COMPLEXITY AND CONFLICT: MODELING CONTESTS WITH EXOGENOUS AND ENDOGENOUS NOISE

by

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A Dissertation

Submitted to the Faculty of Purdue University

In Partial Fulfillment of the Requirements for the degree of

Doctor of Philosophy

Department of Economics

West Lafayette, Indiana

May 2022

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To my wife and kids.

ACKNOWLEDGMENTS

While my name is on the cover of this dissertation, it never would have been possible without the encouragement, confidence, and substantial help of many people. Most significantly, I am deeply grateful to my wife. In my attempt to earn a PhD in economics in less than three years, she has supported me emotionally and physically through late nights, early mornings, and many long, and occasionally frustrating, weeks. She has basically functioned as a single mom to our five kids for the last three years, and she has done an exceptionally good job at it. This degree was very much a team effort, and she deserves as much credit for its accomplishment as I do.

I am also grateful to my children. They have exhibited a level of patience and understanding that is well beyond their ages as they have put up with my absence, both at bedtimes and special events. Moreover, they have been a significant emotional support to me. Their enthusiasm and genuine happiness has lifted and sustained me through exhausting days. It has been fun witnessing their interest in the mathematics, economics, politics, and military strategy that I've been studying, and I've greatly enjoyed discussions with them. I am proud of their intelligence, curiosity, kindness, spirituality, and dependability. They are becoming strong individuals and leaders, and they are already making an impact on the people around them at home, church, and school.

I also owe thanks to my parents for their encouragement and support. My mother encouraged me to get a PhD and attend Purdue, and I know her prayers have been with me over these last few years. I also appreciate the support of my father, who earned his PhD at Indiana University 52 years ago. While he is now gone, his memory and example live with me, as well as his encouragement from when I was younger and his pride in my successes. I strive to live up to their legacy and teachings every day.

My advisor deserves my sincere and special thanks. He was very dedicated to my success, and his attention and mentoring is a major reason for my timely completion. I owe a large portion of my knowledge about game theory, research methods, and research writing to him, and I feel honored to have him as my advisor. I am also very grateful to the other members of my committee for their helpful feedback. They, and the other professors who taught me in class, transmitted a passion for economics and research and exemplified professional excellence.

I owe a debt of gratitude to my classmates in the Krannert PhD cohort of 2019, my officemates, and the older PhD students throughout Krannert, but especially to those in the economics department. We started out as companions on a long journey, and many have become close friends. I also greatly enjoyed my time serving in the Krannert Doctoral Students Association, and as president I benefited tremendously from the relationships and efforts of numerous people, especially my fellow officers, but also our faculty advisors and the members of the Krannert Graduate Programs Office.

My family and I benefited from our relationship with the officers and leaders of the Purdue Military Research Initiative. The leaders and organizers of PMRI created an environment that is extremely welcoming to new military students. I have greatly enjoyed my association with them, their leadership example, and the support network PMRI provides to all active duty students regardless of funding source. On a related note, I thank the overall Purdue community, faculty, staff, and administration for their generous and continuing support of military students and veterans. Purdue's steadfast support to their military students was a primary reason I decided to enroll.

As the Air Force Institute of Technology Liaison Officer to Purdue, I benefited from the leadership examples and guidance of my predecessors in the role, as well as the ROTC Det 220 commander. I was honored to serve with excellent assistant liaison officers, and I admire their proactive efforts to look after the junior officers placed under them. I am also grateful to them for taking a burden off of me as I finished my research and degree.

I am grateful to the Department of Economics and Geosciences at the U.S. Air Force Academy for sponsoring my degree, as well as my supervisors at the Air Force Institute of Technology Civilian Institutions program. They go to great efforts to take care of the students assigned to them, and they have helped me numerous times. My assignment team at the Air Force Personnel Center and the senior leaders of the Air Force 15A career field have been very supportive of my desire to complete a PhD, and they have facilitated my future career progression in ways that I could only hope for when I began my program.

I extend my thanks to our many friends and neighbors which have made this endeavor easier for both me and my family. A few of these include classmates from the Krannert masters programs, loyal neighbors, and several people from church.

Wisdom is the principal thing; therefore get wisdom: and with all thy getting get understanding.

Proverbs 4:7

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ABSTRACT

Contest outcomes often involve some mix of skill and chance. In three essays, I vary the sources of noise and show how player actions either influence, or are influenced by, noise. I begin with a classic multi-battle contest, the Colonel Blotto game. Due to his disadvantage in resources, the weak player in this contest stochastically distributes resources to a subset of battlefields while neglecting all others in an attempt to achieve a positive payoff. In contrast, the strong player evenly distributes his resources in order to defend all battlefields, while randomly assigning extra resources to some. Because the weak player benefits from randomizing over larger numbers of battlefields, a strong player has incentive to decrease the range over which the weak player can randomize. When battlefields are exogenously partitioned into subsets, or *fronts*, he is able to do this by decentralizing his forces to each front in a stage prior to the distribution of forces to battlefields and actual conflict. These allocations are permanent, and each subset of battlefields effectively becomes its own, independent Blotto subgame. I show that there exist parameter regions in which the strong player's unique equilibrium payoffs with decentralization are strictly higher than the unique equilibrium payoffs without decentralization.

In my second paper, I show how sources of exogenous noise, what Clausewitz referred to as the "fog of war," obscure developments on the battlefield from the view of a military leader, while individual inexperience and lack of expertise in a particular situation influence his decisionmaking. I model both forms of uncertainty using the decentralized Colonel Blotto game from the first chapter. To do so, I first test the robustness of allocation-stage subgame perfect equilibria by changing the contest success function to a lottery, then I find the players' quantal response equilibria (QRE) to show how individual decision-making is impacted by bounded rationality and noisy best responses, represented by a range of ψ values in the logit QRE. I find that player actions rely significantly less on decentralization strategies under the lottery CSF compared to the case of the all-pay auction, owing mainly to the increased exogenous noise. Moreover, agent QRE and heterogeneous QRE approximate subgame perfect equilibria for high values of ψ in the case of an all-pay auction, but under

the lottery CSF, QRE is largely unresponsive to changes in ψ due to the increase in exogenous noise.

Finally, I examine a potential method for introducing noise into the all-pay auction (APA) contest success function (CSF) utilized in the Colonel Blotto games of the first two chapters. Many contests are fundamentally structured as APA, yet there is a tendency in the empirical literature to utilize a lottery CSF when stochastic outcomes are possible or the tractability of pure strategy equilibria is desired. However, previous literature has shown that using a lottery CSF sacrifices multiple distinguishing characteristics of the APA, such as the mixed strategy equilibria described by Baye et al. (1996), the *exclusion principle* of Baye et al. (1993), and the *caps on lobbying principle* of Che and Gale (1998). I overcome this by formulating an APA that incorporates noise and retains the defining characteristics of an auction by forming a convex combination of the APA and fair lottery with the risk parameter λ. I prove that equilibria hold by following the proofs of Baye et al. (1996), Baye et al. (1993), and Che and Gale (1998), and I show the new CSF satisfies the axioms of Skaperdas (1996). While player and auctioneer actions, payments, and revenues in the noisy APA adhere closely to the those of the APA for low levels of noise, the effect of discounted expected payoffs results in lower aggregate payments and payoffs when noise is high. Finally, I show the noisy APA is only noise equivalent to the lottery CSF when $\lambda = 0$, i.e., the fair lottery.

1. DECENTRALIZATION IN THE COLONEL BLOTTO GAME[1]

1.1 Introduction

Decentralization has been an an important element in our understanding of economic systems at least since Adam Smith, who used the notion to describe the interactions of numerous independent consumers and firms acting in their own best interest, as opposed to a centralized economy reliant on the wisdom and beneficence of a powerful and authoritative leader. In the realm of contests and conflict decentralization plays a similarly important role. Consider for example the military leader who delegates to lower-level commanders. When this delegation also includes the authority to act autonomously, command is decentralized, and like market systems, contest participants may benefit from the ability to decentralize control to subordinate units.

We define decentralization broadly to be the distribution or delegation of decision-making power away from a central authority or authoritative body to multiple independent, subordinate decision-makers. As illustration, historical examples of decentralization in conflict are plentiful. Alexander the Great is known to have had named officers who led their own units in his conquest of Asia. Likewise, Roman legions were led by a hierarchy of officers, principally the *legati*, who were responsible for maintaining order and military conquest within a certain region of the empire. More recently, in the 16th through 20th centuries, military and naval expeditions were dispatched around the world with very general orders to obtain some particular objective, without specific direction as to how it should be achieved, thus exhibiting a reliance on the subordinate commander's judgment and initiative.[2] Likewise,

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[2]↑For example, consider the orders from Brigadier General Alfred E. Terry to Lieutenant Colonel George A. Custer prior to the Battle of the Little Bighorn. Custer is instructed by Gen Terry's aide to "...proceed up the Rosebud [River] in pursuit of the Indians whose trail was discovered... a few days since. It is, impossible to give you any definite instructions in regard to this movement, and were it not impossible to do so the Department Commander places too much confidence in your zeal, energy, and ability to wish to impose upon you precise orders which might hamper your action when nearly in contact with the enemy. He will, however, indicate to you his own views of what your action should be, and he desires that you should conform to them unless you shall see sufficient reason for departing from them" (Merkel, 2022). Notwithstanding Custer's climactic defeat after receiving them, the orders provide a detailed illustration of decentralization from a

in WWII and even more modern conflicts, when distance or security concerns precluded the use of close communications, forces separated from a main body have had to rely on a set of general orders requiring them to achieve certain objectives while granting the subordinate commander flexibility in the accomplishment of the task.

Modern militaries, including that of the United States, emphasize the importance of centralized planning and decentralized execution, which allows for the accomplishment of strategic objectives while permitting flexibility to the forces responsible for the execution. This approach relies on the initiative, judgment, and discipline of the responsible forces, and it requires clear instructions in the form of a written *commander's intent* and *rules of engagement*. According to U.S. military doctrine,

> Mission command is the conduct of military operations through decentralized execution based upon mission-type orders. It empowers individuals to exercise judgment in how they carry out their assigned tasks and it exploits the human element in joint operations, emphasizing trust, force of will, initiative, judgment, and creativity. Successful mission command demands that subordinate leaders at all echelons exercise disciplined initiative and act aggressively and independently to accomplish the mission. They focus their orders on the purpose of the operation rather than on the details of how to perform assigned tasks. They delegate decisions to subordinates wherever possible, which minimizes detailed control and empowers subordinates' initiative to make decisions based on understanding what the commander wants rather than on constant communications. Essential to mission command is the thorough understanding of the commander's intent at every level of command and a command climate of mutual trust and understanding. (Joint Chiefs of Staff, 2017)

Hence, by communicating and adhering to a commander's intent, unity of command is preserved, but subordinates can act autonomously with speed and responsiveness as circumstances dictate, thus achieving maximum possible effectiveness.

higher authority in regards to military tasks, and they are representative of thousands of other orders like them throughout military history.

Given the prevalence of decentralization in historical and modern conflicts, it is important to include decentralization in our theoretical models of competition, including the Colonel Blotto game. To do this, we begin by acknowledging that contests involve some combination of skill and chance, or alternatively, resources and strategic uncertainty. Moreover, as shown in previous Colonel Blotto literature, it is possible for players in these contests to endogenously alter the combination of skill and chance to achieve a higher payoff. For instance, Roberson (2006) proves the Blotto game has an equilibrium in mixed strategies whereby a weaker player relies on chance for a higher payoff. To see this, recall that this version of the game employs an all-pay auction contest success function, so battlefields are won deterministically by the player who assigns a higher level of resources to them. Yet, if player budget endowments are asymmetric and continuous, an even distribution of force to every battlefield by both players would result in a victory for the strong player and a loss to the weak player on every battlefield. To avoid this, in equilibrium the weak player relies on a stochastic allocation of resources in which his forces are concentrated on a randomly-selected subset of battlefields while neglecting all others, thus securing at least a few victories rather than none. This reliance on chance leads to significant uncertainty for the strong player who is forced to defend all battlefields, while also concentrating additional mass on a randomly-selected subset of battlefields in hopes of intercepting the weak player's attacks somewhere. For a range of player endowments, this strategy guarantees the weak player a higher payoff than an even distribution, even while the weak player's gains are mitigated in part by the strong player's defensive efforts. Thus, although contest outcomes are decided deterministically, the role of chance is endogenous, the weak player benefits from increasing uncertainty around his distribution of force, and the strong player attempts to mitigate the role of chance as much as possible.

This leads us to the phenomenon of decentralization in conflicts. In the constant-sum Colonel Blotto game, if the weak player relies on chance by randomly assigning forces to some battlefields, the strong player has incentive to decrease the role of chance by reducing the number of battlefields over which the weak player can randomize. And, by limiting the range of weak player movement, the strong player forces competition to take place based on the players' relative strengths where he is superior. He does this by effectively breaking the larger conflict into smaller component conflicts along exogenously-defined boundaries, then dividing his forces across these newly-formed "fronts." To each division of force he appoints an independent, subordinate commander who is responsible for further decisions about resource allocations to the individual battlefields within his front. To aid his decision-making, the subordinate commander also acquires intelligence regarding the disposition of weak player forces within his particular sector. If the subordinate commander has reliable information regarding enemy forces within his front, he can execute an optimal response. Thus, while decentralization is a logical response to unpredictable threats posed by a weak adversary, the effectiveness of a decentralization strategy hinges on the presence of quality intelligence. Moreover, if we assume this information is an automatic byproduct of decentralization due the increased focus that decentralization puts on a particular sector of conflict, the contest outcomes in the sector rely less on chance and the strong player achieves relatively more victories as a result of decentralization alone.

Since decentralization leads to more battlefield victories and higher payoffs for the strong player, it consequently results in lower payoffs to the weak player. This means the weak player must respond to the strong player's decentralization efforts as best he can. Because he cannot control the strong player's choice to decentralize, the weak player must operate within the fronts established by the strong player, and he does this by first allocating forces to at least some of the fronts, then randomly distributing forces to the battlefields in those fronts prior to armed conflict. In other words, after the strong player has decentralized, the weak player can no longer operate as if the fronts did not exist, and his own allocations to the fronts become critical to his attacks on battlefields. Also, intelligence about the strong player's allocation of force is important to the weak player, but this information is more readily available due to the strong player's actions to decentralize (e.g., troop movements, patrols, checkpoints, fortifications around government buildings and sensitive targets, etc.), as well as the weak player's collection efforts. Ultimately, knowledge of the strong player's allocations to the fronts informs the weak player's subsequent decisions about where to attack and with what level of force. Finally, if we assume, as in the case of the strong player, that the weak player has quality information as an automatic byproduct of decentralization, we can examine the effects of decentralization alone. This permits us to analyze the role played by decentralization and evaluate the benefit accruing to a strong player that chooses to decentralize his forces away from an authoritative leader.

To picture the concept of decentralization in more vivid terms, consider the following scenario which we use as a foundation for our model: A weak insurgent force confronts a resource-rich government opponent. Seeing that direct confrontation across all battlefields would end in disaster, the insurgents use a guerrilla warfare strategy that concentrates their forces in a subset of battlefields. To the government, the insurgents' choice of battlefields appears to be random and unpredictable, but also potentially damaging. Hence, the government responds by decentralizing its forces and commanders to be more responsive to detected threats. It establishes security sectors throughout the country with armed checkpoints along routes of travel, and each sector is diligently surveilled by military and police forces for insurgent activity, while quick reaction forces are staged in each sector to respond immediately to terrorist attacks or promising intelligence. While it is impossible for the government to conceal its force posture in each security sector, its surveillance activity reveals the presence of insurgent forces in a given sector. Thus, both the insurgents and government force have reliable information regarding the strength of each other's forces in each security sector, and subsequent fighting over specific targets will be conditioned on this knowledge. Insurgent forces will choose a target to attack, and government forces will attempt to defend the site. The side with the higher level of dedicated resources will win any particular battle.

Transitioning this scenario to a model, we demonstrate that the process of decentralization is critical to the players' equilibrium payoffs primarily because of the information that can be obtained by the players about each other's force allocation across fronts. Suppose there are two distinct models: the *centralized game*, or the Colonel Blotto game without decentralization, and the *decentralized game*. Then, to introduce some of our notation for the model, in the centralized game there exists a set of n homogeneously-valued battlefields which we assume are exogenously partitioned into μ symmetrically-sized subsets, or fronts. The players' distribution of force to the battlefields can be visualized as taking place in two stages and four substages. In the first stage, both players S and W simultaneously allocate their budgets X^S and X^W, to one or more fronts, i $\in \{1, \ldots, \mu\}$. Then in the second stage, they simultaneously distribute these forces further to one or more battlefields within each front, x_{ij}^S and x_{ij}^W for $\mathrm{j} \in \{1, \ldots, n_{\mathrm{i}}\}$. At the end of stage two, the players' force distributions to each battlefield are revealed, and the player with a higher level of force on any particular battlefield wins that battlefield with probability one. However, because the players' forces act under centralized authority within each front, they lack local knowledge of their opponent's allocation of force in stage one, and the overall game appears to the players as if it consisted of a single stage of simultaneous moves. As a result, player allocations to fronts, X_{i}^S and X_{i}^W, and distributions to battlefields, x_{ij}^S and x_{ij}^W, are made as if there existed only a single front, the weak player appears to randomize over the entire set of battlefields without regard to fronts, and the strong player appears to defend the entire set of battlefields, as described previously. This is depicted in Figure 1.1.

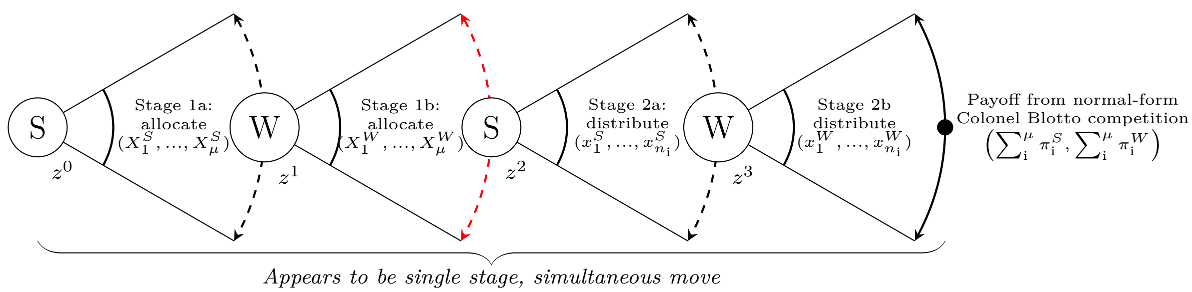


Figure 1.1. The centralized game is extensive-form with imperfect information at every stage. At node z^0, the strong player begins by allocating a tuple of force over a continuous range of μ exogenously-created fronts. At z^1, the weak player similarly allocates a tuple of force over the range of fronts without knowledge of the strong player's move. Then without observing their opponent's stage one allocation (as depicted by the red dashed arc), each player makes an unobservable distribution of force over a continuous range of n battlefields within each front at nodes z^2 and z^3. Note that because the players lack information regarding stage 1a and 1b allocations, the centralized game is functionally equivalent to the single-stage, simultaneous-move Colonel Blotto game.

In the decentralized game, such as in the example of insurgents versus government forces, there exists the same two stages and four substages, but now first-stage player allocations, X_{i}^S and X_{i}^W, become common knowledge. The strong player allocates forces to each front in order to establish security sectors, but these forces are visible to the insurgents. Likewise, any allocation to a front by the insurgents will be discovered through government surveillance efforts. Hence, at the end of stage one, player force allocations are known and second-stage attacks via government quick reaction forces and insurgent guerrillas and are made conditional on the stage one allocations. In this environment, information plays a critical role, and in equilibrium, player allocations in stage one help determine stage two payoffs. This is depicted in Figure 1.2

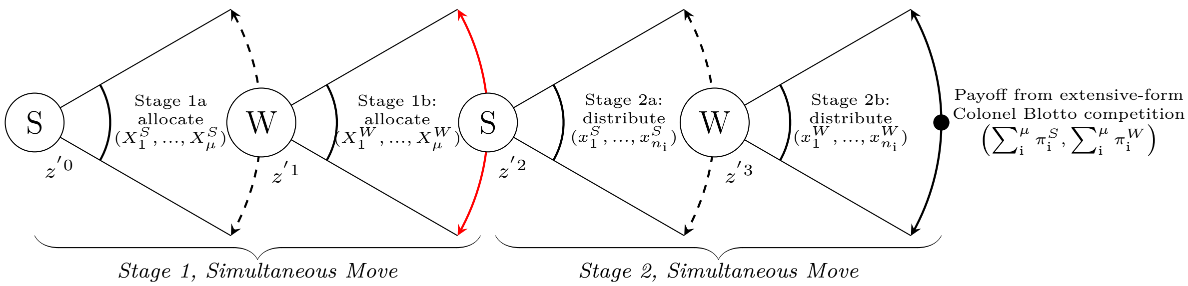


Figure 1.2. The decentralized game is extensive-form with information regarding player allocations at the end of stage one. At node z'^0, the strong player begins by allocating a tuple of force over a continuous range of μ exogenously-created fronts. At z'^1, the weak player similarly allocates a tuple of force over a continuous range of fronts without knowledge of the strong player's move. Each player then observes their opponent's stage one allocation (as indicated by the solid red arc) and makes an unobservable distribution of force to a set of n_i battlefields within each front at z'^2 and z'^3. Because the players have information regarding allocations at z'^0 and z'^1, they condition their stage two distributions on these allocations, and the outcomes of game B will be distinct from those obtained in game A of Figure 1.1.

Note that in the centralized extensive-form game of Figure 1.1, each of the four substages comprise an information set. At z^0, the strong player allocates forces to some combination of μ fronts, denoted by the tuple $(X_1^S, \ldots, X_\mu^S)$, but because this allocation is not observable by the weak player, the weak player only has a single information set. That is, at z^1 he only knows the strong player made an allocation, but he does not know the strong player's μ-tuple, which exists along a μ-dimensional continuum portrayed by the red, dashed curve. Similarly, at z^2, the strong player knows his own stage one allocation, but he does not know the μ-tuple of the weak player and hence at z^2 has one information set for each of their possible stage-one allocations. In contrast, in Figure 1.2, there exists at node z'^2 one information set for each possible pair of stage-one allocations since player allocations to the fronts are revealed, and each player knows his own and his opponent's respective first-stage μ-tuple. This is portrayed by the solid red curve. This information critically influences the distribution of force in stage two and player payoffs at the end of the game. Hence, both players will have subgame perfect equilibrium allocation-stage strategies that may differ from those of the traditional Blotto game portrayed in Figure 1.1. We will discuss this further when we analyze the model.

1.1.1 Literature Review

Since Borel (1921), many variations and applications of the Colonel Blotto game have been introduced. These have ranged from changing the number of players to altering the types, quantities, and values of endowed resources and battlefields. The timing of moves has also been compared using simultaneous and sequential games. While outcomes in the Blotto game are typically decided the auction contest success function, some authors have showed robustness using a lottery CSF; and acknowledging that Colonel Blotto competition implies a system of interconnected battlefields, there is also a large and growing literature on the game as it applies to the attack and defense of networks. We will discuss each of these developments in the paragraphs that follow.

Although slow in its initial evolution, research involving Colonel Blotto competition has advanced rapidly in recent decades. Borel and Ville (1938) solved the game when there are two players, three battlefields, symmetric resources, and an all-pay auction CSF. Gross and Wagner (1950) extended Borel & Ville's results to any finite number of battlefields, while Friedman (1958) partially characterized the solution for n-battlefields and asymmetric resources using the auction CSF, and fully solved the game using a lottery CSF. A. R. Robson (2005) expanded on Friedman's results by testing lottery CSFs for a range of parameters. Roberson (2006) then provided a full characterization of the game with $n \geq 3$ battlefields, asymmetric resources, and an auction CSF, while Macdonell and Mastronardi (2015) solved the game for $n = 2$ battlefields.

In regards to conflict technology, multiple CSFs have been devised, in part to understand rent-seeking behavior, but the findings relate directly to Colonel Blotto contests as well.

Tullock (1980) generalized the ratio-form CSF, which includes both lottery and auction CSFs, by introducing a return-to-effort parameter, r. This parameter dictates the amount of noise or randomness in the system, which directly affects each player's probability of win based on the amount of resources invested. Baye et al. (1994) showed that while players use pure strategies for values of r less than or equal to 2, mixed strategies are used for r-values greater than 2. Ewerhart (2017) then showed that contests with r greater than 2 are payoff-equivalent to a first-price, all-pay auction, or $r = \infty$.

There is also a large literature on resource allocation games in relation to network defense, which is particularly relevant here because CSFs determine the nature of network linkages in a conflict across multiple battlefields. Recent work in this area includes Clark and Konrad (2007) which examines the attack and defense of a weakest-link/best-shot network using a lottery CSF (i.e., a ratio-form CSF with $r = 1$). Their approach results in the interesting finding that a defender and attacker equally disperse their resources across all nodes of the network, but the defender's probability of fighting decreases as the number of fronts increases, even if the defender enjoys a local advantage on each front. Kovenock and Roberson (2018) also model the defense of a weakest-link/best-shot network, but they show equilibrium behavior under an auction CSF. They find that when a weakest-link network is attacked, only one node is targeted, whereas every node in a best-shot network is optimally attacked with positive force levels. In a related vein, Powell (2007) gives the optimal allocation of defense resources when confronting a single, determined terrorist attacker, but his findings are also applied to reducing the threat of damage from non-strategic risk. Dziubiński and Goyal (2017) takes this further by analyzing network defense in the context of specific structures and key nodes, while Haller and Hoyer (2019) look at the pure strategies of group members who are able to build costly links when facing a common enemy trying to disrupt their network.

Dziubiński and Goyal (2017) and Haller and Hoyer (2019) also feature a decentralized defense of a network, but whereas Dziubiński and Goyal examine the optimal strategy of a single defender against an attacker, Haller and Hoyer look at the behavior of a group. This is similar to the network of agents described by Acemoglu et al. (2016), wherein the network experiences cascading failures depending on the type of attack and the investment of resources by the defenders. Because a player's defense preparations affect the vulnerability of his neighbors, externalities influence every player's investment. This is closely related to Kovenock and Roberson (2012) which considers the conditions under which two allies voluntarily share resources when confronting a common adversary on two independent fronts. This is a decentralized defense of two fronts by two players, and the level of resources shared depends on the player's endowments and the nature of any agreements the allies have regarding the *ex post* division of payoffs.

In addition to the literature on network complementarities, is the idea that network formation can be endogenous or exogenous. For example, Jackson and Wolinsky (1996) and Bala and Goyal (2000) look at costly network formation between players, especially in a social network context. Dziubiński and Goyal (2013) and Goyal and Vigier (2014) expand on this by using a centralized model of network formation and defense. A designer builds a network and can establish linkages at a cost, then allocates resources to defend some of the nodes. The attacker likewise invests resources to eliminate selected nodes. Subgame perfect equilibria depend on the costs of defense and linkages, and Goyal and Vigier (2014) find specific network configurations which provide the optimal defense. These approaches relate directly to our model of a decentralized Colonel Blotto game where the strong player has centralized control of defense, but the network structure is determined exogenously.

As a strategic-form game, the basic Colonel Blotto game exhibits simultaneous resource allocation and competition. Yet, when considering the preparations of players competing on exogenously-defined fronts, it is useful to model these moves sequentially and find the resulting subgame perfect equilibrium. This is precisely the approach taken by Kovenock and Roberson (2012) and Kovenock et al. (2010). In these multi-stage games, simultaneous Colonel Blotto competition is preceded by a period of pre-conflict resource allocation, and the authors find equilibria for these stages, while relying on Roberson (2006) for the final-stage Colonel Blotto outcomes. This approach contrasts with Clark and Konrad (2007), whose game structure is similar to that of Kovenock and Roberson (2012), except that player allocation to fronts is part of the simultaneous competition. Likewise, Kovenock and Roberson (2018) examine the defense and attack of weakest-link/best-shot networks as a simultaneous-move, rather than a sequential-move game.

From a theoretical perspective, our research is most closely related to Kovenock and Roberson (2012), who show that in a Colonel Blotto game with asymmetric resource endowments, when two allies face a common enemy they will engage in unilateral resource transfers despite lacking common interests. This "enemy-of-my-enemy is my friend" phenomenon occurs in the form of *ex ante* transfers without any agreement between the allies for *ex post* sharing of rewards, and the results hold for a wide variety of parameters. However, as a benchmark case, they also show that when an *ex post* agreement is in place, the two allies will act as a single entity, sharing resources between themselves so as to maximize their total joint payoff, which results in a "no soft-spot" stance versus the common enemy. The allies still compete in two separate Colonel Blotto games, or "fronts," but their resources are pre-allocated to ensure a best-response across both fronts. In short, the game described by Kovenock and Roberson (2012) is a sequential Colonel Blotto game with multiple defenders, decentralized defense and centralized attack, and an exogenously-defined, general network of battlefields. Competition is decided by an auction CSF.

Our work builds on the results of Kovenock and Roberson (2012) by showing in a two-stage game that a strong player's decision to decentralize by allocating resources to exogenously-defined fronts prior to a conflict can improve his final-stage payoffs. In other words, the strong player can best address the known threat of a weaker opponent by decentralizing his forces across multiple fronts, the action of which forces the weak player to also allocate their resources to fronts. In stage two, Colonel Blotto competition determines the players' payoff in each of the fronts. The game is characterized as one of complete information and recall, and second-stage payoffs are determined by an all-pay auction.

In the sections that follow, we formally introduce the decentralized Colonel Blotto game and describe characteristics of the players and their strategies. We then solve the game by identifying the subgame perfect equilibria for allocations by each player and determine when decentralization results in a higher payoff for the strong player.

1.2 Model

To give a more complete description of the extensive-form game, we introduce the following notation. There are two players, $\{S, W\}$, with endowments $X^S \geq X^W > 0$ of some one-dimensional and continuous resource, and we normalize these budgets so that $X^W = 1$. There also exists an original set of $n \geq 6$ battlefields, exogenously partitioned into $\mu \geq 2$ symmetrically-sized fronts of $n_{\mathrm{i}} = \frac{n}{\mu} \geq 3$ battlefields each. Let each battlefield $\mathrm{j} \in \{1, \ldots, n_{\mathrm{i}}\}$ in front i be homogeneous with an equal payoff of $v \equiv v_{\mathrm{j}} = \frac{1}{n} > 0$ for each player, so that the value of front i is equal to $\phi_{\mathrm{i}} \equiv n_{\mathrm{i}} v = \frac{n_{\mathrm{i}}}{n} = \frac{1}{\mu}$ for both players. The total value of all fronts is therefore $\Phi = \sum_{\mathrm{i}=1}^{\mu} \phi_{\mathrm{i}} = 1$. The players have perfect recall and each player's budget, the number of battlefields and fronts, and the value of each battlefield and front are all common knowledge.

There are two versions of the game, the centralized and the decentralized, and the strong player chooses which game will be played in a pre-game stage we do not include in our model. We will discuss each in detail, but both versions of the game proceed through two stages. In the first stage, the players simultaneously allocate their forces, X^S and X^W, across the μ fronts. This results in a first-stage (or alternatively, an "allocation-stage") μ-tuple of $(X_1^k, \ldots, X_\mu^k)$ for each player $k \in \{S, W\}$. Each front then becomes a Blotto subgame, $G_{\mathrm{i}}(X_{\mathrm{i}}^S, X_{\mathrm{i}}^W)$ for $\mathrm{i} \in \{1, \ldots, \mu\}$. In the second stage of both versions of the game, the players simultaneously distribute the forces they allocated to each front among the battlefields of that front. This means that player k's budget for front i, X_{i}^k, is distributed across the n_{i} battlefields of front i, resulting in the n_{i}-tuple $(x_{\mathrm{i},1}^k, \ldots, x_{\mathrm{i},n_{\mathrm{i}}}^k)$ for player k.

The force allocated to each front and battlefield by the players must be non-negative and unspent resources have no value. We also assume the cost to allocate and distribute forces is zero and that players are risk-neutral and payoff maximizing. Figure 1.3 illustrates the first and second (allocation and distribution) stages of the game described, including the n battlefields partitioned into μ fronts, for both the centralized and decentralized versions of the game. We will discuss differences of the two versions next.

In the centralized game, we suppose that allocation and distribution decisions in each stage are made by a central authority rather than by independent subordinate commanders.

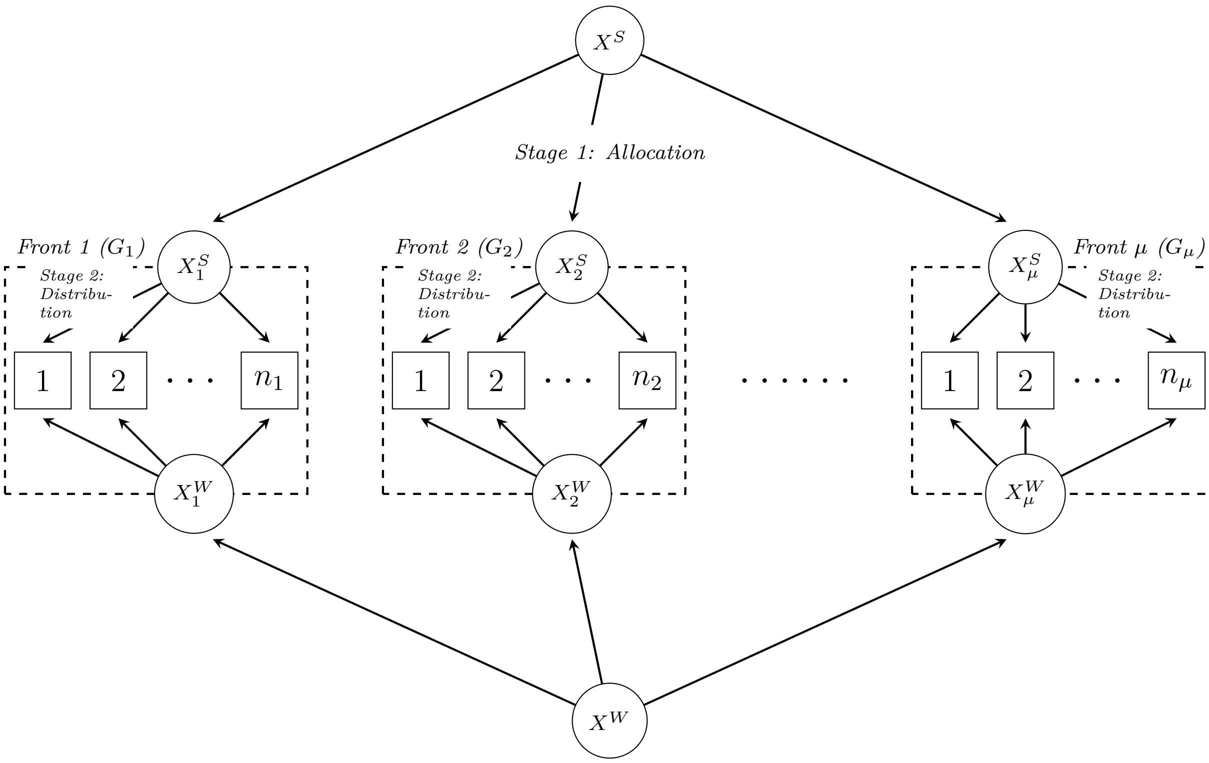


Figure 1.3. Stages of the centralized and decentralized games. Players S and W begin with endowed budgets X^S and X^W respectively, and n battlefields are exogenously partitioned into μ symmetric subsets so that there are ${}^{n}/_{\mu}$ battlefields in each subset. In stage one, each player $k \in \{S, W\}$ allocates $X_{\mathrm{i}}^k \geq 0$ resources to each front G_{i}, $\mathrm{i} \in \{1, \ldots, \mu\}$. In stage two, players distribute the forces available to them on each front, X_{i}^k, to one or more of the n_{i} battlefields in front i. At the end of stage two, force distributions to each battlefield are revealed and the player with the higher number of forces assigned to a battlefield wins that battlefield with probability one. Ties are awarded to player S.

An exogenous partition of battlefields exists, so any distribution of force to the battlefields must be preceded by some allocation to the fronts, but this allocation is made with a focus on the final distribution of force to the n battlefields overall rather than maximizing payoffs in each front individually. In other words, while an allocation to fronts is required, forces assigned to a front do not localize to that front. They do not establish fortifications, conduct patrols, or surveil the area in ways that would reveal a player's position to their opponent or provide information regarding the presence of the other player on that front. Their strategy spans the full range of battlefields and fronts. This means that first-stage player allocations in the centralized game are unobservable to opposing players, and when second-stage distributions are made to the battlefields, they are not conditioned on the opponent's first-stage allocations. Both players can unobservably randomize their placement of forces in either stage one and/or stage two, which has the practical effect of making the centralized game appear to an outside observer as if it were a single-stage, simultaneous-move game.

In contrast, the decentralized game is characterized by a localization of forces to each front. The strong player makes allocation decisions to the fronts in stage one, but this action is accompanied by a delegation of authority to local commanders on each front for subsequent distribution decisions in stage two. The local commander on each front is then able to conduct those actions that discover the presence and strength of their opponent, which also reveals their own presence. Then when forces are distributed to battlefields in a front, the action is conditioned on the allocated strength of both players' forces. In the subgame perfect equilibrium, the availability of this information plays a critical role. In the decentralized version of the game, both players know that allocation decisions will become common knowledge at the end of stage one and influence the outcomes of stage two. This leads each player to choose an allocation strategy that anticipates decentralization in order to achieve the highest payoff possible at the end of the game. Hence, the information available to the players in the decentralized game is fundamentally different than that of the centralized game, and this changes the nature of the contest, its potential payoffs, and ultimately, the players' equilibrium allocation strategies in stage one.

With this context, we now define each version of the game more explicitly. Let $\mathfrak{X}^k$ define the set of second-stage information sets for player k, which is based entirely on the set of player k's own possible first-stage allocations. Then let $\Gamma^C(\{G_{\mathrm{i}}\}_{\mathrm{i}=1}^{\mu}, \{\mathfrak{X}^k, X^k\}_{k\in\{S,W\}})$ denote the overall centralized game composed of the set of subgames, $\{G_1, \ldots, G_\mu\}$, the set of second-stage information sets for each player k in the centralized game, and the players' budgets, X^S and X^W. Henceforth, we refer to the centralized game notationally as Γ^C. Similarly, for the decentralized game the set of possible second-stage information sets is $\mathfrak{X}^S \times \mathfrak{X}^W$, where an element of $\mathfrak{X}^S \times \mathfrak{X}^W$ corresponds to an information set of the decentralized game. Then let the decentralized game be denoted as $\Gamma^D(\{G_{\mathrm{i}}\}_{\mathrm{i}=1}^{\mu}, \mathfrak{X}^S \times \mathfrak{X}^W, \{X^k\}_{k\in\{S,W\}})$, henceforth referred to notationally as Γ^D. We will discuss player strategies at each stage of the game in the next section.

