}\rangle$ with $t_i < a' \leqslant b' < t_{i+1}$, that there exists a semi-algebraic homeomorphism

$$\theta_{a',b'} : S^\star(\bar{\varepsilon})_{[a',b']} \to S^\star(\bar{\varepsilon})_{a'} \times [a', b']$$

such that the following diagram commutes.

$$\begin{tikzcd}
S^\star(\bar{\varepsilon})_{[a',b']} \arrow[rr, "\theta_{a',b'}"] \arrow[dr, "\pi_{k+1}"'] & & S^\star(\bar{\varepsilon})_{a'} \times [a', b'] \arrow[dl, "\pi_{k+1}"] \\
& \mathrm{R}\langle\bar{\varepsilon}\rangle &
\end{tikzcd}$$

Let

$$r_{a',b'} : S^\star(\bar{\varepsilon})_{b'} \times [a', b'] \to S^\star(\bar{\varepsilon})_{a'},$$

be the map defined by

$$
\begin{aligned}
r_{a',b'}(x,t) &= \pi_{[1,k]} \circ \theta_{a',b'}(x,t) \text{ if } t \leqslant P(x), \\
&= x, \text{ else.}
\end{aligned}
$$

Notice, $r_{a',b'}$ is a semi-algebraic continuous map, and moreover for $x \in S^\star(\bar{\varepsilon})_{a'}$, $r_{a',b'}(x,a') = x$. Thus, $r_{a',b'}$ is a semi-algebraic deformation retraction of $S^\star(\bar{\varepsilon})_{b'}$ to $S^\star(\bar{\varepsilon})_{a'}$.

This implies that the inclusion

$$
S^\star(\bar{\varepsilon})_{a'} \hookrightarrow S^\star(\bar{\varepsilon})_{b'} \tag{3.6}
$$

is a semi-algebraic homotopy equivalence.

Now suppose that $a,b \in \mathrm{R}$ with $t_i < a \leqslant b < t_{i+1}$. $S^\star(\bar{\varepsilon})_a$ and $S^\star(\bar{\varepsilon})_b$ are closed and bounded over R, and that $S^\star(\bar{\varepsilon})_a \searrow \mathcal{R}(\phi(\cdot,a))$, $S^\star(\bar{\varepsilon})_b \searrow \mathcal{R}(\phi(\cdot,b))$.

Then, it follows from Lemma 2.3.1 that the inclusions,

$$
\mathrm{ext}(\mathcal{R}(\phi(\cdot,a)), \mathrm{R}\langle\bar{\varepsilon}\rangle) \hookrightarrow S^\star(\bar{\varepsilon})_a, \tag{3.7}
$$

and

$$
\mathrm{ext}(\mathcal{R}(\phi(\cdot,b)), \mathrm{R}\langle\bar{\varepsilon}\rangle) \hookrightarrow S^\star(\bar{\varepsilon})_b, \tag{3.8}
$$

are semi-algebraic homotopy equivalences.

Thus, we have the following commutative diagram of inclusions

$$
\begin{tikzcd}
S^\star(\bar{\varepsilon})_a \arrow[r, hook] & S^\star(\bar{\varepsilon})_b \\
\mathrm{ext}(\mathcal{R}(\phi(\cdot,a)), \mathrm{R}\langle\bar{\varepsilon}\rangle) \arrow[u, hook] \arrow[r, hook] & \mathrm{ext}(\mathcal{R}(\phi(\cdot,b)), \mathrm{R}\langle\bar{\varepsilon}\rangle) \arrow[u, hook]
\end{tikzcd}
$$

in which all arrows other than the bottom inclusion are semi-algebraic homotopy equivalences, and hence so is the bottom arrow. This implies that the inclusion $\mathcal{R}(\phi(\cdot,a)) \hookrightarrow \mathcal{R}(\phi(\cdot,b))$ is a semi-algebraic homotopy equivalence by an application of the Tarski-Seidenberg transfer principle (see for example [2, Chapter 2]).

Now assume that $a = t_i$. Using Lemma 2.3.1 we have that for all small enough $\varepsilon > 0$, the inclusion $\mathcal{R}(\phi(\cdot, a)) \hookrightarrow \mathcal{R}(\phi(\cdot, a + \varepsilon))$ is a semi-algebraic homotopy equivalence. Moreover, from what has been already shown, the inclusion $\mathcal{R}(\phi(\cdot, a + \varepsilon)) \hookrightarrow \mathcal{R}(\phi(\cdot, c))$ is a semi-algebraic homotopy equivalence. It now follows that $\mathcal{R}(\phi(\cdot, a)) \hookrightarrow \mathcal{R}(\phi(\cdot, c))$ is a semi-algebraic homotopy equivalence. This completes the proof. $\square$

Lemma 3.3.4. *Let $\mathcal{G} \subset \mathrm{R}[\bar{\varepsilon}][T]$ be a finite set of non-zero polynomials and*

$$\{t_0, \ldots, t_N\} \subset \bigcup_{G \in \mathcal{G}} \mathrm{Z}(G, \mathrm{R}\langle\bar{\varepsilon}\rangle)$$

with $t_0 < \cdots < t_N$. For $G \in \mathcal{G}$, let $G = \sum_\alpha m_{G,\alpha} G_\alpha$, with $G_\alpha \in \mathrm{R}[T], m_{G,\alpha} \in \mathrm{R}[\bar{\varepsilon}]$, and let $M(G) = \{\alpha \mid m_{G,\alpha} \neq 0\}$. Let $\mathcal{H} = \bigcup_{G \in \mathcal{G}, \alpha \in M(P)} \{G_\alpha\}$, and let

$$\{s_0, \ldots, s_M\} = \bigcup_{H \in \mathcal{H}} \mathrm{Z}(H, \mathrm{R})$$

with $s_0 < s_1 < \cdots < s_M$. Then, for each $i, 0 \leqslant i < M$, there exists $j, 0 \leqslant j < N$, such that $(s_i, s_{i+1}) \subset \mathrm{R}$ is contained in $(t_j, t_{j+1}) \cap \mathrm{R}$.

