CRYSTALLINE CONDITION FOR A_{inf}–COHOMOLOGY AND RAMIFICATION BOUNDS

by

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ABSTRACT

For a prime $p > 2$ and a smooth proper p–adic formal scheme $\mathcal{X}$ over $\mathcal{O}_K$ where K is a p–adic field of absolute ramification degree e, we study a series of conditions (Cr_s), $s \geq 0$ that partially control the G_K–action on the image of the associated Breuil–Kisin prismatic cohomology $\mathsf{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})$ inside the A_{inf}–prismatic cohomology $\mathsf{R}\Gamma_{\Delta}(\mathcal{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}})$. The condition (Cr_0) is a criterion for a Breuil–Kisin–Fargues G_K–module to induce a crystalline representation used by Gee and Liu in [14, Appendix F], and thus leads to a proof of crystallinity of $\mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_p)$ that avoids the crystalline comparison. The higher conditions (Cr_s) are used in an adaptation of a ramification bounds strategy of Caruso and Liu from [11]. As a result, we establish ramification bounds for the mod p representations $\mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathcal{X}_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$ for arbitrary e and i, which extend or improve existing bounds in various situations.

1. INTRODUCTION

1.1 Background: Shafarevich conjectures and Fontaine's ramification bounds

In his highly influential talk at the 1962 International Congress of Mathematicians in Stockholm, I. R. Shafarevich proposed several conjectures regarding complete smooth curves over number fields. In analogy with Hermite's theorem, whose basic version[1] states that a number field K has only finitely many unramified extensions E/F of a fixed degree, Shafarevich conjectured that there are only finitely many isomorphism classes of curves C over F of a given genus $g \geq 1$ that have good reduction everywhere. In the $F = \mathbb{Q}$ case, he moreover conjectured that no such curves exist, in line with Minkowski's theorem that implies the non–existence of everywhere unramified extensions of $\mathbb{Q}$.

Since any prime of good reduction for a curve remains a prime of good reduction for its Jacobian, both of these conjectures were soon generalized to conjectures about abelian varieties. The first one took the form that over a number field F, the set of isomorphism classes of principally polarized g–dimensional abelian varieties with everywhere good reduction is finite, and was proved by Faltings [15][2].

The second conjecture stated that there are no abelian varieties over $\mathbb{Q}$ of dimension $g \geq 1$ with good reduction everywhere or, equivalently by the theory of Néron models, that there is no non–trivial abelian scheme over $\mathbb{Z}$. This conjecture was proved by Fontaine [16] and independently by Abrashkin [3].

The key idea in Fontaine's (as well as Abrashkin's) proof is to consider the potential abelian scheme $\mathcal{A}/\mathbb{Z}$ and the p–divisible group $\mathcal{A}_{p^\infty} = \{\mathcal{A}[p^n]\}_{n\in\mathbb{N}}$ over $\mathbb{Z}$. For suitably chosen p (any prime p with $3 \leq p \leq 17$), Fontaine is able to show that $\mathcal{A}_{p^\infty}$ is a direct sum of the constant and multiplicative p–divisible group; more precisely, that

$$\mathcal{A}_{p^\infty} \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^{\oplus g} \oplus (\mu_{p^\infty})^{\oplus g}. \tag{1.1}$$

[1] ↑Both Hermite's theorem, as well as the corresponding Shafarevich finiteness conjecture over number fields, also have a version where ramification is allowed in a prescribed finite set of places S.

[2] ↑Famously, this result also resolved Mordell's conjecture via the so–called Parshin's trick [32].

This would in particular mean that $\mathcal{A}(\mathbb{Z})$ has infinitely many p^∞–torsion points, which is not possible e.g. since the reduction map $\mathcal{A}(\mathbb{Z}) \to \mathcal{A}(\mathbb{F}_p)$ is injective on torsion points [25, Appendix], while the group $\mathcal{A}(\mathbb{F}_p) = \mathcal{A}_{\mathbb{F}_p}(\mathbb{F}_p)$ is finite.

At the heart of the argument leading to (1.1) is the following bound on ramification of finite flat group schemes, used for (the formal p–completions of) the finite stages $\mathcal{A}[p^n]$. To state it in broader generality, let K be a finite extension of $\mathbb{Q}_p$ with absolute ramification index $e = e(K/\mathbb{Q}_p)$. Let Γ be a finite flat p^n–torsion commutative group scheme over $\mathcal{O}_K$. Consider its splitting field L, that is, $L = K(\Gamma(\overline{K}))$ or, alternatively, $L = \overline{K}^{\mathrm{Ker}\,\rho}$ where $\rho : G_K \to \mathrm{Aut}(\Gamma(\overline{K}))$ is the action of the Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ on $\overline{K}$–points of Γ. Finally, denote $G_K^{(\mu)} = G_K^{\mu-1}$ where G_K^μ denotes the higher ramification subgroup of G_K in the upper numbering (in the standard convention e.g. as in [35]).

Theorem 1.1.1 (Fontaine, [16, Theorem A]). *Denote by $\mathcal{D}_{L/K}$ the different of the extension L/K and by v_K the additive valuation on K normalized by $v_K(K^\times) = \mathbb{Z}$. Then*

(1) $v_K(\mathcal{D}_{L/K}) < e\left(n + \frac{1}{p-1}\right)$.

(2) $G_K^{(\mu)}$ acts trivially on $\Gamma(\overline{K})$ when $\mu > e\left(n + \frac{1}{p-1}\right)$.

The G_K–module $\mathcal{A}[p^n](\overline{K})$ arises naturally as (the dual of) the first étale cohomology of $\mathcal{A}_{\overline{K}}$ with coefficients $\mathbb{Z}/p^n\mathbb{Z}$. It is therefore natural to consider the more general case of p^n–torsion étale cohomology. In this situation, Fontaine conjectured the following pattern of ramification bounds.

Conjecture 1.1.2 (Fontaine, [16]). *Let $\mathcal{X}$ be a proper smooth $\mathcal{O}_K$–scheme. Consider the G_K–module $T = \mathrm{H}^i_{\acute{e}t}(\mathcal{X}_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z})$, and let $L = \overline{K}^{\mathrm{Ker}\,\rho}$ be its splitting field. Then*

(1) $v_K(\mathcal{D}_{L/K}) < e\left(n + \frac{i}{p-1}\right)$,

(2) $G_K^{(\mu)}$ acts trivially on T when $\mu > e\left(n + \frac{i}{p-1}\right)$.

Using Fontaine–Laffaille theory [19], Conjecture 1.1.2 was partially proved by Fontaine in the special case when $n = e = 1$ and $i < p-1$ [18], and by Abrashkin when $e = 1$ an $i < p-1$ ([1]; see also [2]). More precisely, their proofs apply to $\mathbb{Z}/p^n\mathbb{Z}[G_K]$–modules attached to Fontaine–Laffaille modules. Such a module can be attached to mod p^n étale cohomology via the theorem of Fontaine–Messing [20], however, the bound is consequently established also for p^n–torsion G_K–modules of the form Λ/Λ' where $\Lambda' \subseteq \Lambda \subseteq V$ are two G_K–stable lattices in a crystalline $\mathbb{Q}_p$–representation V with Hodge–Tate weights in the range $[-i, 0]$ (equivalently $[0, i]$ by dualizing). For general n, i and e, the conjecture remains open.

1.2 Further results on ramification bounds

Subsequently, Fontaine's and Abrashkin's strategy for obtaining ramification bounds were employed in the semistable context. Under the asumption $i < p-1$ (and arbitrary e), Hattori proved in [22] a ramification bound for p^n–torsion quotients of lattices in semistable representations with Hodge–Tate weights in the range $[-i, 0]$, using (a variant of) Breuil's filtered (ϕ_r, N)–modules. Thanks to a comparison result between log–crystalline and étale cohomology by Caruso [9], this results in a ramification bound for $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z})$ when X is proper with semistable reduction, assuming $ie < p-1$ when $n = 1$ and $(i+1)e < p-1$ when $n \geq 2$ [3].

These results were further extended by Caruso and Liu in [11] for all p^n–torsion quotients of pairs of semistable lattices with Hodge–Tate weights in $[-i, 0]$ (again, equivalently in $[0, i]$ by considering duals), without any restriction on i or e. The proof uses the theory of $(\varphi, \widehat{G})$–modules. Roughly speaking, a $(\varphi, \widehat{G})$–module consists of a free Breuil–Kisin module M and the datum of an action of $\widehat{G} = \mathrm{Gal}(K(\mu_{p^\infty}, \pi^{1/p^\infty})/K)$ on $\widehat{M} = M \otimes_{\mathfrak{S},\varphi} \widehat{\mathcal{R}}$ where $\widehat{\mathcal{R}}$ is a suitable subring of Fontaine's period ring $A_{\inf} = W(\mathcal{O}_{\mathbb{C}_K^\flat})$ and $\pi \in K$ is a fixed choice of a uniformizer. To establish the result, the essential case is that of the $\mathbb{Z}/p^n\mathbb{Z}[G_K]$–module $\Lambda/p^n\Lambda$ where Λ is a G_K–stable lattice in a semistable representation V with Hodge–Tate weights in $[-i, 0]$. By a result of Liu [29], there is a unique $(\varphi, \widehat{G})$–module $\widehat{M}$ associated to Λ via an explicit functor $\widehat{T}$, and moreover, the quotient $\widehat{M}_n := \widehat{M}/p^n\widehat{M}$ is a p^n–torsion version of a $(\varphi, \widehat{G})$–module that is attached to $\Lambda/p^n\Lambda$ via a similar functor $\widehat{T}_n$. The proof then relies on systematic chain of modifications of the module $\widehat{T}_n(\widehat{M}_n)$, with occasional (but crucial) input coming from the knowledge of the free $(\varphi, \widehat{G})$–module $\widehat{M}$.

[3]↑Recently, in [27] Li and Liu extended Caruso's result to the range $ie < p-1$ regardless of n, for $\mathfrak{X}/\mathcal{O}_K$ proper and smooth (formal) scheme. In view of this, results of [22] should apply in these situations as well.

1.3 Case of mod p étale cohomology, and main results

In this thesis, we focus on the case of mod p étale cohomology. More precisely, let $\mathcal{X}$ be a smooth proper p–adic formal scheme over $\mathcal{O}_K$. Denote by $\mathcal{X}_{\overline{\eta}}$ its geometric generic fiber in the sense of adic spaces. As before, let us fix i and let $L = \overline{K}^{\mathrm{Ker}\,\rho}$ be the splitting field of the mod p representation $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$. The main result is the following.

Theorem 1.3.1 (Theorem 6.2.10)**.** *Set*

$$\alpha = \left\lfloor \log_p \left(\max \left\{ \frac{ip}{p-1}, \frac{(i-1)e}{p-1} \right\} \right) \right\rfloor + 1, \quad \beta = \frac{1}{p^\alpha} \left(\frac{iep}{p-1} - 1 \right).$$

(1) Denoting by $\mathcal{D}_{L/K}$ the different of the extension L/K, we have $v_K(\mathcal{D}_{L/K}) < 1 + e\alpha + \beta$.

(2) The group $G_K^{(\mu)}$ acts trivially on $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$ when $\mu > 1 + e\alpha + \max\left\{\beta, \frac{e}{p-1}\right\}$.

In particular, Theorem 1.3.1 applies for arbitrarily large i and e.

Remark 1.3.2. Note, however, that the precise shape of the estimates will depend on the size of i and e relative to p. Concretely, one can draw from Theorem 1.3.1 the following non–optimal, but more tractable consequence: the group $G_K^{(\mu)}$ acts trivially on $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$ when one of the following occurs.

(1) $e \leq p$ and $\mu > 1 + e\left(\left\lfloor \log_p\left(\frac{ip}{p-1}\right)\right\rfloor + 1\right) + e$,

(2) $e > p$ and $\mu > 1 + e\left(\left\lfloor \log_p\left(\frac{ie}{p-1}\right)\right\rfloor + 1\right) + p$,[4]

(3) $i = 1$ (e, p are arbitrary) and $\mu > 1 + e\left(1 + \frac{1}{p-1}\right)$.

The starting point of the proof of Theorem 1.3.1 is the strategy of Caruso and Liu from [11]. To implement their strategy, there are however two main obstacles.

The first obstacle is the fact that p–torsion étale cohomology does not naturally come as a quotient of G_K–stable lattices in a crystalline or semistable representation. More precisely, while every mod p representation admits a crystalline lift by results of Emerton and Gee

[4]↑Strictly speaking, to obtain this precise form one has to replace $(i-1)e$ in α from Theorem 1.3.1 by ie, and modify β appropriately; one can show that such form of Theorem 1.3.1 is still valid.

[14], there does not seem to be enough control on the Hodge–Tate weights of such lifts to be of use in this context. As a consequence, there is no clear way how to attach $(\varphi, \widehat{G})$–modules to $\mathrm{H}^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$ in general.

The type of semilinear data that *is* available in our context are Breuil–Kisin modules and Breuil–Kisin–Fargues modules, an analogue of Breuil–Kisin modules over the base ring A_{inf}. These modules come into the picture as the cohomology groups of the recently developed cohomology theories of Bhatt–Morrow–Scholze and Bhatt–Scholze [5, 6, 7]. Concretely, to a smooth p–adic formal scheme $\mathfrak{X}$, one can associate the "p^n-torsion prismatic cohomology theories"

$$\mathsf{R}\Gamma_{\Delta,n}(\mathfrak{X}/\mathfrak{S}) = \mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S}) \overset{\mathsf{L}}{\otimes} \mathbb{Z}/p^n\mathbb{Z}, \qquad \mathsf{R}\Gamma_{\Delta,n}(\mathfrak{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}}) = \mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}}) \overset{\mathsf{L}}{\otimes} \mathbb{Z}/p^n\mathbb{Z}$$

where $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}}), \mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S})$ are the prismatic avatars of the A_{inf}– and Breuil–Kisin cohomologies from [5] and [6], resp. Taking $M_{\mathrm{BK}} = \mathrm{H}^i_{\Delta,1}(\mathfrak{X}/\mathfrak{S})$ and $M_{\mathrm{inf}} = \mathrm{H}^i_{\Delta,1}(\mathfrak{X}/A_{\mathrm{inf}})$, Li and Liu showed in [27] that M_{BK} is a p–torsion Breuil–Kisin module and M_{inf} is a p–torsion Breuil–Kisin–Fargues module endowed with a compatible G_K–action. These modules recover the étale cohomology group $\mathrm{H}^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$ essentially due to the étale comparison theorem for prismatic cohomology from [7]. The pair $(M_{\mathrm{BK}}, M_{\mathrm{inf}})$ then serves as a suitable replacement of a torsion $(\varphi, \widehat{G})$–module in our context.

The second obstacle in implementing the strategy of Caruso and Liu is of slightly more technical nature. In the course of establishing the ramification bound for representation attached to the torsion $(\varphi, \widehat{G})$–module $(M_n, \widehat{M}_n)$, a crucial step is to provide control on the Galois action on elements of the Breuil–Kisin module M_n inside $\widehat{M}_n$, via a series of conditions of the form

$$\forall g \in \mathrm{Gal}(\overline{K}/K(\pi^{1/p^s}))\ \forall x \in M_n : g(x) - x \in \mathfrak{a}_{n,s}(\widehat{M}_n \otimes_{\widehat{\mathcal{R}}} A_{\mathrm{inf}}) \ , \qquad (1.2)$$

where $\mathfrak{a}_{n,s} \subseteq A_{\mathrm{inf}}$ is a collection of ideals that is descending with increasing s. The proof of this fact utilizes the free $(\varphi, \widehat{G})$–module $\widehat{M}$ and an explicit description of the Galois action in terms of the monodromy operator on the associated Breuil module $\mathcal{D}(\widehat{M})$ (cf. [8], [29,

§3.2]), which is in particular a vector space over a field of characteristic 0. Since our Breuil–Kisin(–Fargues) modules are inherently p–torsion and do not come with any apparent lift to free modules, no such techniques are at our disposal.

To obtain an analogue of (1.2) in our setting, instead we turn to a result of Gee and Liu that characterizes (free) Breuil–Kisin–Fargues G_K–modules whose associated representation is crystalline in terms of conditions of a similar flavor to (1.2).

Theorem 1.3.3 ([14, Appendix F]). *Consider a free Breuil–Kisin–Fargues G_K–module $M_{\inf}$, and assume that it admits a Breuil–Kisin submodule M_{BK} with $M_{\mathrm{BK}} \otimes_{\mathfrak{S}} A_{\inf} \xrightarrow{\sim} M_{\inf}$ and such that $M_{\mathrm{BK}} \subseteq M_{\inf}^{G_\infty}$. Then the representation*

$$V(M_{\inf}) = (M_{\inf} \otimes_{A_{\inf}} W(\mathbb{C}_K^\flat))^{\varphi=1}[1/p]$$

is crystalline if and only if

$$\forall g \in G_K, \;\; \forall x \in M_{\mathrm{BK}} : \;\; g(x) - x \in \varphi^{-1}(\mu)[\underline{\pi}]M_{\inf} \; . \tag{Cr_0}$$

Here $\mu, [\underline{\pi}] \in A_{\inf}$ are certain distinguished elements, and G_∞ denotes the closed subgroup of G_K of all elements fixing a particularly chosen system $\{\pi^{1/p^s}\}_s$ of p^s–th roots of π. We call condition (Cr_0) the *crystalline condition*. The considered formal scheme $\mathfrak{X}$ is assumed to be of good reduction, i.e. smooth over $\mathcal{O}_K$, and therefore the cohomology groups $\mathrm{H}^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Q}_p)$ are crystalline (in this generality, by results of [5]). It is therefore reasonable to expect that the crystalline condition applies to the pair $M_{\mathrm{BK}} = \mathrm{H}^i_{\Delta}(\mathfrak{X}/\mathfrak{S})$ and $M_{\inf} = \mathrm{H}^i_{\Delta}(\mathfrak{X}_{A_{\inf}}/A_{\inf})$, despite the fact that these Breuil–Kisin and Breuil–Kisin–Fargues modules, resp., are not necessarily free.

This is indeed the case and, moreover, it can be shown that the condition even applies at the level of chain complexes, i.e. to the embedding $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S}) \to \mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_{A_{\inf}}/A_{\inf})$. More precisely, one can model the cohomology theories by certain (to an extent) explicit complexes called Čech–Alexander complexes. These were introduced in [7] in the case that $\mathfrak{X}$ is affine, but can be extended to (at least) arbitrary separated smooth p–adic formal schemes. The condition (Cr_0) then can be verified termwise for this pair of complexes. More generally, one can introduce a decreasing series of ideals I_s, $s \geq 0$ where $I_0 = \varphi^{-1}(\mu)[\underline{\pi}]A_{\mathrm{inf}}$, and then formulate and prove the analogue of (Cr_0) for I_s and the action of $\mathrm{Gal}(\overline{K}/K(\pi^{1/p^s}))$. As a consequence, we obtain also the desired conditions for individual cohomology groups:

Theorem 1.3.4 (Theorem 5.1.1, Corollary 5.2.1, Proposition 5.2.3)**.** *Let $\mathfrak{X}$ be a smooth separated p–adic formal scheme over $\mathcal{O}_K$.*

(1) For all $s \geq 0$, the Čech–Alexander complexes $\check{C}^\bullet_{\mathrm{BK}}, \check{C}^\bullet_{\mathrm{inf}}$ that compute $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S})$ and $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}})$, resp., satisfy (termwise) the condition

$$\forall g \in \mathrm{Gal}(\overline{K}/K(\pi^{1/p^s})), \;\; \forall x \in \check{C}^\bullet_{\mathrm{BK}} : \;\; g(x) - x \in I_s \check{C}^\bullet_{\mathrm{inf}} \, . \tag{Cr_s}$$

(2) The associated prismatic cohomology groups satisfy the crystalline condition, that is,

$$\forall g \in G_K, \;\; \forall x \in \mathrm{H}^i_{\Delta}(\mathfrak{X}/\mathfrak{S}) : \;\; g(x) - x \in \varphi^{-1}(\mu)[\underline{\pi}]\mathrm{H}^i_{\Delta}(\mathfrak{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}}) \, .$$

(3) For all pairs of integers s, n with $s+1 \geq n \geq 1$, the p^n–torsion prismatic cohomology groups satisfy the condition

$$\forall g \in \mathrm{Gal}(\overline{K}/K(\pi^{1/p^s})), \;\; \forall x \in \mathrm{H}^i_{\Delta,n}(\mathfrak{X}/\mathfrak{S}) : \;\; g(x) - x \in \varphi^{-1}(\mu)[\underline{\pi}]^{p^{s+1-n}}\mathrm{H}^i_{\Delta,n}(\mathfrak{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}}) \, .$$

In particular, the conditions appearing in Theorem 1.3.4 (3) specialized to $n = 1$ give us an appropriate analogue of (1.2) needed to carry out the proof.

It can be further shown that the "if" part of Theorem 1.3.3 can be generalized to Breuil–Kisin–Fargues G_K–modules that are not necessarily free (Theorem 4.2.5), which is the typical case for the Breuil–Kisin–Fargues modules of the form $M_{\mathrm{inf}} = \mathrm{H}^i_{\Delta}(\mathfrak{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}})$. Consequently, we obtain an alternative proof of the following aforementioned fact.

Corollary 1.3.5 (Corollary 5.2.2)**.** *If $\mathfrak{X}$ is a smooth proper p–adic formal scheme over $\mathcal{O}_K$, then the cohomology groups $\mathrm{H}^i_{\acute{e}t}(X_{\overline{\eta}}, \mathbb{Q}_p)$ are crystalline representations.*

It should be mentioned that the proof of Corollary 1.3.5 thus obtained is not quite independent of the one in [5], as it relies on a large part of the same machinery, namely prismatic cohomology and the étale comparison theorem. On the other hand, the proof avoids the crystalline comparison theorem.

A natural question to ask is how the obtained bounds compare with other bounds from the literature. Roughly speaking, the answer is as follows: in the "semistable cases" of Hattori and Caruso–Liu [22, 11], the known bounds are applicable to mod p étale cohomology only when $ie < p-1$ or when $i = 1$ by a result of Emerton and Gee [13]. In all these cases, the bounds from Theorem 1.3.1 agree with the bounds obtained by these authors. In the "crystalline case" of Fontaine and Abrashkin [16, 1], the bounds apply to mod p étale cohomology only when $e = 1$ and $i < p-1$, however, their bounds are slightly stronger than the bounds from Theorem 1.3.1 (by 1 or $(p-1)/p$ less in terms of the index μ).

A source of ramification bounds not yet mentioned comes from the work of Caruso [10]. Here the bound is given for every $\mathbb{Z}/p^n\mathbb{Z}[G_K]$–module based on its restriction to G_∞, via Fontaine's theory of étale $\mathcal{O}_{\mathcal{E}}$–modules [17]. The observation that the G_K–module $\mathrm{H}^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ has an attached Breuil–Kisin module $\mathrm{H}^i_{\Delta,n}(\mathfrak{X}/\mathfrak{S})$ of height $\leq i$ then makes this bound explicit, as described in more detail in Remark 6.2.1. Due to somewhat different shapes of the estimates, the comparison with Theorem 1.3.1 is not clear–cut and it depends to a large extent on the ground field K: when the absolute ramification of K is small ($e \leq p$), the two bounds are fairly equivalent. When the ramification is tame and large, Caruso's bound becomes more and more favourable, and finally, the bound from Theorem 1.3.1 is stonger when the wild part of the absolute ramification is large (relative to the tame part of the abs. ramification).

1.4 Organization of the thesis

In Chapter 2, we firstly set up some auxiliary commutative algebra: the expository Section 2.1 is devoted to the review of derived I–completeness, I–complete flatness and their relation to the more classical notions of I–adic completeness and flatness. In Section 2.2 we leverage this theory to obtain several auxiliary results about regular sequences on I–complete and I–completely flat modules, with special attention given to modules over the ring A_{inf}.

In Chapter 3, we set up the formalism of prismatic cohomology (Sections 3.1, 3.2), especially for the Breuil–Kisin prism $\mathfrak{S}$ and the Fontaine prism A_{inf}, and the related theory of Breuil–Kisin(–Fargues) modules (Section 3.4). We also establish the Čech–Alexander complexes for computation of prismatic cohomology in the non–affine case (Section 3.3).

Chapter 4 deals with the conditions (Cr_s): after their formal definition and the study of their basic algebraic properties in Section 4.1, we decribe the role of the crystalline condition (Cr_0) in the context of Breuil–Kisin–Fargues G_K–modules in more detail in Section 4.2.

Chapter 5 is devoted to proving Theorem 1.3.4. That is, we prove that the conditions (Cr_s) hold for the embedding of Čech–Alexander complexes (Section 5.1), and deduce consequences for the individual prismatic cohomology groups (Section 5.2).

Finally, in Chapter 6 we use the results from Chapter 5 to deduce the desired ramification bounds, thus proving Theorem 1.3.1. After setting up some additional notation and reviewing Fontaine's formalism for ramification bounds in Section 6.1, the proof itself is carried out in Section 6.2. In the last Section 6.3, we discuss in more detail the comparison of the obtained bounds to other bounds in the literature.

1.5 Basic setup and conventions

Finally, let us describe basic notation and conventions used in this thesis. All rings are always commutative. Let p denote a fixed rational prime, which is assumed to be odd. Let k be a perfect field of characteristic p and let $K' = W(k)[1/p]$ be the associated absolutely unramified p–adic field. We fix a finite totally ramified extension K/K', and set $e = [K : K']$. We further fix a uniformizer $\pi \in \mathcal{O}_K$, and denote by $E(u) \in W(k)[u]$ its minimal (Eisenstein) polynomial. We fix an embedding of K to its algebraic closure $\overline{K}$, and we denote by $\mathbb{C}_K$ the completion of $\overline{K}$.

In $\overline{K}$, we fix a choice $(\pi_s)_{s\geq 0}$ of compatible p^s–th roots of unity of π, meaning that $\pi_0 = \pi$ and $\pi_{s+1}^p = \pi_s$ for all $s \geq 0$. For $s \in \mathbb{N}$, we set $K_s = K(\pi_s)$, and we further denote $K_\infty = \bigcup_s K_s$. We label the corresponding absolute Galois groups in the same manner, that is, G_s denotes the group $\mathrm{Gal}(\overline{K}/K_s)$ for $s \in \mathbb{N} \cup \{\infty\}$. In particular, G_0 is the absolute Galois group of K, which is also denoted by G_K.

Similarly, we fix a non–degenerate compatible system $(\zeta_{p^t})_{t\geq 0}$ of p^t–th roots of 1 in $\overline{K}$. That is, $\zeta_{p^0}=1$, ζ_p satisfies $\zeta_p^p=1$ and $\zeta_p\neq 1$, and for all $t\geq 1$ we have $\zeta_{p^{t+1}}^p=\zeta_{p^t}$. For $t\geq 0$, we denote $K_{p^t}=K(\zeta_{p^t})$, and we further set $K_{p^\infty}=\bigcup_t K_{p^t}$. For $s,t\in\mathbb{N}\cup\{0\}$, $K_{p^t,s}$ denotes the composite $K_{p^t}K_s$. This is a Galois extension of K when $t\geq s$, and in particular it makes sense to consider the Galois groups $\widehat{G}_s=\mathrm{Gal}(K_{p^\infty,\infty}/K_s)$. We further denote $\widehat{G}=\widehat{G}_0$.

The group $\widehat{G}$ is generated by its two subgroups $\mathrm{Gal}(K_{p^\infty,\infty}/K_{p^\infty})$ and $\mathrm{Gal}(K_{p^\infty,\infty}/K_\infty)$ (by [28, Lemma 5.1.2]). The subgroup $\mathrm{Gal}(K_{p^\infty,\infty}/K_{p^\infty})$ is normal, and its element g is uniquely determined by its action on the elements $(\pi_s)_s$, which takes the form $g(\pi_s)=\zeta_{p^s}^{a_s}\pi_s$, with the integers a_s unique modulo p^s and compatible with each other as s increases. It follows that $\mathrm{Gal}(K_{p^\infty,\infty}/K_{p^\infty})\simeq\mathbb{Z}_p$, with a topological generator τ given by $\tau(\pi_n)=\zeta_{p^n}\pi_n$.

For $s\geq 0$, $\widehat{G}_s=\mathrm{Gal}(K_{p^\infty,\infty}/K_s)$ contains $\mathrm{Gal}(K_{p^\infty,\infty}/K_\infty)$, and the intersection of $\widehat{G}_s$ with $\mathrm{Gal}(K_{p^\infty,\infty}/K_{p^\infty})$ is $\mathrm{Gal}(K_{p^\infty,\infty}/K_{p^\infty,s})$. Just as in the $s=0$ case, $\widehat{G}_s$ is generated by these two subgroups, with the subgroup $\mathrm{Gal}(K_{p^\infty,\infty}/K_{p^\infty,s})$ normal and topologically generated by the element τ^{p^s}.

For an integer j, we denote by $\mathbb{C}_K(j)$ the semilinear $\mathbb{C}_K$–representation of G_K given by χ^j where χ is the cyclotomic character. Given a (Hodge–Tate) $\mathbb{Q}_p$–representation V of G_K, we say that j is a Hodge–Tate weight for V if $(V\otimes_{\mathbb{Q}_p}\mathbb{C}_K(j))^{G_K}\neq 0$. In partiular, under this convention the cyclotomic character is of Hodge–Tate weight -1 (rather than 1) and the étale cohomology $\mathrm{H}^i_{\text{ét}}(\mathcal{X}_K,\mathbb{Q}_p)$ of a proper smooth $\mathcal{O}_K$–scheme $\mathcal{X}$ has Hodge–Tate weights contained in the interval $[0,i]$ (rather than $[-i,0]$).

Given a ring A with an endomorphism t, an *A–t–semilinear map* on an A–module M is an additive map $T:M\to M$ such that $T(am)=t(a)T(m)$ for all $a\in A$ and $m\in M$. In this situation, we set $t^*M=A\otimes_{t,A}M$, and define the *linearization of T* to be the map $T_{\mathrm{lin}}:t^*M\to M$ given by $(a\otimes m)\mapsto aT(m)$. When a group G acts on A by ring maps, a semilinar action of G on M is an action such that each $g\in G$ acts on M by an A–g–semilinear map. Note that in this situation, g^*M can be g–semilinearly identified with M via $x\leftrightarrow 1\otimes x$. We refer to the operation of replacing g^*M with M in this manner as "untwisting g^*M".

2. PRELIMINARY COMMUTATIVE ALGEBRA

2.1 Derived completion and complete flatness

In this mostly expository section, we recall the notion of derived I–completion, with emphasis on the case of modules, and of I–complete flatness. Roughly speaking, derived I–completion is a version of I–adic completion that has better homological properties, and it is frequently used in the prismatic setup. On the other hand, in many situations the computations of derived I–completions revert back to the standard I–adic completion; one of the aims of this sections is to explain this in more detail.

The main references for this section are [36, 091N], [7], [33], [34] and [38]. While the terminology is adopted from [36, 091N] and [7], the emphasis on the case of modules results in that some of the arguments and viewpoints resemble more [33], [34] or [38].

For the remainder of this section, let us fix a ring A and an ideal $I \subseteq A$ that is assumed to be finitely generated. Denote by $\mathsf{D}(A)$ the (unbounded) derived category of A–modules.

Definition 2.1.1.

(1) An A–module M is called *derived I–complete* if for every $i \geq 0$ and every $f \in I$, one has $\mathrm{Ext}^i_A(A[1/f], M) = 0$.

(2) An object $C \in \mathsf{D}(A)$ is *derived I–complete* if $\mathsf{R}\mathrm{Hom}_A(A[1/f], C) = 0$.

Remarks 2.1.2.

(1) Let M be an A–module. By [36, proof of 091Q], the set of all $f \in A$ such that $\mathrm{Ext}^i_A(A[1/f], M) = 0$ for all i forms a radical ideal of A. Consequently, to check that M is derived I–complete, it is enough to check the condition $\mathrm{Ext}^i_A(A[1/f], M) = 0$ for f coming from a set of generators of I (or even a set of generators up to radical). Additionally, a derived I–complete module is automatically derived J–complete whenever $J \subseteq A$ is a finitely generated ideal with $J \subseteq \sqrt{I}$.

(2) There is always a two–term free resolution of $A[1/f]$ coming from the presentation $A[1/f] = A[X]/(1 - Xf)$, that is,

$$0 \longrightarrow (1-Xf)A[X] \longrightarrow A[X] \longrightarrow A[1/f] \longrightarrow 0\,.$$

It follows that $\mathrm{Ext}^i_A(A[1/f], -)$ vanishes (on modules) for all $i \neq 0, 1$. Consequently, an A–module M is derived I–complete if and only if for all $f \in I$ (equivalently, for all f taken from a fixed set of generators up to radical as per (2)),

$$\mathrm{Hom}_A(A[1/f], M) = \mathrm{Ext}^1_A(A[1/f], M) = 0.$$

(3) Picking the free basis $\{(1-Xf)X^j\}_{j\geq 0}$ for the first term and $\{X^j\}_{j\geq 0}$ for the second, the above short exact sequence becomes

$$0 \longrightarrow A^{\oplus \mathbb{N}} \xrightarrow{\alpha_f} A^{\oplus \mathbb{N}} \longrightarrow A[1/f] \longrightarrow 0\,,$$

with the map α_f given by

$$\alpha_f : (a_0, a_1, a_2, \dots) \longmapsto (a_0, a_1 - fa_0, a_2 - fa_1, \dots)\,.$$

Applying $\mathrm{Hom}_A(-, M)$, the long exact sequence associated to the short exact sequence above implies that $\mathrm{Ext}^i_A(A[1/f], M) = 0$ for $i = 0, 1$ if and only if the map

$$\begin{aligned} \alpha_f^* : M^{\times \mathbb{N}} &\longrightarrow M^{\times \mathbb{N}} \\ (m_0, m_1, \dots) &\longmapsto (m_0 - fm_1, m_1 - fm_2, \dots) \end{aligned}$$

is an isomorphism. Note that this map is isomorphic to the map

$$\begin{aligned} (X - f) : M[[X]] &\longrightarrow XM[[X]] \\ \sum_{i\geq 0} m_i X^i &\longmapsto \sum_{i\geq 0} (m_i - fm_{i+1}) X^{i+1}\,. \end{aligned}$$

Consequently, M is derived I–complete if and only if the above map is an isomorphism for all $f \in I$ (equivalently, for a generating set of I, possibly up to radical).

