# FACETS OF COMPUTATION PLATFORMS: FROM CONCEPTUAL FRAMEWORKS TO PRACTICAL INSTANTIATIONS <br> by <br> Rishabh Khare 

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To my parents, Suman and Rajendra

## ACKNOWLEDGMENTS

It seems only yesterday when I had taken a shuttle from Chicago airport to West Lafayette to begin my journey as a graduate student at Purdue. My mathematics professor at IISER Kolkata had reminded me that I would be living in a small university town that is surrounded with corn fields and wind mills. Having lived at a similar setup at IISER, West Lafayette seemed quite familiar to me. I still remember missing the first half an hour of my qualifier examinations as I walked from the Purdue Memorial Union to the physics department on a rainy afternoon.

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## TABLE OF CONTENTS

LIST OF FIGURES ..... 10
LIST OF SYMBOLS ..... 13
ABBREVIATIONS ..... 14
ABSTRACT ..... 15
1 INTRODUCTION AND OVERVIEW ..... 16
1.1 Landauer's Principle and Reversible Computing ..... 16
1.2 Topological Quantum Computation and Majorana fermions ..... 19
1.3 Brief Outline ..... 21
2 BALLISTIC FLUXONS IN JOSEPHSON JUNCTION TRANSMISSION LINES ..... 22
2.1 Superconductivity ..... 22
2.2 Josephson Junctions ..... 23
2.3 The RCSJ Model of a Josephson Junction ..... 26
2.3.1 Underdamped and Overdamped Josephson Junctions ..... 29
2.4 Fluxons in Josephson Junction Transmission Lines ..... 31
3 FLUXON-BASED ASYNCHRONOUS BALLISTIC REVERSIBLE COMPUTA- TION ..... 34
3.1 RSFQ Logic: The Original Fluxon-based Logic ..... 34
3.2 Asynchronous Ballistic form of Reversible Logic (ABRL) ..... 35
3.3 ABRL devices ..... 36
3.3.1 Reversible Memory Cell ..... 37
3.3.2 The JJ Rotary ..... 39
3.3.3 Parts of the Rotary Circuit ..... 39
PieceWise Linear pulse (PWL) ..... 40
DC to Single Flux Quantum (DCSFQ) converter ..... 41
The JJ Transmission Lines ..... 41
Rotary Junction ..... 42
3.3.4 Fluxon Path in the Working Rotary Circuit ..... 42
3.4 Time Reversal Property of the Rotary ..... 44
3.5 Rotary Behavior vs. Input Fluxon Energy ..... 48
4 TOWARDS EXACT MODELS OF TOPOLOGICAL QUANTUM COMPUTATION ..... 50
4.1 Majorana Fermions in Particle Physics and Condensed Matter ..... 50
4.2 Kitaev Quantum Wire ..... 51
4.3 An Invitation: Mean Field vs Exact Solution for a Bose Gas ..... 52
4.4 The Lieb-Liniger model ..... 54
4.5 Bethe Ansatz Solution to the Lieb-Liniger Model ..... 56
4.6 Bethe Equations ..... 57
4.7 Ground State of the Lieb-Liniger Model ..... 59
4.8 The Excitation Spectrum of the Lieb-Liniger Model ..... 64
4.8.1 Type I excitation ..... 64
4.8.2 Type II excitation ..... 68
4.9 Experimental Validation of the Lieb-Liniger Solution ..... 69
4.10 Hard Wall Boundary Condition: Bethe Ansatz ..... 70
4.11 Hard Wall Boundary Condition: Bethe Equations ..... 72
4.12 Hard Wall Boundary Condition: Ground State ..... 74
4.13 Fermions With P-wave Interactions ..... 75
4.14 Bound states in the Attractive Lieb-Liniger model ..... 76
4.14.1 Two-string Bound Solutions ..... 79
4.14.2 Future Outlook ..... 80
REFERENCES ..... 81
A IMPEDANCE CALCULATION FOR THE LJJ ..... 90
B PROMISING CIRCUITS ..... 93
B. 1 Rotary with current sources ..... 93
B.1.1 Phases of JJ in the Rotary junction ..... 94
B.1.2 Effect of terminating resistance ..... 95
B.1.3 Addition of a circulating current ..... 96
B.1.4 Replacing the source with circulating current ..... 97
Clockwise ..... 97
Counter-Clockwise ..... 98
B. 2 Symmetric rotary ..... 100
B.2.1 Motivation for junctions ..... 101
B.2.2 Final working circuit ..... 105
B.2.3 Working range of impedance ..... 108
B.2.4 Failure of loop current reset ..... 109
C CODE FOR AUTOMATING THE CIRCUIT SIMULATION ..... 112
C. 1 Using pyWRSpice for automation ..... 112
D SCATTERING CALCULATIONS ..... 118
D. 1 Separation of variables ..... 118
D. 2 Scattering coefficients ..... 119
E LIEB'S CALCULATIONS ..... 121
E. 1 Bethe ansatz in the thermodynamic limit ..... 121
E. 2 Friedholm solver ..... 123
F BETHE'S HARDWALL EQUATIONS FOR 2 PARTICLES ..... 124
G STRING SOLUTION OF TWO PARTICLES ..... 127
G. 1 Single string solution for 3 particles in a hardwall ..... 127