Proof. Notice that it follows from the definition of the set $\{s_0, \ldots, s_M\}$ that for any $i, 0 \leqslant i < M$, the sign condition (cf. Definition 3.3.1) realized by $\mathcal{H}$ at t stays fixed for all $t \in \mathrm{R}$, such that $t \in (s_i, s_{i+1})$.

Since for any $t \in \mathrm{R}$, the sign condition realized by $\mathcal{H}$ at t determines the sign condition of $\mathcal{G}$ realized at t, it follows that the the sign condition (cf. Definition 3.3.1) realized by $\mathcal{G}$ at t also stays fixed for all $t \in \mathrm{R}$, such that $t \in (s_i, s_{i+1})$.

Suppose that $t' \in \mathrm{ext}((s_i, s_{i+1}), \mathrm{R}\langle\bar{\varepsilon}\rangle)$ such that $G(t') = 0$ for some $G \in \mathcal{G}$. We claim that this implies that $\lim_{\bar{\varepsilon}} t' \in \{s_i, s_{i+1}\}$. Suppose not. Then, $\lim_{\bar{\varepsilon}} t' \in (s_i, s_{i+1})$, which contradicts the fact that the sign condition (cf. Definition 3.3.1) realized by $\mathcal{G}$ at t stays fixed for all $t \in \mathrm{R}$, such that $t \in (s_i, s_{i+1})$, since G is a non-zero polynomial.

The lemma now follows from the hypothesis that $\{t_0, \ldots, t_N\} \subset \bigcup_{G \in \mathcal{G}} \mathrm{Z}(G, \mathrm{R}\langle\bar{\varepsilon}\rangle)$. $\square$

Let $S = \mathcal{R}(\Phi)$ and $P, t_0, \ldots, t_N$ as in Proposition 3.3.2, and let $\mathcal{G}, \mathcal{H}$, and $s_0 < \cdots < s_M$ as in Lemma 3.3.4. Let $s_{-1} = -\infty, s_{M+1} = \infty$. Let $\mathcal{F}$ denote the finite filtration of semi-algebraic sets, indexed by the finite ordered set $T = \{s_i \mid -1 \leqslant i \leqslant M+1\}$, with the element of $\mathcal{F}$ indexed by s_i equal to $S_{P \leqslant s_i}$. We have the following proposition.

Proposition 3.3.3. *For each $p \geqslant 0$,*

$$\mathcal{B}_p(S, P) = \mathcal{B}_p(\mathcal{F}).$$

Proof. It follows from Proposition 3.3.2 and Lemma 3.3.4 that for each $i, -1 \leqslant i \leqslant M$ and $s \in (s_i, s_{i+1})$, the inclusion $S_{P \leqslant s_i} \hookrightarrow S_{P \leqslant s}$ is a semi-algebraic homotopy equivalence.

The proposition will now follow from the following two claims.

Claim 3.3.1. *Suppose that $s, t \in [s_{-1}, s_{M+1}], s \leqslant t$. Then, $\mu_p^{s,t}(\mathcal{F}(S, P)) \neq 0 \Rightarrow s, t \in \{s_{-1}, \ldots, s_{M+1}\}$.*

Proof. We consider the following two cases.

1. $s \notin \{s_{-1}, \ldots, s_{M+1}\}$: Without loss of generality we can assume that $s \in (s_i, s_{i+1})$ for some $i, -1 \leqslant i \leqslant M$. Now the inclusion $S_{P \leqslant s'} \hookrightarrow S_{P \leqslant s}$, is a semi-algebraic homotopy equivalence for all $s' \in [s_i, s)$, and hence $i_p^{s',s}$ is an isomorphism for all $s' \in [s_i, s)$.

   It follows that for all $s' \in [s_i, s)$,

$$\mathrm{H}_p^{s',t}(\mathcal{F}(S, P)) = \mathrm{Im}(i_p^{s',t}) = \mathrm{Im}(i_p^{s,t} \circ i_p^{s',s}) = \mathrm{Im}(i_p^{s,t}) = \mathrm{H}_p^{s,t}(\mathcal{F}(S, P)),$$

   which implies that

$$(i_p^{s,t})^{-1}(\mathrm{H}_p^{s',t}(\mathcal{F}(S, P))) = (i_p^{s,t})^{-1}(\mathrm{H}_p^{s,t}(\mathcal{F}(S, P))) = \mathrm{H}_p(S_{P \leqslant s}).$$

   Noting that

$$\bigcup_{s' < s} (i_p^{s,t})^{-1}(\mathrm{H}_p^{s',t}(\mathcal{F}(S, P))) = \bigcup_{s' \in [s_i, s)} (i_p^{s,t})^{-1}(\mathrm{H}_p^{s',t}(\mathcal{F}(S, P))),$$

it now follows that

$$
\begin{aligned}
M_p^{s,t}(\mathcal{F}(S,P)) &= \bigcup_{s'<s} (i_p^{s,t})^{-1}(\mathrm{H}_p^{s',t}(\mathcal{F}(S,P))) \\
&= \mathrm{H}_p(S_{P\leqslant s}), \\
N_p^{s,t}(\mathcal{F}(S,P)) &= \bigcup_{s'<s\leqslant t'<t} (i_p^{s,t'})^{-1}(\mathrm{H}_p^{s',t'}(\mathcal{F}(S,P))) \\
&= \bigcup_{s\leqslant t'<t} (i_p^{s,t'})^{-1}(\mathrm{H}_p^{s,t'}(\mathcal{F}(S,P))) \\
&= \mathrm{H}_p(S_{P\leqslant s}).
\end{aligned}
$$

We have two sub-cases to consider.