1.2.1 Strategies

We begin by outlining the first and second-stage strategies for the players in the centralized game, then we present them for the decentralized game. Let $X_{\mathrm{i}}^S \in [0, X^S]$ be S's first-stage force allocation to G_{i} such that $\sum_{\mathrm{i}=1}^{\mu} X_{\mathrm{i}}^S = X^S$, and let $X_{\mathrm{i}}^W \in [0, X^W]$ be W's force allocation such that $\sum_{\mathrm{i}=1}^{\mu} X_{\mathrm{i}}^W = X^W$ and $X^S \geq X^W$. Then recall that in an extensive-form game, a strategy of player $k \in \{S, W\}$ is a function from each of his information sets to the set of actions available at that information set. So, each player's strategy specifies a first-stage allocation across fronts, and for each of a player's own possible allocations of resources across fronts, a second-stage allocation of the front i resources across the battlefields in front i.

Then for the centralized game and $k \in \{S, W\}$, in the allocation-stage local subgame there is for k's budget, X^k, a probability distribution over the set of k's feasible μ-tuples in the overall Blotto game Γ^C, which we label an *allocation of force* for player k. This is a μ-variate joint distribution function, $P^k : \mathbb{R}_+^{\mu} \to [0, 1]$, with support contained in k's budget, $\mathfrak{X}^k = \{\mathbf{X} \in \mathbb{R}_+^{\mu} | \sum_{\mathrm{i}=1}^{\mu} X_{\mathrm{i}}^k \leq X^k\}$. Then in stage two, there is for each allocated tuple $(X_1^k, \ldots, X_\mu^k)$, a probability distribution over the set of k's feasible n_{i}-tuples in each Blotto subgame G_{i} which we label a *distribution of force* for player k. This is an n_{i}-variate distribution function, $P_{\mathrm{i}}^k : \mathbb{R}_+^{n_{\mathrm{i}}} \to [0, 1]$, with support contained in the set of k's feasible allocations of force, $\widehat{\mathfrak{X}}_{\mathrm{i}}^k = \{\mathbf{x} \in \mathbb{R}_+^{n_{\mathrm{i}}} | \sum_{\mathrm{j}=1}^{n_{\mathrm{i}}} x_{\mathrm{ij}}^k \leq X_{\mathrm{i}}^k\}$, and with a set of univariate marginal distribution functions $\{F^k_{\mathrm{j}}\}^{n_{\mathrm{i}}}_{\mathrm{j}=1}$, one univariate marginal distribution function for each battlefield in the Blotto subgame G_{i}. Hence, in the centralized game players allocate forces to fronts before distributing those forces to battlefields, but both allocations to fronts and distributions to battlefields take place without knowledge of the opponent's allocation of force across fronts.

In the decentralized game, players allocate forces to fronts in stage one according to a behavioral strategy,[3] then condition their second-stage distribution of force on both players' first-stage allocations. So, for player k there is a specification of feasible $\{X^k_{\mathrm{i}}\}^{\mu}_{\mathrm{i}=1}$ in the first stage that we label an *allocation of force*; and in the second stage when forces are distributed to individual battlefields, there is for each allocated tuple $(\{X^S_{\mathrm{i}}, X^W_{\mathrm{i}}\}^{\mu}_{\mathrm{i}=1})$, a probability distribution over the set of k's feasible n_{i}-tuples in each Blotto subgame G_{i}, which we again label a *distribution of force*. This is an n_{i}-variate distribution function, $P^k_{\mathrm{i}} : \mathbb{R}^{n_{\mathrm{i}}}_{+} \to [0, 1]$, with support contained in the set of k's feasible allocations of force, $\widehat{\mathfrak{X}}^k_{\mathrm{i}} = \{\mathbf{x} \in \mathbb{R}^{n_{\mathrm{i}}}_{+} | \sum^{n_{\mathrm{i}}}_{\mathrm{j}=1} x^k_{\mathrm{ij}} \leq X^k_{\mathrm{i}}\}$, and with a set of univariate marginal distribution functions $\{F^k_{\mathrm{j}}\}^{n_{\mathrm{i}}}_{\mathrm{j}=1}$, one univariate marginal distribution function for each battlefield in Blotto subgame G_{i}.

In both versions, the game is constant-sum, and for each battlefield, the player that distributes a higher level of force to that battlefield wins it with certainty. In the case that

[3] Because the decentralized Colonel Blotto game is an extensive-form game with payoffs determined by player actions in each of two stages, it is useful to recall the following definitions regarding player strategies in sequential-move games.

Definition 1.2.1. *(Extensive-Form Strategies) A strategy of player k is a function from each of his information sets to the set of actions available at that information set, i.e.,*

$$s_k : \mathcal{U}_k \to \bigcup_{\mathrm{j}=1}^{J_k} A(U^{\mathrm{j}}_k)$$

where $\mathcal{U}_k = \{U^1_k, \ldots, U^{J_k}_k\}$ *is the collection of player k's information sets, and for each* $U^{\mathrm{j}}_k \in \mathcal{U}_k$

$$s_k(U^{\mathrm{j}}_k) \in A(U^{\mathrm{j}}_k).$$

Note that there are two ways players use mixing in extensive form games. The first is mixed strategies, $\sigma = \{\sigma_k\}_{k\in N}$, or probability distributions over sets of pure strategies. The second is behavioral strategies, $b = \{b_k\}_{k\in N}$, or functions mapping each of a player's information sets to a probability distribution over the set of possible actions at the information sets. Thus, behavioral strategies are independent at each node but mixed strategies identify a player's complete course of action before play begins. Given the equivalence of mixed and behavior strategies in games of perfect recall, the analysis here will be in terms of behavior strategies.

the players assign the same level of force to a given battlefield, the strong player wins with probability one. The specification of this tie-breaking rule does not affect the results of the contest as long as no player has less than $\frac{2}{n_{\mathrm{i}}}$ times the forces of their opponent in a subgame G_{i} (or in the case of the centralized game, $\frac{2}{n}$ times the forces of their opponent in game Γ^C). In the case that this condition does apply, the specification of this tie-breaking rule avoids the need to have the strong player allocate a level of force that is arbitrarily close to, but above, the weak player's maximal allocation of force. A range of tie-breaking rules yields similar results.

1.3 Analysis

We begin by introducing second-stage payoffs, which we call the *Botto CSF*, because they serve as a useful tool for analyzing allocation-stage local equilibrium. These are taken as given based on first-stage allocations and they apply to both versions of the game in a related manner because fronts are exogenously defined. For the centralized game, the Blotto CSF yields payoffs over all n battlefields and μ fronts as a single subgame with resource levels X^S and X^W. For the decentralized game, given any two μ-tuple of first-stage allocations to the fronts, there are μ subgames, and the payoffs from these subgames each result from the Blotto CSF. Blotto CSF payoffs are presented in Theorem 3.2.1, the elements of which were developed and proved by Roberson (2006). They apply to any front $\mathrm{i} \in \{1, \ldots, \mu\}$ which receives some allocation from each player such that $X_{\mathrm{i}}^S \geq 0$, $X_{\mathrm{i}}^W \geq 0$, $X^W = \sum_{\mathrm{i}=1}^{\mu} X_{\mathrm{i}}^W = 1$, $X^S = \sum_{\mathrm{i}=1}^{\mu} X_{\mathrm{i}}^S$, and $X^S > X^W$. To simplify exposition of the theorem, let $\overline{X}_{\mathrm{i}} = \max\{X_{\mathrm{i}}^S, X_{\mathrm{i}}^W\}$ and $\underline{X}_{\mathrm{i}} = \min\{X_{\mathrm{i}}^S, X_{\mathrm{i}}^W\}$ for $\mathrm{i} = 1, \ldots, \mu$. Furthermore, let the player with $\underline{X}_{\mathrm{i}}$ forces be denoted as player k, and the player with $\overline{X}_{\mathrm{i}}$ forces be denoted as player $-k$.

Theorem 1.3.1. ***(Blotto CSF)*** *The unique Nash equilibrium payoffs of the second-stage Colonel Blotto subgame $G_{\mathrm{i}}(X_{\mathrm{i}}^S, X_{\mathrm{i}}^W)$, where $X_{\mathrm{i}}^S \geq 0$, $X_{\mathrm{i}}^W \geq 0$ and $X^W = \sum_{\mathrm{i}=1}^{\mu} X_{\mathrm{i}}^W = 1$, $X^S = \sum_{\mathrm{i}=1}^{\mu} X_{\mathrm{i}}^S$, and $X^S > X^W$, are as follows:*

A. If $\underline{X}_{\mathrm{i}}$ and $\overline{X}_{\mathrm{i}}$ satisfy $\frac{2}{n_{\mathrm{i}}} \leq \frac{\underline{X}_{\mathrm{i}}}{\overline{X}_{\mathrm{i}}} \leq 1$, then the payoff for player k is $\phi_{\mathrm{i}}\left(\frac{\underline{X}_{\mathrm{i}}}{2\overline{X}_{\mathrm{i}}}\right)$ and the payoff for player $-k$ is $\phi_{\mathrm{i}}\left(1 - \frac{\underline{X}_{\mathrm{i}}}{2\overline{X}_{\mathrm{i}}}\right)$.

B. *If* $\underline{X}_{\rm i}$ *and* $\overline{X}_{\rm i}$ *satisfy* $\frac{1}{n_{\rm i}-1} \leq \frac{\underline{X}_{\rm i}}{\overline{X}_{\rm i}} < \frac{2}{n_{\rm i}}$, *then the payoff for player* k *is* $\phi_{\rm i}\left(\frac{2}{n_{\rm i}} - \frac{2\overline{X}_{\rm i}}{n_{\rm i}^2\underline{X}_{\rm i}}\right)$ *and the payoff for player* $-k$ *is* $\phi_{\rm i}\left(1 - \frac{2}{n_{\rm i}} + \frac{2\overline{X}_{\rm i}}{n_{\rm i}^2\underline{X}_{\rm i}}\right)$.

C. *If* $\underline{X}_{\rm i}$ *and* $\overline{X}_{\rm i}$ *satisfy* $\frac{1}{n_{\rm i}} < \frac{\underline{X}_{\rm i}}{\overline{X}_{\rm i}} < \frac{1}{n_{\rm i}-1}$, *then define* $m = \left\lceil \frac{\underline{X}_{\rm i}}{\overline{X}_{\rm i}-\underline{X}_{\rm i}(n_{\rm i}-1)} \right\rceil$, *and note that* $2 \leq m < \infty$. *The payoff for player* k *is* $\phi_{\rm i}\left(\frac{2m-2}{mn_{\rm i}^2}\right)$ *and the payoff for player* $-k$ *is* $\phi_{\rm i}\left(1 - \frac{2m-2}{mn_{\rm i}^2}\right)$.

D. *If* $\underline{X}_{\rm i}$ *and* $\overline{X}_{\rm i}$ *satisfy* $\frac{\underline{X}_{\rm i}}{\overline{X}_{\rm i}} \leq \frac{1}{n_{\rm i}}$, *then the payoff for player* k *is* 0 *and the payoff for player* $-k$ *is* $\phi_{\rm i}$.

Note that Roberson (2006) establishes the existence of equilibrium $n_{\rm i}$-variate distributions which are feasible (i.e., with supports contained in $\{\mathbf{x} \in \mathbb{R}_+^{n_{\rm i}} | \sum_{\rm j=1}^{n_{\rm i}} x_{\rm j} = \underline{X}_{\rm i}\}$ and $\{\mathbf{x} \in \mathbb{R}_+^{n_{\rm i}} | \sum_{\rm j=1}^{n_{\rm i}} x_{\rm j} = \overline{X}_{\rm i}\}$, respectively) and that provide the equilibrium payoffs given in Theorem 3.2.1. Since the game is constant-sum, the uniqueness of the equilibrium payoffs are guaranteed.

Using the Blotto CSF to inform the second-stage payoffs of both versions of the game, we can now solve for subgame perfect equilibrium first-stage allocations in both games using backward induction. We begin with the centralized game.

1.3.1 The Centralized Colonel Blotto Game

Because first-stage allocations are unobservable in the centralized game, there is only one proper subgame. This observation results in payoffs derived from the Blotto CSF of Theorem 3.2.1 with resource levels X^S and X^W. In other words, the centralized game's unobservable allocation across fronts followed by simultaneous distribution across the battlefields within each front does not alter the strategic considerations arising in the traditional (strategic-form) Colonel Blotto game. Hence, the unique equilibrium payoffs for the centralized Blotto game, Γ^C, are equivalent to those of the strategic-form Blotto game when ${\rm i} = \mu = 1$. We state this succinctly in Proposition 1.3.1

Proposition 1.3.1. *In the centralized Colonel Blotto game,* Γ^C, *with resource levels* X^S *and* X^W, *the Nash equilibrium payoffs are equivalent to those of the game with* ${\rm i} = \mu = 1$.

This is captured graphically by Figure 1.4 which depicts the players' respective payoffs in the centralized game as a function of the ratio of the players' resource levels, $\frac{X^W}{X^S}$, as well as the total number of battlefields n. In region A of the figure, player force ratios correspond to part A of Proposition 1.3.1, and the payoffs are linear in the players' resource allocations regardless of the number of battlefields. This changes in regions B and C (corresponding to parts B and C of Proposition 1.3.1), where W and S have increasing and decreasing respective returns to n for any fixed allocations of X^W and X^S. Hence, for a given force ratio $\frac{X^W}{X^S}$, the payoffs to the players depend solely on the number of battlefields, n. The weak player who finds himself in region B or C of the theorem therefore has incentive to increase the number of battlefields, subject to cost constraints, until the players fall in region A (i.e., $\frac{2}{n} \leq \frac{X^W}{X^S}$). This is the result obtained by Kovenock et al. (2010). It also follows that S will have incentive to reduce the number of battlefields on each front.

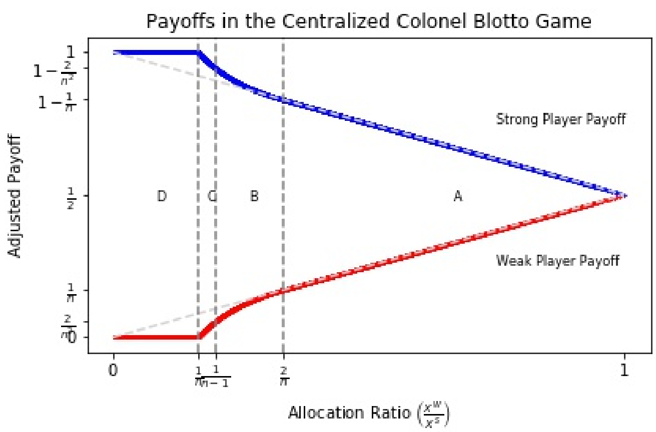


Figure 1.4. Payoffs resulting from the Blotto CSF in the centralized game for n battlefields and μ fronts.

Note that payoff region C of Proposition 1.3.1 differs from regions A and B in that both players have points of discontinuity, which makes analysis considerably more complicated. Therefore, we do not address region C formally, but the basic intuition underlying the analysis is similar to that of region B. Accordingly, the solid curve of region B is extended through the right endpoints (the upper bounds) of the individual steps of region C of Figure 1.4 in order to smooth the players' payoff curves.

1.3.2 The Decentralized Colonel Blotto Game

In the decentralized game the players' first-stage allocations are observable, and this has the potential to alter subgame perfect equilibrium payoffs depending on the particular configuration of parameters $\{X^S, X^W, n, \mu\}$. When a player has an equilibrium strategy that involves allocating a strictly positive level of force to every front, we refer to this as an *interior solution*.

Furthermore, for any endowment of resources in the range $\frac{1}{n} < \frac{X^W}{X^S} < 1$, if decentralization is utilized and there exists a first-stage local equilibrium allocation of resources by player k resulting in an interior solution of $\frac{1}{n_{\mathrm{i}}} < \frac{X_{\mathrm{i}}^W}{X_{\mathrm{i}}^S} < 1$, $\forall$i, then we find that in the first-stage local equilibrium it is necessarily the case that the allocation μ-tuple $(X_1^k, \ldots, X_\mu^k)$ solves $\frac{X_1^k}{\phi_1} = \cdots = \frac{X_\mu^k}{\phi_\mu}$. We call this the *proportional-value allocation*, and it is equivalent to the no soft-spot principle of alliances with complete commitment in Kovenock and Roberson (2012) and Rietzke and Roberson (2013). Obviously, when endowments are such that $\frac{X^W}{X^S} < \frac{1}{n}$, the strong player trivially wins all battlefields on all fronts. The players arrive at the optimal allocations by solving the problem

$$\begin{aligned} \max_{\{X_{\mathrm{i}}^k\}_{\mathrm{i}=1}^{\mu}} \quad & \sum_{\mathrm{i}=1}^{\mu} \pi_{\mathrm{i}}^k \left(X_{\mathrm{i}}^k, X_{\mathrm{i}}^{-k}\right) \\ \text{s.t.} \quad & \sum_{\mathrm{i}=1}^{\mu} X_{\mathrm{i}}^k \leq X^k, \end{aligned} \tag{1.1}$$

and this result is captured in Proposition 1.3.2.

In regards to Table 1.1, because each player's payoff is strictly quasiconcave conditional on entering a fixed number of fronts, it follows that the players proportionally allocating resources across fronts is the unique local equilibrium of the allocation subgame if and only if there are no profitable deviations that involve dropping out of a fixed number of fronts and then proportionally allocating resources across the remaining set of fronts. For a deviation by an arbitrary player k, let ρ denote the number of fronts from which player k drops out. Conversely, $\mu - \rho$ is the number of fronts that player k competes in. Note that for a given parameter configuration it may be the case that only a strict subset of the cases are possible.

Proposition 1.3.2. *In the allocation-stage of the decentralized Colonel Blotto game, if player endowments are such that the conditions of Table 1.1 are satisfied, then the unique allocation-stage local equilibrium is* $\frac{X_1^S}{\phi_1} = \cdots = \frac{X_\mu^S}{\phi_\mu}$ *and* $\frac{X_1^W}{\phi_1} = \cdots = \frac{X_\mu^W}{\phi_\mu}$.

Proof. See Appendix 1.A. $\square$

Table 1.1. Regions defined by the parameters $\{X^S, X^W, n, \mu\}$ for which equilibria in interior solutions exist. For a deviation by an arbitrary player k, let ρ denote the number of fronts that player k drops out of and $\mu - \rho$ be the number of fronts k competes in.

Parameter Range	**Conditions**	**Parameter Range**	**Conditions**
$\frac{\mu}{n} \leq \frac{X^W}{X^S} < \frac{2\mu}{n}$	For each ρ s.t. $\frac{\mu}{n} \leq \frac{X^W(\mu-\rho)}{X^S\mu} < \frac{2\mu}{n}$ $1 - \frac{2\mu}{n} + \frac{2\mu^2 X^S}{n^2 X^W} \geq \left(\frac{\mu-\rho}{\mu}\right)\left(1 - \frac{2\mu}{n} + \frac{2\mu^2 X^S \mu}{n^2 X^W(\mu-\rho)}\right)$	$\frac{2\mu}{n} \leq \frac{X^W}{X^S} \leq 1$	For each ρ s.t. $\frac{2\mu}{n} \leq \frac{X^W(\mu-\rho)}{X^S\mu} \leq 1$ $1 - \frac{X^W}{2X^S} \geq \left(\frac{\mu-\rho}{\mu}\right)\left(1 - \frac{X^W(\mu-\rho)}{2X^S\mu}\right)$
	For each ρ s.t. $\frac{X^W(\mu-\rho)}{X^S\mu} < \frac{\mu}{n}$ $1 - \frac{2\mu}{n} + \frac{2\mu^2 X^S}{n^2 X^W} \geq \frac{\mu-\rho}{\mu}$		For each ρ s.t. $\frac{\mu}{n} \leq \frac{X^W(\mu-\rho)}{X^S\mu} < \frac{2\mu}{n}$ $1 - \frac{X^W}{2X^S} \geq \left(\frac{\mu-\rho}{\mu}\right)\left(1 - \frac{2\mu}{n} + \frac{2\mu^3 X^S}{n^2 X^W(\mu-\rho)}\right)$
	For each ρ s.t. $\frac{\mu}{n} \leq \frac{X^W\mu}{X^S(\mu-\rho)} < \frac{2\mu}{n}$ $\frac{2\mu}{n} - \frac{2\mu^2 X^S}{n^2 X^W} \geq \left(\frac{\mu-\rho}{\mu}\right)\left(\frac{2\mu}{n} - \frac{2\mu^2 X^S(\mu-\rho)}{n^2 X^W \mu}\right)$		For each ρ s.t. $\frac{X^W(\mu-\rho)}{X^S\mu} < \frac{\mu}{n}$ $1 - \frac{X^W}{2X^S} \geq \frac{\mu-\rho}{\mu}$
	For each ρ s.t. $\frac{2\mu}{n} \leq \frac{X^W\mu}{X^S(\mu-\rho)} < 1$ $\frac{2\mu}{n} - \frac{2\mu^2 X^S}{n^2 X^W} \geq \left(\frac{\mu-\rho}{\mu}\right)\left(\frac{X^W\mu}{2X^S(\mu-\rho)}\right)$		For each ρ s.t. $\frac{2\mu}{n} \leq \frac{X^W\mu}{X^S(\mu-\rho)} \leq 1$ $\frac{X^W}{2X^S} \geq \left(\frac{\mu-\rho}{\mu}\right)\left(\frac{X^W\mu}{2X^S(\mu-\rho)}\right)$
	For each ρ s.t. $\frac{2\mu}{n} \leq \frac{X^S(\mu-\rho)}{X^W\mu} \leq 1$ $\frac{2\mu}{n} - \frac{2\mu^2 X^S}{n^2 X^W} \geq \left(\frac{\mu-\rho}{\mu}\right)\left(1 - \frac{X^S(\mu-\rho)}{2X^W\mu}\right)$		For each ρ s.t. $\frac{2\mu}{n} \leq \frac{X^S(\mu-\rho)}{X^W\mu} \leq 1$ $\frac{X^W}{2X^S} \geq \left(\frac{\mu-\rho}{\mu}\right)\left(1 - \frac{X^S(\mu-\rho)}{2X^W\mu}\right)$
	For each ρ s.t. $\frac{\mu}{n} \leq \frac{X^S(\mu-\rho)}{X^W\mu} \leq \frac{2\mu}{n}$ $\frac{2\mu}{n} - \frac{2\mu^2 X^S}{n^2 X^W} \geq \left(\frac{\mu-\rho}{\mu}\right)\left(1 - \frac{2\mu}{n} + \frac{2\mu^2 X^W\mu}{n^2 X^S(\mu-\rho)}\right)$		For each ρ s.t. $\frac{\mu}{n} \leq \frac{X^S(\mu-\rho)}{X^W\mu} \leq \frac{2\mu}{n}$ $\frac{X^W}{2X^S} \geq \left(\frac{\mu-\rho}{\mu}\right)\left(1 - \frac{2\mu}{n} + \frac{2\mu^2 X^W\mu}{n^2 X^S(\mu-\rho)}\right)$
	For each ρ s.t. $\frac{X^S(\mu-\rho)}{X^W\mu} < \frac{\mu}{n}$ $\frac{2\mu}{n} - \frac{2\mu^2 X^S}{n^2 X^W} \geq \frac{\mu-\rho}{\mu}$		For each ρ s.t. $\frac{X^S(\mu-\rho)}{X^W\mu} < \frac{\mu}{n}$ $\frac{X^W}{2X^S} \geq \frac{\mu-\rho}{\mu}$

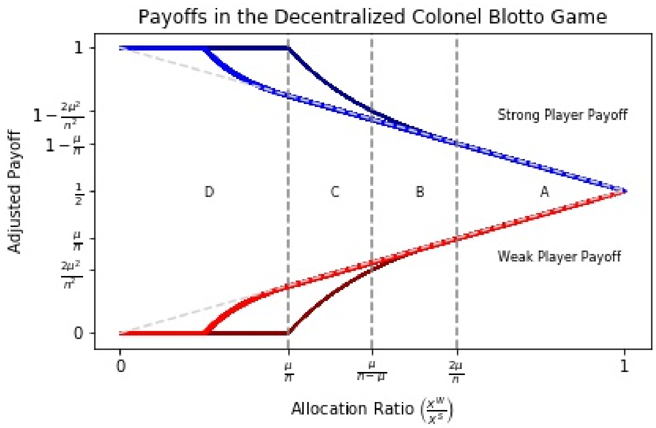


Figure 1.5. Increased payoffs to S in game Γ^D following decentralization over μ fronts. Payoff curves shift right, yielding lower payoffs to player W (dark red) over a wider range of endowment ratios, and higher payoffs to player S (dark blue).

Given the allocations in Proposition 1.3.2, the strong player has incentive to decentralize in a wide range of cases, especially if doing so is costless and the players' endowment ratios are $\frac{X^W}{X^S} < \frac{2\mu}{n}$. Doing so results in payoffs for player S that are equal to or greater than those of the centralized game. To see this, notice that if the strong player decentralizes, there are large regions of $\{X^S, X^W, n, \mu\}$ for which randomizing is suboptimal for the weak player, and in these regions the weak player will choose an interior solution which still results in a higher payoff to the strong player than in the centralized game. Outside of these regions however, the weak player will continue to stochastically allocate forces, as in the centralized game.

Graphically, for the regions where interior solutions exist, decentralization effectively shifts the payoff curves of Figure 1.4 to the right and widens the gap between the players' respective payoffs, as shown by the dark blue and red curves in Figure 1.5. More explicitly, player S has an asymmetric advantage in decentralization if there exists an exogenous partition of battlefields in the game Γ^D, such that $n_1 = \cdots = n_\mu$, $n_{\mathrm{i}} \geq 3\ \forall\mathrm{i}$, and $\frac{\mu}{n} < \frac{X^W}{X^S} < \frac{2\mu}{n}$ and the conditions in Table 1.1 are satisfied. Then the strong player will decentralize and pre-allocate forces to every front during stage one of the two-stage simultaneous Colonel Blotto game.

Because we assume that the fronts are symmetric and homogeneously-valued, one result of the proportional-value allocation in Proposition 1.3.2 is that the ratio of player allocations will be the same on every front, and this is equivalent to the ratio of their endowed budgets, $\alpha = \frac{X^W}{X^S} = \frac{X_\text{i}^W}{X_\text{i}^S}$ for all i, as established in Lemma 1.3.2.4 in the appendix. This also means the payoffs are the same for each player in every Colonel Blotto subgame as a proportion of the total value of those subgames. Then, it is easily verified that for player W this results in region B and C payoffs that are less than or equal to the payoffs of the decentralized game, or $n_1 = n$ original battlefields. In contrast, for player S, payoffs in the decentralized game are greater than or equal to those obtained in the single Colonel Blotto game without decentralization. The payoffs for both players are given in the modified version of Theorem 3.2.1 below.

The Nash equilibrium payoffs of the decentralized Colonel Blotto game Γ^D, where an interior solution exists and $X^W = \sum_{\text{i}=1}^{\mu} X_\text{i}^W = 1$, $X^S = \sum_{\text{i}=1}^{\mu} X_\text{i}^S$, $X^S \geq X^W$, $n_\text{i} = \frac{n}{\mu}$ $\forall$i, and $\Phi = \sum_{\text{i}=1}^{\mu} \phi_\text{i}$ are as follows:

A. If X^W and X^S satisfy $\frac{2\mu}{n} \leq \frac{X^W}{X^S} \leq 1$, then the payoff for player W is $\Phi\left(\frac{X^W}{2X^S}\right)$ and the payoff for player S is $\Phi\left(1 - \frac{X^W}{2X^S}\right)$.

B. If X^W and X^S satisfy $\frac{\mu}{n-\mu} \leq \frac{X^W}{X^S} < \frac{2\mu}{n}$, then the payoff for player W is $\Phi\left(\frac{2\mu}{n} - \frac{2\mu^2 X^S}{n^2 X^W}\right)$ and the payoff for player S is $\Phi\left(1 - \frac{2\mu}{n} + \frac{2\mu^2 X^S}{n^2 X^W}\right)$.

C. If X^W and X^S satisfy $\frac{\mu}{n} < \frac{X^W}{X^S} < \frac{\mu}{n-\mu}$, then the payoff for player W is $\Phi\left(\frac{2\mu}{n} - \frac{2\mu^2 X^S}{n^2 X^W}\right)$ and the payoff for player S is $\Phi\left(1 - \frac{2\mu}{n} + \frac{2\mu^2 X^S}{n^2 X^W}\right)$, as in region B.

D. If X^W and X^S satisfy $\frac{X^W}{X^S} \leq \frac{\mu}{n}$, then the payoff for player W is 0 and the payoff for player S is Φ.

Hence, when a set of battlefields are partitioned into multiple symmetric subgames and $\frac{X^W}{X^S} < \frac{2\mu}{n}$, the strong player has incentive to always decentralize his forces and thereby increase his expected payoff.

Significantly, this result completes the argument made in Lemma 1.3.2.4 of the appendix that the players' respective allocations will be proportional to the value of each front, or $\frac{X_1^k}{\phi_1} = \cdots = \frac{X_\mu^k}{\phi_\mu}$ for $k \in \{S, W\}$. More particularly, Lemma 1.3.2.4 establishes that the strong player is indifferent to decentralization if $\frac{2}{n_i} \leq \frac{X_i^W}{X_i^S} \leq 1$ for all fronts i, but he will decentralize if $\frac{1}{n_i} \leq \frac{X_i^W}{X_i^S} < \frac{2}{n_i}$.

1.4 Discussion

The foregoing analysis shows a strong player's incentive to decentralize, when an exogenous partition exists, by making information about resource allocations public prior to engaging in conflict with a weaker opponent. Decentralization has the effect of reducing the space over which a weaker adversary can act and attack, thus providing an asymmetric advantage to the resource-advantaged strong player. In the context of the Colonel Blotto game, decentralization decreases the number of battlefields over which the weak player can implement a mixed strategy, thus providing a higher payoff to the strong player and a lower payoff to the weak player.

In this way, decentralization is a strong player strategy that alters the dimensions of the Colonel Blotto game in a manner analogous to the weak player's ability to create new battlefields in Kovenock et al. (2010). While not directly addressed here, decentralization by the strong player likely serves as an effective counter-strategy to the weak player's reliance on guerrilla warfare and innovation (i.e., increasing the number of battlefields over which the weak player can use a mixing strategy), the extent of which represents an area for future research.

With regards to the equilibrium allocations made by each player, we find that the unique local equilibrium allocation is the proportional-value allocation, or $\frac{X_1^W}{\phi_1} = \cdots = \frac{X_\mu^W}{\phi_\mu}$ and $\frac{X_1^S}{\phi_1} = \cdots = \frac{X_\mu^S}{\phi_\mu}$. That is, assuming both players value a front equally, the quantity of each player's budget allocated to a front is directly proportional to the value of the front. To do more would incur marginal losses on other fronts that are larger than the marginal gain achieved at the front receiving the additional resources. Conversely, to allocate less would result in marginal losses that are larger than the gains achieved on other fronts. Thus, it is reasonable that each player allocates only that fraction of his budget to a front that is proportional to the value of the front in question.

1.A Appendix: The First Stage Simultaneous Allocation Subgame

We break the proof of Proposition 1.3.2 into the following set of Lemmas. Lemma 1.3.2.1 first establishes the existence of pure strategy local equilibria in the decentralized game that is of the proportional-allocation form. Then Lemmas 1.3.2.2, 1.3.2.3, and 1.3.2.4 prove that when the proportional-value allocation exists, it is the unique allocation-stage equilibria.

LEMMA 1.3.2.1. *(Existence of Pure Strategy Local Equilibria) In the allocation subgame of the decentralized Colonel Blotto game Γ^D, a proportional-value allocation exists if and only if the conditions of Table 1.1 are satisfied.*

Proof. Beginning with the simplest case, if $\frac{1}{n} < \frac{X^W}{X^S} < \frac{\mu}{n}$ then it is not a local equilibrium for the players to allocate resources proportionally across fronts in the first-stage subgame. By way of contradiction, if player S allocates $\frac{X^S}{\mu}$ to each front, then player W can concentrate all X^W resources into a single front, which generates a strictly positive payoff because $\frac{\mu}{n} < \frac{\mu X^W}{X^S}$.

Moving to the remaining case of $\frac{\mu}{n} \leq \frac{X^W}{X^S} \leq 1$ and given that the opposition is using a proportional-value allocation, a proportional-value allocation is a unique best response of the first stage subgame if and only if the conditions of Table 1.1 are satisfied.

To see this, note that, conditional on entering a fixed number of fronts and given that the opposition is using a proportional-value allocation, each player's payoff is strictly quasiconcave. It follows that a proportional-value allocation is a unique best response if and only if there are no profitable deviations that involve dropping out of a fixed number of fronts and then proportionally allocating resources to the remaining set of fronts. For such a deviation by an arbitrary player i, let ρ denote the number of fronts from which player i drops out. Conversely, $\mu - \rho$ is the number of fronts that player i competes in. Then for a given parameter configuration $\{X^S, X^W, n, \mu\}$ and the payoffs available to each player from Theorem 3.2.1, Table 1.1 provides the conditions that identify when a profitable deviation exists. $\square$

In describing the different configurations of player resource allocations that may potentially arise, it will be useful to introduce the notation in Table 1.2. These denote the set of fronts to which one or both players allocates resources. For instance, let $\Omega^N\left(\{X_{\mathrm{i}}^S, X_{\mathrm{i}}^W\}_{\mathrm{i}=1}^{\mu}\right) =$

$\{\mathrm{i} \in \{1,\ldots,\mu\} | X_\mathrm{i}^S = X_\mathrm{i}^W = 0\}$ denote the set of fronts to which neither player allocates a strictly positive level of resources when the players use the allocation-stage local strategy profile $\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right)$. Likewise, let $\Omega^S\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right) = \{\mathrm{i} \in \{1,\ldots,\mu\} | X_\mathrm{i}^S > 0,\ X_\mathrm{i}^W = 0\}$ denote the set of fronts to which only player S allocates resources; $\Omega^W\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right)$ denotes the set of fronts to which only player W allocates resources; and $\Omega^B\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right)$ denotes the set of fronts to which both players allocate resources. It will also be useful to separate $\Omega^B\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right)$ into three subsets. Let $\Omega^{B(S)}\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right)$ denote the set of fronts to which both players allocate resources but that player S wins all of the battlefields in the front. Furthermore, let $\Omega^{B(W)}\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right)$ denote the set of fronts to which both players allocate resources but player W wins all of the battlefields in the front; and let $\Omega^{B(B)}\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right)$ denote the set of fronts to which both players allocate resources and both players win a strictly positive share of battlefields in each of those fronts. For ease of notation in the proofs that follow, we will at times omit the term $\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right)$ when denoting a set of fronts $\Omega^\ell\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right)$. For instance, $\Omega^{B(B)}\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right)$ may be abbreviated to read $\Omega^{B(B)}$ without any change in meaning.

Before proceeding, make note of several additional notational issues: (i) we will use the notation $|\Omega|$ to denote the cardinality of the set Ω; (ii) for each player $k = S, W$, let $p_\mathrm{i}^k : \mathbb{R}_+ \to [0,1]$ denote the Theorem 1 share of front i that player k wins when $X_\mathrm{i}^k = \zeta X_\mathrm{i}^{-k}$ for $\zeta \in \mathbb{R}_+$ and note that $p_\mathrm{i}^k(\zeta) > 0$ for all $\zeta > 1/n_\mathrm{i}$; and (iii) let $\left(\{X_\mathrm{i}^{*S}, X_\mathrm{i}^{*W}\}_{\mathrm{i}=1}^{\mu}\right)$ denote the Proposition 1 allocation-stage local strategy profile. Lastly, recall that $X^S = \alpha X^W$ for some $\alpha \in (\frac{1}{n}, 1]$, and note that for the Proposition 1 allocation-stage local strategy profile $\left(\{X_\mathrm{i}^{*S}, X_\mathrm{i}^{*W}\}_{\mathrm{i}=1}^{\mu}\right)$, it follows that for all fronts $\mathrm{i} \in \{1,\ldots,\mu\}$ we have that $X_\mathrm{i}^k = \alpha X_\mathrm{i}^{-k}$, and player W's equilibrium payoff is $p_\mathrm{i}^W(\alpha)$ and player S's equilibrium payoff is $p_\mathrm{i}^S\left(\frac{1}{\alpha}\right)$.

LEMMA 1.3.2.2. *In the allocation subgame of the decentralized Colonel Blotto game Γ^D, and for those regions where a proportional-value allocation exists, it must be the case that if $\alpha > \frac{\mu}{n}$, then in any pure-strategy allocation-stage local equilibrium $\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}$, $|\Omega^{B(B)}(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu})| > 0$.*

Proof. The proof of Lemma 1.3.2.2, which is by way of contradiction, consists of two steps. Suppose that there exists an allocation-stage local equilibrium $\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}$ in which (using

Table 1.2. Subsets of fronts receiving allocations from S and W

$\Omega^N\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right) = \{\mathrm{i} \in \{1,\ldots,\mu\} \mid X_\mathrm{i}^S = X_\mathrm{i}^W = 0\}$	Set of fronts to which neither player allocates resources
$\Omega^S\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right) = \{\mathrm{i} \in \{1,\ldots,\mu\} \mid X_\mathrm{i}^S > 0,\ X_\mathrm{i}^W = 0\}$	Set of fronts to which only player S allocates resources
$\Omega^W\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right) = \{\mathrm{i} \in \{1,\ldots,\mu\} \mid X_\mathrm{i}^W > 0,\ X_\mathrm{i}^S = 0\}$	Set of fronts to which only player W allocates resources
$\Omega^B\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right) = \{\mathrm{i} \in \{1,\ldots,\mu\} \mid X_\mathrm{i}^S, X_\mathrm{i}^W > 0\}$	Set of fronts to which both players allocate resources
$\Omega^{B(S)}\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right) = \{\mathrm{i} \in \{1,\ldots,\mu\} \mid 0 < \frac{X_\mathrm{i}^W}{X_\mathrm{i}^S} < \frac{1}{n_\mathrm{i}}\}$	Set of fronts to which both players allocate resources, S wins all battlefields
$\Omega^{B(W)}\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right) = \{\mathrm{i} \in \{1,\ldots,\mu\} \mid 0 < \frac{X_\mathrm{i}^S}{X_\mathrm{i}^W} < \frac{1}{n_\mathrm{i}}\}$	Set of fronts to which both players allocate resources, W wins all battlefields
$\Omega^{B(B)}\left(\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}\right) = \{\mathrm{i} \in \{1,\ldots,\mu\} \mid \frac{1}{n_\mathrm{i}} \leq \min\{\frac{X_\mathrm{i}^W}{X_\mathrm{i}^S}, \frac{X_\mathrm{i}^S}{X_\mathrm{i}^W}\} \leq 1\}$	Set of fronts to which both players allocate resources and win a strictly positive share of battlefields

the abbreviated notation) $|\Omega^{B(B)}| = 0$. In the first step, we take advantage of the fact that the allocation-stage subgame is constant-sum and allocation-stage local equilibria are interchangeable. Consider a strategy profile formed by interchanging the assumed allocation-stage local equilibrium $\{X_\mathrm{i}^S, X_\mathrm{i}^W\}_{\mathrm{i}=1}^{\mu}$ in which $|\Omega^{B(B)}| = 0$ with the allocation-stage local strategy profile $(\{X_\mathrm{i}^{*S}, X_\mathrm{i}^{*W}\}_{\mathrm{i}=1}^{\mu})$ from Proposition 1.3.2. In the first step, we show that the resulting interchanged strategy profile forms an equilibrium only if α is below a threshold $\overline{\alpha}$. In the second step, we show that the threshold $\overline{\alpha}$ is less than $\frac{\mu}{n}$, which is a contradiction to the assumption that $\alpha > \frac{\mu}{n}$.

