# LIMITING DISTRIBUTIONS AND DEVIATION ESTIMATES OF RANDOM WALKS IN DYNAMIC RANDOM ENVIRONMENTS

by

**Yongjia Xie**

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# THE PURDUE UNIVERSITY GRADUATE SCHOOL
# STATEMENT OF COMMITTEE APPROVAL


**Dr. Jonathon Peterson, Chair**

Department of Mathematics

**Dr. Rodrigo Banuelos**

Department of Mathematics

**Dr. Christopher Janjigian**

Department of Mathematics

**Dr. Jing Wang**

Department of Mathematics

**Approved by:**

Dr. Plamen Stefanov

To my parents

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# ABSTRACT

This dissertation includes my research works during Ph.D. career about three different kinds of random walks in (dynamical) random environments. It includes my two published papers "Functional weak limit of random walks in cooling random environments" [1] which has been published in electronic communications in probability in 2020, and "Variable speed symmetric random walk driven by the simple symmetric exclusion process" [2] which is the joint work with Peterson and Menezes and has been published in electronic journals of probability in 2021. This dissertation also includes my two other projects, one is the joint work with Janjigian and Emrah about moderate deviation and exit time estimates in integrable directed polymer models. The other one is the joint work with Peterson and Conrado that extends the weak limit of random walks in cooling randon environments with underlying environment is in the transient case with parameter $\kappa \in (0,1)$ or $\kappa = 2$. Previous results show the weak limit in the cases where the environment is recurrent or transient but with $\kappa > 2$ or $\kappa \in (1,2)$.

# 1. INTRODUCTION

Research on random walks in disordered environments has attracted a lot of attention by mathematicians and physicists over the last few decades. The model of random walks in random environments (RWRE) was first studied by Solomon in [3]. In this model the spatial disorder in the environment is random[1], but fixed for all time by the walk. Much of the subsequent interest in this model was driven by the fact that RWREs could exhibit a surprisingly rich array of asymptotic behaviors such as transience with asymptotically zero speed [3], and limiting distributions which are non-Gaussian and have non-diffusive scaling [4]–[6]. These interesting phenomena can be understood as occurring because of the "trapping" effects of the environment. See [7] for an overview of basic results in RWRE.

More recently, there has been interest in a generalization of RWRE called random walks in dynamic random environments (RWDRE) in which the disorder of the environment is random in both space and time. One can see that RWDRE interpolates between simple random walk (SRW) and RWRE: If the dynamics are "frozen", i.e. the environment is not changing after initial set-up, then this is simply a RWRE. On the other hand, if the environment is space-time i.i.d. then it is easy to see that the distribution of the RWDRE (under the annealed measure) is the same as that of a SRW. For RWDRE models which are between these two extremes there is an interplay between the trapping effects introduced by the randomness of the environment and the rate at which the time dynamics of the environment causes these traps to disappear. One might expect therefore that environments with "fast" mixing time dynamics should have similar characteristics as a SRW (e.g. path convergence to Brownian motion) while "slow" mixing time dynamics might retain some of the strange behaviors of RWRE (e.g., non-Gaussian limiting distributions or transience with sublinear speed).

Many of the results thus far in RWDRE have focused on dynamic environments which are in some sense fast mixing, see [8]–[10]. For example, the environment may be assumed to be a Markov chain with uniformly mixing time dynamics or which satifies a Poincaré

[1]↑A common assumption is that the randomness in the environment is i.i.d., though there also results where weaker assumptions are used instead.

inequality. A variety of approaches have been used in these papers, but in all cases one can obtain convergence to Brownian motion after centering and diffusive scaling.

Environments which are more slowly mixing present different problems as the trapping effects of the environment may possibly be stronger. Examples of environments like conservative particle systems have poor mixing rates [11], [12]. A particularly interesting example is the case where the dynamic environment is given by a simple symmetric exclusion process. Avena and Thomann [13] have made conjectures based on simulations that this model can exhibit many of the same strange behaviors as that of RWRE (e.g., transience with zero speed and non-diffusive scaling). However, the results for this model have been limited to some cases where the parameters of the model are near their extremes and in these cases once again the distribution of the walk converges under diffusive scaling to a Brownian motion. Conjecture 3.5 in [13] shows the cases when the parameters are not near extreme, which can allow the walk to have either super or sub diffusive scaling. But no math proof work is done in those cases so far. Other examples of slow mixing environments for which the RWDRE has been shown to converge to Brownian motion are [14]–[16].

All the above results for RWDRE have shown limiting behavior which is like that of a SRW. Recently, however, Avena and den Hollander have introduced a new model of RWDRE, *random walks in cooling random environment*(RWCRE), in which the dynamics can be slow enough that the model retains some of the strange behavior of RWRE [17]. In this model the environment is totally refreshed at some points called resampling times. Results for this model have included a strong law of large numbers, a quenched large deviation principles, sufficient conditions for recurrence/transience, and limiting distributions [17]–[19]. Most relevant to the results of the present thesis, for certain cases of RWCRE they prove that the limiting distributions are Gaussian but with non-diffusive scalings that interpolate between the $(\log n)^2$ scaling of recurrent RWRE and the diffusive $\sqrt{n}$ scaling of SRW [17], [19]. One of the main goal of Section 2 is to determine the appropriate limiting distributions for the path of the walk in these cases.

More recent works about RWCRE in [20] shows the weak limit of the walk when the underlying random environment is such that the RWRE on it is transient. In this case a key parameter $\kappa$ which characterizes the distribution of the environment will make different weak limits of the walk. In the previous paper [19], the case where $\kappa > 2$ is studied, whose weak limit is always a mixture of independent Guassian random variable. While [20] discusses the case when $\kappa \in (1, 2)$, depending on different cooling maps, the weak limit can vary from stable to normal distribution or the mixture of those two kinds. One of the main goals in Section 2 is to extend the previous result and study the cases where $\kappa \in (0, 1)$ and $\kappa = 2$. In brief, when $\kappa \in (0, 1)$ the weak limit is a mixture of independent Mittag-Leffler's distribution and normal distribution due to the $L_2$ convergence property of RWRE under the corresponding environment condition. When $\kappa = 2$, the weak limit is Gaussian under slow and fast cooling regimes, but the variance of the walk may not converge to the one of its limit distribution if the cooling is as slow as polynomial increasing, which reveals that in this case the $L_2$ convergence of RWRE fails.

In Section 3, we prove a quenched functional central limit theorem for a one-dimensional random walk driven by a simple symmetric exclusion process. The model belongs to the class of random walks in dynamical random environments. Recent works have studied examples where the environment is an interacting particle system, including independent random walks [12], the contact process [10] and the simple symmetric exclusion process (SSEP).

To define a random walk driven by the SSEP, one fixes parameters $p_1, p_0, \rho \in [0, 1]$, $\lambda_0, \lambda_1 > 0$ and makes the random walk jump from $x \in \mathbb{Z}$ to $x+1$ at time $t$ at rate $\lambda_1 p_1 \eta_t(x) + \lambda_0 p_0(1 - \eta_t(x))$, where $\eta_t(x)$ is the state of the exclusion process (either 0 or 1) at site $x$ and time $t$, started from equilibrium at density $\rho$. The rate for a jump from $x$ to $x - 1$ is $\lambda_1(1 - p_1)\eta_t(x) + \lambda_0(1 - p_0)(1 - \eta_t(x))$. Several cases were studied. The results in [16] and [15] that we are about to cite were proven for a discrete-time random walk, but we believe that the continuous-time results we state are true as well. In [16], laws of large numbers and Gaussian fluctuations are proven for $\lambda_0 = \lambda_1$ sufficiently large or sufficiently small and appropriate assumptions on $p_0$ and $p_1$. When $\lambda_0 = \lambda_1$, [21] proves that the limiting speed, if any, is strictly between $\lambda_0(2p_0 - 1)$ and $\lambda_1(2p_1 - 1)$. In [15] it is proven that, for $\lambda_0 = \lambda_1 = 1$ the law of large numbers holds for all $\rho$, with only two possible exceptions, and when the speed is not zero a Gaussian central limit theorem holds. Moreover, when $p_0 = 1 - p_1$ (as in [22] and [16]) and $\rho = 1/2$ it was shown in [15] that the speed is zero, but it is an interesting open problem to determine the scale of the fluctuations in this case and there are several competing conjectures: in [23] it is conjectured that under the scaling $t^{3/4}$ the limiting process is a fractional Brownian motion with Hurst index $H = 3/4$; in [24], it is conjectured (for a related continuous model) that the fluctuations are either of order $t^{1/2}$ (for a fast particle) or $t^{2/3}$ (for a slow particle); on the other hand in [25] and [26], it is conjectured that for either fast or slow particle dynamics the fluctuations are always of order $t^{1/2}$ for time $t$ sufficiently large.

Here we allow $\lambda_0 \neq \lambda_1$ but assume $p_0 = p_1 = \frac{1}{2}$. In this setting, the random walk is a time-change of a simple symmetric random walk. The law of large numbers is immediate, and the problem is to prove convergence to Brownian motion and compute the variance of this limiting Brownian motion at time $t$. We perform this computation when the environment starts in equilibrium at density $\rho \in [0, 1]$. With those assumptions, our model falls into the class of balanced dynamic random environments. For this class of models an invariance principle was proved in [27]. In this thesis we give an entirely different proof of the invariance principle for this particular model. Since random walks in balanced environments are martingales, the key to proving an invariance principle is in proving that the quadratic variation grows linearly. In all previous proofs of invariance principles for random walks in (static or dynamic) environments this was accomplished by proving the existence of an invariant measure for the environment viewed from the particle that was absolutely continuous with respect to the initial measure on environments (see e.g., [27]–[30]). In this thesis, however, we are able to prove the linear growth of the quadratic variation without any reference to the existence of invariant measures for the environment viewed from the particle. Not only does this give a simpler proof of the invariance principle for this particular model, but it also enables us to compute explicitly the scaling constant in the invariance principle and allows us to obtain quantitative estimates on the rate of convergence for the quadratic variation, see (3.64).

Since the underlying dynamic environment in our model has only two types of sites (particles/holes), the key to analyzing the growth rate of the quadratic variation is to compute the asymptotic fraction of time, $\lim_{t\to\infty} t^{-1} \int_0^t \eta_{X_s}(s)\,\mathrm{d}s$. We accomplish this by providing an explicit function $\varphi$ and explicit constants $a$ and $b$ such that $L\varphi \approx a\xi_0 + b$, where $\xi_x(t) := \eta_t(x + X_t)$ and $L$ denotes the generator of the process $(\xi(t))_{t\geq 0}$, the environment as seen by the walk. This technique of estimating additive functionals $\int_0^t g(\xi(s))\,\mathrm{d}s$ by solving the equation $g(\xi) \approx a + bu(\xi)$ was introduced in [31]. In the context of random walks in random environments, it has been used in [32], [33], and [34], among other works.

In Section 4, we discuss random polymer models which can also be viewed as a random walk in random environments. The directed polymer in a random environment represents a polymer (a long chain of molecules) by a random path that interacts with a random environment. This model was introduced in statistical physics literature by Huse and Henley [35]. Imbrie and Spencer [36] formulated this model as a random walk in a random environment. The random environment $\omega = (\omega(s,u) : s \in \mathbb{N}, u \in \mathbb{Z}^d)$ puts a real-valued weight $\omega(s,u)$ at space time point $(u,s) \in \mathbb{Z}^d \times \mathbb{N}$. The random path (also can be viewed as a random walk) $X_n$ with length $n$ is given a weight by summing up all the vertex weights on it: $H_n(X_n) = \sum_{k=1}^n \omega(X_k, k)$. The quenched polymer distribution on paths, in environment $\omega$ and an inverse temperature $\beta$, is the probability measure defined by

$$Q_n^\omega(dX_n) = \frac{1}{Z_n^\omega} \exp\{\beta H_n(X_n)\}, \tag{1.1}$$

where the normalizing factor (partition function) $Z_n^\omega = \sum_{|X_n|=n} \exp\{\beta H_n(X_n)\}$. The sum is taken over all directed simple paths of length $n$.

In recent decades, the importance of those models has become more popular due to the belief that it belongs to Kardar-Parisi-Zhang (KPZ) universality class. The KPZ class is characterized by two parameters, the fluctuation exponent $1/3$ and the wandering exponent $2/3$ and certain specified limit distributions on these characteristic scales. See for references [37]–[40]. Tail estimates in models of surface growth in the KPZ class at the characteristic scales and just past them, in the moderate deviation regime, have played a particularly important role in mathematical work seeking to make physically motivated heuristic arguments about random growth models mathematically rigorous. For example, Emrah, Janjigian, and Seppäläinen [41] studied the right tail moderate deviation in the exponential last passage percolation which belongs to corner growth model that is also believed to be a member of the KPZ class.

We study four kinds of 1+1 dimensional directed polymer models. The log-gamma model was introduced in [42]. The strict weak model was introduced in two independent papers, [43], [44], with a slightly different path geometry. The beta polymer was originally introduced in [45] and defines a random walk in a random environment. The inverse-beta polymer was originally introduced in [46]. In [47], Chaumont and Noack define an integrability property called Burke property that shared by log-gamma, strict-weak, beta, and inverse-beta models. Meanwhile, those four models are the only models that possess this property. This property implies a preservation in distribution of ratios of partition functions, as well as allows some exact computations of interest, such as free energy (the log of partition function).

Our main goal is to find the upper and lower bound of the moderate deviation of the free energy under the KPZ fluctuation exponent 1/3. The result shows that the upper bound of the right tail moderate deviation of the free energy has the exponential leading order term $-4/3s^{3/2}$ in the bulk model and $-2/3s^{3/2}$ in the multi-parameter model. This result (in the bulk model) coincides with the leading order of the right tail of the weak limit of the rescaled free energy which is called Tracy-Widom GUE distribution. For the lower bound estimate of the moderate deviation, we also believe that the leading order term is the same as the one of the upper bound. The key step to this conclusion, is to generate a lower bound of the log moment generating function at moment $\lambda$ of the normalized free energy with exponential leading order term $\lambda^3/12$, which is exactly the Legendre transform of $4/3s^{3/2}$.

As a key step to the proof of the lower bound of the moment generation function of the free energy, which is also one of our interests, we discover the tail estimate of the annealed exit time of the polymer. The exit time $\text{Exit}(m, n)$ of a polymer from $(0, 0)$ to $(m, n)$ on a random environment is defined as the time it exits the boundary. So the quenched tail probability $P^\omega(\text{Exit}(m, n) > k)$ will decay to zero as $k \geq m \vee n$. We generate an upper bound of the annealed probability of the exit time that is more than $s(m + n)^{2/3}$, that is, $E[P^\omega(\text{Exit}(m, n) > k)]$ where $k = s(m+n)^{2/3}$. The result shows that the upper bound decays faster that $\exp\{-cs^3\}$, which means it is unlikely for the polymer to leave the boundary later than $(m + n)^{2/3}$ steps. 2/3 also reveals that the model we study belongs to KPZ class.

# 2. WEAK LIMITS OF RANDOM WALKS IN COOLING RANDOM ENVIRONMENT

## 2.1 Random walks in cooling random environment

We will use the same notations as in Avena and den Hollander [17]. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The classical one-dimensional random walk in random environment (RWRE) is defined as follows. Let $\omega = \{\omega(x) : x \in \mathbb{Z}\}$ be an i.i.d. sequence with probability distribution

$$\mu = \alpha^{\mathbb{Z}} \tag{2.1}$$

for some probability distribution $\alpha$ on (0,1). We also assume that $\alpha$ is uniformly elliptic, i.e. there exists $c > 0$, such that

$$P(\omega(0) \in (c, 1-c)) = 1. \tag{2.2}$$

**Remark 2.1.1.** *The uniform ellipticity is not necessarily required for all the results of RWRE throughout this chapter. But for different results, they require some specific assumptions of $\alpha$. For example, proposition 2.2.1 requires $E[\log \rho(0)]$ is well defined, proposition 2.3.1 requires $E[\rho(0)^{\kappa} \log \rho(0)] < \infty$. For simplicity, a uniform ellipticity assumption will satisfy all the extra assumptions of those results. Thus we assume it at the beginning of the introduction of RWRE.*

The random walk in the *spatial* environment $\omega$ is the Markov process $Z = (Z_n)_{n\in\mathbb{N}_0}$ starting at $Z_0 = 0$ with transition probabilities

$$P^{\omega}(Z_{n+1} = x + \mathrm{e} | Z_n = x) = \begin{cases} \omega(x), & \text{if } \mathrm{e} = 1, \\ 1 - \omega(x), & \text{if } \mathrm{e} = -1, \end{cases} \qquad n \in \mathbb{N}_0. \tag{2.3}$$

The properties of $Z$ are well understood, both under the *quenched* law $P^{\omega}(\cdot)$ and the *annealed* law

$$\mathbb{P}_{\mu}(\cdot) = \int_{(0,1)^{\mathbb{Z}}} P^{\omega}(\cdot)\mu(d\omega). \tag{2.4}$$

The random walk in cooling random environment (RWCRE) is a model where $\omega$ is updated along a growing sequence of determined times. Let $\tau: \mathbb{N}_0 \to \mathbb{N}_0$ be a strictly increasing map such that $\tau(0) = 0$ and $\tau(k) \geq k$ for $k \in \mathbb{N}$. Define a sequence of random environments $\Omega = (\omega_n)_{n\in\mathbb{N}_0}$ as follows: At each time $\tau(k)$, $k \in \mathbb{N}_0$, the environment $\omega_{\tau(k)}$ is freshly resampled from $\mu = \alpha^{\mathbb{Z}}$ and does not change during the time interval $[\tau(k), \tau(k+1))$. That is, $\omega_n = \omega_{\tau(k)}$ where $k$ is such that $\tau(k) \leq n < \tau(k+1)$. The random walk in the *space-time* environment $\Omega$ is the Markov process $X = (X_n)_{n\in\mathbb{N}_0}$ starting at $X_0 = 0$ with transition probabilities

$$P^{\Omega,\tau}(X_{n+1} = x + \mathrm{e} | X_n = x) = \begin{cases} \omega_n(x), & \text{if } \mathrm{e} = 1, \\ 1 - \omega_n(x), & \text{if } \mathrm{e} = -1, \end{cases} \qquad n \in \mathbb{N}_0. \tag{2.5}$$

We call $X$ the *random walk in cooling random environment* with *resampling rule* $\alpha$ and *cooling rule* $\tau$. The distribution $P^{\Omega,\tau}$ of the random walk for a given space time environment is called the *quenched law*. The *annealed law* of the walk $\{X_n\}_{n\geq 0}$ is obtained by averaging the quenched with respect to the distribution $\mathbb{Q} = \mathbb{Q}_{\alpha,\tau}$ on $\Omega$.

$$\mathbb{P}^{\tau}(\cdot) = \int_{((0,1)^{\mathbb{Z}})^{\mathbb{N}_0}} P^{\Omega,\tau}(\cdot)\mathbb{Q}(d\omega), \tag{2.6}$$

## 2.2 Random walks in cooling random environment: Sinai's regime

### 2.2.1 Slow and fast cooling: Gaussian fluctuations for recurrent RWRE

In Solomon's seminal paper [3], he showed that the recurrence/transience of a RWRE is determined by the sign of $E_\alpha[\log \rho(0)]$, where

$$\rho(0) = \frac{1 - \omega(0)}{\omega(0)} \tag{2.7}$$

and $E_\alpha[\cdot]$ denotes expectations with respect to the measure $\alpha$. In particular, if $E_\alpha[\log \rho(0)] = 0$ then the RWRE is recurrent. Subsequently, the scaling limit in the recurrent case was identified by Sinai [6] and the explicit form of the limiting distribution by Kesten [5]. Moreover, it was shown by Avena and den Hollander [17] that the convergence also holds in $L^p$.The next proposition summarises their results.

**Proposition 2.2.1.** [[6][5][17], **Scaling limit RWRE: recurrent case**] *Let $\alpha$ be any probability distribution on (0,1) satisfying $E(\log \rho(0)) = 0$ and $\sigma_\mu^2 = E[\log^2 \rho(0)] \in (0, \infty)$. Then, under the annealed law $\mathbb{P}_\mu$, the sequence of random variables*

$$\frac{Z_n}{\sigma_\mu^2 \log^2 n}, \quad n \in \mathbb{N}, \tag{2.8}$$

*converges in distribution and in $L^p$ to a random variable $V$ on $\mathbb{R}$ that is independent of $\alpha$. The law of $V$ has a density $p(x)$, $x \in \mathbb{R}$, with respect to the Lebesgue measure that is given by*

$$p(x) = \frac{2}{\pi} \sum_{k \in \mathbb{N}_0} \frac{(-1)^k}{2k+1} \exp\left[-\frac{(2k+1)^2\pi^2}{8}|x|\right], \qquad x \in \mathbb{R}. \tag{2.9}$$

In their initial paper on RWCRE Avena and den Hollander introduced several kinds of cooling regimes that are interesting to research. For RWCRE in this thesis, following their works, we focus on two kinds of growth regimes for $\tau(k)$. Let $T_k = \tau(k) - \tau(k-1)$,

(R1) *Slow cooling:* $T_k \sim \beta B k^{\beta-1}$, for some $B \in (0, \infty)$ and $\beta \in (1, \infty)$.

(R2) *Fast cooling:* $\log T_k \sim Ck$, for some $C \in (0, \infty)$.

When the distribution $\alpha$ is as in Proposition 2.2.1, Avena and den Hollander [17] proved a limiting distribution for the walk under both the fast and slow cooling regimes. Later in [19] they strengthened this to $L^p$ convergence. The following proposition summaries their results. Note that here and throughout the remainder of this chapter we will use $\mathcal{N}(\mu, \sigma^2)$ to denote a Gaussian random variable with mean $\mu$ and variance $\sigma^2$.

**Proposition 2.2.2.** [[19], **Slow and fast cooling: Gaussian fluctuations for recurrent RWRE**] *Let $\alpha$ be as in Proposition 2.2.1. In regime (R1) and (R2), under the annealed law $\mathbb{P}$,*

$$\frac{X_n - \mathbb{E}(X_n)}{\sqrt{\chi_n(\tau)}} \to^{L^p} \mathcal{N}(0,1), \tag{2.10}$$

*where*

$$\chi_n(\tau) = \begin{cases} (\sigma_\mu^2 \sigma_V)^2 (\frac{\beta-1}{\beta})^4 (\frac{n}{B})^{\frac{1}{\beta}} \log^4 n, & \textit{in regime } (\text{R1}), \\ (\sigma_\mu^2 \sigma_V)^2 (\frac{1}{5C^5}) \log^5 n, & \textit{in regime } (\text{R2}), \end{cases} \tag{2.11}$$

*with* $\sigma_\mu^2$ *the variance of the random variable* $\log \rho(0)$ *and* $\sigma_V^2$ *the variance of the random variable with density* (2.9). *Moreover, in (R2) the centering part can be removed. That is,*

$$\frac{X_n}{\sqrt{\chi_n(\tau)}} \to^{L^p} \mathcal{N}(0,1). \tag{2.12}$$

**Remark 2.2.1.** *In the most recent work [19], the authors have studied more general cooling regimes and have found their limiting behavior. In fact, despite the sequence being always tight, depending on the relative variance weight, the centered walk may not always converge. In short, relative variance weight describes how significant the variance of the walk in a single cooling interval over the variance of* $X_n$. *The results (Theorem 1.9 and Corollary 1.10 in [19]) showed that for general cooling sequences there might be no limiting distribution for* $X_n/\sqrt{\mathbb{V}ar(X_n)}$, *but that one can identify a class of limiting distributions along subsequences which are mixtures of Kesten's distribution and standard Gaussian. See Examples 5 and 6 in [19] for more details.*

### 2.2.2 Functional weak limit under the slow and fast Cooling

In this section we will introduce our main results of the weak limit of $(\tilde{X}_k/\sqrt{\chi_n(\tau)},\ k = 1, 2, ..., n)$ where $\tilde{X}_k = X_k(\omega) - \mathbb{E}(X_k)$ is the centered walk[1] of $X_k$ under both polynomial and exponential cooling. Since the walk $(\tilde{X}_k,\ k = 1, 2, ..., n)$ is a discrete time random walk and we are considering the scaled (under both time and space parameters) weak limit of it, it is reasonable to make this discrete-time-walk a continuous random walk $X_t^n$ within the time interval $t \in [0, 1]$. The simplest way to solve this is to make the process piecewise linear. To this end, define

$$X_t^n(\omega) = \frac{1}{\sqrt{\chi_n(\tau)}} \tilde{X}_{\lfloor tn \rfloor}(\omega) + (tn - \lfloor tn \rfloor) \frac{1}{\sqrt{\chi_n(\tau)}} (\tilde{X}_{\lfloor tn+1 \rfloor}(\omega) - \tilde{X}_{\lfloor tn \rfloor}(\omega)). \tag{2.13}$$

[1] All the ~ signs in this thesis mean the centered random variable under the annealed measure.

Obviously $X_t^n$ is a random function in $C[0,1]$, the space of continuous functions on $[0,1]$, equipped with the uniform topology. The main results are stated as follows.

**Theorem 2.2.2. [Slow cooling: Functional weak limit for recurrent RWRE]** *Let $\alpha$ be as in Proposition 2.2.1. In regime (R1), $X_t^n$ given in (2.13). Under the annealed law $\mathbb{P}$,*

$$(X_t^n,\ t \in [0,1]) \ \Rightarrow_n\ (B_{t^{1/\beta}},\ t \in [0,1]), \qquad in\ regime(R1), \tag{2.14}$$

*where $(B_t, t \in [0,1])$ is a standard Brownian motion. The limit in the right hand side means a time-scaled Brownian motion. The convergence in law holds in the uniform topology on $C[0,1]$.*

In the exponential cooling case, the result is slightly different. The functional weak limit of $X_t^n$ is a random constant function and the law of the random constant is a standard Gaussian distribution.

**Theorem 2.2.3. [Fast cooling: Functional weak limit for recurrent RWRE]** *Let $\alpha$ be as in Proposition 2.2.1. In regime (R2), $X_t^n$ given in (2.13). Under the annealed law $\mathbb{P}$, for any $a \in (0,1]$,*

$$(X_t^n,\ t \in [a,1]) \ \Rightarrow_n\ (N_t,\ t \in [a,1]), \qquad in\ regime(R2), \tag{2.15}$$

*where $N_t = N$ for all $t \in [a,1]$ and $N \sim \mathcal{N}(0,1)$. The convergence in law holds in the uniform topology on $C[a,1]$.*

**Remark 2.2.4.** *In Theorem 2.2.3 the convergence holds in space $C[a,1]$ for any $a \in (0,1]$. In fact, if we want to extend the convergence to then entire time interval $[0,1]$ then neither continuous function space $C[0,1]$ nor the Càdlàg function space $D[0,1]$ (with the Skorohod topology) will be sufficient since the sequence is not tight in either space. Moreover, one can guess the limiting process on $[0,1]$ should be 0 when $t = 0$ and $N_t$ for $t \in (0,1]$, which is not a Càdlàg function. So if we want to extend the convergence to a function space on $[0,1]$ then a wider space would be required, e.g. $L^p[0,1]$, together with a corresponding topology where the weak convergence holds.*

**Remark 2.2.5.** *A heuristic thinking of the result in Theorem 2.2.3 is by the fact the exponential increment is faster than any polynomial increment. If we let $\beta$ go to infinity, then the weak limit in Theorem 2.2.2 will become 0 at time 0 and $B_1$ for $t \in (0,1]$. This also explains the guess in the above remark.*

## 2.3 Random walks in cooling random environment: transient RWRE

In this section we will turn to take a look at RWCRE where $\alpha$ is transient. In this case, the weak limit results have been shown in [19] where $\kappa > 2$ and in [20] where $\kappa \in (1,2)$. We will resume their work by discovering the weak limits for $\kappa \in (0,1)$ and $\kappa = 2$. Briefly speaking, under the case with $\kappa \in (0,1)$, a stronger $L_p$ norm convergence holds for RWRE. While with $\kappa = 2$, the $L_2$ convergence fails and we will discuss more about the tail estimates of RWRE. And then we turn to find the weak limit of the corresponding RWCRE under both slow and fast cooling.

### 2.3.1 Transient RWRE with $\kappa \in (0,1)$

Consider $Z_n$ as the random walk in random environment where the underlying environment has distribution $\mu = \alpha^{\mathbb{N}}$ such that $E_\alpha[\log \rho(0)] < 0$ and $\kappa \in (0,1)$ where $E_\alpha[\rho(0)^\kappa] = 1$. Classic RWRE shows that with some moment assumptions of $\alpha$, under the annealed law $\mathbb{P}$, $Z_n/n^\kappa$ converges weakly to a random variable whose Laplace transform is the Mittag-Leffler function (See [48] for more information about the Mittag-Leffler function). That is,

**Proposition 2.3.1.** **[[4], Weak limit of transient RWRE:** $\kappa \in (0,1)$**]** *Let $\alpha$ be transient with $\kappa \in (0,1)$. Assume further that the support of the distribution of $\log \rho(0)$ is non-lattice and suppose that $E[\rho(0)^\kappa \log \rho(0)] < \infty$. Then under the annealed measure $\mathbb{P}$, there exists a constant $b = b(\alpha) > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{Z_n}{n^\kappa} \le x\right) = [1 - L_{\kappa,b}(x^{-1/\kappa})]1_{x\ge 0}, x \in \mathbb{R} \tag{2.16}$$

*where $L_{\kappa,b}$ is the $\kappa$-stable distribution with scaling parameter $b$, centered at zero and totally skewed to the right. It's characteristic function is*

$$\hat{L}_{\kappa,b}(u) = \exp\left[-b|u|^\kappa \left(1 - \mathrm{i}\frac{u}{|u|}\tan(\frac{\kappa\boldsymbol{\pi}}{2})\right)\right]. \tag{2.17}$$

In order to find the weak limit of RWCRE along arbitrary cooling maps, based on the proof of Theorem 1.9 in [19], it is enough to prove the $L_p$ norm convergence of $Z_n/n^\kappa$. The theorem below shows this result.

**Theorem 2.3.1** **($L_p$ Norm Convergence $\kappa \in (0,1)$).** *Let $\alpha$ be as in Proposition 2.3.1. Assume further that there exists $\epsilon_0 > 0$ such that $E[\rho(0)^{-\epsilon_0}] < \infty$. Denote $\mathfrak{M}$ the weak limit of $Z_n/n^\kappa$. Under the annealed measure $\mathbb{P}$, the convergence holds in the sense of $L_p$ norm for $\forall p > 0$. i.e.*

$$\lim_{n\to\infty} \mathbb{E}\left|\frac{Z_n}{n^\kappa}\right|^p = E[\mathfrak{M}]^p. \tag{2.18}$$

*Moreover, when $p = 1$, the convergence also holds for the expectation of $Z_n/n^\kappa$. Thus $\mathbb{V}ar(Z_n/n^\kappa) \to Var(\mathfrak{M})$.*

**Remark 2.3.2.** *The extra assumption in Theorem 2.3.1 comes from [49].*

**Remark 2.3.3.** *Once the convergence of variance is obtained, one can use coupling to define identically distributed copies of centered $Z_n/n^\kappa$ and $\mathfrak{M}$ to make the convergence (of the centered copy) holds in $L_2$. Then apply Chaper 3.1 in [19] to get the same mixed fluctuation result.*

### 2.3.2 Transient RWRE with $\kappa = 2$

When $\kappa = 2$, it is known that the walk is ballistic with velocity $v$ and there exists $b > 0$ such that $(Z_n - nv)/(b\sqrt{n \log n})$ converges weakly to $\Phi$, a standard Gaussian. The main result in this section shows that the $L_2$ norm convergence holds but not to the variance of the limiting distribution, which means convergence in $L_2$ fails. For $p < 2$, $L_p$ norm convergence holds as well as its limit is also the $L_p$ norm of the standard Gaussian.

**Theorem 2.3.4** (**$L_p$ Norm Convergence** $\kappa = 2$)**.** *Let $\alpha$ be transient with $\kappa = 2$. Under the annealed measure $\mathbb{P}$, there exists $\sigma^2 > 0$ that only depends on $\alpha$, such that*

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{Z_n - vn}{\sqrt{n\log n}}\right]^2 = b^2 + \sigma^2. \tag{2.19}$$

*For $0 < p < 2$,*

$$\lim_{n\to\infty} \mathbb{E}\left|\frac{Z_n - vn}{\sqrt{n\log n}}\right|^p = b^p E|\Phi|^p. \tag{2.20}$$

*When $p = 1$, the expectation of $(Z_n - vn)/\sqrt{n\log n}$ converges to zero.*

### 2.3.3 Slow and fast cooling: Gaussian fluctuations for transient RWRE with $\kappa = 2$

Let $X_n$ be the random walk in cooling random environment and $\tau_k$ be the cooling sequence. Under slow (polynomial) and fast (exponential) cooling, we are able to generate the weak limit result of the walk along the cooling sequence. Let $T_k = \tau_k - \tau_{k-1}$ be the waiting time. In polynomial cooling regime, we assume $T_k \sim \beta k^{\beta-1}$ where $\beta > 1$. In such case, the parameter $\beta$ will affect the variance of the weak limit which is always a Gaussian.

**Theorem 2.3.5** (**Slow cooling: transient when** $\kappa = 2$)**.** *Let $\alpha$ be transient with $\kappa = 2$, and the cooling regime $T_k \sim \beta k^{\beta-1}$. Under the annealed measure $\mathbb{P}$,*

$$\frac{X_{\tau_n} - \mathbb{E}X_{\tau_n}}{\sigma_1\sqrt{n^\beta \log n}} \Rightarrow \Phi \tag{2.21}$$

*where*

$$\sigma_1^2 = \beta(\beta-1)\left[b^2 + \sigma^2\left(\frac{1}{\beta-1} \wedge 1\right)\right]. \tag{2.22}$$

In exponential cooling, $\log T_k \sim Ck$ for some constant $C > 0$. We further assume that $T_k \sim \mathrm{e}^{CK}$to make the scaling parameter below an explicit one. In this case, the theorem below shows that the weak limit of the walk along the cooling sequence converges to a standard normal distribution, and the scaling parameter is exactly $b\sqrt{\sum_{k=1}^n T_k \log T_k}$.

**Theorem 2.3.6** (**Fast cooling: transient when** $\kappa = 2$)**.** *Let $\alpha$ be transient with $\kappa = 2$, and the cooling regime $T_k \sim \mathrm{e}^{CK}$. Under the annealed measure $\mathbb{P}$,*

$$\frac{X_{\tau_n} - \mathbb{E}X_{\tau_n}}{b\sqrt{\sum_{k=1}^{n} T_k \log T_k}} \Rightarrow \Phi. \tag{2.23}$$

*Or equivanlently*

$$\frac{X_{\tau_n} - \mathbb{E}X_{\tau_n}}{\sigma_2\sqrt{Cn\mathrm{e}^{Cn}}} \Rightarrow \Phi \tag{2.24}$$

*where* $\sigma_2^2 = b^2 \frac{\mathrm{e}^C}{\mathrm{e}^C - 1}$.

## 2.4 Proofs

We begin by noting the following useful decomposition property of RWCRE. Let

$$k(n) = \max\{k \in \mathbb{N} : \tau(k) \le n\} \tag{2.25}$$

be the number of resamplings of the environment prior to time $n$. It's easy to see $k(n) \sim (n/B)^{1/\beta}$ in (R1) and $k(n) \sim (1/C)\log n$ in (R2). Furthermore, $X_n$ has a decomposition that will be very useful in the following proof of the theorems,

$$X_n = \sum_{\mathrm{j}=1}^{k(n)} Y_{\mathrm{j}} + \bar{Y}_n, \tag{2.26}$$

where $Y_{\mathrm{j}} = X_{\tau(\mathrm{j})} - X_{\tau(\mathrm{j}-1)}, \mathrm{j} = 1, 2, .., k(n)$, $\bar{Y}_n = X_n - X_{\tau(k(n))}$. A simple fact is that all terms in (2.26) are independent under the annealed measure. Moreover, under the annealed measure, $Y_{\mathrm{j}}$ has the same distribution as $Z_{T_{\mathrm{j}}}$ for $\mathrm{j} \ge 1$, and $\bar{Y}_n$ has the same distribution as $Z_{n-\tau(k(n))}$ for $n \ge 1$, where $\{Z_n\}_{n\ge 0}$ is a RWRE. Since we will deal with the remainder part $\bar{Y}_n$ throughout the proof, we will use the notation $\bar{T}_n = n - \tau(k(n))$ and $\bar{T}_n^c = \tau(k(n)+1) - n$.