(a) If $t < s_{M+1}$:

$$P_p^{s,t}(\mathcal{F}(S,P)) = M_p^{s,t}(\mathcal{F}(S,P))/N_p^{s,t}(\mathcal{F}(S,P)) = 0.$$

(b) If $t = s_{M+1} = \infty$:

$$P_p^{s,\infty}(\mathcal{F}(S,P)) = \mathrm{H}_p(S_{P\leqslant s})/\bigcup_{s\leqslant t} M_p^{s,t}(\mathcal{F}(S,P)) = 0.$$

since

$$\bigcup_{s\leqslant t} M_p^{s,t}(\mathcal{F}(S,P)) = \bigcup_{s\leqslant t} \mathrm{H}_p(S_{P\leqslant s}) = \mathrm{H}_p(S_{P\leqslant s}).$$

2. $t \notin \{s_{-1},\dots,s_{M+1}\}$: Without loss of generality we can assume that $t \in (s_i, s_{i+1})$ for some $i, -1 \leqslant i \leqslant M$. The inclusion $S_{P\leqslant t'} \hookrightarrow S_{P\leqslant t}$, is a semi-algebraic homotopy equivalence for all $t' \in [s_i, t)$, and hence $i_p^{t',t}$ is an isomorphism for all $t' \in [s_i, t)$. This implies that for all $t' \in [s_i, t)$, and $s' < t'$, $\mathrm{Im}(i_p^{s',t'})$ can be identified with $\mathrm{Im}(i_p^{s',t})$ using the isomorphism $i_p^{t',t}$. Furthermore, it is easy to verify that for every fixed $s' < s$ and $s \leqslant t' \leqslant t''$,

$$(i_p^{s,t'})^{-1}(\mathrm{H}_p^{s',t'}(\mathcal{F}(S,P))) \subset (i_p^{s,t'})^{-1}(\mathrm{H}_p^{s',t''}(\mathcal{F}(S,P))),$$

and hence for each fixed $s' < s$,

$$\bigcup_{s \leqslant t' < t} (i_p^{s,t'})^{-1}(\mathrm{H}_p^{s',t'}(\mathcal{F}(S,P))) = \bigcup_{s_i < t' < t} (i_p^{s,t'})^{-1}(\mathrm{H}_p^{s',t'}(\mathcal{F}(S,P))).$$

It follows that for $t \in (s_i, s_{i+1})$

$$\begin{aligned} N_p^{s,t}(\mathcal{F}(S,P)) &= \bigcup_{s' < s \leqslant t' < t} (i_p^{s,t'})^{-1}(\mathrm{H}_p^{s',t'}(\mathcal{F}(S,P))) \\ &= \bigcup_{s' < s} (i_p^{s,t})^{-1}(\mathrm{H}_p^{s',t}(\mathcal{F}(S,P))) \\ &= M_p^{s,t}(\mathcal{F}(S,P)). \end{aligned}$$

We have

$$P_p^{s,t}(\mathcal{F}(S,P)) = M_p^{s,t}(\mathcal{F}(S,P))/N_p^{s,t}(\mathcal{F}(S,P)) = 0.$$

This completes the proof. $\square$

Claim 3.3.2. *For each $i, j, -1 \leqslant i \leqslant j \leqslant M+1$, $\mu_p^{s_i,s_j}(\mathcal{F}(S,P)) = \mu_p^{s_i,s_j}(\mathcal{F})$.*

Proof. It suffices to prove that

$$\begin{aligned} M_p^{s_i,s_j}(\mathcal{F}(S,P)) &= M_p^{s_i,s_j}(\mathcal{F}), \\ N_p^{s_i,s_j}(\mathcal{F}(S,P)) &= N_p^{s_i,s_j}(\mathcal{F}). \end{aligned}$$

To prove the first equality we use the fact that $s' \in [s_{i-1}, s_i)$, the inclusion $S_{P \leqslant s_{i-1}} \hookrightarrow S_{P \leqslant s'}$ is a semi-algebraic homotopy equivalence.

Hence,

$$\begin{aligned} M_p^{s_i,s_j}(\mathcal{F}(S,P)) &= \bigcup_{s' < s_i} (i_p^{s_i,s_j})^{-1}(\mathrm{H}_p^{s',s_j}(\mathcal{F}(S,P))) \\ &= (i_p^{s_i,s_j})^{-1}(\mathrm{H}_p^{s_{i-1},s_j}(\mathcal{F}(S,P))) \\ &= M_p^{s_i,s_j}(\mathcal{F}). \end{aligned}$$

Using additionally the fact that $t' \in [s_{j-1}, s_j)$, the inclusion $S_{P \leqslant s_{j-1}} \hookrightarrow S_{P \leqslant t'}$ is a semi-algebraic homotopy equivalence, we have:

$$\begin{aligned} N_p^{s_i,s_j}(\mathcal{F}(S,P)) &= \bigcup_{s' < s_i \leqslant t' < s_j} (i_p^{s_i,t'})^{-1}(\mathrm{H}_p^{s',t'}(\mathcal{F}(S,P))) \\ &= (i_p^{s_i,s_{j-1}})^{-1}(\mathrm{H}_p^{s_{i-1},s_{j-1}}(\mathcal{F}(S,P))) \\ &= N_p^{s_i,s_j}(\mathcal{F}). \end{aligned}$$

□

This concludes the proof of Proposition 3.3.3. □

Proof of Proposition 3.3.1. Follows immediately from Proposition 3.3.3. □

3.3.2 Persistent multiplicities for finite filtration

In this section, we prove a formula for the persistent multiplicities associated to a finite filtration $\mathcal{F}$, which we later use in Algorithm 7 to obtain the barcodes of a finite filtration. We deduce the formula from our definition of persistent multiplicity (cf. Eqn. (3.1) in Definition 3.2.5). [1]

Proposition 3.3.4. *Let $\mathcal{F}$ denote a finite filtration, given by $X_0 \subset \cdots \subset X_M = X_{M+1} = \cdots = X$, such that rank of $\mathrm{H}_p(X_j)$ is finite for each $p \geqslant 0$. Then for $0 < j < k$,*

$$\mu_p^{j,k}(\mathcal{F}) = \begin{cases} (b_p^{j,k-1}(\mathcal{F}) - b_p^{j,k}(\mathcal{F})) - (b_p^{j-1,k-1}(\mathcal{F}) - b_p^{j-1,k}(\mathcal{F})), & k < \infty, \\ \\ b_p^{j,k}(\mathcal{F}) - b_p^{j-1,k}(\mathcal{F}), & k = \infty. \end{cases} \tag{3.9}$$

[1] ↑This formula already appears in [35, page 152], but what is meant by “independent p-dimensional classes that are born at K_i, and die entering K_j” *loc. cit.* is not totally transparent. See also Remark 11.