Consider the interchanged equilibrium $\{X_\mathrm{i}^S, X_\mathrm{i}^{*W}\}_{\mathrm{i}=1}^{\mu}$ in which player W uses their Proposition 1 allocation-stage local strategy in place of their strategy in the supposed equilibrium. To establish that α must be below a threshold $\overline{\alpha}$, we will make use of the fact that $|\Omega^{B(B)}| = 0$, which we formally establish now. By way of contradiction, suppose that $|\Omega^{B(B)}| \neq 0$. Then consider first the case that $|\Omega^{B(B)}| = \mu$. If $|\Omega^{B(B)}| = \mu$, and given that player W uses his Proposition 1 equilibrium strategy, $\{X_\mathrm{i}^{*W}\}_{\mathrm{i}=1}^{\mu}$, player S has a profitable deviation that equates his marginal payoffs across all fronts. Together with budget balancing for player S, this results in player S's Proposition 1 equilibrium strategy. That is, player S has a profitable deviation from $\{X_\mathrm{i}^S\}_{\mathrm{i}=1}^{\mu}$, because $\{X_\mathrm{i}^{*S}\}_{\mathrm{i}=1}^{\mu}$ is player S's unique best response to $\{X_\mathrm{i}^{*W}\}_{\mathrm{i}=1}^{\mu}$. We know this to be the case given the fact that each player k's objective function in equation 2.4 is strictly concave for every μ-tuple of possible inputs, $(\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{W}\}_{\mathrm{i}=1}^{\mu})$ in $|\Omega^{B(B)}|=0$. [4]

The remaining case involves $|\Omega^{B(B)}| \notin \{0,\mu\}$ with $|\Omega^{B(W)}| \geq 0$, $|\Omega^{B(S)}| \geq 0$ and $|\Omega^{W}| \geq 0$. If there exists an $\mathrm{i} \in \Omega^{B(W)} \cup \Omega^{B(S)}$ then player S has a payoff increasing deviation that involves shifting resources from this front i to a front $\mathrm{i}' \in \Omega^{B(B)}$. Similarly, if there exists an $\mathrm{i} \in \Omega^{W}$ then player W has a payoff increasing deviation that involves shifting resources from this front i to a front $\mathrm{i}' \in \Omega^{B(B)}$. Hence we have a contradiction, and this completes the proof that in the interchanged equilibrium $\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{*W}\}_{\mathrm{i}=1}^{\mu}$ it must be the case that $|\Omega^{B(B)}|=0$.

Next, note that player S's equilibrium payoff in the interchanged equilibrium must be the same as in Proposition 1, namely $p_{\mathrm{i}}^{S}(\frac{1}{\alpha})$ where i is an arbitrary front. In the interchanged equilibrium, we also know that $|\Omega^{N}|=0$ and $|\Omega^{S}|=0$. Because it is also the case that $|\Omega^{B(B)}|=0$, we know that player S obtains a payoff of $p_{\mathrm{i}}^{S}(\frac{1}{\alpha})$ on the set $\Omega^{B(S)}$, which implies that player S's budget satisfies the condition that:

$$X^{S} - |\Omega^{B(S)}(\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{*W}\}_{\mathrm{i}=1}^{\mu})| \frac{nX^{W}}{\mu^{2}} \geq 0 \tag{1.2}$$

and $\Omega^{B(S)}$ satisfies the equal payoff across equilibria condition:

$$\frac{1}{\mu}\sum_{\mathrm{i}=1}^{\mu} p_{\mathrm{i}}^{S}\left(\frac{1}{\alpha}\right) = \frac{|\Omega^{B(S)}(\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{*W}\}_{\mathrm{i}=1}^{\mu})|}{\mu} \tag{1.3}$$

or equivalently,

$$\mu p_{\mathrm{i}}^{S}\left(\frac{1}{\alpha}\right) = |\Omega^{B(S)}(\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{*W}\}_{\mathrm{i}=1}^{\mu})| \geq \frac{\mu}{2}. \tag{1.4}$$

Next, it follows from equations (1.2) and (1.4) that:

$$\frac{2\mu}{n} \geq \frac{\mu^{2}}{n|\Omega^{B(S)}\left(\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{*W}\}_{\mathrm{i}=1}^{\mu}\right)|} \geq \alpha. \tag{1.5}$$

From equation (1.5) we see that the interchanged equilibrium $\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{*W}\}_{\mathrm{i}=1}^{\mu}$ exists only if α is below the threshold $\overline{\alpha} \equiv \frac{\mu^{2}}{n|\Omega^{B(S)}(\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{*W}\}_{\mathrm{i}=1}^{\mu})|}$.

[4] ↑The hessian is a 2x2 matrix with no cross partials and all diagonal elements are negative, therefore we know the function is strictly concave in the region specified.

To complete the proof of Lemma 1.3.2.2, we now move on to the second step in which we show that the threshold $\overline{\alpha}$ is less than $\frac{\mu}{n}$. From, equation (1.5) we know that $\frac{2\mu}{n} \geq \alpha$ and from Theorem 1 it follows that if $\alpha \in (\frac{\mu}{n}, \frac{2\mu}{n}]$ then:

$$p_{\mathrm{i}}^{S}\left(\frac{1}{\alpha}\right) = 1 - \frac{2\mu}{n} + \frac{2\mu^2}{n^2\alpha}. \tag{1.6}$$

Then, inserting equation (1.4) into equation (1.5) we have that:

$$\frac{\mu}{n} \geq \alpha p_{\mathrm{i}}^{S}\left(\frac{1}{\alpha}\right). \tag{1.7}$$

and inserting equation (1.6) into equation (1.7), we have that

$$\frac{\mu}{n} \geq \alpha\left(1 - \frac{2\mu}{n} + \frac{2\mu^2}{n^2\alpha}\right) \tag{1.8}$$

or equivalently,

$$\frac{\mu}{n}\left(1 - \frac{2\mu}{n}\right) \geq \alpha\left(1 - \frac{2\mu}{n}\right) \tag{1.9}$$

and it follows from equation (1.9) that $\frac{\mu}{n} \geq \alpha$ and hence we have a contradiction. This completes the proof of Lemma 1.3.2.2. □

LEMMA 1.3.2.3. *In the allocation subgame of the decentralized Colonel Blotto game Γ^D, and for those regions where a proportional-value allocation exists, it must be the case that in any pure-strategy allocation-stage local equilibrium, all fronts receive a strictly positive allocation of force from both players and both players receive a strictly positive payoff at all fronts, i.e.,* $|\Omega^{B(B)}(\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{W}\}_{\mathrm{i}=1}^{\mu})| = \mu$.

Proof. From Lemma 1.3.2.2 we have that $|\Omega^{B(B)}(\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{W}\}_{\mathrm{i}=1}^{\mu})| > 0$, and it is trivially the case that $|\Omega^{N}(\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{W}\}_{\mathrm{i}=1}^{\mu})| = 0$. Otherwise, either player could reallocate an arbitrarily small level of resources to every $\mathrm{i} \in |\Omega^{N}(\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{*W}\}_{\mathrm{i}=1}^{\mu})| = 0$ and increase their payoff of in i from $\frac{\phi_{\mathrm{i}}}{2}$ to ϕ_{i}. To show that in any allocation-stage local equilibrium $|\Omega^{B(B)(\{X_{\mathrm{i}}^{S}, X_{\mathrm{i}}^{W}\}_{\mathrm{i}=1}^{\mu})}| = \mu$, we will show by way of contradiction that (again using abbreviated notation) $|\Omega^{B(W)}| = 0$, $|\Omega^{B(S)}| = 0$, $|\Omega^{W}| = 0$ and $|\Omega^{S}| = 0$. For each of the following cases, suppose that there exists an allocation-stage local equilibrium in which

(i) $|\Omega^W|> 0$, $|\Omega^S|\geq 0$, $|\Omega^{B(W)}|\geq 0$, $|\Omega^{B(S)}|\geq 0$, $|\Omega^{B(B)}|> 0$, and $|\Omega^N|= 0$

(ii) $|\Omega^S|> 0$, $|\Omega^W|\geq 0$, $|\Omega^{B(W)}|\geq 0$, $|\Omega^{B(S)}|\geq 0$, $|\Omega^{B(B)}|> 0$, and $|\Omega^N|= 0$

(iii) $|\Omega^{B(W)}|> 0$, $|\Omega^{B(S)}|\geq 0$, $|\Omega^W|\geq 0$, $|\Omega^S|\geq 0$, $|\Omega^{B(B)}|> 0$, and $|\Omega^N|= 0$

(iv) $|\Omega^{B(S)}|> 0$, $|\Omega^{B(W)}|\geq 0$, $|\Omega^W|\geq 0$, $|\Omega^S|\geq 0$, $|\Omega^{B(B)}|> 0$, and $|\Omega^N|= 0$.

Beginning with case (i), if $|\Omega^W|> 0$, then there exists an $\epsilon \in (0, \sum_{\mathrm{i}\in\Omega^W} X_{\mathrm{i}}^W)$, and W has a payoff increasing deviation in which W allocates $\delta = \frac{\epsilon}{|\Omega^W|}$ to each front $\mathrm{i} \in \Omega^W$, and allocates $\min\left\{0, \alpha X_{\mathrm{i}}^S - \frac{\delta}{|\Omega^S \cup \Omega^{B(S)}|}\right\}$ to each front $\mathrm{i}' \in \Omega^S \cup \Omega^{B(S)}$. Doing so increases W's payoff from $\sum_{\mathrm{i}\in\Omega^W\cup\Omega^{B(W)}} \phi_{\mathrm{i}} + \alpha \sum_{\mathrm{i}\in\Omega^{B(B)}} \phi_{\mathrm{i}}$ to $\sum_{\mathrm{i}\in\Omega^W\cup\Omega^{B(W)}} \phi_{\mathrm{i}} + \alpha \sum_{\mathrm{i}\in\Omega^{B(B)}\cup\Omega^{B(S)}\cup\Omega^S} \phi_{\mathrm{i}}$. Hence, W has a payoff increasing deviation and there exists no such allocation-stage local equilibrium.

For case (ii), if $|\Omega^S|> 0$, then there exists an $\epsilon \in (0, \sum_{\mathrm{i}\in\Omega^S} X_{\mathrm{i}}^S)$, and S has a payoff increasing deviation in which S allocates $\delta = \frac{\epsilon}{|\Omega^S|}$ to each front $\mathrm{i} \in \Omega^S$, and allocates $\min\left\{0, \frac{X_{\mathrm{i}}^W}{\alpha} - \frac{\delta}{|\Omega^W \cup \Omega^{B(W)}|}\right\}$ to each front $\mathrm{i}' \in \Omega^W \cup \Omega^{B(W)}$. Doing so increases S's payoff from $\sum_{\mathrm{i}\in\Omega^S\cup\Omega^{B(S)}} \phi_{\mathrm{i}} + (1-\alpha) \sum_{\mathrm{i}\in\Omega^{B(B)}} \phi_{\mathrm{i}}$ to $\sum_{\mathrm{i}\in\Omega^S\cup\Omega^{B(S)}} \phi_{\mathrm{i}} + (1-\alpha) \sum_{\mathrm{i}\in\Omega^{B(B)}\cup\Omega^{B(W)}\cup\Omega^W} \phi_{\mathrm{i}}$. Hence, S has a payoff increasing deviation and there exists no case (ii) allocation-stage local equilibrium.

In case (iii), if $|\Omega^{B(W)}|> 0$, then S has a payoff increasing deviation that involves moving all of the resources allocated to $\Omega^{B(W)}$, $\{X_{\mathrm{i}}^S\}_{\mathrm{i}\in\Omega^{B(W)}}$, to $\Omega^{B(B)}$. Doing so strictly increases the share of battlefields won by S in $\Omega^{B(B)}$. Hence, we have a contradiction and case (iii) cannot be an allocation-stage local equilibrium.

For case (iv), we know from the previous cases that $|\Omega^W|= |\Omega^S|= |\Omega^{B(W)}|= 0$. Then, using a similar argument as for case (iii), if $|\Omega^{B(S)}|> 0$ and $|\Omega^{B(B)}|> 0$, W has a payoff increasing deviation that involves reallocating the forces $\{X_{\mathrm{i}}^W\}_{\mathrm{i}\in\Omega^{B(S)}}$. Hence, case (iv) is contradicted, and it must be that $\Omega^{B(B)} = \mu$, guaranteeing the existence of an interior equilibrium. This completes the proof for Lemma 1.3.2.3. □

LEMMA 1.3.2.4. *In the allocation subgame of the decentralized Colonel Blotto game Γ^D, and for those regions where a proportional-allocation exists, each player $k \in \{S, W\}$ allocates a level of force to each front $\mathrm{i} \in \Omega^{B(B)}$ that is proportional to the value of front i, or $\frac{X_1^k}{\phi_1} = \cdots = \frac{X_\mu^k}{\phi_\mu}$, and doing so is a unique pure-strategy allocation-stage local equilibrium.*

Proof. From Lemma 1.3.2.3, we know that in any pure-strategy equilibrium $\Omega^{B(B)} = \mu$. Yet, if $\Omega^{B(B)} = \mu$, then it follows from the arguments in Lemma 1.3.2.1 (namely that equilibria are interchangeable and that the conditions of Table 1.1 specify when a proportional-value allocation is the unique best-response to a proportional-value allocation) that the proportional-value allocation is the unique allocation-stage local equilibrium. $\square$

2. NOISE AND UNCERTAINTY IN THE DECENTRALIZED COLONEL BLOTTO GAME

"War is the realm of uncertainty; three quarters of the factors on which action in war is based are wrapped in a fog of greater or lesser uncertainty." –General Carl von Clausewitz

2.1 Introduction

Since the earliest recorded conflicts, military strategists have sought to obtain intelligence about their enemies, the terrain, and any other factors that could influence the outcome of an engagement. For example, in the Old Testament Moses sends 12 spies to scout the land of Canaan. These report back that the land "floweth with milk and honey" and that it is inhabited by a race of giants. Moreover, they say that "the people that dwell in the land are fierce, and the cities are fortified, and very great" (Numbers 13:1-33). In modern times, militaries expend large sums to predict weather, survey terrain, and acquire intelligence on enemy capabilities, positions, and intentions. Indeed, the United States has entire government agencies dedicated solely to the acquisition of intelligence on foreign nations, the monitoring of electronic communications in other countries, precision-mapping the globe, and monitoring weather worldwide.

These efforts are undertaken, at least in part, with the goal of reducing uncertainty in a conflict and increasing the probability of a favorable outcome. Yet, some forms of uncertainty always remain. Sources of exogenous noise, what Clausewitz referred to as the "fog of war," obscure developments on the battlefield from the view of a military leader, while individual inexperience and lack of expertise in a particular situation influence his decision-making. In this paper, we model both forms of uncertainty using the decentralized Colonel Blotto game introduced in our previous chapter. To do so, we first test the robustness of subgame perfect equilibria in the face of increased exogenous noise by changing the all-pay auction contest success function (CSF) to a lottery CSF, then we find the players' quantal response equilibria (QRE) to show how individual decision-making is impacted by insufficient learning when we lower the "experience parameter," ψ, of the logit QRE. We find that player actions rely significantly less on decentralization strategies under the lottery CSF compared to the case of the all-pay auction, owing mainly to the increased exogenous noise. Similarly, agent QRE and heterogeneous QRE approximate subgame perfect equilibria for high values of ψ in the case of an all-pay auction, but under the lottery CSF, QRE is largely unresponsive to changes in ψ due to the increase in exogenous noise.

The Colonel Blotto game has been used as a model for understanding resource allocation and competition in multi-contest settings since its original formulation by Borel in 1921. In its simplest form, two players, each endowed with a fixed quantity of a one-dimensional resource, simultaneously compete over a set of identical battlefields in order to maximize their respective payoffs. Unspent resources have no value, and the player with the higher allocation of resources to a battlefield wins that battlefield with certainty according to some contest success function. Pure or mixed strategy equilibria may exist depending on the CSF used, the players' relative resource endowments, and the number of battlefields in competition.

As demonstrated in the first chapter with an auction CSF, in a Blotto game where players have asymmetric resource endowments, the weak player will rely on a stochastic distribution of resources to improve his payoff. This entails concentrating resources in a subset of battlefields while neglecting all others, but doing so in a random fashion so that the precise foci of attack are unknown to the strong player in advance. The strong player, knowing his opponent's best response is a stochastic distribution, attempts to defend all battlefields since he does not know where the weak player's attacks will occur. Hence, under the auction CSF, Blotto game outcomes are decided by some combination of exogenously-endowed skill (i.e., resources) and endogenously-generated chance (i.e., strategic uncertainty). The strong player benefits from contests that are determined primarily by skill and superior resources, whereas the weak player benefits from noise and attempts to generate as much uncertainty for the strong player as possible.

In our previous chapter, we also demonstrated that because of this dynamic between skill and chance, the strong player has incentive to reduce the role of chance, and he does so by decentralizing his forces into independent units in the first stage of the game according to exogenously-defined subsets of battlefields we call "fronts." Subordinate commanders assigned to lead each decentralized unit are empowered to make further resource distribution decisions as circumstances require in stage two, and we assumed that the act of decentralization, which permits closer, more localized surveillance of adversary activity, provides the strong player with complete information regarding the weak player's first-stage allocation of forces. Likewise, because decentralization is an overt act characterized by the erection of fortifications and the establishment of patrols and checkpoints, the process of decentralization also reveals the strong player's first-stage force allocations to the weak player. Hence, at the commencement of stage two, each player knows the force disposition of their opponent.

We now include a related element of the Blotto game as introduced by Kovenock et al. (2010). Kovenock et al. demonstrate that under the auction CSF, and because the players' payoffs depend on the number of battlefields for a range of player budget ratios, in an extensive-form game the weak player also has a best response in creating new battlefields, although doing so incurs a cost. This provides him with a greater range of battlefields over which he can randomly concentrate forces, which in turn requires the strong player to disperse resources more broadly to defend the increased number of battlefields. Hence, while the weak player's stochastic distribution strategy relies on increasing the role of chance in the contest, creating new battlefields requires a measure of skill (i.e., resources) to increase strategic uncertainty for the strong player and earn a greater payoff for himself.

Because decentralization on the part of the strong player, and the creation of new battlefields on the part of the weak player, decrease and increase the number of battlefields respectively, we refer to these two strategies as *dimensionality strategies*, and the one is analogous to the other in that they both yield higher payoffs to the strong and weak players. For the analysis in this chapter, we will first show the robustness of decentralization for the strong player in an extensive-form game that is very similar to the model of the first chapter. That is, we will employ a two-stage extensive form game that features a simultaneous allocation of resources to fronts in stage one, followed by a simultaneous distribution of resources to battlefields in stage two, and we will show that decentralization is weakly robust to changes in the CSF from an auction to a lottery.

However, to demonstrate QRE, we incorporate both players' dimensionality strategies in order to show the interaction between the two strategies. We will first demonstrate simultaneous-move dimensionality whereby both players allocate resources to the fronts in stage one as part of the strong player dimensionality strategy of decentralization, but the weak player concurrently creates new battlefields. This is then followed by stage two distribution to the battlefields which is also a simultaneous move by both players. Next, we depict the players' dimensionality strategies as part of a three-stage sequential-move game. In this version, the strong player begins by initiating decentralization which results in a simultaneous allocation of resources to the fronts by both players. Then in stage two, the weak player observes the strong player's disposition of force and chooses to create new battlefields in one or more fronts, or not. Stage three features simultaneous distribution of forces to the battlefields. We use the first version, the simultaneous-move decentralized game, to demonstrate QRE, while the sequential-move game is used to demonstrate agent-QRE and heterogeneous-QRE. These will be discussed in greater detail later, but we mention them now for the purpose of introducing each player's dimensionality strategy. Furthermore, it is interesting to model both the simultaneous and sequential versions of the game since the information available to the players changes substantially according to the version of the game used, as do the players' best responses.

We can now extend our motivating example from the first chapter further. To picture the concept of player dimensionality strategies, consider again the weaker insurgent force that confronts a resource-rich government opponent. Seeing that direct military confrontation across all battlefields would end in certain defeat everywhere, the insurgents use a guerrilla warfare strategy that concentrates their forces in a subset of battlefields while neglecting all others. Then, by employing a dimensionality strategy, the insurgents may also expand the area of conflict by introducing terrorist attacks on select targets not previously considered part of the conflict.[1] In response, the government tries to reduce its losses and the insurgents' gains by imposing added structure on the battlespace through the establishment of "security sectors" monitored by patrols and armed checkpoints along routes of travel. Additionally, each sector is diligently surveilled by military and police forces for insurgent activity, while

[1]↑As an example of insurgent and "strong player" behavior in conflict, including an expansion of targets to civilian sites through terrorist acts, see Horne (2006) for an excellent history of the French-Algerian war of 1954-1962.

quick reaction forces are staged in each sector to respond immediately to terrorist attacks or promising intelligence. However, we assume these activities and fortifications are overt and observable to the insurgents, allowing them to respond with optimal allocations of forces to each sector and carrying out operations in those sectors as mentioned above.

As can be seen from this example, and theoretically from the decentralized Colonel Blotto game itself, the weak player's dimensionality strategy increases the payoffs to using a stochastic distribution of resources, and increasing dimensionality is akin to increasing complexity. Furthermore, in a constant-sum game like the Colonel Blotto game, as the weak player's payoff increases, the payoff to the strong player decreases. Thus, the strong player has incentive to decrease complexity by reducing the dimensionality of the game through decentralization. This concept relates directly to our test of robustness of subgame perfect equilibria in the decentralized Colonel Blotto game.

Our first chapter on the decentralized game utilizes an auction contest success function (CSF), which is the theoretical benchmark throughout the Blotto literature for a range of exogenous noise levels. When presented using the formulation of Tullock (1980), the auction is a special case of ratio-form CSF when the exponent parameter r is set equal to infinity, as shown in Equation 2.1.

$$p_{\mathrm{ij}}^{k}(x_{\mathrm{ij}}^{k}, x_{\mathrm{ij}}^{-k}) = \frac{\left(x_{\mathrm{ij}}^{k}\right)^{r}}{\left(x_{\mathrm{ij}}^{k}\right)^{r} + \left(x_{\mathrm{ij}}^{-k}\right)^{r}} \tag{2.1}$$

In words, the ratio-form CSF says that player k's probability of winning battlefield j in front i is equal to the resources invested in that battlefield by player k, divided by the sum of resources invested in the battlefield by both players.

Of course the key parameter in Equation 2.1 is the exponent r, which is commonly interpreted to represent the level of noise in the competition, or perhaps more appropriately, the returns to effort invested. When r is high, there are high returns to effort, and the probability that k wins battlefield ij becomes more deterministic. In contrast, with r low, returns to effort are low and the probability of k winning becomes more stochastic. Accordingly, the impact of additional investment is affected. As r approaches infinity, k wins with probability one for any amount $\epsilon > 0$ that exceeds the resource investment of the other player. This is the first-price, all-pay auction. Yet, when $r = 1$, the probability that k wins is exactly proportional to the investment made by k relative to the sum of investments from both players. We refer to this as the lottery CSF.

Hence, to show robustness of subgame perfect equilibria in the decentralized Colonel Blotto game, we must demonstrate the results of the strong player's dimensionality strategy as r ranges from zero to two. To do this, we will compare the results of decentralization in our earlier paper which uses an auction CSF to those obtained from a lottery CSF when $r = 1$. For the intermediate ranges, $1 < r < \infty$, we rely on the work of Baye et al. (1994) and Ewerhart (2017), among others, who prove that a ratio-form CSF with $r > 2$ is payoff equivalent to an auction, while any CSF with $r \leq 2$ results in pre-conflict behavior similar that of a lottery.

Returning to our discussion of QRE, players in a conflict face more than exogenous complexity alone. Often, lack of experience and internal uncertainty play a significant role in strategic decision-making, resulting in bounded rationality and noisy best responses. For example, in war, an experienced commander, especially one familiar with his opponent, is expected to fare better on average than a new leader or one that has never before confronted his adversary. Yet, internal uncertainty is expected to diminish as a player learns, either through repeated experience or better information on the intentions and capabilities of his opponent, and this leads to behavioral equilibria that are closer to the Nash equilibria than when decisions were less informed.

As previewed earlier, we model noisy best responses with regard to both players' dimensionality strategies in a variety of settings. First we use the logit QRE described by McKelvey and Palfrey (1995) to demonstrate the players' allocation-stage behavioral equilibrium given a single-stage, simultaneous-move Colonel Blotto game. That is, we find the QRE when players appear to move simultaneously to create new battlefields and decentralize, and we do so using expected payoffs from both the auction and lottery CSFs.

Then, we model noisy best responses in a three-stage sequential setting and find the agent-QRE (AQRE). In this version of the game, there exists an exogenously-formed partition to which both players allocate and decentralize their resources in the first stage. At the end of this stage, as in the model from our first chapter, the strong player announces his allocation to the fronts and discovers that of the weak player so that both players' first-stage allocations become common knowledge. Then in stage two, with complete information of the strong player's first-stage allocation, the weak player creates new battlefields in one or more fronts, if doing so increases his final-stage payoff. As with the single-stage game, third-stage payoffs for both players, in each front, are determined using expected payoffs from both the auction and lottery CSFs, and they are a function of each player's allocation and the number of battlefields in a front.

Given the sequential nature of this extensive-form game, AQRE naturally models the likelihood of using a dimensionality strategy in each subgame, defined at each information set. That is, the strong player decentralizes by announcing his allocation in stage one, or not, but then the weak player can choose to create new battlefields, or not. Thus, there are four distinct outcomes based on the dimensionality strategies employed by the players. This will be discussed in further in our section on 2.3.

Finally, we model the heterogeneous-QRE (HQRE) of the three-stage sequential game wherein the experience level ψ is no longer common to the two players. This is equivalent to saying bounded rationality is unique to each individual, which is a logical assumption for our contest model. The process for modeling the HQRE is nearly identical to that of the AQRE except the probability a player uses a particular behavior strategy is governed by either ψ^S, which we interpret to be the strong player's experience level, or ψ^W, the weak player's experience level. Significantly, QRE, AQRE, and HQRE all approach Nash equilibrium as one or more players become less prone to errors.

Together, testing the effects of changing the external and internal noise parameters r and ψ paints a clearer picture of the role of information in the decentralized Colonel Blotto game. Sufficiently high levels of external complexity diminish the need for dimensionality strategies, while high levels of personal experience lead players to recognize the benefits of dimensionality strategies, even if they only lead to marginal improvements in expected payoffs when r is low. These results will be detailed in the sections that follow.

2.1.1 Literature Review

The basic Colonel Blotto game was formulated by Borel (1921), then expanded upon in Borel and Ville (1938) when it was solved for $n = 3$ battlefields and equal resource endowments between two players. Later advances came with Friedman (1958), who solved the game for $n \geq 3$ battlefields using a lottery CSF, Roberson (2006) who fully characterized the game for $n \geq 3$ battlefields and an auction CSF, and Macdonell and Mastronardi (2015) who solved it for the case of $n = 2$. Especially relevant here is the decentralized Colonel Blotto game developed in our last chapter which is based on the payoffs found in Roberson (2006). This builds on the ideas developed in Kovenock et al. (2010) and Kovenock and Roberson (2012), which, respectively, show that under an auction CSF a weak player will expend resources to create more battlefields, thus increasing the number of battles over which he can randomize and thereby improve his payoff; and Kovenock and Roberson (2012) details the conditions under which two allies will exchange resources when facing a common adversary. If the allies have a binding agreement regarding the *ex post* division of payoffs, the two allies act as a single entity and allocate their resources to both fronts in a manner that leaves no soft spot for the weaker adversary to exploit. Rietzke and Roberson (2013) follow this by showing robustness under a lottery CSF.

In this paper, we compare the results of our previous chapter on the decentralized Colonel Blotto game using an auction CSF to those obtained using Friedman's lottery, and thus show the robustness of subgame perfect equilibria. Alternative CSFs have been proposed, including Hirshleifer (1989)'s difference-form CSF which has the advantage of determining winners based on numerical differences in the contest inputs rather than simply the ratio of resource commitments, which also allows for one-sided submission that in some instances may be in the losing party's best interest. Yet, while this and other CSFs are interesting, we limit our work to the ratio-form and its characteristics, especially in regards to the noise parameter.

Tullock (1980)'s focus was on rent-seeking, but other authors use the ratio-form CSF to explore tournaments, military conflict, and political campaigns, among others. Significant among this work is a wide body of literature seeking to characterize properties of the CSF.

For example, the all-pay auction, which Tullock (1980) showed to be a special case of the ratio-form CSF with $r = \infty$, possesses certain properties identified by Baye et al. (1993) and Che and Gale (1998). Baye et al. (1993) introduces the *exclusion principle*, which is that politicians may benefit from excluding those lobbyists who value a prize most. By excluding those lobbyists with the highest valuations, the politician receives more bids in aggregate because lobbyists with lower valuations of the prize are not discouraged. That is, the lobbyists with lower valuations feel their efforts will be competitive, so they are likely to submit more and higher bids in total. Che and Gale (1998) builds on this result by showing that when caps are put on campaign contributions, this can increase the sum of contributions to a politician because lobbyists with lower valuations of the prize bid more aggressively as their probability of winning increases. Interestingly, as shown by Fang (2002) and highlighted by Rietzke and Roberson (2013), when a lottery CSF is used in these situations (r low), the results found in these papers fail to hold: the exclusion principle does not apply, nor does the perverse effect of caps on lobbying increase aggregate expenditure. As will be shown, these findings are similar to ours regarding dimensionality strategies in the dencentralized Colonel Blotto game. As noise increases, returns to effort decrease, whether in a lobbying environment or on partitioned battlefronts.

Baye et al. (1994) and Baye et al. (1996) explore the role of noise on ratio-form CSFs further. Baye et al. (1994) shows that for $r > 2$, mixed-strategy Nash equilibria exist, and Baye et al. (1996) fully characterizes these equilibria by showing that there exist both a unique symmetric equilibrium and a continuum of asymmetric equilibria, but all are payoff equivalent. Moreover, Ewerhart (2017) makes the important finding that any equilibrium in a probabilistic contest (i.e., a CSF with $r < \infty$), when there exists a unique equilibrium in the corresponding all-pay auction ($r = \infty$), is actually an equilibrium in the all-pay auction, thus making the equilibrium from the probabilistic contest payoff-equivalent to the corresponding all-pay auction. In a similar vein, A. R. Robson (2005) shows that for lottery CSFs of $0 < r < 1$, the equilibrium resource commitments are equivalent to the case of $r = 1$. Together, these two results are highly relevant for our purposes since they effectively limit the range of r over which we must test the ratio-form CSF in the decentralized Colonel Blotto game. In other words, we need only compare subgame perfect equilibria using two

CSFs: the lottery CSF with $r = 1$, and the all-pay auction CSF with $r = \infty$. Since we already demonstrate subgame perfect equilibria under the auction CSF in our first chapter, our focus here is on the lottery CSF.

More recent work includes Kovenock and Arjona (2019), which summarizes best-response functions in Colonel Blotto games with a lottery CSF, and Chowdhury et al. (2021) which looks at Blotto games with lottery CSFs where the battlefields have asymmetric valuations. In this last paper, the salience of particular battlefields results in overbidding relative to the prediction of Nash equilibrium. Also, Drugov and Ryvkin (2020) examines the role of noise in tournaments specifically, but insights can be extended to multi-battle contests as well.

In terms of experimental work on Colonel Blotto games, there is a growing literature which employs the auction CSF (e.g., Avrahami and Kareev (2009), Arad and Rubinstein (2012), Chowdhury et al. (2013), Arad and Rubinstein (2012), Montero et al. (2016), and Mago and Sheremeta (2017)), but fewer feature the lottery CSF. Of these, Chowdhury et al. (2013) is most notable for our purposes as they find the pure strategy equilibrium holds where both players evenly distribute forces to all battlefields.

Experimental evidence of Nash equilibrium resource allocations in the Colonel Blotto game is of particular interest here since we also seek to model the quantal response equilibrium. QRE was first introduced by McKelvey and Palfrey (1995) as a way to find the behavioral equilibrium of a contest given player experience levels. McKelvey and Palfrey primarily rely on the logit QRE, which is the formulation used in this paper, but we also utilize the extensive-form version presented in Mckelvey and Palfrey (1998) as AQRE. QRE was first adapted to accommodate heterogeneity in player skill levels in normal-form games by Mckelvey et al. (2000), but Weizsäcker (2003) extended this by allowing players to have incorrect beliefs regarding the other players' types. For the purpose of showing this heterogeneity of types and beliefs about other players' types, we follow Rogers et al. (2009) which formalizes the ideas for HQRE. Finally, Goeree et al. (2016) contains a compilation of all these advances, and others.

QRE has been applied to the Colonel Blotto game only in a limited fashion. Lim et al. (2010) runs experiments on subject expenditures in a contest with a ratio-form CSF and shows that logit QRE predicts the expenditure distributions. Lim et al. (2014) again uses

QRE to model behavior in experiments on Tullock contest games, and compares these to cognitive hierarchy models. Sheremeta (2011) performs experiments on several forms of contests and concludes that subjects over-dissipate when endowments are higher, and this behavior is predicted by the QRE.

Beyond these there has been very little effort to apply QRE to Colonel Blotto games, and we are not aware of any attempt to apply QRE to an extensive-form Blotto game. Therefore our contributions in this regard are to first, show robustness of equilibria in the decentralized Colonel Blotto game using a lottery CSF and interpret these results; and second, apply AQRE and HQRE to the decentralized Colonel Blotto game and show how behavior tends toward the subgame perfect equilibrium when player experience increases, including in those cases where heterogeneity is present.

In the sections that follow, we begin by reviewing the model of the decentralized Colonel Blotto game. We then solve for the allocation-stage local equilibria of the extensive-form game using backward induction, as in our previous chapter, but now with final-stage payoffs determined by a lottery CSF. We then compare these subgame perfect equilibria to those found in the first chapter using the auction CSF and discuss the robustness of our results to summarize the effects of exogenous noise on player strategies. In the section entitled 2.3, we present the simultaneous QRE and sequential AQRE and HQRE of the players' dimensionality strategies using both the auction and lottery CSFs and show that when experience increases (i.e., noisy best responses over decisions decreases), player actions in the extensive-form game begin to approximate the demonstrated allocation-stage local equilibria.

2.2 Robustness of the Decentralized Colonel Blotto Game

2.2.1 Model

We first give an overview of the Colonel Blotto game with decentralization. There are two players, S and W, with asymmetric resources endowments, $X^S \geq X^W$, such that one player is strong and the other weak. There is also a set of $n^0 \geq 6$ initial battlefields over which the two players compete, separated by an exogenous partition so that the n^0 battlefields form $\mu \geq 2$ symmetric fronts of $n^0_{\mathrm{i}} = \frac{n^0}{\mu} \geq 3$ battlefields each.

The game proceeds in two stages. In the first stage, both players make permanent and simultaneous allocations of force to each front, effectively decentralizing command of their forces into two or more subsets. This means that allocations are private information until both players have completed deploying their resources, at which point both players' allocations are made public. We assume the action of decentralization to be costless, and as we showed in the previous chapter, only the strong player benefits from decentralized subsets of battlefields. Furthermore, when partitioned battlefields exist and the strong player decentralizes his forces, each front in the game becomes its own Colonel Blotto subgame, or G_{i} for $\mathrm{i} = 1, ..., \mu$ fronts. Alternatively, if the strong player chooses to remain centralized, player allocations remain private, the players have no information regarding each other's allocations, and the two-stage game appears to the players as if it were a single-stage, simultaneous move Blotto contest. That is, competition proceeds as if only one subgame exists, G_{i}, where $\mathrm{i} = 1$. In all cases, to utilize the results obtained by Roberson (2006), we require that $n_{\mathrm{i}}^0 \geq 3$, $\mathrm{i} = 1, ..., \mu$. Furthermore, each battlefield $\mathrm{j} \in \{1, ..., n_{\mathrm{i}}^0\}$ in subgame G_{i} has an equal payoff of $v = \frac{1}{n^0}$, making the total value of subgame G_{i} equal to $\phi_{\mathrm{i}} \equiv n_{\mathrm{i}}^0 v = \frac{n_{\mathrm{i}}^0}{n^0}$.

In the second stage of the game, the two players distribute their allocated forces to the battlefields within each front. Their force distributions are then revealed and a player's probability of winning a given battlefield is exactly proportional to the forces he committed to that battlefield, relative to the sum of both players' expenditures, as given in Equation 2.1 when $r = 1$. The game is constant-sum, resources distributed to each battlefield by the players must be nonnegative, and because player payoffs are increasing in expenditure, unspent resources have no value and players have incentive to allocate their full budgets.

2.2.2 Strategies

Each player has a pre-conflict strategy in stage one of the game, and both have conflict strategies during the second stage. In stage one, if the strong player chooses to decentralize, he makes an allocation X_{i}^S to each subgame G_{i}, where $\sum_{\mathrm{i}=1}^{\mu} X_{\mathrm{i}}^S \leq X^S$. Likewise, the weak player makes an allocation X_{i}^W in stage two such that $\sum_{\mathrm{i}=1}^{\mu} X_{\mathrm{i}}^W \leq X^W$. As in chapter one, if the strong player does not to decentralize in stage one, then i $= \mu = 1$, $X_{\text{i}}^S = X^S$, and $X_{\text{i}}^W = X^W$.

Each game in the set $\{G_1(X_1^S, X_1^W), ..., G_\mu(X_\mu^S, X_\mu^W)\}$ represents a final-stage subgame of the overall two-stage Colonel Blotto game, $\Gamma(G_1, ..., G_\mu, X^S, X^W)$, and the players compete simultaneously in each subgame by announcing their distributions of force across the battlefields of the games. Each player's payoff equals the expected value of all battlefields won across all subgames.

The following sections solve the decentralized Colonel Blotto game using backward induction and compare these results to those obtained in chapter one under the all-pay auction CSF. First, we show the Nash equilibrium second-stage force distributions and payoffs for each player under a lottery CSF, then we examine stage one subgame perfect equilibria resulting from the players' allocations.