### 2.4.1 Slow cooling: Functional weak limit for recurrent RWRE

*Proof of Theorem 2.2.2.* We start by finding the weak limit of the finite dimensional random vector $(X_{t_1}^n, X_{t_2}^n, ..., X_{t_k}^n)$. To start with, we will prove the weak convergence under the case

$k = 2$, i.e. the weak limit of $(X_t^n, X_s^n)$ for $0 \le t < s \le 1$. By [17], $\tilde{X}_{\lfloor tn \rfloor}/\sqrt{\chi_{\lfloor tn \rfloor}(\tau)} \Rightarrow_n \mathcal{N}(0,1)$. Obviously $\lim \frac{\chi_{\lfloor tn \rfloor}(\tau)}{\chi_n(\tau)} = t^{1/\beta}$, so $\tilde{X}_{\lfloor tn \rfloor}/\sqrt{\chi_n(\tau)} \Rightarrow_n \mathcal{N}(0, t^{1/\beta})$. If $\psi_{n,t}$ is the rightmost term in (2.13), then $\psi_{n,t} \Rightarrow_n 0$ by the fact that all the numerators are bounded but $\chi_n(\tau)$ goes to infinity. We have

$$(X_t^n, X_s^n - X_t^n) = \frac{1}{\sqrt{\chi_n(\tau)}}(\tilde{X}_{\lfloor tn \rfloor}(\omega),\ \tilde{X}_{\lfloor sn \rfloor}(\omega) - \tilde{X}_{\lfloor tn \rfloor}(\omega)) + (\psi_{n,t},\ \psi_{n,s} - \psi_{n,t}). \tag{2.27}$$

To find the weak limit of $(\tilde{X}_{\lfloor sn \rfloor} - \tilde{X}_{\lfloor tn \rfloor})/\sqrt{\chi_n(\tau)}$, we will follow the approach of [17] in using the following Lyapunov condition.

**Lemma 2.4.1. (Lyapunov condition, Petrov [50])**
*Let $U = (U_k)_{k \in \mathbb{N}}$ be a sequence of independent random variables (at least one of which has a non-degenerate distribution). Let $m_k = E(U_k)$ and $\sigma_k^2 = Var(U_k)$. Define*

$$\chi_n = \sum_{k=1}^{n} \sigma_k^2. \tag{2.28}$$

*Then the Lyapunov condition*

$$\lim_{n \to \infty} \frac{1}{\chi_n^{p/2}} \sum_{k=1}^{n} E(|U_k - m_k|^p) = 0, \tag{2.29}$$

*for some $p > 2$ implies that*

$$\frac{1}{\chi_n} \sum_{k=1}^{n} (U_k - m_k) \Rightarrow_n \mathcal{N}(0,1). \tag{2.30}$$

.

Recall $X_n$ has the decomposition

$$X_n = \sum_{\mathrm{j}=1}^{k(n)} Y_{\mathrm{j}} + \bar{Y}_n. \tag{2.31}$$

Define the variance of $X_{\lfloor sn \rfloor} - X_{\lfloor tn \rfloor}$ (which is also the variance of $\tilde{X}_{\lfloor sn \rfloor} - \tilde{X}_{\lfloor tn \rfloor}$) for any $s < t$ and $n$ large enough

$$\chi_n^{t,s}(\tau) = \sum_{\mathrm{j}=k(\lfloor tn \rfloor)+2}^{k(\lfloor sn \rfloor)} \mathbb{V}ar(Y_{\mathrm{j}}) + \mathbb{V}ar(\bar{Y}_{\lfloor sn \rfloor}) + \mathbb{V}ar(\bar{Y}^c_{\lfloor tn \rfloor}) \tag{2.32}$$

where $\bar{Y}_n^c = X_{\tau(k(n)+1)} - X_n$. Recall that $\tilde{Y}_{\mathrm{j}} = Y_{\mathrm{j}} - \mathbb{E}(Y_{\mathrm{j}})$, $\tilde{\bar{Y}}_n = \bar{Y}_n - \mathbb{E}(\bar{Y}_n)$, and $\tilde{\bar{Y}}_n^c = \bar{Y}_n^c - \mathbb{E}(\bar{Y}_n^c)$. For $p > 2$, let

$$\chi_n^{t,s}(\tau; p) = \sum_{\mathrm{j}=k(\lfloor tn \rfloor)+2}^{k(\lfloor sn \rfloor)} \mathbb{E}\left(|\tilde{Y}_{\mathrm{j}}|^p\right) + \mathbb{E}\left(|\tilde{\bar{Y}}_{\lfloor sn \rfloor}|^p\right) + \mathbb{E}\left(|\tilde{\bar{Y}}^c_{\lfloor tn \rfloor}|^p\right). \tag{2.33}$$

Since $Y_{\mathrm{j}}$ has the same distribution as $Z_{T_{\mathrm{j}}}$, then by Proposition 4 in [17] the following two asymptotic estimates hold as $\mathrm{j} \to \infty$.

$$\mathbb{V}ar(Y_{\mathrm{j}}) \sim (\sigma_\mu^2 \sigma_V)^2 \log^4 T_{\mathrm{j}}, \qquad \mathbb{E}\left(|\tilde{Y}_{\mathrm{j}}|^p\right) = O(\log^{2p} T_{\mathrm{j}}), \quad p > 2. \tag{2.34}$$

Applying these to (2.32) and (2.33) we obtain

$$\begin{aligned} \sum_{\mathrm{j}=k(\lfloor tn \rfloor)+2}^{k(\lfloor sn \rfloor)} \mathbb{V}ar(Y_{\mathrm{j}}) &\sim (\sigma_\mu^2 \sigma_V)^2 \sum_{\mathrm{j}=k(\lfloor tn \rfloor)+2}^{k(\lfloor sn \rfloor)} \log^4 T_{\mathrm{j}}, \\ \sum_{\mathrm{j}=k(\lfloor tn \rfloor)+2}^{k(\lfloor sn \rfloor)} \mathbb{E}\left(|\tilde{Y}_{\mathrm{j}}|^p\right) &= O\left( \sum_{\mathrm{j}=k(\lfloor tn \rfloor)+2}^{k(\lfloor sn \rfloor)} \log^{2p} T_{\mathrm{j}} \right). \end{aligned} \tag{2.35}$$

Moreover, using that $\sum_{\mathrm{j}=1}^{k} \log^{2p} \mathrm{j} \sim \int_1^k \log^{2p} x dx \sim k \log^{2p} k$ for all $p \geq 2$ and that $k(n) \sim (n/B)^{1/\beta}$, we have

$$\begin{aligned}
(\sigma_\mu^2 \sigma_V)^2 \sum_{\mathrm{j}=k(\lfloor tn \rfloor)+2}^{k(\lfloor sn \rfloor)} \log^4 T_{\mathrm{j}} &\sim (\sigma_\mu^2 \sigma_V)^2 (\beta-1)^4 \left[ (\frac{sn}{B})^{1/\beta} \log^4 \left( (\frac{sn}{B})^{1/\beta} \right) - (\frac{tn}{B})^{1/\beta} \log^4 \left( (\frac{tn}{B})^{1/\beta} \right) \right] \\
&\sim (\sigma_\mu^2 \sigma_V)^2 (\beta-1)^4 (\frac{n}{B})^{1/\beta} (\frac{1}{\beta})^4 (s^{1/\beta} - t^{1/\beta}) \log^4 n \\
&= \chi_n(\tau) \left( s^{\frac{1}{\beta}} - t^{\frac{1}{\beta}} \right),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\mathrm{j}=k(\lfloor tn \rfloor)+2}^{k(\lfloor sn \rfloor)} \log^{2p} T_{\mathrm{j}} &\sim (\sigma_\mu^2 \sigma_V)^2 (\beta-1)^{2p} (\frac{n}{B})^{1/\beta} (\frac{1}{\beta})^{2p} (s^{1/\beta} - t^{1/\beta}) \log^{2p} n \\
&= \chi_n^{\frac{p}{2}}(\tau) \left( s^{\frac{1}{\beta}} - t^{\frac{1}{\beta}} \right) \left( \frac{n}{B} \right)^{\frac{2-p}{\beta p}}, \qquad p > 2.
\end{aligned} \tag{2.36}$$

Since $\bar{Y}_n$ has the same distribution as $Z_{\bar{T}_n}$, we can again use Proposition 4 in [17] to obtain that there exists $C^{(2)} > 0, \quad C^{(p)} > 0$, such that

$$\mathbb{V}ar \left( \bar{Y}_n \right) \leq C^{(2)} \log^4 \bar{T}_n, \qquad \mathbb{E} \left( |\tilde{\bar{Y}}_n|^p \right) \leq C^{(p)} \log^{2p} \bar{T}_n. \tag{2.37}$$

These upper bounds will be used to control $\mathbb{V}ar(\bar{Y}_n^c)$ and $\mathbb{E}\left( |\bar{Y}_n^c - \mathbb{E}(\bar{Y}_n^c)|^p \right)$. For $n$ large enough,

$$\begin{aligned}
&\mathbb{V}ar(\bar{Y}_n^c) = \mathbb{V}ar(Y_{k(n)+1} - \bar{Y}_n) \leq 2\mathbb{V}ar(Y_{k(n)+1}) + 2\mathbb{V}ar(\bar{Y}_n) \leq 4 \left[ (\sigma_\mu^2 \sigma_V)^2 + C^{(2)} \right] \log^4 T_{k(n)+1}, \\
&\mathbb{E}\left( |\tilde{\bar{Y}}_n^c|^p \right) = \mathbb{E}\left( |\tilde{Y}_{k(n)+1} - \tilde{\bar{Y}}_n|^p \right) \leq 2^{p-1} \left[ \mathbb{E}\left( |\tilde{Y}_{k(n)+1}|^p \right) + \mathbb{E}\left( |\tilde{\bar{Y}}_n|^p \right) \right] = O(\log^{2p} T_{k(n)+1}).
\end{aligned} \tag{2.38}$$

From (2.37) and (2.38),

$$\begin{aligned}
&\mathbb{V}ar(\bar{Y}_{\lfloor sn \rfloor}) + \mathbb{V}ar(\bar{Y}_{\lfloor tn \rfloor}^c) \leq C^{(2)} \log^4 \bar{T}_{\lfloor sn \rfloor} + 4 \left[ (\sigma_\mu^2 \sigma_V)^2 + C^{(2)} \right] \log^4 T_{k(\lfloor tn \rfloor)+1} = O(\log^4 n), \\
&\mathbb{E}\left( |\tilde{\bar{Y}}_{\lfloor sn \rfloor}|^p \right) + \mathbb{E}\left( |\tilde{\bar{Y}}_{\lfloor tn \rfloor}^c|^p \right) \leq C^{(p)} \log^{2p} \bar{T}_{\lfloor sn \rfloor} + O(\log^{2p} T_{k(\lfloor tn \rfloor)+1}) = O(\log^{2p} n).
\end{aligned} \tag{2.39}$$

By (2.36) and (2.39), we can therefore give the asymptotic of $\chi_n^{t,s}(\tau)$ and $\chi_n^{t,s}(\tau;p)$,

$$\begin{aligned} \chi_n^{t,s}(\tau) &\sim \chi_n(\tau)\left(s^{\frac{1}{\beta}} - t^{\frac{1}{\beta}}\right), \\ \chi_n^{t,s}(\tau;p) &= O\left(\chi_n^{\frac{p}{2}}(\tau)\left(s^{\frac{1}{\beta}} - t^{\frac{1}{\beta}}\right)\left(\frac{n}{B}\right)^{\frac{2-p}{\beta p}}\right), \qquad p>2. \end{aligned} \tag{2.40}$$

From these asymptotics it is easy to check that the Lyapunov condition holds, and thus

$$\frac{\tilde{X}_{\lfloor sn\rfloor} - \tilde{X}_{\lfloor tn\rfloor}}{\sqrt{\chi_n(\tau)}} \Rightarrow_n \mathcal{N}(0, s^{1/\beta} - t^{1/\beta}). \tag{2.41}$$

In order to prove the vector $(X_t^n, X_s^n - X_t^n)$ converges to a 2-d Gaussian vector with independent components, it suffices to show that any linear combination of $X_t^n$ and $X_s^n - X_t^n$ converges to the corresponding linear combination of the components of the limiting Gaussian vector. To this end, the proof is quite similar to what we did above: Decompose $\lambda X_{\lfloor tn\rfloor} + \mu(X_{\lfloor sn\rfloor} - X_{\lfloor tn\rfloor})$ into independent sums and check the Lyapunov condition (2.29). Notice that

$$\begin{aligned} \lambda X_{\lfloor tn\rfloor} + \mu(X_{\lfloor sn\rfloor} - X_{\lfloor tn\rfloor}) &= \lambda \sum_{\mathrm{j}=1}^{k(\lfloor tn\rfloor)} \left(X_{\tau(\mathrm{j})} - X_{\tau(\mathrm{j}-1)}\right) + \lambda\left(X_{\lfloor tn\rfloor} - X_{k(\lfloor tn\rfloor)}\right) \\ &\quad + \mu\left(X_{k(\lfloor tn\rfloor)+1} - X_{\lfloor tn\rfloor}\right) + \mu \sum_{\mathrm{j}=k(\lfloor tn\rfloor)+2}^{k(\lfloor sn\rfloor)} \left(X_{\tau(\mathrm{j})} - X_{\tau(\mathrm{j}-1)}\right) + \mu\left(X_{\lfloor sn\rfloor} - X_{k(\lfloor sn\rfloor)}\right) \\ &= \lambda \sum_{\mathrm{j}=1}^{k(\lfloor tn\rfloor)} Y_{\mathrm{j}} + \left(\lambda \bar{Y}_{\lfloor tn\rfloor} + \mu \bar{Y}^c_{\lfloor tn\rfloor}\right) + \mu \sum_{\mathrm{j}=k(\lfloor tn\rfloor)+2}^{k(\lfloor sn\rfloor)} Y_{\mathrm{j}} + \mu \bar{Y}_{\lfloor sn\rfloor}. \end{aligned} \tag{2.42}$$

The key point to the proof is the expressions of the variance of $\lambda X_{\lfloor tn\rfloor} + \mu(X_{\lfloor sn\rfloor} - X_{\lfloor tn\rfloor})$

$$\begin{aligned} \mathbb{V}ar\left(\lambda X_{\lfloor tn\rfloor} + \mu(X_{\lfloor sn\rfloor} - X_{\lfloor tn\rfloor})\right) = \lambda^2 \sum_{\mathrm{j}=1}^{k(\lfloor tn\rfloor)} \mathbb{V}ar(Y_{\mathrm{j}}) + \mu^2 \sum_{\mathrm{j}=k(\lfloor tn\rfloor)+2}^{k(\lfloor sn\rfloor)} \mathbb{V}ar(Y_{\mathrm{j}}) \\ + \mu^2 \mathbb{V}ar(\bar{Y}_{\lfloor sn\rfloor}) + \mathbb{V}ar\left(\lambda \bar{Y}_{\lfloor tn\rfloor} + \mu \bar{Y}^c_{\lfloor tn\rfloor}\right), \end{aligned} \tag{2.43}$$

and the sum of centered $p$-th moments of the independent components in the above decomposition,

$$\lambda^p \sum_{\mathrm{j}=1}^{k(\lfloor tn \rfloor)} \mathbb{E}\left(|\tilde{Y}_{\mathrm{j}}|^p\right) + \mu^p \sum_{\mathrm{j}=k(\lfloor tn \rfloor)+2}^{k(\lfloor sn \rfloor)} \mathbb{E}\left(|\tilde{Y}_{\mathrm{j}}|^p\right) + \mu^p \mathbb{E}\left(|\tilde{\tilde{Y}}_{\lfloor sn \rfloor}|^p\right) + \mathbb{E}\left(|\lambda \tilde{\tilde{Y}}_{\lfloor tn \rfloor} + \mu \tilde{\tilde{Y}}^c_{\lfloor tn \rfloor}|^p\right). \quad (2.44)$$

The last term in each expression above cannot be separated into two parts because those two random variables are not independent under the annealed measure. But still, we can estimate the last term by the fact that $Var(X+Y) \leq 2(Var(X) + Var(Y))$ (and similarly, $E(|X+Y|^p) \leq 2^{p-1}(E|X|^p + E|Y|^p)$ for the $p$-th moment) for any two random variables $X$ and $Y$. Thus, with the same approach, the last two terms in (2.43) and (2.44) will be dominated by the first two sums. Moreover, the asymptotics of the first two sums in (2.43) and (2.44) can be obtained using the same methods as in the first part of the proof above.

The result is for any $\lambda > 0$, $\mu > 0$, $\lambda X^n_t + \mu(X^n_s - X^n_t)$ converges weakly to $\mathcal{N}(0, \lambda^2 t^{1/\beta} + \mu^2(s^{1/\beta} - t^{1/\beta}))$. This also reveals the independence of the coordinates of the limit random vector, i.e.

$$(X^n_t, X^n_s - X^n_t) \Rightarrow_n (N_1, N_2), \quad (2.45)$$

where $(N_1, N_2)$ is a Gaussian vector with mean $(0,0)$ and variance $(t^{1/\beta}, s^{1/\beta} - t^{1/\beta})$, also $N_1$ and $N_2$ are independent.

It is natural to extend the weak convergence of 2-dimension vector into finite dimension vector $(X^n_{t_1}, X^n_{t_2}, ..., X^n_{t_k})$, $0 \leq t_1 < t_2 < ... < t_k \leq 1$ i.e.

$$(X^n_{t_1}, X^n_{t_2}, ..., X^n_{t_k}) \Rightarrow_n (B_{t_1^{1/\beta}}, B_{t_2^{1/\beta}}, ..., B_{t_k^{1/\beta}}), \quad (2.46)$$

where $(B_t, t \in [0,1])$ is a standard Brownian motion. The proof of this statement follows the same steps as what we did in dimension 2: Decompose $\sum_{\mathrm{i}=1}^k \lambda_{\mathrm{i}}(X_{\lfloor t_{\mathrm{i}} n \rfloor} - X_{\lfloor t_{\mathrm{i}-1} n \rfloor})$ into independent sums where $t_0 = 0$. Then take the variance and the the sum of centered

$p-$th moment of the independent components of the decomposition to check the Lyapunov condition (2.29). The decomposition is

$$\begin{aligned}\sum_{\mathrm{i}=1}^{k}\lambda_{\mathrm{i}}(X_{\lfloor t_{\mathrm{i}}n\rfloor}-X_{\lfloor t_{\mathrm{i}-1}n\rfloor})=\lambda_1\sum_{\mathrm{j}=1}^{k(\lfloor t_1 n\rfloor)}Y_{\mathrm{j}}+\sum_{\mathrm{i}=2}^{k}\lambda_{\mathrm{i}}\sum_{\mathrm{j}=k(\lfloor t_{\mathrm{i}-1}n+2\rfloor)}^{k(\lfloor t_{\mathrm{i}}n\rfloor)}Y_{\mathrm{j}}\\+\left[\sum_{\mathrm{i}=1}^{k-1}(\lambda_{\mathrm{i}}\bar{Y}_{\lfloor t_{\mathrm{i}}n\rfloor}+\lambda_{\mathrm{i}+1}\bar{Y}^{c}_{\lfloor t_{\mathrm{i}}n\rfloor})+\lambda_k\bar{Y}_{\lfloor t_k n\rfloor}\right].\end{aligned}\tag{2.47}$$

So the variance and the sum of centered $p-$th moment of the independent components above are

$$\begin{aligned}\lambda_1^2\sum_{\mathrm{j}=1}^{k(\lfloor t_1 n\rfloor)}\mathbb{V}ar(Y_{\mathrm{j}})+\sum_{\mathrm{i}=2}^{k}\lambda_{\mathrm{i}}^2\sum_{\mathrm{j}=k(\lfloor t_{\mathrm{i}-1}n+2\rfloor)}^{k(\lfloor t_{\mathrm{i}}n\rfloor)}\mathbb{V}ar(Y_{\mathrm{j}})\\+\left[\sum_{\mathrm{i}=1}^{k-1}\mathbb{V}ar\left(\lambda_{\mathrm{i}}\bar{Y}_{\lfloor t_{\mathrm{i}}n\rfloor}+\lambda_{\mathrm{i}+1}\bar{Y}^{c}_{\lfloor t_{\mathrm{i}}n\rfloor}\right)+\lambda_k^2\mathbb{V}ar(\bar{Y}_{\lfloor t_k n\rfloor})\right]\end{aligned}\tag{2.48}$$

and

$$\begin{aligned}\lambda_1^p\sum_{\mathrm{j}=1}^{k(\lfloor t_1 n\rfloor)}\mathbb{E}(|\tilde{Y}_{\mathrm{j}}|^p)+\sum_{\mathrm{i}=2}^{k}\lambda_{\mathrm{i}}^p\sum_{\mathrm{j}=k(\lfloor t_{\mathrm{i}-1}n+2\rfloor)}^{k(\lfloor t_{\mathrm{i}}n\rfloor)}\mathbb{E}(|\tilde{Y}_{\mathrm{j}}|^p)\\+\left[\sum_{\mathrm{i}=1}^{k-1}\mathbb{E}\left(|\lambda_{\mathrm{i}}\tilde{\bar{Y}}_{\lfloor t_{\mathrm{i}}n\rfloor}+\lambda_{\mathrm{i}+1}\tilde{\bar{Y}}^{c}_{\lfloor t_{\mathrm{i}}n\rfloor}|^p\right)+\lambda_k^p\mathbb{E}(|\tilde{\bar{Y}}_{\lfloor t_k n\rfloor}|^p)\right].\end{aligned}\tag{2.49}$$

All the terms in the big brackets are dominated by sums to the left of the brackets. To check the Lyapunov condition holds in this case is nothing new but repeat our works (2.35) and (2.36). The details are tedious and we omit them here.

To complete the proof of the theorem under the slow cooling case, the tightness of the sequence $X^n$ is needed. To this end, by Theorems 7.3 and 7.4 in [51] it is enough to show that for any $\epsilon > 0$, $\eta > 0$, $\exists \delta > 0$ and a sequence of numbers $\{t_{\mathrm{i}}\}$, where $0 = t_0 < t_1 < ... < t_v = 1$, s.t.

$$\min_{1<\mathrm{i}<v}(t_{\mathrm{i}}-t_{\mathrm{i}-1})\geq\delta,\tag{2.50}$$

and $\exists n_0 > 0$, for all $n > n_0$,

$$\sum_{\mathrm{i}=1}^{v} \mathbb{P}[\sup_{t_{\mathrm{i}-1} \le s \le t_{\mathrm{i}}} |X_s^n - X_{t_{\mathrm{i}-1}}^n| \ge \epsilon] < \eta. \tag{2.51}$$

Since $(X_t^n,\ t \in [0,1])$ is the continuous process of $(\tilde{X}_{\lfloor tn \rfloor}/\sqrt{\chi_n(\tau)},\ t \in [0,1])$, the biggest difference in the continuous process within a given interval is, up to an error smaller than $2/\sqrt{\chi_n(\tau)}$, bounded by the biggest difference in the discrete time process. Hence we can check the condition (2.51) by replacing $X_s^n, s \in [t_{\mathrm{i}-1}, t_{\mathrm{i}}]$ and $X_{t_{\mathrm{i}-1}}^n$ by $\tilde{X}_m/\sqrt{\chi_n(\tau)}$, $m \in [t_{\mathrm{i}-1}n, t_{\mathrm{i}}n]$ and $\tilde{X}_{\lfloor t_{\mathrm{i}-1}n \rfloor}/\sqrt{\chi_n(\tau)}$ separately.

Let $m$ be $|\tilde{X}_m - \tilde{X}_{\lfloor t_{\mathrm{i}-1}n \rfloor}| = \sup_{s \in [t_{\mathrm{i}-1}n, t_{\mathrm{i}}n]} |\tilde{X}_s - \tilde{X}_{\lfloor t_{\mathrm{i}-1}n \rfloor}|$, i.e. the exact value of $s$ to make the biggest difference happens. If there are more than one candidates, choose one arbitrarily. We have the following decomposition,

$$\tilde{X}_m - \tilde{X}_{\lfloor t_{\mathrm{i}-1}n \rfloor} = \sum_{\mathrm{j}=\tau(k(\lfloor t_{\mathrm{i}-1}n \rfloor)+1)}^{\tau(k(m))} \tilde{Y}_{\mathrm{j}} + \tilde{\tilde{Y}}_m - \tilde{\tilde{Y}}_{\lfloor t_{\mathrm{i}-1}n \rfloor}, \tag{2.52}$$

or just $\tilde{\tilde{Y}}_m - \tilde{\tilde{Y}}_{\lfloor t_{\mathrm{i}-1}n \rfloor}$ if $k(\lfloor t_{\mathrm{i}-1}n \rfloor) = k(m)$.

Let's deal with the decomposition above in two parts:

- Given $q = \lfloor \beta \rfloor + 1 > 1$, define the martingale $\{M_l\}$ as $M_0 = 0$,

$$M_l = \sum_{\mathrm{j}=\tau(k(\lfloor t_{\mathrm{i}-1}n \rfloor)+1)}^{\tau(k(\lfloor t_{\mathrm{i}-1}n \rfloor)+l)} \tilde{Y}_{\mathrm{j}}, \qquad l \ge 1. \tag{2.53}$$

  Since the function $x^{2q}$ is convex, $\{M_l^{2q}\}$ is a submartingale. By Doob's Maximal Inequality [52], for integer $L > 0$,

$$\mathbb{P}\left(\sup_{l \in [0,L]} \frac{|M_l|}{\sqrt{\chi_n(\tau)}} \ge \frac{\epsilon}{2}\right) \le \frac{\mathbb{E}[M_L^{2q}]}{(\frac{\epsilon}{2})^{2q} \chi_n^q(\tau)}. \tag{2.54}$$

  To estimate the order of $\mathbb{E}[M_L^{2q}]$, notice that if we expand all the terms in $M_L^{2q}$, it is a sum that several terms in it have zero mean. So by counting the number of non-zero terms in $\mathbb{E}[M_L^{2q}]$ will give us the order of it. In fact, any term that has non-zero mean cannot have a factor $\tilde{Y}_{\mathrm{j}}$ of order only one, i.e. either it is not divided by $\tilde{Y}_{\mathrm{j}}$ or it is divided by $\tilde{Y}_{\mathrm{j}}^2$. Thus, a rough upper bound of the number of the non-zero terms in $\mathbb{E}[M_L^{2q}]$ is $\sum_{\mathrm{i}=1}^{q}\binom{L}{\mathrm{i}}\mathrm{i}^{2q}$. Since $q$ is fixed, for $L$ large enough, $\sum_{\mathrm{i}=1}^{q}\binom{L}{\mathrm{i}}\mathrm{i}^{2q} \leq q\binom{L}{q}q^{2q}$.

For any nonzero term in the expansion of $\mathbb{E}[M_L^{2q}]$, by (2.34), it is bounded from above by $C_0 \log^{4q} n$ for some $C_0 > 0$ since we are dealing with the case within the interval $[0, n]$. So

$$\mathbb{E}[M_L^{2q}] \leq C_0 q\binom{L}{q}q^{2q}\log^{4q} n \leq C_0 q^{2q+1}L^q \log^{4q} n. \tag{2.55}$$

Now back to the first part in (2.52),

$$\mathbb{P}\left(\frac{|\sum_{\mathrm{j}=\tau(k(\lfloor t_{\mathrm{i}-1}n\rfloor)+1)}^{\tau(k(m))}\tilde{Y}_{\mathrm{j}}|}{\sqrt{\chi_n(\tau)}} \geq \frac{\epsilon}{2}\right) \leq \mathbb{P}\left(\sup_{l\in[0,k(\lfloor t_{\mathrm{i}}n\rfloor)-k(\lfloor t_{\mathrm{i}-1}n\rfloor)]}\frac{|M_l|}{\sqrt{\chi_n(\tau)}} \geq \frac{\epsilon}{2}\right). \tag{2.56}$$

Combining with (2.54) and (2.55) and recalling $(k(\lfloor t_{\mathrm{i}}n\rfloor)-k(\lfloor t_{\mathrm{i}-1}n\rfloor)) \sim (n/B)^{1/\beta}(t_{\mathrm{i}}^{1/\beta} - t_{\mathrm{i}-1}^{1/\beta})$, we obtain that there exists $C^* > 0$, depending only on $\epsilon$, such that

$$\mathbb{P}\left(\frac{|\sum_{\mathrm{j}=\tau(k(\lfloor t_{\mathrm{i}-1}n\rfloor)+1)}^{\tau(k(m))}\tilde{Y}_{\mathrm{j}}|}{\sqrt{\chi_n(\tau)}} \geq \frac{\epsilon}{2}\right) \leq C^*(t_{\mathrm{i}}^{\frac{1}{\beta}} - t_{\mathrm{i}-1}^{\frac{1}{\beta}})^q. \tag{2.57}$$

- To deal with $\tilde{\tilde{Y}}_m$, notice that $|\tilde{\tilde{Y}}_m|$ is bounded by the maximum of $|\bar{Y}_n - \mathbb{E}(\bar{Y}_n)|$ where $n \in [\tau(k(m)), \tau(k(m)+1)]$. Define $\tilde{Y}_{\mathrm{j}}^* = \max_{n\in[\tau(\mathrm{j}-1),\tau(\mathrm{j})]}|\bar{Y}_n - \mathbb{E}(\bar{Y}_n)|$, then $|\tilde{\tilde{Y}}_m| \leq \tilde{Y}_{k(m)+1}^*$, where $k(m)$ can be from $k(\lfloor t_{\mathrm{i}-1}n\rfloor)$ to $k(\lfloor t_{\mathrm{i}}n\rfloor)$. Hence,

$$\begin{aligned}\mathbb{P}\left(\frac{|\tilde{\tilde{Y}}_m|}{\sqrt{\chi_n(\tau)}} \geq \frac{\epsilon}{4}\right) \leq& \mathbb{P}\left(\sup_{\mathrm{j}\in[k(\lfloor t_{\mathrm{i}-1}n\rfloor)+1,k(\lfloor t_{\mathrm{i}}n\rfloor)+1]}\frac{\tilde{Y}_{\mathrm{j}}^*}{\sqrt{\chi_n(\tau)}} \geq \frac{\epsilon}{4}\right)\\ \leq& \sum_{\mathrm{j}=k(\lfloor t_{\mathrm{i}-1}n\rfloor)+1}^{k(\lfloor t_{\mathrm{i}}n\rfloor)+1}\mathbb{P}\left(\frac{\tilde{Y}_{\mathrm{j}}^*}{\sqrt{\chi_n(\tau)}} \geq \frac{\epsilon}{4}\right).\end{aligned} \tag{2.58}$$

Let $Y_{\mathrm{j}}^* = \max_{n\in[\tau(\mathrm{j}-1),\tau(\mathrm{j})]}|\bar{Y}_n|$, we have

$$\tilde{Y}_{\mathrm{j}}^* = \max_{n\in[\tau(\mathrm{j}-1),\tau(\mathrm{j})]}|\bar{Y}_n - \mathbb{E}(\bar{Y}_n)| \leq \max_{n\in[\tau(\mathrm{j}-1),\tau(\mathrm{j})]}|\bar{Y}_n| + \max_{n\in[\tau(\mathrm{j}-1),\tau(\mathrm{j})]}\mathbb{E}|\bar{Y}_n| \leq Y_{\mathrm{j}}^* + \mathbb{E}Y_{\mathrm{j}}^*. \tag{2.59}$$

Moreover, by the same proof of the Proposition 4 in [17] (both $Z_n > a$ and $Z_n^* > a$ mean $T(a) < n$), for all $p > 0$,

$$\sup_{1 \leq \mathrm{j} \leq k(n)+1} \mathbb{E}\left(\frac{Y_{\mathrm{j}}^*}{\log^2 n}\right)^p \leq \sup_{1 \leq \mathrm{j} \leq k(n)+1} \mathbb{E}\left(\frac{Y_{\mathrm{j}}^*}{\log^2 T_{\mathrm{j}}}\right)^p < \infty. \tag{2.60}$$

From (2.59), Chebyshev's Inequality, and (2.60), there exists $C' > 0$ depending only on $\epsilon$ such that

$$\mathbb{P}\left(\frac{\tilde{Y}_{\mathrm{j}}^*}{\sqrt{\chi_n(\tau)}} \geq \frac{\epsilon}{4}\right) \leq \mathbb{P}\left(\frac{Y_{\mathrm{j}}^* + \mathbb{E}Y_{\mathrm{j}}^*}{\sqrt{\chi_n(\tau)}} \geq \frac{\epsilon}{4}\right) \leq \frac{\mathbb{E}\left(Y_{\mathrm{j}}^* + \mathbb{E}Y_{\mathrm{j}}^*\right)^4}{\left(\frac{\epsilon}{4}\right)^4 \chi_n^2(\tau)} \leq C' n^{-\frac{2}{\beta}}. \tag{2.61}$$

Now the upper bound of (2.58) is clear,

$$\mathbb{P}\left(\frac{|\tilde{\tilde{Y}}_m|}{\sqrt{\chi_n(\tau)}} \geq \frac{\epsilon}{4}\right) \leq \sum_{\mathrm{j}=k(\lfloor t_{\mathrm{i}-1} n \rfloor)+1}^{k(\lfloor t_{\mathrm{i}} n \rfloor)+1} C' n^{-\frac{2}{\beta}} = C'\left(k(\lfloor t_{\mathrm{i}} n \rfloor) - k(\lfloor t_{\mathrm{i}-1} n \rfloor)\right) n^{-\frac{2}{\beta}}. \tag{2.62}$$

The right hand side goes to zero as n goes to infinity since $k(n) \sim (n/B)^{1/\beta}$.

Back to the tightness condition (2.51), for any given $\epsilon > 0, \eta > 0$, let $\delta = 1/K$, and $t_{\mathrm{i}} = \mathrm{i}/K,\ \mathrm{i} = 0, 1, ..., K$, the positive integer $K$ to be determined. By (2.37), (2.52), (2.57), and (2.62), there exists $c > 0$ such that

$$\begin{aligned}
&\sum_{\mathrm{i}=1}^{K} \mathbb{P}\left(\frac{\sup_{s \in [t_{\mathrm{i}-1} n, t_{\mathrm{i}} n]} |\tilde{X}_s - \tilde{X}_{\lfloor t_{\mathrm{i}-1} n \rfloor}|}{\sqrt{\chi_n(\tau)}} \geq \epsilon\right) = \sum_{\mathrm{i}=1}^{K} \mathbb{P}\left(\frac{|\tilde{X}_m - \tilde{X}_{\lfloor t_{\mathrm{i}-1} n \rfloor}|}{\sqrt{\chi_n(\tau)}} \geq \epsilon\right) \\
\leq &\sum_{\mathrm{i}=1}^{K}\left[\mathbb{P}\left(\frac{|\sum_{\mathrm{j}=\tau(k(\lfloor t_{\mathrm{i}-1} n \rfloor)+1)}^{\tau(k(m))} \tilde{Y}_{\mathrm{j}}|}{\sqrt{\chi_n(\tau)}} \geq \frac{\epsilon}{2}\right) + \mathbb{P}\left(\frac{|\tilde{\tilde{Y}}_m|}{\sqrt{\chi_n(\tau)}} \geq \frac{\epsilon}{4}\right) + \mathbb{P}\left(\frac{|\tilde{\tilde{Y}}_{\lfloor t_{\mathrm{i}-1} n \rfloor}|}{\sqrt{\chi_n(\tau)}} \geq \frac{\epsilon}{4}\right)\right] \\
\leq &\sum_{\mathrm{i}=1}^{K}\left[C^*(t_{\mathrm{i}}^{\frac{1}{\beta}} - t_{\mathrm{i}-1}^{\frac{1}{\beta}})^q + C'\left(k(\lfloor t_{\mathrm{i}} n \rfloor) - k(\lfloor t_{\mathrm{i}-1} n \rfloor)\right) n^{-\frac{2}{\beta}} + \frac{16 \mathbb{V}ar(\bar{Y}_{\lfloor t_{\mathrm{i}-1} n \rfloor})}{\epsilon^2 \chi_n(\tau)}\right] \\
\leq &C^* K \sup_{1 \leq \mathrm{i} \leq K}\left[\left(\frac{\mathrm{i}}{K}\right)^{\frac{1}{\beta}} - \left(\frac{\mathrm{i}-1}{K}\right)^{\frac{1}{\beta}}\right]^q + cKn^{-\frac{1}{\beta}} \\
= &C^* K^{1-\frac{q}{\beta}} + cKn^{-\frac{1}{\beta}}.
\end{aligned} \tag{2.63}$$

Since $q > \beta$, by first choosing $K$ large and then choosing $n$ large enough the above bound is less than $\eta$. Hence the tightness condition holds, and (2.14) is proved. $\square$

### 2.4.2 Fast cooling: Functional weak limit for recurrent RWRE

*Proof of Theorem 2.2.3.* We do the proof in the same way as above. First we identify the limits of the finite dimensional distributions and then we prove tightness of the process $(X_t^n,\ t \in [a,1])$. In this case, however, the computation of the limiting finite dimensional distributions is slightly easier. We show that in this case the variance $\chi_n^{t,s}(\tau)$ is of smaller order than $\chi_n(\tau)$ and thus, $(\tilde{X}_{\lfloor sn \rfloor} - \tilde{X}_{\lfloor tn \rfloor})/\sqrt{\chi_n(\tau)} \Rightarrow 0$.