(4) Given $C \in \mathsf{D}(A)$, there is a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_A^p(A[1/f], H^q(C)) \Rightarrow \mathrm{Ext}_A^{p+q}(A[1/f], C) ,$$

[36, 0AVG]. Since $A[1/f]$ is of projective dimension ≤ 1 by the previous remarks, the nonzero entries on the second page are concentrated in the first two columns. Consequently, for every j there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}_A^1(A[1/f], H^{j-1}(C)) \longrightarrow \mathrm{Ext}_A^j(A[1/f], C) \longrightarrow \mathrm{Hom}_A(A[1/f], H^j(C)) \longrightarrow 0 .$$

In particular, C is derived I–complete if and only if $H^j(C)$ is a derived I–complete module for every j.

Proposition 2.1.3.

(1) The category $\mathsf{Mod}_{I-\mathrm{cp}}(A)$ of all derived I–complete A–modules is a full abelian subcategory of $\mathsf{Mod}(A)$ closed under taking kernels, cokernels, and arbitrary direct products. Moreover, it is closed under extensions, hence it forms a weak Serre subcategory of $\mathsf{Mod}(A)$.

(2) The inclusion $\mathsf{Mod}_{I-\mathrm{cp}}(A) \hookrightarrow \mathsf{Mod}(A)$ admits a left adjoint $\widehat{(-)} : \mathsf{Mod}(A) \to \mathsf{Mod}_{I-\mathrm{cp}}(A)$.

Proof. Proof can be found in [33, §1] or [36, 091U], but let us recall the argument for reader's convenience. Stability of $\mathsf{Mod}_{I-\mathrm{cp}}(A)$ under arbitrary direct products is an immediate consequence of the compatibility of $\mathrm{Ext}_A^*(A[1/f], -)$ with direct products. To show that $\mathsf{Mod}_{I-\mathrm{cp}}(A)$ is closed under kernels, cokernels and images of modules, let $g : M \to N$ be a morphism between two derived I–complete A–modules. Fix an arbitrary element $f \in I$.

First we consider the short exact sequence

$$0 \longrightarrow \mathrm{Im}\, g \longrightarrow N \longrightarrow \mathrm{Coker}\, g \longrightarrow 0 ,$$

from which we obtain, using $\mathrm{Hom}_A(A[1/f], -)$, the long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_A(A[1/f], \mathrm{Im}\, g) \longrightarrow \mathrm{Hom}_A(A[1/f], N) \longrightarrow \mathrm{Hom}_A(A[1/f], \mathrm{Coker}\, g) \longrightarrow \\ \quad \longrightarrow \mathrm{Ext}_A^1(A[1/f], \mathrm{Im}\, g) \longrightarrow \mathrm{Ext}_A^1(A[1/f], N) \longrightarrow \mathrm{Ext}_A^1(A[1/f], \mathrm{Coker}\, g) \longrightarrow 0 \end{array}$$

(with the last zero due to $A[1/f]$ having projective dimension ≤ 1). The middle column consists of zeros and thus, we have $\mathrm{Hom}_A(A[1/f], \mathrm{Im}\, g) = \mathrm{Ext}^1_A(A[1/f], \mathrm{Coker}\, g) = 0$ and $\mathrm{Hom}_A(A[1/f], \mathrm{Coker}\, g) \simeq \mathrm{Ext}^1_A(A[1/f], \mathrm{Im}\, g)$.

Repeating the argument with the short exact sequence

$$0 \longrightarrow \mathrm{Ker}\, g \longrightarrow M \longrightarrow \mathrm{Im}\, g \longrightarrow 0\,,$$

we similarly obtain that $\mathrm{Hom}_A(A[1/f], \mathrm{Ker}\, g) = \mathrm{Ext}^1_A(A[1/f], \mathrm{Im}\, g) = 0$ and also that $\mathrm{Hom}_A(A[1/f], \mathrm{Im}\, g) \simeq \mathrm{Ext}^1_A(A[1/f], \mathrm{Ker}\, g)$. The two parts together prove that all these groups vanish, proving the claim. To finish the proof of (1), it remains to prove that given a short exact sequence

$$0 \longrightarrow M \longrightarrow L \longrightarrow N \longrightarrow 0$$

with M and N derived I–complete, the extension L is also derived I–complete. Once again, this immediately follows by invoking the long exact sequence for $\mathrm{Hom}_A(A[1/f], -)$ (for an arbitrary element $f \in I$).

The category $\mathsf{Mod}_{I-\mathrm{cp}}(A)$ is thus in particular closed under kernels and products, hence all limits, formed in $\mathsf{Mod}(A)$. In other words, the natural inclusion $\mathsf{Mod}_{I-\mathrm{cp}}(A) \hookrightarrow \mathsf{Mod}(A)$ preserves limits. The conclusion about left adjoint then follows by the Special adjoint functor theorem (e.g. [36, 0AHQ]). $\square$

There is also a derived variant of the above. As our focus is on the case of modules, we omit the proof.

Proposition 2.1.4 ([36, 091N, 091V])**.**

(1) The full subcategory $\mathsf{D}_{I-\mathrm{cp}}(A)$ of $\mathsf{D}(A)$ consisting of all derived I–complete objects is a saturated triangulated subcategory of $\mathsf{D}(A)$, and also closed under arbitrary products.

(2) The inclusion $\mathsf{D}_{I-\mathrm{cp}}(A) \hookrightarrow \mathsf{D}(A)$ admits a left adjoint $\widehat{(-)} : \mathsf{Mod}(A) \to \mathsf{Mod}_{I-\mathrm{cp}}(A)$.

Definition 2.1.5. The functors $\widehat{(-)}$ from Proposition 2.1.3 and 2.1.4 are called *derived completion as a module* and *derived completion as a complex,* respectively.

Remarks 2.1.6.

(1) If M is an A–module, one can consider the derived completion $\widehat{M}$ as a module, but also as a complex when M is treated as a chain complex concentrated in degree zero. These are in general different, and the completion as a module is equal to H^0 of the completion as a complex.

(2) Let us describe the derived completion as a module explicitly. First let us assume that $I = (f)$ is principal. Then it can be shown, using Remark 2.1.2 (3), that

$$\widehat{M} := M[[X]]/(X - f)M[[X]]$$

satisfies the universal property of derived completion. Moreover, note that $\widehat{M}$ is obtained from M using direct products and cokernels only, since it can be described as the cokernel of the map

$$\begin{aligned} \alpha'_f : M^{\times \mathbb{N}} &\longrightarrow M^{\times \mathbb{N}} \\ (m_1, m_2, \dots) &\longmapsto (-fm_1, m_1 - fm_2, \dots) \, . \end{aligned}$$

Consequently, if M is already derived I'–complete for an ideal I', then $\widehat{M}$ is derived (I', f)–complete. It follows easily that the derived completion for a finitely generated ideal $I = (f_1, f_2, \dots, f_n)$ can be constructed by completing with respect to one generator at a time, ultimately leading to the formula

$$\widehat{M} = M[[X_1, X_2, \dots X_n]]/(X_1 - f_1, X_2 - f_2, \dots, X_n - f_n)M[[X_1, X_2, \dots X_n]] \, . \tag{2.1}$$

(For full proofs of the two assertions, see [33, Theorems 6.4, 7.2].)

(3) A convenient consequence of the completion formula (2.1) is that in the case when $M = R$ is a derived I–complete A–algebra, the isomorphism $R \to \widehat{R}$ picks a preferred representative in R for the power series symbol $\sum_{j_1,\dots,j_n} a_{j_1,\dots,j_n} f_1^{j_1} \dots f_m^{j_n}$ as the preimage of the class represented by $\sum_{j_1,\dots,j_n} a_{j_1,\dots,j_n} X_1^{j_1} \dots X_n^{j_n}$. This gives an algebraically well–

behaved notion of power series summation despite the fact that R is not necessarily I–adically separated[1].

(4) Let us, again without proof, add an explicit description of the derived completion of complexes functor: when $I = (f_1, f_2, \dots, f_n)$, the derived completion is given by

$$\widehat{C} = \mathsf{R}\varprojlim_m C \otimes^{\mathsf{L}}_{\mathbb{Z}[X_1, X_2, \dots X_n]} \mathbb{Z}[X_1, X_2, \dots X_n]/(X_1^m, X_2^m, \dots, X_n^m) ,$$

where C is made into a complex of modules over $\mathbb{Z}[\underline{X}] = \mathbb{Z}[X_1, X_2, \dots X_n]$ by letting X_i act by f_i. Since the sequence $X_1^m, X_2^m, \dots X_n^m$ is regular, a convenient free resolution of $\mathbb{Z}[\underline{X}]/(\underline{X}^m)$ is the Koszul complex $\mathrm{Kos}(\mathbb{Z}[\underline{X}]; X_1^m, X_2^m, \dots X_n^m)$. Since C is a complex of A–modules, it then follows that $C \otimes^{\mathsf{L}}_{\mathbb{Z}[\underline{X}]} \mathbb{Z}[\underline{X}]/(\underline{X}^m)$ can be computed as $C \otimes^{\mathsf{L}}_A \mathrm{Kos}(A; f_1^m, \dots, f_n^m)$. Thus, one has the more explicit formula

$$\widehat{C} = \mathsf{R}\varprojlim_m C \otimes^{\mathsf{L}}_A \mathrm{Kos}(A; f_1^m, \dots, f_n^m) .$$

The proof of validity of this formula can be found e.g. in [36, 0920].

Just as in the case of I–adic completions, a version of the Nakayama lemma (called "derived Nakayama" in the sequel) holds.

Proposition 2.1.7. *If M is a derived I–complete A–module and $M/IM = 0$, then $M = 0$.*

Proof. Once again, this is proved in [36, 09B9], but let us include a proof for convenience, using a slightly different argument. First let us assume that $I = (f)$ is principal. Suppose for contradiction that M is derived I–complete and that $M = fM \neq 0$. Then we may pick a nonzero element $x_0 \in M$. Since $M = fM$, we may write $x_0 = fx_1$ for $x_1 \in M$, and inductively write $x_i = fx_{i+1}$ with $x_{i+1} \in M$ for all i. But this contradicts derived f–completeness of M, since the sequence $(x_0, x_1, \dots)$ gives a nonzero element in the kernel of the map α_f^* from Remark 2.1.2 (3).

[1]↑This operation further leads to the notion of contramodules, discussed e.g. in [33].

In the general case let $I = (f_1, f_2, \dots f_n)$, suppose that $M \neq 0$ and let $k \geq 0$ be the biggest integer such that the module

$$\overline{M} = M/(f_1, f_2, \dots, f_k)M = \operatorname{Coker}\left(M^{\oplus k} \xrightarrow{[f_1, f_2, \dots f_k]} M\right)$$

is nonzero. The cokernel expression shows that $\overline{M}$ is derived I–complete by Proposition 2.1.3. Thus, letting $f = f_{k+1}$, $\overline{M}$ is derived f–complete with $\overline{M}/f\overline{M} = 0$, hence $\overline{M} = 0$ by the previous part. This contradicts the choice of k. $\square$

While Remark 2.1.6 already justifies the name "completion", let us also establish a connection with the more usual I–adic completion.

Notation 2.1.8. From now on, we adopt the following notation and terminology. We say that a module is *classically I–complete* if it is I–adically complete. The I–adic completion of M is called the *classical I–completion of M*, and it is denoted by $\widehat{M}^{\mathrm{cl}}$.

Proposition 2.1.9 ([36, 091R, 091T])**.**

(1) Any classically I–complete module is derived I–complete.

(2) For any A–module M, there is a surjection $\widehat{M} \to \widehat{M}^{\mathrm{cl}}$.

(3) A derived I–complete module M is classically I–complete if and only if $\bigcap_j I^j M = 0$.

Proof. To prove (1), let M be a classically I–complete module. First note that the modules M/I^jM are derived I–complete for any j. Indeed, clearly for any $f \in I$ one has $\operatorname{Hom}_A(A[1/f], M/I^jM) = \operatorname{Ext}^1_A(A[1/f], M/I^jM) = 0$ because multiplication by f is an isomorphism on $A[1/f]$ while nilpotent on M/I^jM. Now, the category of derived I–complete modules is closed under arbitrary limits by Proposition 2.1.3 (1), and thus, in particular, $M \simeq \varprojlim_j M/I^jM$ is derived I–complete.

To prove (2), note that $\widehat{M}^{\mathrm{cl}}$ agrees with the classical I–completion of $\widehat{M}$, since this is just consecutive composition of two left adjoint functors. It is therefore sufficient to show that for a derived I–complete module M, the canonical map $M \to \widehat{M}^{\mathrm{cl}}$ is surjective.

Fix a set of generators $\underline{f} = f_1, f_2, \dots f_n$ of I. An element x of $\varprojlim_j M/I^jM$ is given by a system of elements $(x_i)_i$ of M such that $x_{i+1} - x_i = \sum_{\underline{a}} y_{\underline{a}}\, \underline{f}^{\underline{a}}$, where $\underline{a} = a_1, a_2, \dots, a_n$ runs over sequences of non–negative integers that sum up to i, $y_{\underline{a}} \in M$ are suitable elements and $\underline{f}^{\underline{a}}$ denotes the product $f_1^{a_1} \dots f_n^{a_n}$. Thus, we have

$$x_k = \sum_{\sum a_j = k-1} y_{\underline{a}}\, \underline{f}^{\underline{a}} + x_{k-1} = \sum_{\sum a_j = k-1} y_{\underline{a}}\, \underline{f}^{\underline{a}} + \sum_{\sum a_j = k-2} y_{\underline{a}}\, \underline{f}^{\underline{a}} + x_{k-2} = \dots = \sum_{\sum a_j \leq k-1} y_{\underline{a}}\, \underline{f}^{\underline{a}}$$

(with $y_{\underline{0}} = x_0$). Then the element $\sum_{\underline{a}} y_{\underline{a}}\, \underline{f}^{\underline{a}} \in M$ (where $\underline{a}$ now runs over all length n sequances of non–negative integers), that is, the equivalence class of $\sum_{\underline{a}} y_{\underline{a}}\, \underline{X}^{\underline{a}}$ under the isomorphism $M \simeq M[[X_1, \dots X_n]]/(X_1 - f_1, \dots, X_n - f_n)M[[X_1, \dots X_n]]$, maps onto x.

Moreover, note that the kernel of the map $M \to \widehat{M}^{\mathrm{cl}}$ considered above is precisely $\bigcap_j I^jM$. Thus, part (3) immediately follows. $\square$

Remark 2.1.10. Unlike $\mathsf{Mod}_{I-\mathrm{cp}}(A)$, the category $\mathsf{Mod}_{I-\mathrm{cl}}(A)$ of classically I–complete A–modules is usually not abelian. This results in the key difference between derived and classical I–completion: the functor $\widehat{(-)}$, as any left adjoint between abelian categories, is always right exact, while $\widehat{(-)}^{\mathrm{cl}}$ is not. In fact, in favorable situations (see Proposition 2.1.16 below), the derived completion functor as modules can be identified with $\mathsf{L}_0\widehat{(-)}^{\mathrm{cl}}$, the 0–th left derived functor of classical I–completion.

Example 2.1.11 ([23, Appendix A], [34, 3.6])**.** To demonstrate the difference between classical and derived completion, consider the case $A = \mathbb{Z}$ and $I = (p)$. Let $M = \bigoplus_{n \geq 0} \mathbb{Z}/p^n\mathbb{Z}$, and consider the short exact sequence

$$0 \to \bigoplus_{n \geq 1} \mathbb{Z} \xrightarrow{\bigoplus_n p^n} \bigoplus_{n \geq 1} \mathbb{Z} \to \bigoplus_{n \geq 1} \mathbb{Z}/p^n\mathbb{Z} \to 0 \ .$$

Taking the derived completion as a module then yields the exact sequence

$$\widehat{\bigoplus_{n \geq 1} \mathbb{Z}} \xrightarrow{P} \widehat{\bigoplus_{n \geq 1} \mathbb{Z}} \to \widehat{\bigoplus_{n \geq 1} \mathbb{Z}/p^n\mathbb{Z}} \to 0 \ .$$

Then $M = \widehat{\bigoplus_{n\geq 1}\mathbb{Z}/p^n\mathbb{Z}}$ is a derived p–complete module that is not classically complete. It will be shown in Proposition 2.1.17 that $N := \widehat{\bigoplus_n \mathbb{Z}}$ agrees with the classical p-completion of $\bigoplus_n \mathbb{Z}$. The resulting module is then the module of all sequences in $\mathbb{Z}_p$ that p–adically converge to 0, and the map P is given by $(a_n)_n \mapsto (p^n a_n)_n$. Now consider the sequence $x = (p, p^2, p^3, \dots) \in N$. Then x is not in the image of P, since the sequence $(1, 1, 1, \dots)$ is not in N. On the other hand, for any m we have $x \in \operatorname{Im} P + p^m N$. Thus, the element $\overline{x} \in N/\mathrm{Im}\ P = M$ is non–zero and contained in $p^m M$ for any m. (Note that, since the first two terms are classically p–complete, this example also shows that classical p–completion is not right exact in general.)

Definition 2.1.12. An A–module M is *I–completely (faithfully) flat* if M/IM is a (faithfully) flat A/I–module and $\mathrm{Tor}_i^A(M, A/I) = 0$ for all $i > 0$.

More generally, $C \in \mathsf{D}(A)$ is I–completely (faithfully) flat if $C \otimes^{\mathsf{L}}_A A/I$ is a (faithfully) flat A/I–module (the only cohomology is in degree 0 where it is given by a flat A/I–module).

Clearly flat modules are I–completely flat. The key motivating property for I–complete flatness is that unlike flatness, it is preserved under I–completions.

Proposition 2.1.13. *If F is a (faithfully) flat A–module (more generally, an I–completely (faithfully) flat complex), then the derived I–completion of F as a complex is I–completely (faithfully) flat.*

Proof. Denote by $\mathsf{D}_{I-\mathrm{tor}}(A)$ the full triangulated subcategory of $\mathsf{D}(A)$ consisting of all objects all of whose cohomology groups are I–torsion. Every such complex is derived I–complete, and thus, we have a series of inlusions $\mathsf{D}_{I-\mathrm{tor}}(A) \overset{i}{\hookrightarrow} \mathsf{D}_{I-\mathrm{cp}}(A) \overset{j}{\hookrightarrow} \mathsf{D}(A)$, where the left adjoint to j is derived I–completion and the left adjoint to both i and $j \circ i$ is $- \otimes^{\mathsf{L}}_A A/I$. Therefore

$$F \otimes^{\mathsf{L}}_A A/I \simeq \widehat{F} \otimes^{\mathsf{L}}_A A/I,$$

from which the claim immediately follows. $\square$

Remark 2.1.14. Given an I–completely flat module or complex M, by several dévissage arguments as in [38, Theorem 3.3] one can gradually prove that:

(1) $M \otimes^{\mathsf{L}}_A N$ is concentrated in degree zero for every finitely generated A/I–module N (i.e., $\operatorname{Tor}_i^A(M,N) = 0$ for all $i > 0$ when M is a module),

(2) $M \otimes^{\mathsf{L}}_A N$ is concentrated in degree zero for every A/I–module N (by taking direct limits),

(3) $M \otimes^{\mathsf{L}}_A N$ is concentrated in degree zero for every k and every A/I^k–module N,

(4) $M \otimes^{\mathsf{L}}_A A/I^k$ is a flat A/I^k–module for every k,

(5) $M \otimes^{\mathsf{L}}_A A/J$ is a flat A/J–module for every ideal J with $J \supseteq I^k$ for some k.

In particular, I–complete flatness is equivalent to J–complete flatness whenever I, J are two ideals with $\sqrt{I} = \sqrt{J}$. and implies J'–complete flatness when J' is an ideal with $J' \supseteq I^k$ for some k.

Given an A–module M and $\underline{f} = f_1, \ldots, f_n \in A$, denote by $\operatorname{Kos}(M; \underline{f})$ the usual Koszul complex and, for $m \geq 1$, denote by $\operatorname{Kos}(M; \underline{f}^m)$ the Koszul complex $\operatorname{Kos}(M; f_1^m, f_2^m, \ldots, f_n^m)$. Let $H_j(M; \underline{f}^m)$ denote the j-th Koszul homology of M with respect to $f_1^m, f_2^m, \ldots, f_n^m$. In order to impose cohomological indexing, let us further set $H^j(M; \underline{f}^m) = H_{-j}(M; \underline{f}^m)$ for each m. In other words, we treat $\operatorname{Kos}(M; f_1^m, f_2^m, \ldots, f_n^m)$ as a complex in cohomological degrees $-n$ to 0.

Recall that $\{\operatorname{Kos}(M; \underline{f}^m)\}_m$ naturally forms an inverse system, with transition maps obtained by tensoring together the morphisms $\operatorname{Kos}(A; f_i^{m+1}) \to \operatorname{Kos}(A; f_i^m)$,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f_i^{m+1}"] \arrow[d, "f_i"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & A \arrow[r, "f_i^m"] & A \arrow[r] & 0
\end{tikzcd}$$

(this inverse system was already implicitly used in Remark 2.1.6). We therefore have, for every fixed $j \leq 0$, an inverse system of Koszul homologies $\{H^j(M; \underline{f}^m)\}_m$.

Definition 2.1.15. A sequence $\underline{f} = f_1, \ldots, f_n \in A$ is *weakly pro–regular* if for each $j < 0$, the inverse system $\{H^j(A; \underline{f}^m)\}_m$ is pro–zero, that is, for every m there is $m' > m$ such that the transition map $H^j(A; \underline{f}^{m'}) \to H^j(A; \underline{f}^m)$ is 0. An ideal I is weakly pro–regular if it can be generated by a weakly pro–regular sequence.

Clearly any regular sequence is weakly pro–regular, but the latter is much more general. For example, when A is noetherian, it is a consequence of the Artin–Rees lemma that every finite sequence is weakly pro–regular (see e.g. [37, Lemma 4.3.3]).

Derived completion for a weakly pro–regular ideal is exceptionally well–behaved. For example, the connection between derived and classical completion becomes even stronger:

Proposition 2.1.16 ([39, Theorem 3.11])**.** *If the ideal I is weakly pro–regular, then the derived completion functor (of complexes) is naturally equivalent to $\mathsf{L}\widehat{(-)}^{\mathrm{cl}}$, the (total) left derived functor of the classical I–completion.*

Weak pro–regularity is also beneficial for derived I–completions of flat and I–completely flat modules. The following proposition justifies why in the vast majority of situations considered in this thesis, derived completions end up being equal to the classical completions.

Proposition 2.1.17. *Assume that I is weakly pro–regular, and let $C \in \mathsf{D}^-(A)$ be an I–completely flat complex. Then the derived I–completion of C as a complex is concentrated in degree 0 and it is classically I–complete. In particular, when M is an I–completely flat A–module, then its derived I–completion as a complex is isomorphic to the classical I–completion of M.*

Proposition 2.1.17 might be known to experts. But while there are related results in the literature, such as [38, Theorem 5.9] and [38, Theorem 4.3] (which is a direct consequence of Proposition 2.1.17), we are not aware of any written statement and proof of this fact. Therefore, we include the proof here.

Proof. Fix a weakly pro–regular sequence $\underline{f} = f_1, f_2, \ldots, f_n$ generating I, and denote by K_m the Koszul complex $\mathrm{Kos}(A; \underline{f}^m)$. First we compute the cohomology of $C \otimes^{\mathsf{L}}_A K_m$. Note that there is a spectral sequence

$$E^2_{p,q} = H^p(C \otimes^{\mathsf{L}}_A H^q(A; \underline{f}^m)) \Rightarrow H^{p+q}(C \otimes^{\mathsf{L}}_A K_m)$$

(by [36, 0662]). Since the Koszul homology $H^q(A; \underline{f}^m)$ is an A/I^N–module for big enough N and C is I–completely flat, from Remark 2.1.14 we have that $E^2_{p,q} = 0$ unless $p = 0$, whereas $E^2_{0,q} = H^0(C) \otimes_A H^q(A; \underline{f}^m)$. Thus, the spectral sequence degenerates on the second page and we have $H^q(C \otimes^{\mathsf{L}}_A K_m) = H^0(C) \otimes_A H^q(A; \underline{f}^m)$. In particular, for every $q < 0$, the system $\{H^q(C \otimes^{\mathsf{L}}_A K_m)\}_m$ is pro–zero as well.

Now, the distinguished triangle $\widehat{C} \to \prod_m C \otimes^{\mathsf{L}}_A K_m \to \prod_m C \otimes^{\mathsf{L}}_A K_m \to +1$ defining the derived inverse limit $\widehat{C} = \mathsf{R}\varprojlim_m C \otimes^{\mathsf{L}}_A K_m$ gives a long exact sequence in cohomology

$$\begin{tikzcd} \dots \arrow[r] & \prod_m H^{i-1}(C \otimes^{\mathsf{L}}_A K_m) \arrow[r, "\delta_{i-1}"] & \prod_m H^{i-1}(C \otimes^{\mathsf{L}}_A K_m) \arrow[r] & H^i\left(\widehat{C}\right) \arrow[dll, rounded corners] \\ & \prod_m H^{i}(C \otimes^{\mathsf{L}}_A K_m) \arrow[r, "\delta_{i}"] & \prod_m H^{i}(C \otimes^{\mathsf{L}}_A K_m) \arrow[r] & \dots \end{tikzcd}$$

The map δ_i has $\varprojlim_m H^i(C \otimes^{\mathsf{L}}_A K_m)$ as kernel and $\varprojlim^1_m H^i(C \otimes^{\mathsf{L}}_A K_m)$ as cokernel, and these are both 0 when $i < 0$ since the relevant system of cohomology groups is pro–zero. When $i = 0$, $H^0(C \otimes^{\mathsf{L}}_A K_m) = H^0(C)/(\underline{f}^m)H^0(C)$ is a Mittag–Leffler system defining $\widehat{H^0(C)}^{\mathrm{cl}}$ (in particular, $\varprojlim^1_m H^0(C \otimes^{\mathsf{L}}_A K_m)$ still vanishes). It follows that $H^i(\widehat{C}) = 0$ unless $i = 0$, and therefore

$$\widehat{C} \simeq H^0\left(\widehat{C}\right) \simeq \varprojlim_m H^0(C)/(\underline{f}^m)H^0(C) = \widehat{H^0(C)}^{\mathrm{cl}}.$$

$\square$

Notation 2.1.18. Let A be a commutative ring and $I \subseteq A$ a finitely generated ideal. Given two A–modules M, N, the symbol $M \widehat{\otimes}_A N$ denotes the derived I–completion of $M \otimes_A N$ (leaving the datum of the ideal I implicit) as a module if not stated otherwise. When C, D are two objects of $\mathsf{D}(A)$, the symbol $C \widehat{\otimes}^{\mathsf{L}}_A D$ denotes the derived I–completion of $C \otimes^{\mathsf{L}}_A D$ as a complex.

Corollary 2.1.19. *Let M be an I–completely flat A–module and B an A–algebra. Assume that the ideals $I \subseteq A$, $IB \subseteq B$ are weakly pro–regular. Then $M \widehat{\otimes}^{\mathsf{L}}_A B$ is an IB–completely flat B–module, computed as the classical IB–completion of $M \otimes_A B$.*

Proof. First note that $M \otimes^{\mathsf{L}}_A B$ is an IB–completely flat B–complex since

$$(M \otimes^{\mathsf{L}}_A B) \otimes^{\mathsf{L}}_B B/IB \simeq M \otimes^{\mathsf{L}}_A B/IB \simeq M \otimes_A B/IB \simeq (M \otimes_A A/I) \otimes_{A/I} B/IB$$

(where the second isomorphism used Remark 2.1.14 (2)), and this is a flat B/IB–module. Thus, its derived completion as a complex is IB–completely flat as well, and by the proof of Proposition 2.1.17, we have

$$\widehat{M \otimes^{\mathsf{L}}_A B} \simeq H^0(\widehat{M \otimes^{\mathsf{L}}_A B})^{\mathrm{cl}} = \widehat{M \otimes_A B}^{\mathrm{cl}},$$

as desired. □

2.2 Some regularity results

Next, we discuss regular sequences on derived complete modules in general and then in the particular case of the ring A_{inf} (or, slightly more generally, of the ring of Witt vectors of a perfect complete rank 1 valuation ring of characteristic p). The first lemma is a straightforward generalization of standard facts about Koszul homology (e.g. [30, Theorem 16.5]) and regularity on finitely generated modules.

Lemma 2.2.1. *Let A be a ring, $I \subseteq A$ a finitely generated ideal and let M be a nonzero derived I–complete module. Let $\underline{f} = f_1, f_2, \ldots, f_n \in I$. Then*

(1) $\underline{f}$ forms a regular sequence on M if and only if $H_m(M; \underline{f}) = 0$ for all $m \geq 1$ if and only if $H_1(M; \underline{f}) = 0$.

(2) In this situation, any permutation of $f_1, f_2, \ldots, f_n$ is also a regular sequence on M.

Proof. As Koszul homology is insensitive to the order of the elements $f_1, f_2, \ldots, f_n$, part (2) follows immediately from (1).

To prove (1), the forward implications are standard and hold in full generality (see e.g. [30, Theorem 16.5]). It remains to prove that the sequence $f_1, f_2, \ldots f_n$ is regular on M if $H_1(M; f_1, f_2, \ldots, f_n) = 0$. We proceed by induction on n. The case $n = 1$ is clear

($H_1(M;x) = M[x]$ by definition, and $M/xM \neq 0$ follows by derived Nakayama, Proposition 2.1.7). Let $n \geq 2$, and denote by $\underline{f}'$ the truncated sequence $f_1, f_2, \ldots, f_{n-1}$. Then we have $\mathrm{Kos}(M;\underline{f}) \simeq \mathrm{Kos}(M;\underline{f}') \otimes \mathrm{Kos}(A;f_n)$, which produces a short exact sequence

$$0 \longrightarrow \mathrm{Kos}(M;\underline{f}') \longrightarrow \mathrm{Kos}(M;\underline{f}) \longrightarrow \mathrm{Kos}(M;\underline{f}')[-1] \longrightarrow 0$$

of chain complexes which upon taking homologies results in a long exact sequence

$$\cdots H_1(M;\underline{f}') \xrightarrow{\pm f_n} H_1(M;\underline{f}') \longrightarrow H_1(M;\underline{f}) \longrightarrow M/(\underline{f}')M \xrightarrow{\pm f_n} M/(\underline{f}')M \longrightarrow M/(\underline{f})M \to 0$$

(as in [30, Theorem 7.4]). By assumption, $H_1(M;\underline{f}) = 0$ and thus, $f_n H_1(M;\underline{f}') = H_1(M;\underline{f}')$ where $f_n \in I$. Upon observing that $H_1(M;\underline{f}')$ is obtained from finite direct sum of copies of M by repeatedly taking kernels and cokernels, it is derived I–complete. Thus, derived Nakayama implies that $H_1(M;\underline{f}') = 0$ as well, and by induction hypothesis, $\underline{f}'$ is a regular sequence on M. Finally, the above exact sequence also implies that f_n is injective on $M/(\underline{f}')M$, and $M/(\underline{f})M \neq 0$ is satisfied thanks to derived Nakayama again. This finishes the proof. □

Corollary 2.2.2. *Consider an ideal $I = (\underline{f})$ of A where $\underline{f} = f_1, f_2, \ldots, f_n$ is a regular sequence on A. Let F be a nonzero derived I–complete and I–completely flat A–module. Then $\underline{f}$ is a regular sequence on F and consequently, each f_i is a non–zero divisor on F.*

Proof. By Lemma 2.2.1 (1), $H_m(A;\underline{f}) = 0$ for all $m \geq 1$, hence $\mathrm{Kos}(A;\underline{f})$ is a free resolution of A/I. Thus, the complex $\mathrm{Kos}(F;\underline{f}) = F \otimes_A \mathrm{Kos}(A;\underline{f})$ computes $\mathrm{Tor}^A_*(F, A/I)$, and hence it is acyclic in positive degrees by I–complete flatness. We may thus conclude that $H_i(F;\underline{f}) = 0$ for all $i \geq 1$. By Lemma 2.2.1, $\underline{f}$ is a regular sequence on F, and it remains regular on F after arbitrary permutation. This proves the claim. □

From now until the rest of the section, we specialize to the following case. Let R be a complete valuation ring of rank 1, meaning that its multiplicative valuation $|-|_R$ takes values in $\mathbb{R}_{\geq 0}$, or equivalently that the only nonzero prime ideal of R is maximal. Assume further that R is of characteristic p and perfect. For the rest of the section, we assume that

$A = W(R)$ where $W(-)$ denotes the (p–typical) Witt vectors construction. (The typical example that we consider later on is the Fontaine prism $A = A_{\text{inf}} = W(\mathcal{O}_{\mathbb{C}_K^\flat})$). Note that $R \hookrightarrow \text{Frac}(R)$ induces the injection $A \hookrightarrow W(\text{Frac}(R))$, where the latter is well–known to be a discrete valuation ring (see e.g. [35, II.6]). In particular, A is a domain. The fact that R is perfect implies that A is classically p–complete, since in this case we have

$$\varprojlim_n A/p^n = \varprojlim_n W_n(R) = W(R) = A.$$

Lemma 2.2.3. *For any element $x \in A \setminus (A^\times \cup pA)$ and all $k, l \geq 1$, $p^k A \cap x^l A = p^k x^l A$, and p, x forms a regular sequence. Furthermore, we have that $\sqrt{(p,x)} = (p, W(\mathfrak{m}_R))$ is the unique maximal ideal of A, where m_R denotes the maximal ideal of R. In particular, given two choices x, x' as above, we have $\sqrt{(p,x)} = \sqrt{(p,x')}$.*

Proof. By assumption, the image $\overline{x}$ of x in $A/p = R$ is non–zero and non–unit (non–unit since $x \notin A^\times$ and $p \in \text{rad}(A)$). Thus, x^l is a non–zero divisor both on A and on A/p, hence the claim that $pA \cap x^l A = px^l A$ follows for every l. The element p is itself non–zero divisor on A and thus, p, x is a regular sequence.