2.2.3 Analysis

To begin the backward induction, recall that in the second-stage multi-battle conflicts the CSF determines the probability that player k wins battlefield j in subgame i given the expenditure of resources that each player dedicates to ij in stage one, or $(x_{\text{ij}}^S, x_{\text{ij}}^W)$ for j $= 1, ..., n_{\text{i}}^0$ and i $= 1, ..., \mu$. Note also that $\sum_{\text{j}=1}^{n_{\text{i}}^0} x_{\text{ij}}^k \leq X_{\text{i}}^k$ for $k \in \{S, W\}$ is feasible given the players' budget constraints, $\sum_{\text{i}=1}^{\mu} X_{\text{i}}^S \leq X^S$ and $\sum_{\text{i}=1}^{\mu} X_{\text{i}}^W \leq X^W$. For convenience, the lottery CSF is repeated in Equation 2.2 below where $p_{\text{ij}}^k(x_{\text{ij}}^k, x_{\text{ij}}^{-k})$ represents the probability that player k wins battlefield ij, given his effort x_{ij}^k, the effort of his opponent x_{ij}^{-k}, and the returns to effort $r = 1$.

$$p_{\text{ij}}^k(x_{\text{ij}}^k, x_{\text{ij}}^{-k}) = \begin{cases} \frac{(x_{\text{ij}}^k)}{(x_{\text{ij}}^k)+(x_{\text{ij}}^{-k})}, & \text{if } (x_{\text{ij}}^k, x_{\text{ij}}^{-k}) \neq (0,0) \\ \frac{1}{2}, & \text{if } (x_{\text{ij}}^k, x_{\text{ij}}^{-k}) = (0,0) \end{cases} \tag{2.2}$$

Naturally, it follows that $p_{\text{ij}}^{-k}(x_{\text{ij}}^k, x_{\text{ij}}^{-k}) = 1 - p_{\text{ij}}^k(x_{\text{ij}}^k, x_{\text{ij}}^{-k})$.

Next, because we assume each player seeks to maximize the expected sum of the values of battlefields won, let $\pi_{\mathrm{i}}^k(X_{\mathrm{i}}^k, X_{\mathrm{i}}^{-k})$ denote the expected payoff for k in subgame i, given the resource commitments $(\{X_{\mathrm{i}}^S\}_{\mathrm{i}=1}^{\mu}, \{X_{\mathrm{i}}^W\}_{\mathrm{i}=1}^{\mu})$. This results in

$$\pi_{\mathrm{i}}^k(X_{\mathrm{i}}^k, X_{\mathrm{i}}^{-k}) = v \sum_{\mathrm{j}=1}^{n_{\mathrm{i}}} p_{\mathrm{ij}}^k(x_{\mathrm{ij}}^k, x_{\mathrm{ij}}^{-k})$$

for $\mathrm{i} = 1, ..., \mu$, and the total payoff for player k in $\Gamma(G_1, \ldots, G_\mu, X^S, X^W)$ is

$$\pi^k(X^S, X^W) = \sum_{\mathrm{i}=1}^{\mu} \pi_{\mathrm{i}}^k(X_{\mathrm{i}}^k, X_{\mathrm{i}}^{-k})$$

Now, we can obtain the unique Nash equilibrium expected payoffs for all the second-stage subgames.

From Friedman (1958), we know that a set of pure-strategy Nash equilibrium resource allocations for subgame G_{i} is $x_{\mathrm{ij}}^{k*} = \frac{X_{\mathrm{i}}^k}{n_{\mathrm{i}}^0}$ (Rietzke & Roberson, 2013). This follows intuitively from Equation 2.2 and the assumption that all battlefields in G_{i} have equal value, $v > 0$. So, each player will evenly distribute the forces allocated to front i (i.e., X_{i}^k) to all battlefields in that front. Remembering that $\phi_{\mathrm{i}} \equiv n_{\mathrm{i}}^0 v$, this results in the following unique equilibrium expected payoff,

$$\pi_{\mathrm{i}}^{k*}(X_{\mathrm{i}}^{k*}, X_{\mathrm{i}}^{-k*}) = v \sum_{\mathrm{j}=1}^{n_{\mathrm{i}}} \frac{x_{\mathrm{ij}}^{k*}}{x_{\mathrm{ij}}^{k*} + x_{\mathrm{ij}}^{-k*}} = \phi_{\mathrm{i}} \frac{X_{\mathrm{i}}^k}{X_{\mathrm{i}}^k + X_{\mathrm{i}}^{-k}}.$$

Note that the equilibrium payoffs in each front are the same as if each front was a single battle with a lottery CSF, and where the value of a front is equal to the sum of the battlefield valuations within that front. Following Rietzke and Roberson (2013), this allows us to make an abuse of notation and write each player's unique equilibrium expected payoff in terms of the resource allocations to each front. This is summarized in Theorem 2.2.1 and proved in Friedman (1958).

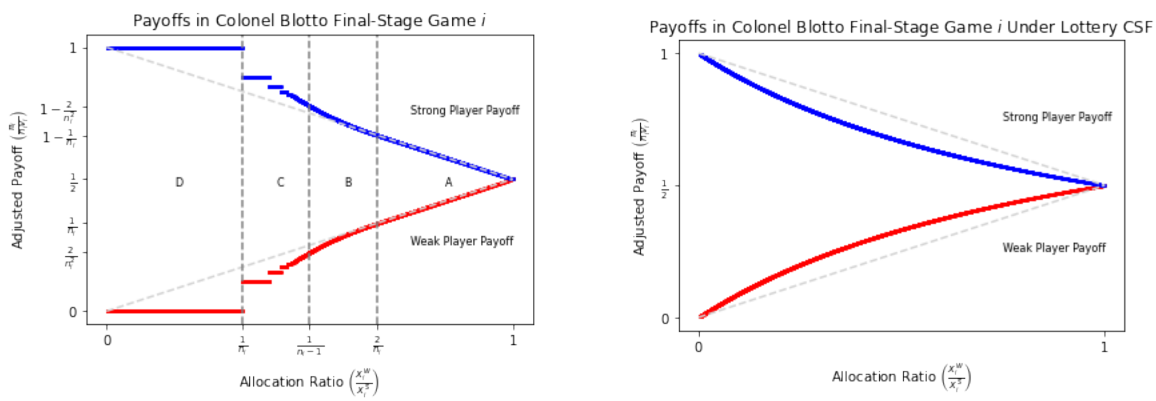
Theorem 2.2.1. *[**Friedman (1958)**] Given the players' resource allocations to subgame* i*,* $(X_{\mathrm{i}}^S, X_{\mathrm{i}}^W)$*, the unique Nash equilibrium expected payoff of the second-stage Colonel Blotto subgame,* $\mathrm{i} = 1, ..., \mu$ *and* $k \in \{S, W\}$*, is*

$$\pi_{\mathrm{i}}^k(X_{\mathrm{i}}^k, X_{\mathrm{i}}^{-k}) = \phi_{\mathrm{i}} \frac{X_{\mathrm{i}}^k}{X_{\mathrm{i}}^k + X_{\mathrm{i}}^{-k}}. \tag{2.3}$$

Obviously, the strong player will always have a higher expected equilibrium payoff than the weak player. However, unlike the case of the auction CSF of our previous chapter, the role of the players has no impact on the structure of the expected payoffs received, which is why we simply refer to players k and $-k$. Given a positive investment of resources, each player receives an expected payoff proportional to the effort exerted.

More importantly, here the number of battlefields has no impact on the players' individual expected payoffs. Theorem 1 of our previous chapter details four distinct payoff regions, each resulting from a particular combination of the weak player's allocation, the strong player's allocation, and the number of battlefields. This is shown graphically in Figure 3.2a. In region A the players' respective payoffs depend only on the resource allocations. Yet, in regions B and C the payoffs depend on allocations and the number of battlefields, and the weak player's payoff increases as the number of battlefields increase. Thus, in a game with the auction CSF, the weak player has incentive to create new battlefields until the combination of allocations and quantity of battlefields yields payoffs in region A, at which point payoffs become linear and independent of the number of battlefields.

In contrast, when we employ a lottery CSF there is a single payoff region and expected payoffs are wholly dependent on the relative resource allocations of the players to game G_{i}. This means that any change to the number of battlefields has no effect on the players' expected payoffs. That is, for any number of battlefields, because the probability of winning is stochastic and only dependent on the level of resources committed to a battlefield, each player has a pure strategy in deploying forces equally to each battlefield in the game. As a result, a player's expected payoff for game G_{i} is in proportion to his endowment, or in the case of exogenously partitioned subsets of battlefields, the amount of resources he has evenly spread to each Colonel Blotto subgame G_{i}. This is depicted in Figure 3.2b where each player's payoff is shown to be exactly proportional to his endowment. Because endowed resources are assumed to be in continuous units, this results in smooth payoff curves and a single payoff region for all ratios of the players' relative resource levels.



(a) $r = \infty$

(b) $r = 1$

Figure 2.1. Comparison of payoffs obtained under the all-pay auction and the lottery CSFs

2.2.4 First Stage

Because the number of battlefields has no impact on the players' equilibrium expected payoffs, there is trivially no incentive for the weak player to create new battlefields, especially if doing so incurs a cost. This is sufficient to indicate the results of Kovenock et al. (2010) do not apply to cases where a lottery CSF is used. Therefore, we can state in regard to W's dimensionality strategy, that $n_{\mathrm{i}}^{W,\mathbb{R}} = 0$ for $\mathrm{i} = 1, ..., \mu$, and $n_{\mathrm{i}} = n_{\mathrm{i}}^0 + n_{\mathrm{i}}^{W,\mathbb{R}} = n_{\mathrm{i}}^0$, where n_{i}^0 is the original number of battlefields in front i and n_{i} is the final number of battlefields. Likewise, without a dimensionality effect the strong player has no incentive to decentralize his forces, and he has no need to counter the weak player's dimensionality strategy. Nonetheless, given the second-stage equilibrium resource distributions in which both players spread their forces evenly to all n^0 battlefields, it should be that any first-stage allocation to exogenously partitioned subgames is equivalent to the case without decentalization, i.e., the centralized game. We will show that this is indeed the case.

To begin the first-stage allocations, consider the player's optimization problem. Each player allocates his forces to solve

$$\begin{aligned} \max_{\{X_i^k\}_{i=1}^{\mu}} \quad & \sum_{i=1}^{\mu} \pi_i^k \left(X_i^k, X_i^{-k} \right) = \sum_{i=1}^{\mu} \phi_i \frac{X_i^k}{X_i^k + X_i^{-k}} \\ \text{s.t.} \quad & \sum_{i=1}^{\mu} X_i^k \leq X^k. \end{aligned} \tag{2.4}$$

And in an environment of private information, we can model the player's allocations as if they take place simultaneously. In Proposition 2.2.1 we show that both players $k \in \{S, W\}$ have an allocation-stage local equilibrium consisting of an interior solution that solves $\frac{X_1^k}{\phi_1} = \cdots = \frac{X_\mu^k}{\phi_\mu}$.

Proposition 2.2.1. *In the allocation stage of the decentralized Colonel Blotto game with a lottery CSF, both players allocate their budget of resources to every front such that* $\frac{X_1^W}{\phi_1} = \cdots = \frac{X_\mu^W}{\phi_\mu}$ *and* $\frac{X_1^S}{\phi_1} = \cdots = \frac{X_\mu^S}{\phi_\mu}$*, and doing so is a unique allocation-stage local equilibrium.*

Proof. The proof of Proposition 2.2.1 consists of two parts. First, we show by way of contradiction that an interior solution exists and is the only feasible allocation. Then, using interchangeability of equilibria, we prove the proportional-value allocation, $\frac{X_1^k}{\phi_1} = \cdots = \frac{X_\mu^k}{\phi_\mu}$, $k \in \{S, W\}$, is the unique interior solution. Let α define the ratio of player resource endowments such that $X^S = \alpha X^W$ for $\alpha \in (0, 1]$.

To begin, consider the set of fronts $\{1, \ldots, \mu\}$ and normalized player budgets $\{X^S, X^W\}$, where $X^S \geq X^W = 1$. Then given that resources have positive shadow values, there obviously exists no allocation-stage local equilibrium in which both players allocate zero resources to all fonts, or $X_i^S = X_i^W = 0$, $\forall$i. For if this were to occur, either player could achieve a payoff increasing deviation by allocating $X_i^k \in (0, X^k]$ to one or more fronts, i $\in \{1, \ldots, \mu\}$, thus capturing the full value of those fronts. Similarly, it also cannot be the case that only one player $k \in \{S, W\}$ allocates zero resources to every front, $X_i^k = 0$, $\forall$i. For by Theorem 2.2.1, any positive allocation to a front results in a strictly positive payoff in expectation, thus constituting a profitable deviation regardless of the opposing player's allocation.

Finally, by the same logic, it cannot be the case that in equilibrium either player allocates zero resources to any front. By way of contradiction, suppose there was an allocation-stage local equilibrium in which player k allocated resources to $\mu - \ell$ fronts and zero resources to the remaining $\ell \in (0, \mu)$ fronts. But then there exists a profitable deviation for player $-k$ to shift resources away from one or more of the ℓ fronts, leaving only an arbitrarily small allocation $\frac{\epsilon}{2} > 0$ on each front while still capturing the full value of the fronts, $\sum_{\mathrm{i}=1}^{\ell} \phi_{\mathrm{i}}$, and increasing his payoff on the other $\mu - \ell$ fronts by $\sum_{\mathrm{i}=1}^{\mu-\ell} \frac{X_{\mathrm{i}}^{-k}+\iota}{X_{\mathrm{i}}^{k}+(X_{\mathrm{i}}^{-k}+\iota)}$ where $\iota = \frac{\sum_{\mathrm{i}=1}^{\ell}(X_{\mathrm{i}}^{-k}-\frac{\epsilon}{2})}{\mu-\ell}$. Hence, a contradiction and it must be the case that in equilibrium both players allocate a positive level of resources to every front, i.e., the interior solution.

Now, we show the proportional-value allocation, $\frac{X_1^k}{\phi_1} = \cdots = \frac{X_\mu^k}{\phi_\mu}$, $k \in \{S, W\}$, is the unique interior solution. To do this, we make use of the fact the allocation-stage subgame is constant-sum and allocation-stage local equilibria are interchangeable. That is, if we interchange an allocation-stage local equilibrium $(\{X_{\mathrm{i}}^k, X_{\mathrm{i}}^{-k}\}_{\mathrm{i}=1}^{\mu})$ in which $\frac{X_{\mathrm{i}}^k}{\phi_{\mathrm{i}}} \neq \frac{X_{\mathrm{j}}^k}{\phi_{\mathrm{j}}}$ for some fronts $\mathrm{i} \neq \mathrm{j}$, with the Proposition 2.2.1 allocation-stage local strategy profile that arises from the proportional-value allocation, $(\{X_{\mathrm{i}}^{k*}, X_{\mathrm{i}}^{-k*}\}_{\mathrm{i}=1}^{\mu})$, the resulting interchanged equilibrium exists only if the payoff resulting from the interchanged equilibria is equal to that resulting from the proportional-value allocation. We will show this cannot be the case.

From Theorem 2.2.1 we know the payoff to player k for making allocation X_{i}^k to front i is $\phi_{\mathrm{i}}\frac{X_{\mathrm{i}}^k}{X_{\mathrm{i}}^k+X_{\mathrm{i}}^{-k}}$. Also, because we assume fronts are symmetrically-sized and homogenously-valued, it must be that $\phi_1 = \phi_2 = \cdots = \phi_\mu = \phi$. Then considering the proportional value allocation, $\frac{X_1^k}{\phi_1} = \cdots = \frac{X_\mu^k}{\phi_\mu}$, we can say $\frac{1}{\phi}(X_1^k) = \cdots = \frac{1}{\phi}(X_\mu^k)$, which implies $X_1^k = \cdots = X_\mu^k = \frac{X^k}{\mu}$. This in turn implies the ratio

$$\frac{X_{\mathrm{i}}^k}{X_{\mathrm{i}}^k + X_{\mathrm{i}}^{-k}} = \frac{X^k/\mu}{X^k/\mu + X^k/\mu} = \frac{X^k}{X^k + X^{-k}}.$$

Then letting $\sum_{\mathrm{i}=1}^{\mu} \phi_{\mathrm{i}} = \mu\phi = \Phi$, we have player k's payoff across all fronts,

$$\pi^k(X^{k*}, X^{-k*}) = \sum_{\mathrm{i}=1}^{\mu} \phi \frac{X^k}{X^k + X^{-k}} = \mu\phi \frac{X^k}{X^k + X^{-k}} = \Phi \frac{X^k}{X^k + X^{-k}}.$$

Now, suppose player k uses their allocation-stage local strategy in the interchanged equilibrium so that $(\{X_{\mathrm{i}}^k, X_{\mathrm{i}}^{-k*}\}_{\mathrm{i}=1}^{\mu})$. This results in an allocation that differs from Proposition 2.2.1 in that $\frac{X_{\mathrm{i}}^k}{\phi_{\mathrm{i}}} \neq \frac{X_{\mathrm{j}}^k}{\phi_{\mathrm{j}}}$ when $\mathrm{i} \neq \mathrm{j}$. Yet, it remains the case that $\phi_{\mathrm{i}} = \phi_{\mathrm{j}} = \phi$, which means that $X_{\mathrm{i}}^k \neq X_{\mathrm{j}}^k$, but $X_{\mathrm{i}}^{-k} = X_{\mathrm{j}}^{-k} = \frac{X^{-k}}{\mu}$. This produces an interchanged equilibrium payoff of

$$\pi^k(X^k, X^{-k*}) = \Phi \sum_{\mathrm{i}=1}^{\mu} \frac{X_{\mathrm{i}}^k}{X_{\mathrm{i}}^k + {X^{-k}}/{\mu}} < \Phi \frac{X^k}{X^k + X^{-k}} = \pi^k(X^{k*}, X^{-k*})$$

when $X_{\mathrm{i}}^k \neq X_{\mathrm{j}}^k$ for some $\mathrm{i} \neq \mathrm{j}$ and $\mathrm{i}, \mathrm{j} \in \{1, \ldots, \mu\}$, and it is due to the diminishing marginal returns of Theorem 2.2.1 when $\alpha \to 1$. Hence, interchangeability of equilibria fails, and $\frac{X_1^k}{\phi_1} = \cdots = \frac{X_\mu^k}{\phi_\mu}$ must be a unique allocation-stage local equilibrium for both players $k \in \{S, W\}$. □

Note that the solution in Proposition 2.2.1 is the classic no soft-spot result from our first chapter, Rietzke and Roberson (2013), Kovenock and Roberson (2012), and others, where a player commits resources to front i according to the value of i and relative to the values of all other fronts. But here, the result applies equally to S and W. More notably, neither player uses a stochastic allocation strategy in the first-stage local subgame. Rather, the equilibrium is in pure strategies.

This pure strategy interior solution for both players is a direct result of the noisy environment induced by the lottery CSF, wherein both players have a positive probability of winning each front they enter, proportional to the investment they make to each front. It is also payoff equivalent to the case where there is no decentralization, indicating the players will be indifferent to any partition in the battlefields, and they will certainly not decentralize their forces in the allocation-stage if doing so incurs a cost. This result is summarized in Proposition 2.2.2.

Proposition 2.2.2. *In the decentralized Colonel Blotto game with a lottery CSF, n^0 original battlefields, and μ exogenously partitioned fronts, players are indifferent to dimensionality strategies when there is no cost to decentralize or create new battlefields; and players will not use dimensionality strategies when doing so incurs a cost.*

Proof. From Theorem 2.2.1 it is trivially obvious that the weak player's payoff is unaffected by the number of battlefields, therefore he will not expend resources to create new ones. As for the strong player, if he chooses not to decentralize in stage one, second-stage competition takes place over a single subgame, or rather $\mathrm{i} = \mu = 1$. Therefore, if all n^0 battlefields have equal value $v = \frac{1}{n^0}$, and recalling that $\phi \equiv n_{\mathrm{i}}^0 v = \frac{n_{\mathrm{i}}^0}{n^0}$ for all i, then any allocation made to this game by player k is equivalent to the total budget of player k, or $X_1^k = X^k$. The pure-strategy Nash equilibrium distribution is $x_{1\mathrm{j}}^k = \frac{X_1^k}{n_1^0} = \frac{X^k}{n^0}$, and the equilibrium expected payoff is

$$\pi^k(X^k, X^{-k}) = \sum_{\mathrm{j}=1}^{n} v \frac{x_{\mathrm{j}}^k}{x_{\mathrm{j}}^k + x_{\mathrm{j}}^{-k}} = \Phi \frac{X^k}{X^k + X^{-k}}.$$

In like manner, by applying the players' best responses in the decentralized game, $\frac{X_1^k}{\phi_1} = \cdots = \frac{X_\mu^k}{\phi_\mu}$, we get the equilibrium expected payoff which is identical to the game without decentralization.

$$\begin{aligned}
\pi^k(X^k, X^{-k}) &= \sum_{\mathrm{i}=1}^{\mu} \sum_{\mathrm{j}=1}^{n_{\mathrm{i}}} v \frac{x_{\mathrm{ij}}^k}{x_{\mathrm{ij}}^k + x_{\mathrm{ij}}^{-k}} \\
&= \sum_{\mathrm{i}=1}^{\mu} \phi_{\mathrm{i}} \frac{X_{\mathrm{i}}^k}{X_{\mathrm{i}}^k + X_{\mathrm{i}}^{-k}} \\
&= \Phi \frac{X^k}{X^k + X^{-k}}.
\end{aligned}$$

□

2.2.5 Discussion

Testing the robustness of our decentralization model from chapter one requires changing the CSF from a deterministic all-pay auction to a stochastic lottery CSF and verifying the previous results still hold. Notationally, this requires changing r in the ratio-form CSF of Equation 2.1 from ∞ to 1, but symbolically this change represents the addition of significant amounts of noise so that the players' returns to effort are much lower. In fact, when $r = 1$, the returns to effort are so low that any strategy affecting the dimensionality of battlefields has no effect on payoffs. So, neither player will have incentive to increase the number of battlefields or decentralize if doing so incurs a cost, and the distribution of forces in the second-stage game are different from those of the auction CSF.

This observation could be perceived as a breakdown of the results obtained in chapter one, but in actuality it is an expected result. To see this, consider the more extreme case of $r = 0$. In this configuration, the CSF represents a fair lottery and each player has a 0.5 chance of winning battlefield j simply by choosing to allocate forces to it. In practical terms, there is so much external interference in the contest that any effort by the players beyond mere participation counts for nothing. Hence, each player has incentive to allocate an equal proportion of resources to each battlefield, however small, when unspent resources are worthless, and there is no incentive to decentralize, increase battlefields, or stochastically distribute forces. Equal allocation to each available battlefield is a pure strategy for both players, and the expected payoff for each battlefield is $0.5v$. Alternatively, if we assume as before that resources are in continuous units, and if unspent resources have positive value, then each player will dedicate an arbitrarily minuscule amount of resources in equal proportions to each battlefield, regardless of the actions of the other player, and receive the same expected payoff of $0.5v$. These results hold for any number of battlefields n^0, exogenous partitions μ, or resource endowments $\{X^S, X^W\}$ greater than zero.

Viewed in this way, it seems obvious that the level of r exists along a continuum, with $r = 0, 1$, and ∞ serving as special cases of the ratio-form CSF, and returns to effort depend on how deterministic the Colonel Blotto game is. It follows that as the CSF becomes more deterministic with increases in $1 < r \leq 2$, the utility of the propositions from chapter one increase for certain endowment ratios as the players are more likely to employ mixed strategies and stochastically distribute forces to battlefields.

The observations in this chapter also contribute to the findings of Fang (2002) and other papers that identify significant departures in equilibria when a lottery CSF is used in place of the all-pay auction. As noted in our review of the literature, the *exclusion principle* and *caps on lobbying principle* fail to hold when a lottery CSF is applied, and equilibria is in pure rather than mixed strategies. In related fashion, our current work shows that while dimensionality strategies constitute Nash equilibria in extensive-form games over multiple battlefields with an auction CSF, this important characteristic does not hold under the lottery CSF.

2.3 Quantal Response Equilibrium

We now examine the role of internal uncertainty in the decentralized Colonel Blotto game. Whereas r in the ratio-form CSF represents exogenous noise that impedes the effectiveness of player resource commitments, ψ in the logit equilibrium of McKelvey and Palfrey (1995) and Mckelvey and Palfrey (1998) symbolizes player experience:

> As a player gains experience playing a particular game and makes repeated observations about the actual payoffs received from different choices, he/she can be expected to make more precise estimates of the expected payoffs from different strategies. ...We refer to this as *learning* (McKelvey & Palfrey, 1995).

In other words, as players learn, ψ increases and expected payoffs improve.

We will model three variations of QRE, beginning with a slight departure from our model in chapter one and the previous section of this paper. Instead of only examining the effects of decentralization on player allocations and payoffs, we now permit both decentralization and the creation of new battlefields in an effort to demonstrate the QRE of dimensionality strategies as they relate to both players. We will study this dynamic using two versions of the extensive-form game, one featuring simultaneous moves and the other sequential moves. We will also show the outcome of each version using the auction and lottery CSFs.

The first version is reminiscent of our earlier model and it features two stages: an allocation stage and a Colonel Blotto conflict stage. The first stage features simultaneous allocations over an exogenous partition. The revelation of these allocations at the conclusion of the first stage is characterized as the strong player's dimensionality strategy since it forces the weak player to decentralize his own efforts in anticipation of the revelation, and compete over multiple independent Blotto subgames where the strong player has a resource advantage given the reduced number of local battlefields. As shown in the first chapter and previous section of this chapter, decentralization leads to reduced payoffs for the weak player under an auction CSF, but there is no change in payoffs under the lottery CSF. Additionally, in stage one the weak player is permitted to create new battlefields in response to the decentralization, potentially increasing his payoff in each subgame under the auction CSF. This is done in the manner outlined by Kovenock et al. (2010) wherein the creation of each new battlefield incurs a cost, and we assume this cost to be constant across all battlefields and fronts, $\mathrm{j} \in \{1, \ldots, n_{\mathrm{i}}^{0}\}$ and $\mathrm{i} \in \{1, \ldots, \mu\}$.

The sequential version of the model features three stages instead of the two. In the first stage, both players make simultaneous allocations to the fronts, which are observable to the players at the conclusion of the first stage if the strong player chooses to decentralize. Then, in the second stage, the weak player can create new battlefields at a cost, as in the simultaneous version. In stage three, the two players engage in simultaneous Colonel Blotto competition across all fronts and battlefields. Again, we model this interaction using both the auction and lottery CSFs.

All three variations of QRE use a logit form for determining the probability of using a dimensionality strategy. The first examines the simultaneous employment of dimensionality strategies based on McKelvey and Palfrey (1995), and we refer to this simply as the QRE. The sequential version of the game uses AQRE developed by Mckelvey and Palfrey (1998) where experience levels are assumed to be the same for both players. Finally, HQRE is modeled using the sequential version of the game and is similar to AQRE but with heterogeneous experience levels. This variation was developed by Rogers et al. (2009). We will discuss each in greater detail below.

Although QRE is not a perfect representation of behavioral equilibrium, and as a purely mathematical expression it often lacks a structural foundation in the phenomena it is used to model, it has proved to be a surprisingly accurate predictor of player behavior in a wide variety of experimental settings. As mentioned previously, a limited number of authors have used it to model behavior in contests with auction CSFs, but to our knowledge it has not been applied to the Colonel Blotto game specifically. Moreover, QRE is not often used in extensive-form games, while HQRE has not yet been applied to extensive-form games even though the application is fairly straightforward.

2.3.1 Model

The QRE of simultaneous player allocation in the first stage uses the same logit formula as that given by McKelvey and Palfrey (1995) for normal-form games,

$$p^k(a) = \frac{e^{\psi u_a^k}}{\sum_{a' \in A^k} e^{\psi u_a^k}}. \tag{2.5}$$

Equation 2.5 says that the probability player k uses his respective dimensionality strategy, a, is equal to the exponential of the expected utility to be gained by that strategy, u_a^k, weighted by the common experience level, ψ, and divided by the sum of the exponential weighted expected utilities of all strategies. We apply this to the two-stage decentralized Colonel Blotto game by taking the final-stage payoffs as given and then calculating the expected utilities of each player for a particular dimensionality strategy in the first stage. In contrast, the AQRE of Mckelvey and Palfrey (1998) models the probability a player will choose a particular behavioral strategy at each node of the sequential-move game based on the expected utility of doing so. For our purposes, we use it to model the probability each player employs a dimensionality strategy in one of the two pre-conflict stages of the sequential, decentralized Colonel Blotto game. We will discuss each of these approaches.

In the version with simultaneous allocation, the game has only two stages but four outcomes, and these payoffs can be collapsed into a single simultaneous-move game if we take the second-stage payoffs as given, as we did when referring to Theorem 1 of chapter one as the Blotto CSF. This first-stage game can then be depicted as in Table 2.1, regardless of whether payoffs are determined by an auction or lottery CSF.

Table 2.1. Strategic-form representation of the simultaneous Decentralized Colonel Blotto game

		Weak Player	
		No Increase	Increase
Strong Player	No Decentralization	Outcome 1	Outcome 2
	Decentralization	Outcome 3	Outcome 4

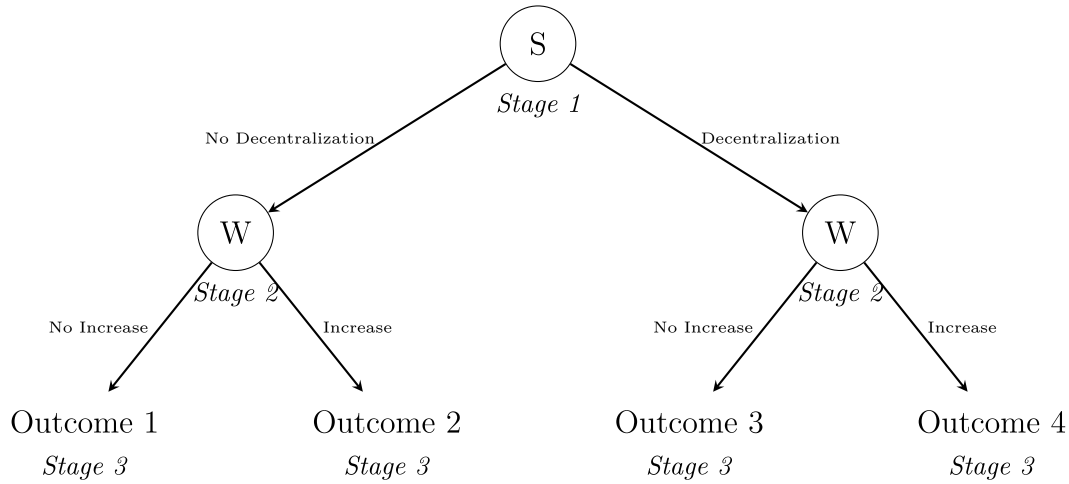


Figure 2.2. Extensive-form representation of the decentralized Colonel Blotto game

The first outcome represents the case when neither player uses a dimensionality strategy. This is the traditional, simultaneous-move Colonel Blotto game, and payoffs are awarded accordingly. The second is when the strong player does not decentralize by making first-stage allocations known, but the weak player does create new battlefields. Depending on the CSF and payoff region of the players, this outcome is likely to yield a higher payoff for the weak player than in the original game of the first outcome. The third is when the strong player decentralizes but the weak player does not increase battlefields, and this likely results in a higher payoff for the strong player than in the original game. The fourth is when both players use their respective dimensionality strategy, at least partially countering the strategy of their opponent.

For the sequential game, we use the same four potential outcomes but there are three decision nodes, each corresponding to one of three information sets. The sequential-move game and attendant outcomes are illustrated in Figure 2.2.

The payoff for each outcome is determined by the players' resource endowments, X^S and X^W; first-stage allocations (if applicable), X_{i}^S and X_{i}^W; the number of battlefields, n_{i}^0, in each third-stage subgame G_{i}, for $\mathrm{i} = 1, ..., \mu$; the number of fronts, μ; and the r-value, which determines the CSF to be used. Following Baye et al. (1994) and Ewerhart (2017), if $r > 2$ we use the auction CSF. If $r \leq 2$, we use the ratio-form CSF (i.e., the lottery CSF), with the input r-value. In accordance with our first chapter and Theorem 2.2.1 and Propositions 2.2.1

and 2.2.2 above, we assume all battlefields have equal value and players allocate their forces evenly to all fronts following the no soft-spot principle. Moreover, the sequential-move game is one of perfect information and players have perfect recall. For the case of the simultaneous QRE and the sequential-move AQRE, players are assumed to be of the same type, so only a single ψ is used. Later, we relax this assumption to analyze the HQRE. Finally, the optimal number of new battlefields created by the weak player, $n_{\text{i}}^{W,\mathbb{R}}$, is found using the first-order condition

$$-\frac{2}{\left(n_{\text{i}}^{0}+n_{\text{i}}^{W,\mathbb{R}}\right)^{2}}+\frac{4X_{\text{i}}^{S}}{X_{\text{i}}^{W}\left(n_{\text{i}}^{0}+n_{\text{i}}^{W,\mathbb{R}}\right)^{3}}-c'\left(n_{\text{i}}^{W,\mathbb{R}}\right)=0,$$

where $c(n_{\text{i}}^{W})$ is the cost of adding new battlefields, which we assume to be linear in n_{i}^{W} (Kovenock et al., 2010). There is no cost to decentralize.

Computing the logit probability for each player of the simultaneous game involves several steps. First, payoff outcomes are determined using either the (auction) Blotto CSF from Theorem 1 of chapter one or the lottery CSF. Then, a vector of notional probabilities are generated for each player, ranging from 0 to 1, representing the likelihood the player will use a dimensionality strategy. Next, using the notional probabilities of the opposing player and the calculated payoffs, we compute the expected utility of a player's dimensionality strategy. The probability that player k uses his dimensionality strategy is then determined using Equation 2.5, and the intersection of this probability with that of the opposing player determines the QRE.

Computing the AQRE and HQRE follows a similar procedure, except now the players use behavioral strategies executed at each information set. The payoff outcomes are determined as before, but only the strong player relies on expected utilities since, in the second stage, the weak player's possible payoffs are known with certainty. Additionally, the strong player must consider the weak player's experience level, ψ^{W}, and corresponding action probabilities when computing his own expected utility. He then selects the action offering the highest expected utility, but the probability he employs this action is influenced by his own experience, ψ^S. This logit probability is given in Equation 2.6.

$$b_l^k(a) = \frac{e^{\psi^k \bar{u}_{la}^k(b)}}{\sum_{a' \in A(h_l^k)} e^{\psi^k \bar{u}_{la}^k(b)}} \tag{2.6}$$

Here, $a \in A$ represents the action taken by player k at information set l within the set of all actions, A. The variable $\bar{u}_a^k(b)$ is the expected payoff for using action a within behavioral strategy b. Naturally, when $\psi^S = \psi^W$, the intersection of the two players' behavioral strategies at a particular information set is the AQRE. When $\psi^S \neq \psi^W$, it is the HQRE.

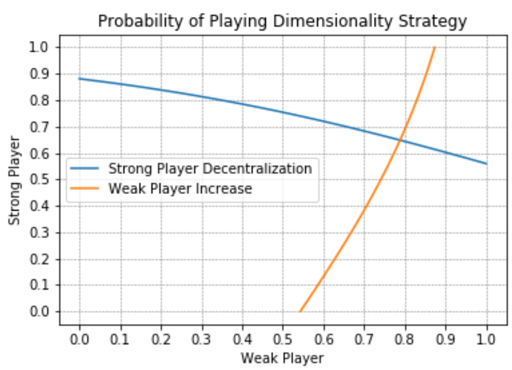
As explained by McKelvey and Palfrey (1995), the QRE represents a behavioral equilibrium, obtained when one of both players lack experience or are prone to committing mistakes and are therefore unable to identify or execute a best response (i.e., a noisy best response). Hence, as the value of ψ tends to infinity, the probability that each player uses a best response increases and the QRE converges to the Nash equilibrium.

2.3.2 Analysis

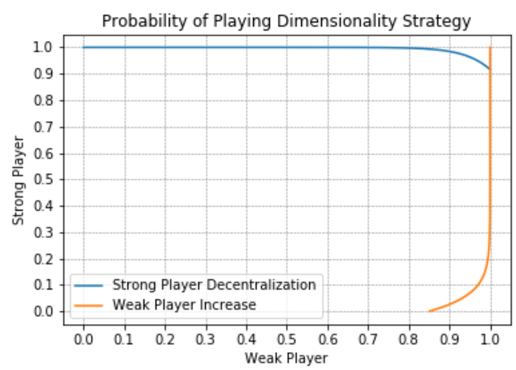
We tested the QRE over a variety of endowment levels, costs, r-values, and ψ-values. All configurations included six original battlefields ($n^0 = 6$) and two exogenous, symmetric fronts of three battlefields each ($\mu = 2$). We found the QRE to be most responsive to changes in cost, c, r, and ψ, while payoffs change as expected given different endowment levels. We demonstrate a sampling of the configurations of simultaneous QRE and sequential AQRE and HQRE in the charts below.

2.3.3 Simultaneous Allocation: QRE

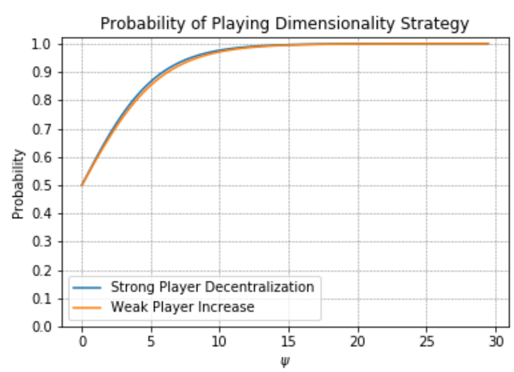
First, the intersection of the two curves in Figure 2.3a illustrates the QRE of the simultaneous allocation by both players. The figure is generated using player endowments of $(X^S, X^W) = (4, 1)$, which puts them in payoff region B of the Blotto CSF. The cost to increase battlefields is $c = 0.01$, and a low level of experience is symbolized by the common $\psi = 3$. Nevertheless, this still results in a high probability that both players use their dimensionality strategies. In Outcome 1 of this configuration, neither player uses a dimensionality strategy, and the weak and strong players receive the payoff tuple (0.67,5.33) from region B. In Outcome 2, the weak player creates one new battlefield, improving his payoff to (0.72,5.28), but the players are still in region B. When the strong player decentralizes in Outcome 3, the payoff is (0,6.0) from region D; and in Outcome 4 the strong player decentralizes, the weak player produces 7 new battlefields in each front, and the players end in region A with total payoffs of (0.65,5.35). The strong player has a probability of approximately 0.65 that he will decentralize, and the weak player will add battlefields at a probability of approximately 0.78.