Proof. We first prove the case where k is finite. By Definition 3.2.4,

$$\begin{aligned}\mu_p^{j,k}(\mathcal{F}) &= \dim P_p^{j,k}(\mathcal{F})\\ &= \dim M_p^{j,k}(\mathcal{F}) - \dim N_p^{j,k}(\mathcal{F}).\end{aligned}$$

Since $\mathcal{F}$ is finite, we have

$$\begin{aligned}M_p^{j,k}(\mathcal{F}) &= (i_p^{j,k})^{-1}(\mathrm{H}_p^{j-1,k}(\mathcal{F})),\\ N_p^{j,k}(\mathcal{F}) &= (i_p^{j,k-1})^{-1}(\mathrm{H}_p^{j-1,k-1}(\mathcal{F})).\end{aligned}$$

Note that $(i_p^{j,k})^{-1}(\mathrm{H}_p^{j-1,k}(\mathcal{F}))$ is a subspace of $\mathrm{H}_p(X_j)$, and hence the linear map $i^{j,k}$: $\mathrm{H}_p(X_j) \to \mathrm{H}_p(X_k)$ factors through a surjection $f : \mathrm{H}_p(X_j) \to \mathrm{H}_p^{j,k}(\mathcal{F})$ followed by an injection $\mathrm{H}_p^{j,k}(\mathcal{F}) \hookrightarrow \mathrm{H}_p(X_k)$ as shown in the following diagram.

$$\begin{tikzcd}
\mathrm{H}_p(X_j) \arrow[rr, "i_p^{j,k}"] \arrow[dr, "f"'] & & \mathrm{H}_p(X_k) \\
& \mathrm{H}_p^{j,k}(\mathcal{F}) \arrow[ur, hook] &
\end{tikzcd} .$$

Now $\mathrm{H}_p^{j-1,k}(\mathcal{F})$ is a subspace of $\mathrm{H}_p^{j,k}(\mathcal{F})$, and let

$$m : \mathrm{H}_p^{j,k}(\mathcal{F}) \to \mathrm{H}_p^{j,k}(\mathcal{F})/\mathrm{H}_p^{j-1,k}(\mathcal{F})$$

be the canonical surjection. Let $g = m \circ f$. Since f and m are both surjective, so is g.

$$\begin{tikzcd}
\mathrm{H}_p(X_j) \arrow[r, "f"] \arrow[rr, bend right, "g\ =\ m\circ f"'] & \mathrm{H}_p^{j,k}(\mathcal{F}) \arrow[r, "m"] & \mathrm{H}_p^{j,k}(\mathcal{F})\ /\ \mathrm{H}_p^{j-1,k}(\mathcal{F})
\end{tikzcd} ,$$

Now notice that

$$\begin{aligned}M_p^{j,k}(\mathcal{F}) &= (i_p^{j,k})^{-1}(\mathrm{H}_p^{j-1,k}(\mathcal{F}))\\ &= f^{-1}(\mathrm{H}_p^{j-1,k}(\mathcal{F}))\\ &= \ker(g).\end{aligned}$$

Since g is surjective,

$$\operatorname{rank}(g) = \dim \mathrm{H}_p^{j-1,k}(\mathcal{F}) - \dim \mathrm{H}_p^{j,k}(\mathcal{F}),$$

and using the rank-nullity theorem we obtain

$$\dim M_p^{j,k}(\mathcal{F}) = b_p(X_j) - (b_p^{j,k}(\mathcal{F}) - b_p^{j-1,k}(\mathcal{F})). \tag{3.10}$$

Using a similar argument we obtain

$$\dim N_p^{j,k}(\mathcal{F}) = b_p(X_j) - (b_p^{j,k-1}(\mathcal{F}) - b_p^{j-1,k-1}(\mathcal{F})). \tag{3.11}$$

Finally,

$$\begin{aligned}
\mu_p^{j,k}(\mathcal{F}) &= \dim M_p^{j,k}(\mathcal{F}) - \dim N_p^{j,k}(\mathcal{F}) \\
&= b_p^{j-1,k}(\mathcal{F}) - b_p^{j,k}(\mathcal{F}) + (b_p^{j,k-1}(\mathcal{F}) - b_p^{j-1,k-1}(\mathcal{F})) \\
&= (b_p^{j,k-1}(\mathcal{F}) - b_p^{j,k}(\mathcal{F})) - (b_p^{j-1,k-1}(\mathcal{F}) - b_p^{j-1,k}(\mathcal{F})).
\end{aligned}$$

If $k = \infty$, then by Definition 3.2.4,

$$\begin{aligned}
\mu_p^{j,k}(\mathcal{F}) &= \dim P_p^{j,k}(\mathcal{F}) \\
&= \dim \mathrm{H}_p(K_j) - \dim \bigcup_{j \leqslant t} M_p^{j,t}(\mathcal{F}).
\end{aligned}$$

Since $M_p^{s,t}(\mathcal{F}) \subset M_p^{s,t'}(\mathcal{F})$ for $t \leqslant t'$, we have

$$M_p^{j,t}(\mathcal{F}) \subset M_p^{j,t+1}(\mathcal{F}) \subset \cdots \subset M_p^{j,M}(\mathcal{F}) = M_p^{j,M+1}(\mathcal{F}) = \cdots = M_p^{j,\infty}(\mathcal{F})$$

$$\bigcup_{t}^{\infty} M_p^{j,t}(\mathcal{F}) = M_p^{j,M}(\mathcal{F})$$

Therefore,

$$\begin{aligned}
\mu_p^{j,k}(\mathcal{F}) &= \dim \mathrm{H}_p(K_j) - \dim M_p^{j,M}(\mathcal{F}) \\
&= b_p(X_j) - (b_p(X_j) - (b_p^{j,M}(\mathcal{F}) - b_p^{j-1,M}(\mathcal{F}))) \\
&= b_p^{j,M}(\mathcal{F}) - b_p^{j-1,M}(\mathcal{F})
\end{aligned}$$

□

3.4 Algorithms and proof of Theorem 3.2.1

In this section we describe our algorithmic results leading to the proof of Theorem 3.2.1. We begin by stating some preliminary mathematical results in Section 3.4.1 that we will need for our algorithms. We describe two technical algorithms that we will need in Section 3.4.2. In Section 3.4.3 we describe Algorithm 6 for reducing the given continuous filtration to a finite one. The proof of correctness of this algorithm relies on Proposition 3.3.3 proved earlier. Finally, in Section 3.4.3 we describe our algorithm for computing the barcode of a semi-algebraic filtration (algorithm 8), prove its correctness and analyze its complexity, thereby proving Theorem 3.2.1.