Given $0 < a \le t \le s \le 1$, recall the variance of $\tilde{X}_{\lfloor sn \rfloor} - \tilde{X}_{\lfloor tn \rfloor}$ is

$$\chi_n^{t,s}(\tau) = \sum_{\mathrm{j}=k(\lfloor tn \rfloor)+2}^{k(\lfloor sn \rfloor)} \mathbb{V}ar(Y_{\mathrm{j}}) + \mathbb{V}ar(\bar{Y}_{\lfloor sn \rfloor}) + \mathbb{V}ar(\bar{Y}^c_{\lfloor tn \rfloor}). \tag{2.64}$$

Again by (2.34), since $t > 0$, for $n$ large enough,

$$\mathbb{V}ar(Y_{\mathrm{j}}) \le 2(\sigma_\mu^2 \sigma_V)^2 \log^4 T_{\mathrm{j}} \le 2(\sigma_\mu^2 \sigma_V)^2 \log^4 n \tag{2.65}$$

holds for $\mathrm{j} \in [k(\lfloor tn \rfloor) + 2, k(\lfloor sn \rfloor)]$. Using the upper bound in (2.39) we obtain

$$\mathbb{V}ar(\bar{Y}_{\lfloor sn \rfloor}) + \mathbb{V}ar(\bar{Y}^c_{\lfloor tn \rfloor}) \le C^{(2)} \log^4 \bar{T}_{\lfloor sn \rfloor} + 4\left[(\sigma_\mu^2 \sigma_V)^2 + C^{(2)}\right] \log^4 T_{k(\lfloor tn \rfloor)+1} = O(\log^4 n). \tag{2.66}$$

In the fast cooling case, since $k(n) \sim (1/C)\log n$, the number of terms in the sum of (2.64) is $(1/C)[\log sn + o(\log sn) - \log tn - o(\log tn)] = (1/C)[\log s/t + o(\log n)]$. Thus,

$$\chi_n^{t,s}(\tau) \le 2(\sigma_\mu^2 \sigma_V)^2 \left[\frac{1}{C}\log\left(\frac{s}{t}\right) + o(\log n)\right] \log^4 n + O(\log^4 n). \tag{2.67}$$

Since $\chi_n(\tau)$ is of order $\log^5 n$, it is obvious $(\tilde{X}_{\lfloor sn \rfloor} - \tilde{X}_{\lfloor tn \rfloor})/\sqrt{\chi_n(\tau)} \Rightarrow_n 0$. Moreover, notice that $\chi_{\lfloor tn \rfloor}(\tau) \sim \chi_n(\tau)$ for any $t \in [a,1]$, and so $\tilde{X}_{\lfloor tn \rfloor}/\sqrt{\chi_n(\tau)} \Rightarrow_n N$, where $N$ is the standard Gaussian random variable. Hence $(X_t^n, X_s^n - X_t^n) \Rightarrow_n (N, 0)$, or equivalently $(X_t^n, X_s^n) \Rightarrow_n$

$(N, N)$. This argument easily extends to the weak convergence of finite dimension vectors $(X_{t_1}^n, X_{t_2}^n, ..., X_{t_k}^n)$, i.e.

$$(X_{t_1}^n, X_{t_2}^n, ..., X_{t_k}^n) \Rightarrow_n (N, N, ..., N), \tag{2.68}$$

where $t_{\mathrm{i}} \in [a, 1], \mathrm{i} = 1, 2, ..., k,\ N \sim \mathcal{N}(0, 1)$.

To check the tightness condition, it is enough to show that for any $\epsilon > 0$,

$$\limsup_{n\to\infty} \mathbb{P}\left(\sup_{a\le s\le t\le 1} |X_s^n - X_t^n| \ge \epsilon\right) = 0, \tag{2.69}$$

which is equivalent to

$$\limsup_{n\to\infty} \mathbb{P}\left(\sup_{\lfloor an\rfloor\le k\le l\le n} \frac{|\tilde{X}_k - \tilde{X}_l|}{\sqrt{\chi_n(\tau)}} \ge \epsilon\right) = 0. \tag{2.70}$$

Since $\sup_{\lfloor an\rfloor\le k\le l\le n} |\tilde{X}_k - \tilde{X}_1| \le 2\sup_{\lfloor an\rfloor\le s\le n} |\tilde{X}_s - \tilde{X}_{\lfloor an\rfloor}|$, we can deal with $|\tilde{X}_s - \tilde{X}_{\lfloor an\rfloor}|$ in the following proof. Let $m$ be $|\tilde{X}_m - \tilde{X}_{\lfloor an\rfloor}| = \sup_{\lfloor an\rfloor\le s\le n} |\tilde{X}_s - \tilde{X}_{\lfloor an\rfloor}|$. the decomposition of it is

$$\tilde{X}_m - \tilde{X}_{\lfloor an\rfloor} = \sum_{\mathrm{j}=\tau(k(\lfloor an\rfloor)+1)}^{\tau(k(m))} \tilde{Y}_{\mathrm{j}} + \tilde{\tilde{Y}}_m - \tilde{\tilde{Y}}_{\lfloor an\rfloor}, \tag{2.71}$$

or just $\tilde{\tilde{Y}}_m - \tilde{\tilde{Y}}_{\lfloor an\rfloor}$ if $k(\lfloor an\rfloor) = k(m)$. Similar to what we did in the proof of slow cooling,

$$\mathbb{P}\left(\frac{|\sum_{\mathrm{j}=\tau(k(\lfloor an\rfloor)+1)}^{\tau(k(m))} \tilde{Y}_{\mathrm{j}}|}{\sqrt{\chi_n(\tau)}} \ge \frac{\epsilon}{2}\right) \le \mathbb{P}\left(\sup_{l\in[0,k(n)-k(\lfloor an\rfloor)]} \frac{|M_l|}{\sqrt{\chi_n(\tau)}} \ge \frac{\epsilon}{2}\right). \tag{2.72}$$

Combining (2.54) and (2.55) under the case $q = 1$, recall that $k(n) - k(\lfloor an\rfloor) \sim -(1/C)\log a$, there exists $C_2 > 0$, depending only on $\epsilon$,

$$\mathbb{P}\left(\frac{|\sum_{\mathrm{j}=\tau(k(\lfloor an\rfloor)+1)}^{\tau(k(m))} \tilde{Y}_{\mathrm{j}}|}{\sqrt{\chi_n(\tau)}} \ge \frac{\epsilon}{2}\right) \le \frac{C_2}{\log n}. \tag{2.73}$$

For $\tilde{\tilde{Y}}_m$, following all the steps from (2.58) to (2.62), there exists $C'' > 0$, depending only on $\epsilon$,

$$\mathbb{P}\left(\frac{|\tilde{\tilde{Y}}_m|}{\sqrt{\chi_n(\tau)}} \ge \frac{\epsilon}{4}\right) \le \frac{C''}{\log^2 n}\left(k(n) - k(\lfloor an\rfloor)\right), \tag{2.74}$$

and obviously the right hand side goes to zero as n goes to infinity. By (2.37), (2.73) and (2.74),

$$\begin{aligned}
&\mathbb{P}\left(\sup_{\lfloor an\rfloor\le s\le n}\frac{|\tilde{X}_s-\tilde{X}_{\lfloor an\rfloor}|}{\sqrt{\chi_n(\tau)}}\ge\epsilon\right)=\mathbb{P}\left(\frac{|\tilde{X}_m-\tilde{X}_{\lfloor an\rfloor}|}{\sqrt{\chi_n(\tau)}}\ge\epsilon\right)\\
\le&\mathbb{P}\left(\frac{|\sum_{\mathrm{j}=\tau(k(\lfloor an\rfloor)+1)}^{\tau(k(m))}\tilde{Y}_{\mathrm{j}}|}{\sqrt{\chi_n(\tau)}}\ge\frac{\epsilon}{2}\right)+\mathbb{P}\left(\frac{|\tilde{\bar{Y}}_m|}{\sqrt{\chi_n(\tau)}}\ge\frac{\epsilon}{4}\right)+\mathbb{P}\left(\frac{|\tilde{\bar{Y}}_{\lfloor an\rfloor}|}{\sqrt{\chi_n(\tau)}}\ge\frac{\epsilon}{4}\right)\\
\le&\frac{C_2}{\log n}+\frac{C''}{\log^2 n}(k(n)-k(\lfloor an\rfloor))+\frac{16\mathbb{V}ar(\bar{Y}_{\lfloor an\rfloor})}{\epsilon^2\chi_n(\tau)}\\
=&O(\frac{1}{\log n}).
\end{aligned}\tag{2.75}$$

The tightness condition holds. Hence (2.15) is proved. □

### 2.4.3 $L_p$ norm convergence for transient RWRE with $\kappa\in(0,1)$

*Proof of theorem 2.3.1.* Fix $M>1$, write

$$\begin{aligned}
\mathbb{E}|Z_n|^p&=\mathbb{E}Z_{n+}^p+\mathbb{E}Z_{n-}^p=\int_0^{n^p}\mathbb{P}(Z_n^p\ge x)dx+\int_0^{n^p}\mathbb{P}(Z_n^p\le -x)dx\\
=&\int_0^{Mn^{p\kappa}}\mathbb{P}(Z_n\ge x^{1/p})dx+\int_{Mn^{p\kappa}}^{n^p}\mathbb{P}(Z_n\ge x^{1/p})dx+\int_0^{n^{p\kappa}}\mathbb{P}(Z_n\le -x^{1/p})dx\\
&+\int_{n^{p\kappa}}^{n^p}\mathbb{P}(Z_n\le -x^{1/p})dx=n^{p\kappa}\int_0^M\mathbb{P}\left(\frac{Z_n}{n^\kappa}\ge y^{1/p}\right)dy+I+II+III.
\end{aligned}\tag{2.76}$$

Our next step is to bound $I,II,III$ from above. Start with $II$,

$$II=n^{p\kappa}\int_0^1\mathbb{P}\left(\frac{Z_n}{n^\kappa}\le -y^{1/p}\right)dy.\tag{2.77}$$

The probability inside the integral converges to 0 for any $y>0$ since $\mathfrak{M}$ is always non-negative. Thus the integral converges to zero since it is an integral over finite range($y\in(0,1)$). And one can conclude easily $II/n^{p\kappa}\to 0$.

Use the backtracking theorem (Theorem 1.4 in [49]) on $III$, we have for $n>N(\alpha,\kappa)$ where $N(\alpha,\kappa)>0$ only depends on $\alpha$ (since $\alpha$ determines $\kappa$),

$$III\le\int_{n^{p\kappa}}^{n^p}\mathbb{P}(Z_n\le -n^\kappa)dx\le n^p\mathrm{e}^{-n^{\kappa/2}}.\tag{2.78}$$

The right hand side converges to zero and thus $III$ is done.

We use regeneration time method to estimate $I$. The regeneration time is defined as follows, let $R_0 = 0$, for $R_k$ where $k > 0$,

$$R_k := \inf\{n > R_{k-1} : \max_{l<n} Z_l < Z_n \leq \min_{m\geq n} Z_m\}. \tag{2.79}$$

Decompose part $I$ by regeneration times,

$$I \leq \int_{Mn^{p\kappa}}^{n^p} \Big[\mathbb{P}(Z_{R_1} \geq an^\kappa) + \mathbb{P}(Z_{R_m} - Z_{R_1} \geq x^{1/p} - an^\kappa) + \mathbb{P}(R_m < n)\Big]\, dx \tag{2.80}$$

where $a < M^{1/p}$ and $m$ to be determined later. For the regeneration time $\{R_k\}_{k\geq 1}$, we know that(see Appendix B in [20] for more detail) $\{R_k - R_{k-1}\}_{k>1}$ is a non-negative i.i.d sequence with polynomial right tail with parameter $\kappa$. Moreover, $\{Z_{R_k} - Z_{R_{k-1}}\}_{k>1}$ is another non-negative i.i.d. sequence with exponential right tail thus LDP holds. $Z_{R_1}$ also has exponential tail but has another distribution. Now we use those properties to identify what value $a$ and $m$ should satisfy in order to get our desired upper bound. Notice that $R_k - R_{k-1}$ is non-negative,

$$\begin{aligned}\mathbb{P}(R_m < n) &\leq \mathbb{P}(R_m - R_1 < n) \leq \mathbb{P}(R_2 - R_1 < n)^{m-1} = [1 - \mathbb{P}(R_2 - R_1 \geq n)]^{m-1}\\ &\leq \Big[1 - Cn^{-\kappa}\Big]^{m-1} \leq \exp\{-\frac{C(m-1)}{n^\kappa}\}\end{aligned} \tag{2.81}$$

for some $C = C(\alpha) > 0$. Meanwhile, by Large deviation principle upper bound, for $a < M^{1/p}$,

$$\mathbb{P}(Z_{R_m} - Z_{R_1} \geq x^{1/p} - an^\kappa) \leq \exp\left\{-(m-1)I\left(\frac{x^{1/p} - an^\kappa}{m-1}\right)\right\} \tag{2.82}$$

where $I(\cdot)$ is the rate function of $Z_{R_2} - Z_{R_1}$. As long as we pick $m - 1 = x^{1/p}/(2\mathbb{E}[Z_{R_2} - Z_{R_1}])$ and $a = M^{1/p}/3$, then for $x \geq Mn^{p\kappa}$,

$$I\left(\frac{x^{1/p} - an^\kappa}{m-1}\right) \geq I\left(\frac{4}{3}\mathbb{E}[Z_{R_2} - Z_{R_1}]\right) > 0. \tag{2.83}$$

Now back to (2.80), $\mathbb{P}(Z_{R_1} \geq an^\kappa) \leq \exp\{-cn^\kappa\}$ for some $c > 0$ and after integration it will still converge to 0. $\mathbb{P}(Z_{R_m} - Z_{R_1} \geq x^{1/p} - an^\kappa) \leq \exp\{-c_0 x^{1/p}\} \leq \exp\{-c_0 M^{1/p} n^\kappa\}$ for some other $c_0 > 0$ and hence after the integration it converges to 0 as well. $\mathbb{P}(R_m < n) \leq \exp\{-\frac{c_1 x^{1/p}}{n^\kappa}\}$ where $c_1 = C/(2\mathbb{E}[Z_{R_2} - Z_{R_1}]) > 0$ and thus

$$\int_{Mn^{p\kappa}}^{n^p} \mathbb{P}(R_m < n)dx \leq n^{p\kappa} \int_M^\infty \exp\{-c_1 y^{1/p}\}dy \tag{2.84}$$

where the last integral converges to 0 as $M$ goes to infinity since $\exp\{-c_1 y^{1/p}\}$ is integrable on $(0, \infty)$.

From decomposition (2.76) and the upper bound estimates of $I, II, III$, we have

$$\limsup_{n\to\infty} \left| \frac{\mathbb{E}[|Z_n|^p]}{n^{p\kappa}} - \int_0^M \mathbb{P}\left(\frac{Z_n}{n^\kappa} \geq y^{1/p}\right) dy \right| \leq \int_M^\infty \exp\{-c_1 y^{1/p}\}dy \tag{2.85}$$

for any $M > 1$. By bounded convergence theorem,

$$\lim_{n\to\infty} \int_0^M \mathbb{P}\left(\frac{Z_n}{n^\kappa} \geq y^{1/p}\right) dy = \int_0^M \mathbb{P}\left(\mathfrak{M} \geq y^{1/p}\right) dy. \tag{2.86}$$

In the end, let $M$ go to infinity and we will get the convergence of the $L_p$ norm. When $p = 1$, the convergence also holds for the expectation since $\mathbb{E}Z_{n-} = II + III = o(n^\kappa)$. $\square$

### 2.4.4 $L_p$ norm convergence for transient RWRE with $\kappa = 2$

*Proof of Theorem 2.3.4.* We will find the left and right tail estimates of $Z_n - vn$. For the right tail upper bound, we use the regeneration time decomposition. Fix $M > 1$, let $M\sqrt{n \log n} < x < (1-v)n$, $a = x/2$ and $m - 1 = (vn + x/4)/(\mathbb{E}(Z_{R_2} - Z_{R_1}))$,

$$\mathbb{P}(Z_n - vn \geq x) \leq \mathbb{P}(Z_{R_1} \geq a) + \mathbb{P}(Z_{R_m} - Z_{R_1} \geq vn + x - a) + \mathbb{P}(R_m - R_1 < n). \tag{2.87}$$

The first part inside the integral decays exponentially as $a$ goes to infinity. For the second part, by the choice of $a$ and $m$,

$$\mathbb{P}(Z_{R_m} - Z_{R_1} \geq vn + x - a) = \mathbb{P}(\tilde{Z}_{R_m} - \tilde{Z}_{R_1} \geq \frac{x}{4}) \tag{2.88}$$

where $\tilde{Z}_{R_k} = Z_{R_k} - \mathbb{E}Z_{R_k}$ means the centered random variable. Since $\tilde{Z}_{R_2} - \tilde{Z}_{R_1}$ has exponential right tail, one can conclude by taking the Taylor expansion of $\Lambda(t) = \log E[\exp\{t(\tilde{Z}_{R_2} - \tilde{Z}_{R_1})\}]$ for some small positive $t$, log of the moment generation function of $\tilde{Z}_{R_2} - \tilde{Z}_{R_1}$, up to the second order to see that as $t \to 0$,

$$\mathbb{E}[\mathrm{e}^{t(\tilde{Z}_{R_2} - \tilde{Z}_{R_1})}] = \mathrm{e}^{\Lambda(t)} = \mathrm{e}^{\frac{ct^2}{2} + o(t^2)} \tag{2.89}$$

where $c = \mathbb{E}(\tilde{Z}_{R_2} - \tilde{Z}_{R_1})^2$. Thus, by applying Theorem 15 in chapter III of [50], there exist $c_0 > 0$, $\delta > 0$ that only depend on $\alpha$ such that $\mathbb{P}(\tilde{Z}_{R_m} - \tilde{Z}_{R_1} \geq \frac{x}{4}) \leq \exp\{-c_0 x^2/(m-1)\}$ for $x \leq \delta(m-1)$. Since $m - 1 \asymp n$, it's fine if we replace $m-1$ by $n$, the inequality still holds for some other $c_0$ and $\delta$.

For the last part in (2.87), notice that $\mathbb{P}(R_m - R_1 < n) = \mathbb{P}(\tilde{R}_m - \tilde{R}_1 < -x/4v)$. In order to bound this left tail, we need an upper bound for $\mathbb{E}(\exp\{-\lambda(\tilde{R}_m - \tilde{R}_1)\})$ which is included in the next lemma.

**Lemma 2.4.2.** *Assume $\xi_1$ has mean zero, is bounded below by $-L$ for some $L > 0$, and has right tail decay $P(\xi_1 > x) = O(x^{-2})$. Then, there exists a constant $C > 0$ that depends on the distribution of $\xi_1$ such that*

$$E[\mathrm{e}^{-\lambda \xi_1}] \leq \mathrm{e}^{C\lambda^2 |\log \lambda|} \tag{2.90}$$

*for all $\lambda \in (0, 1/\mathrm{e})$.*

*Proof.* Define $\hat{\xi}_1 = \xi_1 + L$ so that $\hat{\xi}_1$ is non-negative and $E[\hat{\xi}_1] = L$. Let $K > 0$ such that $P(\hat{\xi}_1 > x) \leq Kx^{-2}$.

$$
\begin{aligned}
\mathrm{e}^{-\lambda L} E[\mathrm{e}^{-\lambda \xi_1}] &= E[\mathrm{e}^{-\lambda \hat{\xi}_1}] = \int_0^\infty \lambda \mathrm{e}^{-\lambda y} P(\hat{\xi}_1 < y) dy = 1 - \int_0^\infty \lambda \mathrm{e}^{-\lambda y} P(\hat{\xi}_1 \geq y) dy \\
\leq & 1 - \int_0^\infty \lambda (1 - \min\{\lambda y, 1\}) P(\hat{\xi}_1 \geq y) dy = 1 - \lambda L + \lambda^2 \int_0^{\lambda^{-1}} y P(\hat{\xi}_1 \geq y) dy + \lambda \int_{\lambda^{-1}}^\infty P(\hat{\xi}_1 \geq y) dy \\
= & 1 - \lambda L + \lambda^2 \int_0^1 y P(\hat{\xi}_1 \geq y) dy + \lambda^2 \int_1^{\lambda^{-1}} y P(\hat{\xi}_1 \geq y) dy + \lambda \int_{\lambda^{-1}}^\infty P(\hat{\xi}_1 \geq y) dy \\
\leq & 1 - \lambda L + \lambda^2 \int_0^1 y dy + K\lambda^2 \int_1^{\lambda^{-1}} y^{-1} dy + K\lambda^2 \int_{\lambda^{-1}}^\infty y^{-2} dy \\
= & 1 - \lambda L + \frac{1}{2}\lambda^2 + K\lambda^2 |\log \lambda| + K\lambda^2 \leq \exp\{-\lambda L + (2K + \frac{1}{2})\lambda^2 |\log \lambda|\}.
\end{aligned}
\tag{2.91}
$$

The last inequality holds since $|\log \lambda| > 1$ and $1 + x \leq \mathrm{e}^x$ for $x \in \mathbb{R}$. $\square$

We use Chebyshev inequality to estimate $\mathbb{P}(\tilde{R}_m - \tilde{R}_1 < -x/4v)$. The parameter $\lambda < 1/\mathrm{e}$ will be determined later. By the above lemma, recall that $\mathbb{P}(R_2 - R_1 > x) \sim Cx^{-2}$,

$$
\begin{aligned}
\mathbb{P}(\tilde{R}_m - \tilde{R}_1 < -x/4v) &\leq \exp\{-\frac{\lambda x}{4v}\} \mathbb{E}[\mathrm{e}^{-\lambda(m-1)(\tilde{R}_2 - \tilde{R}_1)}] \\
&\leq \exp\{-\frac{\lambda x}{4v} + C(m-1)\lambda^2 |\log \lambda|\}.
\end{aligned}
\tag{2.92}
$$

Let $\lambda|\log \lambda| = x/(8Cv(m-1))$. Since the function $\lambda|\log \lambda|$ is increasing for $\lambda \in (0, 1/\mathrm{e})$ and reaches maximum $1/\mathrm{e}$ when $\lambda = 1/\mathrm{e}$, in order to let such $\lambda$ exist, we need $x/(8Cv(m-1)) < 1/\mathrm{e}$. Recall the choice of $m$ and $m - 1 \asymp n$, this can be satisfied if we choose $\delta_1 > 0$ small enough(only depends $\alpha$) and let $x \leq \delta_1 n$. Moreover, With this extra restriction of $x$ and the choice of $\lambda$, $|\log \lambda| < 2|\log[8Cv(m-1)/x]|$ by the monotonicity of $\lambda|\log \lambda|$. Thus,

$$
\mathbb{P}(\tilde{R}_m - \tilde{R}_1 < -x/4v) \leq \exp\{\frac{-x^2}{64Cv^2(m-1)|\log \lambda|}\} \leq \exp\{\frac{-C_1 x^2}{(m-1)|\log \frac{C_2(m-1)}{x}|}\} \tag{2.93}
$$

holds for $C_1, C_2 > 0$. Since $m \asymp n$, we get $\mathbb{P}(\tilde{R}_m - \tilde{R}_1 < -x/4v) \leq \exp\{-c_1 x^2/(n \log n)\}$ for $x \leq \delta_1 n$ and for some other constant $c_1 > 0$.

The following lemma concludes the above results and solves the right tail truncated $L_p$ norm of $Z_n - vn$.

**Lemma 2.4.3 (Right tail estimates).** *Let $\alpha$ be transient with $\kappa = 2$. Under the annealed measure $\mathbb{P}$, fix $M > 0$, there exists $\delta_2 > 0$, $c > 0$, as well as $c_0 > 0$ and $c_1 > 0$ mentioned above, such that for all $x \in (M\sqrt{n \log n}, \delta_2 n)$,*

$$\mathbb{P}(Z_n - vn \geq x) \leq \mathrm{e}^{-cx} + \mathrm{e}^{-c_0 \frac{x^2}{n}} + \mathrm{e}^{-c_1 \frac{x^2}{n \log n}}. \tag{2.94}$$

*By taking $M \to \infty$, the convergence holds for the right tail truncated $L_p$ norm for $p \in (0, 2]$,*

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{1}{(n\log n)^{p/2}} \mathbb{E}\left[|Z_n - vn|^p 1_{\{Z_n - vn \geq M\sqrt{n\log n}\}}\right] = 0. \tag{2.95}$$

*Proof.* Let constant $\delta_2 = \min\{\delta, \delta_1\} > 0$, (2.94) is immediately obtained by the arguments above. Write

$$\begin{aligned}\mathbb{E}\left[(Z_n - vn)^p 1_{\{Z_n - vn \geq M\sqrt{n\log n}\}}\right] = &\int_{M\sqrt{n\log n}}^{\delta_2 n} px^{p-1}\mathbb{P}(Z_n - vn \geq x)dx \\ &+ \int_{\delta_2 n}^{(1-v)n} px^{p-1}\mathbb{P}(Z_n - vn \geq x)dx.\end{aligned} \tag{2.96}$$

Inside the last integral when $x > \delta_2 n$, using LDP upper bound by [53], $\mathbb{P}(Z_n - vn \geq x) \leq \mathbb{P}(Z_n - vn \geq \delta_2 n)$ which decays exponentially in $n$. Thus after taking the integral it will converge to zero as well. For $x \in (M\sqrt{n\log n}, \delta_2 n)$, by (2.94),

$$\begin{aligned}\int_{M\sqrt{n\log n}}^{\delta_2 n} px^{p-1}\mathbb{P}(Z_n - vn \geq x)dx \leq &\int_{M\sqrt{n\log n}}^{\delta_2 n} px^{p-1}\mathrm{e}^{-cx}dx + \int_{M\sqrt{n\log n}}^{\delta_2 n} px^{p-1}\mathrm{e}^{-c_0\frac{x^2}{n}}dx \\ &+ \int_{M\sqrt{n\log n}}^{\delta_2 n} px^{p-1}\mathrm{e}^{-c_1\frac{x^2}{n\log n}}dx \\ \leq &o(1) + n^{p/2}\int_{M\sqrt{\log n}}^{\infty} py^{p-1}\mathrm{e}^{-c_0 y^2}dy \\ &+ (n\log n)^{p/2}\int_{M}^{\infty} py^{p-1}\mathrm{e}^{-c_1 y^2}dy \\ = &o(1) + o(n^{p/2}) + (n\log n)^{p/2}\int_{M}^{\infty} py^{p-1}\mathrm{e}^{-c_1 y^2}dy.\end{aligned} \tag{2.97}$$

So $\limsup_{n\to\infty} \frac{1}{(n\log n)^{p/2}} \mathbb{E}\left[|Z_n - vn|^p 1_{\{Z_n - vn \geq M\sqrt{n\log n}\}}\right] \leq \int_M^\infty py^{p-1}\mathrm{e}^{-c_1 y^2}dy$, and letting $M \to \infty$ finishes the proof. ∎

For the left tail estimates of $Z_n - vn$, we already know some results like general left tail estimates(first inequality of Lemma 4.3 in [20]) and precise left tail estimates(Theorem 1.2 in [54]) which are listed below.

**Proposition 2.4.1** (**General left tail estimates**). *Let $\alpha$ be transient with $\kappa = 2$. Under the annealed measure $\mathbb{P}$, there exists constant $C > 0$, for large $n$ enough,*

$$\mathbb{P}(Z_n - vn \le -t\sqrt{n}) \le Ct^{-2} \tag{2.98}$$

*for $t \le v\sqrt{n}/2$.*

**Proposition 2.4.2** (**Precise left tail estimates**). *Let $\alpha$ be transient with $\kappa = 2$. Under the annealed measure $\mathbb{P}$, there exists constant $K_0(\alpha) > 0$,*

$$\lim_{n\to\infty} \sup_{t\in\Gamma_n} \left| \frac{\mathbb{P}(Z_n - vn \le -t)}{(nv - t)t^{-2}} - K_0 \right| = 0 \tag{2.99}$$

*where $\Gamma_n = (\sqrt{n}\log^3 n, nv - \sqrt{n}\log^3 n)$.*

We will use those tail estimates to show the convergence of the left tail truncated $L_p$ norm, which is the next lemma.

**Lemma 2.4.4.** *Let $\alpha$ be transient with $\kappa = 2$. Under the annealed measure $\mathbb{P}$, there exists $\sigma^2 > 0$ such that for any $M > 0$,*

$$\lim_{n\to\infty} \frac{1}{n\log n} \mathbb{E}\left[(Z_n - vn)^2 1_{\{Z_n - vn \le -M\sqrt{n\log n}\}}\right] = \sigma^2. \tag{2.100}$$

*For $p \in (0, 2)$,*

$$\lim_{n\to\infty} \frac{1}{(n\log n)^{p/2}} \mathbb{E}\left[|Z_n - vn|^p 1_{\{Z_n - vn \le -M\sqrt{n\log n}\}}\right] = 0. \tag{2.101}$$

**Remark 2.4.5.** *Unlike the right tail truncated $L_2$ norm, the left tail away from the standard $M\sqrt{n\log n}$ window contributes to a non zero variance. This is due to the heavy left tail(polynomial decay as $x^{-2}$) of $Z_n - vn$.*

*Proof.* Fix $\delta > 0$, write

$$\mathbb{E}\left[|Z_n - vn|^p 1_{\{Z_n - vn \le -M\sqrt{n\log n}\}}\right] = \int_{M\sqrt{n\log n}}^{(1+v)n} px^{p-1}\mathbb{P}(Z_n - vn \le -x)dx$$
$$= \int_{M\sqrt{n\log n}}^{\sqrt{n}\log^3 n} + \int_{\sqrt{n}\log^3 n}^{nv-\sqrt{n}\log^3 n} + \int_{nv-\sqrt{n}\log^3 n}^{(v+\delta)n} + \int_{(v+\delta)n}^{(v+1)n}. \tag{2.102}$$

Use proposition 2.4.1 in the first integral, there exists constant $C > 0$, such that

$$\begin{aligned}\int_{M\sqrt{n\log n}}^{\sqrt{n}\log^3 n} px^{p-1}\mathbb{P}(Z_n - vn \le -x)dx &= pn^{p/2}\int_{M\sqrt{\log n}}^{\log^3 n} t^{p-1}\mathbb{P}(Z_n - vn \le -t\sqrt{n})dx \\ &\le pn^{p/2}\int_{M\sqrt{\log n}}^{\log^3 n} t^{p-1}Ct^{-2}dt \le O(n^{p/2}\log\log n).\end{aligned} \tag{2.103}$$

The last inequality means that the order of the left hand side is $n\log\log n$ when $p = 2$. If $p < 2$, the order is $n^{p/2}$.

Use large deviation upper bound to the last integral in (2.102) which can imply that it converges to zero exponentially fast. For the third integral, apply proposition 2.4.2 but only on $nv - \sqrt{n}\log^3 n$, since this is the furthest value of $x$ that we can apply such estimate. For the second integral in (2.102), apply proposition 2.4.2 as well. That is, there exists $K_0 = K_0(\alpha) > 0$ such that for $n$ large enough,

$$\begin{aligned}\int_{nv-\sqrt{n}\log^3 n}^{(v+\delta)n} px^{p-1}\mathbb{P}(Z_n - vn \le -x)dx &\le \int_{nv-\sqrt{n}\log^3 n}^{(v+\delta)n} px^{p-1}\mathbb{P}(Z_n \le \sqrt{n}\log^3 n)dx \\ &\le 2K_0\sqrt{n}\log^3 n\frac{(v+\delta)^p n^p}{(nv - \sqrt{n}\log^3 n)^2} = o(n^{p-1}).\end{aligned} \tag{2.104}$$

And for any $\epsilon > 0$,

$$\begin{aligned}&\left|\int_{\sqrt{n}\log^3 n}^{nv-\sqrt{n}\log^3 n} px^{p-1}\mathbb{P}(Z_n - vn \le -x)dx - \int_{\sqrt{n}\log^3 n}^{nv-\sqrt{n}\log^3 n} px^{p-1}K_0(nv-x)x^{-2}dx\right| \\ &\le \int_{\sqrt{n}\log^3 n}^{nv-\sqrt{n}\log^3 n} px^{p-1}\left|\mathbb{P}(Z_n - vn \le -x) - K_0(nv-x)x^{-2}\right|dx \\ &\le \int_{\sqrt{n}\log^3 n}^{nv-\sqrt{n}\log^3 n} px^{p-1}\epsilon(nv-x)x^{-2}dx \le \epsilon O((n\log n)^{p/2})\end{aligned} \tag{2.105}$$

holds for $n$ large enough since the precise left tail estimates holds uniformly on $(\sqrt{n}\log^3 n, nv-\sqrt{n}\log^3 n)$ and since

$$n\int_{\sqrt{n}\log^3 n}^{nv-\sqrt{n}\log^3 n} x^{p-3}dx = O(n\log n)1_{\{p=2\}} + O(n^{p/2}\log^{3(p-2)} n)1_{\{p\in(0,2)\}}. \tag{2.106}$$

Finally, if $p<2$, by the above equation,

$$\lim_{n\to\infty}\frac{1}{(n\log n)^{p/2}}\int_{\sqrt{n}\log^3 n}^{nv-\sqrt{n}\log^3 n} px^{p-1}K_0(nv-x)x^{-2}dx \le \lim_{n\to\infty}\frac{C\log^{3(p-2)} n}{\log^{p/2} n} = 0. \tag{2.107}$$

If $p=2$, define $\sigma^2>0$ as

$$\sigma^2 := \lim_{n\to\infty}\frac{1}{n\log n}\int_{\sqrt{n}\log^3 n}^{nv-\sqrt{n}\log^3 n} 2xK_0(nv-x)x^{-2}dx = vK_0 \tag{2.108}$$

and finish the proof. □

Finally, by combining the truncated $L_p$ norm estimates (2.95), (2.100), and (2.101), together with the same proof techniques where $\kappa\in(0,1)$ by letting $M$ go to infinity, the proof is done. □

### 2.4.5 Further discussion on the tail estimates with $\kappa=2$

Before we move on to solve the weak limit of RWCRE, we need extend our tail estimates to the centered walk $Z_n-\mathbb{E}Z_n$. For the right tail estimates, simply define $m=(\mathbb{E}Z_n+x/4)/(\mathbb{E}[Z_{R_2}-Z_{R_1}])$ instead of $(vn+x/4)/(\mathbb{E}[Z_{R_2}-Z_{R_1}])$. Since $\mathbb{E}Z_n-nv=o(\sqrt{n\log n})$, this new $m\asymp n$ as well. Thus, we can conclude the same version of the right tail bound lemma, that is,

**Lemma 2.4.6 (Centered right tail estimates).** *Let $\alpha$ be transient with $\kappa=2$. Under the annealed measure $\mathbb{P}$, fix $M>0$, there exists $\delta_2>0$, $c>0$, as well as $c_0>0$ and $c_1>0$ mentioned above, such that for all $x\in(M\sqrt{n\log n},\delta_2 n)$,*

$$\mathbb{P}(Z_n-\mathbb{E}Z_n\ge x)\le \mathrm{e}^{-cx}+\mathrm{e}^{-c_0\frac{x^2}{n}}+^{-c_1\frac{x^2}{n\log n}}. \tag{2.109}$$

*By taking $M \to \infty$, the convergence holds for the right tail truncated $L_p$ norm for $p \in (0, 2]$,*

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{1}{(n \log n)^{p/2}} \mathbb{E}\left[|Z_n - \mathbb{E}Z_n|^p 1_{\{Z_n - \mathbb{E}Z_n \geq M\sqrt{n \log n}\}}\right] = 0. \tag{2.110}$$

For the general and precise left tail estimates, we also have the same version of propositions but since $\mathbb{E}Z_n - vn = o(\sqrt{n \log n})$, the range of $t$ needs to be shortened.

**Proposition 2.4.3** (**Centered general left tail estimates**). *Let $\alpha$ be transient with $\kappa = 2$. Under the annealed measure $\mathbb{P}$, there exists constant $C > 0$, such that for any fixed $M > 1$ and for large $n$ enough,*

$$\mathbb{P}(Z_n - \mathbb{E}Z_n \leq -t\sqrt{n}) \leq Ct^{-2} \tag{2.111}$$

*for $t \in (M\sqrt{\log n}, v\sqrt{n}/2 - M\sqrt{\log n})$.*

**Proposition 2.4.4** (**Centered precise left tail estimates**). *Let $\alpha$ be transient with $\kappa = 2$. Under the annealed measure $\mathbb{P}$, there exists constant $K_0(\alpha) > 0$,*

$$\lim_{n\to\infty} \sup_{t\in\Gamma_n} \left| \frac{\mathbb{P}(Z_n - \mathbb{E}Z_n \leq -t)}{(nv - t)t^{-2}} - K_0 \right| = 0 \tag{2.112}$$

*where $\Gamma_n = (\sqrt{n} \log^4 n, nv - \sqrt{n} \log^4 n)$.*

**Remark 2.4.7.** *The constants $K_0$ in precise and centered precise left tail estimates are the same.*

From the above two centered left tail estimates, we also have the centered left tail truncated $L_p$ norm lemma.