(a) Simultaneous QRE, $r = \infty$, $\psi = 3$, $c = 0.01$



(b) Simultaneous QRE, $r = \infty$, $\psi = 30$, $c = 0.01$



(c) Simultaneous QRE, $r = \infty$, $c = 0.01$

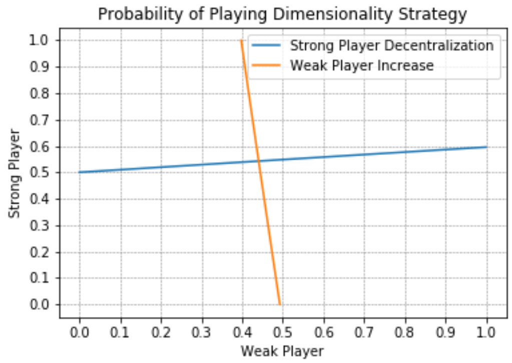
Figure 2.3. QRE as obtained from pre-conflict simultaneous allocation under the auction CSF

If we increase the players' experience level, perhaps through repeated interactions, and raise ψ to 30, the probability of using dimensionality strategies increases markedly, trending toward the Nash equilibrium, as shown in Figure 2.3b. However, the effect of ψ is more obvious in Figure 2.3c where the players' probability of using dimensionality strategies in the simultaneous game are plotted over a range of ψ and the opposing player's probability of using a dimensionality strategy is unknown (i.e., fixed at 0.5). Here, we can see that as experience improves, the players recognize the importance of using a dimensionality strategy, even if expected payoffs are uncertain. In fact, their probability of use goes to nearly 1.0 for $\psi \geq 15$, in accordance with Theorem 2 of McKelvey and Palfrey (1995). This realization suggests that in an experiment with human subjects, repeated play of the game in independent encounters would ultimately show a converging reliance on dimensionality strategies, regardless of role. Moreover, the QRE indicates that both players increase their use of dimensionality strategies at nearly the same rate for increases in a common ψ.

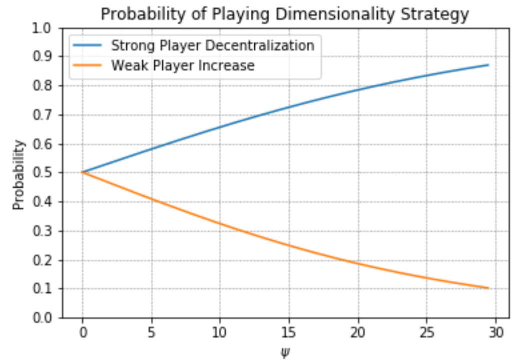
As noted by McKelvey and Palfrey (1995), while logit QRE is not trembling-hand perfect, this is not a drawback of the model. Rather, it is a feature that characterizes QRE as a model for illustrating the impact of learning in contests. In the simultaneous-move game with an auction CSF described above, players have a pure-strategy, allocation-stage local equilibrium in dimensionality strategies, and at the limit of $\psi \to \infty$, the logit equilibrium is not impacted by the magnitude of player payoffs. Yet for intermediate (i.e., low) values of ψ, the QRE can be sensitive to payoffs. Thus, as ψ goes to the limit, strategies with a sufficiently low probability in QRE do not affect players' choices in the rest of the game, and the limiting logit QRE becomes an equilibrium selection mechanism for varying levels of player experience and learning.

Switching to a lottery CSF has a pronounced effect, as the exogenous noise level in the simultaneous allocation game combines with the low experience of the players when we set $\psi = 3$ once more. This is shown in Figure 2.4a. Players are now slower to recognize the value of dimensionality strategies, or due to the highly stochastic nature of the competition with $r = 1$, they are unlikely to find a benefit in decentralization and increasing battlefields, and this is true even as experience improves, as shown in Figure 2.4b.

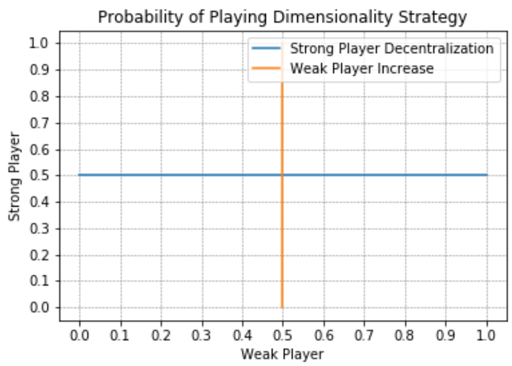
It may seem peculiar in Figure 2.4b that the strong player tends to decentralize more as ψ increases given there is no benefit to doing so under a lottery CSF. Similarly, in Figures 2.3a and 2.3b, there appears to be a downward trend in the probability that the strong



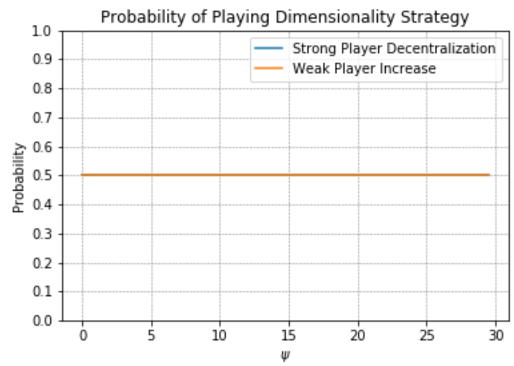
(a) Simultaneous QRE, $r = 1$, $\psi = 3$, $c = 0.01$



(b) Simultaneous QRE, $r = 1$, $c = 0.01$



(c) Simultaneous QRE, $r = 1$, $\psi = 3$, $c = 0$



(d) Simultaneous QRE, $r = 1$, $c = 0$

Figure 2.4. QRE as obtained from pre-conflict simultaneous allocation under the lottery CSF

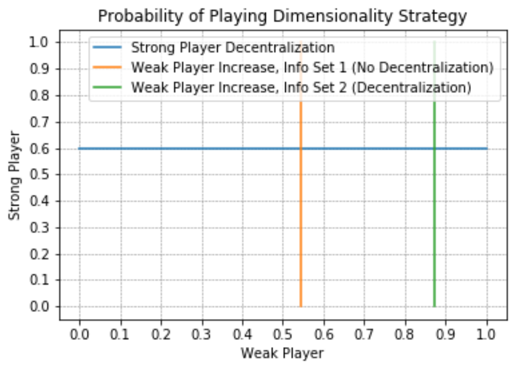
player decentralizes as the probability the weak player increases battlefields goes up. This is simply due to the structure of the game, the calculation of expected utilities, and the weak player's cost of increasing. In Outcomes 2 and 4, the weak player must pay $c = 0.01$ resource units for each new battlefield created, and this has the effect of reducing the weak player's available resources in the third-stage competitions and decreasing his payoff while increasing the payoff for the strong player. Yet, this occurs even while the strong player is decentralizing in Outcome 4 without cost. Thus, as the expected values are calculated, the strong player attributes his higher payoff to decentralizing, when in reality it occurs due to the forced, costly creation of new battlefields by the weak player. To verify this, when we drop the cost of creating battlefields to $c = 0$, we get Figure 2.4c, showing that neither player finds any benefit in dimensionality strategies when $r = 1$. Additionally, an increase in ψ does not result in an increased use of dimensionality strategies, as depicted by the flat curve in Figure 2.4d.

2.3.4 Sequential Allocation: Agent QRE

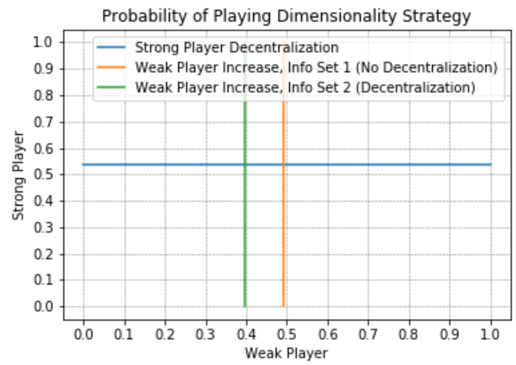
As previously mentioned, AQRE applies to each information set in the extensive-form game. Each player chooses a behavioral strategy given his expected utility and a common ψ value. Unlike the simultaneous game, a player's behavior in the sequential game is not dependent on the notional probabilities of an opponent's move since, in the case of the weak player, the strong player's action is already known; and in the case of the strong player, each of his two possible actions results in only a single expected value. Therefore, the probability of choosing a dimensionality strategy is single-valued at each information set and graphically this results in a flat curve.

Hence, in Figures 2.5a and 2.5b, there are two intersections, each representing the probability the players use a dimensionality strategy at their information sets. In Figure 2.5a, the common ψ is set to three, the cost to increase battlefields is $c = 0.01$, and contests are decided by the Blotto CSF. The strong player then has a probability of 0.60 of decentralizing in Stage 1. The weak player, in turn, has a probability of increasing battlefields of 0.54 if the strong player does not decentralize, and this equivocal response comes due to payoffs of 5.33 and 5.28. In other words, when the strong player does not decentralize, the weak player is unsure whether the cost of increasing the number of battlefields will yield a higher payoff. On the other hand, when the strong player decentralizes, the difference in weak player payoffs for increasing battlefields is more substantial at 6.00 and 5.35, resulting in a 0.87 probability of increasing.

Under a lottery CSF and a common $\psi = 3$, both players are still prone to making mistakes, but the intrinsic complexity of the underlying contest limits the range of these mistakes. With an expected payoff for decentralizing of 4.87, and not decentralizing of 4.80, there is little benefit to decentralizing; and with this low ψ, the probability of decentralizing is only 0.54. Likewise, when the strong player does not decentralize, the probability that the weak player increases battlefields is 0.49, whereas when the strong player does decentralize, the weak player only increases at a probability of 0.40. This indicates that the weak player is aware that there is no benefit to decentralizing when $c > 0$.



(a) AQRE, $r = \infty$, $\psi = 3$, $c = 0.01$



(b) AQRE, $r = 1$, $\psi = 3$, $c = 0.01$

Figure 2.5. AQRE resulting from sequential allocation, obtained under the all-pay auction and the lottery CSFs

2.3.5 Sequential Allocation: Heterogeneous QRE

HQRE is based on the premise that each player is of a particular type, symbolizing different experience levels. In our demonstration below, we first show the case where the weak player is more experienced than the strong player with respective levels of $\psi^S = 3$ and $\psi^W = 15$. Then we reverse this by showing $\psi^S = 15$ and $\psi^W = 3$.

The first case is shown by Figures 2.6a and 2.6b. Here, the probability the strong player decentralizes under the Blotto CSF is little changed from the AQRE case, except that now the probability of decentralizing is slightly lower at 0.55, reflecting the strong player's awareness that the weak player is unlikely to make a mistake. However, the weak player better recognizes the benefits of increasing battlefields, and is more likely to do so at 0.71 and 0.99.

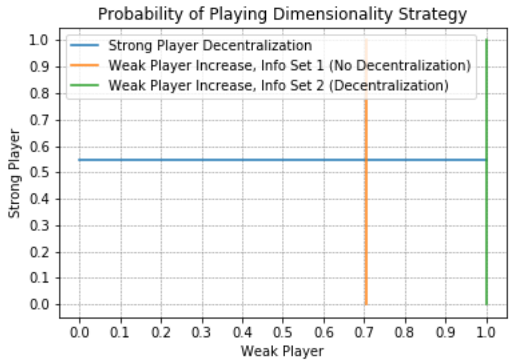
Similarly, under the lottery CSF and $c = 0.01$, the strong player is indifferent to decentralizing, essentially recognizing no benefit to doing so with a probability of 0.51. The weak player is even more unlikely to exercise a dimensionality strategy when there is a positive cost. He only increases battlefields at a probability of 0.46 when the strong player does not decentralize. When the strong player does decentralize, the weak player increases battlefields at a probability of 0.11, indicating a higher knowledge that increasing would leave him worse off, attributable to the cost of increasing with no associated gain in payoff.

In the second case, when $\psi^S = 15$ and $\psi^W = 3$, the strong player is more experienced than the weak player, and this is exhibited by the high probability that the strong player decentralizes under the auction CSF at 0.88. Now, the strong player recognizes that decentralizing is a dominant strategy, especially given the possibility the low-experience weak player may err by not increasing. For his part, the weak player is still equivocal about the benefit of increasing battlefields, but he also benefits from being a second mover. When the strong player does not decentralize, the weak player only increases battlefields with a probability of 0.54, illustrating an inability to distinguish a difference between the payoffs 0.67 and 0.72. When the strong player does decentralize however, the weak player is better able to detect the difference in payoffs, 0.0 and 0.65, and he increases battlefields at a probability of 0.87.

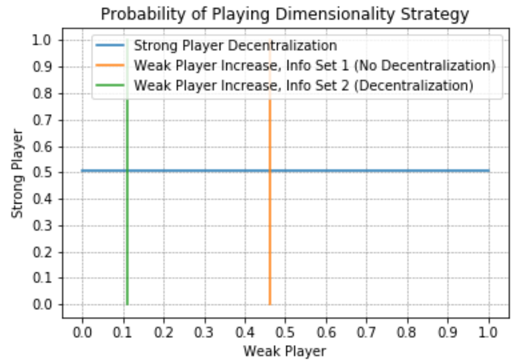
Under the lottery CSF and experience levels of $\psi^S = 15$ and $\psi^W = 3$, the strong player decentralizes at a relatively high rate of 0.68, detecting a difference in expected payoffs of 5.30 and 5.68. The weak player is less able to discern differences in payoffs, but seems to realize that increasing battlefields at a cost lowers his payoff. This is apparent by the probability of increasing battlefields when the strong player does not decentralize at 0.49, and when the there is no decentralization, at 0.40. This second case is shown in Figures 2.6c and 2.6d.

Finally, we show the tendency of the players to use dimensionality strategies in the sequential game as ψ increases to infinity. This is shown for the Blotto CSF in Figure 2.7a, and for the lottery CSF in Figure 2.7b. As with QRE in the simultaneous game and the auction CSF, in the case of AQRE and the sequential game, both players use dimensionality strategies with probability near 1.0 when $\psi^k \geq 15$, except for the weak player when the strong player does not decentralize. In this situation, the weak player must discern between payoffs 0.67 and 0.72, indicating that even at high levels of ψ, it is difficult for the weak player to choose the dominant strategy when increasing battlefields is costly.

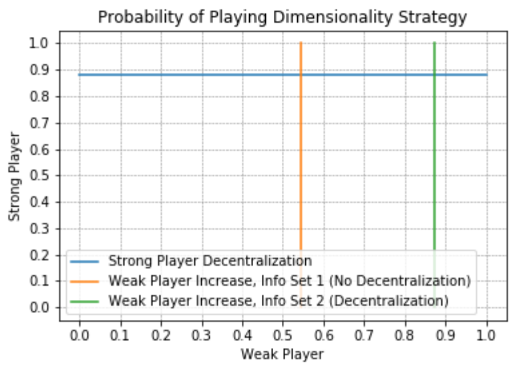
Interestingly, in the case of the lottery CSF, the strong player tends toward decentralization as his experience increases. But like in the case of simultaneous QRE and Figure 2.4b, this only occurs due to the potential that the weak player may make a mistake and



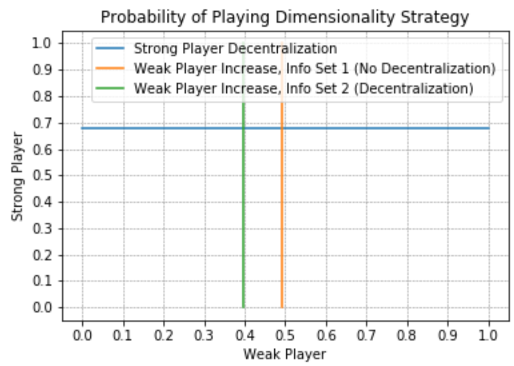
(a) HQRE, $r = \infty$, $\psi^S = 3$, $\psi^W = 15$, $c = 0.01$



(b) HQRE, $r = 1$, $\psi^S = 3$, $\psi^W = 15$, $c = 0.01$



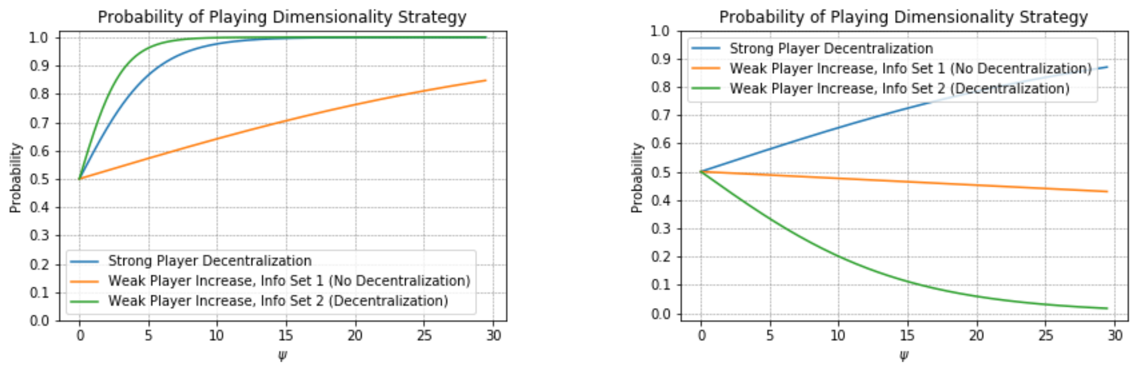
(c) HQRE, $r = \infty$, $\psi^S = 15$, $\psi^W = 3$, $c = 0.01$



(d) HQRE, $r = 1$, $\psi^S = 15$, $\psi^W = 3$, $c = 0.01$

Figure 2.6. HQRE resulting from sequential allocation, obtained under the all-pay auction and lottery CSFs

divert resources to increasing battlefields when the number of battlefields have no impact on payoffs. Hence, due to the structure of the game tree, when the weak player increases, expending $c = 0.01$ per new battlefield, the strong player receives a higher expected payoff after decentralizing, and he comes to learn this as $\psi^S \to \infty$. In contrast, the probability the weak player creates new battlefields decreases steadily as $\psi^W \to \infty$, although this decline is more rapid in the second information set where the difference in payoffs for increasing is more distinct at 1.20 and 1.06. In the first information set, the difference in payoffs is slight at 1.20 and 1.19.



(a) HQRE, $r = \infty$, $c = 0.01$ (b) HQRE, $r = 1$, $c = 0.01$

Figure 2.7. HQRE resulting from sequential allocation, obtained under the all-pay auction and lottery CSFs

2.4 Conclusion

In light of the foregoing discussion on exogenous noise and internal uncertainty, the QRE analysis helps illustrate our overarching thesis: when external noise is sufficiently high, a weak player has no incentive to increase complexity in the game, especially when doing so is costly; and players overcome internal uncertainty regarding dimensionality strategies as their experience grows. When using payoffs generated by the all-pay auction CSF, the Colonel Blotto subgames are very deterministic and the weak player relies on stochastic distributions to improve his payoff. By adding more battlefields, the weak player increases the space over which he can randomize while forcing the strong player to disperse his forces more broadly, resulting in a higher payoff for the weak player at the expense of a lower payoff to the strong player. When this can occur, the strong player recognizes a best response in decentralizing the battlefields and reducing the dimensionality of each subgame, thereby counteracting the weak player's dimensionality strategy and increasing his own payoff while reducing that of the weak player.

Yet, when exogenous noise levels are high, the game is sufficiently stochastic that the weak player has a pure strategy in allocating his force evenly to all battlefields, and the addition of new battlefields does not change his expected equilibrium payoff. The weak player will certainly not expend resources to create new battlefields because doing so decreases the resources available for direct competition against the strong player. Furthermore, because the weak player does not increase battlefields, the strong player is indifferent toward decentralization, and neither player's payoff changes if the strong player decentralizes his forces.

The QRE, AQRE, and HQRE illustrate this dynamic while also showing the role of experience and learning on the players' decisions to use dimensionality strategies prior to conflict. In a deterministic setting with the auction CSF, the players learn with increased experience that dimensionality strategies improve their respective payoffs. When experience is low, their probability of using these strategies is also relatively low, but for values of $\psi \geq 15$ in the simultaneous allocation version of the game, the probability of using them is virtually 1.0. However, this changes when competition switches to a lottery CSF. With high levels of external noise, the weak player is very unlikely to create new battlefields when doing so is costly, and he is indifferent when there is no cost. Similarly, the strong player is indifferent toward decentralization, but recognizes a higher payoff if the weak player chooses to expend resources on new battlefields. Moreover, this behavior changes little with increasing experience. Future research in the form of lab experiments would serve to confirm whether the theoretical predictions and QRE match actual behavior.

3. THE NOISY ALL-PAY-AUCTION

3.1 Introduction

Corchón and Marini (2018) describe a contest as a type of game wherein "contestants exert costly and irretrievable effort in order to obtain one or more prizes with some probability." Examples include advertising campaigns by rival firms, patent races, lobbying and rent-seeking, litigation, war, and political campaigns (Jia et al., 2013). A key element in these games is the contest success function (CSF), which like a production function, maps the efforts of contestants into probabilities of winning, only these efforts are adversarially combined, with each player's probability of reward increasing in their own efforts but decreasing in those of their opponents (Jia et al., 2013).

Two types of CSF are prominent in the literature on contest theory, although others have been proposed. The first is the all-pay auction (APA), characterized by Hillman and Riley (1989) and Baye et al. (1996). By nature, the APA is deterministic with winners selected based only on player expenditures: the player who expends the greatest level of effort wins the contest with probability one. Outside factors play no role in contest outcomes. This leads to sharp competition between players since even small increases of effort can determine winner-take-all outcomes. In many contest environments, this also leads to equilibria in mixed strategies, which is a desirable property for certain models since mixing behavior is also observed in real world conflicts. For example, consider the option play in American football or the guerrilla warfare tactics employed by rebels, insurgents, and weaker combatants in multiple conflicts over centuries. Other desirable characteristics of the APA include its ability to capture phenomena like the *exclusion principle* observed by Baye et al. (1993), the *caps on lobbying principle* discussed by Che and Gale (1998), and as introduced in the preceding two chapters, the existence of dimensionality strategies in extensive-form games over multiple battlefields.

In contrast, the second CSF common in contest theory literature is the ratio-form, or lottery, CSF as formalized by Tullock (1980). This is commonly used to model environments perceived to include high levels of exogenous noise, such as with advertising campaigns, military conflicts, or any situation where random, outside factors may contribute to contest outcomes. As noted by Jia and Skaperdas (2012), the lottery CSF is "the workhorse functional form used in the economics of conflict," particularly due to its ability to incorporate stochastic outcomes, and for technical reasons, the existence of pure strategy equilibria which makes analysis significantly more tractable. Competition under this CSF is characterized as softer since increasing player effort only contributes to an increased *probability* of winning, and marginally higher contributions do not necessarily translate to a guaranteed win.

Additionally, the stochastic nature of the lottery CSF makes it naturally appealing in empirical settings. Jia and Skaperdas (2012), Jia et al. (2013), and S. H. Hwang (2012) note that econometric estimation of contest technologies is difficult owing to the unobservable nature of most effort expenditures and their respective production functions and inputs. Therefore, the ratio-form CSF seems a reasonable approximation of contest technology when noise is obviously present. Yet, some contests are inherently structured as all-pay auctions, despite the presence of noise, and modeling them as lotteries discounts key attributes of the APA while inviting unrealistic behavior and equilibria. For instance, payoff curves resulting from the ratio-form CSF are concave, not discontinuous as in the APA; equilibria is in pure strategies rather than in mixed strategies; and the exclusion principle and caps on lobbying break down in a lottery environment (Fang, 2002). Consider for example the system of litigation in the United States where each party in a lawsuit pays their own legal costs but only one is declared a winner. Despite the possibility that outside factors may influence courtroom outcomes, the intrinsic nature of the contest is an APA and player strategies often rely on mixing in the form of placing apparently disproportionate attention on certain arguments and evidence. So, modeling litigation with a lottery CSF would be an inaccurate portrayal of player considerations and outcomes. Similarly, as observed by Baye et al. (1993), competition between players in a rent-seeking environment is often reduced to a set of finalists, illustrating the exclusion principle, and in the case of Che and Gale (1998), caps on political donations may have the effect of increasing aggregate donations to candidates, even if it succeeds in limiting the contributions of individual donors.

One may reason that the APA is simply a special case of the ratio-form CSF, and that with slight variations of the noise parameter, attributes of the APA can be maintained while increasing exogenous noise. Indeed, the ratio-form CSF, as given in equation 3.1, does equate to the APA when the return-to-effort parameter is set to $r = \infty$.

$$p_{\mathrm{i}}(x_1, x_2, \ldots, x_n) = \frac{x_{\mathrm{i}}^r}{\sum_{\mathrm{j}=1}^{n} x_{\mathrm{j}}^r} \tag{3.1}$$

However, Baye et al. (1994) shows that for values of $r > 2$, equilibria is in mixed strategies, and Ewerhart (2017) proves that the ratio-form CSF with $r > 2$ is payoff equivalent to the APA. Hence, the stochastic nature of the CSF is quickly lost for higher values of r, as is the desired tractability of pure strategies, even as one tries to maintain the characteristics of the APA. Thus, there seems to exist a necessary tradeoff between stochastic outcomes and the foundational attributes that distinguish the APA.

This raises the central question of our research. Given the need for probabilistic outcomes and also the desirability of APA attributes, is it possible to model a noisy APA? We show that such a model can be formed using a convex combination of the APA and fair lottery, which constitutes an APA with stochastic payoffs but with equilibrium characteristics of the APA. Moreover, in certain settings where contest success functions must be conjectured – especially empirical and experimental environments – this structure may be preferred to the lottery CSF. This is reinforced by our demonstration that the noisy APA is microfounded using the axioms of Skaperdas (1996), but it is not noise-equivalent to the lottery CSF.

3.1.1 Literature Review

The literature on contests is extensive, with foundations going back to Bertrand, Cournot, and Nash. Probabilistic choice functions like the ratio-form were first examined in the 1950's by operations researchers, but modern work in contest theory largely begins with the rent-seeking literature of Tullock (1967), Krueger (1974), and Tullock (1980). In particular, Tullock (1980) generalized the ratio-form CSF, which includes the returns-to-effort parameter used in equation 3.1 which many authors use as a representation of exogenous noise. Special cases of the ratio-form CSF include the fair lottery ($r = 0$), the lottery[1] ($r = 1$), and the all-pay auction ($r = \infty$). Fang (2002) described equilibria for the lottery CSF when there are asymmetric valuations among the players, while Hillman and Riley (1989) and Baye et al. (1996) did so for the APA when there is complete information and n-players. Amann and Leininger (1996) found APA equilibria for two players and incomplete information, and Olszewski and Siegel (2016) found approximate equilibrium behavior for large contests with many, possibly heterogeneous, players and prizes. Siegal (2009) generalized models of APAs further and provided a closed-form solution for players' equilibrium payoffs that allows analysis of player efforts.

Alternatives to the ratio-form CSF include the difference-form of Hirshleifer (1989), which has the advantage of capturing the effect of small differences in contestants' resources. This was later augmented by Alcalde and Dahm (2007) who proposed a difference-form CSF that is homogeneous of degree zero (a trait which is lacking in Hirshleifer's original formulation), and Beviá and Corchón (2015) who attempted to unify the properties of lottery and difference-form CSFs. Other work by Baik (1998) and Che and Gale (2000) explore the difference-form in specific settings, and in the latter's case, in combination with the ratio-form where the lottery and APA are extreme cases. Related, is the probit-form used in the rank-order tournaments of Lazear and Rosen (1981), but as pointed out by Jia et al. (2013), the probit-form lacks an analytical functional form, so it has not been used extensively in the contest literature.

Other alternatives include the logit-form CSF of Dixit (1987) which is a more generalized version of the ratio-form. Significantly, this makes Tullock's CSF (including variations like the fair lottery, lottery, and APA) a special case of the logit CSF. This has important implications for our present research since Skaperdas (1996), Clark and Riis (1998), and Jia (2008) all prove microfoundations for logit-form CSFs.

Methods for comparing CSFs include showing payoff, revenue, strategy, and effort equivalence, and while these are featured in many papers, Chowdhury and Sheremeta (2015),

[1] ↑Authors often use the terms "ratio-form," "power-form," "lottery," and "Tullock" CSF interchangeably to refer to the case of $r = 1$. Here, "ratio-form" refers to the general form of the CSF as proposed by Tullock, and "lottery" refers to the case of $r = 1$.

Balart et al. (2017), and Ewerhart (2017) serve as good examples of application. For our present purposes, the proof of noise equivalence in Balart et al. (2017) illustrates a method for comparing the noisy APA to the lottery CSF.

Noise is a critical element to models of competition. Since few contests are solved solely based on deterministic factors, the APA is often not used in empirical settings and the lottery CSF is preferred. For example, S. H. Hwang (2012) notes the lottery CSF to be one of the best known descriptors for military conflicts included in a seventeenth-century data set,[2] but notes that neither the basic difference-form nor ratio-form are good fits to World War II data[3] and so creates a "constant elasticity of augmentation" CSF that has the functional form of a modified logit (i.e., ratio-form) contest. Other examples include Huang and He (2021), who analyze U.S. congressional elections using the ratio-form, and Mildenberger and Pietri (2018) who conclude the lottery CSF performs best when compared to logit and probit models in estimating victories across a dataset of 19,229 battles simulated in virtual worlds. Mildenberger and Pietri (2018) do not consider the APA, presumably because it does not allow for probabilistic outcomes. In Beviá and Corchón (2015), only when player skill levels are considered does the relative difference-form CSF outperform the Tullock, logit and probit CSFs. Again, the APA was not considered.

Efforts to estimate CSFs econometrically, regardless of CSF, are summarized by S.-H. Hwang (2009), Jia and Skaperdas (2012), S. H. Hwang (2012), and Jia et al. (2013), who point to work in areas such as lobbying and political campaigns (Becker (1983); Baron (1994); Skaperdas and Grofman (1995)) and wars (Hirshleifer (1989); Hirshleifer (1995)). More recent literature covers an even broader range, to include advances in the previously mentioned areas, but also litigation (Hirshleifer and Osborne (2001); A. Robson and Skaperdas (2008)), patent races (Reinganum (1989); Baye and Hoppe (2003)), and sports (Szymanski (2003)) (Jia et al., 2013). From an experimental standpoint, Cason et al. (2020) is representative, comparing a deterministic winner-take-all contest, to a probabilistic winner-take-all contest, and a proportional prize contest. They find that subject behavior is largely consistent with theoretical predictions, wherein efforts are highest in deterministic (e.g., APA) contests, and

[2] ↑See Bodart (1908)

[3] ↑See Historical Evaluation and Research Organization (1983)

lower in the stochastic environments (e.g., lotteries). Also, Dechenaux et al. (2014) surveys experimental research on contests, including those with APAs, lotteries, and difference-form CSFs.

There have been few attempts to form a convex combination of different CSFs. Most notably, and in addition to the combination of a lottery and difference-form CSF proposed by Beviá and Corchón (2015), Amegashie (2012) offered a convex combination of the APA and lottery CSFs in an environment of incomplete information and two players, and Balart et al. (2017) formed a convex combination of a lottery and fair lottery when there is complete information and n players. Amegashie's combination takes the form

$$p_{\mathrm{i}}(x_1, x_2) = \lambda \frac{x_{\mathrm{i}}^{\infty}}{x_1^{\infty} + x_2^{\infty}} + (1 - \lambda) \frac{x_{\mathrm{i}}}{x_1 + x_2}, \tag{3.2}$$

where λ represents the exogenous noise and x_{i} represents the effort of player $\mathrm{i} \in \{1, 2\}$. He finds that if λ is sufficiently low, then there is an equilibrium in pure strategies. However, for other ranges of λ, there exist mixed strategy equilibria. Moreover, this CSF is homogeneous of degree zero (i.e., results are scale invariant) and obeys all the axioms of Skaperdas (1996) and Clark and Riis (1998). On the other hand, the convex combination of Balart et al. takes the form

$$p_{\mathrm{i}}(x_1, \ldots, x_n) = \lambda \frac{x_{\mathrm{i}}}{\sum_{\mathrm{j}=1}^{n} x_{\mathrm{j}}} + (1 - \lambda) \frac{1}{n}. \tag{3.3}$$

Balart et al. (2017) shows that their CSF can be effort or noise equivalent to the lottery CSF, it is homogeneous of degree zero, but it fails to satisfy Luce's axiom (i.e., the probability i wins when player k bids $x_k = 0$ is not the same as when $x_k > 0$) due to the inclusion of the fair lottery. Balart et al. (2017) presents this as a tradeoff in which contest designers can choose a method of introducing "tractable" noise depending on the importance of the Luce's axiom.

Our proposed CSF requires a similar tradeoff. While it is homogeneous of degree zero and satisfies all the other axioms of Skaperdas (1996), it fails Luce's axiom for the same reason. Our CSF is also similar to the foregoing two CSFs in functional form since it is a convex combination of the APA and fair lottery. It is based on a formulation suggested in the appendix of Baye et al. (2005), but here we tailor it to fit the more general case of n players rather than their two player litigation model.[4] As given in equation 3.4, the convex combination offers a method for capturing the mixed strategies of the APA while including stochastic outcomes.

$$p_{\mathrm{i}}(x_1, \ldots, x_n) = \lambda \frac{x_{\mathrm{i}}^{\infty}}{\sum_{\mathrm{j}=1}^{n} x_{\mathrm{j}}^{\infty}} + (1-\lambda)\frac{1}{n} \tag{3.4}$$

This results in a range of equilibria in mixed strategies, depending on the value of λ, as well as one degenerate equilibrium in pure strategies when $\lambda = 0$. Finally, while noise-equivalence to the lottery CSF would be beneficial to modelers doing empirical estimation, this does not hold for our convex combination of the APA and fair lottery.

In the sections that follow we will first review the derivation of the noisy APA and contrast its probabilities and payoffs to those of the APA. We will then briefly show the noisy APA satisfies the axioms of Skaperdas (1996) and discuss the range over which equilibria exists. After presenting assumptions for the model, we follow Baye et al. (1996), Baye et al. (1993), and Che and Gale (1998) to show that only the ranges of equilibria vary from that of the APA with n players and complete information, while the exclusion principle and caps on lobbying principle hold. Finally, we demonstrate the noisy APA is not noise equivalent to the lottery CSF, and end with a discussion regarding future research.

3.2 Analysis

Construction of the noisy APA begins with the convex combination of the APA and fair lottery as given in equation 3.4 where the parameter λ serves as an exogenous level of noise. We assume a contest of $n \geq 2$ risk-neutral players and complete information, so λ is known

[4]↑Following the approach of Baye et al. (2005) in their appendix, the convex combination of payoffs in the case of $n = 2$ players and $x_{\mathrm{i}} > x_{\mathrm{j}}\ \forall \mathrm{j} \neq \mathrm{i}$ produces $\lambda[v_{\mathrm{i}} - x_{\mathrm{i}}] + (1-\lambda)\frac{1}{2}[v_{\mathrm{i}} - x_{\mathrm{i}}] + (1-\lambda)\frac{1}{2}[-x_{\mathrm{i}}]$. This form has a certain intuitive appeal since it implies that with probability λ player i wins the contest, but with probability $1-\lambda$ the outcome is decided by random chance. However, this form is easily simplified to $(1+\lambda)\frac{1}{2}v_{\mathrm{i}} - x_{\mathrm{i}}$, and the convex combination results in a linear transformation of APA payoffs, evidenced by the multiplier $(1+\lambda)\frac{1}{2}$. In the two player case, Luce's axiom holds and supports of equilibria of the players' mixed strategies are scaled down to reflect the discounted payoffs.

to the players, valuations of the prize are not necessarily common, and each player tries to maximize their respective payoff. When $\lambda = 0$, the noisy APA converges to the fully random fair lottery, which we treat as a degenerate case with a single pure strategy equilibrium. That is, when $\lambda = 0$, the ratio-form CSF with $r = 0$,

$$p_{\text{i}}^{F}(x_1, \ldots, x_n) = \frac{x_{\text{i}}^0}{\sum_{\text{j}=1}^{n} x_{\text{j}}^0} = \frac{1}{n}, \tag{3.5}$$

implies that any or no allocation of resources to the contest, $x_{\text{i}} \geq 0$, is sufficient to give player i a strictly positive probability of winning, $\frac{1}{n}$. When $\lambda = 1$, the noisy APA converges to the deterministic APA, the equilibria of which are well known. Hence, we know equilibria for the limiting cases of λ, and our analysis of the noisy APA here concerns the range of $\lambda \in (0, 1)$.

To derive the noisy APA, we must first consider the probabilities and payoffs of winning in the APA. As given in Baye et al. (1996), these probabilities are

$$p_{\text{i}}^{APA}(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } x_{\text{i}} > x_{\text{j}} \ \forall \text{j} \neq \text{i} \\ \frac{1}{m}, & \text{if i ties for the high bid with } m-1 \text{ others} \\ 0, & \text{if } \exists \text{j such that } x_{\text{i}} < x_{\text{j}} \end{cases}$$

and the payoffs are

$$u_{\text{i}}^{APA}(x_1, ..., x_n) = \begin{cases} v_{\text{i}} - x_{\text{i}}, & \text{if } x_{\text{i}} > x_{\text{j}} \ \forall \text{j} \neq \text{i} \\ \frac{v_{\text{i}}}{m} - x_{\text{i}}, & \text{if i ties for the high bid with } m-1 \text{ others} \\ -x_{\text{i}}, & \text{if } \exists \text{j such that } x_{\text{i}} < x_{\text{j}}, \end{cases}$$

where v_{i} represents player i's valuation of the prize and $v_1 \geq v_2 \geq \ldots \geq v_n$. The prize is awarded to player i with certainty whenever $x_{\text{i}} > x_{\text{j}}$ $\forall \text{j} \neq \text{i}$, meaning that a winning bid could be achieved if x_{i} is no more than $\epsilon > 0$ greater than an opposing bid x_{j}. As explained in Baye et al. (1996), differences in valuations can be interpreted to arise from differences in abilities. They show this by supposing player i's utility for winning prize W is achieved by exerting effort x_i. Then, $u_i^* = U_i(W) + \beta_i x_i$, where β_i is the marginal cost of effort. Behavior is invariant to affine transformations, so $u_i \equiv \frac{u_i^*}{\beta_i} = \frac{U_i(W)}{\beta_i} - x_i = v_i - x_i$ (Baye et al., 1996). Hence, according to the payoff formulation above, in auction contests where ability matters, minutely small differences in ability can lead to winner-take-all outcomes.

The noisy APA relaxes this requirement so that exogenous circumstances also play a role in contest outcomes. Obviously, in many real-world APA contests factors besides contestant efforts play an important role in determining winners and losers. For instance, political candidates and campaigns may be closely matched in terms of ability, funding, and organization, but unforeseen economic or geopolitical events just prior to an election may turn voters toward one candidate or the other, regardless of the candidates' respective efforts and abilities. The noisy APA captures these occurrences with the parameter λ, which for empirical purposes, could be calibrated for a particular situation and dataset.