3.4.1 Preliminaries

Notation 20 (Derivatives)**.** *Let P be a univariate polynomial of degree p in* $\mathrm{R}[X]$*. We will denote by* $\mathrm{Der}(P)$ *the tuple* $(P, P', \ldots, P^{(p)})$ *of derivatives of* P*.*

The significance of $\mathrm{Der}(P)$ is encapsulated in the following lemma which underlies our representations of elements of R which are algebraic over D (cf. Definition 3.4.1).

Proposition 3.4.1 (Thom's Lemma)**.** *Let* $f \in \mathrm{R}[X]$ *be a univariate polynomial, and, let* σ *be a sign condition on* $\mathrm{Der}(f)$ *Then* $\mathcal{R}(\sigma)$ *is either empty, a point, or an open interval.*

Proof. See [2, Proposition 2.27]. □

Proposition 3.4.1 allows us to specify elements of R which are algebraic over D by means of a pair (f, σ) where $f \in \mathrm{D}[X]$ and $\sigma \in \{0, 1, -1\}^{\mathrm{Der}(f)}$.

Definition 3.4.1. *We say that $x \in \mathrm{R}$ is* associated to the pair (f, σ), *if $\sigma(f) = 0$ and if* $\mathrm{Der}(f)$ *realizes the sign condition σ at x. We call the pair (f, σ) to be a* Thom encoding *specifying x.*

We will also use the notion of a weak sign condition (cf. Definition 3.3.1).

Definition 3.4.2. *A* weak sign condition *is an element of*

$$\{\{0\}, \{0,1\}, \{0,-1\}\}.$$

We say

$$\begin{cases} \mathbf{sign}(x) \in \{0\} & \textit{if and only if } x = 0, \\ \mathbf{sign}(x) \in \{0,1\} & \textit{if and only if } x \geqslant 0, \\ \mathbf{sign}(x) \in \{0,-1\} & \textit{if and only if } x \leqslant 0. \end{cases}$$

A weak sign condition *on $\mathcal{Q}$ is an element of $\{\{0\}, \{0,1\}, \{0,-1\}\}^{\mathcal{Q}}$. If $\sigma \in \{0,1,-1\}^{\mathcal{Q}}$, its* relaxation $\overline{\sigma}$ *is the weak sign condition on $\mathcal{Q}$ defined by $\overline{\sigma}(Q) = \overline{\sigma(Q)}$. The* realization of the weak sign condition τ *is*

$$\mathcal{R}(\tau) = \{x \in \mathrm{R}^k \mid \bigwedge_{Q \in \mathcal{Q}} \mathbf{sign}(Q(x)) \in \tau(Q)\}.$$

Definition 3.4.3. *We say that a set of polynomials $\mathcal{F} \subset \mathrm{R}[X]$ is* closed under differentiation *if $0 \notin \mathcal{F}$ and if for each $f \in \mathcal{F}$ then $f' \in \mathcal{F}$ or $f' = 0$.*

Lemma 3.4.1. *([2, Lemma 5.33]) Let $\mathcal{F} \subset \mathrm{R}[X]$ be a finite set of polynomials closed under differentiation and let σ be a sign condition on the set $\mathcal{F}$. Then*

(a) $\mathcal{R}(\sigma)$ is either empty, a point, or an open interval.

(b) If $\mathcal{R}(\sigma)$ is empty, then $\mathcal{R}(\overline{\sigma})$ is either empty or a point.

(c) If $\mathcal{R}(\sigma)$ is a point, then $\mathcal{R}(\overline{\sigma})$ is the same point.

(d) If $\mathcal{R}(\sigma)$ is an open interval then $\mathcal{R}(\overline{\sigma})$ is the corresponding closed interval.

Remark 18. *In what follows we will allow ourselves to use for* $P \in \mathrm{R}[X_1, \ldots, X_k]$, $\mathbf{sign}(P) = 0$ *(resp.* $\mathbf{sign}(P) = 1$, $\mathbf{sign}(P) = -1$*) in place of the atoms* $P = 0$ *(resp.* $P > 0$, $P < 0$*) in formulas. Similarly, we might write* $\mathbf{sign}(P) \in \bar{\sigma}$*, where* $\bar{\sigma}$ *is a weak sign condition in place of the corresponding weak inequality* $P \geqslant 0$ *or* $P \leqslant 0$*. It should be clear that this abuse of notation is harmless.*

In addition to the mathematical preliminaries described above, we also need two technical algorithmic results that we describe in the next section

3.4.2 Some preliminary algorithms

For technical reasons that will become clear when we describe Algorithm 6, we will need to convert efficiently a given quantifier-free formula defining a closed semi-algebraic set, into a closed formula defining the same semi-algebraic set. This is a non-trivial problem, since the standard quantifier-elimination algorithms in algorithmic semi-algebraic geometry does not guarantee that the output will be a closed formula even if it is known in advance that the semi-algebraic set that the formula is describing is closed. Luckily we only need to deal with formulas in one variable, where the problem is somewhat simpler. Note that even in this case, it is not possible to obtain the description of the given closed semi-algebraic set as a closed formula by merely weakening the inequalities in the original formula.