To obtain $p^k A \cap x^l A = p^k x^l A$ for general k, one can e.g. use induction on k using the fact that p is a non-zero divisor on A (or simply note that one may replace elements in regular sequences by arbitrary positive powers).

To prove the second assertion, note that $\sqrt{(\overline{x})} = \mathfrak{m}_R$ since $A/p = R$ is a rank 1 valuation ring. It follows that $(p, W(\mathfrak{m}_R))$ is the unique maximal ideal of A above (p), hence the unique maximal ideal since $p \in \text{rad}(A)$, and that $\sqrt{(p,x)}$ is equal to this ideal. $\square$

Corollary 2.2.4. *For every two choices x, x' as in Lemma 2.2.3, the (p,x)–adic and (p,x')–adic topologies agree. Furthermore, A is complete with respect to this topology.*

The topology from Corollary 2.2.4 is usually referred to as the *weak topology* on A.

Proof. We need to show only the completeness statement. Pick a non–unit and non–zero element $t \in R$ and denote by $[t]$ its Teichmüller lift. Then $x = [t]$ satisfies the assumptions of Lemma 2.2.3, and so it is enough to show that A is classically $(p, [t])$–complete. By [5, Lemma 3.2 (ii)], A is $[t]$–adically complete, and hence A is derived $(p,[t])$–complete since it is so for p and $[t]$ separately. Finally, since $p,[t]$ is a regular sequence on A (by Lemma 2.2.3 again), it follows that A is classically $(p,[t])$–complete by Proposition 2.1.17. $\square$

Corollary 2.2.5. *Let $x, x' \in A$ be as in Lemma 2.2.3, and let F be a (derived, equivalently classically) (p,x)–complete and (p,x)–completely flat A–module. Then p,x is a regular sequence on F. In particular, for each $k,l > 0$, we have $p^kF \cap (x')^lF = p^k(x')^lF$.*

Consequently, F is a torsion–free A–module.

Proof. By Lemma 2.2.3, A and F are derived (p,x')–complete and F is (p,x')–completely flat over A, and p,x' is a regular sequence on A. Corollary 2.2.2 then proves the claim that p,x' is a regular sequence on F. The sequence $p^k,(x')^l$ is then also regular on F, and the claim $p^kF\cap(x')^lF = p^k(x')^lF$ follows.

To prove the "consequently" part, let y be a non–zero and non–unit element of A. Since A is classically p–complete, we have $\bigcap_n p^nA = 0$, and so there exist n such that $y = p^nx''$ with $x'' \notin pA$. If x'' is a unit, then y is a non–zero divisor on F since so is p^n. Otherwise $x'' \in A\setminus(A^\times \cup pA)$, so p, x'' is a regular sequence on F, and so is x'', p (e.g. by Lemma 2.2.1). In particular p, x'' are both non–zero divisors on F, and hence so is $y = p^nx''$. $\square$

Finally, we record the following consequence on flatness of (p,x)–completely flat modules modulo powers of p that seems interesting on its own.

Corollary 2.2.6. *Let $x \in A\setminus(A^\times\cup pA)$, and let F be a (p,x)–complete and (p,x)–completely (faithfully) flat A–module. Then F is classically p–complete and p–completely (faithfully) flat. In particular, F/p^nF is a flat A/p^n–module for every $n>0$.*

Proof. The fact that F is classically p–complete is clear since it is classically (equivalently, derived) (p,x)–complete. We need to show that F/pF is a flat A/p–module and that $\mathrm{Tor}_i^A(F, A/p) = 0$ for all $i > 0$. The second claim is a consequence of the fact that p is a non–zero divisor on both A and F by Corollary 2.2.5. For the first claim, note that $A/p = R$ is a valuation ring and therefore it is enough to show that F/pF is a torsion–free R–module. This follows again by Corollary 2.2.5.

For the 'faithful' version, note that both the statements that F/pF is faithfully flat over A/p and that $F/(p,x)F$ is faithfully flat over $A/(p,x)$ are now equivalent to the statement $F/\mathfrak{m}_A F \neq 0$ where $\mathfrak{m}_A = (p, W(\mathfrak{m}_R))$ is the unique maximal ideal of A. $\square$

3. PRISMATIC COHOMOLOGY AND BREUIL–KISIN(–FARGUES) MODULES

3.1 δ–rings and prisms

In this and the next section, we recall the apparatus of prismatic cohomology from [7]. First we recall the notion of δ–rings and prisms. The general reference for this material is [7, §2 − 3]. Throughout this section, let p be a fixed prime.

Definition 3.1.1. A δ*–ring* is a commutative ring A together with an operation $\delta : A \to A$ satisfying the following conditions:

(1) $\delta(0) = 0, \delta(1) = 1$.

(2) For all $x, y \in A$, $\delta(xy) = \delta(x)y^p + x^p\delta(y) + p\delta(x)\delta(y)$.

(3) For all $x, y \in A$, $\delta(x + y) = \delta(x) + \delta(y) + \Sigma(x, y)$, where $\Sigma(X, Y) \in \mathbb{Z}[X, Y]$ is the polynomial
$$\Sigma(X, Y) = \frac{(X + Y)^p - X^p - Y^p}{p} .$$

Remarks 3.1.2.

(1) The axioms of a δ–ring A are set up in a manner so that the map
$$\varphi(x) := x^p + p\delta(x)$$
is a ring homomorphism lifting the absolute Frobenius $\mathrm{Fr} : A/p \to A/p$. When A is p–torsion free, to give a δ–ring structure to A is equivalent to giving such a Frobenius lift φ, since δ can be in this case recovered as
$$\delta(x) = \frac{\varphi(x) - x^p}{p}.$$

(2) It is clear from the definition that a δ–ring (for a fixed prime p) is an algebraic structure defined only by universally quantified identities (axioms of commutative rings together

with the four axioms regarding δ). That is, the category of all δ–rings (for the fixed prime p) is a variety of algebras in the sense of universal algebra. Consequently:

- The category $\delta-\mathsf{Ring}_p$ of all δ–rings (with respect to the fixed prime p) has all limits and colimits, and the forgetful functor $\delta-\mathsf{Ring}_p \to \mathsf{Set}$ (and, consequently, also the forgetful functor $\delta-\mathsf{Ring}_p \to \mathsf{CRing}$) preserves limits.
- In particular, $\mathbb{Z}$ is the initial δ–ring, with its unique δ–structure given so that $\varphi = \mathrm{id}_{\mathbb{Z}}$, that is,
$$\delta(x) = \frac{x - x^p}{p}.$$
The same formula defines the unique δ–stucture on $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p$ (unique since the only possible Frobenius lift is in both cases the identity, the only endomorphism of $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$, resp.).
- There are free δ–rings, that is, the forgetful functor to sets has a left adjoint, which we denote by $X \mapsto \mathbb{Z}\{X\}$ (where X is a set). Moreover, the same is true for the forgetful functor to commutative rings. Denote this functor by $(-)^\delta$. That is, R^δ denotes the universal δ–ring associated to a commutative ring R. By a slight abuse of notation, we use the same notation for the adjoint in relative setting: if A is a δ–ring and $A \to B$ is an A–algebra, the universal A–δ–algebra associated to B (obtained as the pushout of $A^\delta \to B^\delta$ along the counit map $A^\delta \to A$) is also denoted by B^δ.

(3) Perhaps more surprisingly, the forgetful functor $\delta-\mathsf{Ring}_p \to \mathsf{CRing}$ admits also a right adjoint, which assigns to a commutative ring R the ring $W(R)$ of its p–typical Witt vectors by a result of Joyal [24], where the δ–operation on $W(R)$ corresponds to the Witt vector Frobenius. Consequently, the forgetful functor to commutative rings commutes also with arbitrary colimits.

Later on, the following consequence of the above discussion will be useful:

Corollary 3.1.3. *Given a δ–ring A and a set X, the free δ–algebra $A\{X\}(= A[X]^\delta$ where $A[X]$ is the polynomial algebra in the variables X) is a polynomial algebra in the variables $\delta^*(X) = \{\delta^i(x) \mid i \geq 0, x \in X\}$.*

Proof. When $A = \mathbb{Z}_{(p)}$ and $X = \{x\}$, this is [7, Lemma 2.11], but the same proof applies to $\mathbb{Z}\{x\}$ as well. When $X = \{x_1, x_2, \ldots, x_n\}$ is finite, comparing universal properties we obtain that $\mathbb{Z}\{X\} \simeq \mathbb{Z}\{x_1\} \otimes_{\mathbb{Z}} \mathbb{Z}\{x_2\} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}\{x_n\}$, which is a polynomial algebra in variables $\delta^*(X)$. When X is infinite, it is a directed union of its finite subsets, and hence $\mathbb{Z}\{X\} = \varinjlim_{X'} \mathbb{Z}\{X'\}$ where X' runs over finite subsets of X, and the direct limit agrees with the direct limit of commutative rings. Upon noting that the transition maps are given by inclusion of variables, the result is again a polynomial algebra on $\delta^*(X)$. Finally, the free A–δ–algebra on X is given as $A \otimes_{\mathbb{Z}} \mathbb{Z}\{X\}$, proving the claim in general. $\square$

Definition 3.1.4. A *prism* (A, I) consists of a δ–ring A and an invertible ideal $I \subseteq A$ satisfying the following:

(1) A is derived (p, I)–complete,

(2) $p \in (I, \varphi(I))$.

A prism (A, I) is called *bounded* if A/I has bounded p^∞–torsion, that is, if there is $N > 0$ such that $A/I[p^\infty] = A/I[p^N]$.

A *morphism of prisms* $f : (A, I) \to (B, J)$ is a morphism of δ–rings such that $f(I) \subseteq J$. A morphism of prisms $f : (A, I) \to (B, J)$ is called *(faithfully) flat* if (B, J) is (p, I)–completely (faithfully) flat (as an A–module).

Remark 3.1.5. We do consider the zero ideal of the zero ring invertible. Consequently, the zero ring with its zero ideal is a bounded prism. This seems to be in accordance with the intention of the original Definition 3.2 of [7], based on some related claims, such as [7, Lemma 3.7 (3)] or [4, Lecture 5, Corollary 5.2], which would break down otherwise.

The importance of the boundedness condition is of a technical nature. In terms of the discussion in Chapter 2, we note the following result of Yekutieli.

Theorem 3.1.6 ([39])**.** *Given a bounded prism (A, I), the ideal (I, p) is weakly pro–regular.*

Let us list some basic properties of prisms from [7] for convenience.

Proposition 3.1.7 ([7, §3])**.** *(1) Given a morphism of prisms $(A, I) \to (B, J)$, one has $J = IB$.*

(2) Given a prism (A, I), a morphism of derived (p, I)–complete δ–rings $A \to B$ can be promoted to a map of prisms $(A, I) \to (B, IB)$ if and only if $B[I] = 0$.

(3) Given a bounded prism (A, I) and a derived (p, I)–complete and (p, I)–completely flat A–complex C, C is concentrated in degree 0 and classically (p, I)–complete. Moreover, $C[I] = 0$ and C/IC has bounded p^∞–torsion. In particular, the bounded prism (A, I) is classically (p, I)–complete.

(4) Consequently, the category of flat prisms over a bounded prism (A, I) is equivalent to the category of (p, I)–complete and (p, I)–completely flat A–δ–algebras via the equivalences $(B, IB) \leftrightarrow B$.

Remark 3.1.8. The claims of Proposition 3.1.7 (3), (4) can be also recovered as a consequence of Theorem 3.1.6 and Proposition 2.1.17.

An important notion for computational purposes is that of a prismatic envelope, which we also recall.

Definition 3.1.9.

(1) A *δ–pair* is the datum of a δ–ring R and an ideal $J \subseteq R$. A morphism of δ–pairs $(R, J) \to (R', J')$ consists of a map of δ–rings $f : R \to R'$ such that $f(J) \subseteq J'$.

(2) Let (A, I) be a prism and $(A, I) \to (R, J)$ a morphism of δ–pairs. The *prismatic envelope of* (R, J) (over (A, I)) is a morphism of δ–pairs $(R, J) \to (B, IB)$ such that

- (B, IB) is a prism over (A, I),

- the map $(R,J)\to(B,IB)$ is universal among maps satisfying the first condition, i.e. any map of δ–pairs $(R,J)\to(C,IC)$ into a prism over (A,I) factors uniquely (via a map of prisms) through $(R,J)\to(B,IB)$.

Proposition 3.1.10. *Let (A,I) be a bounded prism. Consider a map $(A,I)\to(R,J)$ of δ–pairs.*

(1) The prismatic envelope of (R,J) exists.

(2) Assume that R is derived (p,I)–complete. Let $J^\wedge$ denote the (p,I)–complete ideal of R generated by J, i.e. image of the map $\widehat{J}\to R$ where $\widehat{J}$ is the derived (p,I)–completion of J. Then the prismatic envelopes of (R,J) and of $(R,J^\wedge)$ agree.

(3) Let R_0 be a (p,I)–completion of a polynomial A–algebra and $J_0\subseteq R_0$ an ideal such that R_0/J_0 is a (classical) p–completion of a smooth A/I-algebra. Let R be a (p,I)–complete and (p,I)–completely flat δ–ring over R_0. Then the prismatic envelope for (R,JR) (equivalently, for $(R,(JR)^\wedge)$ by the previous part) is a bounded, flat prism over (A,I). Moreover, the formation of prismatic envelopes commutes with (p,I)–completely flat base change on the base prism (A,I).

Proof. Part (1) is [4, Lecture 5, Lemma 5.1], and part (3) is a special case of [7, Proposition 3.13, Example 3.14]. To prove (2), let (B,IB) be the prismatic envelope of (R,J). Since IB is invertible, it is in particular a finitely genrerated projective B–module, hence itself derived (p,I)–complete. The map of δ–pairs therefore induces a commutative square

$$\begin{tikzcd} \widehat{J} \arrow[r] \arrow[d] & R \arrow[d] \\ IB \arrow[r] & B, \end{tikzcd}$$

and thus, a map of δ–pairs $(R,J^\wedge)\to(B,IB)$. The claim now follows easily from the universal property of prismatic envelopes. $\square$

Notation 3.1.11 (The prisms $\mathfrak{S}\hookrightarrow A_{\mathrm{inf}}$)**.** Let us now describe the two prisms that are of central interest in this thesis. Recall the notation from Introduction, namely the perfect field k, finite totally ramified extension K of $W(k)[1/p]$, uniformizer $\pi\in K$, its minimal

(Eisenstein) polynomial $E(u)$, a fixed system $(\pi_s)_s$ of p^s–th roots of π and a fixed system $(\zeta_{p^n})_n$ of primitive p^n–th roots of unity in $\overline{K}$.

(1) The *Breuil–Kisin prism* $\mathfrak{S}$ is given as follows: As a ring, $\mathfrak{S} = W(k)[[u]]$. The δ–structure on $\mathfrak{S}$ is determined by the Frobenius lift $\varphi : \mathfrak{S} \to \mathfrak{S}$ which is given by the Witt vector frobenius on $W(k)$ and by the rule $\varphi(u) = u^p$ (in particular, $\delta(u) = 0$).

   There is a surjective map $\mathfrak{S} \twoheadrightarrow \mathcal{O}_K$ given by $u \mapsto \pi$, whose kernel is the principal ideal $I = (E(u))$. Then it is not hard to verify that $(\mathfrak{S}, (E(u)))$ forms a prism.

(2) Consider the ring of integers $\mathcal{O}_{\mathbb{C}_K} \subseteq \mathbb{C}_K$, and its tilt $\mathcal{O}_{\mathbb{C}_K}^\flat = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p$. The *Fontaine prism* A_{inf} is then given by $A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}_K}^\flat)$ as a δ–ring. In particular, the Witt vector Frobenius φ, given by $(x_i)_i \mapsto (x_i^p)_i$, is an isomorphism since $\mathcal{O}_{\mathbb{C}_K}^\flat$ is perfect.

   Note that one has $\mathcal{O}_{\mathbb{C}_K}^\flat \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}$ as multiplicative monoids, via the map whose inverse is reduction of components modulo p. Consequently, there is a multiplicative map $(-)^\sharp : \mathcal{O}_{\mathbb{C}_K}^\flat \simeq \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K} \xrightarrow{\mathrm{pr}_0} \mathcal{O}_{\mathbb{C}_K}$. This induces a ring map $\theta : A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}$ given, in terms of Teichmüller coordinates on $W(\mathcal{O}_{\mathbb{C}_K}^\flat)$, by the formula

$$\theta\left(\sum_{i=0}^{\infty} p^i [x_i]\right) = \sum_{i=0}^{\infty} p^i x_i^\sharp.$$

   The kernel of this map can be shown to be principal, generated by the element $\omega = \frac{\mu}{\varphi^{-1}(\mu)}$, where $\mu = [\underline{\varepsilon}] - 1$ and $\underline{\varepsilon} \in \mathcal{O}_{\mathbb{C}_K}^\flat$ is given by the system of primite p^n–th roots of unity $(1, \zeta_p, \zeta_{p^2}, \dots)$. Then $(A_{\mathrm{inf}}, (\omega))$ is a prism.

(3) Let $\underline{\pi} \in \mathcal{O}_{\mathbb{C}_K}^\flat$ be the element determined by the system $(\pi, \pi_1, \pi_2, \dots)$. Then the map $\mathfrak{S} \to A_{\mathrm{inf}}$ given by $u \mapsto [\underline{\pi}]$ is injective, and in fact faithfully flat (as an algebra map, i.e. in the classical sense) by [14, Proposition 2.2.13]. Clearly this embedding is compatible with the Frobenius lifts, hence it is a map of δ–rings. Moreover, the fact that $\theta([\underline{\pi}]) = \pi$ shows that $E(u)$ lands in $\operatorname{Ker}\theta$, therefore the map is in fact a map of prisms $(\mathfrak{S}, (E(u))) \to (A_{\mathrm{inf}}, (\omega))$. Consequently, it follows from Proposition 3.1.7 (1) that $E([\underline{\pi}])$ equals ω up to multiplication by an A_{inf}–unit.

(4) As the embedding of prisms $\mathfrak{S} \to A_{\mathrm{inf}}$ remains fixed throughout the whole thesis, to simplify notation we adopt the following convention: we treat the variable u as an element of A_{inf} via identification with $[\underline{\pi}]$, and view $\mathfrak{S}$ as a subring of A_{inf} (and thus, talk about the principal ideal $E(u)A_{\mathrm{inf}}$ etc.). Similarly, instead of the more standard notation μ for the element $[\underline{\varepsilon}] - 1 \in A_{\mathrm{inf}}$, we denote this element by v[1].

3.2 Prismatic cohomology

Next, we recall the (relative) prismatic site and prismatic cohomology from [7, §3-4]. Before stating the definitions, let us briefly discuss conventions regarding formal schemes. In all cases that we consider, a formal scheme will be a p–adic formal scheme over $\mathrm{Spf}(A/I)$ where (A, I) is a bounded prism (note that A/I is in this situation classically p–complete). An affine formal scheme $\mathrm{Spf}(R)$ is called *smooth* over A/I if the (derived, equivalently classically, p–complete) ring R is p–completely flat over A/I and R/p is a smooth $A/(I, p)$–algebra. A result of Elkik [12, Théorème 7] (and its extension to non–noetherian context as sketched in [7, §1.2]) then states that R is of the form $\widehat{R_0}^{\mathrm{cl}}$ for a smooth A/I–algebra R_0 (that is, an algebra such that $\mathrm{Spec}(R_0) \to \mathrm{Spec}(A/I)$ is a smooth map of schemes). A formal scheme $\mathfrak{X}$ is then smooth over A/I if it is locally a smooth affine formal scheme. As usual, $\mathfrak{X}$ is called separated (over A/I) if the diagonal map $\mathfrak{X} \to \mathfrak{X} \times_{A/I} \mathfrak{X}$ is a closed embedding, and proper if $\mathfrak{X} \to \mathrm{Spf}(A/I)$ is separated, of finite type and universally closed.

Definition 3.2.1.

(1) The site $\mathbb{\Delta}$ is given as follows:

- The underlying category is the opposite category of the category of all bounded prisms.
- The covers are given as the opposites of faithfully flat maps of prisms. That is, if $(B, I) \to (C, IC)$ is a faithfully flat map of prisms, the corresponding map $\{(C, IC) \to (B, I)\}$ in $\mathbb{\Delta}$ is a singleton cover.

[1]↑This notation originates in the fact that the embedding $W(k)[[u]] \to A_{\mathrm{inf}}$ can be extended to an embedding $W(k)[[u, v]] \to A_{\mathrm{inf}}$ by mapping v to μ, by [10, Proposition 1.14]. We will, however, not use this fact.

Denote by $\mathcal{O} = \mathcal{O}_{\Delta}$ the presheaf on Δ given by $(B, IB) \mapsto B$.

(2) Let (A, I) be a bounded prism and let $\mathfrak{X}$ be a smooth p–adic formal scheme over A/I. The site $(\mathfrak{X}/A)_{\Delta}$ is given as follows:

- The underlying category has as objects bounded prisms (B, IB) over (A, I) together with a map of formal schemes $\mathrm{Spf}(B/IB) \to \mathfrak{X}$ over A/I. The morphisms are the opposites of maps $(B, IB) \to (C, IC)$ over (A, I) such that the induced map $\mathrm{Spf}(C/IC) \to \mathrm{Spf}(B/IB)$ is compatible with the maps to $\mathfrak{X}$.
- The covers are given by faithfully flat maps of prisms. That is, a morphism in $(\mathfrak{X}/A)_{\Delta}$ is considered a singleton cover if the corresponding map of prisms $(B, IB) \to (C, IC)$ is a faithfully flat map of prisms.

Once again, we denote by $\mathcal{O} = \mathcal{O}_{\Delta}$ the presheaf on $(\mathfrak{X}/A)_{\Delta}$ that assigns to an object of $(\mathfrak{X}/A)_{\Delta}$ the underlying ring B of the corresponding bounded prism.

Proposition 3.2.2. *The data from Definition 3.2.1 define subcanonical sites $\Delta, (\mathfrak{X}/A)_{\Delta}$, and the corresponding presheaves $\mathcal{O}$ are sheaves on these sites (making them into ringed sites).*

Proof. The case of the site Δ is [7, Corollary 3.12], but the same computations verify the claim for $(\mathfrak{X}/A)_{\Delta}$. Alternatively, one can observe that:

(1) The site $(*/A)_{\Delta} := (\mathrm{Spf}(A/I)/A)_{\Delta}$ is a slice site of Δ, identifying $\mathcal{O}$ on the slice site with the restriction of the sheaf $\mathcal{O}$ on Δ,

(2) In the case of general $\mathfrak{X}$, the functor $h_{\mathfrak{X}}$ on $(*/A)_{\Delta}$ given by

$$h_{\mathfrak{X}}(B, IB) := \mathrm{Hom}_{A/I}(\mathrm{Spf}(B/IB), \mathfrak{X})$$

is a sheaf by [7, Remark 4.3]. Then $(\mathfrak{X}/A)_{\Delta}$ can be identified with the "slice site of $(*/A)_{\Delta}$ over $h_{\mathfrak{X}}$," more precisely, as slice site over the site $(*/A)_{\Delta}$ extended by the sheaf $h_{\mathfrak{X}}$ as per [36, 03A1]. Again, the sheaf $\mathcal{O}$ agrees with the corresponding restriction of the sheaf $\mathcal{O}$ on $(*/A)_{\Delta}$.

In all the considered cases, the claim that the topology on the respective site is subcanonical follows from the fact that the corresponding presheaf $\mathcal{O}$ is a sheaf.

□

Definition 3.2.3. Let (A, I) be a bounded prism and let $\mathfrak{X}$ be a smooth p–adic formal scheme over A/I. The *prismatic cohomology of* $\mathfrak{X}$ *over* A is the sheaf cohomology

$$\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A) := \mathsf{R}\Gamma((\mathfrak{X}/A)_{\Delta}, \mathcal{O})$$

(that is, $\mathsf{R}\Gamma(*, \mathcal{O})$ where $*$ denotes the terminal sheaf on $(\mathfrak{X}/A)_{\Delta}$ and where $\Gamma(-,-)$ denotes the bifunctor of morphisms of sheaves). We denote by $\mathrm{H}^*_{\Delta}(\mathfrak{X}/A))$ the individual cohomology groups of $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A)$.

Given $n \geq 0$, we further define the *p^n–torsion prismatic cohomology of* $\mathfrak{X}$ *over* A as

$$\mathsf{R}\Gamma_{\Delta,n}(\mathfrak{X}/A) := \mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A) \otimes^{\mathsf{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}.$$

The individual cohomology groups of $\mathsf{R}\Gamma_{\Delta,n}(\mathfrak{X}/A)$ are denoted by $\mathrm{H}^*_{\Delta,n}(\mathfrak{X}/A)$.

Among the main results of [7], the following are the most relevant for our purposes.

Theorem 3.2.4 ([7, Theorem 1.8])**.** *Let (A, I) be a bounded prism and let $\mathfrak{X}$ be a smooth formal scheme over A/I.*

(1) When $\mathfrak{X}$ is additionally proper, $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A)$ is a perfect complex of A–modules.

(2) $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A)$ is endowed with a φ_A–semilinear endomorphism φ. When $I = (d)$ is principal, the induced linearized homomorphism on the i–th cohomology group,

$$\varphi_{\mathrm{lin}} : H^i(\varphi_A^* \mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A)) \to H^i(\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A)) = \mathrm{H}^i_{\Delta}(\mathfrak{X}/A)$$

admits a map ψ in the opposite direction such that $\varphi_{\mathrm{lin}} \circ \psi = \psi \circ \varphi_{\mathrm{lin}} = d^i$ (in particular, φ_{lin} is an isomorphism after inverting d).

(3) (Base change) If $(A, I) \to (B, IB)$ is a map of bounded prisms, then we have

$$\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A)\widehat{\otimes}^{\mathsf{L}}_{A} B \xrightarrow{\sim} \mathsf{R}\Gamma_{\Delta}(\mathfrak{X}_{B/IB}/B)$$

via the natural map.

(4) (Étale comparison) Assume that A is perfect, i.e. φ_A is an isomorphism, and denote by $\mathfrak{X}_\eta$ the generic fiber of $\mathfrak{X}$ as an adic space. Then there is an isomorphism

$$\mathsf{R}\Gamma_{\acute{e}t}(\mathfrak{X}_\eta, \mathbb{Z}/p^n\mathbb{Z}) \simeq \Big(\mathsf{R}\Gamma_{\Delta,n}(\mathfrak{X}/A)[1/d]\Big)^{\varphi=1}$$

(where $(-)^{\varphi=1}$ stands for taking the homotopy fiber of $\varphi - 1$).

Given a diagram

$$(B_1, IB_1) \to (C, IC) \leftarrow (B_2, IB_2)$$

in Δ or $(\mathfrak{X}/A)_{\Delta}$ (leaving the remaining data implicit in this case), corresponding to a pair of maps of bounded prisms $(C, IC) \to (B_i, IB_i)$, denote the corresponding fibre product (in the variance of Δ, i.e. "pushout" in the variance of prisms) by $(B_1, IB_1) \boxtimes_{(C,IC)} (B_2, IB_2)$. Then we have the following:

Lemma 3.2.5. *Suppose that (B_1, IB_1) is (faithfully) flat over (C, IC). Then the fibre product $(B_1, IB_1) \boxtimes_{(C,IC)} (B_2, IB_2)$ corresponds to the prism given as the classical (p, I)–completion of $B_1 \otimes_C B_2$, and it is a (faithfully) flat prism over (B_2, IB_2).*

Proof. This is essentially the argument in the proof of [7, Corollary 3.12], but let us repeat it for clarity. By Corollary 2.1.19, we have that

$$B_1 \widehat{\otimes}^{\mathsf{L}}_C B_2 = B_1 \widehat{\otimes}_C B_2 = \widehat{B_1 \otimes_C B_2}^{\mathrm{cl}},$$

and the result is (p, I)–completely flat over B_2. In particular, it is a prism by Proposition 3.1.7 (4), and therefore obviously the desired pushout of prisms (i.e. fibre product in Δ). The faithfulness statement is then also clear given that

$$B_1 \widehat{\otimes}_C B_2 \otimes_{B_2} B_2/(p,I)B_2 \simeq B_1/(p,I)B_1 \otimes_{C/(p,I)C} B_2/(p,I)B_2.$$

Note that $B_1 \widehat{\otimes}_C B_2 / I B_1 \widehat{\otimes}_C B_2$ is similarly identified with the classical p–completion of $B_1/IB_1 \otimes_{C/IC} B_2/IB_2$, and therefore it corresponds to the fibre product of $\mathrm{Spf}(B_i/IB_i)$'s over $\mathrm{Spf}(C/IC)$. In particular, in the case of the site $(\mathfrak{X}/A)_{\Delta}$, the implicit map $\mathrm{Spf}(C/IC) \to \mathfrak{X}$ determines the appropriate maps $\mathrm{Spf}(B_i/IB_i) \to \mathfrak{X}$, verifying the fibre product claim in this case. $\square$

3.3 Čech–Alexander complex

We are now ready to describe Čech–Alexander complexes for computing prismatic cohomology, introduced in the affine case in [7, §4.3], in a global setting. For the sake of being explicit, we present the construction in detail and in a very Čech–theoretic way[2].

Throughout this section, let (A, I) be a fixed bounded base prism, and let $\mathfrak{X}$ be a smooth separated p–adic formal scheme over A/I. Given a site $\mathcal{C}$, we denote by $\mathsf{Shv}(\mathcal{C})$ the category (topos) of sheaves of sets on $\mathcal{C}$.

For our purposes a slight modification of the topology on $(\mathfrak{X}/A)_{\Delta}$ is convenient. The following proposition motivates the change.

Proposition 3.3.1. *Let (A, I) be a bounded prism.*

(1) Given a collection of maps of (bounded) prisms $(A, I) \to (B_i, IB_i)$, $i = 1, 2, \dots, n$, the canonical map $(A, I) \to (C, IC) = (\prod_i B_i, I \prod_i B_i)$ is a map of (bounded) prisms.

[2]↑A possible shorter way might be to combine the affine case of the construction from [7] with the fact that prismatic cohomology satisfies Zariski descent. Such an argument would, however, involve computing derived limits of cosimplicial δ–rings, and to make the result explicit enough, we would nonetheless need to establish some auxiliary results, such as Proposition 3.3.11 below.

(2) *(C, IC) is flat over (A, I) if and only if each (B_i, IB_i) is flat over (A, I). In that situation, (C, IC) is faithfully flat prism over (A, I) if and only if the family of maps of formal spectra $\mathrm{Spf}(B_i/IB_i) \to \mathrm{Spf}(A/I)$ is jointly surjective.*

(3) *Let $f \in A$ be an element. Then $(\widehat{A_f}, I\widehat{A_f})$, where $\widehat{(-)}$ stands for the derived (equivalently, classical) (p, I)–completion, is a bounded prism, and the map $(A, I) \to (\widehat{A_f}, I\widehat{A_f})$ is a flat map of prisms*[3].

(4) *Let $f_1, \dots, f_n \in A$ be a collection of elements generating the unit ideal. Then the canonical map $(A, I) \to \left(\prod_i \widehat{A_{f_i}}, I \prod_i \widehat{A_{f_i}}\right)$ is a faithfully flat map of (bounded) prisms.*

Proof. The proof of (1) is more or less formal. The ring $C = \prod_i B_i$ has a unique A–δ–algebra structure since the forgetful functor from δ–rings to rings preserves limits, and C is as product of (p, I)–complete rings (p, I)–complete. Clearly $IC = \prod_i (IB_i)$ is an invertible ideal since each IB_i is. In particular, $C[I] = 0$, hence C is a prism by Proposition 3.1.7 (2). Assuming that all (B_i, IB_i) are bounded, from $C/IC = \prod_i B_i/IB_i$ we have $C/IC[p^\infty] = C/IC[p^k]$ for k big enough so that $B_i/IB_i[p^\infty] = B_i/IB_i[p^k]$ for all i, showing that (C, IC) is bounded.

The ((p, I)–complete) flatness part of (2) is clear. For the faithful flatness statement, note that $C/(p, I)C = \prod_i B_i/(p, I)B_i$, hence $A/(p, I) \to C/(p, I)C$ is faithfully flat if and only if the map of spectra $\coprod_i \mathrm{Spec}(B_i/(p, I)B_i) = \mathrm{Spec}(C/(p, I)C) \to \mathrm{Spec}(A/(p, I))$ is surjective.

Let us prove (3). Since $\widehat{A_f}$ has $p \in \mathrm{rad}(\widehat{A_f})$, the equality $\varphi^n(f^k) = f^{kp^n} + p(\dots)$ shows that $\varphi^n(f^k)$ for each $n, k \geq 0$ is a unit in $\widehat{A_f}$. Consequently, as in [7, Remark 2.16], $\widehat{A_f} = \widehat{S^{-1}A}$ for $S = \{\varphi^n(f^k) \mid n, k \geq 0\}$, and the latter has a unique δ–structure extending that of A by [7, Lemmas 2.15 and 2.17]. In particular, $\widehat{A_f}$ is a (p, I)–completely flat A–δ–algebra, hence $(\widehat{A_f}, I\widehat{A_f})$ is flat prism over (A, I) by Proposition 3.1.7 (4).