**Lemma 2.4.8.** *Let $\alpha$ be transient with $\kappa = 2$. Under the annealed measure $\mathbb{P}$, there exists $\sigma^2 > 0$ such that for any $M > 0$,*

$$\lim_{n\to\infty} \frac{1}{n \log n} \mathbb{E}\left[(Z_n - \mathbb{E}Z_n)^2 1_{\{Z_n - \mathbb{E}Z_n \leq -M\sqrt{n \log n}\}}\right] = \sigma^2. \tag{2.113}$$

*For $p \in (0, 2)$,*

$$\lim_{n\to\infty} \frac{1}{(n \log n)^{p/2}} \mathbb{E}\left[|Z_n - \mathbb{E}Z_n|^p 1_{\{Z_n - \mathbb{E}Z_n \leq -M\sqrt{n \log n}\}}\right] = 0. \tag{2.114}$$

### 2.4.6 Slow cooling: Transient RWRE with $\kappa = 2$

*Proof of Theorem 2.3.5.* Recall the notation $\tilde{X} = X - \mathbb{E}X$. Decompose the cooling random walk $\tilde{X}_{\tau_n}$ into independent sums $\sum_{k=1}^n \tilde{Z}_{T_k}$ and then truncate each term by $A_k = \{|\tilde{Z}_{T_k}| \le \sqrt{T_k^{\beta/(\beta-1)}}\}$. That is,

$$\tilde{X}_{\tau_n} = \sum_{k=1}^n (\tilde{Z}_{T_k} 1_{A_k} + \tilde{Z}_{T_k} 1_{A_k^c}) = \sum_{k=1}^n \left[\tilde{Z}_{T_k} 1_{A_k} - \mathbb{E}\tilde{Z}_{T_k} 1_{A_k}\right] + \sum_{k=1}^n \left[\tilde{Z}_{T_k} 1_{A_k^c} - \mathbb{E}\tilde{Z}_{T_k} 1_{A_k^c}\right]. \quad (2.115)$$

Check the Lindeberg condition of $\sum_{k=1}^n \left[\tilde{Z}_{T_k} 1_{A_k} - \mathbb{E}\tilde{Z}_{T_k} 1_{A_k}\right] / \sqrt{n^\beta \log n}$ in order to apply Lindeberg-Feller CLT. The Lindeberg condition

$$\lim_{n\to\infty} \frac{1}{n^\beta \log n} \sum_{k=1}^n \mathbb{E}\left[\tilde{Z}_{T_k} 1_{A_k} - \mathbb{E}\tilde{Z}_{T_k} 1_{A_k}\right]^2 1_{\left\{|\cdot| \ge \epsilon\sqrt{n^\beta \log n}\right\}} = 0 \quad (2.116)$$

holds for all $\epsilon > 0$ since $|\cdot|$ is of order at most $\sqrt{k^\beta}$ for large $k$. Thus the indicator becomes zero for all large enough $k$ and $k \le n$ which means the sum of the truncated variance is bounded.

To find the limit of the variance of the truncated sum

$$\frac{1}{n^\beta \log n}\left[\sum_{k=1}^n \mathbb{E}[\tilde{Z}_{T_k}^2 1_{A_k}] - \sum_{k=1}^n \left(\mathbb{E}\tilde{Z}_{T_k} 1_{A_k}\right)^2\right], \quad (2.117)$$

first notice that

$$|\mathbb{E}\tilde{Z}_{T_k} 1_{A_k}| = |\mathbb{E}\tilde{Z}_{T_k} 1_{A_k^c}| \le \mathbb{E}|\tilde{Z}_{T_k}| 1_{A_k^c} = o(\sqrt{T_k \log T_k}). \quad (2.118)$$

Since $M^2 T_k \log T_k << T_k^{\beta/(\beta-1)}$ for any $M > 0$ and $T_k$ large enough, by (2.110) and (2.114), the last inequality holds. Therefore, by the choice of $T_k$ and the fact that $\sum_{k=1}^n T_k \log T_k$ in unbounded,

$$\frac{1}{n^\beta \log n}\left[\sum_{k=1}^n \left(\mathbb{E}\tilde{Z}_{T_k} 1_{A_k}\right)^2\right] = \frac{1}{n^\beta \log n} o(\sum_{k=1}^n T_k \log T_k) = \frac{o(n^\beta \log n)}{n^\beta \log n} = o(1). \quad (2.119)$$

It remains to find the limit of the truncated $L_2$ norm $\mathbb{E}\tilde{Z}_{T_k}^2 1_{A_k}$. Fix $M > 1$, decompose the truncated $L_2$ norm into

$$\mathbb{E}\tilde{Z}_{T_k}^2 1_{A_k} = \mathbb{E}\tilde{Z}_{T_k}^2 \left(1_{\{|\tilde{Z}_{T_k}| \le M\sqrt{T_k \log T_k}\}} + 1_{A_k \cap \{\tilde{Z}_{T_k} < -M\sqrt{T_k \log T_k}\}} + 1_{A_k \cap \{\tilde{Z}_{T_k} > M\sqrt{T_k \log T_k}\}}\right). \tag{2.120}$$

By (2.110) the last truncated $L_2$ norm is $o_M(1) T_k \log T_k$, where $o_M(1)$ means it converges to zero as $M \to \infty$.

The middle indicator can be decomposed more precisely as $B_k = \{-\sqrt{T_k^{\beta/(\beta-1)}} \le \tilde{Z}_{T_k} < -\sqrt{T_k} \log^4 T_k\}$ and $C_{k,M} = \{-\sqrt{T_k} \log^4 T_k \le \tilde{Z}_{T_k} < -M\sqrt{T_k \log T_k}\}$. By (2.111), $\mathbb{E}\tilde{Z}_{T_k}^2 1_{C_{k,M}} = o(T_k \log T_k)$. By the similar argument in (2.105), if $\beta > 2$, $\sqrt{T_k^{\beta/(\beta-1)}} << T_k$. Apply (2.112),

$$\begin{aligned}\left|\mathbb{E}\tilde{Z}_{T_k}^2 1_{B_k} - \int_{\sqrt{T_k}\log^4 T_k}^{\sqrt{T_k^{\beta/(\beta-1)}}} 2tK_0(vT_k - t)t^{-2}dt\right| &\le \int_{\sqrt{T_k}\log^4 T_k}^{\sqrt{T_k^{\beta/(\beta-1)}}} 2t\left|\mathbb{P}[\tilde{Z}_{T_k} \le -t] - K_0(vT_k - t)t^{-2}\right|dt \\ &= o(T_k \log T_k).\end{aligned} \tag{2.121}$$

And with simple computation of the integral,

$$\lim_{k\to\infty} \frac{1}{T_k \log T_k} \int_{\sqrt{T_k}\log^4 T_k}^{\sqrt{T_k^{\beta/(\beta-1)}}} 2tK_0(vT_k - t)t^{-2}dt = \frac{vK_0}{\beta - 1}. \tag{2.122}$$

If $\beta \le 2$, $\sqrt{T_k^{\beta/(\beta-1)}} > vT_k - \sqrt{T_k}\log^4 T_k$, then by the same argument in (2.102) (last two integrals), $\mathbb{E}\tilde{Z}_{T_k}^2 1_{B_k \cap \{\tilde{Z}_{T_k} \le -(vT_k - \sqrt{T_k}\log^4 T_k)\}} = o(T_k \log T_k)$ which means in this case the non trivial truncated $L_2$ norm within set $B_k$ is dominant by $\{-(vT_k - \sqrt{T_k}\log^4 T_k) \le \tilde{Z}_{T_k} < -\sqrt{T_k}\log^4 T_k\}$ and its limit we already know is $vK_0$.

Hence,

$$\lim_{M\to\infty} \limsup_{k\to\infty} \frac{1}{T_k \log T_k}\left|\mathbb{E}\tilde{Z}_{T_k}^2 1_{A_k} - vK_0 T_k \log T_k \left(\frac{1}{\beta - 1} \wedge 1\right) - \mathbb{E}\tilde{Z}_{T_k}^2 1_{\{|\tilde{Z}_{T_k}| \le M\sqrt{T_k \log T_k}\}}\right| = 0. \tag{2.123}$$

Together with

$$\lim_{k\to\infty} \frac{1}{b^2 T_k \log T_k}\left|\mathbb{E}\tilde{Z}_{T_k}^2 1_{\{|\tilde{Z}_{T_k}| \le Mb\sqrt{T_k \log T_k}\}} - E[\Phi^2 1_{|\Phi| \le M}]\right| = 0 \tag{2.124}$$

for any fixed $M$, we have

$$\lim_{k\to\infty}\frac{\mathbb{E}\tilde{Z}_{T_k}^2 1_{A_k}}{T_k\log T_k}=b^2+vK_0(\frac{1}{\beta-1}\wedge 1),\tag{2.125}$$

and therefore

$$\lim_{n\to\infty}\frac{1}{\sum_{k=1}^n T_k\log T_k}\sum_{k=1}^n\mathbb{E}[\tilde{Z}_{T_k}^2 1_{A_k}]=b^2+vK_0(\frac{1}{\beta-1}\wedge 1).\tag{2.126}$$

While $T_k\sim\beta k^{\beta-1}$, we have $\sum_{k=1}^n T_k\log T_k\sim\beta(\beta-1)n^\beta\log n$. By (2.126) and (2.119),

$$\lim_{n\to\infty}\frac{1}{n^\beta\log n}\left[\sum_{k=1}^n\mathbb{E}[\tilde{Z}_{T_k}^2 1_{A_k}]-\sum_{k=1}^n\left(\mathbb{E}\tilde{Z}_{T_k}1_{A_k}\right)^2\right]=\beta(\beta-1)\left[b^2+vK_0(\frac{1}{\beta-1}\wedge 1)\right].\tag{2.127}$$

Finally, as long as we show that for any $\epsilon>0$,

$$\lim_{n\to\infty}\mathbb{P}\left[\frac{1}{\sqrt{n^\beta\log n}}\left|\sum_{k=1}^n\left(\tilde{Z}_{T_k}1_{A_k^c}-\mathbb{E}\tilde{Z}_{T_k}1_{A_k^c}\right)\right|\geq\epsilon\right]=0\tag{2.128}$$

which implies the remainder part in (2.115) converges to zero, the proof will be done. To this end, apply $L^1$ norm Chebyshev inequality to (2.128),

$$\begin{aligned}\mathbb{P}\left[\frac{1}{\sqrt{n^\beta\log n}}\left|\sum_{k=1}^n\left(\tilde{Z}_{T_k}1_{A_k^c}-\mathbb{E}\tilde{Z}_{T_k}1_{A_k^c}\right)\right|\geq\epsilon\right]&\leq\frac{\mathbb{E}\left|\sum_{k=1}^n\left(\tilde{Z}_{T_k}1_{A_k^c}-\mathbb{E}\tilde{Z}_{T_k}1_{A_k^c}\right)\right|}{\epsilon\sqrt{n^\beta\log n}}\\&\leq\frac{\sum_{k=1}^n\mathbb{E}\left|\tilde{Z}_{T_k}1_{A_k^c}-\mathbb{E}\tilde{Z}_{T_k}1_{A_k^c}\right|}{\epsilon\sqrt{n^\beta\log n}}\\&\leq\frac{2\sum_{k=1}^n\mathbb{E}|\tilde{Z}_{T_k}|1_{A_k^c}}{\epsilon\sqrt{n^\beta\log n}}.\end{aligned}\tag{2.129}$$

Instead of using the previous estimation $\mathbb{E}|\tilde{Z}_{T_k}|1_{A_k^c}=o(\sqrt{T_k\log T_k})$, we need a sharper bound here. If $\beta<2$, $A_k^c=\phi$ and the proof goes trivially. If $\beta\geq 2$, by the centered right tail estimates and the proof of (2.95), we can easily get rid of the $L_1$ norm truncated to the right of $\delta_2 T_k$. And for the rest of the right tail, using (2.109),

$$\begin{aligned}\int_{\sqrt{T_k^{\beta/(\beta-1)}}}^{\delta_2 T_k} \mathbb{P}(\tilde{Z}_{T_k} \geq x)dx &\leq \int_{\sqrt{T_k^{\beta/(\beta-1)}}}^{\delta_2 T_k} \mathrm{e}^{-cx}dx + \int_{\sqrt{T_k^{\beta/(\beta-1)}}}^{\delta_2 T_k} \mathrm{e}^{-c_0\frac{x^2}{T_k}}dx + \int_{\sqrt{T_k^{\beta/(\beta-1)}}}^{\delta_2 T_k} \mathrm{e}^{-c_1\frac{x^2}{T_k \log T_k}}dx \\ &= o(1) + \sqrt{T_k}\int_{\sqrt{T_k^{1/(\beta-1)}}}^{\infty} \mathrm{e}^{-c_0 y^2}dy + \sqrt{T_k \log T_k}\int_{\sqrt{T_k^{1/(\beta-1)}/\log T_k}}^{\infty} \mathrm{e}^{-c_1 y^2}dy \\ &= o(1)\end{aligned} \tag{2.130}$$

because every term decays stretched exponentially fast.

For the bound of the left tail truncated $L_1$ norm, we only consider the nontrivial case when $\beta \geq 2$. By the centered precise left tail estimates and the proof of (2.101), the truncated $L_1$ norm on $(-(1+v)T_k, -(vT_k - \sqrt{T_k}\log^4 T_k))$ is $o(1)$. For the rest of the left tail, for $k$ large enough, recall $T_k \sim \beta k^{\beta-1}$,

$$\int_{\sqrt{T_k^{\beta/(\beta-1)}}}^{vT_k - \sqrt{T_k}\log^4 T_k} \mathbb{P}(\tilde{Z}_{T_k} \leq -t)dt \leq \int_{\sqrt{T_k^{\beta/(\beta-1)}}}^{vT_k - \sqrt{T_k}\log^4 T_k} 2K_0(vT_k - t)t^{-2}dt \leq \frac{2vK_0T_k}{\sqrt{T_k^{\beta/(\beta-1)}}} = O(k^{(\beta-2)/2}). \tag{2.131}$$

Collect the above bounds of the $L^1$ norm on each truncated pieces, we have $\mathbb{E}|\tilde{Z}_{T_k}|1_{A_k^c} = O(k^{(\beta-2)/2})$. With this sharpened bound, back to (2.129), the last line is $O(1/\sqrt{\log n})$ which indeed decreases to zero. The weak convergence of the remainder term thus holds.

□

### 2.4.7 Fast cooling: Transient RWRE with $\kappa = 2$

*Proof of Theorem 2.3.6.* We will decompose the cooling random walk $\tilde{X}_{\tau_n}$ one more time. As a sum of independent RWRE, $\tilde{X}_{\tau_n}$ will not have a truncated decomposition this time, but will be decomposed into the first $n - M$ terms and the last $M$ term, which means

$$\tilde{X}_{\tau_n} = \sum_{k=1}^{n-M} \tilde{Z}_{T_k} + \sum_{k=n-M-1}^{n} \tilde{Z}_{T_k} \tag{2.132}$$

for some fixed $M > 0$. We want to show that the dominant part of $X_{\tau_n}$ is the last $M$ terms of the sum for large $M$. To start with, we first show that for any $\epsilon > 0$,

$$\lim_{M\to\infty} \limsup_{n\to\infty} \mathbb{P}\left[\left|\frac{\sum_{k=1}^{n-M} \tilde{Z}_{T_k}}{\sqrt{\sum_{k=1}^{n} T_k \log T_k}}\right| \geq \epsilon\right] = 0. \tag{2.133}$$

To see this, by Chebyshev inequality,

$$\mathbb{P}\left[\left|\frac{\sum_{k=1}^{n-M} \tilde{Z}_{T_k}}{\sqrt{\sum_{k=1}^{n} T_k \log T_k}}\right| \geq \epsilon\right] \leq \frac{\sum_{k=1}^{n-M} \mathbb{E}[\tilde{Z}_{T_k}^2]}{\epsilon^2 \sum_{k=1}^{n} T_k \log T_k}. \tag{2.134}$$

We know that $\mathbb{E}[\tilde{Z}_{T_k}^2] \sim (b^2+\sigma^2) T_k \log T_k$ from previous proofs. Therefore, fix $M$ and let $n \to \infty$, the numerator above is $O(\sum_{k=1}^{n-M} T_k \log T_k)$. Recall that $T_k \sim \mathrm{e}^{Ck}$ in exponential cooling, we have $\sum_{k=1}^{n} T_k \log T_k \sim \frac{\mathrm{e}^C}{\mathrm{e}^C - 1} Cn\mathrm{e}^{Cn}$. Hence, the RHS of (2.134) is bounded from above by $O(\mathrm{e}^{-CM})$ which converges to zero as long as $n \to \infty$ first and then $M \to \infty$.

To find the weak limit of the dominant part of the decomposition, notice that

$$\frac{\sum_{k=n-M+1}^{n} \tilde{Z}_{T_k}}{b\sqrt{\sum_{k=1}^{n} T_k \log T_k}} = \sum_{k=n-M+1}^{n} \frac{\tilde{Z}_{T_k}}{b\sqrt{T_k \log T_k}} \sqrt{\frac{T_k \log T_k}{\sum_{k=1}^{n} T_k \log T_k}} \tag{2.135}$$

holds for fixed $M$. As $n \to \infty$, for each $k \in (n-M, n]$, $\tilde{Z}_{T_k}/b(\sqrt{T_k \log T_k})$ converges to a standard Gaussian. Moreover, by the independence of the segments $\tilde{Z}_{T_k}$ in RWCRE, since the sum only takes $M$ terms, the weak limit of the sum is another Gaussian with variance

$$b^2 \lim_{n\to\infty} \frac{\sum_{k=n-M+1}^{n} T_k \log T_k}{\sum_{k=1}^{n} T_k \log T_k} = b^2(1 - \mathrm{e}^{-CM}) \tag{2.136}$$

for $T_k \sim \mathrm{e}^{Ck}$.

Finally, combine (2.134) and (2.136), by Theorem 3.2 in [55], we finish the proof. □

# 3. VARIABLE SPEED SYMMETRIC RANDOM WALK DRIVEN BY SYMMETRIC EXCLUSION

## 3.1 Model and statement of the theorem

Let $\rho, \lambda \in [0,1]$ and $T > 0$ be fixed throughout this section. Denote by $\mu = \bigotimes_{x\in\mathbb{Z}} \mathrm{Ber}(\rho)$ the probability measure on $\{0,1\}^{\mathbb{Z}}$ under which the random variables $\{\eta_x\}_{x\in\mathbb{Z}}$ are i.i.d. of mean $\rho$. We consider a nearest-neighbour random walk on $\mathbb{Z}$, driven by the simple symmetric exclusion process (SSEP) with initial distribution $\mu$. Define the joint law of the random walk and the SSEP by the Markov generator

$$\begin{aligned} L^{\text{joint}} f(\eta, x) = &\sum_{y\in\mathbb{Z}} \left[ f\left(\eta^{y,y+1}, x\right) - f(\eta, x) \right] \\ &+ \left[(1-\lambda)\eta_x + (1-\eta_x)\right]\left[f(\eta, x+1) + f(\eta, x-1) - 2f(\eta, x)\right] \end{aligned} \tag{3.1}$$

acting on local functions $f : \mathbb{Z} \times \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ (a function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ is called *local* if $f(\eta)$ is a function of finitely many of the variables $\{\eta_x\}_{x\in\mathbb{Z}}$). The random walk jumps from a particle at rate $1-\lambda$ and from a hole at rate 1 to one of its neighbors.

For $k \in \mathbb{Z}$ and $\eta \in \{0,1\}^{\mathbb{Z}}$, let $\theta_k\eta$ denote the element of $\{0,1\}^{\mathbb{Z}}$ defined by $(\theta_k\eta)_x = \eta_{x+k}$. We use this to define the environment process viewed from the walk $\xi(t) = \theta_{X_t}\eta(t)$. This is a Markov process, and its generator $L$ acts on local functions as follows:

$$Lf(\xi) = L^{ssep} f(\xi) + \left[(1-\lambda)\xi_0 + (1-\xi_0)\right]\left[f(\theta_1\xi) + f(\theta_{-1}\xi) - 2f(\xi)\right], \tag{3.2}$$

where

$$L^{ssep} f(\xi) := \sum_{y\in\mathbb{Z}} \left[ f\left(\xi^{y,y+1}\right) - f(\xi) \right] \tag{3.3}$$

is the generator of the SSEP with rate 1.

Define the quenched probability $P^\eta(\cdot)$ on $\mathbb{Z} \times [0,\infty)$ as the probability measure of the random walk on underlying environment $\eta = \{\eta_t, t \geq 0\}$. By (3.1), we have for $t, h \geq 0$,

$$P^\eta\left(X_{t+h} - X_t = \pm 1 | X_t\right) = h\left[(1-\lambda)\eta_{X_t}(t) + (1-\eta_{X_t}(t))\right] + o(h). \tag{3.4}$$

Define the annealed measure $\mathbb{P}(\cdot)$ on the same space as

$$\mathbb{P}(\cdot) = \int P_\eta(\cdot)\,\mathrm{d}Q_\mu(\eta) \tag{3.5}$$

where $Q_\mu$ is the distribution of SSEP $\{\eta(t)\}_{t\geq 0}$ with the initial distribution $\eta(0) \sim \mu$,

Our main theorem gives a quenched invariance principle of the walk with explicit scaling parameter(the variance).

**Theorem 3.1.1.** *Let $(X_t, \eta(t))_{t\geq 0}$ be the Markov process generated by $L^{\text{joint}}$, started from $X_0 = 0$ and $\eta(0) \sim \mu$. Then, for $Q_\mu-$ almost every $\eta$, under the quenched measure $P^\eta$, the sequence of processes*

$$\left(\frac{X_{nt}}{\sigma(\rho)\sqrt{n}} : t \in [0,T]\right)_{n\in\mathbb{N}} \tag{3.6}$$

*converges in distribution, with respect to the $J_1$ Skorohod topology, to a standard Brownian motion, where*

$$\sigma^2(\rho) = 2 - \frac{4\lambda\rho}{2-\lambda(1-\rho)}. \tag{3.7}$$

This theorem will follow from the next one, which gives the asymptotic fraction of time that the walk spent on top of particles.

**Theorem 3.1.2.** *Keep the assumptions of Theorem 3.1.1. Let $\xi(t) = \theta_{X_t}\eta(t)$. Then, for $Q_\mu-$almost every $\eta$, under the quenched measure $P^\eta$,*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t (2-\lambda\xi_0(s))(\xi_0(s)-\rho)\,\mathrm{d}s = 0 \text{ in probability.} \tag{3.8}$$

*Or equivalently,*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \xi_0(s)\,\mathrm{d}s = \frac{2\rho}{2-\lambda+\lambda\rho} \text{ in probability.} \tag{3.9}$$

Theorem 3.1.2 shows the convergence under the quenched measure, which automatically implies the same convergence result under the annealed measure. Moreover, the rate of convergence under the annealed measure has an upper bound estimation, which is also a key tool to prove Theorem 3.1.2. This rate of convergence result is shown as follows.

**Theorem 3.1.3.** *Keep the assumptions of Theorem 3.1.1. Let $\xi(t) = \theta_{X_t}\eta(t)$. For any $\epsilon > 0$, there exist $T = T(\epsilon) > 0$ and $C = C(\epsilon) > 0$, such that for any $t > T$,*

$$\mathbb{P}\left[\frac{1}{t}\left|\int_0^t (2-\lambda\xi_0(s))(\xi_0(s)-\rho)\,\mathrm{d}s\right| \geq \epsilon\right] \leq Ct^{-\frac{1}{15}}. \tag{3.10}$$

## 3.2 Proofs

The key observation is that $X_t$ is a mean-zero martingale with respect to the filtration generated by $(X_t, \eta(t))_{t\geq 0}$. Its predictable quadratic variation is given by the formula

$$\langle X\rangle_t = \int_0^t 2 - 2\lambda\xi_0(s)\,\mathrm{d}s. \tag{3.11}$$

More explicitly, we have

$$E^\eta\left[X_t^2 - \langle X\rangle_t \,|(X_r, \eta(r)), r \leq s\right] = X_s^2 - \langle X\rangle_s\,, \quad P^\eta - a.s. \tag{3.12}$$

for any $t \geq s \geq 0$ and all $\eta$.

We claim that if $\lim_{t\to\infty} t^{-1}\langle X\rangle_t \to a$ in probability, for some positive $a > 0$, then the sequence $\left(\frac{X_{nt}}{\sqrt{n}} : t \in [0,T]\right)_{n\in\mathbb{N}}$ converges in distribution to a Brownian motion of variance $a$, with respect to the $J_1$ Skorohod topology on the space $\mathcal{D}([0,T];\mathbb{R})$. This follows from the Martingale Functional Central Limit Theorem, [56] Theorem 7.1.4. Therefore we only need to prove that $\lim_{t\to\infty} t^{-1}\int_0^t \xi_0(s)\,\mathrm{d}s$ exists in probability. This follows from Theorem 3.1.2, since if (3.8) holds, then

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t (2-\lambda+\lambda\rho)\,\xi_0(s)\,\mathrm{d}s = 2\rho \text{ in probability}, \tag{3.13}$$

whence $\lim_{t\to\infty} t^{-1}\int_0^t \xi_0(s)\,\mathrm{d}s = \frac{2\rho}{2-\lambda+\lambda\rho}$.

Although in Theorem 3.1.2 the convergence holds quenched, we will prove the convergence in the annealed measure first. Our proof will yield a estimate on the rate of convergence that is strong enough that allows us to deduce the quenched convergence from it.

Before we start our proofs, we remind the readers that there are some technical lemmas that will be used throughout the proofs. Those lemmas are introduced in section 4 as well as their proofs. But we will use them in section 3 without mentioning too much in order to make the proof less tedious.

### 3.2.1 Proof of the asymptotic limit of $\xi(t)$ under the annealed measure

Our goal is to prove the following theorem.

**Theorem 3.2.1.** *Under the assumptions of Theorem 3.1.1, under the annealed measure $\mathbb{P}$,*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t (2-\lambda\xi_0(s))(\xi_0(s)-\rho)\,\mathrm{d}s = 0 \text{ in probability.} \tag{3.14}$$

Given $x \in \mathbb{Z}$ and $\ell \in \mathbb{N}$, denote

$$\overrightarrow{\xi}_x^{\ell} := \frac{\xi_{x+1}+\cdots+\xi_{x+\ell}}{\ell}, \quad \overleftarrow{\xi}_x^{\ell} := \frac{\xi_{x-1}+\cdots+\xi_{x-\ell}}{\ell}. \tag{3.15}$$

For any choice of positive integers $\ell$ and $n$ one can write

$$\frac{1}{t}\int_0^t (2-\lambda\xi_0(s))(2\xi_0(s)-2\rho)\,\mathrm{d}s \tag{3.16}$$

$$=\frac{1}{t}\int_0^t (2-\lambda\xi_0(s))(2\xi_0(s)-\xi_n(s)-\xi_{-n}(s))\,\mathrm{d}s \tag{3.17}$$

$$+\frac{1}{t}\int_0^t (2-\lambda\xi_0(s))(\xi_n(s)-\overrightarrow{\xi}_n^{\ell}(s)+\xi_{-n}(s)-\overleftarrow{\xi}_{-n}^{\ell}(s))\,\mathrm{d}s \tag{3.18}$$

$$+\frac{1}{t}\int_0^t (2-\lambda\xi_0(s))(\overrightarrow{\xi}_n^{\ell}(s)+\overleftarrow{\xi}_n^{\ell}(s)-2\rho)\,\mathrm{d}s \tag{3.19}$$

We are going to choose $n$ and $\ell$ depending on $t$ in such a way that all three integrals on the right-hand side converge to 0 in probability, as $t\to\infty$. It turns out one can choose

$$n = \lfloor t^{\alpha} \rfloor \text{ for some } \alpha \in \left(\frac{1}{2},\frac{2}{3}\right), \quad 1 \ll \ell \ll \frac{t}{n}. \tag{3.20}$$

**Proposition 3.2.1.** *Under the assumption of Theorem 3.1.1, assume (3.20). Under the annealed measure*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t (2-\lambda\xi_0(s))(2\xi_0(s)-\xi_n(s)-\xi_{-n}(s))ds = 0 \tag{3.21}$$

*in probability.*

The proof strategy is to show that the integrand is in the range of the generator and use this to rewrite the integral as the sum of a martingale and a vanishing term. The martingale is then shown to vanish too, by means of an explicit bound on its quadratic variation.

Thus we seek a function $\psi_{n,\ell}$ such that $L\psi_{n,\ell}(\xi) = (2-\lambda\xi_0)(2\xi_0 - \xi_n - \xi_{-n})$. We start the search by computing

$$\begin{aligned} L\xi_x &= [\xi_{x+1}+\xi_{x-1}-2\xi_x]\,[\xi_0(1-\lambda)+(1-\xi_0)] + (\xi_{x+1}-\xi_x) + (\xi_{x-1}-\xi_x) \\ &= (2-\lambda\xi_0)(\xi_{x+1}+\xi_{x-1}-2\xi_x). \end{aligned} \tag{3.22}$$

Let $k>0$. Sum from $x=-k+1$ to $x=k-1$ to get

$$L\left(\sum_{x=-k+1}^{k-1} \xi_x\right) = (2-\lambda\xi_0)\left(\xi_k - \xi_{k-1} + \xi_{-k} - \xi_{-k+1}\right). \tag{3.23}$$

Sum from $k=1$ to $k=n$ to get

$$L\left(\sum_{k=1}^{n}\sum_{x=-k+1}^{k-1} \xi_x\right) = (2-\lambda\xi_0)\left(-2\xi_0 + \xi_n + \xi_{-n}\right). \tag{3.24}$$

Define

$$\psi_{n,\ell}(\xi) := -\sum_{k=1}^{n}\sum_{x=-k+1}^{k-1} (\xi_x - \rho), \tag{3.25}$$

the following process is a mean zero martingale with respect to the filtration generated by $\xi(s)_{s\geq 0}$:

$$M_s(\psi_{n,\ell}) := \psi_{n,\ell}(\xi(s)) - \psi_{n,\ell}(\xi(0)) - \int_0^s (2-\lambda\xi_0(r))(2\xi_0(r)-\xi_n(r)-\xi_{-n}(r))\,\mathrm{d}r. \tag{3.26}$$

We need separate arguments to control the terms $\frac{\psi_{n,\ell}(\xi(t))-\psi_{n,\ell}(\xi(0))}{t}$ and $\frac{1}{t}M_t(\psi_{n,\ell})$.

**Lemma 3.2.2.** *Under the assumptions of Theorem 3.1.1, assume (3.20). With $\psi_{n,\ell}$ given by (3.25),*

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}|\psi_{n,\ell}(\xi(t))| = 0.$$

*Proof.* Rewrite $\psi_{n,\ell}(\xi) = n(\xi_0-\rho)+\sum_{k=1}^{n}(n-k)(\xi_k+\xi_{-k}-2\rho)$. It suffices to prove

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\Big|\sum_{k=1}^{n}(n-k)(\xi_k(t)-\rho)\Big| = 0. \tag{3.27}$$

Notice that the trivial pointwise bound is of order $n^2$, which is much bigger than $t$. The idea is that when $k$ is large the variables $\xi_k(t)-\rho$ are approximately independent and have mean zero. Recall that $\xi_x(t)=\eta_{x+X_t}(t)$, where $\eta(t)$ is a stationary SSEP and $X_t$ is the random walk. Then

$$\mathbb{E}\Big|\sum_{k=1}^{n}(n-k)(\xi_k(t)-\rho)\Big| \le n^2\mathbb{P}\left(|X_t|>n\right)+\mathbb{E}\Big|\sup_{|\mathrm{j}|\le n}\sum_{k=1}^{n}(n-k)\left(\eta_{k+\mathrm{j}}(t)-\rho\right)\Big|. \tag{3.28}$$

By Lemma 3.3.3, the first term is of order $t^3n^{-4}$. It then follows from our assumption (3.20) that $\lim_{t\to\infty}t^{-1}n^2\mathbb{P}\left(|X_t|>n\right)=0$, as we need.

To bound the second term, write

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}\Big|\sup_{|\mathrm{j}|\le n}\sum_{k=1}^{n}(n-k)\left(\eta_{k+\mathrm{j}}(t)-\rho\right)\Big| \\
=&\int_0^\infty \mathbb{P}\left(\Big|\sup_{|\mathrm{j}|\le n}\sum_{k=1}^{n}(n-k)\left(\eta_{k+\mathrm{j}}(t)-\rho\right)\Big|>\beta t\right)\,\mathrm{d}\beta \\
\le&\delta+\sum_{|\mathrm{j}|\le n}\int_\delta^\infty \mathbb{P}\left(\Big|\sum_{k=1}^{n}(n-k)\left(\eta_{k+\mathrm{j}}(t)-\rho\right)\Big|>\beta t\right)\,\mathrm{d}\beta \\
\le&\delta+2\sum_{|\mathrm{j}|\le n}\int_\delta^\infty \exp\left(-\frac{t^2}{n^3}\frac{\beta^2}{2}\right)\,\mathrm{d}\beta \\
\le&\delta+12\frac{n^{\frac{5}{2}}}{t}\cdot\exp\left(-\frac{t^2}{n^3}\frac{\delta^2}{2}\right) \\
=&\delta+12\,t^{\frac{5\alpha}{2}-1}\cdot\exp\left(-\frac{\delta^2t^{2-3\alpha}}{2}\right).
\end{aligned} \tag{3.29}$$

The fourth line is by Lemma 3.3.1, the fifth line is by lemma 3.75, and the last line is by (3.20).

Now choose $\delta = t^{-(\frac{2}{3}-\alpha)}$, we get an upper bound of $\mathbb{E}\Big|\sum_{k=1}^{n}(n-k)(\xi_k(t)-\rho)\Big|$ as

$$\frac{1}{t}\mathbb{E}\Big|\sum_{k=1}^{n}(n-k)(\xi_k(t)-\rho)\Big| \leq c_0\left(t^{2-4\alpha}+t^{\alpha-\frac{2}{3}}\right) \tag{3.30}$$

for some constant $c_0 > 0$ and $t$ large enough. Let $t \to \infty$, the right hand side converges to zero, this finishes the proof of (3.27). $\square$

The next lemma controls $\frac{1}{t}M_t(\psi_{n,\ell})$.

**Lemma 3.2.3.** *Under the assumptions of Theorem 3.1.1, assume* (3.20). *With $\psi_{n,\ell}$ given by* (3.25) *and $M_t(\psi_{n,\ell})$ given by* (3.26),

$$\lim_{t\to\infty} t^{-2}\mathbb{E}\left[M_t^2(\psi_{n,\ell})\right] = 0. \tag{3.31}$$

*Proof.* There is an explicit formula for the predictable quadratic variation of $M_t(\psi_{n,\ell})$:

$$\langle M_\cdot(\psi_{n,\ell})\rangle_t = \int_0^t \sum_{x\in\mathbb{Z}} \left[\psi_{n,\ell}\left(\xi^{x,x+1}(s)\right) - \psi_{n,\ell}\left(\xi(s)\right)\right]^2 \mathrm{d}s \tag{3.32}$$

$$+ \int_0^t (1-\lambda\xi_0(s))\left[\psi_{n,\ell}\left(\theta_1\xi(s)\right) - \psi_{n,\ell}\left(\xi(s)\right)\right]^2 \mathrm{d}s \tag{3.33}$$

$$+ \int_0^t (1-\lambda\xi_0(s))\left[\psi_{n,\ell}\left(\theta_{-1}\xi(s)\right) - \psi_{n,\ell}\left(\xi(s)\right)\right]^2 \mathrm{d}s. \tag{3.34}$$

Our goal is to prove $\lim_{t\to\infty} t^{-2}\mathbb{E}\left\langle M_\cdot(\psi_{n,\ell})\right\rangle_t = 0$. To bound the first term, notice that $\left[\psi_{n,\ell}\left(\xi^{x,x+1}\right) - \psi_{n,\ell}\left(\xi\right)\right]^2 = 0$ if $|x| > n$ and no greater than 1 if $|x| \leq n$, so the integrand is much smaller than $2tn$. The second term demands more work while the third term has the similar proof as the second one. To start, we compute

$$-\psi_{n,\ell}(\theta_1\xi) + \psi_{n,\ell}(\xi) = \sum_{k=1}^{n}\xi_k - \sum_{k=-n+1}^{0}\xi_k. \tag{3.35}$$

It is enough to prove

$$\lim_{t\to\infty} \sup_{s\le t} t^{-1}\mathbb{E}\left[\left(\sum_{k=1}^{n} \xi_k(s) - \sum_{k=-n+1}^{0} \xi_k(s)\right)^2\right] = 0. \tag{3.36}$$

The expectation above is small by the same reason that (3.27) is small: the random variables $\xi_k(s)$, for large $k$, are approximately independent of mean $\rho$. We follow the same method of proof.

$$\begin{aligned} & t^{-1}\mathbb{E}\left[\left(\sum_{k=1}^{n} \xi_k(s) - \sum_{k=-n+1}^{0} \xi_k(s)\right)^2\right] \\ & \le \frac{n^2}{t}\mathbb{P}\left(|X_s| > n\right) + t^{-1}\mathbb{E}\left[\sup_{|\mathrm{j}|\le n}\left(\sum_{k=1}^{n} \eta_{k+\mathrm{j}}(s) - \sum_{k=-n+1}^{0} \eta_{k+\mathrm{j}}(s)\right)^2\right] \end{aligned} \tag{3.37}$$

By Lemma 3.3.3, the first term is of order $\frac{t^2}{n^4}$, so it vanishes as $t \to \infty$. The second term is bounded, for any $\delta > 0$, by

$$\begin{aligned} & \delta + \sum_{|\mathrm{j}|\le n} \int_{\delta}^{\infty} \mathbb{P}\left(\left[\sum_{k=1}^{n} \eta_{k+\mathrm{j}}(s) - \sum_{k=-n+1}^{0} \eta_{k+\mathrm{j}}(s)\right]^2 \ge \beta t\right) \mathrm{d}\beta \\ \le & \, \delta + 2\sum_{|\mathrm{j}|\le n} \int_{\delta}^{\infty} \exp\left(-\frac{\beta t}{10n}\right) \mathrm{d}\beta \\ \le & \, \delta + \frac{60n^2}{t} \exp\left(-\frac{\delta t}{10n}\right) \\ = & \, \delta + 60t^{2\alpha-1} \exp\left(-\frac{\delta t^{1-\alpha}}{10}\right). \end{aligned} \tag{3.38}$$

The second line is by Lemma 3.3.1. Choose $\delta = t^{-\frac{1-\alpha}{2}}$, we then get an upper bound

$$t^{-1}\mathbb{E}\left[\left(\sum_{k=1}^{n} \xi_k(s) - \sum_{k=-n+1}^{0} \xi_k(s)\right)^2\right] \le c_1\left(\frac{t^2}{n^4} + t^{\frac{\alpha-1}{2}}\right) \tag{3.39}$$

for some constant $c_1 > 0$ and $t$ large enough.