For example, consider the formula $\phi(X) := (X^2(X-1) > 0) \wedge ((X \geqslant 2) \vee (X \leqslant 0))$. Then, $\mathcal{R}(\phi) = [2, \infty)$ is a closed semi-algebraic set, but the formula obtained by weakening the inequality $X^2(X-1) > 0$, namely

$$\widetilde{\phi} := (X^2(X-1) \geqslant 0) \wedge ((X \geqslant 2) \vee (X \leqslant 0)),$$

has as its realization the set $\{0\} \cup [2, \infty)$ which is strictly bigger than $\mathcal{R}(\phi)$.

Nevertheless, using Lemma 3.4.1 we have the following algorithm to achieve the above mentioned task efficiently.

Proof of correctness. The correctness of the algorithm follows from the correctness of Algorithm 13.1 (Computing realizable sign conditions) in [2], and Lemma 3.4.1. □

Algorithm 4 (Make closed)

Input:

A quantifier-free formula $\theta(Y)$ with coefficients in D, in one free variable Y, such that $\mathcal{R}(\theta)$ is closed.

Output:

A closed formula $\psi(Y)$ equivalent to $\theta(Y)$.

Procedure:

1: Let $\theta(Y) = \bigvee_{1 \leqslant i \leqslant M} \bigwedge_{1 \leqslant j \leqslant N_i} (\mathbf{sign}(F_{i,j}) = \sigma_{i,j})$.
2: **for** each (i, j) such that $\sigma_{i,j} \neq 0$ **do**
3: Call Algorithm 13.1 (Computing realizable sign conditions) in [2] with input $\mathrm{Der}(F_{i,j})$, and obtain the set $\Sigma_{i,j}$ of realizable sign conditions of $\mathrm{Der}(F_{i,j})$.
4: $\Sigma'_{i,j} \leftarrow \{\sigma \in \Sigma_F \mid \sigma(F_{i,j}) = \sigma_{i,j}\}$.
5: $\overline{\Sigma_{i,j}} \leftarrow \{\bar{\sigma} \mid \sigma \in \Sigma'_{i,j}\}$.
6: **end for**
7: **return** the formula

$$\psi(Y) = \bigvee_{1 \leqslant i \leqslant M} (\bigwedge_{\sigma_{i,j}=0} (\mathbf{sign}(F_{i,j}) = 0) \wedge \bigwedge_{\sigma_{i,j} \neq 0} \bigvee_{\bar{\sigma} \in \overline{\Sigma_{i,j}}} (\mathbf{sign}(F_{i,j}) \in \bar{\sigma}).$$

Complexity: The complexity of the algorithm is bounded by $(sd)^{O(1)}$ where s is the number of polynomials appearing θ and d a bound on their degrees.

Complexity analysis. The complexity bound follows from the complexity of Algorithm 13.1 (Computing realizable sign conditions) in [2]. □

We will also need an algorithm that takes as input a finite set of polynomials $\mathcal{G}$ in one variable with coefficients in $\mathrm{D}[\bar{\varepsilon}]$, and outputs a set of Thom encodings whose set of associated points $\{s_0, \ldots, s_M\}$ satisfy the property stated in Lemma 3.3.4.

Algorithm 5 (Removal of infinitesimals)

Input:

A finite set $\mathcal{G} \subset \mathrm{D}[\bar{\varepsilon}][T]$ such that each $P \in \mathcal{G}$ depends on at most $k+1$ of the ε_i's.

Output:

A finite set of Thom encodings $\mathcal{F} = \{(f_i, \sigma_i) \mid 0 \leqslant i \leqslant N\}$, with $f_i \in \mathrm{D}[T]$ with associated points $s_0 < \cdots < s_M$, such that letting $s_{-1} = -\infty, s_{M+1} = \infty$, for each $i, 0 \leqslant i < M$, there exists $j, 0 \leqslant j < N$, such that $(s_i, s_{i+1}) \subset \mathrm{R}$ is contained in $(t_j, t_{j+1}) \cap \mathrm{R}$, where $\{t_0, \ldots, t_N\} = \bigcup_{G \in \mathcal{G}} \mathrm{Z}(G, \mathrm{R}\langle\bar{\varepsilon}\rangle)$, with $t_0 < \cdots < t_N$.

Procedure:

1: **for** $G \in \mathcal{G}$ **do**

2: $0 \leqslant i_0 < \cdots < i_h \leqslant s+1$ be such that $G \in D[\varepsilon_{i_0}, \ldots, \varepsilon_{i_h}][T]$.

3: Write $G = \sum_\alpha m_{G,\alpha}(\varepsilon_{i_0}, \ldots, \varepsilon_{i_h}) G_\alpha$, with $G_\alpha \in \mathrm{D}[T], m_{G,\alpha} \in D[\varepsilon_{i_0}, \ldots, \varepsilon_{i_h}]$.

4: Let $M(G) = \{\alpha \mid m_{G,\alpha} \neq 0\}$.

5: **end for**

6: Let $\mathcal{H} = \bigcup_{G \in \mathcal{G}, \alpha \in M(G)} \{G_\alpha\}$.

7: Use Algorithm 10.17 from [2] with $\mathcal{H}$ as input to obtain an ordered list of Thom encodings $\mathcal{F}$.

8: **return** $\mathcal{F}$.

Complexity: The complexity of the algorithm is bounded by $sD^{O(k)}$, where $s = \mathrm{card}(\mathcal{G})$ and D is a bound on the degrees of the polynomials in $\mathcal{G}$ in $\bar{\varepsilon}$ and in T.

Proof of correctness. The correctness of the algorithm follows from Lemma 3.3.4 and the correctness of Algorithm 10.17 from [2]. □

Complexity analysis. The complexity bound follows from the complexity bound of Algorithm 10.17 from [2]. □

3.4.3 Algorithm for reducing to a finite filtration

We are now in a position to describe our algorithm for reducing the problem of computing the barcode of a filtration of a semi-algebraic set S by the sub-level sets of a polynomial P, to the problem of computing the barcode of a finite filtration.

Algorithm 6 computes a finite subset of R, as Thom encodings (cf. Definition 3.4.1), such that it includes the values of P at which the homotopy type of the sub-level sets of S changes. The algorithm has singly exponentially bounded complexity.