Part (4) follows formally from parts (1)–(3). $\square$

Construction 3.3.2. Denote by $(\mathfrak{X}/A)^{\amalg}_{\Delta}$ the site whose underlying category is $(\mathfrak{X}/A)_{\Delta}$. The covers on $(\mathfrak{X}/A)^{\amalg}_{\Delta}$ are given by the opposites of finite families $\{(B, IB) \to (C_i, IC_i)\}_i$ of flat maps of prisms such that the associated maps $\{\mathrm{Spf}(C_i/IC_i) \to \mathrm{Spf}(B/IB)\}$ are jointly surjective. Let us call these "faithfully flat families" for short. The covers of the initial object

[3]↑Note that it can happen that $\widehat{A_f} = 0$, which occurs e.g. when $f \in (p, I)$.

$\varnothing$ [4] are the empty cover and the identity. We similarly extend $\mathbb{\Delta}$ to $\mathbb{\Delta}^{\amalg}$, that is, we proclaim the identity cover and the empty cover to be covers of $\varnothing$, and generally proclaim (finite) faithfully flat families to be covers.

Clearly isomorphisms as well as composition of covers are covers in both cases. To check that $(\mathfrak{X}/A)^{\amalg}_{\mathbb{\Delta}}$ and $\mathbb{\Delta}^{\amalg}$ are sites, it thus remains to check the base change axiom. This is trivial for situations involving $\varnothing$, so it remains to check that given a faithfully flat family $\{(B, IB) \to (C_i, IC_i)\}_i$ and a morphism corresponding to a map of bounded prisms $(B, IB) \to (D, ID)$, the fibre products $(C_i, IC_i) \boxtimes_{(B,IB)} (D, ID)$ exist and the collection $\{(D, ID) \to (C_i, IC_i) \boxtimes_{(B,IB)} (D, ID)\}_i$ is a faithfully flat family. The existence and flatness follows from Lemma 3.2.5, and the claim about faithfully flat family follows from the faithully flat version together with the fact that

$$\left(\prod_i (C_i, IC_i)\right) \boxtimes_{(B,IB)} (D, ID) = \prod_i \left((C_i, IC_i) \boxtimes_{(B,IB)} (D, ID)\right),$$

again thanks to Lemma 3.2.5 (and using Remark 3.3.3 (1) below).

Remarks 3.3.3.

(1) Note that for a finite family of objects (C_i, IC_i) in $(\mathfrak{X}/A)_{\mathbb{\Delta}}$, the structure map of the product $(A, I) \to \prod_i (C_i, IC_i)$ together with the map of formal spectra (induced from the maps for individual i's)

$$\operatorname{Spf}(\prod_i C_i/IC_i) = \coprod_i \operatorname{Spf}(C_i/IC_i) \to \mathfrak{X}$$

makes $(\prod_i C_i, I \prod_i C_i)$ into an object of $(\mathfrak{X}/A)_{\mathbb{\Delta}}$ that is easily seen to be the coproduct of (C_i, IC_i)'s. In view of Proposition 3.3.1 (2), one thus arrives at the equivalent formulation

$$\{Y_i \to Z\}_i \text{ is a } (\mathfrak{X}/A)^{\amalg}_{\mathbb{\Delta}}\text{–cover } \Leftrightarrow \coprod_i Y_i \to Z \text{ is a } (\mathfrak{X}/A)_{\mathbb{\Delta}}\text{–cover.}$$

That is, $(\mathfrak{X}/A)^{\amalg}_{\mathbb{\Delta}}$ is the (finitely) superextensive site having covers of $(\mathfrak{X}/A)_{\mathbb{\Delta}}$ as singleton covers. (Similar considerations apply to $\mathbb{\Delta}$ and $\mathbb{\Delta}^{\amalg}$.)

[4]↑That is, $\varnothing$ corresponds to the zero ring, which we consider to be a prism as per Remark 3.1.5.

(2) The two sites are honestly different in that they define different categories of sheaves. Namely, for every finite coproduct $Y = \coprod_i Y_i$, the collection of the canonical maps $\{Y_i \to \coprod_i Y_i\}_i$ forms a $(\mathcal{X}/A)^{\amalg}_{\Delta}$–cover, and the sheaf axiom forces upon $\mathcal{F} \in \mathsf{Shv}((\mathcal{X}/A)^{\amalg}_{\Delta})$ the identity $\mathcal{F}(\coprod_i Y_i) = \prod_i \mathcal{F}(Y_i)$, which is not automatic[5]. In fact, $\mathsf{Shv}((\mathcal{X}/A)^{\amalg}_{\Delta})$ can be identified with the full subcategory of $\mathsf{Shv}((\mathcal{X}/A)_{\Delta})$ consisting of all sheaves compatible with finite disjoint unions in the sense above. In particular, the structure presheaf $\mathcal{O} = \mathcal{O}_{\Delta} : (B, IB) \mapsto B$ is a sheaf for the $(\mathcal{X}/A)^{\amalg}_{\Delta}$–topology. (Again, the same is true for Δ and $\Delta^{\amalg}$, including the fact that $\mathcal{O} : (B, IB) \mapsto B$ is a sheaf.)

Despite the above fine distinction, for the purposes of prismatic cohomology, the two topologies are interchangeable. This is a consequence of the following lemma.

Lemma 3.3.4. *Given an object $(B, IB) \in (\mathcal{X}/A)^{\amalg}_{\Delta}$, one has $\mathrm{H}^i((B, IB), \mathcal{O}) = 0$ for $i > 0$.*

Proof. The sheaf $\mathcal{O} : (B, I) \mapsto B$ on $\Delta^{\amalg}$ has vanishing positive Čech cohomology essentially by the proof of [7, Corollary 3.12]: one needs to show acyclicity of the Čech complex for any $\Delta^{\amalg}$–cover $\{(B, I) \to (C_i, IC_i)\}_i$, but the resulting Čech complex is identical to that for the Δ–cover $(B, I) \to \prod_i (C_i, IC_i)$ (by Lemma 3.2.5), for which the acyclicity is proved in [7, Corollary 3.12]. By a general result (e.g. [36, 03F9]), this implies vanishing of $\mathrm{H}^i_{\Delta^{\amalg}}((B, I), \mathcal{O})$ for all bounded prisms (B, I) and all $i > 0$.

Now we make use of the fact that cohomology of an object can be computed as the cohomology of the corresponding slice site, [36, 03F3]. Let $(B, IB) \in (\mathcal{X}/A)^{\amalg}_{\Delta}$. After forgetting structure, we may view (B, IB) as an object of $\Delta^{\amalg}$ as well, and then [36, 03F3] implies that for every i, we have the isomorphisms

$$\mathrm{H}^i_{(\mathcal{X}/A)^{\amalg}_{\Delta}}((B, IB), \mathcal{O}) \simeq \mathrm{H}^i((\mathcal{X}/A)^{\amalg}_{\Delta}/(B, IB), \mathcal{O}\mid_{(B,IB)}),$$

$$\mathrm{H}^i_{\Delta^{\amalg}}((B, IB), \mathcal{O}) \simeq \mathrm{H}^i((\Delta^{\amalg}/(B, IB), \mathcal{O}\mid_{(B,IB)})$$

[5]↑For example, every constant presheaf is a sheaf for a topology given by singleton covers only, which is not the case for $(\mathcal{X}/A)^{\amalg}_{\Delta}$.

(where $\mathcal{C}/c$ for a site $\mathcal{C}$ and $c \in \mathcal{C}$ denotes the slice site). Upon noting that the slice sites $(\mathfrak{X}/A)^{\mathrm{II}}_{\Delta}/(B, IB)$, $\Delta^{\mathrm{II}}/(B, IB)$ are equivalent sites (in a manner that identifies the two versions of the sheaf $\mathcal{O}\mid_{(B,IB)}$), the claim follows. $\square$

Corollary 3.3.5. *One has*

$$\mathsf{R}\Gamma((\mathfrak{X}/A)_{\Delta}, \mathcal{O}) = \mathsf{R}\Gamma((\mathfrak{X}/A)^{\mathrm{II}}_{\Delta}, \mathcal{O}).$$

Proof. The coverings of $(\mathfrak{X}/A)^{\mathrm{II}}_{\Delta}$ contain the coverings of $(\mathfrak{X}/A)_{\Delta}$, so we are in the situation of [36, 0EWK], namely, there is a morphism of sites $\varepsilon : (\mathfrak{X}/A)^{\mathrm{II}}_{\Delta} \to (\mathfrak{X}/A)_{\Delta}$ given by the identity functor of the underlying categories, where the sheaf pushforward functor $\varepsilon_* : \mathsf{Shv}((\mathfrak{X}/A)^{\mathrm{II}}_{\Delta}) \to \mathsf{Shv}((\mathfrak{X}/A)_{\Delta})$ is the natural inclusion and the (exact) inverse image functor $\varepsilon^{-1} : \mathsf{Shv}((\mathfrak{X}/A)_{\Delta}) \to \mathsf{Shv}((\mathfrak{X}/A)^{\mathrm{II}}_{\Delta})$ is the sheafification with respect to the "II"-topology. One has

$$\Gamma((\mathfrak{X}/A)^{\mathrm{II}}, -) = \Gamma((\mathfrak{X}/A), -) \circ \varepsilon_*$$

(where in this context, ε_* denotes the inclusion of abelian sheaves only), hence

$$\mathsf{R}\Gamma((\mathfrak{X}/A)^{\mathrm{II}}, \mathcal{O}) = \mathsf{R}\Gamma((\mathfrak{X}/A), \mathsf{R}\varepsilon_*\mathcal{O}),$$

and to conclude it is enough to show that $\mathsf{R}^i\varepsilon_*\mathcal{O} = 0\ \forall i > 0$. But $\mathsf{R}^i\varepsilon_*\mathcal{O}$ is the sheafification of the presheaf given by $(B, IB) \mapsto \mathrm{H}^i((B, IB), \mathcal{O})$ ([36, 072W]), which is 0 by Lemma 3.3.4. Thus, $\mathsf{R}^i\varepsilon_*\mathcal{O} = 0$, which proves the claim. $\square$

For an open p–adic formal subscheme $\mathcal{V} \subseteq \mathfrak{X}$, denote by $h_{\mathcal{V}}$ the functor sending an object$(B, IB) \in (\mathfrak{X}/A)_{\Delta}$ to the set of factorizations of the implicit map $\mathrm{Spf}(B/IB) \to \mathfrak{X}$ through $\mathcal{V} \hookrightarrow \mathfrak{X}$; that is,

$$h_{\mathcal{V}}((B, IB)) = \begin{cases} * & \text{if the image of } \mathrm{Spf}(B/IB) \to \mathfrak{X} \text{ is contained in } \mathcal{V}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $(B, IB) \to (C, IC)$ correspond to a morphism in $(\mathcal{X}/A)_{\Delta}$. If $\mathrm{Spf}(B/IB) \to \mathcal{X}$ factors through $\mathcal{V}$, then so does $\mathrm{Spf}(C/IC) \to \mathrm{Spf}(B/IB) \to \mathcal{X}$. It follows that $h_{\mathcal{V}}$ forms a presheaf on $(\mathcal{X}/A)_{\Delta}$ (with transition maps $h_{\mathcal{V}}((B, IB)) \to h_{\mathcal{V}}((C, IC))$ given by $* \mapsto *$ when $h_{\mathcal{V}}((B, IB)) \neq \emptyset$, and the empty map otherwise). Note that $h_{\mathcal{X}}$ is the terminal sheaf.

Proposition 3.3.6. *$h_{\mathcal{V}}$ is a sheaf on $(\mathcal{X}/A)^{\amalg}_{\Delta}$.*

Proof. Consider a cover in $(\mathcal{X}/A)^{\amalg}_{\Delta}$ given by a faithfully flat family $\{(B, IB) \to (C_i, IC_i)\}_i$. One needs to check that the sequence

$$h_{\mathcal{V}}((B, IB))) \to \prod_i h_{\mathcal{V}}((C_i, IC_i)) \rightrightarrows \prod_{i,j} h_{\mathcal{V}}((C_i, IC_i) \boxtimes_{(B,IB)} (C_j, IC_j))$$

is an equalizer sequence. All the terms have at most one element; consequently, there are just two cases to consider, depending on whether the middle term is empty or not. In both cases, the pair of maps on the right necessarily agree, and so one needs to see that the map on the left is an isomorphism. This is clear in the case when the middle term is empty (since the only map into an empty set is an isomorphism). It remains to consider the case when the middle term is nonempty, which means that $h_{\mathcal{V}}((C_i, IC_i)) = *$ for all i. In this case we need to show that $h_{\mathcal{V}}((B, IB)) = *$. Since the maps $\mathrm{Spf}(C_i/IC_i) \to \mathrm{Spf}(B/IB)$ are jointly surjective and each $\mathrm{Spf}(C_i/IC_i) \to \mathcal{X}$ lands in $\mathcal{V}$, it follows that so does the map $\mathrm{Spf}(B/IB) \to \mathcal{X}$. Thus, $h_{\mathcal{V}}((B, IB)) = *$, which finishes the proof. $\square$

Construction 3.3.7 (Čech–Alexander cover of $\mathcal{V}$)**.** Let us now assume additionally that $\mathcal{V} = \mathrm{Spf}(R)$ is affine, and choose a surjection $P_{\mathcal{V}} \to R$ where $P_{\mathcal{V}} = \widehat{A[\underline{X}]}$ is a (p, I)–completed free A–algebra. Denote by $J_{\mathcal{V}}$ the kernel of the surjection. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_{\mathcal{V}} & \longrightarrow & P_{\mathcal{V}} & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \widehat{J_{\mathcal{V}} P^{\delta}_{\mathcal{V}}} & \longrightarrow & \widehat{P^{\delta}_{\mathcal{V}}} & \longrightarrow & R \widehat{\otimes_{P_{\mathcal{V}}}} P^{\delta}_{\mathcal{V}} & \longrightarrow & 0, \end{array}$$

where $\widehat{(-)}$ stands for derived (p, I)–completion. Note, however, that $P^{\delta}_{\mathcal{V}}$ is a flat (actually free) $P_{\mathcal{V}}$–algebra by Corollary 3.1.3 and consequently, it is equivalently classical

(p,I)–completion (classical p–completion on the right–hand side term) by Proposition 2.1.17. Denote by $J_{\mathcal{V}}^{\delta,\wedge} \subseteq \widehat{P_{\mathcal{V}}^{\delta}}$ the image of the map $\widehat{J_{\mathcal{V}} P_{\mathcal{V}}^{\delta}} \to \widehat{P_{\mathcal{V}}^{\delta}}$, i.e. the (p,I)–complete ideal of $\widehat{P_{\mathcal{V}}^{\delta}}$ topologically generated by $J_{\mathcal{V}}$. Then we have a short exact sequence

$$0 \longrightarrow J_{\mathcal{V}}^{\delta,\wedge} \longrightarrow \widehat{P_{\mathcal{V}}^{\delta}} \longrightarrow R\,\widehat{\otimes_{P_{\mathcal{V}}}}\,P_{\mathcal{V}}^{\delta} \longrightarrow 0.$$

Let $(\check{C}_{\mathcal{V}}, I\check{C}_{\mathcal{V}})$ be the prismatic envelope of $(\widehat{P_{\mathcal{V}}^{\delta}}, J_{\mathcal{V}}^{\delta,\wedge})$. It follows from Proposition 3.1.10 that $(\check{C}_{\mathcal{V}}, I\check{C}_{\mathcal{V}})$ exists and is given by a flat prism over (A,I). The map

$$R \to R\,\widehat{\otimes_{P_{\mathcal{V}}}}\,P_{\mathcal{V}}^{\delta} = \widehat{P_{\mathcal{V}}^{\delta}}/J_{\mathcal{V}}^{\delta,\wedge} \to \check{C}_{\mathcal{V}}/I\check{C}_{\mathcal{V}}$$

of p–complete rings corresponds to the map of formal schemes $\mathrm{Spf}(\check{C}_{\mathcal{V}}/I\check{C}_{\mathcal{V}}) \to \mathcal{V} \hookrightarrow \mathcal{X}$. This defines an object of $(\mathcal{X}/A)^{\amalg}_{\mathbb{A}}$, which we call a *Čech–Alexander cover of* $\mathcal{V}$.

Remarks 3.3.8.

(1) Note that, by Proposition 3.1.10, $(\check{C}_{\mathcal{V}}, I\check{C}_{\mathcal{V}})$ is equivalently the prismatic envelope of $(\widehat{P_{\mathcal{V}}^{\delta}}, J_{\mathcal{V}}\widehat{P_{\mathcal{V}}^{\delta}})$. Moreover, when the ideal $J_{\mathcal{V}}$ is finitely generated, one has, in fact, the equality $J_{\mathcal{V}}^{\delta,\wedge} = J_{\mathcal{V}}\widehat{P_{\mathcal{V}}^{\delta}}$.

(2) Since the algebra R in Construction 3.3.7 is a p–completion of a finitely presented A/I–algebra (as discussed at the beginning of Section 3.2), it follows that the map $P_{\mathcal{V}} \to R$ may be chosen so that $P_{\mathcal{V}}$ is the (derived) (p,I)–completion of a polynomial A–algebra of finite type and the kernel $J_{\mathcal{V}}$ is finitely generated. While such a choice may be preferable, we formulate the construction without imposing it, as it may be convenient to allow non–finite–type free algebras in the construction e.g. for the reasons of functoriality (see the remark at the end of [7, Construction 4.17]).

The following proposition justifies the "cover" part of the name.

Proposition 3.3.9. *Denote by $h_{\check{C}_{\mathcal{V}}}$ the sheaf represented by the object $(\check{C}_{\mathcal{V}}, I\check{C}_{\mathcal{V}}) \in (\mathcal{X}/A)^{\amalg}_{\mathbb{A}}$. There exists a unique map of sheaves $h_{\check{C}_{\mathcal{V}}} \to h_{\mathcal{V}}$, and it is an epimorphism.*

Proof. If $(B, IB) \in (\mathcal{X}/A)_{\mathbb{A}}$ is an object with $h_{\check{C}_{\mathcal{V}}}((B,IB)) \neq \emptyset$, this means that the map $\mathrm{Spf}(B/IB) \to \mathcal{X}$ factors through $\mathcal{V}$ since it factors through $\mathrm{Spf}(\check{C}_{\mathcal{V}}/I\check{C}_{\mathcal{V}})$. Thus, we also have $h_{\mathcal{V}}((B, IB)) = *$, and so the (necessarily unique) map $h_{\check{C}_{\mathcal{V}}}((B, IB)) \to h_{\mathcal{V}}((B, IB))$ is defined. When $h_{\check{C}_{\mathcal{V}}}((B, IB))$ is empty, the map $h_{\check{C}_{\mathcal{V}}}((B, IB)) \to h_{\mathcal{V}}((B, IB))$ is still defined and unique, namely given by the empty map. Thus, the claimed morphism of sheaves exists and is unique.

We show that this map is an epimorphism. Let $(B, IB) \in (\mathcal{X}/A)_{\mathbb{A}}$ be an object such that $h_{\mathcal{V}}((B, IB)) = *$, i.e. such that $\mathrm{Spf}(B/IB) \to \mathcal{X}$ factors through $\mathcal{V}$, and consider the map $R \to B/IB$ associated to the map $\mathrm{Spf}(B/IB) \to \mathcal{V}$. Since $P_{\mathcal{V}}$ is a (p, I)–completed free A–algebra surjecting onto R and B is (p, I)–complete, the map $R \to B/IB$ admits a lift $P_{\mathcal{V}} \to B$. This induces an A–δ–algebra map $\widehat{P^{\delta}_{\mathcal{V}}} \to B$ which gives a morphism of δ–pairs $(\widehat{P^{\delta}_{\mathcal{V}}}, J^{\delta,\wedge}_{\mathcal{V}}) \to (B, IB)$, and further the map of prisms $(\check{C}_{\mathcal{V}}, I\check{C}_{\mathcal{V}}) \to (B, IB)$ using the universal properties of objects involved. It is easy to see that this is indeed (the opposite of) a morphism in $(\mathcal{X}/A)_{\mathbb{A}}$. This shows that $h_{\check{C}_{\mathcal{V}}}((B, IB))$ is nonempty whenever $h_{\mathcal{V}}((B, IB))$ is. Thus, the map is an epimorphism. $\square$

Let $\mathfrak{V} = \{\mathcal{V}_j\}_{j \in J}$ be an affine open cover of $\mathcal{X}$. For an integer $n \geq 1$ and a multi–index $(j_1, j_2, \ldots, j_n) \in J^n$, denote by $\mathcal{V}_{j_1,\ldots j_n}$ the intersection $\mathcal{V}_{j_1} \cap \cdots \cap \mathcal{V}_{j_n}$. As $\mathcal{X}$ is assumed to be separated, each $\mathcal{V}_{j_1,\ldots j_n}$ is affine and we write $\mathcal{V}_{j_1,\ldots j_n} = \mathrm{Spf}(R_{j_1,\ldots,j_n})$.

Remark 3.3.10 (Binary products in $(\mathcal{X}/A)_{\mathbb{A}}$). For $(B, IB), (C, IC) \in (\mathcal{X}/A)_{\mathbb{A}}$, let us denote their binary product by $(B, IB) \boxtimes (C, IC)$. Let us describe it explicitly at least under the additional assumptions that

(1) at least one of $(B, IB), (C, IC)$ is a flat prism over (A, I),

(2) there are affine opens $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X}$ such that $h_{\mathcal{U}}((B, IB)) = * = h_{\mathcal{V}}((C, IC))$.

Set $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$ and denote the rings corresponding to the affine open sets $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ by R, S and T, resp. Then any object $(D, ID) \in (\mathcal{X}/A)_{\mathbb{A}}$ with maps both to (B, IB) and (C, IC) lives over $\mathcal{W}$, i.e. satisfies $h_{\mathcal{W}}((D, ID)) = *$. This justifies the following construction. Consider the following commutative diagram, where $\urcorner$ denotes the pushout of p–complete commutative rings, i.e. taking the classically p–completed tensor product $\widehat{\otimes}$ (and $B\widehat{\otimes}_A C$ is the derived, but equivalently classical, (p, I)–completion of $B \otimes_A C$):

$$
\begin{tikzcd}
& & B\widehat{\otimes}_A C \arrow[d] & & \\
B \arrow[urr] \arrow[dd] & & & & C \arrow[ull] \arrow[dd] \\
& & (B/IB\widehat{\otimes}_R T)\widehat{\otimes}_T(C/IC\widehat{\otimes}_S T) \arrow[d, phantom, "\vee", very near start] & & \\
B/IB \arrow[r] & B/IB\widehat{\otimes}_R T \arrow[ur] \arrow[dl, phantom, "\llcorner", very near start] & & C/IC\widehat{\otimes}_S T \arrow[ul] \arrow[dr, phantom, "\lrcorner", very near start] & C/IC \arrow[l] \\
R \arrow[u] \arrow[r] & T \arrow[u] \arrow[rr, equal] & & T \arrow[u] & S \arrow[u] \arrow[l]
\end{tikzcd}
$$

Let $J \subseteq B\widehat{\otimes}_A C$ be the kernel of the map $B\widehat{\otimes}_A C \to (B/IB\widehat{\otimes}_R T)\widehat{\otimes}_T(C/IC\widehat{\otimes}_S T)$. Then $(B, IB) \boxtimes (C, IC)$ is given by the prismatic envelope of the δ–pair $(B\widehat{\otimes}_A C, J)$.

Proposition 3.3.11. *The Čech–Alexander covers can be chosen so that for all indices $j_1, \ldots, j_n$ we have*

$$(\check{C}_{\mathcal{V}_{j_1,\ldots j_n}}, I\check{C}_{\mathcal{V}_{j_1,\ldots j_n}}) = (\check{C}_{\mathcal{V}_{j_1}}, I\check{C}_{\mathcal{V}_{j_1}}) \boxtimes (\check{C}_{\mathcal{V}_{j_2}}, I\check{C}_{\mathcal{V}_{j_2}}) \boxtimes \cdots \boxtimes (\check{C}_{\mathcal{V}_{j_n}}, I\check{C}_{\mathcal{V}_{j_n}}).$$

Proof. Clearly it is enough to show the statement for binary products. More precisely, given two affine opens $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathfrak{X}$ and an arbitrary initial choice of $(\check{C}_{\mathcal{V}_1}, I\check{C}_{\mathcal{V}_1})$ and $(\check{C}_{\mathcal{V}_2}, I\check{C}_{\mathcal{V}_2})$, we show that $P_{\mathcal{V}_{12}} \to R_{12}$ can be chosen so that the resulting Čech–Alexander cover $(\check{C}_{\mathcal{V}_{12}}, I\check{C}_{\mathcal{V}_{12}})$ of $\mathcal{V}_{12}$ is equal to $(\check{C}_{\mathcal{V}_1}, I\check{C}_{\mathcal{V}_1}) \boxtimes (\check{C}_{\mathcal{V}_2}, I\check{C}_{\mathcal{V}_2})$. For the purposes of this proof, let us refer to a prismatic envelope of a δ–pair (S, J) also as "the prismatic envelope of the arrow $S \to S/J$".

Consider $\alpha_i : P_{\mathcal{V}_i} \twoheadrightarrow R_i,\ i = 1, 2$ as in Construction 3.3.7, and set $P_{\mathcal{V}_{12}} = P_{\mathcal{V}_1}\widehat{\otimes}_A P_{\mathcal{V}_2}$. Then one has the induced surjection $\alpha_1 \otimes \alpha_2 : P_{\mathcal{V}_{12}} \to R_1\widehat{\otimes}_{A/I} R_2$, which can be followed by the induced map $R_1\widehat{\otimes}_{A/I} R_2 \to R_{12}$. This latter map is surjective as well since $\mathfrak{X}$ is separated, and therefore the composition of these two maps $\alpha_{12} : P_{\mathcal{V}_{12}} \to R_{12}$ is surjective, with the kernel $J_{\mathcal{V}_{12}}$ that contains $(J_{\mathcal{V}_1}, J_{\mathcal{V}_2})P_{\mathcal{V}_{12}}$. We may construct a diagram analogous to the one from Remark 3.3.10, which becomes the diagram

$$
\begin{tikzcd}[column sep=small]
 & & \widehat{P^\delta_{\mathcal{V}_{12}}} \arrow[dd] & & \\
\widehat{P^\delta_{\mathcal{V}_1}} \arrow[urr] \arrow[dd] & & & & \widehat{P^\delta_{\mathcal{V}_2}} \arrow[ull] \arrow[dd] \\
 & & R_{12}\widehat{\otimes}_{P_{\mathcal{V}_{12}}}(\widehat{P^\delta_{\mathcal{V}_{12}}}) \arrow[d, phantom, "\vee"] & & \\
R_1\widehat{\otimes}_{P_{\mathcal{V}_1}}\widehat{P^\delta_{\mathcal{V}_1}} \arrow[r] \arrow[dr, phantom, "\llcorner"] & R_{12}\widehat{\otimes}_{P_{\mathcal{V}_1}}\widehat{P^\delta_{\mathcal{V}_1}} \arrow[ur] & {} & R_{12}\widehat{\otimes}_{P_{\mathcal{V}_2}}\widehat{P^\delta_{\mathcal{V}_2}} \arrow[ul] & R_2\widehat{\otimes}_{P_{\mathcal{V}_2}}\widehat{P^\delta_{\mathcal{V}_2}} \arrow[l] \arrow[dl, phantom, "\lrcorner"] \\
R_1 \arrow[r] \arrow[u] & R_{12} \arrow[rr, equal] \arrow[u] & & R_{12} \arrow[u] & R_2, \arrow[l] \arrow[u]
\end{tikzcd}
$$

where the expected arrow in the central column is replaced by an isomorphic one, namely the map obtained from the surjection $P_{\mathcal{V}_{12}} \to R_{12}$ by the procedure as in Construction 3.3.7. Now $(\check{C}_{\mathcal{V}_{12}}, I\check{C}_{\mathcal{V}_{12}})$ is obtained as the prismatic envelope of this composed central arrow, while $(\check{C}_{\mathcal{V}_1}, I\check{C}_{\mathcal{V}_1}) \boxtimes (\check{C}_{\mathcal{V}_2}, I\check{C}_{\mathcal{V}_2})$ is obtained the same way, but only after replacing the downward arrows on the left and right by their prismatic envelopes. Comparing universal properties, one easily sees that the resulting central prismatic envelope remains unchanged[6], proving the claim. $\square$

Remark 3.3.12. Suppose that for each j, the initial choice of the map $P_{\mathcal{V}_j} \to R_j$ has been made as in Remark 3.3.8 (2), that is, $P_{\mathcal{V}_j}$ is the (p, I)–completion of a finite type free A–algebra and the ideal $J_{\mathcal{V}_j}$ is finitely generated. If now $P_{\mathcal{V}_{j_1,j_2,\dots j_n}}$ is the (p, I)–completed free A–algebra for $\mathcal{V}_{j_1,j_2,\dots j_n}$ obtained by iterating the procedure in the proof of Proposition 3.3.11, it is easy to see that in this case, the algebra $P_{\mathcal{V}_{j_1,j_2,\dots j_n}}$ is still the (p, I)–completion of a finite type free A–algebra, and it can be shown that the corresponding ideal $J_{\mathcal{V}_{j_1,j_2,\dots j_n}}$ is finitely generated.

[6]↑In more detail, one can consider the category $(\mathcal{X}/A)_\delta$ defined in the same manner as $(\mathcal{X}/A)_{\Delta}$, except the basic objects are given by (the opposites of) all posible (p, I)–complete δ–pairs (P, J) over (A, I) and not just bounded prisms. Then $(\mathcal{X}/A)_{\Delta}$ forms a full subcategory of $(\mathcal{X}/A)_\delta$, and the inclusion admits a right adjoint induced by the operation of taking prismatic envelope. The diagram in the proof shows that $(\check{C}_{\mathcal{V}_{12}}, I\check{C}_{\mathcal{V}_{12}})$ corresponds to the universal object of $(\mathcal{X}/A)_{\Delta}$ admitting maps to the objects associated to $(\widehat{P^\delta_{\mathcal{V}_i}}, J^{\delta,\wedge}_{\mathcal{V}_i})$, $i = 1, 2$, but every such pair of maps factors through their prismatic envelopes $(\check{C}_{\mathcal{V}_i}, I\check{C}_{\mathcal{V}_i})$, which establishes a map $(\check{C}_{\mathcal{V}_1}, I\check{C}_{\mathcal{V}_1}) \boxtimes (\check{C}_{\mathcal{V}_2}, I\check{C}_{\mathcal{V}_2}) \to (\check{C}_{\mathcal{V}_{12}}, I\check{C}_{\mathcal{V}_{12}})$ in $(\mathcal{X}/A)_{\Delta}$. The inverse map is defined similarly, using the pair of maps $(\check{C}_{\mathcal{V}_1}, I\check{C}_{\mathcal{V}_1}) \boxtimes (\check{C}_{\mathcal{V}_2}, I\check{C}_{\mathcal{V}_2}) \to (\check{C}_{\mathcal{V}_i}, I\check{C}_{\mathcal{V}_i}) \to (\widehat{P^\delta_{\mathcal{V}_i}}, J^{\delta,\wedge}_{\mathcal{V}_i})$, $i = 1, 2$, in $(\mathcal{X}/A)_\delta$.

In more detail, given a ring B and a finitely generated ideal $J \subseteq B$, Let us call a B–algebra C *J–completely finitely presented* if C is derived J–complete and there exists a map $\alpha : B[\underline{X}] \to C$ from the polynomial ring in finitely many variables $\underline{X} = \{X_1, \ldots, X_n\}$ such that the derived J–completed map $\widehat{\alpha} : \widehat{B[\underline{X}]} \to C$ is surjective and with a finitely generated kernel. Then the algebra $R_{j_1,j_2,\ldots j_n}$ corresponding to $\mathcal{V}_{j_1,j_2,\ldots j_n}$ is (p, I)–completely finitely presented by Remark 3.3.8 (2), and since $P_{\mathcal{V}_{j_1,j_2,\ldots j_n}}$ is the (p, I)–completion of a finite type polynomial A–algebra, the following lemma shows that $J_{\mathcal{V}_{j_1,j_2,\ldots j_n}}$ is finitely generated.

Lemma 3.3.13. *Let C be a J–completely finitely presented B–algebra, and consider a map $\beta : B[\underline{Y}] \to C$ from a polynomial algebra in finitely many variables $\underline{Y} = \{Y_1, \ldots, Y_m\}$ such that $\widehat{\beta}$ is surjective. Then the kernel of $\widehat{\beta}$ is finitely generated.*

Proof. The proof is an adaptation of the proof of [36, 00R2], which is a similar assertion about finitely presented algebras. Consider α as in Remark 3.3.12, and additionally let us fix a generating set $(f_1, f_2, \ldots, f_k) \subseteq \widehat{B[\underline{X}]}$ of $\operatorname{Ker} \widehat{\alpha}$.

For $i = 1, \ldots, m$, let us choose $g_i \in \widehat{B[\underline{X}]}$ such that $\widehat{\alpha}(g_i) = \beta(Y_i)$. Then one can define a surjective map

$$\theta_0 : \widehat{B[\underline{X}]}[\underline{Y}] \to C, \quad \theta_0 \mid_{\widehat{B[\underline{X}]}} = \widehat{\alpha}, \;\; \theta_0(Y_i) = \beta(Y_i),$$

and it is easy to see that $\operatorname{Ker} \theta_0 = (f_1, \ldots, f_k, Y_1 - g_1, \ldots, Y_m - g_m)$. That is, we have an exact sequence

$$(\widehat{B[\underline{X}]}[\underline{Y}])^{\oplus k+m} \to \widehat{B[\underline{X}]}[\underline{Y}] \xrightarrow{\theta_0} C \to 0,$$

where the map on the left is a module map determined by the finite set of generators of $\operatorname{Ker} \theta_0$. After taking the derived J–completion, the sequence becomes the exact sequence

$$\widehat{B[\underline{X}, \underline{Y}]}^{\oplus k+m} \to \widehat{B[\underline{X}, \underline{Y}]} \xrightarrow{\theta} C \to 0.$$

That is, we have a surjective map $\theta : \widehat{B[\underline{X}, \underline{Y}]} \to C$ determined on topological generators by $\theta(X_j) = \alpha(X_j), \theta(Y_i) = \beta(Y_i)$, and the kernel of θ is $(f_1, \ldots, f_k, Y_1 - g_1, \ldots, Y_m - g_m)$.