Collect all the above upper bounds we have

$$\frac{1}{t^2}\mathbb{E}\left[M_t^2(\psi_{n,\ell})\right] \le \frac{2n}{t} + c_1\left(\frac{t^2}{n^4} + t^{\frac{\alpha-1}{2}}\right) = 2t^{\alpha-1} + c_1(t^{2-4\alpha} + t^{\frac{\alpha-1}{2}}). \tag{3.40}$$

By the assumption 3.20, the upper bound vanishes as $t \to \infty$.

$\square$

*Proof of Proposition 3.2.1.* By Chebyshev inequality, for any $\epsilon > 0$, notice that $\xi(0) = \eta(0)$, there exists some constant $c_2 > 0$ such that

$$\mathbb{P}\left[\left|\frac{\psi_{n,\ell}(\xi(0))}{t}\right| \geq \epsilon\right] \leq \frac{\mathbb{E}\left[\psi^2_{n,\ell}(\xi(0))\right]}{\epsilon^2 t^2} = \frac{\mathbb{E}\left[\left(\sum_{k=1}^n (n-k)(\eta_k(0)-\rho)\right)^2\right]}{\epsilon^2 t^2} \leq \frac{c_2}{\epsilon^2} t^{2\alpha-2}. \quad (3.41)$$

The last inequality uses the fact that $\{\eta_k(0) - \rho\}_{k\in\mathbb{Z}}$ is an i.i.d mean zero sequence. The cross terms above will vanish after taking the expectation.

Use this upper bound, together with (3.26), (3.30), and (3.40) for any $\epsilon > 0$,

$$\mathbb{P}\left[\frac{1}{t}\left|\int_0^t (2-\lambda\xi_0(s))(2\xi_0(s)-\xi_n(s)-\xi_{-n}(s))\,\mathrm{d}s\right| \geq \epsilon\right] \leq C_0(\epsilon)t^{\gamma_1} \quad (3.42)$$

where constant $C_0(\epsilon) > 0$ and

$$\gamma_1 = \max\left\{2\alpha-2, \alpha-1, \frac{\alpha-1}{2}, 2-4\alpha, \alpha-\frac{2}{3}\right\} < 0 \quad (3.43)$$

due to assumption (3.20). Hence Proposition 3.2.1 is proved. $\square$

The next proposition shows the limit of the second part of the decomposition (3.16).

**Proposition 3.2.2.** *Under the assumption of Theorem 3.1.1, assume* (3.20)*. Under the annealed measure,*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t (2-\lambda\xi_0(s))(\xi_n(s) - \overrightarrow{\xi}^{\ell}_{n}(s) + \xi_{-n}(s) - \overleftarrow{\xi}^{\ell}_{-n}(s))\,\mathrm{d}s = 0 \quad (3.44)$$

*in probability.*

*Proof.* We show that the integrand is in the range of the generator and split the integral into a martingale term plus a vanishing term. Notice that

$$\xi_x - \overrightarrow{\xi}^{\ell}_{x} = \sum_{\mathrm{j}=0}^{\ell-1}\frac{\ell-\mathrm{j}}{\ell}(\xi_{x+\mathrm{j}} - \xi_{x+\mathrm{j}+1}) \quad (3.45)$$

and

$$\xi_x - \overleftarrow{\xi}^{\ell}_x = \sum_{\mathrm{j}=0}^{\ell-1} \frac{\ell-\mathrm{j}}{\ell}(\xi_{x-\mathrm{j}} - \xi_{x-\mathrm{j}-1}). \tag{3.46}$$

From (3.23), we get

$$(2-\lambda\xi_0)\left(\xi_n - \overrightarrow{\xi}^{\ell}_n + \xi_{-n} - \overleftarrow{\xi}^{\ell}_{-n}\right) = -L\varphi_{n,\ell}(\xi), \tag{3.47}$$

where

$$\varphi_{n,\ell}(\xi) := \sum_{\mathrm{j}=0}^{\ell-1} \frac{\ell-\mathrm{j}}{\ell} \sum_{x=-n-\mathrm{j}}^{n+\mathrm{j}} \xi_x. \tag{3.48}$$

The process

$$M_s(\varphi_{n,\ell}) := \varphi_{n,\ell}(\xi(s)) - \varphi_{n,\ell}(\xi(0)) - \int_0^s L\varphi_{n,\ell}(\xi(r))\,\mathrm{d}r \tag{3.49}$$

is a martingale with respect to the filtration generated by $(\xi(s))_{s\geq 0}$. To prove (3.44), we show that $|\varphi_{n,\ell}| \ll t$ and $\langle M.(\varphi_{n,\ell})\rangle_t \ll t^2$. For the first term,

$$|\varphi_{n,\ell}(\xi)| \leq \sum_{\mathrm{j}=0}^{\ell-1} \frac{\ell-\mathrm{j}}{\ell}(2n+2\mathrm{j}+1) \leq C\left(\ell n + \ell^2\right) \tag{3.50}$$

for some $C>0$, so it follows from (3.20) that $\lim_{t\to\infty} t^{-1}|\varphi_{n,\ell}(\xi)| = 0$ for any $\xi \in \{0,1\}^{\mathbb{Z}}$.

It remains to prove that $t^{-1}M_t(\varphi_{n,\ell}) \to 0$ in probability. We prove this by controlling the second moment of $M_t(\varphi_{n,\ell})$ through its predictable quadratic variation

$$\begin{aligned}\langle M.(\varphi_{n,\ell})\rangle_t = &\int_0^t \sum_{x\in\mathbb{Z}} \left[\varphi_{n,\ell}\left(\xi^{x,x+1}(s)\right) - \varphi_{n,\ell}(\xi)(s)\right]^2 \mathrm{d}s + \\ &+ \int_0^t (1-\lambda\xi_0(s))\left[\varphi_{n,\ell}(\theta_1\xi(s)) - \varphi_{n,\ell}(\xi(s))\right]^2 \mathrm{d}s \\ &+ \int_0^t (1-\lambda\xi_0(s))\left[\varphi_{n,\ell}(\theta_{-1}\xi(s)) - \varphi_{n,\ell}(\xi(s))\right]^2 \mathrm{d}s.\end{aligned} \tag{3.51}$$

We claim that,

$$\lim_{t\to\infty} t^{-2}\langle M.(\varphi_{n,\ell})\rangle_t = 0. \tag{3.52}$$

Let $a_k := \sum_{\mathrm{j}=k}^{\ell-1} \frac{\ell-\mathrm{j}}{\ell}$. Then

$$\varphi_{n,\ell}(\xi) = a_0 \sum_{\mathrm{j}=-n}^{n} \xi_{\mathrm{j}} + \sum_{k=1}^{\ell-1} a_k \left(\xi_{n+k} + \xi_{-n-k}\right). \tag{3.53}$$

It's easy to see that

$$\left[\varphi_{n,\ell}\left(\xi^{x,x+1}\right) - \varphi_{n,\ell}\left(\xi\right)\right]^2 \le \sum_{k=0}^{\ell-1} 1_{|x|=n+k}(a_k - a_{k+1})^2 \le \ell \tag{3.54}$$

and

$$\left[\varphi_{n,\ell}\left(\theta_1\xi\right) - \varphi_{n,\ell}\left(\xi\right)\right]^2 \le (2a_0)^2 \le C\ell^2 \tag{3.55}$$

for some $C > 0$ independent of $\ell$ and $n$.

These bounds imply

$$\langle M_\cdot(\varphi_{n,\ell})\rangle_t \le 3\left(tl + Ct\ell^2\right). \tag{3.56}$$

Hence, by (3.49), (3.50), (3.56), for any $\epsilon > 0$,

$$\mathbb{P}\left[\frac{1}{t}\left|\int_0^t (2-\lambda\xi_0(s))(\xi_n(s) - \overrightarrow{\xi}^{\,\ell}_{n}(s) + \xi_{-n}(s) - \overleftarrow{\xi}^{\,\ell}_{-n}(s))\,\mathrm{d}s\right| \ge \epsilon\right] \le C_1(\epsilon)\frac{l + ln + l^2}{t} \tag{3.57}$$

for some $C_1(\epsilon) > 0$. By (3.20) the right hand side of (3.57) indeed converges to zero. □

**Remark 3.2.4.** *Proposition 3.2.2 gives the convergence under the annealed measure. But one can see from the key upper bounds (3.50) and (3.56) are deterministic. This implies that the convergence holds not only in the annealed sense, but also in the quenched sense, i.e. under $P^\eta$ for all $\eta \in [0,1]^{\mathbb{N}} \times \mathbb{R}^+$.*

**Proposition 3.2.3.** *Under the assupmtion of Theorem 3.1.1, assume (3.20). Under the annealed measure*

$$\frac{1}{t}\int_0^t (2-\lambda\xi_0(s))(\overrightarrow{\xi}^{\,\ell}_{n}(s) - \rho)\,\mathrm{d}s \to 0 \tag{3.58}$$

*in probability as $t \to \infty$. The same holds if $\overrightarrow{\xi}^{\,\ell}_{n}$ is replaced by $\overleftarrow{\xi}^{\,\ell}_{-n}$.*

*Proof.* Define, for $m > 0$, the event

$$A_m := \left\{\max_{s\le t} |X_s| < m\right\}. \tag{3.59}$$

Then

$$\begin{aligned}&\mathbb{E}\left[\left(\frac{1}{t}\int_0^t(2-\lambda\xi_0(s))(\overrightarrow{\xi}^{\ell}_{n}(s)-\rho)\,\mathrm{d}s\right)^2\right]\\&\leq 4\mathbb{P}(A_m^c)+\frac{1}{t}\int_0^t\mathbb{E}\left[1_{A_m}(2-\lambda\xi_0(s))^2(\overrightarrow{\xi}^{\ell}_{n}(s)-\rho)^2\right]\,\mathrm{d}s\end{aligned}\tag{3.60}$$

We will prove that, for $t^{\frac{1}{2}} \ll m \ll n$, the upper bound in the last equation vanishes as $t \to \infty$. To bound the second term, we apply the Lateral Decoupling Lemma ([15], Proposition 4.1). To do so, we need the random variable inside the expectation to be a function of the exclusion process only. Thus we rewrite the expectation as

$$\begin{aligned}&\mathbb{E}\left[1_{A_m}(2-\lambda\xi_0(s))^2(\overrightarrow{\xi}^{\ell}_{n}(s)-\rho)^2\right]\\&=\sum_{|k|<m}\mathbb{E}\left[(2-\lambda\eta_k(s))^2\mathbb{E}\left(1_{A_m,X_s=k}|\mathcal{F}_t\right)(\overrightarrow{\eta}^{\ell}_{n+k}(s)-\rho)^2\right],\end{aligned}\tag{3.61}$$

where $\mathcal{F}_t$ is the filtration generated by $(\eta_s)_{s\in[0,t]}$. If $m \ll n$ and (3.20) holds, we can apply Proposition 3.3.1 with $H = t$, $f_1(\eta) = (2-\lambda\eta_k(s))^2\frac{1}{2}\mathbb{E}\left(1_{A_m,X_s=k}|\mathcal{F}_t\right)$ and $f_2(\eta) = (\overrightarrow{\eta}^{\ell}_{n+k}(s)-\rho)^2$ for all $|k| < m$. Note that the support of $f_1$ is contained in $[-m, m]\times[0, t] \subset [m-t, m]\times[0, t]$, and the support of $f_2$ is contained in $[n+k, n+k+\ell]\times[0, t] \subset [n+k, n+k+t]\times[0, t]$, and by (3.20) and the assumption that $t^{1/2} \ll m \ll n$ the horizontal separation of these boxes is $n+k-m \gg t^{\alpha'}$ for any $\alpha' \in (\frac{1}{2}, \alpha)$. Therefore, applying Proposition (3.3.1) it holds that

$$\begin{aligned}&\mathbb{E}\left[1_{A_m}(2-\lambda\xi_0(s))^2(\overrightarrow{\xi}^{\ell}_{n}(s)-\rho)^2\right]\\&=\sum_{|k|<m}\mathbb{E}\left[(2-\lambda\eta_k(s))^2\mathbb{E}\left(1_{A_m,X_s=k}|\mathcal{F}_t\right)(\overrightarrow{\eta}^{\ell}_{n+k}(s)-\rho)^2\right]\\&\leq 4m\cdot\exp\left(-t^{2\alpha'-1}\right)+\sum_{|k|<m}\mathbb{E}\left[1_{A_m,X_s=k}(2-\lambda\eta_k(s))^2\right]\mathbb{E}\left[(\overrightarrow{\eta}^{\ell}_{n+k}(s)-\rho)^2\right]\\&\leq 4m\cdot\exp\left(-t^{2\alpha'-1}\right)+\frac{4}{\ell}.\end{aligned}\tag{3.62}$$

Using this bound in (3.61), together with Lemma (3.3.3), we get

$$\mathbb{E}\left[\left(\frac{1}{t}\int_0^t(2-\lambda\xi_0(s))(\overrightarrow{\xi}^{\ell}_{n}(s)-\rho)\,\mathrm{d}s\right)^2\right]\leq 4m\cdot\exp\left(-t^{2\alpha'-1}\right)+\frac{4}{\ell}+\frac{c_3t^3}{m^6}\tag{3.63}$$

for some $c_3 > 0$ and $t$ large enough. If $t^{\frac{1}{2}} \ll m \ll n$ and (3.20) holds then the upper bound vanishes as $t \to \infty$. By Chebyshev inequality, this finishes the proof.

□

One can get Theorem 3.2.1 immediately from propositions 3.2.1, 3.2.2, and 3.2.3.

### 3.2.2 Proof of the asymptotic limit of $\xi(t)$ under the quenched measure

First recall (3.42), (3.57) and (3.63). By choosing adequate $\alpha, \ell$ and $m$, one can get an explicit upper bound on the rate of convergence in (3.8).

*Proof of Theorem 3.1.3.* Let $\alpha = 0.6$, $\ell = t^{0.2}$ and $m = t^{0.55}$. Then,for any $\epsilon > 0$ and for large enough $t$, one can check

$$\mathbb{P}\left[\frac{1}{t}\left|\int_0^t (2-\lambda\xi_0(s))(\xi_0(s)-\rho)\,\mathrm{d}s\right| \geq \epsilon\right] \leq C(\epsilon)t^{-\frac{1}{15}} \tag{3.64}$$

for some $C(\epsilon) > 0$. □

The next lemma shows how to get the convergence in probability under the quenched measure $Q_\mu - a.s.$ from the annealed measure.

**Lemma 3.2.5.** *Under the assumptions of Theorem 3.1.1, let* $Y_t = \int_0^t (2-\lambda\xi_0(s))(\xi_0(s)-\rho)\,\mathrm{d}s$ *for* $t > 0$ *and* $Q_\mu$ *defined in section 2. Then for any* $\epsilon, \delta > 0$*, there exists* $t_\eta(\epsilon,\delta) > 0$ *such that*

$$Q_\mu\left[\{P^\eta\left(|Y_t| \geq \epsilon t\right) < \delta\} \text{ for } \forall t > t_\eta(\epsilon,\delta)\right] = 1. \tag{3.65}$$

*Proof.* Define a sequence $\{t_k\}_{k\geq 1}$ as $t_k = k^{16}$. By (3.64), we have for $k$ large enough,

$$\mathbb{P}\left[|Y_{t_k}| \geq \epsilon t_k\right] \leq C(\epsilon)k^{-\frac{16}{15}}. \tag{3.66}$$

By Chebyshev inequality,

$$Q_\mu\left[P^\eta\left[|Y_{t_k}| \geq \epsilon t_k\right] \geq \delta\right] \leq \frac{1}{\delta}\mathbb{P}\left[|Y_{t_k}| \geq \epsilon t_k\right] \leq \frac{C(\epsilon)}{\delta}k^{-\frac{16}{15}}. \tag{3.67}$$

the upper bound is summable for $k$. Thus by Borel-Cantelli lemma,

$$Q_\mu\left[\left\{P^\eta\left[|Y_{t_k}| \geq \epsilon t_k\right] \geq \delta\right\} \text{i.}o.\right] = 0. \tag{3.68}$$

For any $t \geq 1$, it must lie in the interval $[t_k, t_{k+1})$ for some $k$. Notice that $Y_t$ has bounded increments, which means $|Y_s - Y_r| \leq 2|s - r|$ for any $s, r > 0$. This gives the upper bound

$$\frac{|Y_t|}{t} \leq \frac{|Y_{t_k}| + 2(t_{k+1} - t_k)}{t_k}. \tag{3.69}$$

Let $k_\epsilon > 0$ satisfy $2(t_{k_\epsilon+1} - t_{k_\epsilon})t_{k_\epsilon}^{-1} < \epsilon$, for any $k > k_\epsilon$ and $t \in [t_k, t_{k+1})$, $\{|Y_t|/t \geq 2\epsilon\}$ implies $\{|Y_{t_k}|/t_k \geq \epsilon\}$. Define $A_{\epsilon,\delta}$ that $A_{\epsilon,\delta}^c = \left\{\left\{P^\eta\left[|Y_{t_k}| \geq \epsilon t_k\right] \geq \delta\right\} \text{i.}o.\right\}$. Choose any $\eta \in A_{\epsilon,\delta}$, there exists $k_\eta(\epsilon, \delta)$ such that for all $k > k_\eta(\epsilon, \delta)$

$$P^\eta\left[|Y_{t_k}| \geq \epsilon t_k\right] < \delta. \tag{3.70}$$

Pick $t_\eta(2\epsilon, \delta) = t_{k_\epsilon} \vee t_{k_\eta(\epsilon,\delta)}$ then by the above argument we have for all $t \in [t_k, t_{k+1}), k \geq k_\epsilon \vee k_\eta(\epsilon, \delta)$,

$$P^\eta\left[|Y_t| \geq 2\epsilon t\right] \leq P^\eta\left[|Y_{t_k}| \geq \epsilon t_k\right] < \delta \tag{3.71}$$

which finishes the proof since $P_\mu\left(A_{\epsilon,\delta}\right) = 1$. □

In the last part of this section we prove Theorem 3.1.2.

*Proof of Theorem 3.1.2.* From Lemma 3.2.5, we just need one more step to reach our final goal. To see this, for any $\epsilon > 0$, let

$$A_\epsilon = \bigcap_{n=1}^{\infty} A_{\epsilon,\frac{1}{n}}. \tag{3.72}$$

We have $P_\mu\left(A_\epsilon\right) = 1$ since it is a intersection of countably many sets while each has probability 1. Choose any $\eta \in A_\epsilon$, for any $n \geq 1$,

$$P^\eta\left[|Y_t| \geq \epsilon t\right] < \frac{1}{n} \tag{3.73}$$

holds for all $t > t_\eta(\epsilon, \frac{1}{n})$. Thus $t^{-1}|Y_t|$ converge to zero in probability under $P^\eta$. $\square$

### 3.3 Technical lemmas

**Lemma 3.3.1** ([57], Theorem 2.8)**.** *Let $\zeta_1, \zeta_2, \ldots$ be i.i.d. random variables with $|\zeta_1| \leq 1$ and $\mathbb{E}\zeta_1 = 0$. Then, for any $\lambda > 0$,*

$$\mathbb{E}\left(\left|\sum_{\mathrm{j}\leq n} b_{\mathrm{j}}\zeta_{\mathrm{j}}\right| > \lambda\right) \leq 2 \cdot \exp\left(-\frac{\lambda^2}{2\sum_{\mathrm{j}\leq n} b_{\mathrm{j}}^2}\right). \tag{3.74}$$

**Lemma 3.3.2.** *For any $\delta > 0$,*

$$\int_\delta \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x \leq \sqrt{2\pi\sigma^2} \cdot \exp\left(-\frac{\delta^2}{2\sigma^2}\right). \tag{3.75}$$

*Proof.* For any $\lambda > 0$,

$$\begin{aligned}\int_\delta \exp\left(-\frac{x^2}{2\sigma^2}\right) \mathrm{d}x &\leq \mathrm{e}^{-\lambda\delta}\int_\delta^\infty \exp\left(\lambda x - \frac{x^2}{2\sigma^2}\right) \mathrm{d}x \\ &\leq \exp\left(-\lambda\delta + \frac{\lambda^2\sigma^2}{2}\right)\int_{-\infty}^\infty \exp\left(-\frac{(x-\lambda\sigma^2)^2}{2\sigma^2}\right) \mathrm{d}x \\ &= \sqrt{2\pi\sigma^2}\exp\left(-\lambda\delta + \frac{\lambda^2\sigma^2}{2}\right).\end{aligned} \tag{3.76}$$

Choosing $\lambda = \delta/\sigma^2$ gives the desired bound. $\square$

**Lemma 3.3.3.** *For any positive $\gamma$ and $t$,*

$$\mathbb{P}\left(\sup_{s\leq t}|X_s| \geq \gamma\right) = O\left(\frac{t^3}{\gamma^6}\right). \tag{3.77}$$

*Proof.* The first observation is that $X$ is a martingale, so Doob's $L^p-$inequality gives

$$\mathbb{P}\left(\sup_{s\leq t}|X_s| \geq \gamma\right) \leq \left(\frac{6}{5}\right)^6 \frac{\mathbb{E}\left(X_t^6\right)}{\gamma^6}. \tag{3.78}$$

To bound the sixth moment, we compare our random walk with a simple symmetric walk: let $Y_1, \ldots, Y_n$ be i.i.d. random variables with $\mathbb{P}\left(Y_1 = \pm 1\right) = 1/2$ and let $J_t$ denote the number of times that $X$ jumps during the time interval $[0,t]$. Then $X_t = \sum_{k=1}^{J_t} Y_k$ in distribution, whence

$$\begin{aligned}\mathbb{E}\left(X_t^6\right) &= \mathbb{E}\left(\sum_{\mathrm{i}\le J_t} Y_{\mathrm{i}}^6 + 15 \sum_{\mathrm{i}<\mathrm{j}\le J_t} Y_{\mathrm{i}}^2 Y_{\mathrm{j}}^4 + 90 \sum_{\mathrm{i}<\mathrm{j}<k\le J_t} Y_{\mathrm{i}}^2 Y_{\mathrm{j}}^2 Y_k^2\right) \\ &\le \mathbb{E}\left(J_t + 15 J_t^2 + 90 J_t^3\right).\end{aligned} \tag{3.79}$$

Since $J_t$ is stochastically dominated by a mean $t$ Poisson random variable, the last expectation is bounded by a multiple of $t^3$. □

The next lemma comes from [15]. To get the version stated below, one only needs to change the last line of the original proof, using (3.75).

**Proposition 3.3.1** (Lateral Decoupling, [15] Proposition 4.1)**.** *Let $f_1, f_2 : \{0,1\}^{\mathbb{Z}} \times \mathbb{R}_+ \to [0,1]$ be measurable functions and $H, y, \alpha > 0$. Let $B_1 = [-H, 0] \times [0, H] \subset \mathbb{R}^2$ and $B_2 = [y, y+H] \times [0, H] \subset \mathbb{R}^2$. Assume $f_1$ is supported on $B_1$, that is, if the trajectories $\eta, \eta' : \mathbb{Z} \times \mathbb{R}_+ \to \{0,1\}$ satisfy $\eta_x(s) = \eta'_x(s)$ for all $(x,s) \in B_1$ then $f_1(\eta) = f_1(\eta')$. Assume $f_2$ is supported on $B_2$. Finally, denote by $\mathbb{P}_\rho$ the law of SSEP started from equilibrium at density $\rho \in (0,1)$, that is, started from the product measure $\otimes_{x\in\mathbb{Z}} \mathrm{Ber}(\rho)$. Let $\mathbb{E}_\rho$ be the expectation with respect to $\mathbb{P}_\rho$.*

*Then $y \ge H^\alpha$ implies*

$$\mathbb{E}_\rho\left[f_1 f_2\right] \le \mathbb{E}_\rho\left[f_1\right] \mathbb{E}_\rho\left[f_2\right] + C\,\mathrm{e}^{-H^{2\alpha-1}} \tag{3.80}$$

*for some $C > 0$.*

# 4. MODERATE DEVIATION AND EXIT POINT ESTIMATES FOR SOLVABLE DIRECTED POLYMER MODELS

## 4.1 Model settings and basic properties

Consider 1+1 dimensional polymer models which can be embedded into $\mathbb{N}^2$ lattice. We will denote vectors and collections of vectors with bold letters and numbers with non-bold letters. The standard coordinate basis of $\mathbb{R}^2$ is denoted by $\mathsf{e}_1 = (1,0)$ and $\mathsf{e}_2 = (0,1)$. The origin is denoted by $\mathbf{0} = (0,0)$. The site $(1,1) = \mathsf{e}_1 + \mathsf{e}_2$ is denoted by $\mathbf{1}$. Given $\mathbf{x} \in \mathbb{Z}^2$, we denote the quadrant rooted at $\mathbf{x}$ via $\mathbb{Z}^2_{\geq \mathbf{x}} = \{\mathbf{y} \in \mathbb{Z}^2 : \mathbf{y} \cdot \mathsf{e}_1 \geq \mathbf{x} \cdot \mathsf{e}_1, \mathbf{y} \cdot \mathsf{e}_2 \geq \mathbf{x} \cdot \mathsf{e}_2\}$. We associate to each pair of nearest-neighbor edges of the form $(\mathbf{x} - \mathsf{e}_{\mathrm{i}}, \mathbf{x})$, where $\mathbf{x} \in \mathbb{Z}^2_{>0}$, a weight $W_{\mathbf{x}-\mathsf{e}_{\mathrm{i}},\mathbf{x}} = W^{\mathrm{i}}_{\mathbf{x}}$. To each edge of the form $(\mathbf{x} - \mathsf{e}_1, \mathbf{x})$ where $\mathbf{x} \in \mathbb{Z}_{>0} \times \{0\}$, we assign weights $I_{\mathbf{x}-\mathsf{e}_1,\mathbf{x}} = I_{\mathbf{x}}$; to each edge of the form $(\mathbf{x} - \mathsf{e}_2, \mathbf{x})$, where $\mathbf{x} \in \{0\} \times \mathbb{Z}_{>0}$, we assign weights $J_{x-\mathsf{e}_2,x} = J_x$. In each of the models we consider, the collections of weights $\{(W^1_x, W^2_x), I_{\mathbf{y}}, J_{\mathbf{z}} : \mathbf{x} \in \mathbb{Z}^2_{>0}, \mathbf{y} \in \mathbb{Z}_{>0} \times \{0\}, \mathbf{z} \in \{0\} \times \mathbb{Z}_{>0}\}$ will be mutually independent.

We initially consider four families of weights, which are indexed by parameters $\mu, \nu > 0$ and $z, w \in \mathcal{J}$, where $\mathcal{J}$ is a model-dependent collection of parameters. One of these families, the Beta polymer is split into two cases. The four models are as follows, the definition of gamma and beta distributions are introduced in the appendix.

1. **Log Gamma** In the multi-parameter log Gamma model, abbreviated LG, the distributions of the weights are given for $z, w \in \mathcal{J} = (0, \mu)$ by
$$(W^1, W^2) \sim (X, X), \quad X \sim \mathrm{InvGa}(\mu, \nu), \quad I \sim \mathrm{InvGa}(\mu - w, \nu), \quad J \sim \mathrm{InvGa}(z, \mu).$$

2. **Inverse Beta** In the multi-parameter Inverse Beta model, abbreviated IB, the distributions of the weights are given for $z, w \in (0, \mu) = \mathcal{J}$ by
$$(W^1, W^2) \sim (X, X - 1), X \sim \mathrm{InvBe}(\mu, \nu), I \sim \mathrm{InvBe}(\mu - w, \nu), J \sim \mathrm{Be}(z, \mu + \nu - z) - 1.$$

3. **Gamma** Also called strict-weak model. In the multi-parameter Gamma model, abbreviated G, the distributions of the weights are given for $z, w \in \mathcal{J} = (0, \infty)$ by

$$(W^1, W^2) \sim (X, 1), \quad X \sim \mathrm{Ga}(\mu, \nu), \quad I \sim \mathrm{Ga}(\mu + w, \nu), \quad J \sim \mathrm{InvBe}(z, \mu).$$

4. **Beta** We consider two families of multi-parameter Beta models, parametrized by $z, w \in (0, \infty) = \mathcal{J}$:

   (a) In the first family of multi-parameter Beta models, which we abbreviate by BI,

   $$(W^1, W^2) \sim (X, 1 - X), \quad X \sim \mathrm{Be}(\mu, \nu), \quad I \sim \mathrm{Be}(\mu + w, \nu), \quad J \sim \mathrm{InvBe}(z, \mu).$$

   (b) In the second family of multi-parameter Beta models, which we abbreviate by BII,

   $$(W^1, W^2) \sim (X, 1 - X), \quad X \sim \mathrm{Be}(\mu, \nu), \quad I \sim \mathrm{InvBe}(z, \nu), \quad J \sim \mathrm{Be}(\nu + w, \mu).$$

When $z = w$ in the previous settings, we call the model *increment-stationary* instead of multi-parameter because of the stationary increment property of the partition function which will be introduced below. We will, at times, study the model in which the boundary weights are replaced by the $X$ variables instead of the $I$ and $J$ variables. This model is known as the *bulk* model.

The directed polymer is an up-right path on $\mathbb{Z}^2$ usually starts from $\mathbf{0}$. The weight of a directed polymer is the product of all edge weights that it contains. The partition function $Z^{w,z}_{(a,b)}(m, n)$ is the sum of weights of all admissible directed polymers from $(a, b)$ to $(m, n)$. Write $Z^{w,z}(m, n)$ if the staring point is $\mathbf{0}$ and write $Z^{w}_{(a,b)}(m, n)$ if $w = z$. In the bulk model, $Z_{(a,b)}(m, n)$ without superscript parameters is the partition function. The free energy is then defined as the log of the partition function. Denote the increment of the partition function to be

$$\tilde{I}_{\mathbf{x}} = \frac{Z^{w,z}(\mathbf{x})}{Z^{w,z}(\mathbf{x} - \mathsf{e}_1)}, \quad \tilde{J}_{\mathbf{x}} = \frac{Z^{w,z}(\mathbf{x})}{Z^{w,z}(\mathbf{x} - \mathsf{e}_2)}. \tag{4.1}$$

Note that when $\mathbf{x}$ is on x-axis, $\tilde{I}_{\mathbf{x}} = I_{\mathbf{x}}$ and when $\mathbf{x}$ is on y-axis, $\tilde{J}_{\mathbf{x}} = J_{\mathbf{x}}$. By the definition of the partition function, we have iterate relations of the increments as follows,

$$\tilde{I}_{\mathbf{x}} = W^1_{\mathbf{x}} + W^2_{\mathbf{x}} \frac{\tilde{I}_{\mathbf{x}-\mathbf{e}_2}}{\tilde{J}_{\mathbf{x}-\mathbf{e}_1}}, \quad \tilde{J}_{\mathbf{x}} = W^1_{\mathbf{x}} \frac{\tilde{J}_{\mathbf{x}-\mathbf{e}_1}}{\tilde{I}_{\mathbf{x}-\mathbf{e}_2}} + W^2_{\mathbf{x}}. \tag{4.2}$$

From [47] and the similar argument (corner flipping method) of Theorem 3.1 in [58], we know the following stationary-increment property holds for all the four models mentioned above.

**Proposition 4.1.1** ([47])**.** *Given the above four models, when the parameters satisfy $w = z$, we have that*

$$(\tilde{I}_{\mathbf{x}}, \tilde{J}_{\mathbf{x}}) \stackrel{d}{=} (\tilde{I}_{\mathbf{x}-\mathbf{e_2}}, \tilde{J}_{\mathbf{x}-\mathbf{e_1}}) \tag{4.3}$$

*for $\mathbf{x} \in \mathbb{Z}^2_{>0}$. And alongside any down-right path $\mathcal{Y}$, all the increment functions on edge segments of the path are mutually independent.*

With the above proposition in hand, we are able to conclude the law of large numbers result of the free energy in increment-stationary models as well as the log of the moment generation function at certain degree of the free energy in multi-parameter models. We omit the proof here since it's pure computing. The law of large number results are discovered by different people in different independent papers.

**Proposition 4.1.2** (LLN of the free energy)**.** *Fix a direction $(s,t)$ where $s,t \geq 0$ and $s+t > 0$. The following limit holds for $z \in \mathcal{J}$,*

$$\lim_{N\to\infty} \frac{\log Z^z(Ns, Nt)}{N} = \gamma^z(s,t) \quad a.s.. \tag{4.4}$$

*The explicit formulas for the shape function $\gamma^z(s,t) = sE[\log I] + tE[\log J]$ in four models where $z \in \mathcal{J}$ and $s,t > 0$ are listed below, where $\Psi_0 = \Gamma'/\Gamma$ is the digamma function.*

- *Log-gamma model:* $\gamma^z(s,t) = s\,[\log \nu - \Psi_0(\mu - z)] + t\,[\log \nu - \Psi_0(z)]$.
- *Strict-weak model:* $\gamma^z(s,t) = s\,[\Psi_0(\mu + z) - \log \nu] + t\,[\Psi_0(\mu + z) - \Psi_0(z)]$.

- *Beta model:* $\gamma^z(s,t) = s[\Psi_0(\mu+z) - \Psi_0(\mu+\nu+z)] + t[\Psi_0(\mu+z) - \Psi_0(z)]$ *in BI* *or* $\gamma^z(s,t) = s[\Psi_0(\nu+z) - \Psi_0(z)] + t[\Psi_0(\nu+z) - \Psi_0(\mu+\nu+z)]$ *in BII.*

- *Inverse-beta model:* $\gamma^z(s,t) = s[\Psi_0(\mu+\nu-z) - \Psi_0(\mu-z)] + t[\Psi_0(\mu+\nu-z) - \Psi_0(z)]$.

Define $L^{w,z}(m,n) = \log E\left[(Z^{w,z}(m,n))^{z-w}\right]$ in all models but BII. In BII, $L^{w,z}(m,n) = \log E\left[(Z^{w,z}(m,n))^{w-z}\right]$. By Proposition 4.1.1 and a change of measure argument, the explicit formulas for $L^{w,z}(m,n)$ are shown below.

- Log-gamma model: $L^{w,z}(m,n) = m\log\frac{\Gamma(\mu-z)}{\Gamma(\mu-w)}\nu^{z-w} + n\log\frac{\Gamma(w)}{\Gamma(z)}\nu^{z-w}$.

- Strict-weak model: $L^{w,z}(m,n) = m\log\frac{\Gamma(\mu+z)}{\Gamma(\mu+w)}\nu^{w-z} + n\log\frac{\Gamma(\mu+z)\Gamma(w)}{\Gamma(\mu+w)\Gamma(z)}$.

- Beta model: $L^{w,z}(m,n) = m\log\frac{\Gamma(\mu+z)\Gamma(\mu+\nu+w)}{\Gamma(\mu+w)\Gamma(\mu+\nu+z)} + n\log\frac{\Gamma(\mu+z)\Gamma(w)}{\Gamma(\mu+w)\Gamma(z)}$ in BI or $L^{w,z}(m,n) = m\log\frac{\Gamma(\nu+w)\Gamma(z)}{\Gamma(\nu+z)\Gamma(w)} + n\log\frac{\Gamma(\nu+w)\Gamma(\mu+\nu+z)}{\Gamma(\nu+z)\Gamma(\mu+\nu+w)}$ in BII.

- Inverse-beta model: $L^{w,z}(m,n) = m\log\frac{\Gamma(\mu+\nu-w)\Gamma(\mu-z)}{\Gamma(\mu+\nu-z)\Gamma(\mu-w)} + n\log\frac{\Gamma(\mu+\nu-w)\Gamma(w)}{\Gamma(\mu+\nu-z)\Gamma(z)}$.

By another simple computation, one can find the relation between the shape function $\gamma^z(s,t)$ and the l.m.g.f. $L^{w,z}(m,n)$.

$$\int_w^z \gamma^t(m,n)dt = L^{w,z}(m,n) \tag{4.5}$$

in all but BII. In BII,

$$\int_z^w \gamma^t(m,n)dt = L^{w,z}(m,n). \tag{4.6}$$

Given the above key integrable equation, it is necessary to discover more properties of the shape function in all four models. Since those properties are as important as our main moderate deviation theorems, the next section shows the discussion of the shape function, and the deviation estimates will come after it.