## LIST OF FIGURES

1.1 Energy targets from the 2015 ITRS roadmap ..... 18
2.1 Josephson Junction ..... 24
2.2 RCSJ diagram ..... 27
2.3 Analogy between gauge-invariant phase difference and damped motion of a particle ..... 29
2.4 Overdamped and Underdamped Josephson Junctions ..... 30
2.5 JJ transmission lines ..... 31
3.1 RSFQ transmission line ..... 34
3.2 Synchronous vs Asynchronous ..... 36
3.3 Reversible Memory cell ..... 37
3.4 WRSpice simulation for RM cell ..... 38
3.5 Rotary and Flipping Rotary ..... 39
3.6 Rotary circuit ..... 40
3.7 PieceWise Linear Pulse generator ..... 41
3.8 DC to SFQ converter ..... 42
3.9 JJ transmission line unit ..... 43
3.10 DLJJ 20 ..... 43
3.11 Rotary junction closeup ..... 44
3.12 Fluxon path ..... 45
3.13 Fluxon path for longer times ..... 46
3.14 Antifluxon input ..... 46
3.15 Fluxon input with flipped lines near the Rotary junction ..... 47
3.16 Antifluxon input with flipped lines near the Rotary junction ..... 47
3.17 Varying Fluxon Energy ..... 48
3.18 Energy vs Resistance ..... 49
4.1 Majorana bound states in Kitaev wires ..... 52
4.2 Excitations in an Interacting One-dimensional Bose gas ..... 53
4.3 Partitioning the Hilbert Space ..... 55
4.4 Scattering in one dimension ..... 56
4.5 Scattering in a ring ..... 58
4.6 Ground state energy ..... 63
4.7 Excitations in a Bose gas ..... 65
4.8 Making a hole in the distribution of the roots ..... 65
4.9 Ground state re-assignment for $N=6$ ..... 66
4.10 Ground state re-assignment for $N=8$ ..... 68
4.11 Excited state spectrum ..... 70
4.12 Hard wall boundary condition ..... 71
4.13 Scattering with hard walls ..... 72
4.14 Ground state energy in the presence of hard walls ..... 74
A. 1 Recursive relation in LJJ ..... 91
A. 2 Impedance matching ..... 92
B. 1 Rotary circuit with current sources ..... 93
B. 2 Rotary in action ..... 94
B. 3 Critical value of bias ..... 95
B. 4 Terminating with resistors ..... 96
B. 5 Circulating current in the loop ..... 97
B. 6 Loop current ..... 98
B. 7 Clockwise current ..... 99
B. 8 Anticlockwise current ..... 100
B. 9 Built-in chirality ..... 101
B. 10 Symmetric circuit ..... 102
B. 11 Symmetric junction closeup ..... 103
B. 12 Wustmann junction ..... 103
B. 13 JJK2 rotary ..... 104
B. 14 All JJK2 junction ..... 104
B. 15 Addition of loop current ..... 105
B. 16 Capacitance correction ..... 106
B. 17 Working symmetric circuit ..... 107
B. 18 Fluxon path for long times ..... 109
B. 19 Trapped fluxon ..... 110
B. 20 Junction node diagram ..... 110