Next, we choose elements $h_j \in \widehat{B[\underline{Y}]}$ such that $\widehat{\beta}(h_j) = \alpha(X_j)$ for each j. Then we have a surjective map $\psi : \widehat{B[\underline{X}, \underline{Y}]} \to \widehat{B[\underline{Y}]}$ given by $X_j \mapsto h_j$ and $Y_i \mapsto Y_i$, which has the property that $\widehat{\beta} \circ \psi = \theta$. That is,

$$\operatorname{Ker} \theta = \operatorname{Ker} (\widehat{\beta} \circ \psi) = \psi^{-1}(\operatorname{Ker} (\widehat{\beta})),$$

and therefore $\psi(\operatorname{Ker} \theta) = \operatorname{Ker} \widehat{\beta}$ since ψ is surjective. But $\operatorname{Ker} \theta$ is finitely generated by the previous, and hence so is $\operatorname{Ker} \widehat{\beta}$. □

Let us now return to the situation before Remark 3.3.12. The next step is to show that an initial choice of the Čech–Alexander covers $\check{C}_{\mathcal{V}_j}$ for an affine open cover $\{\mathcal{V}_j\}_j$ of $\mathfrak{X}$ determines, in a suitable sense, a global cover of $\mathfrak{X}$.

Proposition 3.3.14. *The map $\coprod_j h_{\mathcal{V}_j} \to h_{\mathfrak{X}} = *$ (where $\coprod$ denotes the coproduct in $\mathsf{Shv}((\mathfrak{X}/A)^{\amalg}_{\mathbb{\Delta}})$) to the final object is an epimorphism, hence so is the map $\coprod_j h_{\check{C}_{\mathcal{V}_j}} \to *$.*

Proof. It is enough to show that for a given object $(B, IB) \in (\mathfrak{X}/A)^{\amalg}_{\mathbb{\Delta}}$, there is a faithfully flat family $(B, IB) \to (C_i, IC_i)$ in $(\mathfrak{X}/A)^{\amalg,\mathrm{op}}_{\mathbb{\Delta}}$ such that $\coprod^{\mathrm{pre}}_j h_{\mathcal{V}_j}((C_i, IC_i)) \neq \emptyset$ for all i where $\coprod^{\mathrm{pre}}$ denotes the coproduct of presheaves.

With that aim, let us first consider the preimages $\mathcal{W}_j \subseteq \operatorname{Spf}(B/IB)$ of each $\mathcal{V}_j$ under the map $\operatorname{Spf}(B/IB) \to \mathfrak{X}$. This is an open cover of $\operatorname{Spf}(B/IB)$ that corresponds to an open cover of $\operatorname{Spec} B/(p, I)B$. One can then choose $f_1, f_2, \ldots, f_m$ such that $\{\operatorname{Spec}(B/(p, I)B)_{f_i}\}_i$ refines this cover, i.e. every $\operatorname{Spec}(B/(p, I)B)_{f_i}$ corresponds to an open subset of $\mathcal{W}_{j(i)}$ for some index $j(i)$.

The elements $f_1, \ldots, f_m$ generate the unit ideal of B since they do so modulo (p, I) which is contained in $\operatorname{rad}(B)$. Thus, the family

$$(B, IB) \to (C_i, IC_i) := (\widehat{B_{f_i}}, I\widehat{B_{f_i}}) \quad i = 1, 2, \ldots, m$$

is easily seen to give the desired faithfully flat family, with each $\coprod^{\mathrm{pre}}_j h_{\mathcal{V}_j}((C_i, IC_i))$ nonempty, since each $\operatorname{Spf}(C_i/IC_i) \to \mathfrak{X}$ factors through $\mathcal{V}_{j(i)}$ by construction. □

Remark 3.3.15. The proof of Proposition 3.3.14 is the one step where we used the relaxation of the topology, namely the fact that the faithfully flat cover $(B, IB) \to \prod_i (C_i, IC_i)$ can be replaced by the family $\{(B, IB) \to (C_i, IC_i)\}_i$.

Finally, we obtain the Čech–Alexander complexes in the global case.

Proposition 3.3.16. $\mathsf{R}\Gamma((\mathfrak{X}/A)_{\mathbb{A}}, \mathcal{O})$ *is modelled by the Čech–Alexander complex*

$$0 \longrightarrow \prod_j \check{C}_{\mathcal{V}_j} \longrightarrow \prod_{j_1,j_2} \check{C}_{\mathcal{V}_{j_1,j_2}} \longrightarrow \prod_{j_1,j_2,j_3} \check{C}_{\mathcal{V}_{j_1,j_2,j_3}} \longrightarrow \dots \tag{$\check{C}^\bullet_{\mathfrak{V}}$}$$

Proof. By [36, 079Z], the epimorphism of sheaves $\coprod_j h_{\check{C}_{\mathcal{V}_j}} \to *$ from Proposition 3.3.14 implies that there is a spectral sequence with E_1–page

$$E_1^{p,q} = \mathrm{H}^q\Big(\Big(\coprod_j h_{\check{C}_{\mathcal{V}_j}}\Big)^{\times p}, \mathcal{O}\Big) = \mathrm{H}^q\Big(\coprod_{j_1,\dots,j_p} h_{\check{C}_{\mathcal{V}_{j_1,\dots,j_p}}}, \mathcal{O}\Big) = \prod_{j_1,\dots,j_p} \mathrm{H}^q((\check{C}_{\mathcal{V}_{j_1,\dots,j_p}}, I\check{C}_{\mathcal{V}_{j_1,\dots,j_p}}), \mathcal{O})$$

converging to $\mathrm{H}^{p+q}(*, \mathcal{O}) = \mathrm{H}^{p+q}((\mathfrak{X}/A)^{\amalg}_{\mathbb{A}}, \mathcal{O}) = \mathrm{H}^{p+q}((\mathfrak{X}/A)_{\mathbb{A}}, \mathcal{O})$, where we implicitly used Corollary 3.3.5 and the fact that $h_{\check{C}_{\mathcal{V}_{j_1}}} \times h_{\check{C}_{\mathcal{V}_{j_2}}} = h_{\check{C}_{\mathcal{V}_{j_1}} \boxtimes \check{C}_{\mathcal{V}_{j_2}}} = h_{\check{C}_{\mathcal{V}_{j_1,j_2}}}$ as in Proposition 3.3.11, and similarly for higher multi–indices.

By Lemma 3.3.4, $\mathrm{H}^q((\check{C}_{\mathcal{V}_{j_1,\dots,j_n}}, I\check{C}_{\mathcal{V}_{j_1,\dots,j_n}}), \mathcal{O}) = 0$ for every $q > 0$ and every multi–index $j_1, \dots, j_n$. The first page is therefore concentrated in a single row of the form $\check{C}^\bullet_{\mathfrak{V}}$ and thus, the spectral sequence collapses on the second page. This proves that the cohomologies of $\mathsf{R}\Gamma((\mathfrak{X}/A)_{\mathbb{A}}, \mathcal{O})$ are computed as cohomologies of $\check{C}^\bullet_{\mathfrak{V}}$, but in fact, this yields a quasi–isomorphism of the complexes themselves. (For example, analyzing the proof of [36, 079Z] via [36, 03OW], the double complex $E_0^{\bullet\bullet}$ of the above spectral sequence comes with a natural map $\alpha : \check{C}^\bullet_{\mathfrak{V}} \to \mathrm{Tot}(E_0^{\bullet\bullet})$, and a natural quasi–isomorphism $\beta : \mathsf{R}\Gamma((\mathfrak{X}/A)_{\mathbb{A}}, \mathcal{O}) \to \mathrm{Tot}(E_0^{\bullet\bullet})$; when the spectral sequence collapses as above, α is also a quasi–isomorphism). $\square$

Remarks 3.3.17.

(1) The formation of Čech–Alexander complexes is termwise compatible with flat base–change on the base prism. That is, given a flat map of bounded prisms $(A, I) \to (B, IB)$, a smooth and separated formal scheme $\mathfrak{X} \to \mathrm{Spf}(A/I)$ and a Čech–Alexander complex $\check{C}^\bullet = (\{\check{C}^m\}_m, \partial)$ corresponding to the affine open cover $\mathfrak{V} = \{\mathcal{V}_j\}_j$, the com-

plex $(\{\check{C}^m\widehat{\otimes}_A B\}_m, \partial\otimes\mathrm{id})$ (which, by a slight abuse of notation, will be denoted by $\check{C}^\bullet\widehat{\otimes}_A B$ from now on) is a Čech–Alexander complex for $\mathfrak{X}_{B/IB}$ corresponding to the cover $\mathfrak{V}_{B/IB}=\{(\mathcal{V}_j)_{B/IB}\}_j$. Indeed, the claim immediately reduces to the case of Čech–Alexander cover for a single affine open $\mathcal{V}$. The initial p–completed free algebras can be clearly chosen compatibly, i.e. so that $P_{\mathcal{V}_{B/IB}}=P_{\mathcal{V}}\widehat{\otimes}_A B$. From that point on, every step of the construction is base–change compatible (for taking prismatic envelopes, this is thanks to Proposition 3.1.10 (3)).

(2) Let now (A,I) be the prism $(A_{\mathrm{inf}},\operatorname{Ker}\theta)$ and let $\mathfrak{X}$ be of the form $\mathfrak{X}=\mathfrak{X}^0\times_{\mathcal{O}_K}\mathcal{O}_{\mathbb{C}_K}$ where $\mathfrak{X}^0$ is a smooth separated formal $\mathcal{O}_K$–scheme. A convenient way to describe the G_K–action on $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A_{\mathrm{inf}})$ is via base–change: given $g\in G_K$, action of g on A_{inf} gives a map of prisms $g:(A_{\mathrm{inf}},(E(u)))\to(A_{\mathrm{inf}},(E(u)))$, and $g^*\mathfrak{X}=\mathfrak{X}$ since $\mathfrak{X}$ comes from $\mathcal{O}_K$. The base–change theorem for prismatic cohomology (Theorem 3.2.4 (3)) then gives an A_{inf}–linear map $g^*\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A_{\mathrm{inf}})\to\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A_{\mathrm{inf}})$ (moreover, since g is an isomorphism it is clear that $g^*\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A_{\mathrm{inf}})=\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A_{\mathrm{inf}})\widehat{\otimes}^{\mathsf{L}}_{A_{\mathrm{inf}},g}A_{\mathrm{inf}}$ agrees with the "termwise base–change" description from previous remark). Untwisting by g on the left, this gives an A_{inf}–g–semilinear action map $g:\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A_{\mathrm{inf}})\to\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/A_{\mathrm{inf}})$. The exact same procedure defines the G_K–action on the the Čech–Alexander complexes modelling the cohomology theories since they are base–change compatible.

3.4 Breuil–Kisin and Breuil–Kisin–Fargues modules

An abstract avatar of the prismatic cohomology theories $\mathsf{R}\Gamma_{\Delta}(-/\mathfrak{S}),\mathsf{R}\Gamma_{\Delta}(-/A_{\mathrm{inf}})$ are the Breuil–Kisin and Breuil–Kisin–Fargues modules, which we now briefly recall.

Definition 3.4.1.

(1) A *Breuil–Kisin module* is a finitely generated $\mathfrak{S}$–module M together with an isomorphism of $\mathfrak{S}[1/E(u)]$–modules

$$\varphi=\varphi_{M[1/E]}:(\varphi_{\mathfrak{S}}^*M)[1/E(u)]\xrightarrow{\sim}M[1/E(u)].$$

For a positive integer i, the Breuil–Kisin module M is said to be *of height* $\leq i$ if $\varphi_{M[1/E]}$ is induced (by linearization and localization) by a φ–$\mathfrak{S}$–semilinear map $\varphi_M : M \to M$ such that, denoting $\varphi_{\mathrm{lin}} : \varphi^*M \to M$ its linearization, there exists an $\mathfrak{S}$–linear map $\psi : M \to \varphi^*M$ such that both the compositions $\psi \circ \varphi_{\mathrm{lin}}$ and $\varphi_{\mathrm{lin}} \circ \psi$ are multiplication by $E(u)^i$. A Breuil–Kisin module is *of finite height* if it is of height $\leq i$ for some i.

(2) A *Breuil–Kisin–Fargues module* is a finitely presented A_{inf}–module M such that $M[1/p]$ is a free $A_{\mathrm{inf}}[1/p]$–module, together with an $A_{\mathrm{inf}}[1/E(u)]$–linear isomorphism

$$\varphi = \varphi_{M[1/E]} : (\varphi^*_{A_{\mathrm{inf}}} M)[1/E(u)] \xrightarrow{\sim} M[1/E(u)].$$

Similarly, the Breuil–Kisin–Fargues module is called *of height* $\leq i$ if $\varphi_{M[1/E]}$ comes from a semilinear map $\varphi_M : \varphi^*M \to M$ such that there exist an A_{inf}–linear map $\psi : M \to \varphi^*M$ such that $\psi \circ \varphi_{\mathrm{lin}}$ and $\varphi_{\mathrm{lin}} \circ \psi$ are multiplication maps by $E(u)^i$, where φ_{lin} is the inearization of φ_M. A Breuil–Kisin–Fargues module is *of finite height* if it is of height $\leq i$ for some i.

(3) A *Breuil–Kisin–Fargues G_K–module* (of height $\leq i$, of finite height, resp.) is a Breuil–Kisin–Fargues module (of height $\leq i$, of finite height, resp.) that is additionally endowed with an A_{inf}–semilinear G_K–action that makes $\varphi_{M[1/E]}$ G_K–equivariant (that makes also φ_M G_K–equivariant in the finite height cases).

That is, the definition of a Breuil–Kisin module that we use agrees with the one in [5], and M_{inf} is a Breuil–Kisin–Fargues module in the sense of the above definition if and only if $\varphi^*_{A_{\mathrm{inf}}} M_{\mathrm{inf}}$ is a Breuil–Kisin–Fargues module in the sense of [5][7]. The notion of Breuil–Kisin module of height $\leq i$ agrees with what is called "(generalized) Kisin modules of height i" in [27]. The above notion of finite height Breuil–Kisin–Fargues modules agrees with the one from [14, Appendix F] except that the modules are not assumed to be free. Also note that under these definitions, for a Breuil–Kisin module M_{BK} (of height $\leq i$, resp.), the

[7]↑This is to account for the fact that while Breuil–Kisin–Fargues modules in the sense of [5] appear as A_{inf}–cohomology groups of smooth proper formal schemes in their original definition, Breuil–Kisin–Fargues modules in the above sense appear as *prismatic* A_{inf}–cohomology groups of smooth proper formal schemes. Since the two theories differ by a φ–twist, so does the notion of a Breuil–Kisin–Fargues module.

A_{inf}–module $M_{\mathrm{inf}} = A_{\mathrm{inf}} \otimes_{\mathfrak{S}} M_{\mathrm{BK}}$ is a Breuil–Kisin–Fargues module (of height $\leq i$, resp.), without the need to twist the embedding $\mathfrak{S} \to A_{\mathrm{inf}}$ by φ.

Remark 3.4.2. Note that Definition 3.4.1 allows Breuil–Kisin(–Fargues) modules to have p– and u–torsion in general, and in particular, we do not require these modules to be free. The reason is that the modules coming from cohomology are generally of this form. In particular, we also allow entirely p^n–torsion Breuil–Kisin(–Fargues) modules.

However, given a Breuil–Kisin module M_{BK} in the sense above, it is always related to a free one: by [5, Propostition 4.3] there is a functorial exact sequence

$$0 \longrightarrow M_{\mathrm{BK,tor}} \longrightarrow M_{\mathrm{BK}} \longrightarrow M_{\mathrm{BK,free}} \longrightarrow \overline{M_{\mathrm{BK}}} \longrightarrow 0$$

where $M_{\mathrm{BK,free}}$ is a free Breuil–Kisin module, $M_{\mathrm{BK,tor}}$ is a p^n-torsion module for some n and $\overline{M_{\mathrm{BK}}}$ is supported at the maximal ideal (p,u). An analogous result holds also in the Breuil–Kisin–Fargues case by [5, Propostition 4.13]. (Of course, in the case of p^n–torsion Breuil–Kisin(–Fargues) modules, these exact sequences degenerate.)

Definition 3.4.3 (étale realizations)**.** Denote by $\mathbb{C}_K^\flat$ the fraction field of $\mathcal{O}_{\mathbb{C}_K}^\flat$.

(1) Given a Breuil–Kisin module M_{BK}, we define its étale realization by

$$V(M_{\mathrm{BK}}) := \left(M_{\mathrm{BK}} \otimes_{\mathfrak{S}} W(\mathbb{C}_K^\flat)\right)^{\varphi=1}[1/p],$$

its integral étale realization by

$$T(M_{\mathrm{BK}}) := \left(M_{\mathrm{BK}} \otimes_{\mathfrak{S}} W(\mathbb{C}_K^\flat)\right)^{\varphi=1},$$

and its p^n–torsion étale realization by

$$T_n(M_{\mathrm{BK}}) := \left(M_{\mathrm{BK}} \otimes_{\mathfrak{S}} W_n(\mathbb{C}_K^\flat)\right)^{\varphi=1}.$$

(where $W_n(\mathbb{C}_K^\flat)$ denotes the truncated Witt vectors $W(\mathbb{C}_K^\flat)/p^n$). All the étale realizations are considered as G_∞–modules via the G_∞–action on the $W(\mathbb{C}_K^\flat)$–component.

(2) Given a Breuil–Kisin–Fargues module M_{inf}, we define its étale realization by

$$V^{\mathrm{inf}}(M_{\mathrm{inf}}) := \left(M_{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} W(\mathbb{C}_K^\flat)\right)^{\varphi=1}[1/p],$$

its integral étale realization by

$$T^{\mathrm{inf}}(M_{\mathrm{inf}}) := \left(M_{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} W(\mathbb{C}_K^\flat)\right)^{\varphi=1},$$

and its p^n–torsion étale realization by

$$T_n^{\mathrm{inf}}(M_{\mathrm{inf}}) := \left(M_{\mathrm{inf}} \otimes_{A_{\mathrm{inf}}} W_n(\mathbb{C}_K^\flat)\right)^{\varphi=1}.$$

When M_{inf} additionally carries the structure of a Breuil–Kisin–Fargues G_K–module, the étale realizations are naturally G_K–modules, via the G_K–action on both components of the tensor product.

The following consequence of Theorem 3.2.4 (due to Bhatt, Morrow and Scholze [5, 6, 7]) relates prismatic cohomology, Breuil–Kisin(–Fargues) modules and their étale realizations. The p^n–torsion variant was established by Li and Liu [27, §7.1].

Proposition 3.4.4. *Let $\mathcal{X}$ be a proper smooth formal $\mathcal{O}_K$–scheme and $\mathcal{X}'$ a proper smooth formal $\mathcal{O}_{\mathbb{C}_K}$–scheme.*

(1) For every $i \geq 0$ and every $n \geq 0$, the modules $M_{\mathrm{inf}} = \mathrm{H}^i_{\Delta}(\mathcal{X}'/A_{\mathrm{inf}})$, $M_{\mathrm{inf},n} = \mathrm{H}^i_{\Delta,n}(\mathcal{X}'/A_{\mathrm{inf}})$ are Breuil–Kisin–Fargues modules of height $\leq i$ (with $M_{\mathrm{BK},n}$ being p^n–torsion Breuil–Kisin–Fargues module).

(2) For every $i \geq 0$ and every $n \geq 0$, the modules $M_{\mathrm{BK}} = \mathrm{H}^i_{\Delta}(\mathcal{X}/\mathfrak{S})$, $M_{\mathrm{BK},n} = \mathrm{H}^i_{\Delta,n}(\mathcal{X}/\mathfrak{S})$ are Breuil–Kisin modules of height $\leq i$ (with $M_{\mathrm{BK},n}$ being p^n–torsion Breuil–Kisin module).

(3) Denote by $\mathcal{X}'_\eta$ the generic fiber of $\mathcal{X}'$ as an adic space. Then the étale realizations satisfy

$$V^{\mathrm{inf}}(M_{\mathrm{inf}}) \simeq \mathrm{H}^i_{\acute{e}t}(\mathcal{X}'_\eta, \mathbb{Q}_p), \quad T^{\mathrm{inf}}(M_{\mathrm{inf}}) \simeq \mathrm{H}^i_{\acute{e}t}(\mathcal{X}'_\eta, \mathbb{Z}_p), \quad T_n^{\mathrm{inf}}(M_{\mathrm{inf},n}) \simeq \mathrm{H}^i_{\acute{e}t}(\mathcal{X}'_\eta, \mathbb{Z}/p^n\mathbb{Z}).$$

(4) *Denote by* $\mathfrak{X}_{\overline{\eta}}$ *the geometric generic fiber of* $\mathfrak{X}$ *as an adic space. Then the étale realizations satisfy*

$$V(M_{\mathrm{BK}}) \simeq \mathrm{H}^i_{\acute{e}t}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Q}_p), \quad T(M_{\mathrm{BK}}) \simeq \mathrm{H}^i_{\acute{e}t}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}_p), \quad T_n(M_{\mathrm{BK},n}) \simeq \mathrm{H}^i_{\acute{e}t}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$$

as G_∞*–modules.*

(5) *Assume that* $\mathfrak{X}' = \mathfrak{X}_{\mathcal{O}_{\mathbb{C}_K}}$. *Then* $M_{\mathrm{inf}} = M_{\mathrm{BK}} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ *and* $M_{\mathrm{inf},n} = M_{\mathrm{BK},n} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ *as Breuil–Kisin–Fargues modules. Moreover, they are naturally Breuil–Kisin–Fargues* G_K*–modules, and we have*

$$V^{\mathrm{inf}}(M_{\mathrm{inf}}) \simeq \mathrm{H}^i_{\acute{e}t}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Q}_p), \quad T^{\mathrm{inf}}(M_{\mathrm{inf}}) \simeq \mathrm{H}^i_{\acute{e}t}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}_p), \quad T^{\mathrm{inf}}_n(M_{\mathrm{inf},n}) \simeq \mathrm{H}^i_{\acute{e}t}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$$

as G_K*–modules.*

4. THE CONDITIONS (Cr_s) AND THE CRYSTALLINE CONDITION

4.1 Definition and basic properties

In this section we define and discuss the conditions (Cr_s), which is in a sense the key technical part of this work. Recall the field–theoretic setup from Introduction, namely the Kummer tower $\{K_s = K(\pi_s)\}_s$ and the corresponding sequence of absolute Galois groups $G_s = \mathrm{Gal}(\overline{K}/K_s)$.

There is a natural G_K–action on $A_{\mathrm{inf}} = W(\mathcal{O}_{\mathbb{C}_K}^{\flat})$ inbduced by the G_K–action on $\mathcal{O}_{\mathbb{C}_K}^{\flat}$. This action makes the map $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_K}$ G_K–equivariant, in particular, the kernel $E(u)A_{\mathrm{inf}}$ is G_K–stable. The G_K–action on the G_K–closure of $\mathfrak{S}$ in A_{inf} factors through $\widehat{G} = \mathrm{Gal}(K_{p^\infty,\infty}/K)$. Note that the subgroup $\mathrm{Gal}(K_{p^\infty,\infty}/K_\infty)$ of $\widehat{G}$ acts trivially on elements of $\mathfrak{S}$, and the action of the subgroup $\mathrm{Gal}(K_{p^\infty,\infty}/K_{p^\infty})$ is determined by the equality $\tau(u) = (v+1)u$ (where τ is a topological generator of $\mathrm{Gal}(K_{p^\infty,\infty}/K_{p^\infty})$ as in Introduction).

Notation 4.1.1. For an integer $s \geq 0$ and i between 0 and s, denote by $\xi_{s,i}$ the element

$$\xi_{s,i} = \frac{\varphi^s(v)}{\omega\varphi(\omega)\dots\varphi^i(\omega)} = \varphi^{-1}(v)\varphi^{i+1}(\omega)\varphi^{i+2}(\omega)\dots\varphi^s(\omega)$$

(recall that $\omega = v/\varphi^{-1}(v)$ is one choice of a generator of the principal ideal $\mathrm{Ker}\,\theta \subseteq A_{\mathrm{inf}}$), and set

$$I_s = \left(\xi_{s,0}u, \xi_{s,1}u^p, \dots, \xi_{s,s}u^{p^s}\right) \subseteq A_{\mathrm{inf}}.$$

For convenience of notation, we further set $I_\infty = 0$ and $\varphi^\infty(v)u = 0$.

We are concerned with the following conditions.

Definition 4.1.2. Let M_{inf} be an A_{inf}–module endowed with a G_K–A_{inf}–semilinear action, let M_{BK} be an $\mathfrak{S}$–module and let $M_{\mathrm{BK}} \to M_{\mathrm{inf}}$ be an $\mathfrak{S}$–linear map. Let $s \geq 0$ be an integer or ∞.

(1) An element $x \in M_{\text{inf}}$ is called a (Cr_s)*–element* if for every $g \in G_s$,

$$g(x) - x \in I_s M_{\text{inf}}.$$

(2) We say that the pair $M_{\text{BK}} \to M_{A_{\text{inf}}}$ *satisfies the condition* (Cr_s) if for every element $x \in M_{\text{BK}}$, the image of x in M_{inf} is (Cr_s).

(3) An element $x \in M_{\text{inf}}$ is called a (Cr'_s)*–element* if for every $g \in G_s$, there is an element $y \in M_{\text{inf}}$ such that

$$g(x) - x = \varphi^s(v)uy.$$

(4) We say that the pair $M_{\text{BK}} \to M_{A_{\text{inf}}}$ *satisfies the condition* (Cr'_s) if for every element $x \in M_{\text{BK}}$, the image of x in M_{inf} is (Cr'_s).

(5) Aditionally, we call (Cr_0)–elements *crystalline elements* and we call the condition (Cr_0) *the crystalline condition.*

Remarks 4.1.3.

(1) Since $I_0 = \varphi^{-1}(v)uA_{\text{inf}}$, the crystalline condition equivalently states that for all $g \in G_K$ and all x in the image of M_{BK},

$$g(x) - x \in \varphi^{-1}(v)uM_{\text{inf}}.$$

The reason for the extra terminology in the case $s = 0$ is that the condition is connected with a criterion for certain representations to be crystalline, as discussed in Section 4.2. The higher conditions (Cr_s) will on the other hand find application in computing bounds on ramification of p^n–torsion étale cohomology. The conditions (Cr'_s) serve an auxiliary purpose. Clearly for every s (Cr'_s) implies (Cr_s).

(2) Strictly speaking, one should talk about the crystalline condition (or (Cr_s)) for the map f, but we choose to leave the datum of the map f implicit. This is because typically we consider the situation that M_{BK} is an $\mathfrak{S}$–submodule of M_{inf} and $M_{\text{BK}} \otimes_{\mathfrak{S}} A_{\text{inf}} \simeq M_{A_{\text{inf}}}$

via the natural map (or the derived $(p, E(u))$–completed variant, $M_{\mathrm{BK}}\widehat{\otimes}_{\mathfrak{S}}A_{\inf} \simeq M_{A_{\inf}}$). Also note that $f : M_{\mathrm{BK}} \to M_{\inf}$ satisfies (Cr_s) if and only if $f(M_{\mathrm{BK}}) \subseteq M_{\inf}$ does.

(3) When $s = \infty$, both (Cr_∞) and (Cr'_∞) simply state that for an element $x \in M_{\mathrm{BK}}$, $f(x)$ lands in the set $M_{\inf}^{G_\infty}$ of G_∞–fixed points of $M_{\inf}$.

For the rest of this section, we discuss several basic algebraic properties of the conditions (Cr_s) and (Cr'_s) that will be useful later on in Chapter 5.

Lemma 4.1.4. *For any integer s, the ideals $\varphi^s(v)uA_{\inf}$ and I_s are G_K–stable.*

Proof. It is enough to prove that the ideals $uA_{\inf}$ and $vA_{\inf}$ are G_K–stable. Note that the G_K–stability of $vA_{\inf}$ implies G_K–stability of $\varphi^s(v)A_{\inf}$ for any $s \in \mathbb{Z}$ since φ is a G_K–equivariant automorphism of $A_{\inf}$. Once we know this, we know that $g\varphi^s(v)$ equals to $\varphi^s(v)$ times a unit for every g and s, the same is then true of $\varphi^i(\omega) = \varphi^i(v)/\varphi^{i-1}(v)$, hence also of all the elements $\xi_{s,i}$ and it follows that I_s is G_K–stable.

Given $g \in G_K$, $g(\pi_n) = \zeta_{p^n}^{a_n}\pi_n$ for an integer a_n unique modulo p^n and such that $a_{n+1} \equiv a_n \pmod{p^n}$. It follows that $g(u) = [\underline{\varepsilon}]^a u$ for a p–adic integer $a(= \lim_n a_n)$. (The $\mathbb{Z}_p$–exponentiation used here is defined by $[\underline{\varepsilon}]^a = \lim_n [\underline{\varepsilon}]^{a_n}$ and the considered limit is with respect to the weak topology.) Thus, $uA_{\inf}$ is G_K–stable.

Similarly, we have $g(\zeta_{p^n}) = \zeta_{p^n}^{b_n}$, for integers b_n coprime to p, unique modulo p^n and compatible with each other as n grows. It follows that $g([\underline{\varepsilon}]) = [\underline{\varepsilon}]^b$ for $b = \lim_n b_n$, and so $g(v) = (v+1)^b - 1 = \lim_n((v+1)^{b_n} - 1)$. The resulting expression is still divisible by v. To see that, fix the integers b_n to have all positive representatives. Then the claim follows from the formula

$$(v+1)^{b_n} - 1 = v((v+1)^{b_n-1} + (v+1)^{b_n-2} + \cdots + 1),$$

upon noting that the sequence of elements $((v+1)^{b_n-1}+(v+1)^{b_n-2}+\cdots+1) = ((v+1)^{b_n}-1)/v$ is still (p, v)–adically (i.e. weakly) convergent thanks to Lemma 2.2.3. $\square$

In view of the above lemma, the following is a convenient restatement of the conditions (Cr_s), (Cr'_s).

Lemma 4.1.5. *Given $f : M_{\mathrm{BK}} \to M_{\mathrm{inf}}$ as in Definition 4.1.2, the pair $(M_{\mathrm{BK}}, M_{\mathrm{inf}})$ satisfies the condition* (Cr_s) *((Cr'_s), resp.) if and only if the image of M_{BK} in $\overline{M_{\mathrm{inf}}} := M_{\mathrm{inf}}/I_s M_{\mathrm{inf}}$ ($\overline{M_{\mathrm{inf}}} := M_{\mathrm{inf}}/\varphi^s(v)uM_{\mathrm{inf}}$, resp.) lands in $\overline{M_{\mathrm{inf}}}^{G_s}$.*

Proof. Upon noting that the G_K–action is well–defined on $\overline{M_{\mathrm{inf}}}$ thanks to Lemma 4.1.4, this is just a direct reformulation of the conditions (Cr_s) or (Cr'_s). □

Assume that $f(M_{\mathrm{BK}}) \subseteq M_{\mathrm{inf}}^{G_\infty}$, that is, the condition (Cr'_∞) for the pair $(M_{\mathrm{BK}}, M_{\mathrm{inf}})$. Then the G_K–closure of $f(M_{\mathrm{BK}})$ in M_{inf} is contained in the G_K–submodule $M^{G_{K_{p^\infty,\infty}}}$, and thus, the G_K–action on it factors through $\widehat{G}$. Under relatively mild assumptions on M_{inf}, the G_s–action on the elements of $f(M_{\mathrm{BK}})$ is ultimately determined by τ^{p^s}, the topological generator of $\mathrm{Gal}(K_{p^\infty,\infty}/K_{p^\infty,s})$. Consequently, the remaining conditions (Cr'_s) can be checked for the action of this single element:

Lemma 4.1.6. *Let $f : M_{\mathrm{BK}} \to M_{\mathrm{inf}}$ be as in Definition 4.1.2, and assume that (Cr'_∞) holds. Additionally assume that M_{inf} is classically $(p, E(u))$–complete and $(p, E(u))$–completely flat, and that the G_K–action on M_{inf} is continuous with respect to this topology. Then the action of $\widehat{G}$ on elements of $f(M_{\mathrm{BK}})$ makes sense, and for any finite s, the pair $(M_{\mathrm{BK}}, M_{\mathrm{inf}})$ satisfies the condition (Cr'_s) if and only if*

$$\forall x \in f(M_{\mathrm{BK}}) : \tau^{p^s}(x) - x \in \varphi^s(v)uM_{\mathrm{inf}}.$$

Proof. Clearly the stated condition is necessary. To prove sufficiency, assume the above condition for τ^{p^s}. By the fixed–point interpretation of the condition (Cr'_s) as in Lemma 4.1.5, it is clear that the analogous condition holds for every element $g \in \langle \tau^{p^s} \rangle$.

Next, assume that $g \in \mathrm{Gal}(K_{p^\infty,\infty}/K_{p^\infty,s}) = \overline{\langle \tau^{p^s} \rangle}$. This means that $g = \lim_n \tau^{p^s a_n}$ with the sequence of integers (a_n) p–adically convergent. Then, for $x \in f(M_{\mathrm{BK}})$, by continuity we have $g(x) - x = \lim_n(\tau^{p^s a_n}(x) - x)$, which equals to $\lim_n \varphi^s(v)uy_n$ with $y_n \in M_{\mathrm{inf}}$. Upon noting that the sequence (y_n) is still convergent (using the fact that the $(p, E(u))$–adic topology is the $(p, \varphi^s(v)u)$–adic topology, and that $p, \varphi^s(v)u$ is a regular sequence on M_{inf} by Lemma 2.2.5), we have that $g(x) - x = \varphi^s(v)uy$ where $y = \lim_n y_n$.