All edge weights for different parameters $w, z$ on the same single edge discussed in this chapter are assumed to be coupled together using the natural monotone coupling of random variable, which we call the inverse-CDF coupling. In this coupling, first generate a uniform random variable $U$ on $(0,1)$, one realizes a real random variable $X$ as $X = G_X(U)$, $G_X(u) = \inf\{x : F_X(x) \geq u\}$, and $F_X(\cdot)$ is the cumulative distribution function of $X$. When the joint distribution of the random variables matters, we will elaborate on the coupling. In this way, in any of the four models, by the lemma of the Radon-Nykodym derivative of the Gamma and Beta distributions that is shown below, for the partition function of different parameters, $Z(m,n) \le Z^{w,z}(m,n)$ holds for all $w, z \in \mathcal{J}$ and $(m,n)$ in the first quadrant. Moreover, for $w < z$, we can also get $Z^{w,z}(m,n) < Z^{w}(m,n)$ and $Z^{w,z}(m,n) < Z^{z}(m,n)$ in all models but BII. While in BII, the inequality goes the opposite direction.

**Lemma 4.1.1.** *Fix $w \in \mathbb{R}$ and $z > 0$. The following hold:*

1. *If $0 \le x < y$, $X_1 \sim Ga(x,z)$, and $X_2 \sim Ga(y,z)$, then $P(X_1 \ge w) \le P(X_2 \ge w)$. Therefore, if $Y_1 \sim InvGa(x,z)$ and $Y_2 \sim InvGa(y,z)$, then $P(Y_1 \ge w) \ge P(Y_2 \ge w)$.*

2. *If $0 \le x < y$, $X_1 \sim Be(x,z)$, and $X_2 \sim Be(y,z)$, then $P(X_1 \ge w) \le P(X_2 \ge w)$. Therefore, if $Y_1 \sim InvBe(x,z)$ and $Y_2 \sim InvBe(y,z)$, then $P(Y_1 \ge w) \ge P(Y_2 \ge w)$.*

3. *If $0 < x < y < z$, $X_1 \sim Be(x, z-x)$, and $X_2 \sim Be(y, z-y)$, then $P\left(X_1 \ge w\right) \le P\left(X_2 \ge w\right)$. Therefore, if $Y_1 \sim InvBe(x, z-x)$ and $Y_2 \sim InvBe(y, z-y)$, then $P\left(Y_1 \ge w\right) \ge P\left(Y_2 \ge w\right)$.*

*Proof.* A sufficient condition for the claimed stochastic dominance is that the listed distributions are likelihood ratio ordered, meaning in this case that the associated Radon-Nikodym derivative is monotone where it is non-zero. See, for example, [59, Theorem 1.C.1]. Monotonicity follows from the computation below,

$$\begin{aligned}
\frac{d\mathrm{Ga}(y,z)}{d\mathrm{Ga}(x,z)}(t) &= z^{y-x}\frac{\Gamma(x)}{\Gamma(y)}t^{y-x}1_{(0,\infty)}(t), \\
\frac{d\mathrm{Be}(y,z)}{d\mathrm{Be}(x,z)} &= \frac{\beta(x,z)}{\beta(y,z)}t^{y-x}1_{(0,1)}(t), \text{ and} \\
\frac{d\mathrm{Be}(y,z-y)}{d\mathrm{Be}(x,z-x)} &= \frac{\beta(x,z-x)}{\beta(y,z-y)}\left(\frac{t}{1-t}\right)^{y-x}1_{(0,1)}(t)
\end{aligned}$$

are increasing functions of $t$ where they are nonzero. The secondary claims in each part of the statement follow from the primary claims and can also be proven directly with the same argument. $\square$

## 4.2 Properties of the shape function

In all but Beta models, the shape function $\gamma^z(s,t)$ has a unique minimum $\gamma(s,t) := \gamma^\theta(s,t) = \inf_{z\in\mathcal{J}} \gamma^z(s,t)$ for all $(s,t)$ in the first quadrant, where $\theta(s,t)$ is where shape function reaches its minimum. In BI and BII, this unique minimum exists for $(s,t)$ in part of the first quadrant. We also want our local minimum point away from the boundary, thus we define $S_\delta = \{(x,y) : x > 0, y > 0, x/y > \delta, y/x > \delta\}$ and assume $s,t \in S_\delta$. The lemmas are shown below.

**Lemma 4.2.1** (log-gamma)**.** *Let $\delta > 0$,*

1. *There exists $\epsilon = \epsilon(\delta) > 0$ such that for all $(x,y) \in S_\delta$, there exists a unique $\theta(x,y) \in (\epsilon, \mu - \epsilon)$.*
2. $c(x+y) \le \frac{d^2}{dt^2}\left[\gamma^t(x,y)\right]\Big|_{t=\xi} = -x\Psi_2(\mu-\xi) - y\Psi_2(\xi) \le C(x+y)$ *for* $\xi \in (\epsilon, \mu-\epsilon)$, $(x,y) \in S_\delta$, *and some positive constants* $c, C$ *which only depend on* $\delta$.
3. $\frac{d^3}{dt^3}\left[\gamma^t(x,y)\right]\Big|_{t=\xi} = x\Psi_3(\mu-\xi) - y\Psi_3(\xi) \le C_1(x+y)$ *for* $\xi \in (\epsilon, \mu-\epsilon)$ *and constant* $C_1 = C_1(\delta) > 0$.

**Lemma 4.2.2** (strict-weak)**.** *Let $\delta > 0$,*

1. *There exist $0 < \epsilon = \epsilon(\delta) < \mathcal{E} = \mathcal{E}(\delta) < \infty$ such that for all $(x,y) \in S_\delta$, there exists a unique $\theta(x,y) \in (\epsilon, \mathcal{E})$.*
2. $c(x+y) \le \frac{d^2}{dt^2}\left[\gamma^t(x,y)\right]\Big|_{t=\theta(x,y)} = x\Psi_2(\mu+\theta) + y\left[\Psi_2(\mu+\theta) - \Psi_2(\theta)\right] \le C(x+y)$ *for* $(x,y) \in S_\delta$, *for some positive constants* $c, C$ *which only depend on* $\delta$.
3. $\frac{d^3}{dt^3}\left[\gamma^t(x,y)\right]\Big|_{t=\xi} = x\Psi_3(\mu+\xi) + y\left[\Psi_3(\mu+\xi) - \Psi_3(\xi)\right] \le C_1(x+y)$ *for* $\xi \in (\epsilon, \infty)$ *and constant* $C_1 = C_1(\delta) > 0$.

**Lemma 4.2.3** (inverse beta)**.** *Let $\delta > 0$,*

1. *There exists $\epsilon = \epsilon(\delta) > 0$ such that for all $(x,y) \in S_\delta$, there exists a unique $\theta(x,y) \in (\epsilon, \mu - \epsilon)$.*

2. $c(x+y) \leq \frac{d^2}{dt^2}\left[\gamma^t(x,y)\right]\Big|_{t=\theta(x,y)} = x\left[\Psi_2(\mu+\nu-\theta) - \Psi_2(\mu-\theta)\right] + y\left[\Psi_2(\mu+\nu-\theta) - \Psi_2(\theta)\right] \leq C(x+y)$ *for* $(x,y) \in S_\delta$, *and some positive constants* $c, C$ *which only depend on* $\delta$.

3. $\frac{d^3}{dt^3}\left[\gamma^t(x,y)\right]\Big|_{t=\xi} = x\left[\Psi_3(\mu-\xi) - \Psi_3(\mu+\nu-\xi)\right] - y\left[\Psi_3(\xi) + \Psi_3(\mu+\nu-\xi)\right] \leq C_1(x+y)$ *for* $\xi \in (\epsilon, \mu-\epsilon)$ *and constant* $C_1 = C_1(\delta) > 0$.

Define $\xi^* = (\xi_1^*, \xi_2^*) = (\frac{\mu}{\mu+\nu}, \frac{\nu}{\mu+\nu})$, and $B_\delta^+(\xi^*) = \{(x,y),\ \frac{x}{x+y} \geq \xi_1^* + \delta\}$, $B_\delta^-(\xi^*) = \{(x,y),\ \frac{x}{x+y} \leq \xi_1^* - \delta\}$. For BI and BII models,

**Lemma 4.2.4** (beta)**.** *Let* $\delta > 0$,

1. *There exist* $0 < \epsilon = \epsilon(\delta) < \mathcal{E} = \mathcal{E}(\delta) < \infty$, *such that a unique* $\theta(x,y) \in (\epsilon, \mathcal{E})$ *exists, if* $(x,y) \in S_\delta \cap B_\delta(\xi^*)^+$ *in BI or* $(x,y) \in S_\delta \cap B_\delta(\xi^*)^-$ *in BII.*

2. $c(x+y) \leq \frac{d^2}{dt^2}\left[\gamma^t(x,y)\right]\Big|_{t=\theta(x,y)} \leq C(x+y)$ *for some positive constants* $c, C$ *which only depend on* $\delta$, *if* $(x,y) \in S_\delta \cap B_\delta(\xi^*)^+$ *in SI Beta I or* $(x,y) \in S_\delta \cap B_\delta(\xi^*)^-$ *in SI Beta II.*

3. $\frac{d^3}{dt^3}\left[\gamma^t(x,y)\right]\Big|_{t=\xi} \leq C_1(x+y)$ *for* $\xi \in (\epsilon, \infty)$ *and constant* $C_1 = C_1(\delta) > 0$.

Although there are differences between those lemmas, the major properties that we care about are the same: Unique minimum and bounded third moment derivative away from the boundary, bounded (from above and below) second moment derivative at the minimum.

Define $\mathcal{L}^\lambda(m,n) = \inf_{w,z\in\mathcal{J}, z-w=\lambda} L^{w,z}(m,n)$ and $\mathcal{I}_s(m,n) = \sup_{\lambda\geq 0}\{\lambda s - \mathcal{L}^\lambda(m,n)\}$. By the exponential Markov inequality and the pathwise inequality $Z(m,n) \leq Z^{w,z}(m,n)$,

$$\log P\left(\log Z(m,n) \geq s\right) \leq -\lambda s + \mathcal{L}^\lambda(m,n). \tag{4.7}$$

This inequality holds for all $\lambda \geq 0$, so

$$\log P\left(\log Z(m,n) \geq s\right) \leq -\mathcal{I}_s(m,n). \tag{4.8}$$

$\mathcal{I}_s(m,n)$ also has a geometric meaning. Notice that $\mathcal{I}_s(m,n) = \sup_{\lambda\geq 0}\{\lambda s - \mathcal{L}^\lambda(m,n)\} = \sup_{\lambda\in\mathcal{J}}\{\lambda s - \mathcal{L}^\lambda(m,n)\} = \sup_{w,z\in\mathcal{J}, z-w=\lambda}\{(z-w)s - L^{w,z}(m,n)\} = \sup_{w,z\in\mathcal{J}}\{(z-w)s -$

$\int_w^z \gamma^t(m,n)dt\}$. Therefore $\mathcal{I}_s(m,n)$ is the area bounded by $y = s$ and $y = \gamma^t(m,n)$ as a function of $t$.

Since we always consider $t$ away from the boundary, let $\mathcal{J}_\epsilon$ be $(\epsilon, \mu - \epsilon)$ in LG and IB, $(\epsilon, \infty)$ in gamma and beta models. In the next lemma we give the lower bound estimation of $\mathcal{I}_s(m,n)$ by using Taylor expansion centered at $\theta$ up to the 2nd order of $\gamma^t(m,n)$.

**Lemma 4.2.5.** *Let $\delta > 0$ and $\epsilon = \epsilon(\delta) > 0,\ C_1 = C_1(\delta)$ shown in the above lemmas. For $(m,n) \in S_\delta \cap \mathbb{Z}^2_{>0}$ (in BI and BII $(m,n)$ should also be in their corresponding area) we have*

$$\left|\gamma^t(m,n) - M - \sigma^3(t-\theta)^2\right| \leq C_1(m+n)|t-\theta|^3 \tag{4.9}$$

*for $t \in \mathcal{J}_\epsilon$ and*

$$\mathcal{I}_s(m,n) \geq \sup_{w,z\in\mathcal{J}_\epsilon, w\leq\theta\leq z} \left\{(s-M)(z-w) - \frac{\sigma^3}{3}\left[(z-\theta)^3 - (w-\theta)^3\right] - C_1(m+n)\left[(z-\theta)^4 + (w-\theta)^4\right]\right\}, \tag{4.10}$$

*where $M = \gamma(m,n)$ and $2\sigma^3 = \frac{d^2}{dt^2}\left[\gamma^t(m,n)\right]\Big|_{t=\theta(m,n)}$.*

*Proof.* The Taylor expansion centered at $\theta$ up to the 2nd order of $\gamma^t(m,n)$ is

$$\gamma^t(m,n) = M + \sigma^3(t-\theta)^2 + \frac{1}{6}\frac{d^3}{dt^3}\left[\gamma^t(m,n)\right]\Big|_{t=\xi}(t-\theta)^3 \tag{4.11}$$

for some $\xi$ lies in between $t$ and $\theta$. By assumption both $t$ and $\theta$ are in the interval $\mathcal{J}_\epsilon$, so does $\xi$. Together with the above lemmas, (4.9) holds. (4.10) can be checked by integrating $\gamma^t(m,n)$ from $w$ to $z$ which are both in $\mathcal{J}_\epsilon$. $\square$

Before we show the main theorems, let's define the quenched exit time of the random polymer. Define the probability measure $Q^{w,z}(\cdot)$ on the set of path $\prod_{(0,0),(m,n)}$ as, for $x. = \{x_0, x_1, ..., x_{(m+n)}\} \in \prod_{(0,0),(m,n)}$ where $x_0 = (0,0)$ and $x_{m+n} = (m,n)$,

$$Q^{w,z}(\{x.\}) = \frac{\prod_{\mathrm{i}=1}^{m+n}\omega_{(x_{\mathrm{i}-1},x_{\mathrm{i}})}}{Z^{w,z}_{(0,0)}(m,n)}. \tag{4.12}$$

$\omega_{e_i}$ is the weight of the edge $e_i$. Then we are able to define the distribution of the quenched exit time from the boundary Exit = Exit$(m,n)$ which takes value in $\mathbb{Z}$ through

$$Q^{w,z}(\text{Exit} \geq k) = \frac{\left[\prod_{i=1}^{k} I_{(i,0)}\right] Z^{w}_{(k,0)}(m,n)}{Z^{w,z}_{(0,0)}(m,n)} \tag{4.13}$$

and

$$Q^{w,z}(\text{Exit} \leq -k) = \frac{\left[\prod_{k=1}^{k} J_{(0,j)}\right] Z^{z}_{(0,k)}(m,n)}{Z^{w,z}_{(0,0)}(m,n)}. \tag{4.14}$$

i.e. the exit time takes integer values. With the $(m,n)$ dependence suppressed, the event $\{\text{Exit} = k\}$ consists of admissible paths from $(0,0)$ to $(m,n)$ which exit the horizontal boundary at $(k,0)$. Similarly, the event $\{\text{Exit} = -k\}$ consists of those paths which exit the vertical boundary at $(0,k)$.

### 4.3 Moderate deviation and exit time estimates

Our main result relies on a log moment generating function identity, which is an analogue of an identity originally developed in the zero temperature setting by Rains [60, Corollaries 3.3-3.4] using integrable probability. Our methods extend the arguments of [41] to the setting of solvable polymer models. This section marks the first time that identities of this type.

Our first result gives the upper bound of the right tail deviations of the free energy in bulk models. The proof is very short so we present it right after the statement of the theorem.

**Theorem 4.3.1.** *Fix $\delta > 0$. There exist constant $c = c(\delta), C = C(\delta) > 0$ so that, in the models other than the Beta model, for all $(m,n) \in S_\delta$ and all $s \in [0, c(m+n)^{2/3}]$,*

$$\log P\left(\log Z(m,n) \geq \gamma(m,n) + \sigma s\right) \leq -\frac{4}{3}s^{\frac{3}{2}} + Cs^2(m+n)^{-\frac{1}{3}}. \tag{4.15}$$

*In BI, the same holds for $(m,n) \in S_\delta \cap \mathbb{Z}^2_{>0} \cap B_\delta(\xi^*)^+$, in BII, the same holds for $(m,n) \in S_\delta \cap \mathbb{Z}^2_{>0} \cap B_\delta(\xi^*)^-$.*

*Proof.* Consider $\mathcal{I}_{\gamma(m,n)+\sigma s}(m,n)$ and take $w = \theta - \sqrt{s}/\sigma$, $z = \theta + \sqrt{s}/\sigma$. We can find $c > 0$ such that as long as $s \le c(m+n)^{2/3}$, $\theta \pm \sqrt{s}/\sigma \in \mathcal{J}_\epsilon$ since $\sigma$ is of order $(m+n)^{1/3}$. Therefore by (4.10),

$$\mathcal{I}_{\gamma(m,n)+\sigma s}(m,n) \ge \frac{4s^{3/2}}{3} - \frac{2C_1(m+n)s^2}{\sigma^4}. \tag{4.16}$$

Recall $\sigma$ is of order $(m+n)^{1/3}$ again, there exists $C > 0$, such that $2C_1(m+n)/\sigma^4 \le C/(m+n)^{1/3}$. By (4.8), the theorem is proved. $\square$

Next theorem shows the upper bound of the right tail of the free energy $\log Z^{w,z}(m,n)$ in multi-parameter models. $w, z$ are required to be close to the local minimum $\theta$ in the theorem.

**Theorem 4.3.2.** *Fix $\delta > 0$, $K \ge 0$, $p > 0$, and $s_0 > 0$. There exists constants $C_0 = C_0(\delta)$, $c_0 = c_0(\delta)$, and $N_0 = N_0(\delta, K, p, s_0)$ such that, with $t = \min\{s, c(m+n)^{2/3}\}$,*

$$\log P\left[\log Z^{w,z}(m,n) \ge \gamma(m,n) + \sigma s + \log 2\right] \le -\frac{2t^{3/2}}{3} - (s-t)\sqrt{t} + \log 2 + \frac{C_0 K t}{(m+n)^p} + \frac{C_0 t^2}{(m+n)^{1/3}} \tag{4.17}$$

*whenever $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\ge N_0}$ (also in $B_\delta(\xi^*)^+$ in BI and $B_\delta(\xi^*)^-$ in BII), $c \in [s_0(m+n)^{-2/3}, c_0]$, $s \in \mathbb{R}_{\ge s_0}$, and $w, z \in \mathcal{J}$ with*

$$\max\{|w - \theta(m,n)|, |z - \theta(m,n)|\} \le K(m+n)^{-1/3-p}. \tag{4.18}$$

For the quenched exit time bound, we focus on estimating the decay of the annealed exit time event $|\text{Exit}| > s(m+n)^{2/3}$, that is, the annealed probability of the random polymer that exit the boundary later than $s(m+n)^{2/3}$ steps.

**Theorem 4.3.3.** *For $\delta > 0$, $K \ge 0$ there exist finite constants $c = c(\delta) > 0$, $N_0 = N_0(\delta, K) > 0$, $s_0 = s_0(\delta, K) > 0$, and $\mathrm{e}_0 = \mathrm{e}_0(\delta, K) > 0$ such that*

$$E\left[Q^{w,z}(|\text{Exit}| > s(m+n)^{\frac{2}{3}})\right] \le exp\{-cs^3\} \tag{4.19}$$

*for $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\ge N_0}$ (also in $B_\delta(\xi^*)^+$ in BI and $B_\delta(\xi^*)^-$ in BII), $s \ge s_0$, and $w, z \in (0,\infty)$ such that $\max\{|w - \theta(m,n)|, |z - \theta(m,n)|\} \le K(m+n)^{-1/3}$ and $|w - z| \le \mathrm{e}_0(m+n)^{-1/3}$.*

The last theorem gives the lower bound of l.m.g.f. of the free energy $\log Z(m,n)$ in bulk models. This is the key step to figure out the lower bound of the moderate deviation of the right tail of the free energy.

**Theorem 4.3.4.** *Fix $\delta > 0$, there exist positive constants $C_0 = C_0(\delta)$, $c_0 = c_0(\delta)$, $N_0 = N_0(\delta)$, $K_0 = K_0(\delta)$, such that*

$$\log E\left[(Z(m,n))^{\lambda}\right] \geq \mathcal{L}^{\lambda}(m,n) - K_0\lambda^{\frac{9}{4}}(m+n)^{\frac{3}{4}}\max\{1, \lambda(m+n)^{\frac{1}{4}}\} \tag{4.20}$$

*or equivalently,*

$$\log E\left[(Z(m,n))^{\lambda}\right] \geq \lambda\gamma(m,n) + \frac{\lambda^3\sigma^3}{12} - K_0\lambda^{\frac{9}{4}}(m+n)^{\frac{3}{4}}\max\{1, \lambda(m+n)^{\frac{1}{4}}\} \tag{4.21}$$

*whenever $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\geq N_0}$ (also in $B_\delta(\xi^*)^+$ in BI and $B_\delta(\xi^*)^-$ in BII), and $\lambda \in [C_0(m+n)^{-1/3}, c_0]$.*

## 4.4 Proofs

### 4.4.1 Proof of properties of the shape function

*Proof of Lemma 4.2.1.* Recall $\gamma^t(x,y) = s\left[\log\nu - \Psi_0(\mu - t)\right] + t\left[\log\nu - \Psi_0(t)\right]$, in order to find the critical point, take derivative of $t$ on both sides, we have $\theta$ solves the equation

$$s\Psi_1(\mu - \theta) = t\Psi_1(\theta) \tag{4.22}$$

where $\Psi_1(x)$ is trigamma function such that $\Psi_1 = \Psi_0'$. Moreover, $\Psi_1$ is a positive decreasing function with limit 0 for $x \to \infty$ and limit $\infty$ for $x \to 0$. Therefore, for $f(x) = \Psi_1(\mu - x)/\Psi_1(x)$, it is increasing and $f(0) = 0$, $f(\mu) = \infty$ (in the limit sense). So when $(x,y) \in S_\delta$, there always exists a unique $\theta$ such that the equation holds. And it is indeed a global minimum point by the monotonicity of $f(x)$. The second and third statements in Lemma 4.2.1 is simply by the fact that both $\Psi_2(x)$ and $\Psi_3(x)$ are bounded (from above and away from zero from below) when $x \in (\epsilon, \mu - \epsilon)$. $\square$

*Proof of Lemma 4.2.2.* The upper bounds for both part 2 and 3 are obvious since if $(x, y) \in S_\delta$, then by part 1, $\theta \in (\epsilon, \mathcal{E})$ which is also the range of $\xi$. This will give the upper bound estimation since the absolute value of $\Psi_2$ and $\Psi_3$ are all decreasing to zero.

In order to prove part 1, since $\theta(x, y)$ satisfies

$$\frac{x}{y} = \frac{\Psi_1(\theta)}{\Psi_1(\theta+\mu)} - 1, \tag{4.23}$$

it is enough to show that the function $g_\mu(z) = \frac{\Psi_1(z)}{\Psi_1(z+\mu)}$ for $z \in (0, \infty)$ is a $C^\infty$ decreasing function and

$$\lim_{z\to 0^+} g_\mu(z) = \infty, \quad \lim_{z\to\infty} g_\mu(z) = 1. \tag{4.24}$$

Notice that $\Psi_3(z)\Psi_1(z) > \Psi_2^2(z)$ by the integral representation of poly gamma functions and C-S inequality(the equation can not be satisfied), we have

$$\frac{d}{dz} g_\mu(z) = \frac{\Psi_1(z)}{\Psi_1(z+\mu)} \left[\frac{\Psi_2(z)}{\Psi_1(z)} - \frac{\Psi_2(z+\mu)}{\Psi_1(z+\mu)}\right] < 0 \tag{4.25}$$

for all $z \in (0, \infty)$. Moreover, since $\lim_{z\to 0^+} \Psi_1(z) = \infty$, $\lim_{z\to 0^+} g_\mu(z) = \infty$ can be get immediately. In order to check the second half of (4.24), for poly gamma functions, we know that

$$\Psi_1(z) = \Psi_1(z+1) + \frac{1}{z^2}. \tag{4.26}$$

Hence, $\Psi_1(z) = \Psi_1(z + \lceil\mu\rceil) + \sum_{\mathrm{i}=z}^{z+\lceil\mu\rceil-1} \frac{1}{\mathrm{i}^2} \leq \Psi_1(z + \lceil\mu\rceil) + \frac{\lceil\mu\rceil}{z^2}$. Use this upper bound, together with the fact that $z^2\Psi_1(z)$ grows linearly as $z \to \infty$, we have

$$1 \leq g_\mu(z) = \frac{\Psi_1(z)}{\Psi_1(z+\mu)} \leq \frac{\Psi_1 z}{\Psi_1(z+\lceil\mu\rceil)} \leq 1 + \frac{\lceil\mu\rceil}{z^2\Psi_1(z) - \lceil\mu\rceil} \to 1 \tag{4.27}$$

as $z \to \infty$, which finishes the proof of part 1 of the lemma.

To get the lower bound estimation in part 2, assume $y = 1$. Since $\theta(x, 1)$ solves

$$x\Psi_1(\theta+\mu) + \Psi_1(\theta+\mu) - \Psi_1(\theta) = 0, \tag{4.28}$$

take derivative over $\theta$ on both side, we have(treat $x$ as a function of $\theta$)

$$\frac{dx}{d\theta} = \frac{(x+1)\Psi_2(\theta+\mu) - \Psi_2(\theta)}{-\Psi_1(\theta+\mu)} = \frac{\left[M^\theta(x,1)\right]''}{-\Psi_1(\theta+\mu)}. \tag{4.29}$$

Meanwhile, recall the definition of $g$ we have $g_\mu(z) = x+1$ when $y=1$, so $\frac{d}{d\theta}g_\mu(\theta) = \frac{dx}{d\theta}$. Since $(x,y) \in S_\delta$, $\theta \in (\epsilon, \mathcal{E})$ by part 1, there exists $c_0 = c_0(\delta) > 0$ such that

$$\frac{\left[M^\theta(x,1)\right]''}{-\Psi_1(\theta+\mu)} = \frac{d}{d\theta}g_\mu(\theta) \le -c_0 < 0. \tag{4.30}$$

In the end, $\left[M^\theta(x,1)\right]'' \ge c_0\Psi_1(\theta+\mu) \ge c_0\Psi_1(\mathcal{E}+\mu) \ge \frac{c_0\Psi_1(\mathcal{E}+\mu)}{(\mathcal{E}+1)}(x+1) > 0$ for $x \in (\epsilon, \mathcal{E})$ and y=1. The general case can be proved by scaling $(x,y)$. □

*Proof of Lemma 4.2.3.* Once again, the upper bound in part 2 and part 3 is simple by the same reason in the proof of Lemma 4.2.2. To see part 1 is true, define

$$g_{\mu,\nu}(z) = \frac{\Psi_1(z) + \Psi_1(\mu+\nu-z)}{\Psi_1(\mu-z) - \Psi_1(\mu+\nu-z)}. \tag{4.31}$$

It is easy to see $\lim_{z\to 0^+} g_{\mu,\nu}(z) = +\infty$ and $\lim_{z\to\mu^-} g_{\mu,\nu}(z) = 0$. We need to check this function is strictly decreasing for $z \in (0,\mu)$. Take the derivative of $g_{mu,\nu}$ over $z$, after a couple of steps of simplification, we have

$$\begin{aligned}&[\Psi_1(\mu-z) - \Psi_1(\mu+\nu-z)]^2 \frac{d}{dz}g_{\mu,\nu}(z) = \Psi_2(z)\left[\Psi_1(\mu-z) - \Psi_1(\mu+\nu-z)\right] \\ &+ \Psi_1(z)\left[\Psi_2(\mu-z) - \Psi_2(\mu+\nu-z)\right] + \frac{1}{\Psi_1(\mu-z)\Psi_1(\mu+\nu-z)}\left[\frac{\Psi_2(\mu-z)}{\Psi_1(\mu-z)} - \frac{\Psi_2(\mu+\nu-z)}{\Psi_1(\mu+\nu-z)}\right].\end{aligned} \tag{4.32}$$

For all three terms on the right hand side in the above equation, it's easy to see they are all negative valued, thus the derivative is negative for all $z \in (0,\mu)$.

The proof of the lower bound in part 2 is similar to the proof of Lemma 4.2.2, thus we omit here. □

*Proof of Lemma 4.2.4.* The proof of the upper bound in part 2 and part 3 is again obvious since both $\theta$ and $\xi$ are in $(\epsilon, \infty)$ and make the coefficients of $x$ and $y$ have finite upper bounds. To see part 1 is true, since $\theta(x, y)$ solves the equation

$$\frac{x}{x+y} = \frac{\Psi_1(\theta) - \Psi_1(\mu+\theta)}{\Psi_1(\theta) - \Psi_1(\mu+\nu+\theta)} \tag{4.33}$$

in SI Beta I model or

$$\frac{x}{x+y} = \frac{\Psi_1(\mu+\nu+\theta) - \Psi_1(\nu+\theta)}{\Psi_1(\mu+\nu+\theta) - \Psi_1(\theta)} = 1 - \frac{\Psi_1(\theta) - \Psi_1(\nu+\theta)}{\Psi_1(\theta) - \Psi_1(\mu+\nu+\theta)} \tag{4.34}$$

in SI Beta II model. Define the function

$$g(z) = \frac{\Psi_1(z) - \Psi_1(\mu+z)}{\Psi_1(z) - \Psi_1(\mu+\nu+z)} \tag{4.35}$$

which is a decreasing function for $z$ in $(0, \infty)$. Moreover, $\lim_{z\to 0} g(z) = 1$ and $\lim_{z\to\infty} g(z) = \xi_1^*$. $g(z)$ is decreasing can be seen by taking the derivative and using the convexity of the poly gamma functions. The limit of $g$ at zero is obvious. The limit of $g$ at infinity is $\xi_1^* = \mu/(\mu+\nu)$ can be seen by taking the Taylor expansion

$$\begin{aligned} \Psi_1(\mu+z) &= \Psi_1(z) + \mu\Psi_2(z) + o(\Psi_2(z)) \\ \Psi_1(\mu+\nu+z) &= \Psi_1(z) + (\mu+\nu)\Psi_2(z) + o(\Psi_2(z)) \end{aligned}. \tag{4.36}$$

Hence, notice that $\Psi_2(z) \to 0$ when $z \to \infty$, we have

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \frac{\mu + o(1)}{\mu+\nu+o(1)} = \frac{\mu}{\mu+\nu} = \xi_1^* \tag{4.37}$$

finish proving our statements. The lower bound in part 2 has the similar proof as the one of Lemma 4.2.2. The only thing that needs to be clarified is both $\Psi_1(\mu+\theta) - \Psi_1(\mu+\nu+\theta)$ and $\Psi_1(\nu+\theta) - \Psi_1(\mu+\nu+\theta)$ are positive and has a lower bound since $\theta \in (\epsilon, \mathcal{E})$. $\square$

### 4.4.2 Proof of Theorem 4.3.2

We will prove the theorem only in the log gamma model case. Since what matters the proof is the properties of the shape function that we present before and will be given later that all four models except BII share. For BII, it can be viewed as a coordinate-flipping model of BI, thus it can be modeled as we replace the $x, y$ coordinates in BI, the corresponding proof also can be obtained in the same way.

Define

$$\mathcal{L}^{\lambda,w,hor}(x,y) = \inf_{u,v\in(w,\mu),v-u=\lambda} \log E\left[(Z^{u,v}(x,y))^{\lambda}\right], \tag{4.38}$$

$$\mathcal{L}^{\lambda,z,ver}(x,y) = \inf_{u,v\in(0,z),v-u=\lambda} \log E\left[(Z^{u,v}(x,y))^{\lambda}\right], \tag{4.39}$$

$$\mathcal{L}^{\lambda,w,z}(x,y) = \mathcal{L}^{\lambda,w,hor}(x,y) \vee \mathcal{L}^{\lambda,z,ver}(x,y). \tag{4.40}$$

For $\lambda \in [0,\mu)$ write $\zeta_\lambda^-(x,y)$ and $\zeta_\lambda^+(x,y)$ in $(0,\mu)$ satisfying

$$\zeta_\lambda^+ - \zeta_\lambda^- = \lambda,\ \gamma^{\zeta_\lambda^-}(x,y) = \gamma^{\zeta_\lambda^+}(x,y). \tag{4.41}$$

Those definitions are well-defined since the shape function is continuous and decreasing up to $\theta$ and then increasing all the way after $\theta$. These notations characterise when the infimum is obtained in (4.38), see the lemma below.

**Lemma 4.4.1.** *Let $x, y \in \mathbb{R}_{>0}$ and $\lambda \in [0,\mu)$, $w \in [0,\mu)$ and $z \in (0,\mu]$,*

1. $\mathcal{L}^{\lambda,w,hor}(x,y) = \log E\left[\left(Z^{\zeta_\lambda^-,\zeta_\lambda^+}(x,y)\right)^{\lambda}\right]$ *if* $w \le \zeta_\lambda^-$ *and* $\mathcal{L}^{\lambda,w,hor}(x,y) = \log E\left[\left(Z^{w,w+\lambda}(x,y)\right)^{\lambda}\right]$ *if* $\zeta_\lambda^- < w < \mu-\lambda$, *otherwise* $\mathcal{L}^{\lambda,w,hor}(x,y) = +\infty$.

2. $\mathcal{L}^{\lambda,z,ver}(x,y) = \log E\left[\left(Z^{\zeta_\lambda^-,\zeta_\lambda^+}(x,y)\right)^{\lambda}\right]$ *if* $z \ge \zeta_\lambda^+$ *and* $\mathcal{L}^{\lambda,z,ver}(x,y) = \log E\left[\left(Z^{z-\lambda,z}(x,y)\right)^{\lambda}\right]$ *if* $\lambda < z < \zeta_\lambda^+$, *otherwise* $\mathcal{L}^{\lambda,z,ver}(x,y) = +\infty$.

*Proof.* To see the first argument is true, recall that

$$L^{u,u+\lambda}(m,n) = \int_u^{u+\lambda} \gamma^t(m,n)dt, \tag{4.42}$$

the derivative of $L^{u,u+\lambda}(m,n)$ over u is just $\gamma^{u+\lambda}(m,n) - \gamma^{u}(m,n)$. To find the minimum of $L^{u,u+\lambda}(m,n)$ for $u \in (w,\mu)$, simply solve $\gamma^{u+\lambda}(m,n) - \gamma^{u}(m,n) = 0$. The unique solution to this equation is $u = \zeta_\lambda^-$. If $w \leq \zeta_\lambda^-$, then $u = \zeta_\lambda^-$ is admissible. To see it is the minimum point, notice that $\gamma^{u+\lambda}(m,n) - \gamma^{u}(m,n)$ is negative for $u < \zeta_\lambda^-$ and positive for $u > \zeta_\lambda^-$. If $\zeta_\lambda^- < w < \mu - \lambda$, then the global minimum cannot be obtained and the conditional minimum is obtained when $u = w$ since the derivative of $L^{u,u+\lambda}(m,n)$, $\gamma^{u+\lambda}(m,n) - \gamma^{u}(m,n)$ is positive for $u > \zeta_\lambda^-$. The second argument is proved in the same way. $\square$

Next lemma show the upper bound of l.m.g.f. of the free energy starts from $(0,1)$ or $(1,0)$. Since only $w$ matters the free energy if the polymer starts from $(1,0)$ and $z$ matters the free energy if the polymer starts from $(0,1)$, we only use one superscript when writing the free energy.

**Lemma 4.4.2.** *Let $m,n \in \mathbb{Z}_{>0}$ and $\lambda \in \mathbb{R}_{\geq 0}$, $w \in [0,\mu)$ and $z \in (0,\mu]$, then*

1. $\log E\left[\left(I_{(1,0)} Z^{w}_{(1,0)}(m,n)\right)^{\lambda}\right] \leq \mathcal{L}^{\lambda,w,hor}(m,n)$,
2. $\log E\left[\left(J_{(0,1)} Z^{z}_{(0,1)}(m,n)\right)^{\lambda}\right] \leq \mathcal{L}^{\lambda,z,ver}(m,n)$.

*Proof.* To check the first inequality, we can only consider the case where $\lambda < \mu - w$, or the right hand side is $+\infty$ makes the inequality trivial.

Notice that $I_{(1,0)} Z^{w}_{(1,0)}(m,n) \leq Z^{w,w+\lambda}(m,n)$ and $I_{(1,0)} Z^{w}_{(1,0)}(m,n) \leq I_{(1,0)} Z^{\zeta_\lambda^-}_{(1,0)}(m,n)$ when $w \leq \zeta_\lambda^-$, we have

$$\log E\left[\left(I_{(1,0)} Z^{w}_{(1,0)}(m,n)\right)^{\lambda}\right] \leq \log E\left[\left(Z^{w,w+\lambda}(m,n)\right)^{\lambda}\right] \tag{4.43}$$

and

$$\log E\left[\left(I_{(1,0)} Z^{w}_{(1,0)}(m,n)\right)^{\lambda}\right] \leq \log E\left[\left(Z^{\zeta_\lambda^-,\zeta_\lambda^+}(m,n)\right)^{\lambda}\right] \tag{4.44}$$

when $w \leq \zeta_\lambda^-$. Combine with part 1 in lemma 4.4.1, we have the first inequality holds. The proof of the second inequality is almost the same.