Combining the probabilities of the APA with that of the fair lottery as in equation 3.4, we get the following probabilities for the noisy APA:

$$p_i(x_1, \ldots, x_n) = \begin{cases} \frac{\lambda(n-1)+1}{n}, & \text{if } x_i > x_j \ \forall j \neq i \\ \frac{\lambda(n-m)+m}{nm}, & \text{if i ties for the high bid with } m-1 \text{ others, and } m \leq n \\ \frac{1-\lambda}{n}, & \text{if } \exists j \text{ such that } x_i < x_j. \end{cases} \tag{3.6}$$

We can then apply these probabilities to payoffs and get

$$u_i(x_1, ..., x_n) = \begin{cases} \left(\frac{\lambda(n-1)+1}{n}\right) v_i - x_i, & \text{if } x_i > x_j \ \forall j \neq i \\ \left(\frac{\lambda(n-m)+m}{nm}\right) v_i - x_i, & \text{if i ties for the high bid with } m-1 \text{ others, and } m \leq n \\ \left(\frac{1-\lambda}{n}\right) v_i - x_i, & \text{if } \exists j \text{ such that } x_i < x_j, \end{cases} \tag{3.7}$$

which we will utilize in our analysis. Note that for the first case, $x_i > x_j \ \forall j \neq i$, and $\lambda \in (0,1)$, expected payoffs to player i lie in the range $u_i(x_1, ..., x_n) = \left(\frac{1}{n} v_i - x_i, v_i - x_i\right)$. Likewise, for the third case, $x_i < x_j$, and $\lambda \in (0,1)$, $u_i(x_1, ..., x_n) = \left(-x_i, \frac{1}{n} v_i - x_i\right)$. Hence, depending on the value of λ, even when a player does not have a winning bid he can still receive a positive expected payoff. This has important implications to the range of equilibria over which players employ mixed strategies, as will be discussed later.

The probabilities and payoffs in equations 3.6 and 3.7 are not complicated considering that in any contest, the values for n and λ are fixed, and m is constant in the APA based on the number of players with equal valuations. Indeed, the probabilities of the noisy APA only serve as multipliers on player i's valuation of the prize. Hence, equilibrium strategies in the noisy APA should closely resemble those of the APA, albeit with efforts that reflect the discounted payoffs. Also, a continuous set of equilibria exist within a range that varies according to the level of λ.

It is also important to point out that the noisy APA adheres to the axioms for CSFs given by Skaperdas (1996) and Clark and Riis (1998), thus giving it microfoundations that support its use in empirical settings. These axioms are easily verified, as follows:

(A1) (*Imperfect Discrimination*) $\sum_{\mathrm{i}=1}^{n} p_{\mathrm{i}}(x_1, \ldots, x_n) = 1$ and $p_{\mathrm{i}}(x_1, \ldots, x_n) \geq 0$ for all $\mathrm{i} \in \{1, \ldots, n\}$ and all $x \in \{x_1, \ldots, x_n\}$; if $x_{\mathrm{i}} > 0$ then $p_{\mathrm{i}}(x_1, \ldots, x_n) > 0$.

(A2) (*Monotonicity*) For all $\mathrm{i} \in \{1, \ldots, n\}$, $p_{\mathrm{i}}(x_1, \ldots, x_n)$ is increasing in x_{i} and decreasing in x_{j} for all $\mathrm{i} \neq \mathrm{j}$.

(A3) (*Anonymity*) For any permutation π of $\{1, \ldots, n\}$, we have $p_{\pi(\mathrm{i})}(x_1, \ldots, x_n) = p(x_{\pi 1}, \ldots, x_{\pi n})$ for all $\mathrm{i} \in \{1, \ldots, x_n\}$.

(A4) (*Independence of Irrelevant Alternatives [Luce's Axiom]*) Suppose that one player k does not participate in the contest ($x_k = 0$). The probability that a competing player i wins the reduced contest is $p_{\mathrm{i}}(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) = \frac{p_{\mathrm{i}}(x_1,\ldots,x_n)}{1-p_k(x_1,\ldots,x_n)} \forall k \neq \mathrm{i}$. In other words, the probability that i wins if player k does not participate is equal to the probability that i wins when k participates given that k does not win. (Clark & Riis, 1998)

(A5) (*Homogeneity of Degree Zero*) $p_{\mathrm{i}}(x_1, \ldots, x_n) = p_{\mathrm{i}}(\phi x_1, \ldots, \phi x_n), \forall \mathrm{i}, \phi > 0$.

Clearly, the noisy APA of equation 3.6 satisfies axioms (A1) - (A3) and (A5), but it fails (A4) due to the inclusion of the fair lottery term. This is not surprising, and the failure is a known trait of the fair lottery. However, satisfying Luce's axiom is also not crucial for our purposes.

3.2.1 Characterization of Equilibria

We now show that APA equilibria from Baye et al. (1996) hold in the case of the noisy APA. Baye et al. characterize equilibria in the APA using three theorems. The first of these establishes equilibria when there is a set of players with homogeneous valuations, or $v_1 = \cdots = v_m > v_{m+1} \geq \ldots \geq v_n$ and $m \geq 2$. Similarly, the second addresses the case of a strong player competing against a set of weaker players with equal valuations of the prize, $v_1 > v_2 = \cdots = v_m > v_{m+1} \geq \ldots \geq v_n$ and $3 \leq m \leq n$. Theorem 3, which Baye et al. borrow from Hillman and Riley (1989), covers the case of heterogeneous valuations, or $v_1 > v_2 > v_3 \geq \ldots \geq v_n$. We replicate these theorems here for the noisy APA by including the convex combination of risk captured by $\lambda \in (0, 1)$ and show that the ranges of effort over which equilibria exist change proportionally to the noise parameter λ, even if total player effort decreases as a result of the imposed risk. Note that in the following theorems, and the proofs in the appendix, we follow the original authors' language and notation closely.

Baye et al. (1996) proves that x_{i} is determined by a Nash equilibrium in mixed strategies, denoted by $G_{\mathrm{i}}(x)$, the cumulative distribution function (cdf) for each $\mathrm{i} \in \{1, \ldots, n\}$. This cdf has upper and lower supports, $\overline{s}_{\mathrm{i}}$ and $\underline{s}_{\mathrm{i}}$, based on the value of exogenously given v_{i} $\forall \mathrm{i}$, so that $v_{\mathrm{i}} \geq \overline{s}_{\mathrm{i}} \geq \underline{s}_{\mathrm{i}} \geq 0$. Therefore, if we assume player i has information regarding the probability that random chance will determine his payoff, we can define $v'_{\mathrm{i}} \equiv \left(\frac{\lambda(n-1)+1}{n}\right) v_{\mathrm{i}} \leq v_{\mathrm{i}}$ for $\lambda \in (0, 1)$, and x_{i}, $\overline{s}_{\mathrm{i}}$, and $\underline{s}_{\mathrm{i}}$ will also scale down in adjustment to v'_{i}. Additionally, the maximum expected payoff for the winner of the prize is $u_{\mathrm{i}} = \left(\frac{\lambda(n-1)+1}{n}\right) v_{\mathrm{i}} - x_{\mathrm{i}}$, and since payoffs are decreasing in x_{i}, bidding $x_{\mathrm{i}} = 0$ and losing still guarantees a payoff of $\left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}} > 0$ for all players $\mathrm{i} \in \{1, \ldots, n\}$. These possible outcomes have a significant impact on equilibrium behavior in the noisy APA: no player will bid an amount greater than $\left(\frac{\lambda(n-1)+1}{n}\right) v_{\mathrm{i}} - \left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}} = \lambda v_{\mathrm{i}}$, and any bid equal to or less than λv_{i} will secure a payoff of *at least* $\left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}}$. This means the efforts of players with high valuations will decrease in response to their lower expected payoffs while the payoffs to players with lower valuations (both those with losing bids and nonparticipating players) scale up as $\lambda \to 0$. Since payoffs are decreasing in x_i, low-valuation players have no incentive to increase their efforts, the lower bound on all players' effort remains at zero, and the upper support for all players shifts down, resulting in lower revenue to the auctioneer overall. Hence, by making the convex combination of risk in the APA a primitive of the model, we achieve a linear transformation of bids and payoffs, and the envelope over which equilibrium distribution functions exists becomes smaller than in the APA, while there is no equilibrium in pure strategies for $\lambda > 0$.[5] We will show this in detail through the following theorems.

Theorem 3.2.1. *When $v_1 = \cdots = v_m > v_{m+1} \geq \ldots \geq v_n$, $m \geq 2$, and $\lambda \in (0, 1)$:*

(A) If $m = 2$, the Nash equilibrium is unique and symmetric. If $3 \leq m \leq n$, there is a unique symmetric Nash equilibrium, as well as a continuum of asymmetric Nash equilibria. In any equilibrium players $m + 1$ through n bid zero with probability one, and at least two players randomize continuously on $[0, \lambda v_1]$. Each other player $\mathrm{i} \in \{1, \ldots, m\}$ randomizes continuously on $[b_i, \lambda v_1]$, where $b_i \geq 0$ is a free parameter, and bids zero with positive probability if $b_i > 0$. When two or more players randomize continuously on a common interval, their corresponding cdf's are identical over that interval.

(B) In any equilibrium, the expected payoff to each player is $\left(\frac{1-\lambda}{n}\right) v_i$.

(C) All equilibria are revenue equivalent: the expected sum of the bids in any equilibrium is λv_1.

Proof. See appendix for Proof of Theorem 3.2.1. □

Each theorem in this section allows us to characterize the algebraic form of the family of equilibrium mixed strategies for a set of player valuations. For the case of Theorem 3.2.1 and $v_1 = \cdots = v_m > v_{m+1} \geq \ldots \geq v_n$, we get the distribution functions summarized in Table 3.1 for player i. To describe Theorem 3.2.1 in short, let $v = v_1 = \cdots = v_m$.

[5] ↑Of course, when $\lambda = 0$, the contest is a fair lottery and the only equilibrium is in pure strategies. Each player bids $x_i = 0$.

Then players $m+1$ through n are low-valuation players who bid zero with probability one, which also makes them non-participatory. Next, suppose without loss of generality that players $\mathrm{i} = 1, 2, \ldots, h$, $2 \leq h \leq m$, randomize continuously over $[0, \lambda v]$, but players $\mathrm{i} = h+1, \ldots, m$ randomize continuously only over the region $[b_{\mathrm{i}}, v]$, while also putting mass at 0, and $b_{h+1} \leq b_{h+2} \leq \ldots \leq b_m \leq v$. The b_{i}'s are arbitrary and unique to each player $\mathrm{i} \in \{1, \ldots, m\}$, and varying the b_{i}'s generates the continuum of equilibria. The distributions of randomly placed bids constitute the cdf's of Table 3.1.

Table 3.1. Theorem 3.2.1 equilibrium strategies

$\forall x \in [b_m, \lambda v]$:	$G_{\mathrm{i}}(x) = \left[\frac{x}{\lambda v}\right]^{\frac{1}{m-1}}$	$\mathrm{i} = 1, \ldots, m$
$\forall x \in [b_{\mathrm{j}}, b_{\mathrm{j}+1})$:	$G_{\mathrm{i}}(x) = \left[\frac{x}{\lambda v}\right]^{\frac{1}{\mathrm{j}-1}} \left[\prod_{k>\mathrm{j}} G_k(b_k)\right]^{-\frac{1}{\mathrm{j}-1}}$	$\mathrm{i} = 1, \ldots, \mathrm{j}$, $\mathrm{j} \in \{h+1, \ldots, m-1\}$
	$G_k(x) = G_k(b_k)$	$k = \mathrm{j}+1, \ldots, m$
$\forall x \in [0, b_{h+1})$:	$G_{\mathrm{i}}(x) = \left[\frac{x}{\lambda v}\right]^{\frac{1}{h-1}} \left[\prod_{k>\mathrm{j}} G_k(b_k)\right]^{-\frac{1}{h-1}}$	$\mathrm{i} = 1, \ldots, h$
	$G_k(x) = G_k(b_k)$	$k = h+1, \ldots, m$

These equilibria stand in contrast to those of the APA in that the expected payoffs are lower for every value of $\lambda < 1$. This is illustrated graphically in figure 3.1, which is based on a scenario given by Baye et al. (1996) and replicated here with the noisy APA. For the case of $n = 3$ players, $v \equiv v_1 = v_2 = v_3 = 1$, and an arbitrarily chosen b, only two players submit bids over the interval $[0, b]$, while all three compete over $[b, v]$. This results in symmetric cdf's $G_1 = G_2 = G_3 = \left(\frac{x}{\lambda v}\right)^{1/2}$ for $x \in [b, v]$. For $x \in [0, b]$, player 3 does not compete since the probability of a winning bid in this region is equal to zero, but instead places all mass at 0. Players 1 and 2 are competitive over this region however, with the symmetric cdf's $G_1 = G_2 = \left(\frac{x}{\lambda v}\right)\left(\frac{b}{\lambda v}\right)^{-1/2}$. This separation in strategies is marked by a kink in the plot of cdf's for player 1 who competes over the entire interval $[0, \lambda v]$. As shown, in the APA case ($\lambda = 1$), player $\mathrm{i} = 1$ competes over $[0, v]$, but as payoffs become more risky, say $\lambda = 0.5$, i's maximum bid falls to $\lambda v = 0.5$. As the contest nears a fair lottery with $\lambda = 0.1$, player strategies require a much smaller upper bound on resources than when outcomes are more certain, or $\lambda v = 0.1$.

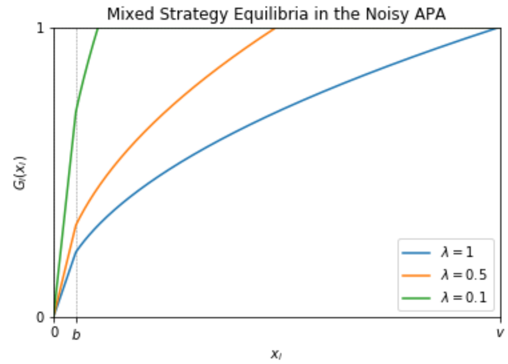


Figure 3.1. Plot of equilibrium strategies in the case of $n = m = 3$ players, $v \equiv v_1 = v_2 = v_3 = 1$, and $\lambda = 0.1, 0.5$, and 1.

The setting for our next theorem is similar to that of the first in that it includes a group of players with homogeneous valuations, excluding the player with the highest value, $v_1 > v_2 = \cdots = v_m$. In this environment, player 1 has no incentive to bid above the discounted value of player 2, λv_2, but always bids λv_2 with some positive probability. So, aggressive competition takes place among the players with the highest valuations over the interval $[0, \lambda v_2]$. As with Theorem 3.2.1, equilibria from the APA can be extended to the noisy APA, but it applies to a smaller range of bids and total payoffs are lower. We modify Theorem 3.2.2 of Baye et al. (1996) as follows:

Theorem 3.2.2. *When $v_1 > v_2 = \cdots = v_m > v_{m+1} \geq \ldots \geq v_n$, $3 \leq m \leq n$, and $\lambda \in (0, 1)$:*

(A) There exists a continuum of Nash equilibria. In any equilibrium, player 1 randomizes continuously on the interval $[0, \lambda v_2]$ and players $m+1$ through n bid 0 with probability one. Each player $\mathrm{i} \in \{2, \ldots, m\}$, employs a strategy G_i with support contained in $[0, \lambda v_2]$ that has an atom $\alpha_\mathrm{i}(0)$ at 0. The size of the atom may differ across players, but $\prod_{\mathrm{i}=2}^{m} \alpha_\mathrm{i}(0) = {}^{(v_1 - \lambda v_2)}/_{v_1}$. Each G_i is characterized by a number $b_\mathrm{i} \geq 0$, where $b_\mathrm{i} = 0$ for at least one $\mathrm{i} \neq 1$, such that $G_\mathrm{i}(x) = G_\mathrm{i}(0) = \alpha_\mathrm{i}(0)$ $\forall x \in [0, b_\mathrm{i}]$ and player i randomizes continuously on $(b_\mathrm{i}, \lambda v_2]$. Furthermore, when two or more players in the set $\{2, \ldots, m\}$ randomize continuously on a common interval, their cdf's are identical on that interval.

(B) In any equilibrium player 1 earns an expected payoff of $\left(\frac{\lambda(n-1)+1}{n}\right) v_1 - \lambda v_2$*, while each of the players 2 through* n *earns an expected payoff of* $\left(\frac{1-\lambda}{n}\right) v_{\text{i}}$*.*

(C) There is not revenue equivalence. In particular, the expected sum of the bids is

$$\sum Ex_{\text{i}} = \frac{v_2}{v_1}\lambda v_2 + \left[1 - \frac{v_2}{v_1}\right] E_1 x_1 \tag{3.8}$$

where Ex_1 *varies across the continuum of equilibria, is minimized when symmetric players use symmetric strategies, and is maximized when only one of the players 2 through* m *is active (i.e., submits positive bids with positive probability).*

Proof. See appendix for Proof of Theorem 3.2.2. □

Similar to Theorem 3.2.1, when $v_1 > v_2 = \cdots = v_m > v_{m+1} \geq \ldots \geq v_n$, players $m+1$ through n bid zero with probability one, while a set of players in $\{2, \ldots, m\}$, namely $\text{i} = 2, \ldots, h, h \geq 2$, randomize continuously over the interval $(0, \lambda v_2]$. Then players $\text{i} = h+1, \ldots, m$ randomize continuously over $(b_{\text{i}}, \lambda v_2]$, with $b_{h+1} \leq b_{h+2} \leq \ldots \leq b_m \leq \lambda v_2$. Here again, b_{i} is arbitrary for each player i, and varying b_{i} generates the continuum of equilibria. Unlike Theorem 3.2.1 however, in this scenario player 1 randomizes continuously over $[0, \lambda v_2]$ and earns an expected payoff of $\lambda(v_1 - v_2)$. This creates a distinct set of cdf's that are summarized in Table 3.2.

Finally, we modify Theorem 3 of Baye et al. (1996) as follows:

Theorem 3.2.3. *(Baye et al., 1996) If* $v_1 > v_2 > v_3 \geq \ldots \geq v_n$*:*

(A) The Nash equilibrium is unique.

(B) In equilibrium, player 1 randomizes continuously on $[0, \lambda v_2]$*. Player 2 randomizes continuously on* $(0, \lambda v_2]$*, placing an atom of size* $\alpha_2(0) = \frac{v_1 - v_2}{v_1}$ *at zero. Players 3 through* n *bid zero with probability one.*

(C) Player 1's equilibrium payoff is $u_1^* = \left(\frac{\lambda(n-1)+1}{n}\right) v_1 - \lambda v_2$*, while players 2 through* n *earn expected payoffs of* $\left(\frac{1-\lambda}{n}\right) v_{\text{i}}\ \forall \text{i} \neq 1$*.*

Table 3.2. Theorem 3.2.2 equilibrium strategies

<table>
<tr><td rowspan="2">$\forall x \in [b_m, \lambda v_2]$:</td><td>$G_{\mathrm{i}}(x) = \left[\frac{\lambda(v_1-v_2)+x}{\lambda v_1}\right]^{\frac{1}{m-1}}$</td><td>$\mathrm{i} = 2, \ldots, m$</td></tr>
<tr><td>$G_1(x) = \frac{x}{\lambda v_2}\left[\frac{\lambda(v_1-v_2)+x}{\lambda v_1}\right]^{\frac{2-m}{m-1}}$</td><td></td></tr>
<tr><td rowspan="3">$\forall x \in [b_{\mathrm{j}}, b_{\mathrm{j}+1}]$:</td><td>$G_{\mathrm{i}}(x) = \left[\frac{\lambda(v_1-v_2)+x}{\lambda v_1}\right]^{\frac{1}{\mathrm{j}-1}}\left[\prod_{k>\mathrm{j}} G_k(b_k)\right]^{-\frac{1}{\mathrm{j}-1}}$</td><td>$\mathrm{i} = 2, \ldots, \mathrm{j}$,
$\mathrm{j} \in \{h+1, \ldots, m-1\}$</td></tr>
<tr><td>$G_k(x) = G_k(b_k)$</td><td>$k = \mathrm{j}+1, \ldots, m$</td></tr>
<tr><td>$G_1(x) = \frac{x}{\lambda v_2}\left[\frac{\lambda(v_1-v_2)+x}{\lambda v_1}\right]^{\frac{2-\mathrm{j}}{\mathrm{j}-1}}\left[\prod_{k>\mathrm{j}} G_k(b_k)\right]^{-\frac{1}{\mathrm{j}-1}}$</td><td></td></tr>
<tr><td rowspan="3">$\forall x \in [0, b_{h+1}]$:</td><td>$G_{\mathrm{i}}(x) = \left[\frac{\lambda(v_1-v_2)+x}{\lambda v_1}\right]^{\frac{1}{h-1}}\left[\prod_{k>\mathrm{j}} G_k(b_k)\right]^{-\frac{1}{h-1}}$</td><td>$\mathrm{i} = 2, \ldots, h$</td></tr>
<tr><td>$G_k(x) = G_k(b_k)$</td><td>$k = h+1, \ldots, m$</td></tr>
<tr><td>$G_1(x) = \frac{x}{\lambda v_2}\left[\frac{\lambda(v_1-v_2)+x}{\lambda v_1}\right]^{\frac{2-h}{h-1}}\left[\prod_{k>h} G_k(b_k)\right]^{-\frac{1}{h-1}}$</td><td></td></tr>
</table>

(D) (Hillman & Riley, 1989) The expected sum of the bids is $E(x_1 + x_2) = \frac{\lambda v_2}{2}\left(1 + \frac{v_2}{v_1}\right)$.

Proof. See appendix for Proof of Theorem 3.2.3. □

It is now apparent that while equilibria under the noisy APA vary from those of the APA only in the interval over which players mix, these differences are expected and result directly from the inclusion of risk. It is natural that players should reduce their efforts when there is a possibility of loss despite a winning bid, and the model captures this by abbreviating the region over which distributions exist when λ approaches zero, while not sacrificing the structure of the equilibria themselves. In fact, the reduction results primarily from decreases to the upper bounds of player supports, leading to lower individual bids. As a result, total aggregate revenue to the players remains unchanged (a winning player i still receives v_{i}), but expected payoffs to winning players decrease.

However, the two CSFs are not revenue equivalent, and this is also expected. As individual efforts decrease with λ, expected payments to the auctioneer are likewise reduced. In the homogeneous cases of $v_1 = \cdots = v_m > v_{m+1} \geq \ldots \geq v_n$ and $v_1 > v_2 = \cdots = v_m > v_{m+1} \geq \ldots \geq v_n$, total expected payments to the auctioneer sum to λv_1 versus v_1 in the APA, and $\sum Ex_{\mathrm{i}} = \frac{v_2}{v_1}\lambda v_2 + \left[1 - \frac{v_2}{v_1}\right] E_1 x_1$ versus $\sum Ex_{\mathrm{i}} = \frac{v_2}{v_1} v_2 + \left[1 - \frac{v_2}{v_1}\right] E_1 x_1$, respectively. Similarly, in the heterogeneous case of Theorem 3.2.3 and $v_1 > v_2 > v_3 \geq \ldots \geq v_n$, payments to the auctioneer are equal to only the discounted sum of efforts of the two players with the highest valuations, $E(x_1 + x_2) = \frac{\lambda v_2}{2}\left(1 + \frac{v_2}{v_1}\right)$ verus $E(x_1 + x_2) = \frac{v_2}{2}\left(1 + \frac{v_2}{v_1}\right)$ in the APA.

Lower expected revenue is also a more accurate reflection of reality. Few contests are determined by differences in player efforts alone, even if the contests feature many characteristics of an APA. Rather, most are influenced by random exogenous occurrences, and the inclusion of risk is an important consideration in these contests. The noisy APA is therefore able to capture both elements of a realistic contest by retaining the intrinsic structure and equilibria of an APA, while including elements of risk that serve to lower payoffs and revenue in expected ways. In the two sections that follow, we will demonstrate further that the noisy APA retains the distinguishing characteristics of an APA, even if these characteristics are diminished by the inclusion of noise.

3.2.2 The Exclusion Principle

Baye et al. (1993) shows that an auctioneer can increase his own expected payoffs in a contest by excluding those players who value the prize most. This counter-intuitive result derives from the notion that the inclusion of players with a high valuation of the prize discourages competition among those who value the prize less. By eliminating players with high valuations, the auctioneer creates an incentive for the other players to compete more aggressively, thereby increasing expenditures in the aggregate.

This outcome is driven entirely by the ranking of player valuations, which for players $\{1, \ldots, n\}$ naturally take the form of $v_1 \geq v_2 \geq \ldots \geq v_n$. Like in the case of APA equilibria presented earlier, when we include the convex combination of risk in the noisy APA, player payoffs for these valuations simply shift lower, resulting in decreased bids from each player and overall. However, this does nothing to negate the exclusion principle and the findings of Baye et al. (1993) still hold, as we show next. To do so, we closely follow the proofs and explanation of Baye et al. (1993).

Theorem 1 of Baye et al. (1993) gives the expected rents accruing to an auctioneer in any Nash equilibrium (note that equation 3.9 is identical to equation 3.8 in the previous section).

This theorem is crucial to showing the *exclusion principle* since it is used to determine the set of finalists that maximize the auctioneer's rents. We modify it as follows.

Theorem 3.2.4. *Let $v_1 \geq v_2 \geq \ldots \geq v_n$ denote the valuations of lobbyists $\{1, 2, \ldots, n\}$ in the stage-2 lobbying game. Let $E_1 x_1$ denote the expected bid of a lobbyist with the highest valuation. Then in any Nash equilibrium,*

$$W \equiv \sum E_{\mathrm{i}} x_{\mathrm{i}} = \frac{v_2}{v_1} \lambda v_2 + \left[1 - \frac{v_2}{v_1}\right] E_1 x_1 \leq \lambda v_2. \tag{3.9}$$

Proof. Let $G_{\mathrm{i}}(x_{\mathrm{i}})$ denote the cdf of player i in a mixed strategy equilibrium, and let S_{i} denote the support of the cdf. Player i must earn constant expected payoffs almost everywhere (a.e.) in S_{i}. From equation 3.7 we know that for player 1 this constant must equal $u_{\mathrm{i}}^* = \left(\frac{\lambda(n-1)+1}{n}\right) v_1 - \lambda v_2$, and for players $2, 3, \ldots, n$, this constant is $\left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}}$. Hence, the following conditions must hold:

$$\begin{aligned} B_1(x_1) = \prod_{\mathrm{i} \neq 1}^{n} G_{\mathrm{i}}(x_{\mathrm{i}}) \left[\left(\frac{\lambda(n-1)+1}{n}\right) v_1 - x_1\right] + \left[1 - \prod_{\mathrm{i} \neq 1}^{n} G_{\mathrm{i}}(x_{\mathrm{i}})\right] \left[\left(\frac{1-\lambda}{n}\right) v_1 - x_1\right] \\ = \left(\frac{\lambda(n-1)+1}{n}\right) v_1 - \lambda v_2 \end{aligned} \tag{3.10}$$

a.e. on S_{i}, and

$$\begin{aligned} B_{\mathrm{i}}(x_{\mathrm{i}}) = \prod_{\mathrm{j} \neq \mathrm{i}}^{n} G_{\mathrm{j}}(x_{\mathrm{i}}) \left[\left(\frac{\lambda(n-1)+1}{n}\right) v_{\mathrm{i}} - x_{\mathrm{i}}\right] + \left[1 - \prod_{\mathrm{j} \neq \mathrm{i}}^{n} G_{\mathrm{j}}(x_{\mathrm{i}})\right] \left[\left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}} - x_{\mathrm{i}}\right] \\ = \left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}} \end{aligned} \tag{3.11}$$

a.e. on S_{i}, $\mathrm{i} \neq 1$.

Let $A_{\mathrm{i}}(x_{\mathrm{i}}) \equiv \prod_{\mathrm{j} \neq \mathrm{i}}^{n} G_{\mathrm{j}}(x_{\mathrm{i}})$ denote the probability that player i wins the prize, conditional on his bid and the strategies employed by the other $n-1$ players. Then, since equations

3.10 and 3.11 hold almost everywhere in their respective supports, taking the expectations of these equations reveals that

$$\lambda P_1 v_1 - E_1 X_1 = \lambda v_1 - \lambda v_2 \tag{3.12}$$

and

$$\lambda P_{\text{i}} v_{\text{i}} - E_{\text{i}} X_{\text{i}} = 0, \quad \forall \text{i} \neq 1 \tag{3.13}$$

where E denotes the expectation with respect to player j's equilibrium mixed strategy and $P_{\text{j}} \equiv E_{\text{j}} p_{\text{j}}(x_{\text{j}})$. Summing over equations 3.12 and 3.13 yields

$$W \equiv \sum_{\text{j}=1}^{n} E_{\text{j}} x_{\text{j}} = \lambda[(P_1 - 1)v_1 + \sum_{\text{i}\neq 1} P_{\text{i}} v_{\text{i}} + v_2] \tag{3.14}$$

Then for $\sum_{\text{j}=1}^{n} P_{\text{j}}$, it follows that $\sum_{\text{j}=1}^{n} P_{\text{j}} = 1$. Furthermore, if $v_2 > v_{\text{i}}$, $\text{i} > 2$, then $P_{\text{i}} = 0$, as in Theorems 3.2.1, 3.2.2, and 3.2.3. Hence,

$$W = \lambda[(P_1 - 1)v_1 + (2 - P_1)v_2] \tag{3.15}$$

Rearranging, equation 3.12, we get

$$P_1 = \frac{\lambda v_1 - \lambda v_2 + E_1 x_1}{\lambda v_1}$$

which, inserted into equation 3.15 yields our desired result,

$$W = \frac{v_2}{v_1} \lambda v_2 + \left[1 - \frac{v_2}{v_1}\right] E_1 x_1.$$

□

Baye et al. note that in the case of the APA, if two or more players value the prize at a common level v, the expected sum of the bids results in full rent dissipation. This is also the case with the noisy APA. Likewise, if $v_1 > v_2$, the expected sum of bids are strictly less than λv_2, or an under dissipation of rents. Hence, player participation occurs at the same rate for any given prize, despite the value of the prize being discounted.

Baye et al. borrow their Lemma 1 from Hillman and Riley (1989), which we modify to state that in the unique Nash equilibrium for $v_1 \geq v_2 > v_3 \geq v_4 \geq \ldots \geq v_n$, it must be that $Ex_1 = \frac{\lambda v_2}{2}$. Combining this with equation 3.9 implies that when two players value the prize more than all other players, the expected rents accruing to the auctioneer are

$$W(v_1, v_2) = \left[1 + \frac{v_2}{v_1}\right] \frac{\lambda v_2}{2}. \tag{3.16}$$

Accordingly, when $v_1 > v_2 > v_3$, expected rents are increasing in v_2 but decreasing in v_1. Or rather, as player 1's valuation increases, competition becomes more unequal and player 2 reduces his expected payment to the auctioneer for any given level of λ. Hence, total expected bids decline both when risk is not present and when it is, and there is incentive for an auctioneer to exclude the player with the highest valuation.

Finalists are selected according to Proposition 1 of Baye et al., which we modify for the noisy APA:

Proposition 3.2.1. *If $\{\hat{1}, \ldots, \hat{m}\}$ is a rent-maximizing set of finalists (with valuations $\hat{v}_1 \geq \ldots \geq \hat{v}_m$), then expected rents are*

$$W(\hat{v}_1, \hat{v}_2) = \left[1 + \frac{\hat{v}_2}{\hat{v}_1}\right] \frac{\lambda \hat{v}_2}{2}. \tag{3.17}$$

Proof. We must show that if $\{\hat{1}, \ldots, \hat{m}\}$ is a set of finalists that maximizes expected rents (and corresponding valuations are $\hat{v}_1 \geq \ldots \geq \hat{v}_m$), then expected rents are $W(\hat{v}_1, \hat{v}_2)$. We will do so by way of contradiction. The proposition is clearly true if $m = 2$; hence suppose $m > 2$. If $\hat{v}_1 = \hat{v}_2 \equiv \hat{v}$, equation 3.9 reveals that $W = \lambda\hat{v} = W(\hat{v}_1, \hat{v}_2)$. If $\hat{v}_1 > \hat{v}_2 > \hat{v}_3$, equation 3.9 agains shows that $W = W(\hat{v}_1, \hat{v}_2)$. Finally, if $\hat{v}_1 > \hat{v}_2 = \hat{v}_3 \equiv \hat{v}$, expected rents increase by excluding player $\hat{1}$, since by Theorem 3.2.4 $W(\hat{v}_1, \hat{v}_2) < \lambda\hat{v}_2 = W(\hat{v}_2, \hat{v}_3)$. However, this contradicts the hypothesis that the set $\{\hat{1}, \ldots, \hat{m}\}$ maximizes expected rents. Hence, we conclude that any rent-maximizing set of finalists generates expected rents of $W(\hat{v}_1, \hat{v}_2)$. ∎

While equation 3.16 does not hold for all possible configurations of values (e.g., when $\hat{v}_1 > \hat{v}_2 = \hat{v}_3 \equiv \hat{v}$), it does hold when the set of finalists is selected to maximize expected rents, as in equation 3.17. This allows us to determine the set of finalists that maximizes the auctioneer's expected rents. Since equation 3.17 is decreasing in $\hat{v}_1$ and increasing in $\hat{v}_2$, it does not pay to exclude a player with a valuation that lies between those of any two players who are in the set of finalists, meaning that the expected rent-maximizing set of finalists will be determined by considering all pair-wise combinations of adjacent players. This is done until players k and $k+1$ are found such that

$$W(v_k, v_{k+1}) = \max_{\mathrm{i}} W(v_{\mathrm{i}}, v_{\mathrm{i}+1}).$$

This is summarized by modifying Baye et al. (1993)'s Proposition 2.

Proposition 3.2.2. *Suppose $v_1 \geq v_2 \geq v_3 \geq \ldots \geq v_n$, then the auctioneer maximizes expected rents by constructing a set of finalists that excludes players with valuations strictly greater than v_k, where k is such that*

$$\left(1 + \frac{v_{k+1}}{v_k}\right) \frac{\lambda v_{k+1}}{2} \geq \left(1 + \frac{v_{\mathrm{i}+1}}{v_{\mathrm{i}}}\right) \frac{\lambda v_{\mathrm{i}+1}}{2}, \; \forall \mathrm{i}. \tag{3.18}$$

Baye et al. conclude by discussing the implications of Proposition 3.2.2 via two corollaries. The first says that if $v_1 = v_2 \geq v_3 \geq \ldots \geq v_n$, then the auctioneer does not gain by constructing an agenda that excludes some players from the game. This is intuitive since the two players with the highest valuations will compete aggressively for the prize, resulting in full rent dissipation. The second corollary states that if $v_1 > v_2 = v_3 \geq \ldots \geq v_n$, then the auctioneer maximizes total expected bids by excluding the player with the highest valuation from the set of finalists.

For our purposes, the foregoing is sufficient to demonstrate that the exclusion principle holds in the noisy APA for all values of $\lambda \in (0, 1)$. As in the case of equilibria in Baye et al. (1996), adding the λ parameter only results in shifted payoffs and equilibria, but as a linear transformation it does not affect core behavior of the APA. We will demonstrate this further in the next section by showing that the caps on lobbying principle still applies.

3.2.3 The Caps on Lobbying Principle

We continue by demonstrating that the basic results of the caps on lobbying principle of Che and Gale (1998) are unchanged by the convex combination of risk in the noisy APA. This principle states that placing a cap on individual player expenditures in the APA may actually increases total expenditures accruing to the auctioneer. We find that under the reduced payoffs resulting from the inclusion of risk, placing limits on players' bids results in higher total expenditures for some values of λ, even if these are lower than in the original formulation. This finding is detailed below, along with several lemmas, that closely follow those given in Che and Gale (1998).

First, we alter our model slightly to make it compatible with that of Che and Gale. Specifically, let there be $n = m = 2$ players with valuations $v_1 > v_2 > 0$, and let ω denote the maximum allowable bid. Of course we could show the principle using the same n players as before, but doing so would require a significant expansion of Che and Gale's result to include the three cases of Baye et al. (1996) (i.e., $v_1 = \cdots = v_m > v_{m+1} \geq \ldots \geq v_n$, for $m \geq 2$; $v_1 > v_2 = \cdots = v_m > v_{m+1} \geq \ldots \geq v_n$, for $3 \leq m \leq n$; and $v_1 > v_2 > v_3 \geq \ldots \geq v_n$), and doing it is beyond the scope of our present purpose.

As before in the noisy APA, the maximum expected payoff is $u_{\mathrm{i}} = \left(\frac{1+\lambda}{2}\right) v_{\mathrm{i}} - x_{\mathrm{i}}$ and payoffs are decreasing in x_{i}, so bidding $x_{\mathrm{i}} = 0$ guarantees a payoff of $\left(\frac{1-\lambda}{2}\right) v_{\mathrm{i}}$. Therefore, no player will bid an amount greater than $\left(\frac{1+\lambda}{2}\right) v_{\mathrm{i}} - \left(\frac{1-\lambda}{2}\right) v_{\mathrm{i}} = \lambda v_{\mathrm{i}}$, and doing so will secure a payoff of *at least* $\left(\frac{1-\lambda}{2}\right) v_{\mathrm{i}} > 0$. In a setting where bidding is capped and $v_1 > v_2$, it is then reasonable to set the maximum allowable bid such that $\omega < \lambda v_2$. Otherwise a cap on expenditures is meaningless as both players' maximum bid would not exceed λv_2, as proved earlier.

The first lemma of Che and Gale (1998) states that neither player has a mass point at any bid $x \in (0, \omega)$, while at most one player has a mass point at zero. Since this follows directly for the convex combination without change, we omit the proof but state the lemma below for the sake of convenience. However, its implication is important for the noisy APA. Since there is zero probability that player j will bid $x \in (0, \omega)$ when player i does so, player i's payoff is

$$\begin{aligned} u_{\mathrm{i}}(x) &= \lambda \left(v_{\mathrm{i}} G_{\mathrm{j}}(x)\right) + \left(\frac{1-\lambda}{2}\right) v_{\mathrm{i}} - x \\ &= \frac{v_{\mathrm{i}}}{2} \left(2\lambda G_{\mathrm{j}}(x) + 1 - \lambda\right) - x. \end{aligned} \tag{3.19}$$

Like in previous sections where similar expected payoffs were found for player i and $n-1$ opponents, this fact will be useful in the analysis that follows. Also, there cannot be an equilibrium in pure strategies unless both players bid ω, which we demonstrate is not the case.

LEMMA 3.2.2.1. *Neither player has a mass point at any bid $x \in (0, \omega)$. At most one player has a mass point at zero.*

Lemma 3.2.2.2 establishes the lower limit of bids made in equilibrium, along with the implication that $x_1 = x_2 = \omega$ when $\omega < \frac{\lambda v_2}{2}$.