To conclude, note that every element of $\widehat{G}_s$ is of the form g_1g_2 with $g_1 \in \mathrm{Gal}(K_{p^\infty,\infty}/K_{p^\infty,s})$ and $g_2 \in \mathrm{Gal}(K_{p^\infty,\infty}/K_\infty)$. Then for $x \in f(M_{\mathrm{BK}})$, by the assumption $f(M_{\mathrm{BK}}) \subseteq M_{\mathrm{inf}}^{G_\infty}$ we have $g_1g_2(x) - x = g_1(x) - x$, and so the condition (Cr'_s) is proved by the previous part. $\square$

Let us now discuss some basic algebraic properties of the conditions (Cr_s) and (Cr'_s). The basic situation when they are satisfied is the inclusion $\mathfrak{S} \hookrightarrow A_{\mathrm{inf}}$ itself.

Lemma 4.1.7. *The pair $\mathfrak{S} \hookrightarrow A_{\mathrm{inf}}$ satisfies the conditions (Cr'_s) (hence also (Cr_s)) for all $s \geq 0$.*

Proof. This is clear for $s = \infty$, and so from now on assume that s is finite. By Lemma 4.1.6, it is enough to consider the action of the element $\tau^{p^s} \in \widehat{G}_s$. For an element $f = \sum_i a_i u^i \in \mathfrak{S}$ we have

$$\tau^{p^s}(f) - f = \sum_{i\geq 0} a_i((v+1)^{p^s}u)^i - \sum_{i\geq 0} a_i u^i = \sum_{i\geq 1} a_i((v+1)^{p^s i} - 1)u^i,$$

and thus,

$$\frac{\tau^{p^s}(f) - f}{\varphi^s(v)u} = \sum_{i\geq 1} a_i \frac{(v+1)^{p^s i} - 1}{\varphi^s(v)} u^{i-1} = \sum_{i\geq 1} a_i \frac{(v+1)^{p^s i} - 1}{(v+1)^{p^s} - 1} u^{i-1}$$

Since $\varphi^s(v) = (v+1)^{p^s} - 1$ divides $(v+1)^{p^s i} - 1$ for each i, the obtained series has coefficients in A_{inf}, showing that $\tau^{p^s}(f) - f \in \varphi^s(v)uA_{\mathrm{inf}}$, as desired. $\square$

The following lemma shows that in various contexts, it is often sufficient to verify the conditions (Cr_s), (Cr'_s) on generators.

Lemma 4.1.8. *Fix $s \in \mathbb{N} \cup \{\infty\}$. Let (C) be either the condition (Cr_s) or (Cr'_s).*

(1) Let M_{inf} be an A_{inf}–module with a G_K–A_{inf}–semilinear action. The set of all (C)–elements forms an $\mathfrak{S}$–submodule of M_{inf}.

(2) Let C_{inf} be an A_{inf}–algebra endowed with a G_K–semilinear action. The set of (C)–elements of C_{inf} forms an $\mathfrak{S}$–subalgebra of C_{inf}.

(3) If the algebra C_{inf} from (2) is additionally an A_{inf}–δ–algebra such that G_K acts by δ–maps (i.e. $\delta g = g\delta$ for all $g \in G_K$) then the set of all (C)–elements forms a $\mathfrak{S}$–δ–subalgebra of C_{inf}.

(4) If the algebra C_{inf} as in (2) is additionally derived $(p, E(u))$–complete, the G_K–action on it is continuous with respect to the $(p, E(u))$–adic topology and $C_{\mathrm{BK}} \to C_{\mathrm{inf}}$ is a map of $\mathfrak{S}$–algebras that satisfies the condition (C)*, then so does $\widehat{C_{\mathrm{BK}}} \to C_{\mathrm{inf}}$, where $\widehat{C_{\mathrm{BK}}}$ is the derived $(p, E(u))$–completion of C_{BK}. In particular, the set of all* (C)*–elements in C_{inf} forms a derived $(p, E(u))$–complete $\mathfrak{S}$–subalgebra of C_{inf}.*

Proof. Let J be the ideal I_s if (C)=(Cr_s) and the ideal $\varphi^s(v)uA_{\mathrm{inf}}$ if (C)=(Cr'_s). In view of Lemma 4.1.5, the sets described in (1),(2) are obtained as the preimages of $(M_{\mathrm{inf}}/JM_{\mathrm{inf}})^{G_s}$ (ring $(C_{\mathrm{inf}}/JC_{\mathrm{inf}})^{G_s}$, resp.) under the canonical map $M_{\mathrm{inf}} \to M_{\mathrm{inf}}/JM_{\mathrm{inf}}$ ($C_{\mathrm{inf}} \to C_{\mathrm{inf}}/JC_{\mathrm{inf}}$, resp.). As these G_s–fixed points form an $\mathfrak{S}$–module ($\mathfrak{S}$–algebra, resp.) by Lemma 4.1.7, this proves (1) and (2).

Similarly, to prove (3) we need to prove only that the ideal JC_{inf} is a δ–ideal (that is, an ideal closed under the δ–operation) and therefore the canonical projection $C_{\mathrm{inf}} \to C_{\mathrm{inf}}/JC_{\mathrm{inf}}$ is a map of δ–rings. This is clear for $s = \infty$, so let us consider finite s.

Let us argue first in the case (Cr'_s). As $\delta(u) = 0$, we have

$$\delta(\varphi^s(v)u) = \delta(\varphi^s(v))u^p = \frac{\varphi(\varphi^s(v)) - (\varphi^s(v))^p}{p}u^p = \frac{\varphi^{s+1}(v) - (\varphi^s(v))^p}{p}u^p.$$

Recall that $\varphi^s(v) = [\underline{\varepsilon}]^{p^s} - 1$ divides $\varphi^{s+1}(v) = ([\underline{\varepsilon}]^{p^s})^p - 1$. The numerator of the last fraction is thus divisible by $\varphi^s(v)$ and since $\varphi^s(v)A_{\mathrm{inf}} \cap pA_{\mathrm{inf}} = \varphi^s(v)pA_{\mathrm{inf}}$ by Lemma 2.2.3, $\varphi^s(v)$ divides the whole fraction $(\varphi^{s+1}(v) - (\varphi^s(v))^p)/p$ in A_{inf}. (We note that this is true for *every* integer s, in particular $s = -1$, as well.)

Let us now prove that the ideal $J = I_s$ is a δ–ideal. For any i between 0 and $s - 1$, we have

$$\begin{aligned}\delta\left(\xi_{s,i}\right) &= \delta(\varphi^{-1}(v)\varphi^{i+1}(\omega)\dots\varphi^s(\omega)) = \\ &= \frac{\varphi^{-1}(v)\omega\varphi^{i+2}(\omega)\dots\varphi^{s+1}(\omega) - \varphi^{-1}(v)^p\varphi^{i+1}(\omega)^p\dots\varphi^s(\omega)^p}{p}.\end{aligned}$$

The numerator is divisible by $\xi_{s,i+1}$, hence so is the fraction again by Lemma 2.2.3. Thus, we have that $\delta(\xi_{s,i}u^{p^i}) = \delta(\xi_{s,i})u^{p^{i+1}}$ is a multiple of $\xi_{s,i+1}u^{p^{i+1}}$. Finally, when $i = s$, we have $\xi_{s,s} = \varphi^{-1}(v)$, and $\delta(\xi_{s,s})$ is thus a multiple of $\xi_{s,s}$ by the previous argument. Consequently,

$\delta(\xi_{s,s}u^{p^s}) = \delta(\xi_{s,s})u^{p^{s+1}}$ is divisible by $\xi_{s,s}u^{p^s}$. This shows that I_s (hence also I_sC_{inf}) is a δ–ideal.

Finally, let us prove (4). Note that $E(u) \equiv u^e \pmod{p\mathfrak{S}}$, hence $\sqrt{(p, E(u))} = \sqrt{(p, u^e)} = \sqrt{(p, u)}$ even as ideals of $\mathfrak{S}$; consequently, the derived $(p, E(u))$–completion agrees with the derived (p, u)–completion both for $\mathfrak{S}$– and A_{inf}–modules. We may therefore replace $(p, E(u))$–completions with (p, u)–completions throughout.

Since C_{inf} is derived (p, u)–complete, any power series of the form

$$f = \sum_{i,j} c_{i,j} p^i u^j$$

with $c_{i,j} \in C_{\mathrm{inf}}$ defines a unique[1] element $f \in C_{\mathrm{inf}}$, and f comes from $\widehat{C_{\mathrm{BK}}}$ if and only if the coefficients $c_{i,j}$ may be chosen in the image of the map $C_{\mathrm{BK}} \to C_{\mathrm{inf}}$. Assuming this, for $g \in G_s$ we have

$$g(f) - f = \sum_{i,j} g(c_{i,j}) p^i (\gamma u)^j - \sum_{i,j} c_{i,j} p^i u^j =$$

$$= \sum_{i,j} \Big(g(c_{i,j})\gamma^j - g(c_{i,j}) + g(c_{i,j}) - c_{i,j} \Big) p^i u^j,$$

where γ is the A_{inf}–unit such that $g(u) = \gamma u$. Thus, it is clearly enough to show, assuming the condition (C) for $(C_{\mathrm{BK}}, C_{\mathrm{inf}})$, that the terms $(g(c_{i,j})\gamma^j - g(c_{i,j}))\, p^i u^j$ and $(g(c_{i,j}) - c_{i,j})\, p^i u^j$ are in JC_{inf} when $g \in G_s$. (Note that here we rely on the fact that an element $d = \sum_{i,j} d_{i,j} p^i u^j$ with $d_{i,j} \in JC_{\mathrm{inf}}$ is itself in JC_{inf}, a fact that holds thanks to J being finitely generated.)

We have $g(c_{i,j}) - c_{i,j} \in JC_{\mathrm{inf}}$ by assumption, so it remains to treat the term $g(c_{i,j})(\gamma^j - 1)$. Note that $(\gamma^j - 1)$ is divisible by $\gamma - 1$, which is divisible by $\varphi^s(v)$ by Lemma 4.1.7, and so the terms $g(c_{i,j})(\gamma^j - 1)p^i u^j$ are divisible by $\varphi^s(v)u$ when $j \geq 1$; thus, they belong to JC_{inf} in both considered cases. When $j = 0$, these terms become 0 and there is nothing to prove.

To prove the second assertion of (4), let now $C_{\mathrm{BK}} \subseteq C_{\mathrm{inf}}$ be the $\mathfrak{S}$–subalgebra of all crystalline elements. By the previous, the map $\widehat{C_{\mathrm{BK}}} \to C_{\mathrm{inf}}$ satisfies (C), and hence the image $C_{\mathrm{BK}}^{\wedge}$ of this map consists of (C)–elements. Thus, we have $C_{\mathrm{BK}} \subseteq C_{\mathrm{BK}}^{\wedge} \subseteq C_{\mathrm{BK}}$, and hence, C_{BK} is derived $(p, E(u))$–complete since so is $C_{\mathrm{BK}}^{\wedge}$. $\square$

[1] ↑Here we are using the preferred representatives of power series as mentioned in Remark 2.1.6 (3).

Remark 4.1.9. One consequence of Lemma 4.1.8 is that the $\mathfrak{S}$–subalgebra $\mathfrak{C}$ of A_{inf} formed by all crystalline elements (or even (Cr_0')–elements) forms a prism, with the distinguished invertible ideal $I = E(u)\mathfrak{C}$.

For future use in applications to p^n–torsion modules, we consider the following simplification of the ideals I_s appearing in the conditions (Cr_s).

Lemma 4.1.10. *Consider integers s, n with $s \geq 0, n \geq 1$. Set $t = \max\{0, s+1-n\}$. Then the image of the ideal I_s in the ring $W_n(\mathcal{O}_{\mathbb{C}_K^\flat}) = A_{\mathrm{inf}}/p^n$ is contained in the ideal $\varphi^{-1}(v)u^{p^t}W_n(\mathcal{O}_{\mathbb{C}_K^\flat})$. That is, we have $I_s + p^n A_{\mathrm{inf}} \subseteq \varphi^{-1}(v)u^{p^t}A_{\mathrm{inf}} + p^n A_{\mathrm{inf}}$.*

Proof. When $t = 0$ there is nothing to prove, therefore we may assume that $t = s+1-n > 0$. In the definition of I_s, we may replace the elements

$$\xi_{s,i} = \varphi^{-1}(v)\varphi^{i+1}(\omega)\varphi^{i+2}(\omega)\dots\varphi^{s}(\omega)$$

by the elements

$$\xi'_{s,i} = \varphi^{-1}(v)\varphi^{i+1}(E(u))\varphi^{i+2}(E(u))\dots\varphi^{s}(E(u)),$$

since the quotients $\xi_{s,i}/\xi'_{s,i}$ are A_{inf}–units.

It is thus enough to show that for every i with $0 \leq i \leq s$, the element

$$\vartheta_{s,i} = \frac{\xi'_{s,i}u^{p^i}}{\varphi^{-1}(v)} = \varphi^{i+1}(E(u))\varphi^{i+2}(E(u))\dots\varphi^{s}(E(u))u^{p^i}$$

taken modulo p^n is divisible by $u^{p^{s+1-n}}$.

This is clear when $i \geq s+1-n$, and so it remains to discuss the cases when $i \leq s-n$. Write $\varphi^j(E(u)) = (u^e)^{p^j} + px_j$ (with $x_j \in \mathfrak{S}$). Then it is enough to show that

$$\frac{\vartheta_{s,i}}{u^{p^i}} = ((u^e)^{p^{i+1}} + px_{i+1})((u^e)^{p^{i+2}} + px_{i+2})\dots((u^e)^{p^s} + px_s) \qquad (*)$$

taken modulo p^n is divisible by

$$u^{p^{s+1-n}-p^i} = u^{p^i(p-1)(1+p+\dots+p^{s-n-i})}.$$

Since we are interested in the product ($*$) only modulo p^n, in expanding the brackets we may ignore the terms that use the expressions of the form px_j at least n times. Each of the remaining terms contains the product of at least $s-i-n+1$ distinct terms from the following list:

$$(u^e)^{p^{i+1}}, (u^e)^{p^{i+2}}, \dots, (u^e)^{p^s}.$$

Thus, each of the remaining terms is divisible by (at least)

$$(u^e)^{p^{i+1}+p^{i+2}+\cdots+p^{s-n+1}} = (u^e)^{p^i\cdot(p)\cdot(1+p+\cdots+p^{s-n-i})},$$

which is more than needed. This finishes the proof. □

4.2 Breuil–Kisin–Fargues G_K–modules and the crystalline condition

The connection between Breuil–Kisin modules, Breuil–Kisin–Fargues G_K–modules and the crystalline condition (that also justifies the name) is the following theorem.

Theorem 4.2.1 ([14, Appendix F], [21]). *Consider a free Breuil–Kisin–Fargues G_K–module M_{inf} which admits as an $\mathfrak{S}$–submodule a free Breuil–Kisin module $M_{\mathrm{BK}} \subseteq M_{\mathrm{inf}}$ of finite height, such that $A_{\mathrm{inf}} \otimes_{\mathfrak{S}} M_{\mathrm{BK}} \xrightarrow{\sim} M_{\mathrm{inf}}$ (as Breuil–Kisin–Fargues modules) via the natural map, and such that the pair $(M_{\mathrm{BK}}, M_{\mathrm{inf}})$ satisfies (Cr_∞) and the crystalline condition. Then the étale realization $V^{\mathrm{inf}}(M_{\mathrm{inf}})$ of M_{inf} is a crystalline representation.*

Remarks 4.2.2.

(1) Theorem 4.2.1 is actually an equivalence: If $V^{\mathrm{inf}}(M_{\mathrm{inf}})$ is crystalline, it can be shown that the pair $(M_{\mathrm{BK}}, M_{\mathrm{inf}})$ satisfies the crystalline condition. We state the theorem in the one direction since this is the one that we use. However, the converse direction motivates why it is resonable to expect the crystalline condition for prismatic cohomology groups that is discussed in Chapter 5.

(2) Strictly speaking, in [14, Appendix F] one assumes extra conditions on the pair M_{inf} ("satisfying all descents"); however, these extra assumptions are used only for a semistable version of the statement. Theorem 4.2.1 in its equivalence form is therefore only implicit

in the proof of [14, Theorem F.11], which is based on a related crystallinity criterion of Ozeki [31, Theorem 3.8] in the context of $(\varphi, \widehat{G})$–modules.

(3) On the other hand, Theorem 4.2.1 in the one–sided form as above is a consequence of [21, Proposition 7.11] that essentially states that $V^{\mathrm{inf}}(M_{\mathrm{inf}})$ is crystalline if and only if the much weaker[2] condition

$$\forall g \in G_K : \quad (g-1)M_{\mathrm{BK}} \subseteq \varphi^{-1}(v)W(\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_K^\flat}})M_{\mathrm{inf}}$$

is satisfied. We note a related result of *loc. cit.*: $V^{\mathrm{inf}}(M_{\mathrm{inf}})$ is semistable if and only if

$$\forall g \in G_K : \quad (g-1)M_{\mathrm{BK}} \subseteq W(\mathfrak{m}_{\mathcal{O}_{\mathbb{C}_K^\flat}})M_{\mathrm{inf}}.$$

This is interesting for at least two reasons: Firstly, the proof of [14, Theorem F.11] is based on arguments of [31] that make heavy use of the fact that for any $r \geq 0$, the sequence u^{p^n}/p^{nr} converges p–adically to 0 in A_{cris}, the p–completion of $A_{\mathrm{inf}}^{\mathrm{pd}} = A_{\mathrm{inf}}[(E(u)^n/n!)_n]$. In particular, in this approach u is crucial and $\varphi^{-1}(v)$ is essentially irrelevant, which appears to be the complete opposite of the situation in [21]. Secondly, the semistable criterion above might be a good starting point in generalizing the results of Chapters 5 and 6 to the case of semistable reduction, using the log–prismatic cohomology developed in [26]. Thus, a natural question to ask is: Similarly to how the crystalline condition is a stronger version of the crystallinity criterion from [21], what is an analogous stronger (while still generally valid) version of the semistability criterion from [21]?

It will be convenient later to have version of Theorem 4.2.1 that applies to not necessarily free Breuil–Kisin and Breuil–Kisin–Fargues modules. Consider, for a Breuil–Kisin module M_{BK}, the exact sequence

$$0 \longrightarrow M_{\mathrm{BK,tor}} \longrightarrow M_{\mathrm{BK}} \longrightarrow M_{\mathrm{BK,free}} \longrightarrow \overline{M_{\mathrm{BK}}} \longrightarrow 0$$

from Remark 3.4.2. Taking the base–change to A_{inf}, one obtains the analogous exact sequence

[2]↑“Weaker” for the purposes of controlling the G_K–action on the submodule M_{BK} inside M_{inf}.

$$0 \longrightarrow M_{\mathrm{inf,tor}} \longrightarrow M_{\mathrm{inf}} \longrightarrow M_{\mathrm{inf,free}} \longrightarrow \overline{M_{\mathrm{inf}}} \longrightarrow 0$$

where $M_{\mathrm{inf,free}}$ is a free Breuil–Kisin–Fargues module. Clearly the maps $M_{\mathrm{BK}} \to M_{\mathrm{free}}$ and $M_{\mathrm{inf}} \to M_{\mathrm{inf,free}}$ become isomorphisms after inverting p.

Now, let us assume that M_{inf} is endowed with a G_K–action that makes it a Breuil–Kisin–Fargues G_K–module. The functoriality of the latter exact sequence implies that the G_K–action on M_{inf} induces a G_K–action on $M_{\mathrm{inf,free}}$, endowing it with the structure of a free Breuil–Kisin–Faruges G_K–module. In more detail, given $\sigma \in G_K$, the semilinear action map $\sigma : M_{\mathrm{inf}} \to M_{\mathrm{inf}}$ induces the A_{inf}–linear map $\sigma_{\mathrm{lin}} : \sigma^* M_{\mathrm{inf}} \to M_{\mathrm{inf}}$. As σ is an isomorphism that fixes p, $E(u)$ up to unit and the ideal $(p, u)A_{\mathrm{inf}}$, it is easy to see that $\sigma^* M_{\mathrm{inf}}$ is itself a Breuil–Kisin–Fargues module, and the exact sequence as above but for $\sigma^* M_{\mathrm{inf}}$ can be identified with the upper row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & \sigma^* M_{\mathrm{inf,tor}} \arrow[r] \arrow[d, "\sigma_{\mathrm{lin}}"] & \sigma^* M_{\mathrm{inf}} \arrow[r] \arrow[d, "\sigma_{\mathrm{lin}}"] & \sigma^* M_{\mathrm{inf,free}} \arrow[r] \arrow[d, "\sigma_{\mathrm{lin}}"] & \sigma^* \overline{M_{\mathrm{inf}}} \arrow[r] \arrow[d, "\sigma_{\mathrm{lin}}"] & 0 \\
0 \arrow[r] & M_{\mathrm{inf,tor}} \arrow[r] & M_{\mathrm{inf}} \arrow[r] & M_{\mathrm{inf,free}} \arrow[r] & \overline{M_{\mathrm{inf}}} \arrow[r] & 0,
\end{tikzcd}$$

where the second vertical map is the linearization of σ and the rest is induced by functoriality of the sequence. Finally, untwisting $\sigma^* M_{\mathrm{inf,free}}$, the third vertical map σ_{lin} induces a semilinear map $\sigma : M_{\mathrm{inf,free}} \to M_{\mathrm{inf,free}}$. Note that the module $M_{\mathrm{inf}}[1/p] \simeq M_{\mathrm{inf,free}}[1/p]$ inherits the G_K–action from M_{inf}; it is easy to see that the G_K-action on $M_{\mathrm{inf,free}}$ agrees with the one on $M_{\mathrm{inf}}[1/p]$ when viewing $M_{\mathrm{inf,free}}$ as its submodule.

Proposition 4.2.3. *Assume that the pair $M_{\mathrm{BK}} \hookrightarrow M_{\mathrm{inf}}$ satisfies the crystalline condition. Then so does the pair $M_{\mathrm{BK,free}} \hookrightarrow M_{\mathrm{inf,free}}$.*

Proof. Notice that the crystalline condition is satisfied for $M_{BK}[1/p] \to M_{\mathrm{inf}}[1/p]$ and by [5, Propositions 4.3, 4.13], this map can be identified with $M_{\mathrm{BK,free}}[1/p] \hookrightarrow M_{\mathrm{inf,free}}[1/p]$. Thus, the following lemma finishes the proof. □

Lemma 4.2.4. *Let F_{inf} be a free A_{inf}–module endowed with A_{inf}–semilinear G_K–action and let $F_{BK} \subseteq F_{\mathrm{inf}}$ be a free $\mathfrak{S}$–submodule such that $F_{BK}[1/p] \hookrightarrow F_{\mathrm{inf}}[1/p]$ satisfies the crystalline condition. Then the pair $F_{BK} \hookrightarrow F_{\mathrm{inf}}$ satisfies the crystalline condition.*

Proof. Fix an element $a \in F_{BK}$ and $g \in G_K$. The crystalline condition holds after inverting p, and so

$$b := (g-1)a = \varphi^{-1}(v)u\frac{c}{p^k}$$

with $c \in F_{\text{inf}}$. In other words (using that p^k is a non-zero divisor on F_{inf}), we have

$$p^k b = \varphi^{-1}(v)uc \in p^k F_{\text{inf}} \cap \varphi^{-1}(v)uF_{\text{inf}} = p^k\varphi^{-1}(v)uF_{\text{inf}},$$

where the last equality follows by Lemma 2.2.3 since F_{inf} is a free module. In particular,

$$p^k b = p^k\varphi^{-1}(v)ud$$

for yet another element $d \in F_{\text{inf}}$. As p^k is a non–zero divisor on A_{inf}, hence on F_{inf}, we may cancel out to conclude

$$(g-1)a = b = \varphi^{-1}(v)ud \in \varphi^{-1}(v)uF_{\text{inf}},$$

as desired. $\square$

Combining Theorem 4.2.1 and Proposition 4.2.3, we arrive at the following theorem.

Theorem 4.2.5. *The "free" assumption in Theorem 4.2.1 is superfluous. That is, given a Breuil–Kisin–Fargues G_K–module M_{inf} together with its Breuil–Kisin–$\mathfrak{S}$–submodule M_{BK} of finite height such that $A_{\text{inf}} \otimes_{\mathfrak{S}} M_{\text{BK}} \xrightarrow{\sim} M_{\text{inf}}$ and such that the pair $(M_{\text{BK}}, M_{\text{inf}})$ satisfies the condition (Cr_∞) and the crystalline condition, the representation $V^{\text{inf}}(M_{\text{inf}})$ is crystalline.*

Proof. With the notation as above, upon realizing that $V^{\text{inf}}(M_{\text{inf}})$ and $V^{\text{inf}}(M_{\text{inf,free}})$ agree, the result is a direct consequence of Proposition 4.2.3. $\square$

5. THE CONDITIONS (Cr_s) FOR COHOMOLOGY

5.1 (Cr_s) for Čech–Alexander complexes

Let $\mathfrak{X}$ be a smooth separated p–adic formal scheme over $\mathcal{O}_K$. Denote by $\check{C}^{\bullet}_{\mathrm{BK}}$ a Čech–Alexander complex that models $\mathsf{R}\Gamma_{\Delta}(\mathfrak{X}/\mathfrak{S})$ as described in Section 3.3, and let us set $\check{C}^{\bullet}_{\mathrm{inf}} = \check{C}^{\bullet}_{\mathrm{BK}}\widehat{\otimes}_{\mathfrak{S}}A_{\mathrm{inf}}$ (computed termwise). The next goal is to prove the following theorem, which is a precise version of the first part of Theorem 1.3.4 from Introduction.

Theorem 5.1.1. *For every $m \in \mathbb{N}$ and every $s \in \mathbb{N}\cup\{\infty\}$, the pair $\check{C}^{m}_{\mathrm{BK}} \to \check{C}^{m}_{\mathrm{inf}}$ satisfies the condition* (Cr_s).

Let $\mathrm{Spf}(R) = \mathcal{V} \subseteq \mathfrak{X}$ be an affine open formal subscheme. Then it is enough to prove the content of Theorem 5.1.1 for $\check{C}_{\mathrm{BK}} \to \check{C}_{\mathrm{inf}}$ where $\check{C}_{\mathrm{BK}}$ and $\check{C}_{\mathrm{inf}} = \check{C}_{\mathrm{BK}}\widehat{\otimes}_{\mathfrak{S}}A_{\mathrm{inf}}$ are the Čech–Alexander covers of $\mathcal{V}$ and $\mathcal{V}' = \mathcal{V}\times_{\mathfrak{S}}A_{\mathrm{inf}}$ with respect to the base prism $\mathfrak{S}$ and A_{inf}, respectively, since the Čech–Alexander complexes termwise consist of products of such covers. Let $R' = R\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_{\mathbb{C}_K}(= R\widehat{\otimes}_{\mathfrak{S}}A_{\mathrm{inf}})$.

Let us fix a choice of the free $\mathfrak{S}$–algebra $P_0 = \mathfrak{S}[\{X_i\}_{i\in I}]$ whose $(p, E(u))$–completion is the algebra $P = P_{\mathcal{V}}$ as in Construction 3.3.7, with J being the kernel of the surjection $P \to R$. Then the corresponding choices at the A_{inf}–level are $P_0' = P_0\otimes_{\mathfrak{S}}A_{\mathrm{inf}}$ and $P' = P\widehat{\otimes}_{\mathfrak{S}}A_{\mathrm{inf}}$, and the associated $(p, E(u))$–completed "δ–envelopes" are also related by the completed base change; that is, we have a diagram with exact rows

$$\begin{tikzcd}
\widehat{JP^{\delta}} \arrow[r] \arrow[d] & \widehat{P^{\delta}} \arrow[r] \arrow[d] & \widehat{R\otimes_P P^{\delta}} \arrow[r] \arrow[d] & 0 \\
\widehat{J(P')^{\delta}} \arrow[r] & \widehat{(P')^{\delta}} \arrow[r, "\alpha"] & \widehat{R'\otimes_{P'}(P')^{\delta}} \arrow[r] & 0.
\end{tikzcd} \quad -\widehat{\otimes}_{\mathfrak{S}}A_{\mathrm{inf}} \tag{5.1}$$

By Remark 3.3.12, we may and do assume that the set of variables $\{X_i\}_i$ is finite, and that the ideal J is finitely generated. Consequently, after replacing the maps on the left by their respective images (and invoking Remark 3.3.8 (1)), diagram (5.1) becomes

$$\begin{tikzcd}
0 \arrow[r] & J\widehat{P^\delta} \arrow[r] \arrow[d] & \widehat{P^\delta} \arrow[r] \arrow[d, "-\widehat{\otimes}_{\mathfrak{S}}A_{\mathrm{inf}}"] & \widehat{R \otimes_P P^\delta} \arrow[r] \arrow[d, "-\widehat{\otimes}_{\mathfrak{S}}A_{\mathrm{inf}}"] & 0 \\
0 \arrow[r] & J\widehat{(P')^\delta} \arrow[r] & \widehat{(P')^\delta} \arrow[r, "\alpha"] & R' \widehat{\otimes_{P'} (P')^\delta} \arrow[r] & 0
\end{tikzcd} \tag{5.2}$$

where the rows are exact. The middle and the last term of the second row is given the G_K–action on A_{inf}–coefficients, that is, $g(x \otimes a) = x \otimes g(a)$ for every $g \in G_K$, $a \in A_{\mathrm{inf}}$ and x coming from the first row[1]. In summary, we have:

(1) On $\widehat{(P')^\delta}$, whose topological generators are $\{\delta^j(X_i)\}$, the action takes the simple form $g(\delta^j(X_i)) = \delta^j(X_i)$. It is further an action by δ–maps (as it comes, functorially by functors of δ–rings, from the action on A_{inf}, which is given by δ–maps).

(2) The map α is G_K–equivariant and thus, the ideal $J\widehat{(P')^\delta}$ is G_K–stable.

(3) Furthermore, statements analogous to (1) and (2) hold without taking the δ–envelopes. That is, P' can be given a semilinear A_{inf}–action determined by $g(X_i) = X_i$ on variables, and $R' = R\widehat{\otimes}_{\mathfrak{S}}A_{\mathrm{inf}}$ can be given an action on the second factor; the resulting map $P' \to R'$ is then G_K–equivariant and therefore the kernel, which is JP', is G_K–stable. Moreover, these actions are compatible with the ones on $\widehat{(P')^\delta}$ and $R' \widehat{\otimes_{P'} (P')^\delta}$, resp., in the obvious manner.

Thanks to the G_K–equivariance statement (2), the G_K–action extends to the prismatic envelope $(\check{C}_{\mathrm{inf}}, I\check{C}_{\mathrm{inf}})$ where the action obtained this way agrees with the one indicated in Remark 3.3.17. Setting $(\check{C}_{\mathrm{BK}}, I\check{C}_{\mathrm{BK}})$ to be the prismatic envelope of $(\widehat{P^\delta}, J\widehat{P^\delta})$, we arrive at the situation $\check{C}_{\mathrm{BK}} \hookrightarrow \check{C}_{\mathrm{inf}} = \check{C}_{\mathrm{BK}}\widehat{\otimes}_{\mathfrak{S}}A_{\mathrm{inf}}$ for which we wish to verify the conditions (Cr_s).

With the goal of understanding the G_K–action on $\check{C}_{\mathrm{inf}}$ even more explicitly, in similar spirit to the proof of [7, Proposition 3.13] we employ the following approximation of the prismatic envelope.

[1]↑Note that this description indeed makes sense: the algebra $\widehat{(P')^\delta}$ is the $(p, E(u))$–completion of the free A_{inf}–algebra $P_0^\delta \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$, and so the (semilinear) action on A_{inf}–coefficients is well–defined. The ring on the right–hand side is in fact isomorphic to $R \otimes_P P^\delta \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{\mathbb{C}_K}$ and thus, the G_K–action on the last factor makes sense and is linear.

Definition 5.1.2. Let B be a δ–ring, $J \subseteq B$ an ideal with a generating set $\underline{x} = \{x_i\}_{i\in\Lambda}$, and let $b \in J$ be an element. Denote by K_0 the kernel of the B–algebra map

$$B[\underline{T}] = B[\{T_i\}_{i\in\Lambda}] \longrightarrow B\left[\frac{1}{b}\right]$$
$$T_i \longmapsto \frac{x_i}{b},$$

and let K be the δ–ideal in $B\{\underline{T}\}$ generated by K_0. Then we denote by $B\{\frac{\underline{x}}{b}\}$ the δ–ring $B\{\underline{T}\}/K$, and call it the *weak δ–blowup algebra of $\underline{x}$ and b.*

That is, the above construction adjoins (in δ–sense) the fractions x_i/b to B together with all relations among them that exist in $B[1/b]$, making it possible to naturally compute with the fractions as opposed to possibly simpler constructions such as $B\{\underline{T}\}/(T_ib - x_i)_\delta$.

Note that if $B \to C$ is a map of B–δ–algebras such that $JC = bC$ and this ideal is invertible, the fact that the localization map $C \to C[\frac{1}{b}]$ is injective shows that there is a unique map of B–δ–algebras $B\{\frac{\underline{x}}{b}\} \to C$. In fact, if b happens to be a non–zero divisor on $B\{\frac{\underline{x}}{b}\}$, then $B\{\frac{\underline{x}}{b}\}$ is initial among all such B–δ–algebras. This justifies the name 'weak δ–blowup algebra'.

The purpose of the construction is the following.

Proposition 5.1.3. *Let (A, I) be a bounded prism with $I = (d)$ principal. Consider a map of δ–pairs $(A, I) \to (B, J)$ and assume that (C, IC) is the prismatic envelope for (B, J) and that it is classically (p, I)–complete. Let $\underline{x} = \{x_i\}_{i\in\Lambda}$ be a system of generators of J. Then there is a surjective map of δ–rings $\widehat{B\{\frac{\underline{x}}{d}\}}^{\mathrm{cl}} \to C$, where $\widehat{(-)}^{\mathrm{cl}}$ denotes the classical (p, I)–completion.*

Note that the assumptions apply to a Čech–Alexander cover in place of (C, IC) since it is (p, I)–completely flat over the base prism, hence classically (p, I)–complete.