$\square$

Define

$$\mathcal{I}_s^{w,hor}(x,y) = \sup_{u,v\in(w,\mu),u\leq v} \left\{(v-u)s - \int_u^v \gamma^t(x,y)dt\right\} \tag{4.45}$$

$$\mathcal{I}_s^{z,ver}(x,y) = \sup_{u,v\in(0,z),u\leq v} \left\{(v-u)s - \int_u^v \gamma^t(x,y)dt\right\} \tag{4.46}$$

$$\mathcal{I}_s^{w,z}(x,y) = \mathcal{I}_s^{w,hor}(x,y) \wedge \mathcal{I}_s^{z,ver}(x,y). \tag{4.47}$$

By this definition, together with Lemma 4.4.1, we have $\mathcal{I}_s^{w,hor}(x,y) + \mathcal{L}^{\lambda,w,hor}(x,y) \geq \lambda s$ and $\mathcal{I}_s^{z,ver}(x,y) + \mathcal{L}^{\lambda,z,ver}(x,y) \geq \lambda s$ immediately.

We also care about when the equations hold for the above two inequalities. For $s > \gamma(x,y)$, define $\xi_s^-(x,y)$ and $\xi_s^+(x,y)$ in $(0,\mu)$ such that

$$\xi_s^- < \xi_s^+,\ \gamma^{\xi_s^-}(x,x) = \gamma^{\xi_s^+}(x,y) = s, \tag{4.48}$$

i.e. $\xi_s^-$ is the smaller root of the equation $\gamma^t(x,y) = s$ and $\xi_s^+$ is the larger root. The definition of $\xi_s^-$ is well-defined for all $s > \gamma(x,y)$, while for $\xi_s^+$ it is defined for all $s > \theta$ in all but beta models. In beta model (BI and BII with corresponding range of $(x,y)$), $\xi_s^+$ is only defined for $\gamma(x,y) < s < 0$ since the shape function $\gamma^t(x,y)$ has limit zero as $t \to \infty$.

**Lemma 4.4.3.** *Let $x,y \in \mathbb{R}_{>0}$, $\lambda \in \mathbb{R}_{\geq 0}$, $w \in [0,\mu)$, $z \in (0,\mu]$, and $s \in \mathbb{R}$,*

1. *$\mathcal{I}_s^{w,hor}(x,y) + \mathcal{L}^{\lambda,w,hor}(x,y) \geq \lambda s$ with equality holds iff $s > \inf_{t\in(w,\mu)} \gamma^t(x,y)$, $\lambda = \xi_s^+ - \xi_s^- \vee w$ or $s \leq \inf_{t\in(w,\mu)} \gamma^t(x,y)$, $\lambda = 0$.*

2. *$\mathcal{I}_s^{z,ver}(x,y) + \mathcal{L}^{\lambda,z,ver}(x,y) \geq \lambda s$ with equality holds iff $s > \inf_{t\in(0,z)} \gamma^t(x,y)$, $\lambda = \xi_s^+ \wedge z - \xi_s^-$ or $s \leq \inf_{t\in(0,z)} \gamma^t(x,y)$, $\lambda = 0$.*

*Proof.* We prove part 1 of the lemma and part 2 is thus similarly proved. The sufficient condition "$\Leftarrow$" is easy to check. To see the other direction of the statement is true, if $s \leq \inf_{t\in(w,\mu)} \gamma^t(x,y)$, then inside the sup of $\mathcal{I}_s^{w,hor}(x,y)$ is always non-positive and takes value 0 when $u = v$. While $\mathcal{L}^{\lambda,w,hor}(x,y) > \lambda s$ as long as $\lambda > 0$ since $\mathcal{L}^{\lambda,w,hor}(x,y)$ is an integral of $\gamma^t(x,y)$ on some interval of $t$ with length $\lambda$ by Lemma 4.4.1 and the integrable equation (4.5). So the only value for $\lambda$ to make equation holds in the case $s \leq \inf_{t\in(w,\mu)} \gamma^t(x,y)$ is 0.

If $s > \inf_{t\in(w,\mu)} \gamma^t(x,y)$, then $u, v \in (w,\mu)$ should satisfy $u \geq \xi_s^-$ and $v \leq \xi_s^+$ since if not, we can adjust $u$ or $v$ closer to $\theta$ to make the sup larger since we subtract a negative value. If $w \leq \xi_s^-$ which means $u = \xi_s^-$ and $v = \xi_s^+$ are admissible, then

$$\mathcal{I}_s^{w,hor}(x,y) = (\xi_s^+ - \xi_s^-)s - \int_{\xi_s^-}^{\xi_s^+} \gamma^t(x,y)dt. \tag{4.49}$$

And the equation of the lemma holds when $\mathcal{L}^{\lambda,w,hor}(x,y) = \int_{\xi_s^-}^{\xi_s^+} \gamma^t(x,y)dt$, which means $\lambda = \xi_s^+ - \xi_s^-$ by Lemma 4.4.1. If $w > \xi_s^-$, then the best value of $u$ we can choose is $u = w$. Therefore

$$\mathcal{I}_s^{w,hor}(x,y) = (\xi_s^+ - w)s - \int_{w}^{\xi_s^+} \gamma^t(x,y)dt. \tag{4.50}$$

And the equation of the lemma holds when $\mathcal{L}^{\lambda,w,hor}(x,y) = \int_{w}^{\xi_s^+} \gamma^t(x,y)dt$, which means $\lambda = \xi_s^+ - w$ by Lemma 4.4.1. □

This lemma tells us for any $s \in \mathbb{R}$, we can find $\lambda^{hor}(s)$ and $\lambda^{ver}(s)$ such that two equations hold. We are now ready to give the upper bound of the right tail of the free energy.

**Lemma 4.4.4.** *Let $m, n \in \mathbb{Z}_{>0}$, $w \in [0,\mu)$, $z \in (0,\mu]$, and $s \in \mathbb{R}$,*

1. $\log P\left[\log\left(I_{(1,0)}Z^w_{(1,0)}(m,n)\right) \geq s\right] \leq -\mathcal{I}_s^{w,hor}(m,n)$.
2. $\log P\left[\log\left(J_{(0,1)}Z^z_{(0,1)}(m,n)\right) \geq s\right] \leq -\mathcal{I}_s^{z,ver}(m,n)$.
3. $\log P\left[\log Z^{w,z}(m,n) \geq s + \log 2\right] \leq -\mathcal{I}_s^{w,z}(m,n) + \log 2$.

*Proof.* The first two inequalities hold by taking exponential Markov inequality for $\lambda = \lambda^{hor}(s)$ and $\lambda = \lambda^{ver}(s)$ then using Lemma 4.4.2 and lemma 4.4.3. To check the third inequality,

$$\begin{aligned}
&\log P\left[\log Z^{w,z}(m,n) \geq s + \log 2\right] && (4.51)\\
=&\log P\left\{\log\left[I_{(1,0)}Z^w_{(1,0)}(m,n) + J_{(0,1)}Z^z_{(0,1)}(m,n)\right] \geq s + \log 2\right\} && (4.52)\\
\leq&\log\left\{P\left[\log\left(I_{(1,0)}Z^w_{(1,0)}(m,n)\right) \geq s\right] + P\left[\log\left(J_{(0,1)}Z^z_{(0,1)}(m,n)\right) \geq s\right]\right\} && (4.53)\\
\leq&\log 2 + \log P\left[\log\left(I_{(1,0)}Z^w_{(1,0)}(m,n)\right) \geq s\right] \vee \log P\left[\log\left(J_{(0,1)}Z^z_{(0,1)}(m,n)\right) \geq s\right] && (4.54)\\
\leq&\log 2 + (-\mathcal{I}_s^{w,hor}(m,n)) \vee (-\mathcal{I}_s^{z,ver}(m,n)) = \log 2 - \mathcal{I}_s^{w,z}(m,n). && (4.55)
\end{aligned}$$

$\square$

Finally, to prove the main theorem in log-gamma model, let $w' = w \vee \theta(m,n)$, $z' = z \wedge \theta(m,n)$. Define $c_0 = c_0(\delta)$ small enough and $N_0 = N_0(\delta, K, p, s_0)$ large enough such that for $s_0 \le s \le c_0(m+n)^{2/3}$ and $m, n \ge N_0$, the following condition holds,

$$\frac{K}{(m+n)^{1/3+p}} \le \frac{\sqrt{s}}{\sigma} < (z' - \epsilon) \wedge (\mu - \epsilon - w'). \tag{4.56}$$

From the above condition, one can conclude that both $w'$ and $w' + \sqrt{s}/\delta$ are in $(\epsilon, \mu - \epsilon)$, which means we can apply Lemma 4.2.1 and Lemma 4.2.5 to the shape function $\gamma^t(m,n)$ for $t \in [w', w' + \sqrt{s}/\delta]$. Notice that

$$\begin{aligned} \mathcal{I}^{w,hor}_{M+\sigma s}(m,n) &= \sup_{u,v \in (w,\mu), u \le v} \left\{ (v-u)(M+\sigma s) - \int_u^v \gamma^t(m,n)dt \right\} \\ &\ge \frac{\sqrt{s}}{\sigma}(M+\sigma s) - \int_{w'}^{w'+\sqrt{s}/\sigma} \gamma^t(m,n)dt \end{aligned} \tag{4.57}$$

holds by choosing $u = w'$ and $v = w' + \sqrt{s}/\delta$. Thus

$$\begin{aligned} \mathcal{I}^{w,hor}_{M+\sigma s}(m,n) &\ge \frac{\sqrt{s}}{\sigma}(M+\sigma s) - \int_{w'}^{w'+\sqrt{s}/\sigma} \gamma^t(m,n)dt && (4.58)\\ &\ge s^{3/2} - \frac{\sigma^3}{3}\left\{ (w' + \frac{\sqrt{s}}{\sigma} - \theta)^3 - (w' - \theta)^3 \right\} && (4.59)\\ &- C_1(m+n)\left\{ (w' + \frac{\sqrt{s}}{\sigma} - \theta)^4 + (w' - \theta)^4 \right\} && (4.60)\\ &= \frac{2s^{3/2}}{3} - \frac{\sigma^3}{3}\left\{ (w' + \frac{\sqrt{s}}{\sigma} - \theta)^3 - (w' - \theta)^3 - (\sqrt{s}/\sigma)^3 \right\} && (4.61)\\ &- C_1(m+n)\left\{ (w' + \frac{\sqrt{s}}{\sigma} - \theta)^4 + (w' - \theta)^4 \right\} && (4.62)\\ &\ge \frac{2s^{3/2}}{3} - C_0\left( \frac{Ks}{(m+n)^p} + \frac{s^2}{(m+n)^{1/3}} \right) && (4.63) \end{aligned}$$

for some constant $C_0 = C_0(\delta, c_0(\delta), C_1(\delta)) > 0$. The last inequality holds since $0 \le w' - \theta \le \sqrt{s}/\delta$.

When $s > S = c(m+n)^{2/3}$ for some $c \in [s_0(m+n)^{-2/3}, c_0]$, notice that

$$\mathcal{I}^{w,hor}_{M+\sigma s}(m,n) \geq \frac{\sqrt{S}}{\sigma}(M+\sigma s) - \int_{w'}^{w'+\sqrt{S}/\sigma} \gamma^t(m,n)dt \tag{4.64}$$

$$= \sqrt{S}(s-S) + \frac{\sqrt{S}}{\sigma}(M+\sigma S) - \int_{w'}^{w'+\sqrt{S}/\sigma} \gamma^t(m,n)dt \tag{4.65}$$

$$\geq \sqrt{S}(s-S) + \frac{2S^{3/2}}{3} - C_0\left(\frac{KS}{(m+n)^p} + \frac{S^2}{(m+n)^{1/3}}\right). \tag{4.66}$$

Combine the above two bounds together, for $t = s \wedge S$,

$$\mathcal{I}^{w,hor}_{M+\sigma s}(m,n) \geq \sqrt{S}(s-t) + \frac{2t^{3/2}}{3} - C_0\left(\frac{Kt}{(m+n)^p} + \frac{t^2}{(m+n)^{1/3}}\right). \tag{4.67}$$

The same bound holds for $\mathcal{I}^{z,ver}_{M+\sigma s}(m,n)$ via similar arguments. By Lemma 4.4.4, the desired result is proved.

### 4.4.3 Proof of Theorem 4.3.3

In this subsection we try to give a uniform proof that can cover all models (SI Beta II excluded, since it is a coordinate system-flip model of SI Beta I). First, three core lemmas that are required in the main proof are given below.

**Lemma 4.4.5.** *In all four models, let $(x,y), (x+\Delta, y) \in S_{\delta/2}$, then there exist $c_0 = c_0(\delta) > 0$, $C_0 = C_0(\delta) > 0$ such that*

$$c_0\Delta(x+y)^{-1} \leq |\theta(x+\Delta, y) - \theta(x,y)| \leq C_0\Delta(x+y)^{-1}. \tag{4.68}$$

*Proof.* Since $(x,y),\ (x+\Delta,y) \in S_{\delta/2}$, by part 1 in Lemma 4.2.1 in LG or analogue lemmas in the other models, there exist $0 < \epsilon = \epsilon(\delta/2) < \mathcal{E} = \mathcal{E}(\delta/2) < \infty$ such that $\theta(x,y),\ \theta(x+\Delta,y) \in (\epsilon, \mu-\epsilon)$ or $\theta(x,y),\ \theta(x+\Delta,y) \in (\epsilon, \mathcal{E})$, depending on different models. Recall the definition of $\theta$ in every model,

$$h_{\mu,\nu}(\theta(x,y)) = \frac{y}{x},\ \textit{and}\ h_{\mu,\nu}(\theta(x+\Delta)) = \frac{y}{x+\Delta}. \tag{4.69}$$

where functions $h_{\mu,\nu}(z)$ in different models are shown below.

| Model | $h_{\mu,\nu}(z)$ |
|---|---|
| LG | $\frac{\Psi_1(\mu-z)}{\Psi_1(z)}$ |
| G | $\frac{\Psi_1(\mu+z)}{\Psi_1(z)-\Psi_1(\mu+z)}$ |
| BI | $\frac{\Psi_1(\mu+z)-\Psi_1(\mu+\nu+z)}{\Psi_1(z)-\Psi_1(\mu+z)}$ |
| IB | $\frac{\Psi_1(\mu-z)-\Psi_1(\mu+\nu-z)}{\Psi_1(\mu+\nu-z)+\Psi_1(z)}$ |

Notice that for all $z \in (\epsilon, \mu-\epsilon)$ in LG, IB or $z \in (\epsilon, \mathcal{E})$ in gamma and beta models, there exist $c = c(\delta) > 0$, $C = C(\delta) > 0$,

$$C(\delta) \leq \Big|h_{\mu,\nu}(z)\Big| \leq c(\delta). \tag{4.70}$$

Thus, there exists $c_0 = c_0(\delta) > 0$, $C_0 = C_0(\delta) > 0$ such that

$$c_0\Delta(x+y)^{-1} \leq \left(\frac{y}{x} - \frac{y}{x+\Delta}\right)\frac{1}{c} \leq |\theta(x,y)-\theta(x+\Delta,y)| \leq \left(\frac{y}{x} - \frac{y}{x+\Delta}\right)\frac{1}{C} \leq C_0\Delta(x+y)^{-1}. \tag{4.71}$$

□

**Lemma 4.4.6.** *Let $(x,y) \in S_{\delta/2}$ and $\epsilon = \epsilon(\delta)$ given in part 1 of Lemma 4.2.1 in LG or analogue lemmas in other models, then there exists $C_1 = C_1(\delta) > 0$ and $\Lambda = \Lambda(\delta) > 0$ such that for all $0 < \lambda < \epsilon \wedge \Lambda$,*

$$|\zeta_\lambda^-(x,y) - \theta(x,y) + \frac{\lambda}{2}| = |\zeta_\lambda^+(x,y) - \theta(x,y) - \frac{\lambda}{2}| \leq C_1\lambda^2 \tag{4.72}$$

*Proof.* First notice that $\zeta_\lambda^+ - \zeta_\lambda^- = (\theta+\lambda/2) - (\theta-\lambda/2) = \lambda$, so either $\theta + \lambda/2 \leq \zeta_\lambda^+$ or $\zeta_\lambda^- \leq \theta - \lambda/2$ holds. Assume $\theta + \lambda/2 \leq \zeta_\lambda^+$ holds, since $(x,y) \in S_{\delta/2}$ and $\lambda < \epsilon$, we have $\theta(x,y) \pm \lambda/2 \in (\epsilon/2, \mathcal{E} + \epsilon/2)$ in gamma and beta models, or $\theta(x,y) \pm \lambda/2 \in (\epsilon/2, \mu - \epsilon/2)$ in LG and IB. The following inequality holds

$$\gamma^{\theta+\lambda/2}(x,y) \leq \gamma^{\zeta_\lambda^+}(x,y) = \gamma^{\zeta_\lambda^-}(x,y) \leq \gamma^{\theta-\lambda/2}(x,y). \tag{4.73}$$

Moreover, by Lemma 4.2.5, there exists $C = C(\delta) > 0$,

$$\left|\gamma^{\theta+\lambda/2}(x,y) - \gamma^{\theta-\lambda/2}(x,y)\right| \le C(x+y)\lambda^3. \tag{4.74}$$

Recall that $\theta + \lambda/2 \le \zeta_\lambda^+$ and the derivative of $\gamma^t(x,y)$ is increasing in $(\theta, \infty)$, we have

$$|\zeta_\lambda^+ - \theta - \lambda/2| \frac{d}{dt}\left(\gamma^t(x,y)\right)\Big|_{t=\theta+\lambda/2} \le \left|\gamma^{\theta+\lambda/2}(x,y) - \gamma^{\zeta_\lambda^+}(x,y)\right| \le C(x+y)\lambda^3. \tag{4.75}$$

Next, we need a lower bound for $[\gamma^t(x,y)]''$, notice that for all $\epsilon < t < \mu - \epsilon$,

$$\left|\left[\gamma^t(x,y)\right]'' - \left[\gamma^\theta(x,y)\right]''\right| \le 3(x+y)|t-\theta|\left[|\Psi_3(\epsilon)|\right]. \tag{4.76}$$

Thus, let $\Lambda = \Lambda(\delta) > 0$ small enough such that as long as $|t - \theta| < \Lambda$, $\left|[\gamma^t(x,y)]'' - \left[\gamma^\theta(x,y)\right]''\right| \le c(x+y)/2$ where c is defined as the lower bound constant in Lemma 4.2.1 in LG or analogue lemmas in other models. This also implies

$$\frac{d}{dt}\left(\gamma^t(x,y)\right)\Big|_{t=\theta+\lambda/2} \ge \frac{c}{2}(x+y)\left[\theta + \frac{\lambda}{2} - \theta\right] = \frac{c(x+y)\lambda}{4}. \tag{4.77}$$

Combine (4.75) and (4.77), we have proved the lemma. □

**Lemma 4.4.7.** *Under the same assumption as in Lemma 4.4.6, there exists $C_2 = C_2(\delta) > 0$, such that*

$$\left|\mathcal{L}^\lambda(x,y) - \lambda\gamma(x,y) - \frac{\lambda^3\sigma^3}{12}\right| \le C_2(x+y)\lambda^4. \tag{4.78}$$

*Proof.* Since $\mathcal{L}^\lambda(x,y) = \int_{\zeta_\lambda^-}^{\zeta_\lambda^+} \gamma^t(x,y)dt$, the lemma can be checked straightforwardly from Lemma 4.2.5 and Lemma 4.4.6. □

Now we turn back to the proof of Theorem 4.3.3. Let $N_0 = N_0(\delta, K) > 0$ such that whenever $(x,y) \in S_{\delta/2} \cap \mathbb{R}^2_{\ge N_0}$, $|w - \theta(x,y)| \le K(x+y)^{-1/3}$ implies $w \in (\epsilon, \mu - \epsilon)$ or $w \in (\epsilon, \mathcal{E})$, where $0 < \epsilon = \epsilon(\delta/4) < \mathcal{E} = \mathcal{E}(\delta/4) < \infty$ are functions of $\delta/4$ and are determined by Lemma 4.2.1 in LG, Lemma 4.2.2 in gamma model, Lemma 4.2.3 in IB, or Lemma 4.2.4 in beta model. Let $s_0 = s_0(\delta, K)$ large enough(to be determined later) satisfies (this is one of the condition) $C_0 s_0 \geq 2K$ where $C_0$ is the constant shown in Lemma 4.4.5. Notice that if $s \geq (m+n)^{1/3}$ then the theorem is trivial. We will prove the theorem first under the case $s_0 \leq s \leq c_0(m+n)^{1/3}$ where $c_0 = c_0(\delta) > 0$ small enough such that for all $k = s(m+n)^{2/3}$, $(m-k, n) \in S_{\delta/2}$.

Let $\lambda = \eta s(m+n)^{-1/3} < 1$ where $\eta \in (0,1)$ is small in a manner to be specified later. $0 < \mathrm{e}_0 = \mathrm{e}_0(\eta, s_0) \leq \eta s$ will also be specified later. Notice that as long as $|w - z| \leq \mathrm{e}_0(m+n)^{-1/3} \leq \lambda$,

$$Q^{w,z}(\text{Exit} > k) \leq Q^{z+\lambda,z}(\text{Exit} > k). \tag{4.79}$$

This is by the definition (4.13),

$$Q^{w,z}(\text{Exit} > k) = \prod_{\mathrm{j}=1}^{k+1} \frac{I_{(\mathrm{j},0)} Z^{w}_{(\mathrm{j},0)}(m,n)}{Z^{w,z}_{(\mathrm{j}-1,0)}(m,n)} = \prod_{\mathrm{j}=1}^{k+1} \left[ 1 - \frac{\omega_{((\mathrm{j}-1,0),(\mathrm{j}-1,1))} Z^{z}_{(\mathrm{j}-1,1)}(m,n)}{Z^{w,z}_{(\mathrm{j}-1,0)}(m,n)} \right]. \tag{4.80}$$

If $w$ is increased, then $Z^{w,z}_{(\mathrm{j}-1,0)}(m,n)$ is also increased, so is $Q^{w,z}(Exit > k)$ by the inverse-CDF coupling. By Chebyshev inequality, Hölder's inequality, and the independence of the weights on edges, we have the following bound for the annealed exit time tail.

$$\begin{aligned} E\left[Q^{w,z}(\text{Exit} > k)\right] &\leq E\left[Q^{z+\lambda,z}(\text{Exit} > k)\right] = \int_0^1 P\left( \frac{\left[\prod_{\mathrm{i}=1}^{k+1} I_{(\mathrm{i},0)}\right] Z^{z+\lambda}_{(k+1,0)}(m,n)}{Z^{z+\lambda,z}_{(0,0)}(m,n)} \geq t \right) dt \\ &\leq \int_0^1 t^{-\frac{\lambda}{2}} dt \sqrt{E\left[\left(\prod_{\mathrm{i}=1}^{k} I_{(\mathrm{i},0)}\right)^{\lambda}\right]} \sqrt{E\left[\left(I_{(k+1,0)} Z^{z+\lambda}_{(k+1,0)}(m,n)\right)^{\lambda}\right]} \sqrt{E\left[\left(Z^{z+\lambda,z}_{(0,0)}(m,n)\right)^{-\lambda}\right]}. \end{aligned} \tag{4.81}$$

By shift invariance property, shift the coordinate to the left by $k$ units, the second square root term is the same as $\sqrt{E\left[\left(I_{(1,0)} Z^{z+\lambda}_{(1,0)}(m-k,n)\right)^{\lambda}\right]}$. Moreover, by part 1 in Lemma 4.4.2 in LG and analogue lemmas in other models, the explicit formula of l.m.g.f in section 1, and the distribution of $I$, we are able to solve the value or bound the above terms as follows,

$$\begin{aligned} 2\log E\left[Q^{w,z}(\text{Exit} > k)\right] &\leq -2\log\left(1 - \frac{\lambda}{2}\right) + k\left(\log E[I^{\lambda}] + E[I^{-\lambda}]\right) \\ &\quad + \mathcal{L}^{\lambda, z+\lambda, hor}(m-k,n) + L^{z+\lambda,z}(m-k,n). \end{aligned} \tag{4.82}$$

The first term in the above inequality requires $1 - \lambda/2 > 0$. Also a simple inequality holds $-\log(1-a) \leq 2a$ when $a \leq 1/2$, this can be achieved by taking $\eta < 1$. Hence, the first term on the right hand side is bounded by $2\lambda$ for $\eta$ sufficiently small.

The second term in the above inequality can be bounded by $b_0 k\lambda^2 = b_0\eta^2 s^3$ for some $b_0 = b_0(\delta) > 0$ and for $\lambda \leq \eta$ small enough (depending only on $\delta$). This has already been checked in the previous subsections.

To estimate the third and last term in (4.82), first we claim that for $\eta$ small enough, $z + \lambda \leq \zeta_\lambda^-(m-k, n)$. To check this, by the definition of $N_0$ and the choice of $c_0$, both $(m, n)$ and $(m-k, n)$ are in $S_{\delta/2}$. Apply Lemma 4.4.5,

$$|\theta(m,n) - \theta(m-k,n)| \geq C_0 k(m+n)^{-1} = C_0 s(m+n)^{-\frac{1}{3}}. \tag{4.83}$$

Recall $C_0 s_0 \geq 2K$ and $s \geq s_0$, we have

$$|z - \theta(m-k,n)| \geq \frac{C_0 s}{2(m+n)^{\frac{1}{3}}}. \tag{4.84}$$

Moreover, it's easy to see $\theta(m-k,n) \geq \theta(m,n)$, so $|z - \theta(m-k,n)| = \theta(m-k,n) - z$. Let $\eta < C_0/4$, then

$$\theta(m-k,n) - z - 2\lambda \geq \left(\frac{C_0}{2} - 2\eta\right) \frac{s}{(m+n)^{\frac{1}{3}}} > 0. \tag{4.85}$$

Now combine Lemma 4.4.6 and let $\eta < 1/(2C_1)$, we have

$$z + \lambda < \theta(m-k,n) - \lambda \leq \zeta_\lambda^-(m-k,n). \tag{4.86}$$

By Lemma 4.4.1 in LG or analogue lemmas in other models, (4.86) shows that $\mathcal{L}^{\lambda,z+\lambda,hor}(m-k,n) = \mathcal{L}^{\lambda}(m-k,n)$. Next, use the estimations in Lemma 4.2.5 and lemma 4.4.7. There exists constant $C_3 = C_3(\delta) > 0$, such that

$$\begin{aligned}
&\mathcal{L}^{\lambda}(m-k,n) + L^{z+\lambda,z}(m-k,n) \leq \left(\lambda\gamma(m-k,n) + \frac{\lambda^3\sigma^3(m-k,n)}{12}\right) \\
&-\left[\lambda\gamma(m-k,n) + \frac{\sigma^3(m-k,n)}{3}\left((\theta(m-k,n)-z)^3 - (\theta(m-k,n)-z-\lambda)^3\right)\right] + C_3(m+n)\lambda^4 \\
&\leq \sigma^3(m-k,n)\left[-\lambda(\theta(m-k,n)-z)^2 + \lambda^2(\theta(m-k,n)-z)\right] + C_3(m+n)\lambda^4 \\
&\leq \frac{\sigma^3(m-k,n)}{2}\left[-\lambda\left(\theta(m-k,n)-z\right)^2\right] + C_3(m+n)\lambda^4.
\end{aligned} \tag{4.87}$$

Last inequality holds since $\theta(m-k,n) - z \geq 2\lambda$. Recall $k = s(m+n)^{2/3} \leq c_0(m+n)$, the definition of $\sigma(m-k,n)$, and the order of $\sigma(m-k,n)$ is $(m-k+n)^{1/3}$, there exists $c_1 = c_1(\delta) > 0$ that only depends on $\delta$, let $\eta < c_1$, we have

$$\mathcal{L}^{\lambda}(m-k,n) + L^{z+\lambda,z}(m-k,n) \leq -4C_4(m+n)\lambda\left(\theta(m-k,n)-z\right)^2 \leq -C_4C_0^2\eta s^3 \tag{4.88}$$

for some $C_4 = C_4(\delta) > 0$. Last inequality holds since (4.84).

Collect all the estimations of the upper bound of the right hand side in (4.82) together, we have

$$2\log E\left[Q^{w,z}(\text{Exit} > k)\right] \leq 2\lambda + b(\delta)\eta^2 s^3 - C_4C_0^2\eta s^3. \tag{4.89}$$

Let $\eta$ small enough, such that $2b\eta^2 \leq C_4C_0\eta$. Meanwhile, collect all the previous restrictions on $\eta$, one can find that there exist $\eta_0 = \eta_0(\delta) > 0$ that only depends on $\delta$, such that as long as $\eta \leq \eta_0$, all the previous restrictions on $\eta$ are satisfied. By this choice of $\eta$, we have

$$2\log E\left[Q^{w,z}(\text{Exit} > k)\right] \leq 2\lambda - \frac{1}{2}C_4C_0^2\eta s^3 \leq 2\eta - \frac{1}{2}C_4C_0^2\eta s^3 = \frac{1}{2}\eta\left(4 - C_4C_0^2 s^3\right). \tag{4.90}$$

Now, let $s_0(\delta, K) = \max\{2/\sqrt[3]{C_4C_0^2}, 2K/C_0\}$, and let $c = c(\delta) = C_4C_0^2\eta_0/8 > 0$, (4.90) becomes

$$\log E\left[Q^{w,z}(\text{Exit} > k)\right] \leq -cs^3 \tag{4.91}$$

for $s_0 \le s \le c_0(m+n)^{1/3}$ and for $|w-z| \le \lambda$. Finally, let $\mathrm{e}_0 = \mathrm{e}_0(\eta_0, s_0) = \eta_0 s_0$, then obviously, $|w-z| \le \mathrm{e}_0(m+n)^{-1/3} \le \lambda$.

When $c_0(m+n)^{1/3} \le s \le (m+n)^{1/3}$, notice that

$$\log E\left[Q^{w,z}(Exit > s(m+n)^{\frac{2}{3}})\right] \le \log E\left[Q^{w,z}(Exit > c_0(m+n))\right] \le -cc_0^3(m+n) \le -cc_0^3 s^3 \tag{4.92}$$

where $cc_0^3 > 0$ is a constant only depends on $\delta$. In the end, the upper bound for the horizontal exit time is proved for all $s \ge s_0$.

The analogous upper bound for the vertical exit time can be proved by the similar method. Notice that the value of $\mathrm{e}_0 = \eta_0 s_0$ might be changed, but it will be okay if we finally choose the minimum one. Then a union bound completes the proof.

### 4.4.4 Proof of Theorem 4.3.4

Our aim is to find the lower bound of $\log E[(Z(m,n))^\lambda]$. To reach our final goal, first we need lower bound estimations of $\log E[(Z^{w,z}(m,n))^\lambda]$, $\log E[(Z^{w,z,k,l}(m,n))^\lambda]$ where $Z^{w,z,k,l}$ is the partition function of a truncated multi-parameter model, and upper bound estimations of the exit time event $E[Q^w(\text{Exit} > 0)]$ and $E[Q^w(\text{Exit} < 0)]$. We try to write one version of all lemmas that can cover all 4 models(BII can be treated as a coordinate-flip model of BI, thus we omit here since many conditions in BII is totally the reverse side of BI so that it's tedious to repeat it again).

**Lemma 4.4.8 (first step exit time upper bound).** *Fix $\delta > 0$. Let $(m,n) \in S_\delta \cap \mathbb{Z}^2_{>0}$ (in Beta models extra assumption $(m,n) \in B_\delta(\xi^*)^+$ in SI Beta I is needed, this extra assumption will only be mentioned here and will be omitted in the rest of this section), $\theta = \theta(m,n)$ and $w \in (0,\mu)$ or $w \in (0,\infty)$, depending on the models. There exists a constant $c = c(\delta) > 0$ and $C = C(\delta) > 0$ such that*

$$\begin{aligned} &\log E[Q^w(\text{Exit} > 0)] \le -c(m+n)(\theta - w)^3 \;\; if \;\; w + \frac{C}{(m+n)^{\frac{1}{3}}} < \theta, \; and \\ &\log E[Q^w(\text{Exit} < 0)] \le -c(m+n)(w-\theta)^3 \;\; if \;\; w - \frac{C}{(m+n)^{\frac{1}{3}}} > \theta. \end{aligned} \tag{4.93}$$

*Proof.* The proof is similar to the one in the previous subsection, and it's even more simple than that. To check the first inequality above, first we consider the case when $w \geq \epsilon/2$, where $\epsilon = \epsilon(\delta)$ is the lower bound of $\theta$ that is defined in all models previously. Under this extra condition, we are able to use Taylor expansion estimations for $\gamma^t(m,n)$ when $t \geq w$. Follow the previous step in the proof of exit time upper bound, let $\lambda = \eta(\theta - w)$ where $\eta = \eta(\delta) > 0$ is a small manner which will be determined later. We have

$$
\begin{aligned}
&E\left[Q^w(\text{Exit} > 0)\right] \leq E\left[Q^{w+\lambda,w}(\text{Exit} > 0)\right] = \int_0^1 P\left(\frac{\left[I_{(1,0)}\right] Z^{w+\lambda}_{(1,0)}(m,n)}{Z^{w+\lambda,w}_{(0,0)}(m,n)} \geq t\right) dt \\
\leq &\int_0^1 t^{-\frac{\lambda}{2}} dt \sqrt{E\left[\left(I_{(1,0)} Z^{w+\lambda}_{(1,0)}(m,n)\right)^{\lambda}\right]} \sqrt{E\left[\left(Z^{w+\lambda,w}_{(0,0)}(m,n)\right)^{-\lambda}\right]}.
\end{aligned} \tag{4.94}
$$

Combine with the estimations on the two terms inside the square root, we have

$$
2\log E\left[Q^w(\text{Exit} > 0)\right] \leq -2\log\left(1 - \frac{\lambda}{2}\right) + \mathcal{L}^{\lambda, w+\lambda, hor}(m,n) + L^{w+\lambda,w}(m,n). \tag{4.95}
$$

Now, let $\eta < 1/2$, then $\zeta_\lambda^- \geq \theta - \lambda \geq w + \lambda$ and thus $\mathcal{L}^{\lambda, w+\lambda, hor}(m,n) = \mathcal{L}^\lambda(m,n)$. Use the fact that $\mathcal{L}^\lambda(m,n) \leq L^{\theta,\theta+\lambda}(m,n)$, there exists constant $C_3 = C_3(\delta) > 0$ such that

$$
\begin{aligned}
&\mathcal{L}^\lambda(m,n) + L^{w+\lambda,w}(m,n) \leq L^{\theta,\theta+\lambda}(m,n) - L^{w,w+\lambda}(m,n) \\
\leq &\left(\lambda M + \frac{\lambda^3\sigma^3}{3}\right) - \left[\lambda M + \frac{\sigma^3}{3}\left((\theta - w)^3 - (\theta - w - \lambda)^3\right)\right] + C_3(m+n)\lambda^4 \\
\leq &\sigma^3\left[-\lambda(\theta - w)^2 + \lambda^2(\theta - w)\right] + C_3(m+n)\lambda^4 \leq -\frac{\sigma^3}{2}\lambda(\theta - w)^2 + C_3(m+n)\lambda^4 \\
\leq &- C_4(m+n)\eta(\theta - w)^3.
\end{aligned} \tag{4.96}
$$

Last inequality holds since the order of $\sigma$ is $(m+n)^{1/3}$ and for $\eta$ small enough. Let $\eta$ small enough (e.g. $\eta < 1/(2\mu)$ in LG and IB or $\eta < 1/(2\mathcal{E})$ in gamma and beta models) such that $-\log(1 - \lambda/2) \leq \lambda$. Combine with the above upper bound estimation, let $\eta = \eta_0 = \eta_0(\delta) > 0$ that satisfies all the above requirements of $\eta$, we have

$$
2\log E\left[Q^w(\text{Exit} > 0)\right] \leq 2\lambda - C_4(m+n)\eta_0(\theta - w)^3 \leq \eta_0\left[2(\mu \vee \mathcal{E}) - C_4(m+n)(\theta - w)^3\right]. \tag{4.97}
$$

Finally, let $(\theta - w)^3(m+n) > C = C(\delta) > 0$ such that $2(\mu \vee \mathcal{E}) - C_4(m+n)(\theta - w)^3 \leq -C_4(m+n)(\theta - w)^3/2$. This finishes the proof for $w \geq \epsilon/2$. When $0 \leq w \leq \epsilon/2$, notice that $\theta - w \leq \mu \vee \mathcal{E}$, by the monotonicity of $Q^w$,

$$\begin{aligned} \log E\left[Q^w(\text{Exit} > 0)\right] &\leq \log E\left[Q^{\frac{\epsilon}{2}}(\text{Exit} > 0)\right] \\ &\leq -c(m+n)(\theta - \frac{\epsilon}{2})^3 \leq -c\left(\frac{\epsilon}{2}\right)^3 \frac{1}{(\mu \vee \mathcal{E})^3}(m+n)(\theta - w)^3 \end{aligned} \tag{4.98}$$

the desired inequality still holds after adjusting $c$ by a constant factor that depends only on $\delta$.