Algorithm 6 (Reducing to a finite filtration)

Input:

(a) $\ell \in \mathbb{Z}_{\geqslant 0}$.

(b) $R \in \mathrm{D}, R > 0$.

(c) A finite set $\mathcal{P} = \{P_1, \ldots, P_s\} \subset \mathrm{D}[X_1, \ldots, X_k]$.

(d) A $\mathcal{P}$-closed formula ϕ.

(e) A polynomial $P \in \mathrm{R}[X_1, \ldots, X_k]$.

Output:

(a) A finite set of Thom encodings $\mathcal{F} = \{(f_i, \sigma_i) \mid 0 \leqslant i \leqslant N\}$, with $f_i \in \mathrm{D}[T]$ with associated points $t_0 < \cdots < t_N$, such that for $t \in \mathrm{R}$, denoting by $S_t = \mathcal{R}(\phi) \cap \overline{B_k(0, R)} \cap \{x \mid P(x) \leqslant t\}$, for each $i, 0 \leqslant i \leqslant N-1$, and all $t \in [t_i, t_{i+1})$ the inclusion maps $S_{t_i} \hookrightarrow S_t$ are homological equivalences.

(b) A filtration of finite simplicial complexes

$$K_0 \subset K_1 \subset \cdots \subset K_N$$

such that $\mathbf{Simp}^{[N]}(S_{t_0}, \ldots, S_{t_N})$ is homologically ℓ-equivalent to $\mathbf{Simp}^{[N]}(|K_0|, \ldots, |K_N|)$.

Procedure:

$P_0 \leftarrow \sum_{i=1}^{k} X_i^2 - R$.

$P_{s+1} \leftarrow P - Y$.

$$\mathcal{P}^\star(\bar{\varepsilon}) \leftarrow \bigcup_{0 \leqslant i \leqslant s+1} \{P_i + \varepsilon_i, P_i - \varepsilon_i\}.$$

Denote by $\phi^\star(\bar{\varepsilon})$, the $\mathcal{P}^\star(\bar{\varepsilon})$-closed formula obtained by replacing each occurrence of $P_i \geqslant 0$ in ϕ by $P_i + \varepsilon_i \geqslant 0$ (resp. $P_i \leqslant 0$ in ϕ by $P_i - \varepsilon_i \leqslant 0$) for $0 \leqslant i \leqslant s+1$.

for $\mathcal{Q} \subset \mathcal{P}^\star(\varepsilon), \mathrm{card}(\mathcal{Q}) \leqslant k$ **do**

$$Jac(\mathcal{Q}) \leftarrow \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{\mathrm{card}(\mathcal{Q}')} \leqslant k} \det\left(\left(\frac{\partial Q}{\partial X_{i_j}}\right)_{Q \in \mathcal{Q}, 1 \leqslant i \leqslant k}\right)$$

end for

for $\mathcal{Q} \subset \mathcal{P}^\star(\varepsilon), \mathrm{card}(\mathcal{Q}) = k+1$ **do**

$$\Sigma(\mathcal{Q}') \leftarrow \sum_{Q \in \mathcal{Q}} Q^2.$$

end for

$$\mathcal{H} \leftarrow \{Jac(\mathcal{Q}) \mid \mathcal{Q} \subset \mathcal{P}^\star(\varepsilon), \mathrm{card}(\mathcal{Q}) \leqslant k\} \cup \{\Sigma(\mathcal{Q}) \mid \mathcal{Q} \subset \mathcal{P}^\star(\varepsilon), \mathrm{card}(\mathcal{Q}) = k+1\}.$$

Call Algorithm 14.1 (Block Elimination) from [2] with the block of variables $(X_1, \ldots, X_k)$ and $\mathcal{H}$ as input, and obtain $\mathcal{G} = \mathrm{BElim}_X(\mathcal{F})$ (following the same notation as in [2, Algorithm 14.1 (Block Elimination)]).

Call Algorithm 5 with $\mathcal{G}$ as input and obtain an ordered list of Thom encodings $\mathcal{F} = ((f_0, \sigma_0), \ldots, (f_N, \sigma_N))$.

for $0 \leqslant i \leqslant N$ **do**

Call Algorithm 14.5 (Quantifier Elimination) [2] with input the formula

$$\widetilde{\psi}(Y) := \forall Z((f_i(Z) = 0) \wedge (\mathbf{sign}(\mathrm{Der}(f_i))(Z) = \sigma_i)) \Rightarrow (Y \leqslant Z)$$

to obtain an equivalent quantifier-free formula $\widetilde{\psi}_i(Y)$.

Call Algorithm 4 with $\widetilde{\psi}_i(Y)$ as input to obtain a closed formula $\psi_i(Y)$.

$\phi_i \leftarrow \widetilde{\phi} \wedge \psi_i(Y)$.

$\mathcal{Q}_i \leftarrow$ the set of polynomials appearing in ψ_i.

end for

Call Algorithm for simplicial replacement with input: the closed formulas $\phi_0, \ldots, \phi_N$, R and ℓ, and output the simplicial complexes $K_i, 0 \leqslant i \leqslant N$.

Complexity: The complexity of the algorithm is bounded by $(sd)^{k^{O(\ell)}}$, where $s = \mathrm{card}(\mathcal{P})$, and $d = \max_{P \in \mathcal{P}} \deg(P)$.

Proof of correctness. The correctness of the algorithm follows from Proposition 3.3.3, and the correctness of the following algorithms: Algorithm 14.1 (Block Elimination) in [2]), Algorithm 5, Algorithm 14.5 (Quantifier Elimination) in [2], Algorithm 4, and the Algorithm for simplicial replacement Theorem 2.2.1. □

Complexity analysis. The complexity bound follows from the complexity bounds of Algorithm 14.1 (Block Elimination) in [2]), Algorithm 5, Algorithm 14.5 (Quantifier Elimination) in [2], Algorithm 4, and the Algorithm 3. □

3.4.4 Computing barcodes of semi-algebraic filtrations

We can now describe our algorithm for computing the barcode of the filtration of a semi-algebraic set by the sub-level sets of a polynomial. First we need an algorithm for computing barcodes of finite filtrations of finite simplicial complexes.