Proof. Since $JC = dC$ and and d is a non–zero divisor on C, there is an induced map $B\{\frac{\underline{x}}{d}\} \to C$ and hence a map of δ–rings $\widehat{B\{\frac{\underline{x}}{d}\}}^{\mathrm{cl}} \to C$ (using [7, Lemma 2.17]).

To see that this map is surjective, let C' denote its image in C, and denote by ι the inclusion of C' into C. Then C' is (derived, and, consequently, clasically) (p, I)–complete

A–δ–algebra with $C'[d]=0$. It follows that $(C', IC') = (C', (d))$ is a prism by Proposition 3.1.7 (2). Thus, by the universal property of C, there is a map of B–δ–algebras $r : C \to C'$ which is easily seen to be right inverse to ι. In particular, ι is surjective, proving the claim. $\square$

Finally, we are ready to prove the following proposition which, as noted above, proves Theorem 5.1.1.

Proposition 5.1.4. *$\check{C}_{\mathrm{BK}} \to \check{C}_{\mathrm{inf}}$ satisfies the condition* (Cr_s) *for every* $s \in \mathbb{N} \cup \{\infty\}$.

Proof. Fix a generating set $\{y_j\}_j$ of J. We obtain a commutative diagram

$$\begin{array}{ccc} \widehat{\widehat{P}^{\delta}\{\frac{y}{E(u)}\}} & \longrightarrow & \widehat{\widehat{(P')}^{\delta}\{\frac{y}{E(u)}\}} \\ \big\downarrow & & \big\downarrow \\ \check{C}_{\mathrm{BK}} & \longrightarrow & \check{C}_{\mathrm{inf}}, \end{array} \tag{5.3}$$

where the vertical maps are the surjective maps from Proposition 5.1.3, and the horizontal maps come from the classically $(p, E(u))$–completed base change $-\widehat{\otimes}_{\mathfrak{S}} A_{\mathrm{inf}}$.

The G_K–action on $\widehat{(P')}^{\delta}$ naturally extends to $\widehat{\widehat{(P')}^{\delta}\{\frac{y}{E(u)}\}}$ by the rule on topological generators (as a δ–ring)

$$g\left(\frac{y_j}{E(u)}\right) = \frac{g(y_j)}{g(E(u))} = \gamma^{-1}\frac{g(y_j)}{E(u)}$$

where γ is the A_{inf}–unit such that $g(E(u)) = \gamma E(u)$ (note that the fraction on the right–hand side makes sense as $g(y_j) \in JP'$). It is easy to see that this makes the right vertical map G_K–equivariant.

It is therefore enough to prove the validity of (Cr_s) for the pair $(\widehat{\widehat{P}^{\delta}\{\frac{y}{E(u)}\}}, \widehat{\widehat{(P'_0)}^{\delta}\{\frac{y}{E(u)}\}})$. By Lemma 4.1.8 (3),(4), it is enough to check the conditions for the topological generators of $\widehat{\widehat{P}\{\frac{y}{E(u)}\}}$ as an $\mathfrak{S}$–δ–algebra, which are the generators $\{X_i\}_i$ (i.e. the variables originally from P_0) and $\{y_j/E(u)\}_j$.

Fix $s \in \mathbb{N} \cup \{\infty\}$. Firstly, note that the elements X_i satisfy $g(X_i) - X_i = 0$ for every $g \in G_s$; consequently, by Lemma 4.1.8 the pair $P \to P'$ satisfies the stronger condition (Cr'_s). In particular, (Cr'_s) holds for these generators, and since the elements y_j all come from P, it follows that these are (Cr'_s)–elements as well. Thus, upon fixing an index j and an element $g \in G_s$, we may write $g(y_j) - y_j = \varphi^s(v)uz_j$ for some $z_j \in P'$. Similarly, we have $g^{-1}(E(u)) - E(u) = (\gamma^{-1} - 1)E(u) = \varphi^s(v)ua$ with $a \in A_{\mathrm{inf}}$ and an A_{inf}–unit γ satisfying $g(E(u)) = \gamma E(u)$. We may thus write

$$g\left(\frac{y_j}{E(u)}\right) - \frac{y_j}{E(u)} = \frac{\gamma^{-1}g(y_j) - y_j}{E(u)} = \frac{\gamma^{-1}g(y_j) - \gamma^{-1}y_j + \gamma^{-1}y_j - y_j}{E(u)} =$$

$$= \gamma^{-1}\frac{g(y_j) - y_j}{E(u)} + (\gamma^{-1} - 1)\frac{y_j}{E(u)}.$$

When $s = \infty$, we have $\gamma = 1, g(y_j) = y_j$ and thus, the right–hand side equal to 0, as desired. So let us consider $s < \infty$. Then we have $g(y_j) - y_j = \varphi^s(v)uz_j$ and since ω and $E(u)$ are equal up to an A_{inf}–unit, we may write $g(y_j) - y_j = \xi_{s,0}uE(u)\tilde{z}_j$ (where $\tilde{z}_j$ equals z_j up to a unit). Similarly, we have, in A_{inf} and hence in any A_{inf}–δ–G_K–algebra, $(\gamma^{-1} - 1) = \xi_{s,0}u\tilde{a}$ (where $\tilde{a}$ equals a up to a unit). Thus, we obtain

$$g\left(\frac{y_j}{E(u)}\right) - \frac{y_j}{E(u)} = \xi_{s,0}u\gamma^{-1}\tilde{z}_j + \xi_{s,0}u\tilde{a}\frac{y_j}{E(u)} \in I_s(P')^{\delta}\{\frac{\underline{y}}{E(u)}\},$$

and we are done. $\square$

5.2 Consequences for cohomology groups

Let us now use Theorem 5.1.1 to draw some conclusions for individual cohomology groups, and thus finish the proof of Theorem 1.3.4 from Introduction.

The first one is the crystalline condition for the prismatic cohomology groups and its consequence for p–adic étale cohomology. As before, let $\mathfrak{X}$ be a separated smooth p–adic formal scheme over $\mathcal{O}_K$. Denote by $\mathfrak{X}_{A_{\mathrm{inf}}}$ the base change $\mathfrak{X} \times_{\mathcal{O}_K} \mathcal{O}_{\mathbb{C}_K} = \mathfrak{X} \times_{\mathfrak{S}} A_{\mathrm{inf}}$, and by $\mathfrak{X}_{\overline{\eta}}$ the geometric generic adic fiber.

Corollary 5.2.1. *For any $i \geq 0$, the pair $\mathrm{H}^i_{\Delta}(\mathfrak{X}/\mathfrak{S}) \to \mathrm{H}^i_{\Delta}(\mathfrak{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}})$ satisfies the crystalline condition, and the image of $\mathrm{H}^i_{\Delta}(\mathfrak{X}/\mathfrak{S})$ is contained in $\mathrm{H}^i_{\Delta}(\mathfrak{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}})^{G_\infty}$.*

Proof. By the results of Section 3.3, we may model the cohomology theories by the Čech–Alexander complexes

$$\check{C}^{\bullet}_{\mathrm{BK}} \to \check{C}^{\bullet}_{\mathrm{inf}} = \check{C}^{\bullet}_{\mathrm{BK}} \widehat{\otimes}_{\mathfrak{S}} A_{\mathrm{inf}},$$

and by Theorem 5.1.1 the conditions (Cr_0) and (Cr_∞) termwise hold for this pair. In particular, the claim that $\mathrm{H}^i_{\Delta}(\mathfrak{X}/\mathfrak{S}) \subseteq \mathrm{H}^i_{\Delta}(\mathfrak{X}/A_{\mathrm{inf}})^{G_\infty}$ thus follows immediately.

Each of the terms $\check{C}^i_{\mathrm{inf}}$ is $(p, E(u))$–completely flat over A_{inf}, which means in particular that the terms $\check{C}^i_{\mathrm{inf}}$ are torsion–free by Corollary 2.2.5. Denote the differentials on $\check{C}^\bullet_{\mathrm{BK}}, \check{C}^\bullet_{\mathrm{inf}}$ by ∂ and ∂', resp.

To prove the crystalline condition for cohomology groups, it is clearly enough to verify the condition at the level of cocycles. Given $x \in Z^i(\check{C}^\bullet_{\mathrm{BK}})$, denote by x' its image in $Z^i(\check{C}^\bullet_{\mathrm{inf}})$. For $g \in G_K$ we have $g(x') - x' = \varphi^{-1}(v)uy'$ for some $y' \in \check{C}^i_{\mathrm{inf}}$. As $g(x') - x' \in Z^i(\check{C}^\bullet_{\mathrm{inf}})$, we have

$$\varphi^{-1}(v)u\partial'(y') = \partial'(\varphi^{-1}(v)uy') = 0,$$

and the torsion–freeness of $\check{C}^{i+1}_{\mathrm{inf}}$ implies that $\partial'(y') = 0$. Thus, $y' \in Z^i(\check{C}_{\mathrm{inf}})$ as well, showing that $g(x') - x' \in \varphi^{-1}(v)uZ^i(\check{C}^\bullet_{\mathrm{inf}})$, as desired. □

When $\mathfrak{X}$ is proper over $\mathcal{O}_K$, we use the previous results to reprove the result from [5] that the étale cohomology groups $\mathrm{H}^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Q}_p)$ are in this case crystalline representations.

Corollary 5.2.2. *Assume that $\mathfrak{X}$ is additionally proper over $\mathcal{O}_K$. Then for any $i \geq 0$, the p–adic étale cohomology $\mathrm{H}^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Q}_p)$ is a crystalline representation.*

Proof. This is now a direct consequence of Corollary 5.2.1, together with Theorem 4.2.5 and Proposition 3.4.4. □

For the purposes of ramification bounds discussed in the next chapter, let us establish the consequences of the conditions (Cr_s) in the case of torsion prismatic cohomology.

Proposition 5.2.3. *Consider a pair of integers $s \geq 0, n \geq 1$. Set $t = \max\{0, s+1-n\}$. Then the torsion prismatic cohomology groups $\mathrm{H}^i_{\Delta,n}(\mathfrak{X}/\mathfrak{S}) \to \mathrm{H}^i_{\Delta,n}(\mathfrak{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}})$ satisfy the condition (Cr_∞), as well as the following condition:*

$$\forall g \in G_s : \quad (g-1)\mathrm{H}^i_{\Delta,n}(\mathfrak{X}/\mathfrak{S}) \subseteq \varphi^{-1}(v)u^{p^t}\mathrm{H}^i_{\Delta,n}(\mathfrak{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}}).$$

Proof. The proof is a slightly refined variant of the proof of Corollary 5.2.1. Consider again the associated Čech–Alexander complexes over $\mathfrak{S}$ and A_{inf},

$$\check{C}^{\bullet}_{\mathrm{BK}} \to \check{C}^{\bullet}_{\mathrm{inf}} = \check{C}^{\bullet}_{\mathrm{BK}} \widehat{\otimes}_{\mathfrak{S}} A_{\mathrm{inf}}.$$

Both of these complexes are given by torsion–free, hence $\mathbb{Z}$–flat, modules by Corollary 2.2.5. Consequently, $\mathsf{R}\Gamma_{\mathbb{A},n}(\mathfrak{X}/\mathfrak{S})$ is modelled by $\check{C}^{\bullet}_{\mathrm{BK},n} := \check{C}^{\bullet}_{\mathrm{BK}}/p^n\check{C}^{\bullet}_{\mathrm{BK}}$, and similarly for tha A_{inf}–cohomology $\mathsf{R}\Gamma_{\mathbb{A},n}(\mathfrak{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}})$ and $\check{C}^{\bullet}_{\mathrm{inf},n} = \check{C}^{\bullet}_{\mathrm{inf}}/p^n\check{C}^{\bullet}_{\mathrm{inf}}$. That is, the considered maps between cohomology groups are obtained as the maps on cohomologies for the base–change map of chain complexes

$$\check{C}^{\bullet}_{\mathrm{BK},n} \to \check{C}^{\bullet}_{\mathrm{inf},n} = \check{C}^{\bullet}_{\mathrm{BK},n} \widehat{\otimes}_{\mathfrak{S}} A_{\mathrm{inf}},$$

and it follows from Theorem 5.1.1 that the conditions (Cr_s) hold termwise for this pair for every $s \in \mathbb{N} \cup \{\infty\}$. The condition (Cr_∞) once again follows immediately. In order to prove the condition from the statement for finite s, just as in the proof of Corolary 5.2.1, it is enough to establish the desired condition for the respective groups of cocycles.

Set $\alpha = \varphi^{-1}(v)u^{p^t}$. Note that by Lemma 4.1.10, the condition (Cr_s) for the pair of complexes $\check{C}^{\bullet}_{\mathrm{BK},n} \to \check{C}^{\bullet}_{\mathrm{inf},n}$ implies the condition

$$\forall g \in G_s : \quad (g-1)\check{C}^{\bullet}_{\mathrm{BK},n} \subseteq \alpha \check{C}^{\bullet}_{\mathrm{inf},n}$$

(meant termwise as usual), and since the terms of the complex $\check{C}^{\bullet}_{\mathrm{inf}}$ are $(p, E(u))$–complete and $(p, E(u))$–completely flat, α is a non–zero divisor on the terms of $\check{C}^{\bullet}_{\mathrm{inf},n}$ by Corollary 2.2.5.

So pick any element $x \in Z^i(\check{C}^{\bullet}_{\mathrm{BK},n})$. The image x' of x in $\check{C}^{i}_{\mathrm{inf},n}$ lies in $Z^i(\check{C}^{\bullet}_{\mathrm{inf},n})$ and for any chosen $g \in G_s$ we have $g(x') - x' = \alpha y'$ for some $y' \in \check{C}^{i}_{\mathrm{inf},n}$. Now $g(x') - x'$ lies in $Z^i(\check{C}^{\bullet}_{\mathrm{inf},n})$, so $\alpha y' = g(x') - x'$ satisfies

$$0 = \partial'(\alpha y') = \alpha \partial'(y').$$

Since α is a non–zero divisor on $\check{C}^{i+1}_{\mathrm{inf},n}$, it follows that $\partial'(y') = 0$, that is, y' lies in $Z^i(\check{C}^{\bullet}_{\mathrm{inf},n})$. We thus infer that $g(x') - x' = \alpha y' \in \alpha Z^i(\check{C}^{\bullet}_{\mathrm{inf},n})$, as desired. □

6. RAMIFICATION BOUNDS

6.1 Fontaine's strategy for ramification bounds

We are ready to discuss the implications to the question of ramification bounds for p–torsion étale cohomology groups $\mathrm{H}^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$ when $\mathfrak{X}$ is smooth and proper p–adic formal scheme over $\mathcal{O}_K$. In this section, we set up the required notation and review the key condition (P_m) established by Fontaine in [16].

We define an additive valuation $v^\flat$ on $\mathcal{O}_{\mathbb{C}_K}^\flat$ by $v^\flat(x) = v(x^\sharp)$ where v is the valuation on $\mathcal{O}_{\mathbb{C}_K}$ normalized so that $v(\pi) = 1$, and $(-)^\sharp : \mathcal{O}_{\mathbb{C}_K}^\flat \to \mathcal{O}_{\mathbb{C}_K}$ is the multiplicative lift from Notation 3.1.11. This way, we have $v^\flat(\underline{\pi}) = 1$ and $v^\flat(\underline{\varepsilon} - 1) = pe/(p-1)$. For a real number $c \geq 0$, denote by $\mathfrak{a}^{>c}$ ($\mathfrak{a}^{\geq c}$, resp.) the ideal of $\mathcal{O}_{\mathbb{C}_K}^\flat$ formed by all elements x with $v^\flat(x) > c$ ($v^\flat(x) \geq c$, resp.).

Similarly, we fix an additive valuation v_K of K normalized by $v_K(\pi) = 1$. Then for an algebraic extension L/K and a real number $c \geq 0$, we denote by $\mathfrak{a}_L^{>c}$ the ideal consisting of all elements $x \in \mathcal{O}_L$ with $v_K(x) > c$ (and similarly, for '$\geq$' as well).

For a finite extensions $M/F/K$ and a real number $m \geq 0$, let us recall (a version of[1]) Fontaine's property $(P_m^{M/F})$:

$(P_m^{M/F})$ For any algebraic extension E/F, the existence of an $\mathcal{O}_F$–algebra map $\mathcal{O}_M \to \mathcal{O}_E/\mathfrak{a}_E^{>m}$ implies the existence of an F–injection of fields $M \hookrightarrow E$.

We also recall the upper ramification numbering in the convention used in [11]. For $G = \mathrm{Gal}(M/F)$ and a non–negative real number λ, set

$$G_{(\lambda)} = \{g \in G \mid v_M(g(x) - x) \geq \lambda \ \ \forall x \in \mathcal{O}_M\},$$

where v_M is again the additive valuation of M normalized by $v_M(M^\times) = \mathbb{Z}$.

For $t \geq 0$, set

$$\phi_{M/F}(t) = \int_0^t \frac{\mathrm{d}t}{[G_{(1)} : G_{(t)}]}$$

[1]↑Fontaine's original condition uses the ideals $\mathfrak{a}_E^{\geq m}$ instead. Up to changing some inequalities from '<' to '≤' and vice versa, the conditions are fairly equivalent.

(which makes sense as $G_{(t)} \subseteq G_{(1)}$ for all $t > 0$). Then $\phi_{M/F}$ is a piecewise–linear increasing continuous concave function. Denote by $\psi_{M/F}$ its inverse, and set $G^{(\mu)} = G_{(\psi_{M/F}(\mu))}$.

Denote by $\lambda_{M/F}$ the infimum of all $\lambda \geq 0$ such that $G_{(\lambda)} = \{\mathrm{id}\}$, and by $\mu_{M/F}$ the infimum of all $\mu \geq 0$ such that $G^{(\mu)} = \{\mathrm{id}\}$. Clearly one has $\mu_{M/F} = \phi_{M/F}(\lambda_{M/F})$.

Remark 6.1.1. Let us compare the indexing conventions with [35] and [16], as the results therein are (implicitly or explicitly) used. If $G^{\text{S-}(\mu)}, G^{\text{F-}(\mu)}$ are the upper–index ramification groups in [35] and [16], resp., and similarly for $G_{\text{S-}(\lambda)}$ and $G_{\text{F-}(\lambda)}$ in the case of lower–index ramification groups, then we have

$$G^{(\mu)} = G^{\text{S-}(\mu-1)} = G^{\text{F-}(\mu)}, \quad G_{(\lambda)} = G_{\text{S-}(\lambda-1)} = G_{\text{F-}(\lambda/\tilde{e})},$$

where $\tilde{e} = e_{M/F}$ is the ramification index of M/F.

In particular, since the enumeration differs from the one in [35] only by a shift by one, the claims that lower indexing is compatible with restrictions to subgroups and upper indexing is compatible with passing to quotients remain valid. Thus, it make sense to set

$$G_F^{(\mu)} = \varprojlim_{M'/F} \mathrm{Gal}(M'/F)^{(\mu)}$$

where M'/F varies over finite Galois extensions M'/F contained in a fixed algebraic closure $\overline{K}$ of K (and $G_F = \varprojlim_{M'/F} \mathrm{Gal}(M'/F)$ is the absolute Galois group).

Regarding μ, the following transitivity formula is useful.

Lemma 6.1.2 ([11, Lemma 4.3.1])**.** *Let $N/M/F$ be a pair of finite extensions with both N/F and M/F Galois. Then we have $\mu_{N/F} = \max(\mu_{M/F}, \phi_{M/F}(\mu_{N/M}))$.*

The property $(P_m^{M/F})$ is connected with the ramification of the field extension M/F as follows.

Proposition 6.1.3. *Let $M/F/K$ be finite extensions of fields with M/F Galois and let $m > 0$ be a real number. If the property $(P_m^{M/F})$ holds, then:*

(1) ([40, Proposition 3.3]) $\mu_{M/F} \leq e_{F/K}m$. *In fact,* $\mu_{M/F}/e_{F/K}$ *is the infimum of all* $m > 0$ *such that* $(P_m^{M/F})$ *is valid.*

(2) ([11, Corollary 4.2.2]) $v_K(\mathcal{D}_{M/F}) < m$, *where* $\mathcal{D}_{M/F}$ *denotes the different of the field extension* M/F.

Corollary 6.1.4. *Both the assumptions and the conclusions of Proposition 6.1.3 are insensitive to replacing* F *by any unramified extension of* F *contained in* M.

Proof. Let F'/F be an unramified extension such that $F' \subseteq M$. The fact that $(P_m^{M/F})$ is equivalent to $(P_m^{M/F'})$ is proved in [40, Proposition 2.2]. To show that also the conclusions are the same for F and F', it is enough to observe that $e_{F'/K} = e_{F/K}, e_{M/F'} = e_{M/F}$, $v_K(\mathcal{D}_{M/F'}) = v_K(\mathcal{D}_{M/F})$ and $\mu_{M/F'} = \mu_{M/F}$. The first two equalities are clear since F'/F is unramified. The third equality follows from $\mathcal{D}_{M/F} = \mathcal{D}_{M/F'}\mathcal{D}_{F'/F}$ upon noting that $\mathcal{D}_{F'/F}$ is the unit ideal. Finally, by Lemma 6.1.2, we have $\mu_{M/F} = \max(\mu_{F'/F}, \phi_{F'/F}(\mu_{M/F'}))$. As F'/F is unramified, we have $\mu_{F'/F} = 0$ and $\phi_{F'/F}(t) = t$ for all $t \geq 0$. The fourth equality thus follows as well. $\square$

6.2 Ramification bounds for mod p étale cohomology

Finally, we proceed to the proof of ramification bounds. Let $\mathfrak{X}$ be a proper and smooth p–adic formal scheme over $\mathcal{O}_K$. Fix the integer i, and set $T' = \mathrm{H}^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$. Let L be the splitting field of T', i.e. $L = \overline{K}^{\operatorname{Ker}\rho}$ where $\rho : G_K \to \operatorname{Aut}_{\mathbb{F}_p}(T')$ is the associated representation. The goal is to provide an upper bound on $v_K(\mathcal{D}_{L/K})$, and a constant $\mu_0 = \mu_0(e, i, p)$ such that $G_K^{(\mu)}$ acts trivially on T' for all $\mu > \mu_0$.

As described in Introduction, to achieve this we follow rather closely the strategy of [11]. The main difference is that the input of $(\varphi, \widehat{G})$–modules attached to the discussed G_K–respresentations in [11] is in our situation replaced by a p–torsion Breuil–Kisin module and a Breuil–Kisin–Fargues G_K–module that arise as the p–torsion prismatic $\mathfrak{S}$– and $A_{\inf}$–cohomology, resp. (and, more importantly, unlike in [11] we also do not have any lifts of these Breuil–Kisin(–Fargues) modules to free ones).

Let us therefore lay out the strategy, referring to proofs in [11] whenever possible, and describe the needed modifications where necessary. To facilitate this approach further, the notation used will usually reflect the notation of [11], except for mostly omitting the index n throughout (which in our situation is always equal to 1).

Let $M^0_{\mathrm{BK}} = \mathrm{H}^i_{\Delta,1}(\mathfrak{X}/\mathfrak{S})$ and $M^0_{\inf} = \mathrm{H}^i_{\Delta,1}(\mathfrak{X}_{A_{\inf}}/A_{\inf})$, so that, by Proposition 3.4.4, we have

$$M^0_{\inf} = M^0_{\mathrm{BK}} \otimes_{\mathfrak{S}} A_{\inf} \quad \text{and} \quad T_1(M^0_{\mathrm{BK}}) = T_1^{\inf}(M^0_{\inf}) = \mathrm{H}^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z}).$$

Observe further that, since u is a unit of $W_1(\mathbb{C}_K^\flat) = \mathbb{C}_K^\flat$, we have $T_1(M^0_{\mathrm{BK}}) = T_1(M_{\mathrm{BK}})$ and $T_1^{\inf}(M^0_{\inf}) = T_1^{\inf}(M_{\inf})$, where $M_{\mathrm{BK}} = M^0_{\mathrm{BK}}/M^0_{\mathrm{BK}}[u^\infty]$ and $M_{\inf} = M^0_{\inf}/M^0_{\inf}[u^\infty]$ are again a Breuil–Kisin module and a Breuil–Kisin–Fargues G_K–module, resp., of height $\leq i$. Since $\mathfrak{S} \hookrightarrow A_{\inf}$ is faithfully flat, it is easy to see that the isomorphism $M_{\inf} \simeq M_{\mathrm{BK}} \otimes_{\mathfrak{S}} A_{\inf}$ remains true. Furthermore, the pair $(M_{\mathrm{BK}}, M_{\inf})$ satisfies the conditions

$$\forall g \in G_s \;\; \forall x \in M_{\mathrm{BK}} : \quad g(x) - x \in \varphi^{-1}(v) u^{p^s} M_{\inf} \tag{6.1}$$

for all $s \geq 0$, since the pair $(M^0_{\mathrm{BK}}, M^0_{\inf})$ satisfies the analogous conditions by Proposition 5.2.3. Finally, the module M_{BK} is finitely generated and u–torsion–free $k[[u]]$–module, hence a finite free $k[[u]]$–module (and, consequently, $M_{\inf}$ is a finite free $\mathcal{O}_{\mathbb{C}_K^\flat}$–module).

Instead of referring to $T_1^{\inf}(M_{\inf}) = H^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ directly, we will discuss the ramification bound for

$$T := T_1^{*,\inf}(M_{\inf}) = \mathrm{Hom}_{A_{\inf},\varphi}(M_{\inf}, \mathcal{O}_{\mathbb{C}_K^\flat}) \simeq H^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})^\vee,$$

which is equivalent, as the splitting field of T is still equal to L. Also note that we have $T \simeq T_1^*(M_{\mathrm{BK}}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(M_{\mathrm{BK}}, \mathcal{O}_{\mathbb{C}_K^\flat})$ as a $\mathbb{Z}/p\mathbb{Z}[G_\infty]$–module.

Remark 6.2.1 (Ramification bounds of [10])**.** Similarly to the discussion above we may take, for any $n \geq 1$, $M^0_{\mathrm{BK}} = \mathrm{H}^i_{\Delta,n}(\mathfrak{X}/\mathfrak{S})$, and $M_{\mathrm{BK}} = M^0_{\mathrm{BK}}/M^0_{\mathrm{BK}}[u^\infty]$. Then the G_∞–module $T := T_n^*(M_{\mathrm{BK}}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(M_{\mathrm{BK}}, W_n(\mathcal{O}_{\mathbb{C}_K^\flat}))$ is the restriction of $\mathrm{H}^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p^n\mathbb{Z})^\vee$ to G_∞. Denoting by $\mathcal{O}_{\mathcal{E}}$ the p–adic completion of $\mathfrak{S}[1/u]$, $M_{\mathcal{E}} := M_{\mathrm{BK}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathcal{E}}$ then becomes an étale

φ–module over $\mathcal{O}_{\mathcal{E}}$ in the sense of [17, §A], with the natural map $M_{\mathrm{BK}} \to M_{\mathcal{E}}$ injective; thus, in terminology of [10], M_{BK} serves as a φ–lattice of height dividing $E(u)^i$. Upon observing that T is the G_∞–respresentation associated with $M_{\mathcal{E}}$ (see e.g. [10, §2.1.3]), Theorem 2 of [10] implies the ramification bound

$$\mu_{L/K} \leq 1 + c_0(K) + e\left(s_0(K) + \log_p(ip)\right) + \frac{e}{p-1}.$$

Here $c_0(K), s_0(K)$ are constants that depend on the field K and that generally grow with increasing e. (Their precise meaning is described in Section 6.3.)

We employ the following approximations of the functors T_1^* and $T_1^{*,\mathrm{inf}}$.

Notation 6.2.2. For a real number $c \geq 0$, we define

$$J_c(M_{\mathrm{BK}}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(M_{\mathrm{BK}}, \mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>c}),$$

$$J_c^{\mathrm{inf}}(M_{\mathrm{inf}}) = \mathrm{Hom}_{A_{\mathrm{inf}},\varphi}(M_{\mathrm{inf}}, \mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>c}).$$

We further set $J_\infty(M_{\mathrm{BK}}) = T_1^*(M_{\mathrm{BK}})$ and $J_\infty^{\mathrm{inf}}(M_{\mathrm{inf}}) = T_1^{*,\mathrm{inf}}(M_{\mathrm{inf}})$. Given $c, d \in \mathbb{R}^{\geq 0} \cup \{\infty\}$ with $c \geq d$, we denote by $\rho_{c,d} : J_c(M_{\mathrm{BK}}) \to J_d(M_{\mathrm{BK}})$ ($\rho_{c,d}^{\mathrm{inf}} : J_c^{\mathrm{inf}}(M_{\mathrm{inf}}) \to J_d^{\mathrm{inf}}(M_{\mathrm{inf}})$, resp.) the map induced by the quotient map $\mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>c} \to \mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>d}$.

Since $M_{\mathrm{inf}} \simeq M_{\mathrm{BK}} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ as φ–modules, it is easy to see that for every $c \in \mathbb{R}^{\geq 0} \cup \{\infty\}$, we have a natural isomorphism $\theta_c : J_c(M_{\mathrm{BK}}) \xrightarrow{\simeq} J_c^{\mathrm{inf}}(M_{\mathrm{inf}})$ of abelian groups; the biggest point of distinction between the two is that $J_c^{\mathrm{inf}}(M_{\mathrm{inf}})$ naturally attains the action of G_K from the one on M_{inf}, by the usual rule

$$g(f)(x) := g(f(g^{-1}(x))), \;\; g \in G_K, \; f \in J_c^{\mathrm{inf}}(M_{\mathrm{inf}}), \; x \in M_{\mathrm{inf}}.$$

As for $J_c(M_{\mathrm{BK}})$, there is a natural action given similarly by the formula $g(f)(x) := g(f(x))$ where $f \in J_c(M_{\mathrm{BK}})$ and $x \in M_{\mathrm{BK}}$. However, in order for this action to make sense, one needs that each $g(f)$ defined this way is still an $\mathfrak{S}$–linear map, which boils down to the requirement that $g(u) = u$ (that is, $g(\underline{\pi}) = \underline{\pi}$) in the ring $\mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>c}$. This is certainly true for $g \in G_\infty$, but also for possibly bigger subgroups of G_K, depending on c. The concrete result is the following.

Proposition 6.2.3 ([11, Proposition 2.5.3])**.** *Let s be a non–negative integer such that $s > \log_p(\frac{c(p-1)}{ep})$. Then the natural action of G_s on $\mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>c}$ induces an action of G_s on $J_c(M_{\mathrm{BK}})$. Furthermore, when $d \leq c$, the map $\rho_{c,d} : J_c(M_{\mathrm{BK}}) \to J_d(M_{\mathrm{BK}})$ is G_s–equivariant, and when $s' \geq s$, the $G_{s'}$–action on $J_c(M_{\mathrm{BK}})$ defined in this manner is the restriction of the G_s–action to $G_{s'}$.*

The crucial link to establish is the connection between the actions on $J_c(M_{\mathrm{BK}})$ and $J_c^{\inf}(M_{\inf})$. This is done via the input of the conditions (Cr_s) (and their consequences).

Proposition 6.2.4. *For*

$$s > \max\left\{\log_p\left(\frac{c(p-1)}{ep}\right), \log_p\left(c - \frac{e}{p-1}\right)\right\},$$

the natural isomorphism $\theta_c : J_c(M_{\mathrm{BK}}) \xrightarrow{\simeq} J_c^{\inf}(M_{\inf})$ is G_s–equivariant.

Proof. Identifying $M_{\inf}$ with $M_{\mathrm{BK}} \otimes_{\mathfrak{S}} A_{\inf}$, θ_c takes the form $f \mapsto \widetilde{f}$ where $\widetilde{f}(x \otimes a) := af(x)$ for $x \in M_{\mathrm{BK}}$ and $a \in A_{\inf}$. Note that we have $\varphi^{-1}(v)u^{p^s}\mathcal{O}_{\mathbb{C}_K^\flat} = \mathfrak{a}^{\geq p^s + e/(p-1)}$. The condition (6.1) then states that for all $x \in M_{\mathrm{BK}}$ and all $g \in G_s$, $g(x \otimes 1) - x \otimes 1$ lies in $\mathfrak{a}^{\geq p^s+e/(p-1)}M_{\inf}$ and therefore in $\mathfrak{a}^{>c}M_{\inf}$ thanks to the assumption on s. It then follows that for every $\widetilde{f} \in J_c^{\inf}(M_{\inf})$, $\widetilde{f}(g(x \otimes 1)) = \widetilde{f}(x \otimes 1)$, and hence

$$g(\widetilde{f})(x \otimes a) = g\left(\widetilde{f}(g^{-1}(x \otimes a))\right) = g\left(g^{-1}(a)\widetilde{f}(g^{-1}(x \otimes 1))\right) = ag\left(\widetilde{f}(x \otimes 1)\right) = ag(f(x))$$

for every $g \in G_s$, $x \in M_{\mathrm{BK}}$ and $a \in A_{\inf}$. Thus, we have that $g(\widetilde{f}) = \widetilde{g(f)}$ for every $g \in G_s$ and $f \in J_c(M_{\mathrm{BK}})$, proving the equivariance of θ_c. $\square$

From now on, set $b := ie/(p-1)$ and $a := iep/(p-1)$. Then T is determined by $J_a(M), J_b(M)$ in the following sense.

Proposition 6.2.5.

(1) The map $\rho_{\infty,b}: T_1^*(M_{\mathrm{BK}}) \to J_b(M_{\mathrm{BK}})$ *is injective, and* $\rho_{\infty,b}(M_{\mathrm{BK}}) = \rho_{a,b}(J_a(M_{\mathrm{BK}}))$.