The vertical first step exit time upper bound uses the similar proof. Under LG and IB, consider $w \leq \mu - \epsilon/2$ first. Under the s-w and beta model, there is no need to separate the range of $w$ because the Taylor expansion estimation for $\gamma^t(m,n)$ holds for all $t \in (\epsilon, \infty)$. $\square$

**Lemma 4.4.9 (lower bound multi-para model).** *Let $(m,n) \in \mathbb{Z}_{>0}$, $w, z \in (0,\mu)$ or $(0,\infty)$, depending on different models. $\lambda > 0$ such that $\lambda \geq z - w$. Then*

$$\log E\left[(Z^{w,z}(m,n))^\lambda\right] \geq \mathcal{L}^{\lambda,w,z}(m,n). \tag{4.99}$$

*Proof.* First consider the case when $\lambda < \min\{\mu - w, z\}$ in LG and IB or just $\lambda < z$ in gamma and beta model, by the monotonicity of $Z^{w,z}$ and Lemma 4.4.1 for LG and the analogue lemmas for all other models, we have

$$\begin{aligned} \log E\left[(Z^{w,z}(m,n))^\lambda\right] &\geq \max\{\log E\left[\left(Z^{w,w+\lambda}(m,n)\right)^\lambda\right], \log E\left[\left(Z^{z-\lambda,z}(m,n)\right)^\lambda\right]\} \\ &= \max\{L^{w,w+\lambda}(m,n), L^{z-\lambda,z}(m,n)\} \\ &\geq \max\{\mathcal{L}^{\lambda,w,hor}(m,n), \mathcal{L}^{\lambda,z,ver}(m,n)\} = \mathcal{L}^{\lambda,w,z}(m,n). \end{aligned} \tag{4.100}$$

When $\lambda \geq \min\{\mu - w, z\}$ or $\lambda \geq z$, it's easy to see the left hand side is infinity, which makes the inequality trivial. $\square$

Define the truncated multi-parameter model as follows,

$$
\begin{aligned}
Z^{w,k,hor}(m,n) &= \sum_{\mathrm{i}=1}^{k}\Big[Z^w(\mathrm{i},0)\omega_{(\mathrm{i},0),(\mathrm{i},1)}Z_{(\mathrm{i},1)}(m,n)\Big] = I_{(1,0)}Z^w_{(1,0)}(m,n) - \prod_{\mathrm{i}=1}^{k+1} I_{(\mathrm{i},0)}Z^w_{(k+1,0)}(m,n)\\
Z^{z,l,ver}(m,n) &= \sum_{\mathrm{j}=1}^{l}\Big[Z^z(0,\mathrm{j})\omega_{(0,\mathrm{j}),(1,\mathrm{j})}Z_{(1,\mathrm{j})}(m,n)\Big] = J_{(0,1)}Z^z_{(0,1)}(m,n) - \prod_{\mathrm{j}=1}^{l+1} I_{(0,\mathrm{j})}Z^z_{(0,\mathrm{j}+1)}(m,n)\\
Z^{w,z,k,l}(m,n) &= Z^{w,k,hor}(m,n) + Z^{z,l,ver}(m,n)
\end{aligned}
\tag{4.101}
$$

**Lemma 4.4.10** (**lower bound truncated m-p model**). *Fix $\delta > 0$, There exist positive constants $C_0 = C_0(\delta) > 1$, $c_0 = c_0(\delta) < 1$, $K_0 = K_0(\delta)$, and $N_0 = N_0(\delta)$ such that*

$$\log E\left[\left(Z^{w,z,k,l}(m,n)\right)^{\lambda}\right] \geq \mathcal{L}^{\lambda}(m,n) - \log 3 \tag{4.102}$$

*whenever $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\geq N_0}$, $\lambda \in [C_0(m+n)^{-1/3}, c_0]$, $w = \zeta^-_\lambda(m,n)$, $z = \zeta^+_\lambda(m,n)$, and $k \leq m$, $l \leq n$ with*

$$\min\{k,l\} \geq K_0(m+n)^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\max\{1, \lambda^2(m+n)^{\frac{1}{2}}\}. \tag{4.103}$$

*Proof.* Set $C_0 = C_0(\delta)$, $c_0 = c_0(\delta) < 1$, $K_0 = K_0(\delta)$, and $N_0 = N_0(\delta)$ to be determined later. Let $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\geq N_0}$, $\lambda \in [C_0(m+n)^{-1/3}, c_0]$, $w = \zeta^-_\lambda(m,n)$, $z = \zeta^+_\lambda(m,n)$. $N_0$ large enough to make sure the preceding intervals are non-empty. Define

$$R_{m,n,\lambda} = K_0(m+n)^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\max\{1, \lambda^2(m+n)^{\frac{1}{2}}\}, \tag{4.104}$$

and consider $k, l \in [R_{m,n,\lambda}, 2R_{m,n,\lambda}]$. Let $c_0 = c_0(\delta, K_0)$ small enough, $N_0 = N_0(\delta, C_0, K_0)$ large enough such that both $(m-k,n), (m,n-l)$ are in $S_{\delta/2}$.

Notice that

$$Z^{w,z}(m,n) = Z^{w,z,k,l}(m,n) + \prod_{\mathrm{i}=1}^{k+1} I_{(\mathrm{i},0)}Z^w_{(k+1,0)}(m,n) + \prod_{\mathrm{j}=1}^{l+1} J_{(0,\mathrm{j})}Z^z_{(0,l+1)}(m,n), \tag{4.105}$$

together with Lemma 4.4.9, Lemma 4.4.2 for l-g model and analogue lemmas for other three models, recall $\lambda \le c_0 < 1$ we have

$$\begin{aligned}
&\mathcal{L}^{\lambda,w,z}(m,n) \le \log 3 + \max\{\log E\left[\left(Z^{w,z,k,l}(m,n)\right)^{\lambda}\right], \\
&k\log E[I^{\lambda}] + \log E\left[\left(I_{(k+1,0)}Z^{w}_{(k+1,0)}(m,n)\right)^{\lambda}\right], l\log E[J^{\lambda}] + \log E\left[\left(J_{(0,l+1)}Z^{z}_{(0,l+1)}(m,n)\right)^{\lambda}\right]\} \\
\le& \max\left\{\log E\left[\left(Z^{w,z,k,l}(m,n)\right)^{\lambda}\right], k\log E[I^{\lambda}] + \mathcal{L}^{\lambda,w,hor}(m-k,n), l\log E[J^{\lambda}] + \mathcal{L}^{\lambda,z,ver}(m,n-l)\right\} \\
&+\log 3.
\end{aligned} \tag{4.106}$$

Next step is to show both $k\log E[I^{\lambda}]+\mathcal{L}^{\lambda,w,hor}(m-k,n)$ and $l\log E[J^{\lambda}]+\mathcal{L}^{\lambda,z,ver}(m,n-l)$ are less than $\mathcal{L}^{\lambda,w,z}(m,n) - \log 3$ for big enough $k,l$ and small enough $\lambda$. To check this, first notice that if $w = \zeta_{\lambda}^{-}(m,n)$ and $z = \zeta_{\lambda}^{+}(m,n)$, then $\mathcal{L}^{\lambda,w,z}(m,n) = \mathcal{L}^{\lambda}(m,n) = L^{w,z}(m,n)$, and $L^{w,z}(m,n) - k\log E[I^{\lambda}] = L^{w,z}(m-k,n)$ as well as $L^{w,z}(m,n) - l\log E[J^{\lambda}] = L^{w,z}(m,n-l)$.

In order to get $\mathcal{L}^{\lambda,w,hor}(m-k,n) = \mathcal{L}^{\lambda}(m-k,n)$, we need $w = \zeta_{\lambda}^{-}(m,n) \le \zeta_{\lambda}^{-}(m-k,n)$. For all the four models ,there is a momotonicity for $\theta$ with different parameters $(m,n)$, that is

$$\theta(m-k,n) \ge \theta(m,n) \ge \theta(m,n-l) \tag{4.107}$$

as long as $k,l \ge 0$. This can be easily seen in the proof of the uniqueness of $\theta$, that $g$ is a decreasing function (This does not include the case in BII, but as we mentioned at the beginning, BII is just a coordinate system-flip BI, hence we don't consider it here).

Moreover, by Lemma 4.4.6, for $\lambda$ small enough ($c_0 = c_0(\delta)$ small enough), $\zeta_{\lambda}^{-} \in [\theta - \lambda/2 - C_1\lambda^2, \theta - \lambda/2 + C_1\lambda^2]$. Thus, to reach our goal, we need $\theta(m-k,n) - \theta(m,n) \ge 2C_1\lambda^2$. By Lemma 4.4.5, we need

$$\frac{k}{m+n} \ge c\lambda^2 \tag{4.108}$$

for some $c = c(\delta) > 0$. This can be satisfied by letting $\lambda \le c_0 = c_0(\delta, K_0)$ small enough since $R_{m,n,\lambda}/(m+n) \ge K_0\lambda^{3/2}$.

Write $\theta = \theta(m, n)$, $\tilde{\theta} = \theta(m-k, n)$, and $\tilde{\sigma} = \sigma(m-k, n)$. The last step is to check

$$\mathcal{L}^{\lambda}(m-k, n) - L^{w,z}(m-k, n) < -\log 3. \tag{4.109}$$

In order to control $w - \tilde{\theta}$ and $z - \tilde{\theta}$, by Lemma 4.4.5 again, there exists $C = C(\delta) > 0$, as long as

$$\frac{k}{m+n} \le C\lambda, \tag{4.110}$$

then $\tilde{\theta} - \theta \le \lambda$ and thus $\max\{|w - \tilde{\theta}|, |z - \tilde{\theta}|\} \le 2\lambda$. (4.110) can be satisfied as long as $C_0 = C_0(\delta, K_0)$ large enough and $c_0 = c_0(\delta, K_0)$ small enough since

$$\frac{2R_{m,n,\lambda}}{(m+n)\lambda} \le \max\{\frac{2K_0}{(m+n)^{\frac{1}{2}}\lambda^{\frac{3}{2}}}, 2K_0\lambda^{\frac{1}{2}}\} \le \max\{2K_0C_0^{-\frac{3}{2}}, 2K_0c_0^{\frac{1}{2}}\} \tag{4.111}$$

By Lemmas 4.2.5, 4.4.5, 4.4.6, and 4.4.7, we have

$$\begin{aligned}
&\mathcal{L}^{\lambda}(m-k, n) - L^{w,z}(m-k, n) \\
\le& \frac{\tilde{\sigma}^3}{3}\left[\frac{\lambda^3}{4} - \left(\tilde{\theta} - w\right)^3 + \left(\tilde{\theta} - z\right)^3\right] + C_2(m+n)\lambda^4 \\
\le& \frac{\tilde{\sigma}^3}{3}\left[\frac{\lambda^3}{4} - \left(\tilde{\theta} - \theta + \frac{\lambda}{2}\right)^3 + \left(\tilde{\theta} - \theta - \frac{\lambda}{2}\right)^3\right] + C_3(m+n)\lambda^4 \\
=& -\tilde{\sigma}^3\lambda(\tilde{\theta} - \theta)^2 + C_3(m+n)\lambda^4 \\
\le& -C_4\frac{k^2\lambda}{m+n} + C_3(m+n)\lambda^4 \le -\frac{C_4}{2}K_0^2
\end{aligned} \tag{4.112}$$

for some positive constants $C_2 = C_2(\delta),\ C_3 = C_3(\delta),\ C_4 = C_4(\delta)$. The last inequality holds since if $\lambda \le (m+n)^{-1/4}$, then let $K_0 = K_0(\delta)$ large enough, $-C_4K_0^2 + C_3 \le -C_4K_0^2/2$. If $\lambda > (m+n)^{-1/4}$, then

$$-C_4\frac{k^2\lambda}{m+n} + C_3(m+n)\lambda^4 \le \left[-C_4K_0^2 + C_3\right](m+n)\lambda^4 \le -\frac{C_4K_0^2}{2} \tag{4.113}$$

for $K_0 = K_0(\delta)$ large enough and $C_0 > 1$. Finally, let $K_0(\delta)$ large enough such that $b_0K_0^2 > \log 3$, which will finish our proof under the case $k, l \in [R_{m,n,\lambda}, 2R_{m,n,\lambda}]$. When $k, l \ge 2R_{m,n,\lambda}$, by the monotonicity of $Z^{w,z,k,l}$, (4.102) still holds. □

Now we turn to prove Theorem 4.3.4. First, let $C_0, c_0, K_0, N_0$ be constants that are determined in Lemma 4.4.10, their values will be adjusted later in the proof. Introduce parameter $q \in [1,2]$ that will be determined later. Pick $(m,n) \in S_\delta \cap \mathbb{Z}^2_{\geq N_0}$ and $\lambda \in [C_0(m+n)^{-1/3}q, c_0 q]$. Let $\lambda_q = \lambda/q$, abbreviate $\zeta^- = \zeta^-_{\lambda_q}(m,n)$, $\zeta^+ = \zeta^+_{\lambda_q}(m,n)$ and

$$k = \lceil K_0(m+n)^{\frac{1}{2}} \lambda_q^{-\frac{1}{2}} \max\{1, \lambda_q^2 (m+n)^{\frac{1}{2}}\} \rceil. \tag{4.114}$$

In the proof of Lemma 4.4.10, k is such a integer that both $(m-k,n)$ and $(m,n-k)$ are in $S_{\delta/2}$. By the same lemma, we have

$$\begin{aligned} \mathcal{L}^{\lambda_q}(m,n) - \log 3 &\leq \log E\left[\left(Z^{\zeta^-,\zeta^+,k,k}(m,n)\right)^{\lambda_q}\right] \\ &\leq \max\{\log E\left[\left(Z^{\zeta^-,k,hor}(m,n)\right)^{\lambda_q}\right], \log E\left[\left(Z^{\zeta^+,k,ver}(m,n)\right)^{\lambda_q}\right]\} + \log 2. \end{aligned} \tag{4.115}$$

The last inequality holds since $\lambda_q \leq c_0 \leq 1$. Due to symmetry, we can assume that

$$\log E\left[\left(Z^{\zeta^-,k,hor}(m,n)\right)^{\lambda_q}\right] \geq \mathcal{L}^{\lambda_q}(m,n) - \log 6 \tag{4.116}$$

holds.

Next step is to estimate the ratio between $Z^{\zeta^-,k,hor}(m,n)$ and $Z(m,n)$. Introduce increment-stationary model $Z^z(m,n)$ where $z = \theta - R\lambda_q$, $R = R(\delta) \geq 1$ to be determined below. Couple this model with SI model $Z^{\zeta^-}(m,n)$ as follows. For $(\mathrm{i},\mathrm{j}) \in \mathbb{N}^2_{\leq (m,n)}$,

$$Z^{\zeta^-}_{(\mathrm{i},\mathrm{j})}(m,n) = Z_{(\mathrm{i},\mathrm{j})}(m,n) = Z^z_{(1,1)}(m+1-\mathrm{i}, n+1-\mathrm{j}). \tag{4.117}$$

i.e. couple all the bulk weights inside the first quadrant in the way $\omega^{\zeta^-}_{\mathrm{e}} = \omega^z_{(m+1,n+1)-\mathrm{e}}$. Moreover, the boundary weights in $Z^z$ is independent of the ones in SI model $Z^{\zeta^-}$. Decrease $c_0 = c_0(R)$ if necessary to make sure $z \in (0,\theta)$. Define the event

$$E_z = \{I_{(1,0)} Z^z_{(1,0)}(m,n) \leq J_{(0,1)} Z^z_{(0,1)}(m,n)\}. \tag{4.118}$$

Then let $C_0 = C_0(R)$ large enough to apply Lemma 4.4.8, we have

$$\begin{aligned} P(E_z^c) &= P\left(I_{(1,0)} Z^z_{(1,0)}(m,n) > J_{(0,1)} Z^z_{(0,1)}(m,n)\right) \\ &= P\left(Q^z(\text{Exit} > 0) > \frac{1}{2}\right) \leq 2E\left[Q^z(\text{Exit} > 0)\right] \leq 2exp\{-c(m+n)(\theta - z)^3\}. \end{aligned} \tag{4.119}$$

With the event $E_z$ defined above, simply separate the LHS in (4.116) into two parts,

$$\mathcal{L}^{\lambda_q}(m,n) - \log 12 \leq \max\{\log E\left[1_{E_z}\left(Z^{\zeta^-,k,hor}(m,n)\right)^{\lambda_q}\right], \log E\left[1_{E_z^c}\left(Z^{\zeta^-,k,hor}(m,n)\right)^{\lambda_q}\right]\}. \tag{4.120}$$

We will prove, under the event $E_z$, $Z^{\zeta^-,k,hor}(m,n)$ and $Z(m,n)$ are close to each other. While under the event $E_z^c$,

$$\log E\left[1_{E_z^c}\left(Z^{\zeta^-,k,hor}(m,n)\right)^{\lambda_q}\right] < \mathcal{L}^{\lambda_q}(m,n) - \log 12. \tag{4.121}$$

To check (4.121), apply Holder's inequality with $p = 1 + R^{-3}$, together with Lemma 4.4.1 and 4.4.2,

$$\begin{aligned} \log E\left[1_{E_z^c}\left(Z^{\zeta^-,k,hor}(m,n)\right)^{\lambda_q}\right] &\leq \frac{1}{p}\log E\left[\left(I_{(1,0)} Z^{\zeta^-}_{(1,0)}(m,n)\right)^{p\lambda_q}\right] + \frac{p-1}{p}\log P(E_z^c) \\ &\leq \frac{1}{p}\mathcal{L}^{p\lambda_q,\zeta^-,hor}(m,n) - \frac{1}{p}c(m+n)\lambda_q^3 \\ &= \frac{1}{p}L^{\zeta^-,\zeta^-+p\lambda_q}(m,n) - \frac{1}{p}c(m+n)\lambda_q^3. \end{aligned} \tag{4.122}$$

The last equation holds since $\zeta^-_{p\lambda_q} \leq \zeta^-_{\lambda_q} = \zeta^-$. Apply Taylor expansion estimation on $L$ by taking $c_0 = c_0(\delta)$ small enough, together with Lemma 4.4.6, we have

$$\begin{aligned} &\log E\left[1_{E_z^c}\left(Z^{\zeta^-,k,hor}(m,n)\right)^{\lambda_q}\right] \leq \frac{1}{p}L^{\zeta^-,\zeta^-+p\lambda_q}(m,n) - \frac{1}{p}c(m+n)\lambda_q^3 \\ &\leq \lambda_q M + \frac{\sigma^3}{3p}\left[(\theta-\zeta^-)^3 - (\theta-\zeta^- - p\lambda_q)^3\right] + C_1 p^3 \lambda_q^4(m+n) - \frac{1}{p}c(m+n)\lambda_q^3 \\ &\leq \lambda_q M + \frac{\sigma^3 {\lambda_q}^3}{12} + \frac{\sigma^3 {\lambda_q}^3}{6}(2p-1)(p-1) + C_2\lambda_q^4(m+n) - \frac{1}{p}c(m+n)\lambda_q^3 \\ &\leq \mathcal{L}^{\lambda_q}(m,n) + a_0 R^{-3}{\lambda_q}^3(m+n) + C_3\lambda_q^4(m+n) - \frac{1}{2}c(m+n)\lambda_q^3. \end{aligned} \tag{4.123}$$

Let $R = R(\delta)$ large enough, then $\lambda_q \leq c_0 = c_0(\delta, R)$ small enough, and $C_0 = C_0(\delta)$ large enough, we have

$$\begin{aligned}\log E\left[1_{E_z^c}\left(Z^{\zeta^-,k,hor}(m,n)\right)^{\lambda_q}\right] &\leq \mathcal{L}^{\lambda_q}(m,n) - \frac{1}{8}c(m+n)\lambda_q^3 \\ &\leq \mathcal{L}^{\lambda_q}(m,n) - \frac{1}{8}cC_0^3 < \mathcal{L}^{\lambda_q}(m,n) - \log 12,\end{aligned} \tag{4.124}$$

which finishes the proof of (4.121).

The last step of the proof is to bound $Z^{\zeta^-,k,hor}(m,n)/Z(m,n)$ under the event $E_z$. To this end, notice that the following inequalities hold,

$$\begin{aligned}\frac{Z^{\zeta^-,k,hor}}{Z_{(1,1)}(m,n)} &= \sum_{\mathrm{i}=1}^{k} \frac{Z^{\zeta^-}(\mathrm{i},0)\omega_{(\mathrm{i},0)(\mathrm{i},1)}Z_{(\mathrm{i},1)}(m,n)}{Z_{(1,1)}(m,n)} \\ &= \sum_{\mathrm{i}=1}^{k} Z^{\zeta^-}(\mathrm{i},0)\omega_{(\mathrm{i},0)(\mathrm{i},1)}\frac{Z^z_{(1,1)}(m+1-\mathrm{i},n)}{Z^z_{(1,1)}(m,n)} \\ &\leq \sum_{\mathrm{i}=1}^{k} Z^{\zeta^-}(\mathrm{i},0)\omega_{(\mathrm{i},0)(\mathrm{i},1)}\frac{Z^z_{(0,1)}(m+1-\mathrm{i},n)}{Z^z_{(0,1)}(m,n)} \\ &\leq 2\sum_{\mathrm{i}=1}^{k} Z^{\zeta^-}(\mathrm{i},0)\omega_{(\mathrm{i},0)(\mathrm{i},1)}\frac{Z^z_{(0,0)}(m+1-\mathrm{i},n)}{Z^z_{(0,0)}(m,n)}\end{aligned} \tag{4.125}$$

The second line is by the coupling of $Z^{\zeta^-}$ and $Z^z$, the third line is by "crossing lemma"

**Lemma 4.4.11** (**Crossing lemma for the positive temperature model**). *Let $Z(m,n)$ be the partition function of any positive temperature model(i.e. $Z^{w,z}(m,n)$ is also allowed). For any 4 integer points in the first quadrant $v_1$, $v_2$, $v_3$, $v_4$, as long as if we run the loop $v_1 \to v_2 \to v_3 \to v_4 \to v_1$, the loop does not intersect with itself and form a quadrilateral, the following inequality holds.*

$$Z_{v_1}(v_3)Z_{v_2}(v_4) \leq Z_{v_1}(v_2)Z_{v_3}(v_4). \tag{4.126}$$

The last inequality of (4.125) holds since in the event $E_z$, $2J_{(0,1)}Z^z_{(0,1)}(m,n) \geq Z(m,n)$.

From (4.125), recall $Z^z$ is the partition function of a stationary- increment model, for $1 \leq \mathrm{i} \leq m$, define

$$M_{\mathrm{i}} = \log Z^{\zeta^-}_{(1,0)}(\mathrm{i}, 0) - (\mathrm{i}-1) E\left[\log I^{(\zeta^-)}\right] + \log Z^z(m+1-\mathrm{i}, n) - \log Z^z(m, n) + (\mathrm{i}-1) E\left[\log I^{(z)}\right] \tag{4.127}$$

where $I^{(\zeta^-)}$ and $I^{(z)}$ means the horizontal boundary weights in $Z^{\zeta^-}$ and $Z^z$, respectively. Then

$$\frac{Z^{\zeta^-,k,hor}}{Z_{(1,1)}(m,n)} \leq 2 I^{(\zeta^-)}_{(1,0)} \mathrm{e}^{\max_{1\leq \mathrm{i}\leq k} M_{\mathrm{i}} + k\left|E\left[\log I^{(z)}\right] - E\left[\log I^{(\zeta^-)}\right]\right|} \sum_{\mathrm{i}=1}^{k} \omega_{(\mathrm{i},0)(\mathrm{i},1)}. \tag{4.128}$$

Moreover, $I^{(\zeta^-)}_{(0,1)}$, $\log Z_{(1,1)}(m,n) + \max_{1\leq \mathrm{i}\leq k} M_{\mathrm{i}}$, and $\omega_{(\mathrm{i},0)(\mathrm{i},1)}$ are independent of each other (this is due to the coupling we construct). Combine (4.120), (4.121), and (4.128), recall $\lambda_q \leq \lambda \leq 1$ then apply Jensen's inequality, we have

$$\begin{aligned} \mathcal{L}^{\lambda_q}(m,n) - \log 12 \leq & \log E\left[\left(Z_{(1,1)}(m,n)\right)^{\lambda_q} \mathrm{e}^{\lambda_q \max_{1\leq \mathrm{i}\leq k} M_{\mathrm{i}}}\right] + \log 2 \\ & + \lambda_q k \left| E\left[\log I^{(z)}\right] - E\left[\log I^{(\zeta^-)}\right]\right| + \log E\left[\left(I^{(\zeta^-)}\right)^{\lambda_q}\right] \\ & + \log k + \log E\left[\left(\omega_{(1,0)(1,1)}\right)^{\lambda_q}\right]. \end{aligned} \tag{4.129}$$

The terms $\lambda_q k\left|E\left[\log I^{(z)}\right] - E\left[\log I^{(\zeta^-)}\right]\right| + \log E\left[\left(I^{(\zeta^-)}\right)^{\lambda_q}\right] + \log E\left[\left(\omega_{(1,0)(1,1)}\right)^{\lambda_q}\right]$ can be bounded by $A_0 k R {\lambda_q}^2$ for some $A_0 = A_0(\delta)$ large enough and $c_0 = c_0(\delta)$ small enough such that $|\zeta^- - \theta| \leq \lambda_q$. One can check with explicit distribution functions of all those weights in all four models to see it is true. Using mean value theorem the first term is bounded by $A_0 \lambda_q k |\zeta^- - z|$. The last two terms are of order $\lambda_q$, but with $k\lambda_q >> 1$ they can also be bounded of order $k\lambda_q^2$.

By the definition of $k$ it's easy to check $k\lambda_q^2 >> (m+n)^p$ for some $p > 0$. Thus, let $N_0 = N_0(c_0, C_0, R, \delta)$ large enough, $\log k < A_0 k R \lambda_q^2$. Finally, with all the bounds we get, apply Holder's inequality to the first term on the RHS in (4.129), together with simple inequality $Z_{(1,1)}(m,n) \le Z(m,n)$, we have

$$\mathcal{L}^{\lambda_q}(m,n) - A_0(1 + kR\lambda_q^2) \le \frac{1}{q} \log E\left[(Z(m,n))^{\lambda}\right] + \frac{q-1}{q} \log E\left[\mathrm{e}^{\frac{\lambda}{q-1}\max_{1\le \mathrm{i} \le k} M_{\mathrm{i}}}\right]. \quad (4.130)$$

Using Lemma 4.4.12 which is an estimate of the right tail of the maximum of a martingale which will be shown at the very end of the proof, there exist $b_0 = b_0(\delta)$ and $B_0 = B_0(\delta)$ such that

$$E\left[\mathrm{e}^{\frac{\lambda}{q-1}\max_{1\le \mathrm{i}\le k} M_{\mathrm{i}}}\right] \le B_0 + \frac{B_0 k^{\frac{1}{2}}\lambda}{q-1}\mathrm{e}^{\frac{B_0 k\lambda^2}{(q-1)^2}} \le B_0 \mathrm{e}^{\frac{B_0 k\lambda^2}{(q-1)^2}} \quad (4.131)$$

provided that $\lambda/(q-1) \le b_0$. We will show later in this page with our choice of $q$, this inequality will be satisfied. The last inequality holds after increase $B_0$ and let $N_0 = N_0(c_0, \delta)$ large enough.

Combine the above two inequalities together, and apply Lemma 4.4.7,

$$\begin{aligned} \frac{1}{q}\log E\left[(Z(m,n))^{\lambda}\right] &\ge \mathcal{L}^{\lambda_q}(m,n) - A_0(1+kR\lambda_q^2) - \frac{B_0 k\lambda^2}{q-1} \\ &\ge \lambda_q M + \frac{\lambda_q^3\sigma^3}{12} - A_0\left(\lambda^4(m+n) + 1 + kR\lambda^2\right) - \frac{B_0 k\lambda^2}{q-1} \end{aligned} \quad (4.132)$$

after replacing $kR\lambda_q^2$ by $kR\lambda^2$ since $\lambda_q \le \lambda \le 2\lambda_q$ and adjusting $A_0 = A_0(\delta)$. Multiply both sides by $q \in (1, 2]$, apply Lemma 4.4.7 again with parameter $\lambda$, and use the choice of $k$ to obtain

$$\begin{aligned} &\log E\left[(Z(m,n))^{\lambda}\right] - \mathcal{L}^{\lambda}(m,n) \ge -D_0\left[\lambda^4(m+n) + 1 + \frac{k\lambda^2}{q-1} + (q-1)\lambda^3(m+n)\right] \\ &\ge -D_0\left[\lambda^4(m+n) + 1 + (q-1)\lambda^3(m+n) + \frac{\lambda^{\frac{3}{2}}(m+n)^{\frac{1}{2}}}{q-1}\max\{1, \lambda^2(m+n)^{\frac{1}{2}}\}\right] \end{aligned} \quad (4.133)$$

for some constant $D_0 = D_0(\delta, A_0, B_0, K_0, R) > 0$.

Now choose the value of $q$ and check it indeed satisfies our previous conditions. Let

$$q = 1 + \lambda^{-\frac{3}{4}}(m+n)^{-\frac{1}{4}}\max\{1, \lambda(m+n)^{\frac{1}{4}}\}, \quad (4.134)$$

then $q > 1$ is obvious. Moreover, $q \leq 1 + \max\{(C_0)^{-3/4}, (qc_0)^{1/4}\}$ will guarantee $q < 2$ once $C_0$ large enough and $c_0$ small enough. After we 've checked $q \in (1,2)$, $q \geq 1 + \lambda^{1/4} \geq 1 + \lambda(2c_0)^{-3/4}$ will let $\lambda/(q-1) \leq b_0$ hold by choosing $c_0 = c_0(\delta)$ small enough. Thus (4.134) gives an admissible value of $q$ that we use throughout the proof.

Back to (4.133), with the value of $q$ we choose, the last two terms inside the bracket on the last line are equal, with lower bounds

$$\lambda^{\frac{9}{4}}(m+n)^{\frac{3}{4}} \max\{1, \lambda(m+n)^{\frac{1}{4}}\} \geq \lambda^{\frac{13}{4}}(m+n) \geq \lambda^4(m+n) \tag{4.135}$$

provided $c_0 < 1$ and

$$\lambda^{\frac{9}{4}}(m+n)^{\frac{3}{4}} \max\{1, \lambda(m+n)^{\frac{1}{4}}\} \geq (C_0)^{\frac{9}{4}} \geq 1 \tag{4.136}$$

provided $C_0 > 1$. Hence,

$$\log E\left[(Z(m,n))^{\lambda}\right] - \mathcal{L}^{\lambda}(m,n) \geq -4D_0\lambda^{\frac{9}{4}}(m+n)^{\frac{3}{4}} \max\{1, \lambda(m+n)^{\frac{1}{4}}\} \tag{4.137}$$

for $\lambda \in [2C_0(m+n)^{-1/3}, c_0]$. The equivalent statement in the theorem holds since (4.133) still holds if we replace $\mathcal{L}^{\lambda}(m,n)$ by $\lambda M + \lambda^3\sigma^3/12$ and adjusting $D_0$ by applying Lemma 4.4.7.

The lemma that we mentioned above without clarify is

**Lemma 4.4.12.** *Let $n \geq 0$ and $\{X_{\mathrm{i}}\}_{\mathrm{i}=1}^{n}$ be a collection of i.i.d. random variables with tail distribution $P(|X_1| \geq x) \leq C\mathrm{e}^{-xI}$ for some positive constants $C$, $I$ and for all $x > 0$. Define $M_0 = 0$ and*

$$M_k = \sum_{\mathrm{i}=1}^{k} [X_{\mathrm{i}} - E(X_{\mathrm{i}})] \tag{4.138}$$

*for $0 \leq k \leq n$. Then the following inequalities holds for some $c = c(C, I) > 0$.*

$$P\left(\max_{0 \leq k \leq n} M_k \geq x\right) \leq \mathrm{e}^{-cx\min\{\frac{x}{n},1\}} \tag{4.139}$$

*for* $x \geq 0$ *and*

$$E\left[\mathrm{e}^{\lambda \max_{0\leq k\leq n} M_k}\right] \leq 1 + \lambda\sqrt{\frac{n}{c}}\mathrm{e}^{\frac{n\lambda^2}{4c}} + \frac{\lambda}{c-\lambda}\mathrm{e}^{-(c-\lambda)n} \tag{4.140}$$

*for* $0 \leq \lambda < c$.

*Proof.* Let $\lambda < I$, since the tail distribution of $X_1$ is bounded of order $\mathrm{e}^{-Ix}$, then $E[\mathrm{e}^{\lambda X_1}]$ is finite as well as $E[X_1^m]$ for all $m \geq 1$. Therefore,

$$E\left[\mathrm{e}^{\lambda M_k}\right] = \left[E(\mathrm{e}^{\lambda X_1})\right]^k \mathrm{e}^{-kE[X_1]} < \infty. \tag{4.141}$$

The sequence $\{\mathrm{e}^{\lambda M_k}\}_{0\leq k\leq n}$ is a submartingale. Applying Doob's maximal inequality leads to

$$\begin{aligned} P\left(\max_{0\leq k\leq n} M_k \geq x\right) &\leq E\left[\mathrm{e}^{\lambda M_k}\right]\mathrm{e}^{-\lambda x} \\ &= \exp\{n\left[\log E[\mathrm{e}^{\lambda X_1}] - \lambda E[X_1]\right] - \lambda x\}. \end{aligned} \tag{4.142}$$

Consider $\log E[\mathrm{e}^{\lambda X_1}]$ as a function of $\lambda$, then use Taylor expansion up to the second order, there exists $\Lambda = \Lambda(C, I) > 0$ such that for $0 < \lambda \leq \Lambda$

$$\log E[\mathrm{e}^{\lambda X_1}] - \lambda E[X_1] \leq \frac{Var(X_1)+1}{2}\lambda^2. \tag{4.143}$$

Hence, write $V = Var(X_1) < \infty$,

$$P\left(\max_{0\leq k\leq n} M_k \geq x\right) \leq \exp\{\frac{n(V+1)}{2}\lambda^2 - \lambda x\} \tag{4.144}$$

given $\lambda \leq \min\{\Lambda, I/2\}$. Let $\lambda = \min\{\Lambda, I/2, x/(nV+n)\}$, then there exists $c = c(C, I) > 0$ such that

$$P\left(\max_{0\leq k\leq n} M_k \geq x\right) \leq -\frac{1}{2}\min\{\Lambda x, \frac{xI}{2}, \frac{x^2}{n(V+1)}\} \leq -c\min\{\frac{x}{n}, 1\}. \tag{4.145}$$

The last inequality needs $V \leq E[X_1^2] \leq 2C/I^2$ that can be checked by the tails bound of $X_1$.

(4.140) can be checked simply by taking integral of (4.139) from $x = 0$ to $\infty$ (since $M_0 = 0$, for $x < 0$ the probability is always equal to 1, that's the reason why a 1 appears in (4.140)). □

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# A. Gamma and Beta distributions

We denote $\mathrm{Ga}(\alpha, \beta)$ for $\alpha, \beta > 0$ be the gamma distribution which has probability density function

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} \mathrm{e}^{-\beta x} \tag{A.1}$$

supported on $(0, \infty)$, where $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \mathrm{e}^{-y} dy$ is the gamma function. We denote $\mathrm{InvGa}(\alpha, \beta)$ for $\alpha, \beta > 0$ be the inverse gamma distribution which means if $X \sim \mathrm{Ga}(\alpha, \beta)$, then $X^{-1} \sim \mathrm{InvGa}(\alpha, \beta)$. The probability density function of $\mathrm{InvGa}(\alpha, \beta)$ is

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{-\alpha-1} \mathrm{e}^{\frac{\beta}{x}} \tag{A.2}$$

supported on $(0, \infty)$.

We denote $\mathrm{Be}(\alpha, \beta)$ for $\alpha, \beta > 0$ be the beta distribution which has probability density function

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} \tag{A.3}$$

supported on $(0, 1)$. We denote $\mathrm{InvBe}(\alpha, \beta)$ for $\alpha, \beta > 0$ be the inverse beta distribution which means if $X \sim \mathrm{Be}(\alpha, \beta)$, then $X^{-1} \sim \mathrm{InvBe}(\alpha, \beta)$. The probability density function of $\mathrm{InvBe}(\alpha, \beta)$ is

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\gamma(\beta)} \left(\frac{1}{x}\right)^{\alpha+\beta} (x-1)^{\beta-1} \tag{A.4}$$

supported on $(1, \infty)$.

## VITA

Yongjia Xie was born in June 11, 1993. He got his bachelor degree of science in mathematics as well as a minor bachelor degree in economics in May 2015 at Peking university. He then got his master degree of science in mathematics in June 2017 at Peking university under the supervision of Dr. Dayue Chen. He earned his Ph.D. degree at Purdue university in May 2022, whose advisor is Dr. Jonathon Peterson. His research interests during Ph.D. career was random walks in dynamic random environments, which includes random walks in cooling random environment, random walks in symmetric exclusion processes, and random polymer models. He will start working for Huawei technologies as an algorism engineer in Shenzhen from August 2022.