LEMMA 3.2.2.2. *If $\omega \in \left(\frac{\lambda v_2}{2}, \lambda v_2\right)$, both players have an infimum bid of zero. If $\omega < \frac{\lambda v_2}{2}$, both players have an infimum of ω.*

Proof. See appendix for proof of Lemma 3.2.4.2. □

Now, we can show Lemma 3.2.2.3 which proves there is a gap in the set of possible equilibrium bids when $\omega \in (\frac{\lambda v_2}{2}, \lambda v_2)$, and that both players have mass points at ω.

LEMMA 3.2.2.3. *Suppose that $\omega \in (\frac{\lambda v_2}{2}, \lambda v_2)$. There exists a constant x' such that both players place nonzero density on every $x \in (0, x']$ and zero density on every $x \in (x', \omega)$. Both players have mass points at ω.*

Following Che and Gale's method, Lemmas 3.2.2.1, 3.2.2.2 and 3.2.2.3 provide necessary conditions for equilibrium distribution functions, and from these we can find the set of distributions existing across the ranges identified, $x = 0$, $x \in (0, x']$, $x \in (x', \omega)$, and $x = \omega$. We will also find the exact value of x', given the players' valuations, v_1 and v_2, and the cap, ω. To begin, we must first determine the distribution functions that make the players indifferent among all bids in $(0, x']$ and a set of possible caps $\{\omega\}$, which we denote as $(0, x'] \bigcup \{\omega\}$.

Since player 1 must be indifferent among all bids in $(0,x']\bigcup\{\omega\}$, each bid in that set must yield the same expected payoff. That is, player 1's expected payoff for a bid in the region $(0,x']$ must be equal to his expected payoff at ω, given player 2's distribution function, $G_2(x)$.

$$\left(\frac{1+\lambda}{2}\right)v_1G_2(x)+\left(\frac{1-\lambda}{2}\right)v_1(1-G_2(x))-x=\lambda v_1\left[G_2(x')+\left(\frac{1-G_2(x')}{2}\right)\right]+\frac{1-\lambda}{2}v_1-\omega$$
$$\frac{v_1}{2}\left(2\lambda G_2(x)+1-\lambda\right)-x=\frac{v_1}{2}\left(\lambda G_2(x')+1\right)-\omega. \tag{3.20}$$

The left-hand side of equation 3.20 gives the expected payoff for either winning or losing when $x\in(0,x']$; and the right-hand side is the expected payoff when $x=\omega$. If player 1 bids ω, there is probability $1-G_2(x')$ that player 2 also bids ω, resulting in a tie that is broken in player 1's favor with probability 1/2. Likewise, with probability $G_2(x')$, player 2 bids less than ω, and player 1 wins the discounted prize $\left(\frac{1+\lambda}{2}\right)v_1$. Similarly, a bid $x\in(0,x']$ yields player 2

$$\frac{v_2}{2}\left(2\lambda G_1(x)+1-\lambda\right)-x=\frac{v_2}{2}\left(\lambda G_1(x')+1\right)-\omega. \tag{3.21}$$

We can now use equations 3.20 and 3.21 to show that player 2 has mass at zero. Rearranging 3.20 gives

$$\omega-x=\frac{\lambda v_1}{2}\left(G_2(x')-2G_2(x)+1\right), \tag{3.22}$$

and this also implies

$$\omega-x'=\frac{\lambda v_1}{2}\left(G_2(x')-2G_2(x')+1\right). \tag{3.23}$$

Then adding 3.22 and 3.23 together and simplifying yields

$$2\omega-x-x'=\lambda v_1(1-G_2(x)). \tag{3.24}$$

By analogy, we also get

$$2\omega - x - x' = \lambda v_2(1 - G_1(x)), \tag{3.25}$$

and equations 3.24 and 3.25 together imply

$$v_1(1 - G_2(x)) = v_2(1 - G_1(x)), \quad \forall x \in (0, x']. \tag{3.26}$$

Lemma 3.2.2.1 states that the players cannot both have mass points at zero, so either $G_1(0) = 0$ or $G_2(0) = 0$. Since $v_1 > v_2$, equation 3.26 implies $1 - G_2(0) < 1 - G_1(0)$ when $\lim_{x\downarrow 0} G_\mathrm{i}(x) = G_\mathrm{i}(0)$, so it must be that $G_1(0) = 0$ and $G_2(0) = 1 - \frac{v_2}{v_1}$. Hence, we have $G_\mathrm{i}(0)$ for $\mathrm{i} = 1, 2$.

We will now find $G_\mathrm{i}(x)$ for $x > 0$. From equation 3.19 we know player 2's equilibrium expected payoff is $\frac{1-\lambda}{2}v_2$ when $x = 0$, so 3.21 implies

$$\frac{v_2}{2}(2\lambda G_1(x) + 1 - \lambda) - x = \frac{1-\lambda}{2}v_2 \tag{3.27}$$

$$\lambda v_2 G_1(x) - x = 0 \tag{3.28}$$

for all $x \in (0, x']$. Thus, player 1's distribution function satisfies $G_1(x) = \frac{x}{\lambda v_2}$ in that range. Then, because neither player has density in $x \in (x', \omega)$ by Lemma 3.2.2.3, this implies $G_1(x) = G_1(x') = \frac{x'}{\lambda v_2}$ for all $x \in (x', \omega)$. Finally, $G_1(\omega) = 1$ by definition, so we have the piecewise defined distributions for player 1.

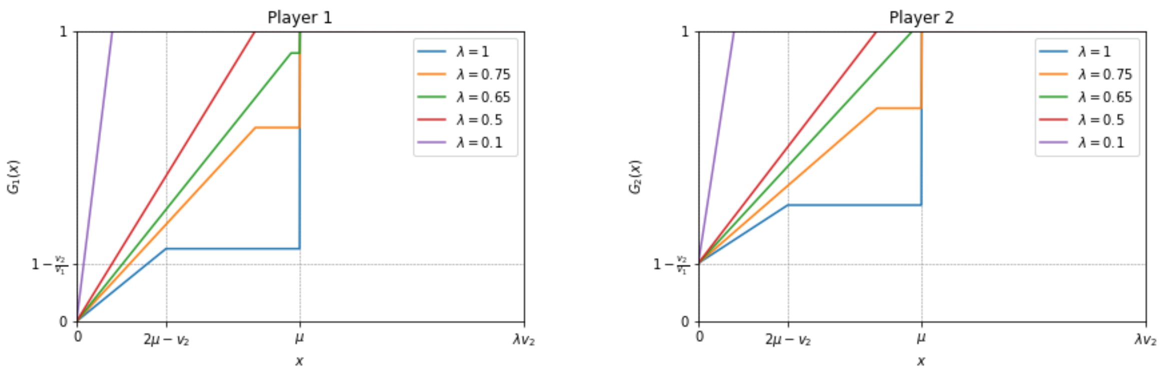
To obtain the set of distributions for player 2, we know that $\lambda v_1 G_2(0) = \lambda v_1 \left(1 - \frac{v_2}{v_1}\right) = \lambda(v_1 - v_2) > 0$, and because $\lim_{x\downarrow 0} \lambda v_1 G_2(x) - x = \lambda v_1 G_2(0)$, we can also say that $\lambda v_1 G_2(x) - x = \lambda(v_1 - v_2)$ for $x \in (0, x']$. Hence, $G_2(x) = 1 - \frac{(v_2 - x/\lambda)}{v_1}$ for $x \in [0, x']$. Then, $G_2(x) = 1 - \frac{(v_2 - x'/\lambda)}{v_1}$ for $x \in (x', \omega)$, and $G_2(\omega) = 1$. Finally, to find x', recall equation 3.21 and the fact that $G_1(x') = \frac{x'}{\lambda v_2}$ when $x \in (0, x']$. Together these give

$$\frac{v_2}{2}(2\lambda G_1(x') - \lambda) - x' = \frac{v_2}{2}(\lambda G_1(x')) - \omega \tag{3.29}$$

$$x' = 2\omega - \lambda v_2 \tag{3.30}$$

Table 3.3. Noisy APA equilibrium distribution functions when there exists a cap on bids.

$x = 0$:	$G_1(x) = 0$	$G_2(x) = 1 - \frac{v_2}{v_1}$
$\forall x \in (0, 2\omega - \lambda v_2]$:	$G_1(x) = \frac{x}{\lambda v_2}$	$G_2(x) = 1 - \frac{v_2 - x/\lambda}{v_1}$
$\forall x \in (2\omega - \lambda v_2, \omega)$:	$G_1(x) = \frac{2\omega}{\lambda v_2} - 1$	$G_2(x) = 1 - \frac{2}{v_1}\left(v_2 - \frac{\omega}{\lambda}\right)$
$x = \omega$:	$G_1(x) = 1$	$G_2(x) = 1$



(a) Player 1

(b) Player 2

Figure 3.2. Equilibrium distribution functions for different levels of risk, λ, when $v_1 > v_2$ and there is a cap on bids ω.

Thus, we have the equilibrium distributions for both players across the range $[0, \omega]$ when λ describes exogenous noise. These are summarized in table 3.3 and illustrated graphically in figure 3.2. Like the equilibrium distributions for the case without caps on bids in figure 3.1, λ has a significant effect. When $\lambda = 1$, we have the APA which clearly exhibits a piecewise function defined over three distinct regions of x. For higher levels of noise, say when $\lambda = 0.5$ or 0.1, these regions still exist, but their slopes are steeper over the range $(0, 2\omega - \lambda v_2]$, highlighting the players' likelihood to make lower bids overall and to reach these bids sooner than in the case of no risk. Furthermore, as noise and random outcomes increase, the region $(2\omega - \lambda v_2, \omega)$ gradually collapses, indicating an incentive for the players to use mixed strategies over a greater range of bids. In fact, when $\lambda = \frac{\omega}{v_2}$, the players mix over the full range $[0, \omega]$, and they bid less than the cap when $\lambda < \frac{\omega}{v_2}$.

We now find the auctioneer's expected revenue. Player 1's *ex ante* probability of winning the prize is

$$p_1^*(x) = \int_0^{x'} \frac{1}{\lambda v_2} G_2(x)\, dx + \left[1 - \frac{x'}{\lambda v_2}\right] \left[G_2(x') + \frac{1 - G_2(x')}{2}\right] = 1 - \frac{v_2}{2v_1},$$

where $\frac{1}{\lambda v_2}$ is the density for player 1, the integral gives the probability of winning given a bid in $(0, x']$, and the second term is the probability of winning with a bid of ω. This results in an expected payment from player 1 that is the difference between gross and net expected payoffs,

$$v_1 \left(\frac{1+\lambda}{2}\right) \left(1 - \frac{v_2}{2v_1}\right) - \lambda(v_1 - v_2) = v_1 \left(\frac{1-\lambda}{2}\right) - v_2 \left(\frac{1-3\lambda}{4}\right)$$

Similarly, player 2's *ex ante* probability of winning is $p_2^*(x) = 1 - p_1^*(x) = 1 - \left(1 - \frac{v_2}{2v_2}\right) = \frac{v_2}{2v_2}$, and his expected payment is

$$(1+\lambda)\frac{v_2^2}{4v_1} - (1-\lambda)\frac{v_2}{2}$$

This results in expected revenue to the auctioneer of

$$\begin{aligned} &\left[v_1 \left(\frac{1-\lambda}{2}\right) - v_2 \left(\frac{1-3\lambda}{4}\right)\right] + \left[(1+\lambda)\frac{v_2^2}{4v_1} - (1-\lambda)\frac{v_2}{2}\right] \\ &\qquad = (1-\lambda)\frac{v_1}{2} - (3-5\lambda)\frac{v_2}{4} + (1+\lambda)\frac{v_2^2}{4v_1}. \end{aligned} \tag{3.31}$$

Now consider $\omega < \frac{\lambda v_2}{2}$. By Lemma 3.2.2.2 we know $x_1 = x_2 = \omega$ in any equilibrium. This results in expected payoffs of $\frac{v_i}{2} - \omega > 0$ for both players, i $= 1, 2$. This is obviously equilibrium behavior since higher bids are not allowed due to the cap, and lower bids lose with probability one. Hence, in equilibrium, each player wins with probability $\frac{1}{2}$, and the auctioneer's expected revenue is 2ω. This is summarized in Che and Gale's Proposition 1, which we modify here for the noisy APA.

Proposition 3.2.3. *If $\omega \in \left(\frac{\lambda v_2}{2}, \lambda v_2\right)$, player 1 wins with probability $1 - \frac{v_2}{2v_1}$, and the auctioneer's expected revenue is $(1-\lambda)\frac{v_1}{2} - (3-5\lambda)\frac{v_2}{4} + (1+\lambda)\frac{v_2^2}{4v_1}$. If $\omega < \frac{\lambda v_2}{2}$, player 1 wins with probability $\frac{1}{2}$, and the expected revenue is 2ω.*

Significantly, a cap $\omega \in \left(\frac{\lambda v_2}{2}, \lambda v_2\right)$ results in the same expected revenue to the auctioneer as in the case without a cap. However, if

$$\omega \in \left((1-\lambda)\frac{v_1}{4} - (3-5\lambda)\frac{v_2}{8} + (1+\lambda)\frac{v_2^2}{8v_1}, \frac{\lambda v_2}{2}\right),$$

then expected revenue is $2\omega > (1-\lambda)\frac{v_1}{2} - (3-5\lambda)\frac{v_2}{4} + (1+\lambda)\frac{v_2^2}{4v_1}$, which exceeds the revenue without a cap. Within this region, the cap is low enough to remove player 1's advantage but large enough that the increase in player 2's bid outweighs any decrease in that of player 1. Finally, for a ω lower than $(1-\lambda)\frac{v_1}{4} - (3-5\lambda)\frac{v_2}{8} + (1+\lambda)\frac{v_2^2}{8v_1}$, the expected revenue is strictly lower with the cap than without.

The presence of risk in the noisy APA changes the effect of a cap. For $\lambda = 1$ the probabilities of winning, expected payments, and the auctioneer's expected revenue are all equivalent to those of Che and Gale's. Yet, when $\lambda = 0.5$, the benefit of a cap on bids disappears and the region $\left((1-\lambda)\frac{v_1}{4} - (3-5\lambda)\frac{v_2}{8} + (1+\lambda)\frac{v_2^2}{8v_1}, \frac{\lambda v_2}{2}\right)$ collapses to 0. Moreover, when $\lambda < 0.5$, a cap on bids creates a situation in which players could theoretically obtain a surplus from the auctioneer by bidding below the cap. Hence, a cap on bidding is only beneficial to an auctioneer when exogenous noise levels are relatively low. Figure 3.3 captures these observations graphically.

The foregoing has interesting implications regarding the effect of noise on political contributions. Empirically, it is difficult ascertain the effect of campaign contribution limits on lobbying efforts. In their paper, Che and Gale (1998) cite the fact that aggregate spending on congressional races doubled over the period of 1976 to 1992 after the Federal Election Campaign Act was passed, and they cite other studies pointing to correlations between such things as campaign contributions and roll-call votes, or committee assignments and political action committee contributions. Yet, more recent studies, such as Pastine and Pastine (2010) and Bombardini and Trebbi (2020), describe the difficulty of establishing causation

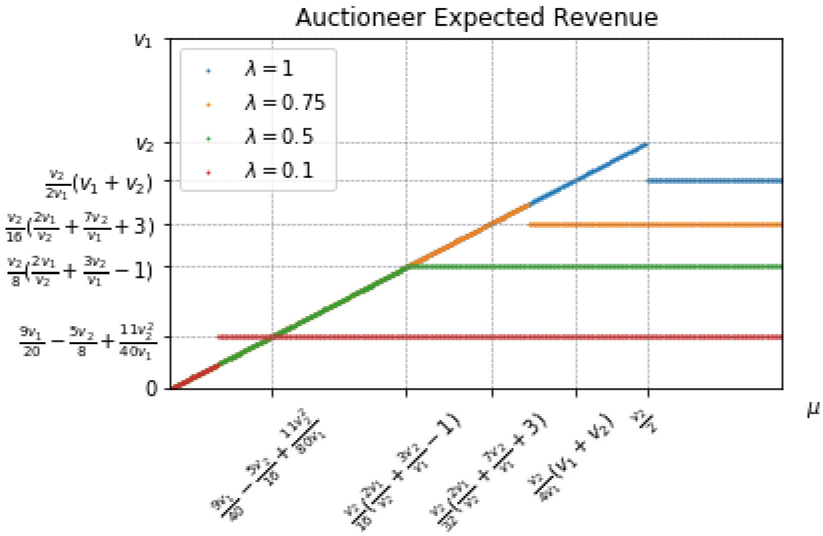


Figure 3.3. Plot of auctioneer revenue versus varying caps, ω, and levels of risk, λ, for the case of $n = m = 2$ players and $v_1 > v_2$.

while also citing papers that indicate either restrictions on campaign contributions tend to reduce individual spending, or that there is no significant effect on total campaign spending. In light of our analysis here, a cap in the noisy APA only yields increased aggregate spending when noise is low. Hence, it could be that while maximum allowable bids are set below the players' true valuations of a prize, the presence of noise has a substantial discounting effect. If this is the case, then we would expect to see situations where a cap on spending has no observable impact on contributions. In other words, lobbying efforts would already be only weakly effective, so a cap on such contributions would not have a serious or detectable impact, thereby explaining why, in the presence of a cap, aggregate contributions sometimes seem to increase, and other times they do not.

3.2.4 Noise Equivalence

Finally, we evaluate whether the noisy APA is noise equivalent to the lottery CSF, which we show is not the case, except for when $\lambda = 0$. Balart et al. (2017) employ a method devised by Alcalde and Dahm (2010) that compares the effort-elasticities of the probabilities of winning. By their definition, two CSFs are noise equivalent if and only if their effort-elasticities are identical when effort is held constant. Effort elasticities are defined as

$$v_{\mathrm{i}}(x_1, \ldots, x_n) = \frac{\partial f_{\mathrm{i}}(x_1, \ldots, x_n)}{\partial x_{\mathrm{i}}} \frac{x_{\mathrm{i}}}{f_{\mathrm{i}}(x_1, \ldots, x_n)}, \tag{3.32}$$

and noise equivalence is provided in definition 3.2.1.

Definition 3.2.1. *(Balart et al., 2017) Two CSFs s and t are noise equivalent if and only if $v_{\mathrm{i}}^s(x_1^s, \ldots, x_n^s) = v_{\mathrm{i}}^t(x_1^t, \ldots, x_n^t)$ for all $\mathrm{i} = 1, \ldots, n$; whenever $x_1^s = x_2^s = \cdots = x_n^s$ and $x_1^t = x_2^t = \cdots = x_n^t$.*

The method for comparing the effort-elasticities is straightforward. For the noisy APA we have

$$v_{\mathrm{i}}^A(x_1^A, \ldots, x_n^A) = \frac{(1-\lambda) r \sum_{\mathrm{j} \neq \mathrm{i}} x_{\mathrm{j}}^r}{x_{\mathrm{i}}^r + \sum_{\mathrm{j} \neq \mathrm{i}} x_{\mathrm{j}}^r},$$

and for the lottery CSF,

$$v_{\mathrm{i}}^L(x_1^L, \ldots, x_n^L) = \frac{r \sum_{\mathrm{j} \neq \mathrm{i}} x_{\mathrm{j}}^r}{x_{\mathrm{i}}^r + \sum_{\mathrm{j} \neq \mathrm{i}} x_{\mathrm{j}}^r}.$$

Setting the two CSFs equal to each other and holding effort constant (i.e., $x = x_1 = x_2 = \cdots = x_n$) produces

$$v_{\mathrm{i}}^A(x) = \frac{r(1-\lambda)(n-1)}{n} = \frac{r(n-1)}{n} = v_{\mathrm{i}}^k(x)$$
$$\lambda = 0$$

Thus, the noisy APA is only noise equivalent to the lottery when $\lambda = 0$ (i.e., the fair lottery) which naturally only holds when $r = \lambda = 0$, and the noisy APA is distinct from the lottery CSF in all other cases.

3.3 Conclusion

Many contests are fundamentally structured as all-pay auctions, yet casual and empirical observation indicates that they are not completely deterministic. Rather, depending on the setting, there are often significant levels of exogenous noise present. When modeling these environments, the empirical literature often defaults to the lottery CSF, presumably because it allows for stochastic outcomes and equilibria in pure strategies, which makes analysis more tractable. However, applying the lottery CSF risks sacrificing the distinguishing characteristics of an auction for increased noise alone. When contest attributes such as mixed strategies and level playing field issues are important, the noisy APA may be a more reasonable choice.

By creating a convex combination of the APA and fair lottery, we achieve a CSF with equilibria in mixed strategies and characteristics very similar to the traditional APA. While not revenue equivalent to the APA, this is primarily due to lower expected payoffs in the presence of risk. When a winning player faces the possibility of randomly losing the prize despite their efforts, equilibrium efforts are lower, leading to reduced payments to the auctioneer. This also leads to smaller regions over which mixed strategies exist, and these decrease in proportion to the increase in noise. Nonetheless, the exclusion principle still holds for values of $\lambda > 0$, and for $\lambda > 0.50$, the caps on lobbying principle holds.

Finally, we show that the noisy APA is not noise equivalent to the lottery CSF, but this result is expected, and for empirical use, the λ parameter could be calibrated to match the observed level of noise in a game, thereby serving as a suitable CSF for APA-like games where noise is present.

3.A Appendix

3.A.1 Proof of Theorem 3.2.1

We first prove Part (A) of the theorem. Let $\alpha_{\mathrm{i}}(x)$ denote the mass that player i places on bid x.

LEMMA 3.2.1.1. *For all* i *and* $\lambda \in (0,1)$, $\lambda v \geq \overline{s}_{\mathrm{i}} \geq \underline{s}_{\mathrm{i}} \geq 0$.

Proof. Because payoffs are decreasing in x_{i}, players can set $x_{\mathrm{i}} = 0$ and guarantee a payoff of at least $\left(\frac{1-\lambda}{n}\right) v$. Winning bids cannot go above $x_{\mathrm{i}} = \left(\frac{\lambda(n-1)+1}{n}\right) v - \left(\frac{1-\lambda}{n}\right) v = \lambda v$ without resulting in payoffs less than $\left(\frac{1-\lambda}{n}\right) v$. Thus, bids cannot be greater than λv and payoffs will be at least $\left(\frac{1-\lambda}{n}\right) v$. Bids less than 0 are ruled out a priori. □

LEMMA 3.2.1.2. *If* $\exists \mathrm{i}$ *such that* $\underline{s}_{\mathrm{i}} \geq \underline{s}_{\mathrm{j}}$ *and* $\alpha_{\mathrm{i}}(\underline{s}_{\mathrm{j}}) = 0$, *then* $\underline{s}_{\mathrm{j}} = 0$ *and* $G_{\mathrm{j}}(0) = \lim_{x \uparrow \underline{s}_{\mathrm{i}}} G_{\mathrm{j}}(x)$. *If in addition,* $\alpha_{\mathrm{i}}(\underline{s}_{\mathrm{i}}) = 0$, *then* $G_{\mathrm{j}}(0) = G_{\mathrm{j}}(\underline{s}_{\mathrm{i}})$.

Proof. Let $u_{\mathrm{j}}(x_{\mathrm{j}}, G_{-\mathrm{j}})$ denote j's payoff to bidding x_{j} when strategies $G_{-\mathrm{j}}$ are employed by the other $n-1$ players. Then $u_{\mathrm{j}}(\underline{s}_{\mathrm{j}}, G_{-\mathrm{j}}) = -\underline{s}_{\mathrm{i}} < 0$ for $\underline{s}_{\mathrm{j}} > 0$. Since the same holds for $u_{\mathrm{j}}(x_{\mathrm{j}}, G_{-\mathrm{j}})$ for $x_{\mathrm{j}} < \underline{s}_{\mathrm{i}}$, and $x_{\mathrm{j}} = \underline{s}_{\mathrm{i}}$ if $\alpha_{\mathrm{i}}(\underline{s}_{\mathrm{i}}) = 0$, the claim follows. □

LEMMA 3.2.1.3. *If* $\underline{s}_1 = \cdots = \underline{s}_m > \underline{s}_{m+1} \geq \ldots \geq \underline{s}_n$, *for* $n \geq m \geq 2$, *then* $\exists \mathrm{i} \leq m$ *such that* $\alpha_{\mathrm{i}}(\underline{s}_{\mathrm{i}}) = 0$.

Proof. By way of contradiction, suppose this were not the case. Then any $\mathrm{i} \leq m$ has incentive to raise their $\underline{s}_{\mathrm{i}}$ by some ϵ small and win the prize. □

LEMMA 3.2.1.4. *If* $\underline{s}_1 = \cdots = \underline{s}_m > \underline{s}_{m+1} \geq \ldots \geq \underline{s}_n$, *for* $n \geq m \geq 2$, *then* $\underline{s}_{\mathrm{i}} = 0$ $\forall \mathrm{i}$.

Proof. Follows immediately from Lemmas 3.2.1.2 and 3.2.1.3. □

LEMMA 3.2.1.5. *There exists no player* i *such that* $\underline{s}_{\mathrm{i}} > \underline{s}_{\mathrm{j}}$ $\forall \mathrm{j} \neq \mathrm{i}$.

Proof. By way of contradiction, suppose such a player did exist. If $\alpha_{\mathrm{i}}(\underline{s}_{\mathrm{i}}) = 0$, from Lemma 3.2.1.2 $G_{\mathrm{j}}(0) = G_{\mathrm{j}}(\underline{s}_{\mathrm{i}})$ $\forall \mathrm{j} \neq \mathrm{i}$, which implies that $u_{\mathrm{i}}(\underline{s}_{\mathrm{i}}, G_{-\mathrm{i}}) < \lim_{x_{\mathrm{i}} \downarrow 0} u_{\mathrm{i}}(x_{\mathrm{i}}, G_{-\mathrm{i}})$. Hence, a contradiction. If the claim held and $\alpha_{\mathrm{i}}(\underline{s}_{\mathrm{i}}) > 0$ then $\forall \mathrm{j} \neq \mathrm{i}$, $\alpha_{\mathrm{j}}(\underline{s}_{\mathrm{i}}) = 0$, so $G_{\mathrm{j}}(0) = \lim_{x_{\mathrm{j}} \uparrow \underline{s}_{\mathrm{i}}} G_{\mathrm{j}}(x_{\mathrm{j}})$ leads to a similar contradiction. □

LEMMA 3.2.1.6. $\underline{s}_{\mathrm{i}} = 0\ \forall \mathrm{i}$.

Proof. Immediate from Lemmas 3.2.1.4 and 3.2.1.5. $\square$

LEMMA 3.2.1.7. $u_{\mathrm{i}}^* = u_{\mathrm{j}}^*\ \forall \mathrm{i}, \mathrm{j}$.

Proof. Without loss of generality, suppose $u_{\mathrm{i}}^* < u_{\mathrm{j}}^*$. Let $\overline{s}_{\mathrm{j}}$ be the upper bound of j's support. But then $u_{\mathrm{i}}^* < u_{\mathrm{j}}^* = u_{\mathrm{j}}(\overline{s}_{\mathrm{j}}, G_{-\mathrm{j}}) \leq \lim_{x_{\mathrm{i}} \downarrow \overline{s}_{\mathrm{j}}} u_{\mathrm{i}}(x_{\mathrm{i}}, G_{-\mathrm{i}})$, a contradiction. $\square$

LEMMA 3.2.1.8. $u_{\mathrm{i}}^* = \left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}}\ \forall \mathrm{i}$.

Proof. If $\alpha_{\mathrm{i}}(\underline{s}_{\mathrm{i}}) = 0$, $\forall \mathrm{i}$ we are through. If $\exists \mathrm{j}$ such that $\alpha_{\mathrm{j}}(\underline{s}_{\mathrm{j}}) > 0$, then $u_{\mathrm{j}}^* = \left(\frac{1-\lambda}{n}\right) v_{\mathrm{j}}$ from Lemmas 3.2.1.3 and 3.2.1.6, and with players receiving equal utility from Lemma 3.2.1.7, $u_{\mathrm{i}}^* = \left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}}\ \forall \mathrm{i}$. $\square$

LEMMA 3.2.1.9. $\exists \mathrm{i}, \mathrm{j}$ *such that* $\overline{s}_{\mathrm{i}} = \overline{s}_{\mathrm{j}} = \lambda v$.

Proof. By way of contradiction, suppose this were not the case. Let $\overline{s}_{\mathrm{i}}$ be the second highest $\overline{s}_{\mathrm{j}}$. Then the player with the highest $\overline{s}_{\mathrm{j}}$ can bid slightly above $\overline{s}_{\mathrm{i}}$ and earn $u_{\mathrm{j}} = \left(\frac{\lambda(n-1)+1}{n}\right) v - \overline{s}_{\mathrm{i}} > u_{\mathrm{j}}^*$. $\square$

The nine lemmas above establish that $\underline{s}_{\mathrm{i}} = 0\ \forall \mathrm{i}$; there exist two i's, say $\mathrm{i} = 1, 2$, such that $\overline{s}_1 = \overline{s}_2 = \lambda v$; and $u_{\mathrm{i}}^* = \left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}}\ \forall \mathrm{i}$. We now pin down the equilibrium distributions. Let $W(x_{\mathrm{i}}) = \left(\frac{\lambda(n-1)+1}{n}\right) v - x_{\mathrm{i}}$, $L(x_{\mathrm{i}}) = \left(\frac{1-\lambda}{n}\right) v - x_{\mathrm{i}}$, $A_{\mathrm{i}} = \prod_{\mathrm{j}=1, \mathrm{j} \neq \mathrm{i}}^{n} G_{\mathrm{j}}$, $A_{\mathrm{ij}} = \prod_{k=1, k \neq \mathrm{j}, \mathrm{i}}^{n} G_k$, and $A_{\mathrm{ij}m} = \prod_{h=1, h \neq \mathrm{j}, \mathrm{i}, m}^{n} G_h$.

LEMMA 3.2.1.10. *There are no point masses on the half open interval* $(0, \lambda v]$.

Proof. Suppose one of the cdfs, say G_{i}, has a mass point at $x_{\mathrm{i}} \in (0, \lambda v]$. By lemma 3.2.1.6, $\forall x \in (0, \lambda v]$ $A_{\mathrm{ij}} G_{\mathrm{i}} > 0$, and hence $A_{\mathrm{ij}} G_{\mathrm{i}}$ has an upward jump at x_{i}, $\forall \mathrm{j} \neq \mathrm{i}$. This follows directly from the monotonicity of the cdfs. For $x_{\mathrm{i}} < \lambda v$ this implies that is is worthwhile for j to transfer mass from an ϵ-neighborhood below x_{i} to some δ-neighborhood above x_{i}. At $x_{\mathrm{i}} = \lambda v$ it pays for j to transfer mass from an ϵ-neighborhood below x_{i} to 0. Thus, there would be an ϵ-neighborhood below x_{i} in which no other player j would put mass. But then it is not an equilibrium strategy for player i to put mass at x_{i}. $\square$

LEMMA 3.2.1.11. *The integrand*

$$B_\mathrm{i}(x_\mathrm{i}) \equiv W(x_\mathrm{i})A_\mathrm{i}(x_\mathrm{i}) + L(x_\mathrm{i})(1 - A_\mathrm{i}(x_\mathrm{i})) \tag{3.33}$$

is constant and equal to zero at the points of increase of G_i in the half-open interval $(0, \lambda v]$ $\forall \mathrm{i}$.

Proof. By Lemma 3.2.1.10, there are no point masses in $(0, \lambda v]$. Thus, $B_\mathrm{i}(x_\mathrm{i})$ is the expected payoff to player i from bidding $x_\mathrm{i} \in (0, \lambda v]$. If x_i is a point of increase of G_i, then player i must make its equilibrium payoff at x_i. □

LEMMA 3.2.1.12. *Suppose x is a point of increase of G_i and G_j in $(0, \lambda v]$. Then $G_\mathrm{i} = G_\mathrm{j}$ at x.*

Proof. By Lemma 3.2.1.8, $B_\mathrm{i}(x) = B_\mathrm{j}(x) = 0$. From equation 3.33 we have

$$W(x)G_\mathrm{j}(x)A_\mathrm{ij}(x) + L(x)[1 - G_\mathrm{j}(x)A_\mathrm{ij}(x)] = 0.$$

This implies $G_\mathrm{j}(x)A_\mathrm{ij}(x) = \frac{-L(x)}{W(x)-L(x)} = G_\mathrm{i}(x)A_\mathrm{ji}(x)$. Division by $A_{\mathrm{ij}(x)} = A_\mathrm{ji}(x) > 0$ gives $G_\mathrm{j}(x) = G_\mathrm{i}(x)$. □

LEMMA 3.2.1.13. *For every* i *and every point of increase x of G_i in $(0, \lambda v]$, there is at least one G_j, $\mathrm{j} \neq \mathrm{i}$, such that G_j is increasing at x.*

Proof. Because $B_\mathrm{i}(x)$ is constant in a neighborhood about x by Lemma 3.2.1.11, $dB_\mathrm{i}(x) = 0$. Suppose contrary to the hypothesis that $dA_\mathrm{i}(x) = 0$. Totally differentiating $B_\mathrm{i}(x)$ then gives $A_\mathrm{i}dW + (1 - A_\mathrm{i})dL = 0$. However, both dW and dL are negative and $A_\mathrm{i}(x) \in (0, 1]$. Hence, for dB_i to be zero, dA_i is necessarily positive. By the monotonicity of the G_j's, at least one has to increase. □

LEMMA 3.2.1.14. *If G_i is strictly increasing on some open interval (a, b), $0 < a < b < \lambda v$, then G_i is strictly increasing on $(a, \lambda v]$.*

Proof. Without loss of generality, suppose, to the contrary, that G_i were constant on (b, c), $b < c \leq \lambda v$. Then from Lemma 3.2.1.10, $G_\mathrm{i}(b) = G_\mathrm{i}(c)$. It is evident that there exists an

$\epsilon > 0$ such that on the interval $(b, b+\epsilon)$ there exist at least two players, say h and k, with strictly increasing cdf's over the interval (otherwise mass would be moved down to b by some player). Thus, for every $x \in (b, b+\epsilon)$, $B_h(x) = B_k(x) = 0$. Furthermore, since there are no mass points in the interval $(0, \lambda v]$, $B_h(b) = B_k(b) = B_{\mathrm{i}}(b) = 0$ which, from arguments similar to those used in proving Lemma 3.2.1.12, implies that $G_h(b) = G_k(b) = G_{\mathrm{i}}(b) > 0$. But with $B_{\mathrm{i}}(b) = B_h(b) = B_h(x)\ \forall x \in (b, b+\epsilon)$, it must be that $B_{\mathrm{i}}(x) \le B_h(x)\ \forall x \in (b, b+\epsilon)$, since such values of x do not lie in i's support. But this implies that $A_{\mathrm{i}}(x) \le A_h(x)$, and hence that $G_h(x) \le G_{\mathrm{i}}(x)$, a contradiction to the fact that $G_{\mathrm{i}}(b) = G_h(b)$, $G_h(x)$ is increasing on $(b, b+\epsilon)$, and $G_{\mathrm{i}}(x)$ is constant on $(b, b+\epsilon)$. □

LEMMA 3.2.1.15. *At least two players randomize continuously on* $[0, \lambda v]$.

Proof. Three cases are possible at 0: (i) all players allocate all mass at 0, (ii) all players have $G_{\mathrm{i}}(x_{\mathrm{i}}) = 0$ at some $x_{\mathrm{i}} > 0$, or (iii) there is at least one player with $G_{\mathrm{i}}(x_{\mathrm{i}}) > 0$ for all $x_{\mathrm{i}} > 0$ and $G_{\mathrm{i}}(0) < 1$. Cases (i) and (ii) are easily ruled out by previous lemmas. For the third case, by Lemmas 3.2.1.3 and 3.2.1.6 at least one of the players has $G_{\mathrm{i}}(0) = 0$. Lemmas 3.2.1.12, 3.2.1.13, and 3.2.1.14 then imply that there are at least two players that randomize continuously over $[0, \lambda v]$. □

LEMMA 3.2.1.16. *Once* G_{i} *is constant on a subset* (a, b), $0 < a < b \le \lambda v$, *it is constant on* $[0, b)$ *and has a mass point at 0.*

Proof. The first part follows immediately from Lemma 3.2.1.14. The second part follows from Lemma 3.2.1.6. □

We now prove Parts (B) and (C) of the theorem.

Proof. By Lemma 3.2.1.8, $E(u_{\mathrm{i}}) = \left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}}\ \forall \mathrm{i}$, and hence $E\left(\sum_{\mathrm{i}=1}^{n} u_{\mathrm{i}}\right) = v_{\mathrm{i}} - \lambda v_{\mathrm{i}}$. This satisfies Part (B). As u_{i} equals the expected revenue to player i minus the bid x_{i}, for $v_1 = \cdots = v_m$ we can write

$$E\left(\sum_{\mathrm{i}=1}^{n} u_{\mathrm{i}}\right) = E\left[\left(\frac{\lambda(n-1)+1}{n}\right) v_1 - x_{\mathrm{i}} + (n-1)\left(\frac{1-\lambda}{n}\right) v_1 - \sum_{\mathrm{j}=1}^{n-1} x_{\mathrm{j}}\right] = v_1 - \lambda v_1,$$

and the right-hand side reduces to $E\left[\sum_{i=1}^{n} x_i\right] = \lambda v_1$. Hence, the sum of expected bids is λv_1, which is less than expected revenue v_1 for $\lambda \in (0,1)$. This satisfies Part (C) and concludes the proof of Theorem 3.2.1. □

3.B.2 Proof of Theorem 3.2.2

We first prove part (A), beginning with with Lemmas 3.2.2.1 and 3.2.2.2, which establish the lower bounds of support for each player's mixed strategy. As with Theorem 3.2.1, if $\overline{s}_i$ and $\underline{s}_i$ are the upper and lower bounds of the support of player i's mixed strategy, then $\forall i$, $v_i \geq \lambda v_i \geq \overline{s}_i \geq \underline{s}_i \geq 0$. Also, $\alpha_i(x)$ is the mass placed at x by player i's mixed strategy.

LEMMA 3.2.2.1. *If* $\exists i$ *such that* $\underline{s}_i \geq \underline{s}_j$ *and* $\alpha_i(\underline{s}_j) = 0$, *then* $\underline{s}_j = 0$ *and* $G_j(0) = \lim_{x\uparrow\underline{s}_j} G_j(x)$. *If, in addition,* $\alpha_i(\underline{s}_i) = 0$, *then* $G_j(0) = G_j(\underline{s}_i)$.

Proof. Let $u_j(x_j, G_{-j})$ denote player j's payoff to bidding x_j when strategies G_{-j} are employed by the other $n-1$ players. Now $u_j(\underline{s}_j, G_{-j}) = -\underline{s}_j < 0$ for $\underline{s}_j > 0$. Since the same holds for $u_j(x_j, G_{-j})$ for any $x_j < \underline{s}_i$, and also for $x_j = \underline{s}_i$ if $\alpha_i(\underline{s}_i) = 0$, the claim follows. □

LEMMA 3.2.2.2. $\underline{s}_i = 0\ \forall i$.