(2) The map $\rho_{\infty,b}^{\mathrm{inf}}: T_1^{*,\mathrm{inf}}(M_{\mathrm{inf}}) \to J_b^{\mathrm{inf}}(M_{\mathrm{inf}})$ *is injective, and* $\rho_{\infty,b}^{\mathrm{inf}}(M_{\mathrm{inf}}) = \rho_{a,b}^{\mathrm{inf}}(J_a^{\mathrm{inf}}(M_{\mathrm{inf}}))$.

(3) For $s > \log_p(i)$, $T_1^*(M_{\mathrm{BK}})$ *has a natural action of* G_s *that extends the usual* G_∞*–action.*

(4) For $s > \max\left(\log_p(i), \log_p((i-1)e/(p-1))\right)$, *the action from (3) agrees with* $T\mid_{G_s}$.

Proof. Part (1) is proved in [11, Proposition 2.3.3]. Then $T_1^*(M_{\mathrm{BK}})$ attains the action of G_s with $s > \log_p(i)$ by identification with $\rho_{a,b}(J_a(M_{\mathrm{BK}}))$ and using Proposition 6.2.3 (see also [11, Theorem 2.5.5]), which proves (3). Finally, the proof of (2),(4) is analogous to [11, Corollary 3.3.3] and [11, Theorem 3.3.4]. Explicitly, consider the commutative diagram

$$\begin{array}{ccccc} T_1^*(M_{\mathrm{BK}}) & \xrightarrow{\rho_{\infty,a}} & J_a(M_{\mathrm{BK}}) & \xrightarrow{\rho_{a,b}} & J_b(M_{\mathrm{BK}}) \\ \sim\Big\downarrow\theta_\infty & & \sim\Big\downarrow\theta_a & & \sim\Big\downarrow\theta_b \\ T_1^{*,\mathrm{inf}}(M_{\mathrm{inf}}) & \xrightarrow{\rho_{\infty,a}^{\mathrm{inf}}} & J_a^{\mathrm{inf}}(M_{\mathrm{inf}}) & \xrightarrow{\rho_{a,b}^{\mathrm{inf}}} & J_b^{\mathrm{inf}}(M_{\mathrm{inf}}), \end{array}$$

where the composition of the rows are $\rho_{\infty,b}$ and $\rho_{\infty,b}^{\mathrm{inf}}$, resp. This immediately proves (2) using (1). Finally, the map $\rho_{\infty,b}^{\mathrm{inf}}$ is G_K–equivariant and the map $\rho_{\infty,b}$ is tautologically G_s–equivariant for $s > \log_p(i)$ by the proof of (3), and both maps are injective. Since θ_b is G_s–equivariant when $s > \log_p((i-1)e/(p-1))$ by Proposition 6.2.4, it follows that so is θ_∞, which proves (4). □

We employ further approximations of $J_c(M_{\mathrm{BK}})$ defined as follows.

Notation 6.2.6. Let s be a non–negative integer, consider a real number $c \in [0, ep^s)$ and an algebraic extension E/K_s. We consider the ring

$$(\varphi_k^s)^*\mathcal{O}_E/\mathfrak{a}_E^{>c/p^s} = k \otimes_{\varphi_k^s,k} \mathcal{O}_E/\mathfrak{a}_E^{>c/p^s}$$

(note that the condition on c implies that $p \in \mathfrak{a}_E^{>c/p^s}$, making $\mathcal{O}_E/\mathfrak{a}_E^{>c/p^s}$ a k–algebra). We endow this ring with an $\mathfrak{S}$–algebra structure via $\mathfrak{S} \overset{\mathrm{mod}\, p}{\to} k[[u]] \overset{\alpha}{\to} (\varphi_k^s)^*\mathcal{O}_E/\mathfrak{a}_E^{>c/p^s}$ where α extends the k–algebra structure map by the rule $u \mapsto 1 \otimes \pi_s$. Then we set

$$J_c^{(s),E}(M_{\mathrm{BK}}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(M_{\mathrm{BK}}, (\varphi_k^s)^*\mathcal{O}_E/\mathfrak{a}_E^{>c/p^s}).$$

Note that the fact that $g(\pi_s) = \pi_s$ for all $g \in G_s$ implies that $J_c^{(s),E}(M_{\mathrm{BK}})$ attains a G_s–action induced by the G_s–action on $\mathcal{O}_E/\mathfrak{a}_E^{>c/p^s}$, assuming that E/K_s is Galois.

When c, d are two real numbers satisfying $ep^s > c \geq d \geq 0$, there is a transition map $\rho_{c,d}^{(s),E}(M_{\mathrm{BK}}) : J_c^{(s),E}(M_{\mathrm{BK}}) \to J_d^{(s),E}(M_{\mathrm{BK}})$ which is G_s–equivariant.

The relation to $J_c(M_{\mathrm{BK}})$ is the following.

Proposition 6.2.7. *Let s, c be as above. Then*

(1) Given an algebraic extension E/K_s, $J_c^{(s),E}(M_{\mathrm{BK}})$ naturally embeds into $J_c(M_{\mathrm{BK}})$ (as a G_s–submodule if E/K_s is Galois).

(2) Given a tower of algebraic extensions $F/E/K_s$, $J_c^{(s),E}(M_{\mathrm{BK}})$ naturally embeds into $J_c^{(s),F}(M_{\mathrm{BK}})$ (as a G_s–submodules if $E/K_s, F/K_s$ are Galois).

(3) $J_c^{(s),\overline{K}}(M_{\mathrm{BK}})$ is naturally isomorphic to $J_c(M_{\mathrm{BK}})$ as a G_s–module.

Proof. Part (2) is immediate upon observing that the inclusion $\mathcal{O}_E \hookrightarrow \mathcal{O}_F$ induces the map $\mathcal{O}_E/\mathfrak{a}_E^{>c/p^s} \to \mathcal{O}_F/\mathfrak{a}_F^{>c/p^s}$ which is still injective (and clearly G_s–equivariant in the Galois case). Similarly, part (3) follows from the fact that $\mathrm{pr}_s : \mathcal{O}_{\mathbb{C}_K^\flat} = \varprojlim_{s,\varphi} \mathcal{O}_{\overline{K}}/p \to \mathcal{O}_{\overline{K}}/p$ induces a (G_s–equivariant) isomorphism $\mathcal{O}_{\mathbb{C}_K^\flat}/\mathfrak{a}^{>c} \to (\varphi_k^s)^*\mathcal{O}_{\overline{K}}/\mathfrak{a}_{\overline{K}}^{>c/p^s}$ when $s > \log_p(c/e)$ (so a fortiori when $s > \log_p(c)$), which is proved in [11, Lemma 2.5.1]. Part (1) is then obtained as a direct combination of (2) and (3). $\square$

For a non–negative integer s, denote by L_s the composite of the fields K_s and L. The following adaptation of Theorem 4.1.1 of [11] plays a key role in establishing the ramification bound.

Theorem 6.2.8. *Let s be an integer satisfying*

$$s > M_0 := \max\left\{\log_p\left(\frac{a}{e}\right), \log_p\left(b - \frac{e}{p-1}\right)\right\} = \max\left\{\log_p\left(\frac{ip}{p-1}\right), \log_p\left(\frac{(i-1)e}{(p-1)}\right)\right\},$$

and let E/K_s be an algebraic extension. The inclusion $\rho_{a,b}^{(s),E}(J_a^{(s),E}(M_{\mathrm{BK}})) \hookrightarrow \rho_{a,b}(J_a(M_{\mathrm{BK}}))$, facilitated by the inclusions $J_a^{(s),E}(M_{\mathrm{BK}}) \hookrightarrow J_a(M_{\mathrm{BK}})$ and $J_b^{(s),E}(M_{\mathrm{BK}}) \hookrightarrow J_b(M_{\mathrm{BK}})$ from Proposition 6.2.7, is an isomorphism if and only if $L_s \subseteq E$.

Proof. The proof of [11, Theorem 4.1.1] applies in our context as well, as we now explain. Using just the fact that M_{BK} is a Breuil–Kisin module that is free over $k[[u]]$ together with the assumption $s > \log_p(a/e)$, for every F/K_s algebraic, an auxiliary set $\widetilde{J}_1^{(s),F}(M_{\mathrm{BK}})$ is constructed, together with maps of sets $\widetilde{\rho}_c^{(s),F} : \widetilde{J}_1^{(s),F}(M_{\mathrm{BK}}) \to J_c^{(s),F}(M_{\mathrm{BK}})$ for every $c \in (0, ep^s)$. When F is Galois over K, this set is naturally a G_s–set and the maps are G_s–equivariant. Moreover, the sets have the property that $\left(\widetilde{J}_1^{(s),F}(M_{\mathrm{BK}})\right)^{G_{F'}} = \widetilde{J}_1^{(s),F'}(M_{\mathrm{BK}})$ when $F/F'/K_s$ is an intermediate extension.

Subsequently, it is shown in [11, Lemma 4.1.4] that

$$\widetilde{\rho}_b^{(s),F} \text{ is injective and its image is } \rho_{a,b}^{(s),F}(J_a^{(s),F}(M_{\mathrm{BK}})), \tag{$*$}$$

where the only restriction on s is again $s > \log_p(a/e)$.

Finally, one obtains a series of G_s–equivariant bijections:

$$\begin{aligned} \widetilde{J}_1^{(s),\overline{K}}(M_{\mathrm{BK}}) &\simeq \rho_{a,b}^{(s),\overline{K}}(J_a^{(s),\overline{K}}(M_{\mathrm{BK}})) && \text{(by } (*)\text{)} \\ &\simeq \rho_{a,b}(J_a(M_{\mathrm{BK}})) && \text{(Proposition 6.2.7 (3))} \\ &\simeq \rho_{a,b}^{\mathrm{inf}}(J_a^{\mathrm{inf}}(M_{\mathrm{inf}})) && \text{(Proposition 6.2.4)} \\ &\simeq T && \text{(Proposition 6.2.5 (2))} \end{aligned}$$

(where the third isomorphism relies on the assumption $s > \log_p(b - e/(p-1))$). Applying $(-)^{G_E}$ to both sides and using $(*)$ again then yields

$$\rho_{a,b}^{(s),E}(J_a^{(s),E}(M_{\mathrm{BK}})) \simeq T^{G_E}.$$

Therefore, we may replace the inclusion from the statement of the theorem by the inclusion $T^{G_E} \subseteq T$, and the claim now easily follows. □

Finally, we are ready to establish the desired ramification bound. Let $N_s = K_s(\zeta_{p^s})$ be the Galois closure of K_s over K, and set $M_s = L_s N_s$. Then we have

Proposition 6.2.9. *Let s be as in Theorem 6.2.8, and set $m = a/p^s$. Then the properties $(P_m^{L_s/K_s})$ and $(P_m^{M_s/N_s})$ hold.*

Proof. The proof of $(P_m^{L_s/K_s})$ is the same as in [11], which refers to an older version of [22] for parts of the proof. Let us therefore reproduce the argument for convenience. By Corollary 6.1.4, it is enough to prove $(P_m^{L_s/K_s^{un}})$ where K_s^{un} denotes the maximal unramified extension of K_s in L_s.

Let E/K_s^{un} be an algebraic extension and $f : \mathcal{O}_{L_s} \to \mathcal{O}_E/\mathfrak{a}_K^{\geq m}$ be an $\mathcal{O}_{K_s^{un}}$–algebra map. Setting $c = a$ or $c = b$, one may consider an induced map $f_c : \mathcal{O}_{L_s}/\mathfrak{a}_{L_s}^{>c/p^s} \to \mathcal{O}_E/\mathfrak{a}_K^{>c/p^s}$, and we claim is that this map is well–defined and injective.

Indeed, let ϖ be a uniformizer of L_s, satisfying the relation

$$\varpi^{e'} = c_1\varpi^{e'-1} + c_2\varpi^{e'-2} + \cdots + c_{e'-1}\varpi + c_{e'},$$

where $P(T) = T^{e'} - \sum_i c_i T^{e'-i}$ is an Eisenstein polynomial over K_s^{un}. Applying f one thus obtains $t^{e'} = \sum_i c_i t^{e'-i}$ in $\mathcal{O}_E/\mathfrak{a}_K^{\geq m}$ where $t = f(\varpi)$, and thus, lifting t to $\widetilde{t} \in \mathcal{O}_E$, we obtain the equality

$$\widetilde{t}^{e'} = c_1\widetilde{t}^{e'-1} + c_2\widetilde{t}^{e'-2} + \cdots + c_{e'-1}\widetilde{t} + c_{e'} + r$$

with $v_K(r) > m > 1/p^s = v_K(c_{e'})$. It follows that $v_K(\widetilde{t}) = v_K(\varpi) = 1/p^s e'$, and thus, $\varpi^n \in \mathfrak{a}_{L_s}^{>c/p^s}$ if and only if $\widetilde{t}^n \in \mathfrak{a}_E^{>c/p^s}$, proving that f_c is both well–defined and injective.

The map f_c induces an injection of k–algebras $(\varphi_k^s)^*\mathcal{O}_{L_s}/\mathfrak{a}_{L_s}^{>c/p^s} \hookrightarrow (\varphi_k^s)^*\mathcal{O}_E/\mathfrak{a}_E^{>c/p^s}$ which in turn gives an injection $J_c^{(s),L_s}(M_{\mathrm{BK}}) \to J_c^{(s),E}(M_{\mathrm{BK}})$, where $c = a$ or $c = b$; consequently, we obtain an injection

$$\rho_{a,b}^{(s),L_s}(J_a^{(s),L_s}(M_{\mathrm{BK}})) \hookrightarrow \rho_{a,b}^{(s),E}(J_a^{(s),E}(M_{\mathrm{BK}})).$$

Combining this with Propositions 6.2.5 and 6.2.7, we have the series of injections

$$\rho_{a,b}^{(s),L_s}(J_b^{(s),L_s}(M_{\mathrm{BK}})) \hookrightarrow \rho_{a,b}^{(s),E}(J_b^{(s),E}(M_{\mathrm{BK}})) \hookrightarrow \rho_{a,b}^{(s),\overline{K}}(J_b^{(s),\overline{K}}(M_{\mathrm{BK}})) \hookrightarrow \rho_{a,b}(J_b(M_{\mathrm{BK}})) \simeq T.$$

Since $\rho_{a,b}^{(s),L_s}(J_b^{(s),L_s}(M_{\mathrm{BK}})) \simeq T$ by Theorem 6.2.8, the result is in fact an injective map $T \hookrightarrow T$ and therefore an isomorphism since T is finite. In particular, the natural morphism $\rho_{a,b}^{(s),E}(J_b^{(s),E}(M_{\mathrm{BK}})) \hookrightarrow \rho_{a,b}(J_b(M_{\mathrm{BK}}))$ is an isomorphism, and Theorem 6.2.8 thus implies that $L_s \subseteq E$. This finishes the proof of (1).

Similarly as in [11], the property $(P_m^{M_s/N_s})$ is deduced from $(P_m^{L_s/K_s})$ as follows. Given an algebraic extension E/N_s and an $\mathcal{O}_{N_s}$–algebra morphism $\mathcal{O}_{M_s} \to \mathcal{O}_E/\mathfrak{a}_E^{>m}$, by restriction we obtain an $\mathcal{O}_{K_s}$–algebra morphism $\mathcal{O}_{L_s} \to \mathcal{O}_E/\mathfrak{a}_E^{>m}$, hence there is a K_s–injection $L_s \to E$. Since $N_s \subseteq E$, this can be extended to a K_s–injection $M_s \to E$, and upon noting that the extension M_s/K_s is Galois, one obtains an N_s–injection $M_s \to E$ by precomposing with a suitable automorphism of M_s. This proves $(P_m^{M_s/N_s})$. □

Everything is now ready for the final proof of Theorem 1.3.1 from Introduction.

Theorem 6.2.10. *Let*

$$\alpha = \lfloor M_0 \rfloor + 1 = \left\lfloor \log_p \left(\max \left\{ \frac{ip}{p-1}, \frac{(i-1)e}{p-1} \right\} \right) \right\rfloor + 1.$$

Then

(1)

$$v_K(\mathcal{D}_{L/K}) < 1 + e\alpha + \frac{iep}{p^\alpha(p-1)} - \frac{1}{p^\alpha}.$$

(2) For any μ satisfying

$$\mu > 1 + e\alpha + \max \left\{ \frac{iep}{p^\alpha(p-1)} - \frac{1}{p^\alpha}, \frac{e}{p-1} \right\},$$

$G_K^{(\mu)}$ acts trivially on T.

Proof. We may set $s = \alpha$ as the condition $s > M_0$ is then satisfied. Propositions 6.1.3 and 6.2.9 then imply that $v_K(\mathcal{D}_{L_s/K_s}) < a/p^s$ (where $a = iep/(p-1)$) and thus

$$v_K(\mathcal{D}_{L_s/K}) = v_K(\mathcal{D}_{K_s/K}) + v_K(\mathcal{D}_{L_s/K_s}) < 1 + es - \frac{1}{p^s} + \frac{a}{p^s} = 1 + e\alpha + \frac{a-1}{p^\alpha}.$$

Similarly, we have $v_K(\mathcal{D}_{L/K}) = v_K(\mathcal{D}_{L_s/K}) - v_K(\mathcal{D}_{L_s/L}) \leq v_K(\mathcal{D}_{L_s/K})$, and the claim (1) thus follows.

To prove (2), let M_s and N_s be as in Proposition 6.2.9. The fields N_s and $M_s = LN_s$ are both Galois over K, hence Lemma 6.1.2 applies and we thus have

$$\mu_{M_s/K} = \max\left\{\mu_{N_s/K}, \phi_{N_s/K}(\mu_{M_s/N_s})\right\}.$$

By [22, Remark 5.5], we have

$$\mu_{N_s/K} = 1 + es + \frac{e}{p-1}.$$

As for the second argument, Proposition 6.1.3 gives the estimate

$$\mu_{M_s/N_s} \leq e_{N_s/K} m = \frac{e_{N_s/K}}{p^s} a.$$

The function $\phi_{N_s/K}(t)$ is concave and has a constant slope $1/e_{N_s/K}$ beyond $t = \lambda_{N_s/K}$, where it attains the value $\phi_{N_s/K}(\lambda_{N_s/K}) = \mu_{N_s/K} = 1 + es + e/(p-1)$. Thus, $\phi_{N_s/K}(t)$ can be estimated linearly from above as follows:

$$\phi_{N_s/K}(t) \leq 1 + es + \frac{e}{p-1} + \frac{1}{e_{N_s/K}}\left(t - \lambda_{N_s/K}\right) = 1 + es + \frac{t}{e_{N_s/K}} - \frac{\lambda_{N_s/K}}{e_{N_s/K}} + \frac{e}{p-1}$$

There is an automorphism $\sigma \in \mathrm{Gal}(N_s/K)$ with $\sigma(\pi_s) = \zeta_p \pi_s$. That is, we have that $v_K(\sigma(\pi_s) - \pi_s) = e/(p-1) + 1/p^s$, showing that

$$\lambda_{N_s/K} \geq e_{N_s/K}\left(\frac{e}{p-1} + \frac{1}{p^s}\right),$$

and combinig this with the estimate of $\phi_{N_s/K}(t)$, we obtain

$$\phi_{N_s/K}(t) \leq 1 + es + \frac{t}{e_{N_s/K}} - \frac{1}{p^s}.$$

Plugging in the estimate for μ_{M_s/N_s} then yields

$$\phi_{N_s/K}(\mu_{M_s/N_s}) \leq 1 + es + \frac{a}{p^s} - \frac{1}{p^s} = 1 + es + \frac{\frac{iep}{p-1} - 1}{p^s}.$$

Thus, we have

$$\mu_{L/K} \leq \mu_{M_s/K} \leq 1 + e\alpha + \max\left\{\frac{iep}{p^\alpha(p-1)} - \frac{1}{p^\alpha}, \frac{e}{p-1}\right\},$$

which finishes the proof of part (2). □

6.3 Comparisons of bounds

Finally, let us compare the bounds obtained in Theorem 6.2.10 with other results from the literature. These are summarized in Table 6.1 below.

Table 6.1. : Comparisons of estimates of $\mu_{L/K}$

	$\mu_{L/K} \leq \cdots$	
Theorem 6.2.10	$1 + e + e\left\lfloor\log_p\left(\max\left\{\frac{ip}{p-1}, \frac{(i-1)e}{p-1}\right\}\right)\right\rfloor + \max\left\{\beta, \frac{e}{p-1}\right\},$	$\beta < \min(e, 2p)$ [2]
[11]	$1 + e + e\left\lfloor\log_p\left(\frac{ip}{p-1}\right)\right\rfloor + \max\left\{\beta, \frac{e}{p-1}\right\},$	$\beta < e$ [3]
[10]	$1 + c_0(K) + e\left(s_0(K) + \log_p(ip)\right) + \frac{e}{p-1}$	
[22]	$\begin{cases} 1 + e + \frac{e}{p-1}, & i = 1, \\ 1 + e + \frac{ei}{p-1} - \frac{1}{p}, & i > 1, \end{cases}$	when $ie < p - 1$
[18], [1]	$1 + \frac{i}{p-1}$	when $\begin{matrix} e = 1, \\ i < p - 1 \end{matrix}$

[2] ↑More precisely: When $i = 1$, it is easy to see that $\beta = (eip/(p-1)-1)/p^\alpha$ is smaller than $e/(p-1)$, and hence does not have any effect. When $i > 1$, one can easily show using $p^\alpha > ip/(p-1), p^\alpha > (i-1)e/(p-1)$ that $\beta < e$ and $\beta < pi/(i-1) \leq 2p$.

[3] ↑The number β here has different meaning than the number β of [11, Theorem 1.1].

Comparison with [22]. If we assume $ie < p-1$, then the first maximum in the estimate of $\mu_{L/K}$ is realized by $ip/(p-1) \in (1,p)$; that is, in Theorem 6.2.10 one has $\alpha = 1$ and thus,

$$\mu_{L/K} \leq 1 + e + \max\left\{\frac{ei}{p-1} - \frac{1}{p}, \frac{e}{p-1}\right\},$$

which agrees precisely with the estimate [22].

Comparison with [18], [1]. Specializing to $e = 1$ in the previous case, the bound becomes

$$\mu_{L/K} \leq \begin{cases} 2 + \frac{1}{p-1}, & i = 1, \\ 2 - \frac{1}{p} + \frac{i}{p-1} & i > 1. \end{cases}$$

This is clearly a slightly worse bound than that of [18] and [1] (by 1 and $(p-1)/p$, respectiely).

Comparison with [11]. From the shape of the bounds it is clear that the bounds are equivalent when $(i-1)e \leq ip$, that is, when $e \leq p$ and some "extra" cases that include the case when $i = 1$ (more precisely, these extra cases are when $e > p$ and $i \leq e/(e-p)$), and in fact, the terms β in such situation agree. In the remaining case when $(i-1)e > ip$, our estimate becomes gradually worse compared to [11].

Remark 6.3.1.

(1) It should be noted that the bounds from [11] do not necessarily apply to our situation as it is not clear when $\mathrm{H}^i_{\text{ét}}(\mathfrak{X}_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$ can be obtained as a quotient of two G_K–stable lattices in a semi–stable representation with Hodge–Tate weights in $[0,i]$. To our knowledge the only result along these lines is [13, Theorem 1.3.1] that states that this is indeed the case when $i = 1$ (and X is a proper smooth variety over K with semistable reduction). Interestingly, in this case the bound from Theorem 6.2.10 always agrees with the one from [11].

(2) Let us also point out that the verbatim reading of the bound from [11] as described in Theorem 1.1 of *loc. cit.* would have the term $\lceil \log_p(ip/(p-1)) \rceil$ (i.e. upper integer part) instead of the term $\lfloor \log_p(ip/(p-1)) \rfloor + 1$ as in Table 6.1, but this version seems to be the correct one. Indeed, the proof of Theorem 1.1 in [11] (in the case $n = 1$)

ultimately relies on the objects $J_{1,a}^{(s),E}(\mathfrak{M})$ that are analogous to $J_a^{(s),E}(M_{\mathrm{BK}})$, where $s = \lceil \log_p(ip/(p-1)) \rceil$. In particular, Lemma 4.2.3 of *loc. cit.* needs to be applied with $c = a$, and the implicitly used fact that the ring $\mathcal{O}_E/\mathfrak{a}_E^{>a/p^s}$ is a k–algebra (i.e. of characteristic p) relies on the *strict* inequality $e > a/p^s$, equivalently $s > \log_p(ip/(p-1))$. In the case that $ip/(p-1)$ happens to be equal to p^t for some integer t, one therefore needs to take $s = t+1$ rather than $s = t$. This precisely corresponds to the indicated change.

Comparison with [10]. Let us explain the constants $s_0(K), c_0(K)$ that appear in the estimate. The integer $s_0(K)$ is the smallest s such that $1 + p^s\mathbb{Z}_p \subseteq \chi(\mathrm{Gal}(K_{p^\infty}/K))$ where χ denotes the cyclotomic character. The rational number $c_0(K) \geq 0$ is the smallest constant c such that $\psi_{K/K_0}(1+t) \geq 1 + et - c$ (this exists, as the last slope of $\psi_{K/K_0}(t)$ is e)[4].

In the case when K/K_0 is tamely ramified, the estimate from [10] becomes

$$\mu_{L/K} \leq 1 + e\left(\log_p(ip) + 1\right) + \frac{e}{p-1},$$

which is fairly equivalent to the bound from Theorem 6.2.10 when $e < p$ (and again also in some extra cases, e.g. when $i = 1$ for any e and p), with the difference of estimates being approximately

$$e\left(\log_p\left(\frac{p}{p-1}\right) - \frac{1}{p-1}\right) \in \left(-\frac{e}{4\sqrt{p}}, 0\right).$$

In general, when e is big and coprime to p, the bound in [10] becomes gradually better unless, for example, $i = 1$.

In the case when K has relatively large wild absolute ramification, we expect that the bound from Theorem 6.2.10 generally becomes stronger, especially if K contains p^n–th roots of unity for large n, as can be seen in the following examples (where we assume $i > 1$; for $i = 1$, our estimate retains the shape of the tame ramification case and hence the difference between the estimates becomes even larger).

[4]↑To make sense of this in general, one needs to extend the definition of the functions $\psi_{L/M}, \varphi_{L/M}$ to the case when the extension L/M is not necessarily Galois. This is done e.g. in [10, §2.2.1].

Example 6.3.2.

(1) When $K = \mathbb{Q}_p(\zeta_{p^n})$ for $n \geq 2$, one has $e = (p-1)p^{n-1}$, $s_0(K) = n$ and from the classical computation of $\psi_{K/\mathbb{Q}_p}$ (e.g. as in [35, IV §4]), one obtains

$$c_0(K) = [(n-1)(p-1)-1]p^{n-1} + 1.$$

Then the difference between the two estimates is approximately $ne - p^{n-1} + 1 > (n-1)e$.

(2) When $K = \mathbb{Q}_p(p^{1/p^n})$ for $n \geq 3$, one has $e = p^n$ and $s_0(K) = 1$. The description of $\psi_{K/\mathbb{Q}_p}$ in [11, §4.3] implies that $c_0(K) = np^n = ne$. The difference between the two estimates is thus approximately

$$e\Big(1 + \log_p(i) - \log_p(i-1) + \log_p(p-1)\Big) \approx 2e.$$

(In the initial cases $n = 1, 2$, one can check that the difference is still positive, in both cases bigger than p.)

REFERENCES

[1] V. Abrashkin, "Ramification in étale cohomology," *Inventiones mathematicae*, vol. 101, no. 1, pp. 631–640, 1990.

[2] ——, "Ramification estimate for Fontaine–Laffaille Galois modules," *Journal of Algebra*, vol. 427, pp. 319–328, 2015.

[3] ——, "Group schemes of period p over the ring of Witt vectors," *Doklady Akademii Nauk SSSR*, vol. 283, no. 6, pp. 1289–1294, 1985.

[4] B. Bhatt, "Geometric aspects of p–adic Hodge theory," lecture notes, available online at http://www-personal.umich.edu/~bhattb/teaching/prismatic-columbia/, 2018.

[5] B. Bhatt, M. Morrow, and P. Scholze, "Integral p–adic hodge theory," *Publications mathématiques de l'IHÉS*, vol. 128, no. 1, pp. 219–397, 2018.

[6] ——, "Topological Hochschild homology and integral p–adic Hodge theory," *Publications mathématiques de l'IHÉS*, vol. 129, no. 1, pp. 199–310, 2019.

[7] B. Bhatt and P. Scholze, "Prisms and prismatic cohomology," *arXiv preprint arXiv:1905.08229*, 2019.

[8] C. Breuil, "Représentations p–adiques semi–stables et transversalité de Griffiths," *Mathematische Annalen*, vol. 307, no. 2, pp. 191–224, 1997.

[9] X. Caruso, "Conjecture de l'inertie modérée de Serre," *Inventiones mathematicae*, vol. 171, no. 3, pp. 629–699, 2008.

[10] ——, "Représentations galoisiennes p-adiques et (φ, τ)–modules," *Duke Mathematical Journal*, vol. 162, no. 13, pp. 2525–2607, 2013.

[11] X. Caruso and T. Liu, "Some bounds for ramification of p^n–torsion semi–stable representations," *Journal of Algebra*, vol. 325, no. 1, pp. 70–96, 2011.

[12] R. Elkik, "Solutions d'équations ā coefficients dans un anneau hensélien," in *Annales scientifiques de l'École Normale Supérieure*, vol. 6, no. 4, 1973, pp. 553–603.

[13] M. Emerton and T. Gee, “p-adic Hodge-theoretic properties of étale cohomology with mod p coefficients, and the cohomology of Shimura varieties,” *Algebra & Number Theory*, vol. 9, no. 5, pp. 1035–1088, 2015.

[14] ——, “Moduli stacks of étale (φ, Γ)–modules and the existence of crystalline lifts,” *arXiv preprint arXiv:1908.07185*, 2019.

[15] G. Faltings, “Endlichkeitssätze für abelsche Varietäten über Zahlkörpern,” *Inventiones Mathematicae*, vol. 73, no. 3, pp. 349–366, 1983.

[16] J.-M. Fontaine, “Il n'y a pas de variété abélienne sur $\mathbb{Z}$,” *Inventiones mathematicae*, vol. 81, no. 3, pp. 515–538, 1985.

[17] ——, “Représentations p-adiques des corps locaux I.” in *The Grothendieck Festschrift.* Birkhäuser, Boston, MA, 1990, pp. 249–309.

[18] ——, “Schémas propres et lisses sur $\mathbb{Z}$,” in *Proceedings of the Indo–French Conference on Geometry.* New Delhi: Hindustan Book Agency, 1993, pp. 43–56.

[19] J.-M. Fontaine and G. Laffaille, “Construction de représentations p-adiques,” in *Annales scientifiques de l'École Normale Supérieure*, vol. 15, no. 4, 1982, pp. 547–608.

[20] J.-M. Fontaine and W. Messing, “p-adic periods and p-adic étale cohomology,” in *Current trends in arithmetical algebraic geometry*, ser. Contemp. Math., vol. 67. Arcata, Calif., 1985: Amer. Math. Soc., 1987, pp. 179–207.

[21] H. Gao, “Breuil-Kisin modules and integral p-adic Hodge theory,” *arXiv preprint arXiv:1905.08555*, 2019.

[22] S. Hattori, “On a ramification bound of torsion semi-stable representations over a local field,” *Journal of Number Theory*, vol. 129, no. 10, pp. 2474–2503, 2009.

[23] M. Hovey and N. P. Strickland, *Morava K-theories and localisation.* American Mathematical Soc., 1999, vol. 666.

[24] A. Joyal, "δ-anneaux et vecteurs de Witt," *CR Math. Rep. Acad. Sci. Canada*, vol. 7, no. 3, pp. 177–182, 1985.

[25] N. M. Katz, "Galois properties of torsion points on abelian varieties," *Inventiones Mathematicae*, vol. 62, no. 3, pp. 481–502, 1980.

[26] T. Koshikawa, "Logarithmic prismatic cohomology I," *arXiv preprint arXiv:2007.14037*, 2020.

[27] S. Li and T. Liu, "Comparison of prismatic cohomology and derived de Rham cohomology," *arXiv preprint arXiv:2012.14064*, 2020.

[28] T. Liu, "On lattices in semi–stable representations: a proof of a conjecture of Breuil," *Compositio Mathematica*, vol. 144, no. 1, pp. 61–88, 2008.

[29] ——, "A note on lattices in semi-stable representations," *Mathematische Annalen*, vol. 346, no. 1, pp. 117–138, 2010.

[30] H. Matsumura, *Commutative ring theory.* Cambridge university press, 1989, vol. 8.

[31] Y. Ozeki, "Lattices in crystalline representations and Kisin modules associated with iterate extensions," *Doc. Math.*, vol. 23, pp. 497–541, 2018.

[32] A. N. Parshin, "Algebraic curves over function fields. I," *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, vol. 32, no. 5, pp. 1191–1219, 1968.

[33] L. Positselski, "Contraadjusted modules, contramodules, and reduced cotorsion modules," *Moscow Mathematical Journal*, vol. 17, no. 3, pp. 385–455, 2017.

[34] C. Rezk, "Analytic completion," available online at https://conf.math.illinois.edu/~rezk/analytic-paper.pdf.

[35] J.-P. Serre, *Local fields.* Springer, 2013, vol. 67.

[36] T. Stacks Project Authors, "Stacks project," available online at http://stacks.math.columbia.edu, 2022.

[37] J. R. Strooker, *Homological questions in local algebra.* Cambridge University Press, 1990, vol. 145.

[38] A. Yekutieli, "Flatness and completion revisited," *Algebras and Representation Theory*, vol. 21, no. 4, pp. 717–736, 2018.

[39] ——, "Weak proregularity, derived completion, adic flatness, and prisms," *Journal of Algebra*, vol. 583, pp. 126–152, 2021.

[40] M. Yoshida, "Ramification of local fields and Fontaine's property (P_m)," *Journal of Mathematical Sciences, the University of Tokyo*, vol. 17, no. 3, pp. 247–265, 2010.

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