Algorithm 7 (Barcode of a finite filtration of finite simplicial complexes)

Input:

1. $\ell \in \mathbb{Z}_{\geqslant 0}$.
2. A finite filtration $\mathcal{F}$, $K_0 \subset \cdots \subset K_N$ of finite simplicial complexes.

Output:

$\mathcal{B}_p(\mathcal{F}), 0 \leqslant p \leqslant \ell$.

Procedure:

$K_{-1} \leftarrow \varnothing$.

$K_{N+1} \leftarrow K_N$.

for $-1 \leqslant i \leqslant j \leqslant N+1$ **do**

  Use Gaussian elimination to compute the persistent Betti numbers $b_p^{i,j}(\mathcal{F})$.

end for

for $0 \leqslant p \leqslant \ell, 0 \leqslant i \leqslant j \leqslant N+1$ **do**

  if $j = N+1$ **then**

$$\mu_p^{i,j} \leftarrow b_p^{i,j}(\mathcal{F}) - b_p^{i-1,j}(\mathcal{F})$$

  else

$$\mu_p^{i,j} \leftarrow (b_p^{i,j-1}(\mathcal{F}) - b_p^{i,j}(\mathcal{F})) - (b_p^{i-1,j-1}(\mathcal{F}) - b_p^{i-1,j}(\mathcal{F}))$$

  end if

  (cf. Eqn. (3.9)).

  end for

  for $0 \leqslant p \leqslant \ell$ **do**

    Output

$$\mathcal{B}_p(\mathcal{F}) = \{(i, j, \mu_p^{i,j}) \mid 0 \leqslant i \leqslant j \leqslant N, \mu_p^{i,j} > 0\} \cup$$
$$\{(i, \infty, \mu_p^{i,j}) \mid 0 \leqslant i \leqslant j = N+1, \mu_p^{i,j} > 0\}.$$

  end for

Complexity: The complexity of the algorithm is bounded polynomially in N times the number of simplices appearing in the complex K_N.

Proof of correctness. The correctness of the algorithm follows from Eqn. (3.9). □

Complexity analysis. The complexity of the algorithm follows from the complexity of Gaussian elimination. □

Algorithm 8 (Computing persistent homology barcodes of semi-algebraic sets)

Input:

A. A $\mathcal{P}$-closed formula ϕ, with $\mathcal{P}$ a finite subset of $\mathrm{D}[X_1, \ldots, X_k]$, such that $\mathcal{R}(\phi, \mathrm{R}^k)$ is bounded.

B. A polynomial $P \in \mathrm{D}[X_1, \ldots, X_k]$.

C. $\ell, 0 \leqslant \ell \leqslant k$.

Output:

For each $p, 0 \leqslant p \leqslant \ell$, $\mathcal{B}_p(S, P)$, where $S = \mathcal{R}(\phi)$.

Procedure:

1: $\mathcal{P}' \leftarrow \mathcal{P} \cup \{\varepsilon(X_1^2 + \cdots + X_k^2) - 1\}$.

2: $\phi' \leftarrow \phi \wedge \varepsilon^2(X_1^2 + \cdots + X_k^2) - 1 \leqslant 0)$.

3: $\mathrm{R} \leftarrow \mathrm{R}\langle\varepsilon\rangle$, $\mathrm{D} \leftarrow \mathrm{D}[\varepsilon]$.

4: Call Algorithm 6 with input $\ell, 1/\varepsilon, \mathcal{P}', \phi', P$, to obtain a finite ordered set of Thom encodings $(f_0, \sigma_0), \ldots, (f_N, \sigma_N)$, and a finite filtration $\mathcal{F} = (K_0 \subset \cdots \subset K_N)$, where K_N is a finite simplicial complex.

5: Call Algorithm 7 with input ℓ and the finite filtration $\mathcal{F}$, and output for each $p, 0 \leqslant p \leqslant \ell$, $\mathcal{B}_p(\mathcal{F})$.

6: **for each** $p, 0 \leqslant p \leqslant \ell$

Output

$$\mathcal{B}_p(S, P) = \bigcup_{(i,j,\mu) \in \mathcal{B}_p(\mathcal{F}), 0 \leqslant i \leqslant j \leqslant N} \{((f_i, \sigma_i), (f_j, \sigma_j), \mu)\} \cup \bigcup_{(i,\infty,\mu) \in \mathcal{B}_p(\mathcal{F})} \{((f_i, \sigma_i), \infty, \mu)\}.$$

Complexity: The complexity of the algorithm is bounded by $(sd)^{k^{O(\ell)}}$, where $s = \mathrm{card}(\mathcal{P})$, and $d = \max_{Q \in \mathcal{P} \cup \{P\}} \deg(Q)$.

Proof of correctness. The correctness of the algorithm follows from the correctness of Algorithms 6 and 7. □

Complexity analysis. The complexity bound follows from the complexity bounds of Algorithms 6 and 7. □

Proof of Theorem 3.2.1. The theorem follows from the correctness and the complexity analysis of Algorithm 8. □

3.5 Future work and open problems

We conclude this chapter by stating some open problems and possible future directions of research in this area.

1. It would be very interesting (and challenging) to obtain an algorithm with singly exponential complexity that computes the entire barcode of a semi-algebraic filtration, and not restricted to dimension up to ℓ. This would imply also an algorithm with singly exponential complexity for computing all the Betti numbers of a given semi-algebraic set, which is a challenging problem on its own [5].

2. Another open problem is to extend Algorithm 8 to the case of non-proper semi-algebraic maps using the proposed definition of barcodes for non-proper semi-algebraic maps (see Definition 3.2.8).

3. One very active topic in the area of persistent homology is the theory of multi-dimensional persistent homology [36]. In our setting this would imply studying the sub-level sets of two or more real polynomial functions simultaneously. While the so called persistence modules and associated barcodes can be defined analogously to the one-dimensional situation (see for example [36]), an analog of Proposition 3.3.1 is missing. It is thus an open problem to give an algorithm with singly exponential complexity to compute the barcodes of "higher dimensional" semi-algebraic filtrations.

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