Proof. Clearly, $\lambda v_i \geq \underline{s}_i \geq 0\ \forall i$, so it is sufficient to show that no player employs a mixed strategy that has a support with a strictly positive lower bound. By way of contradiction, suppose $S \equiv \{i | \underline{s}_i > 0\}$ is nonempty, i.e., $\underline{s}_i > 0$ for at least one i.

If S consists of a single player i, then $\underline{s}_i > \underline{s}_j = 0\ \forall j \neq i$. In this case, if $\alpha_i(\underline{s}_i) = 0$, Lemma 3.2.2.1 implies that $G_j(0) = G_j(\underline{s}_i)\ \forall j \neq i$, which in turn implies that $u_i(\underline{s}_i, G_{-i}) < \lim_{x_i\downarrow 0} u_i(x_i, G_{-i})$. This contradicts the hypothesis that $\underline{s}_i > 0$. If $\alpha_i(\underline{s}_i) > 0$, then $\forall j \neq i$, $\alpha_j(\underline{s}_i) = 0$, so $G_j(0) = \lim_{x_j\uparrow\underline{s}_i} G_j(x_j)$ leads to a similar contradiction.

If S contains more than one player, then an argument similar to that just made implies $\underline{s}_i = \underline{s}_j > 0\ \forall i, j \in S$. At least one player $i \in S$ must employ a mixed strategy with $\alpha_i(\underline{s}_i) = 0$, for otherwise any $j \in S$ could gain by increasing $\underline{s}_j$ by a small $\epsilon > 0$ (unless $\underline{s}_j = \lambda v_j$, in which case j has incentive to reduce the bid λv_j to 0). But this means that there exist $i, j \in S$ such that $\underline{s}_i = \underline{s}_j > 0$ and $\alpha_i(\underline{s}_i) = 0$, a contradiction to Lemma 3.2.2.1.

Thus, we conclude that $\underline{s}_i = 0$ for all i. □

Lemma 3.2.2.3 illustrates that, in any mixed-strategy equilibrium, each player 2 through n must employ a strategy that places an atom at 0, while player 1 with his higher valuation does not have an atom at 0. This implies, along with Lemma 2, that players 2 through n earn equilibrium expected payoffs, $u_i^* = \left(\frac{1-\lambda}{n}\right) v_i$.

LEMMA 3.2.2.3. *(a)* $\alpha_1(0) = 0$*; (b)* $\forall i \neq 1$, $\alpha_i(0) > 0$*; (c)* $u_i^* = \left(\frac{1-\lambda}{n}\right) v_i$ $\forall i \neq 1$

Proof. We begin with part (a). Since player i can never receive a payoff greater than $\left(\frac{\lambda(n-1)+1}{n}\right) v_i$, he would never use a strategy that puts mass on $(\lambda v_i, \infty)$ (setting the bid equal to zero strictly dominates such a strategy). Similarly, player 1 has no incentive to use a strategy that puts mass in the interval $(\lambda v_2, v_1]$ since player 2 would never bid higher than λv_2. Hence, $\forall i$, $\bar{s}_i \leq \lambda v_2 < v_1$, which guarantees that player 1 must have an equilibrium payoff u_1^* of at least $\left(\frac{\lambda(n-1)+1}{n}\right) v_1 - \lambda v_2 > 0$. Recognizing this and the fact that not all players can use mixed strategies that have an atom at 0, implies that player 1's mixed strategy cannot place an atom at 0.

For part (b), we know from part (a) that $u_1^* > \left(\frac{1-\lambda}{n}\right) v_1$ in every neighborhood above 0, so player 1 must outbid every other player with a probability that is bounded away from zero. Thus, every player but player 1 must use a strategy that has an atom at 0.

For part (c), we know from part (a) that player 1's mixed strategy does not have an atom at 0, so it follows from part (b) that all other players receive an equilibrium payoff of 0, i.e., $\forall i \neq 1$, $u_i^* = u_i(0, G_{-i}) = \left(\frac{1-\lambda}{n}\right) v_i$. □

We have now established that zero is the lower bound of the support of each player's equilibrium mixed strategy, that all players except player 1 must employ equilibrium strategies that contain an atom at 0, and that the equilibrium payoffs for players 2 through n are zero. Lemma 4 will show that at least two players have λv_2 as the upper bound of the support of their mixed strategies.

LEMMA 3.2.2.4. $\bar{s}_i \leq \lambda v_2$ $\forall i$*, with strict equality for at least two players.*

Proof. From the proof of Lemma 3.2.2.3, $\bar{s}_i \leq \lambda v_2$ $\forall i$. By way of contradiction, suppose that $\bar{s}_i < \lambda v_2$ for all i. By bidding above $\bar{s} \equiv \max_k\{\bar{s}_k\}$ by an arbitrarily small amount, player 2 can earn arbitrarily close to $\left(\frac{\lambda(n-1)+1}{n}\right) v_2 - \bar{s} > \left(\frac{1-\lambda}{n}\right) v_2 = u_2^*$, which contradicts Lemma

3.2.2.3(c). Thus, it must be that $\overline{s}_{\mathrm{i}} = \lambda v_2$ for at least one i. Another player, $\mathrm{j} \neq \mathrm{i}$ must also have $\overline{s}_{\mathrm{j}} = \lambda v_2$, for otherwise player i could gain by reducing $\overline{s}_{\mathrm{i}}$ by a small $\epsilon > 0$. $\square$

The next five lemmas provide the rough characterization of the equilibrium strategies of players $\{2, \ldots, n\}$ stated in Theorem 3.2.2A. For notational convenience, we define $A_{\mathrm{i}}(x) \equiv \prod_{\mathrm{j}\neq \mathrm{i}} G_{\mathrm{j}}(x)$, $A_{\mathrm{i}k}(x) \equiv \prod_{\mathrm{j}\neq \mathrm{i},k} G_{\mathrm{j}}(x)$, and $A_{\mathrm{i}kh}(x) \equiv \prod_{\mathrm{j}\neq \mathrm{i},k,h} G_{\mathrm{j}}(x)$. Also, let $B_{\mathrm{i}}(x_{\mathrm{i}}) \equiv \left[\left(\frac{\lambda(n-1)+1}{n}\right) v_{\mathrm{i}} - x_{\mathrm{i}}\right] A_{\mathrm{i}}(x_{\mathrm{i}}) + \left[\left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}} - x_{\mathrm{i}}\right] (1 - A_{\mathrm{i}}(x_{\mathrm{i}})) = \left(\frac{\lambda(nA_{\mathrm{i}}(x_{\mathrm{i}})-1)+1}{n}\right) v_{\mathrm{i}} - x_{\mathrm{i}}$.

LEMMA 3.2.2.5. *For all* $\mathrm{j} \in \{1, \ldots, n\}$, G_{j} *contains no atoms in the half open interval* $(0, \lambda v_2]$.

Proof. Suppose one of the cdf's say G_{i}, has an atom at $x_{\mathrm{i}} \in (0, \lambda v_2]$. Lemma 3.2.2.2 implies that $\forall x \in (0, \lambda v_2]$, $A_{\mathrm{ij}} G_{\mathrm{i}} > 0$, and hence $A_{\mathrm{ij}} G_{\mathrm{i}}$ has an upward jump at x_{i}, $\forall \mathrm{j} \neq \mathrm{i}$. This follows directly from the monotonicity of the cdf's. For $x_{\mathrm{i}} < \lambda v_{\mathrm{j}}$ this implies that player j can gain by transferring mass from an ϵ-neighborhood below x_{i} to some δ-neighborhood above x_{i}. At $x_{\mathrm{i}} = \lambda v_{\mathrm{j}}$ it pays for j to transfer mass from an ϵ-neighborhood below x_{i} to zero. Thus, there would be an ϵ-neighborhood below x_{i} in which no other player's mixed strategy puts mass. But then it is not an equilibrium strategy for player i to put mass at x_{i}. $\square$

LEMMA 3.2.2.6. $B_{\mathrm{i}}(x_{\mathrm{i}})$ *is constant and equal to* u_{i}^* *at the points of increase of* G_{i} *on* $(0, \lambda v_2]$ $\forall \mathrm{i}$. $B_{\mathrm{i}}(x_{\mathrm{i}}) \leq u_{\mathrm{i}}^*$ *if* x_{i} *is not a point of increase of* G_{i} *on* $(0, \lambda v_2]$.

Proof. By Lemma 3.2.2.5 there are no atoms in $(0, \lambda v_2]$. Thus, $B_{\mathrm{i}}(x_{\mathrm{i}})$ is the expected payoff to player i from bidding $x_{\mathrm{i}} \in (0, \lambda v_2]$. If x_{i} is a point of increase of G_{i}, player i must make his equilibrium payoff at x_{i}. $\square$

LEMMA 3.2.2.7. *For all* $x \in (0, \lambda v_2]$, *there exists* $\mathrm{i}_1, \mathrm{i}_2$ *such that* $\forall \epsilon > 0$: $G_{\mathrm{i}}(x + \epsilon) - G_{\mathrm{i}}(x - \epsilon) > 0$, $\mathrm{i} = \mathrm{i}_1, \mathrm{i}_2$.

Proof. Immediate. $\square$

LEMMA 3.2.2.8. $\overline{s}_{\mathrm{i}} = 0$ $\forall \mathrm{i} > m$.

Proof. Without loss of generality, assume $\overline{s}_{m+1} = \max_{\mathrm{i}>m}\{\overline{s}_{\mathrm{i}}\}$. Suppose $\overline{s}_{m+1} \neq 0$. Then there exists an interval $(\overline{s}_{m+1} - \epsilon, \overline{s}_{m+1}]$ in which G_{m+1} increases and in which $B_{m+1}(x) =$

$u^*_{m+1} = \left(\frac{1-\lambda}{n}\right) v_{m+1} = \left(\frac{\lambda(nA_{m+1}(x)-1)+1}{n}\right) v_{m+1} - x$. Thus, $\lambda v_{m+1} = \frac{x}{A_{m+1}}$ $\forall x \in (\overline{s}_{m+1} - \epsilon, \overline{s}_{m+1}]$. From Lemma 3.2.2.7, $\forall x \in (\overline{s}_{m+1}, \lambda v_2]$ $\exists \mathrm{i} \in \{2, \ldots, m\}$ such that G_{i} is increasing at x. Since there are no atoms in $(\overline{s}_{m+1}, \lambda v_2]$, for each $x \in (\overline{s}_{m+1}, \lambda v_2]$ there is a player $\mathrm{i} \in \{2, \ldots, m\}$ such that $\lambda v_{\mathrm{i}} = \frac{x}{A_{\mathrm{i}}(x)}$. This implies that for any $x > \overline{s}_{m+1}$, but arbitrarily close to $\overline{s}_{m+1}$, there exists an $\mathrm{i} \in \{2, \ldots, m\}$ such that $A_{\mathrm{i}}(x) = \prod_{\mathrm{j} \neq \mathrm{i}} G_{\mathrm{j}}(x) > \prod_{\mathrm{j} \neq \mathrm{i}} G_{\mathrm{j}}(\overline{s}_{m+1}) > \prod_{\mathrm{j} \neq m+1} G_{\mathrm{j}}(\overline{s}_{m+1}) = A_{m+1}(\overline{s}_{m+1})$, a contradiction to the fact that $v_{m+1} < v_{\mathrm{i}}$. Thus, $\overline{s}_{m+1} = 0$. □

Lemma 3.2.2.8 demonstrates that when $n > m$, players $m+1$ through n bid zero with probability one. The following lemmas characterize the equilibrium strategies of players 1 through m.

LEMMA 3.2.2.9. *Suppose $x \in (0, \lambda v_2]$ is a point of increase in G_{i} and g_{j} for $\mathrm{i}, \mathrm{j} \in \{2, \ldots, m\}$. Then $G_{\mathrm{i}} = G_{\mathrm{j}}$ at x.*

Proof. By Lemmas 3.2.2.3(c) and 3.2.2.6, $B_{\mathrm{i}}(x) = B_{\mathrm{j}}(x) = 0$, which may be written as

$$\left[\left(\frac{\lambda(n-1)+1}{n}\right) v_2 - x\right] G_{\mathrm{j}}(x) A_{\mathrm{ij}}(x) + \left[\left(\frac{1-\lambda}{n}\right) v_2 - x\right] (1 - G_{\mathrm{j}}(x) A_{\mathrm{ij}}(x)) = 0.$$

This implies that $G_{\mathrm{j}}(x) A_{\mathrm{ij}}(x) = \frac{x}{\lambda v_2} = G_{\mathrm{i}}(x) A_{\mathrm{ji}}(x)$. Division by $A_{\mathrm{ij}} = A_{\mathrm{ji}} > 0$ gives $G_{\mathrm{j}}(x) = G_{\mathrm{i}}(x)$. □

LEMMA 3.2.2.10. *If G_{i}, $\mathrm{i} \in \{2, \ldots, m\}$ is strictly increasing on some open subset (a, b), where $0 < a < b < \lambda v_2$, then G_{i} is strictly increasing on the entire interval $(a, \lambda v_2]$. Furthermore, at least one of the players $\{2, \ldots, m\}$ randomizes continuously on the interval $(0, \lambda v_2]$.*

Proof. Suppose to the contrary that G_{i} were constant on (b, c), $b < c \leq \lambda v_2$. Then from Lemma 3.2.2.5, $G_{\mathrm{i}}(b) = G_{\mathrm{i}}(c)$. By Lemma 3.2.2.7, there exists an $\epsilon > 0$ such that on the interval $(b, b + \epsilon)$ there exist at least two players, h and k, with strictly increasing cdf's over the interval. At least one of these players, say h, must be an element of $\{2, \ldots, m\}$. Since the mixed strategies contain no atoms in the interval $(0, \lambda v_2]$, from Lemma 3.2.2.9 $G_h(b) = G_{\mathrm{i}}(b) > 0$. But from Lemmas 3.2.2.3(c) and 3.2.2.6, $B_{\mathrm{i}}(b) = B_h(b) = B_h(x)$ $\forall x \in (b, b + \epsilon)$. Hence, $B_{\mathrm{i}}(x) \leq B_h(x)$ $\forall x \in (b, b + \epsilon)$, since such values of x do not lie in i's support. But this implies that $A_i(x) \leq A_h(x)$, and hence $G_h(x) \leq G_i(x)$, a contradiction to the fact that $G_i(b) = G_h(b)$, G_h is increasing on $(b, b+\epsilon)$, and G_i is constant on $(b, b+\epsilon)$. The second statement follows from the first part of Lemma 3.2.2.10 and from Lemmas 3.2.2.4 and 3.2.2.7. $\square$

Lemma 3.2.2.10 shows that, in equilibrium, at least one of the players $\{2, \ldots, m\}$ randomizes continuously on $(0, \lambda v_2]$, where the only change is in the valuation of player 2. Notice that, by Lemma 3.2.2.3, the mixed strategies of players $\{2, \ldots, m\}$ contain an atom at 0, but by 3.2.2.5 no player's mixed strategy places an atom in the half-open interval $(0, \lambda v_2]$. Lemma 3.2.2.10 thus implies that if G_i, $i \in \{2, \ldots, m\}$ is increasing over any interval (a, b), $0 < a < b < \lambda v_2$, then G_i must be strictly increasing on the interval $(a, \lambda v_2]$. Hence, any gap in the support of player i's mixed strategy must be of the form $(0, b_i]$ for some $b_i > 0$. Furthermore, from Lemma 3.2.2.9, for any point of increase $x \in (0, \lambda v_2]$ of G_i and G_j, $i, j \in \{2, \ldots, m\}$, these distribution functions must take identical values.

LEMMA 3.2.2.11. *(a) $\bar{s}_1 = \lambda v_2$. Furthermore, for every bid $0 < x < \lambda v_2$ in the support of G_1, $G_1(x) < G_i(x)$, $i \in \{2, \ldots, m\}$. (b) $u_1^* = \left(\frac{\lambda(n-1)+1}{n}\right) v_1 - \lambda v_2$.*

Proof. (a) From Lemma 3.2.2.10, at least one $j \in \{2, \ldots, m\}$ randomizes continuously on $(0, \lambda v_2]$. Without loss of generality, suppose player 2 is such a player. From Lemma 3.2.2.3(a), player 1's mixed strategy does not have an atom at 0, and from Lemma 3.2.2.5, no player's mixed strategy has an atom in $(0, \lambda v_2]$. Thus, there exists some point $x \in (0, \lambda v_2]$ at which $G_1(x)$ is increasing. At any such point, $B_1(x) \geq \left(\frac{\lambda(n-1)+1}{n}\right) v_1 - \lambda v_2$, since the right-hand side represents what player 1 can obtain by bidding λv_2 with probability one. Rearranging this expression we obtain $A_1(x) \geq \frac{\lambda v_1 - \lambda v_2 + x}{\lambda v_1}$. From Lemmas 3.2.2.3 and 3.2.2.6, $A_2(x) = \frac{x}{\lambda v_2}$. Then, subtracting A_1 from A_2 gives

$$A_2(x) - A_1(x) \leq \frac{1}{\lambda v_1}\left(\frac{v_1 x}{v_2} - \lambda v_1 + \lambda v_2 - x\right) < 0,$$

where the strict right-hand side inequality follows from the assumption that $\lambda v_2 > x$ and $v_1 > v_2$. Thus, at any point of increase of G_1 in $(0, \lambda v_2)$, $A_1 > A_2$. This directly implies that $G_2 > G_1$ for any such point. But since G_2 has support $[0, \lambda v_2]$ and G_1 has no atoms, this implies $\overline{s}_1 = \lambda v_2$. Furthermore, since for any other player $\mathrm{i} \in \{2, \ldots, m\}$ and for any $x \in [0, \lambda v_2]$, $G_2(x) \leq G_{\mathrm{i}}(x)$, we have the second claim.

To prove part (b), part (a) and Lemma 3.2.2.6 imply that player 1's equilibrium payoff is $u_1^* = \left(\frac{\lambda(n-1)+1}{n}\right) v_1 - \lambda v_2)$. $\square$

This completes the proof to part (B) of Theorem 3.2.2. To complete the proof to part (A) we must show Lemma 3.2.2.12.

LEMMA 3.2.2.12. *(a) Player 1 randomizes continuously on support* $[0, \lambda v_2]$. *(b)* $\prod_{\mathrm{i}=2}^{m} \alpha_{\mathrm{i}}(0) = \frac{(v_1 - v_2)}{v_1}$.

Proof. To show part (a), we know that $\overline{s}_1 = \lambda v_2$ and $\underline{s}_1 = 0$. Suppose there is a gap (a, b) in which $G_1(x)$ is constant, $0 < a < b < \lambda v_2$. By Lemmas 3.2.2.6, 3.2.2.7, and 3.2.2.8, we know that at $x = a$ there are at least two players $\mathrm{i}, k \in \{2, \ldots, m\}$ such that $A_{\mathrm{i}}(x) = A_k(x) = \frac{x}{\lambda v_2}$. At $x = b$ this holds as well. In addition, since a and b are in the support of G_1, $A_1(x) = \frac{\lambda v_1 - \lambda v_2 + x}{\lambda v_1}$, $x = a, b$. Thus we have

$$G_1(x) G_k(x) A_{\mathrm{i}k1}(x) = \frac{x}{\lambda v_2}, \qquad x = a, b \tag{3.34}$$

$$G_{\mathrm{i}}(x) G_k(x) A_{\mathrm{i}k1}(x) = \frac{\lambda v_1 - \lambda v_2 + x}{\lambda v_1}, \qquad x = a, b. \tag{3.35}$$

Now, because we assume $G_1(a) = G_1(b)$, and by Lemma 3.2.2.9 $G_{\mathrm{i}}(x) = G_k(x)$ for $x \in [a, b]$, equation 3.34 implies

$$\frac{[G_k(a) A_{\mathrm{i}k1}(a)]}{[G_k(b) A_{\mathrm{i}k1}(b)]} = \frac{a}{b}, \tag{3.36}$$

and equation 3.35 implies

$$\frac{[[G_k(a)]^2 A_{\mathrm{i}k1}(a)]}{[[G_k(b)]^2 A_{\mathrm{i}k1}(b)]} = \frac{\lambda v_1 - \lambda v_2 + a}{\lambda v_1 - \lambda v_2 + b} \tag{3.37}$$

If we let $\theta \equiv \lambda(v_1 - v_2) > 0$, and combine 3.36 and 3.37 we get

$$\begin{aligned}
\frac{[G_k(a)]^2 A_{\mathrm{i}k1}(a)}{[G_k(a)]^2 A_{\mathrm{i}k1}(a)} &= \frac{G_k(a)}{G_k(b)}\left(\frac{a}{b}\right) \\
\frac{\lambda(\theta + a)}{\lambda(\theta + b)} &= \frac{G_k(a)}{G_k(b)}\left(\frac{a}{b}\right) \\
G_k(a) &= \frac{G_k(b)(\theta + a)b}{(\theta + b)a}.
\end{aligned}$$

Since $\frac{b}{a} > \frac{\theta+b}{\theta+a}$, this implies that $G_k(a) > G_k(b)$, which contradicts the fact that $b > a$. Thus, player 1's mixed strategy distributes positive mass to every open interval in $[0, \lambda v_2]$. This, along with Lemmas 3.2.2.3(a) and 3.2.2.5, implies that player 1's mixed strategy contains no atoms and has a strictly increasing cdf on its support, $[0, \lambda v_2]$. Part (b) follows from part (a), Lemma 3.2.2.6, and Lemma 3.2.2.11(b). $\square$

Hence, in any equilibrium: (1) player 1 earns an expected payoff of $\left(\frac{\lambda(n-1)+1}{n}\right) v_1 - \lambda v_2$, while all other players earn expected payoffs of $\left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}}\ \forall \mathrm{i} \neq 1$; (2) player 1's mixed strategy contains no atoms or gaps in its support, and thus G_1 is strictly increasing on its support, $[0, \lambda v_2]$; (3) players $m + 1$ through n bid zero with probability one; and (4) all other players $\mathrm{j} \in \{2, \ldots, m\}$ play a mixed strategy that has an atom at zero and a strictly increasing cdf on some interval of the form $(b_{\mathrm{j}}, \lambda v_2]$, where $b_{\mathrm{j}} \geq 0\ \forall \mathrm{j}$, with strict equality for at least one j. Lemma 3.2.2.9 guarantees that in subintervals of $(0, \lambda v_2]$ where the mixed strategies of any subset of the players $\{2, \ldots, m\}$ apply a positive mass, the players have the same value of their cdf's. The system of equations given by $B_{\mathrm{i}}(x) = u^*_{\mathrm{i}}$ for $\mathrm{i} \in \{2, \ldots, m\}$ in Lemma 3.2.2.6 thus determines the equilibrium mixed strategies, $G_{\mathrm{i}}(x)$, for any given nonnegative vector $(b_2, b_3, \ldots, b_m)$ for which at least one $b_{\mathrm{i}} = 0$. These are given in table 3.2 in the text. Recursive application of Lemma 3.2.2.9 for given b_{i}'s implies that these constitute all the equilibria.

Proof of Theorem 2C. We showed in Theorem 3.2.4, and equation 3.9 specifically, that in any Nash equilibrium, $\sum Ex_{\mathrm{i}} = \frac{v_2}{v_1}\lambda v_2 + \left[1 - \frac{v_2}{v_1}\right] E_1 x_1 \leq \lambda v_2$. Similar to Baye et al. (1996), we will show that

(a) Ex_1 is maximized in an equilibrium in which all but one of players 2 through m bid zero with probability one, and

(b) Ex_1 is minimized when players 2 through m play symmetric strategies.

Note that if cdf F stochastically dominates cdf G, then $E_F[x] > E_G[x]$.

To show part (a), recall from Lemma 3.2.2.10 that at least one of the players $\{2, \ldots, m\}$ randomizes continuously on $(0, \lambda v_2]$. Suppose this is player i. Then by Lemma 3.2.2.6, $B_{\mathrm{i}}(x) = 0 \ \forall x \in (0, \lambda v_2]$. Isolating the cdf of player 1, G_1, in the expression for A_{i} yields $G_1(x) = \left[\frac{x}{\lambda v_2 \prod_{\mathrm{j} \neq 1,\mathrm{i}} G_{\mathrm{j}}(x)}\right]$. Hence, across all equilibria, $G_1(x)$ is minimized for each $x \in (0, \lambda v_2]$ when the denominator is maximized. This implies that $G_1(x)$ is minimized when $\prod_{\mathrm{j} \neq 1,\mathrm{i}} G_{\mathrm{j}}(x) = 1$ (i.e., in the equilibrium where only players 1 and i actively bid). But this means that G_1 in this asymmetric equilibrium stochastically dominates the corresponding G_1's that arise in the other equilibria, which implies Ex_1 is maximized in this equilibrium.

For part (b), suppose player i $\in \{2, \ldots, m\}$ randomizes continuously on $(0, \lambda v_2]$. Then $G_1(x)$ is maximized for each $x \in [0, \lambda v_2]$ across equilibria when $\prod_{\mathrm{j} \neq 1,\mathrm{i}} G_{\mathrm{j}}(x)$ is minimized. By Lemma 3.2.2.12, in any equilibrium player 1 randomizes continuously over $(0, \lambda v_2]$. This implies by Lemma 3.2.2.6 that in any equilibrium $A_1(x) = \frac{\lambda(v_1 - v_2) + x}{\lambda v_1}$. Since $A_1(x) = \prod_{\mathrm{j} \neq 1} G_{\mathrm{j}}(x)$ is constant across equilibria, $\prod_{\mathrm{j} \neq 1,\mathrm{i}} G_{\mathrm{j}}(x)$ is minimized in an equilibrium in which $G_{\mathrm{i}}(x)$ is maximized. But by Lemmas 3.2.2.5, 3.2.2.9, and 3.2.2.10, in any equilibrium and for every $\mathrm{j} \in \{2, \ldots, m\}$, $\mathrm{j} \neq \mathrm{i}$, $G_{\mathrm{i}}(x) \leq G_{\mathrm{j}}(x) \ \forall x \in [0, \lambda v_2]$. Hence, maximizing $G_1(x)$ across equilibria requires maximizing the minimum of the $G_k(x)$'s, $k \in \{2, \ldots, m\}$. Since for each $x \in (0, \lambda v_2]$, $A_1(x)$ is constant across equilibria, this is done by setting $G_k(x) = G_{\mathrm{j}}(x) \ \forall k, \mathrm{j} \in \{2, \ldots, m\}$ on $[0, \lambda v_2]$. □

This completes the proof of Theorem 3.2.4, which shows that equilibria under the APA with a convex combination of risk is largely identical to traditional APA, with the exception that each player's valuations are reduced by a multiple of λ, and this creates an upper bound for each player that is carried through to the equilibria.

3.B.3 Proof of Theorem 3.2.3

LEMMA 3.2.3.1. *For all* i, $\lambda v_{\text{i}} \geq \overline{s}_{\text{i}} \geq \underline{s}_{\text{i}} \geq 0$.

Proof. Same as proof for Lemma 3.2.1.1, but insert v_{i} in place of v. ☐

LEMMA 3.2.3.2. *Same as Lemma 3.2.1.2.*

LEMMA 3.2.3.3. *If* $\underline{s}_{\text{i}} = \cdots = \underline{s}_m > \underline{s}_{m+1}, \ldots, \underline{s}_n$ *for* $n \geq m \geq 2$ *then* $\exists$ i $\leq m$ *such that* $\alpha_{\text{i}}(\underline{s}_{\text{i}}) = 0$.

Proof. Suppose not. Then any i $\leq m$ has incentive to raise the bid $\underline{s}_{\text{i}}$ by ϵ small, unless $\underline{s}_{\text{i}} = \lambda v_{\text{i}}$, in which case i has incentive to reduce the bid λv_{i} to 0. ☐

LEMMA 3.2.3.4. *Same as Lemma 3.2.1.4.*

LEMMA 3.2.3.5. *Same as Lemma 3.2.1.5.*

LEMMA 3.2.3.6. *Same as Lemma 3.2.1.6.*

In the analysis that follows let $\overline{s}$ be the upper bound of the union of the supports of the players' equilibrium bid distributions.

LEMMA 3.2.3.7. $\overline{s} \leq \lambda v_2$.

Proof. Player i would never put mass above λv_{i} since setting the bid equal to 0 strictly dominates such a strategy. Also, player 1 clearly has no incentive to put mass in the interval $(\lambda v_2, v_1]$. ☐

LEMMA 3.2.3.8. *All players other than player 1 must place a mass point at 0.*

Proof. By Lemma 3.2.3.6, $\underline{s}_{\text{i}} = 0$ $\forall$i. Since $\overline{s} \leq \lambda v_2 \leq \left(\frac{\lambda(n-1)+1}{n}\right) v_1$ player 1 must have an equilibrium payoff u_1^* of at least $\left(\frac{\lambda(n-1)+1}{n}\right) v_1 - \lambda v_2 > 0$. Thus, player 1 cannot have a mass point at 0. This follows from Lemma 3.2.3.3, i.e., some player must put no mass at 0, in which case player 1 with probability 1 would not submit the high bid at 0, and would have payoff $u_1 = \left(\frac{1-\lambda}{n}\right) v_1$ there. Since $u_1^* > \left(\frac{1-\lambda}{n}\right) v_1$, in every neighborhood above 0 player 1 must outbid every other player with a probability that is bounded away from zero. Thus, every player but player 1 must put a mass point at 0. ☐

LEMMA 3.2.3.9. *For all* $\mathrm{i} \neq 1$, $u_{\mathrm{i}}^* = \left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}}$.

Proof. Immediate from Lemmas 3.2.3.3 and 3.2.3.8. □

LEMMA 3.2.3.10. $\overline{s} = \lambda v_2$ *and* $\overline{s}_1 = \overline{s}_2 = \lambda v_2$.

Proof. From Lemma 3.2.3.7, $\overline{s} \leq \lambda v_2$. Suppose $\overline{s} < \lambda v_2$. By bidding above $\overline{s}$ by an arbitrarily small amount, player 2 can earn arbitrarily close to $\left(\frac{\lambda(n-1)+1}{n}\right) v_2 - \overline{s} > \left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}} = u_2^*$, a contradiction. Thus, $\overline{s} = \lambda v_2$. The second part of the claim is straightforward. □

LEMMA 3.2.3.11. *There are no mass points on the half open interval* $(0, \lambda v_2]$.

Proof. Similar to the proof of Lemma 3.2.1.10, inserting v_2 for v the first two times that v appears in the proof, and v_{j} for v the last two times it appears. □

LEMMA 3.2.3.12. $B_{\mathrm{i}}(x_{\mathrm{i}}) \equiv \left[\left(\frac{\lambda(n-1)+1}{n}\right) v_{\mathrm{i}} - x_{\mathrm{i}}\right] A_{\mathrm{i}}(x_{\mathrm{i}}) + \left[\left(\frac{1-\lambda}{n}\right) v_{\mathrm{i}} - x_{\mathrm{i}}\right] (1 - A_{\mathrm{i}}(x_{\mathrm{i}}))$ *is constant and equal to* u_{i}^* *at the points of increase of* G_{i} *in* $(0, \lambda v_2]$ *for all* i. $B_{\mathrm{i}}(x_{\mathrm{i}}) \leq u_{\mathrm{i}}^*$ *if* x_{i} *is not a point of increase in* $(0, \lambda v_2]$.

Proof. Similar to Lemma 3.2.1.11. □

LEMMA 3.2.3.13. *For all* $x \in (0, \lambda v_2]$ $\exists \mathrm{i}_1, \mathrm{i}_2$ *such that* $\forall \epsilon > 0$: $G_{\mathrm{i}}(x + \epsilon) - G_{\mathrm{i}}(x - \epsilon) > 0$, $\mathrm{i} = \mathrm{i}_1, \mathrm{i}_2$.

Proof. Immediate. □

LEMMA 3.2.3.14. $\overline{s}_{\mathrm{i}} = 0$ $\forall \mathrm{i} > 2$.

Proof. Without loss of generality, assume $\overline{s}_3 = \max_{\mathrm{i} \geq 3} \overline{s}_{\mathrm{i}}$. Suppose $\overline{s}_3 \neq 0$. Then there exists an interval of increase $(\overline{s}_3 - \epsilon, \overline{s}_3]$ in which $B_3(x) = u_3^* = \left(\frac{1-\lambda}{n}\right) v_3 = \left[\left(\frac{\lambda(n-1)+1}{n}\right) v_3 - x\right] A_3(x) + \left[\left(\frac{1-\lambda}{n}\right) v_3 - x\right] (1 - A_3(x))$. Thus, $v_3 = \frac{x}{\lambda A_3(x))}$ $\forall\, x \in (\overline{s}_3 - \epsilon, \overline{s}_3]$. But as G_1 and G_2 are increasing on $(\overline{s}_3, \lambda v_2]$, $v_2 = \frac{\overline{s}_3}{\lambda A_3(\overline{s}_3))}$. Since for $\overline{s}_3 > 0$, $A_2(\overline{s}_3) = \prod_{\mathrm{j} \neq 2} G_{\mathrm{j}}(\overline{s}_3) > \prod_{\mathrm{j} \neq 3} G_{\mathrm{j}}(\overline{s}_3) = A_3(\overline{s}_3)$, we have a contradiction to the fact that $v_3 < v_2$. Thus, $\overline{s}_3 = 0$. □

This concludes the proof of Theorem 3.2.3.

3.B.4 Proofs of Lemmas 3.2.2.2 and 3.2.2.3 for caps on lobbying using the noisy APA

LEMMA 3.2.4.2. *If $\omega \in \left(\frac{\lambda v_2}{2}, \lambda v_2\right)$, both players have an infimum bid of zero. If $\omega < \frac{\lambda v_2}{2}$, both players have an infimum of ω.*

Proof. Let $x^* \equiv \inf\{z|G_1(z) > 0\}$ denote the infimum of player 1's bids. We first show that only zero or ω can be infimum bids in equilibrium. Suppose instead that player 1's infimum bid is $x^* \in (0, \omega)$. If player 2 makes a bid in $(0, x^*)$, he loses with probability one. Since a bid of zero is better, player 2 must have zero density in $(0, x^*)$. This means that player 1 could profitably move density in $(x^*, x^* + \epsilon^*)$ arbitrarily close to zero. For $x \in (x^*, x^* + \epsilon^*)$, the payment would drop by x. However, the probability of winning would drop by only $G_2(x) - G_2(0) = G_2(x) - G_2(x^*) < G_2(x^* + \epsilon^*) - G_2(x^*)$. This last term is of order ϵ^*, by Lemma 1. It follows that moving the density raises player 1's expected payoff, for some $\epsilon^* > 0$. Since a profitable deviation exists, an infimum bid of $x^* \in (0, \omega)$ cannot occur in equilibrium. The symmetric argument shows that player 2 cannot have an infimum in $(0, \omega)$ either, so only zero and ω are possible infimum bids in equilibrium.

The remainder of the theorem consists of two cases. In the first case, suppose that $\omega \in (\frac{\lambda v_2}{2}, \lambda v_2)$. Next, by way of contradiction, suppose that $x^* = \omega$, which implies that player 1 bids $x_1 = \omega$. Then player 2 bids zero or ω, or randomizes between the two, since a bid of zero strictly dominates any $x \in (0, \omega)$. Bidding zero is obviously not an equilibrium since $x_1 = \omega$ is not optimal if $x_2 = 0$. Yet, bidding ω or randomizing between 0 and ω can only be optimal for player 2 if $\frac{\lambda v_2}{2} - \omega \geq 0$, since a bid of ω results in a tie. However, this restriction on ω contradicts $\omega \in (\frac{\lambda v_2}{2}, \lambda v_2)$, so $x^* = \omega$ cannot occur in equilibrium. A similar argument shows that player 2 cannot have an infimum bid of ω, so the common infimum must be zero.

For the second case, and again by way of contradiction, we show that both players have an infimum of ω. Suppose $\omega < \frac{\lambda v_2}{2}$. Bidding ω ensures at least a tie, so player i must receive an expected payoff of at least $\frac{\lambda v_i}{2} - \omega > 0$. If player i has an infimum bid of zero, then a bid near zero must be as good as a bid of ω for player i. But if player j does not have mass at zero, then player i receives less than $\frac{\lambda v_i}{2} - \omega$ if he bids near zero. Player j must therefore have mass at zero. Since player j's infimum is also zero, the same argument implies that player i must have mass at zero, by Lemma 3.2.2.1, so the infimum must equal ω for both players. $\square$

LEMMA 3.2.4.3. *Suppose that $\omega \in (\frac{\lambda v_2}{2}, \lambda v_2)$. There exists a constant x' such that both players place nonzero density on every $x \in (0, x']$ and zero density on every $x \in (x', \omega)$. Both players have mass points at ω.*

Proof. We first show that both players have mass points at ω. We know by Lemma 3.2.2.2 that the common infimum is zero, while Lemma 3.2.2.1 shows that at least one player has no mass at zero. Suppose player i does not have mass at zero. Then if player j bids arbitrarily close to zero, his expected payoff is strictly below $\lambda v_j - \omega > 0$. Since his infimum is zero, a bid near zero must be as good as a bid of ω for player j. However, if player i does not have mass at ω, then a bid of ω would yield player j an expected payoff of $\lambda v_j - \omega$. Player i must therefore have mass at ω. We conclude that at least one player has mass at ω, since at least one player has no mass at zero.

Suppose that player i has mass $\alpha_i(\omega) > 0$ at ω. If player j has nonzero density in $(\omega - \epsilon', \omega)$, he could profitably move it to ω, for some $\epsilon' > 0$. For $x \in (\omega - \epsilon', \omega)$, the payment would rise by only $\omega - x < \epsilon'$, but the probability of winning would rise by at least ${}^{\alpha_i(\omega)}/_2$, since player j would now tie if player i bids ω. Since moving the density up raises player j's expected payoff, player j must have zero density in $(\omega - \epsilon', \omega)$. If player j has no mass at ω, then player i could profitably take mass from ω and move it lower. We conclude that both players have mass points at ω.

The presence of mass points at ω for both players implies that both players have zero density in $(\omega - \epsilon'', \omega)$, for some $\epsilon'' > 0$. This demonstrates the existence of $x^* \in [0, \omega]$ such that both players have zero density in (x^*, ω). Let x' denote the smallest $x^* \in [0, \omega]$ such that both players place zero density on every bid in (x^*, ω). We now show that both players place nonzero density on every $x \in (0, x']$. By the argument used in the proof of Lemma 3.2.2.2, if player i has zero density in an interval $(a, b) \subset (0, x']$, then so must player j. But if both players have zero density in (a, b) then either player could profitably move density from $(b, b+\epsilon^*)$ down to a, for some $\epsilon^* > 0$. Thus, both players must have nonzero density on every $x \in (0, x']$. This concludes the proof for Lemma 3.2.2.3. $\square$

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