## LIST OF SYMBOLS

| $\Psi, \psi$ | wavefunction |
| :--- | :--- |
| $\hbar$ | reduced Planck's constant |
| $\mu$ | chemical potential |
| e | electron charge |
| $\rho$ | electron density |
| $V$ | potential difference |
| $\phi$ | superconducting phase |
| $\delta$ | phases difference in a Josephson Junction |
| $J$ | current density |
| $I$ | current |
| $\lambda_{J}$ | Josephson penetration depth |
| $I_{c}$ | critical current |
| $\Phi_{0}$ | flux quantum |
| $R$ | resistance |
| $C$ | capacitance |
| $L$ | inductance |
| $U$ | potential energy |
| $M$ | mass |
| $\eta$ | damping |
| $\beta_{C}$ | Stewart-McCumber parameter |
| $D$ | torque |
| $g$ | acceleration due to gravity |
| $c$ | interaction strength |
| $k_{\mathrm{j}}$ | wavevector |
| $E$ | energy |

## ABBREVIATIONS

| JJ | Josephson Junction |
| :--- | :--- |
| LJJ | Long Josephson Junction |
| DLJJ | Discrete Long Josephson Junction |
| RCSJ | Resistively and Capacitively Shunted Junction |
| LCR | Inductor, Capacitance and Resistor |
| ABRL | Asynchronous Ballistic Reversible Logic |
| ABRC | Asynchronous Ballistic Reversible Computing |
| RM | Reversible Memory |
| SFQ | Single Flux Quantum |
| DCSFQ | DC to SFQ |
| 1D | 1 Dimension |


#### Abstract

We live in an age in which computation touches upon every aspect of our lives in ever increasing ways. To meet the demand for increased computing power and ability, new computation strategies are continually being proposed. In this dissertation, we consider two research projects related to two such cutting edge paradigms. We first consider developing superconducting devices that implement asynchronous reversible ballistic computation. This paradigm was developed to circumvent Landauer's principle of a minimum energy required per bitwise computation operation. We report the design of a new device, the rotary, which is a critical step towards developing universal computation gates in the scheme of synchronous reversible ballistic computation. Next, we turn to the consideration of anyons which have been predicted to enable topological quantum computing, a quantum computing paradigm that is relatively immune to environmental noise. We consider initial steps in the development of a Bethe ansatz solvable model that will help decipher the many-body properties of Majorana zero modes in superconducting Kitaev wires.


## 1. INTRODUCTION AND OVERVIEW

In this dissertation, we will explore research topics related to two computation paradigms at the cutting edge of research - ballistic reversible computing and topological quantum computation. These have been proposed as future alternatives to conventional computing approaches: ballistic reversible computing is expected to lead to much lower energy consumption because it circumvents Landauer's principle mandating a minimum energy dissipation per elementary computation step; topological quantum computation is hoped to implement a decoherence-resistant form of quantum computing, a new paradigm that can outperform classical algorithms in certain kinds of important tasks.

### 1.1 Landauer's Principle and Reversible Computing

Rolf Landauer [1] applied thermodynamical reasoning to digital computers, arguing that an elementary computation on a bit, which is conventionally accompanied by erasure of the input bit and writing of the output bit to memory, requires expenditure of $\Delta W \geq k T \ln 2$ amount of work which eventually gets lost as heat. The act of erasure requires increasing entropy by $\Delta S=k \ln 2$, where $k=k_{B}$ is the Boltzmann's constant. The subsequent process of writing leads to the loss of this entropy that needs to be supplied by work done on the system: $\Delta W \geq k T \ln 2$, where $T$ is the temperature of the surroundings. This lower dissipation bound is a consequence of the way traditional computation is performed, in a thermodynamically irreversible fashion.

A simple demonstration of Landauer's principle comes from considering an elementary bit memory to be the position of a classical particle in a double well. A simple erasure technique is to lower the double well barrier to zero and then lift it back up again slowly. At the end of this process, if we started with a written bit, i.e., we knew that the particle was in a particular well, the entropy of the memory would have increased by $\Delta S=k \ln 2$. This 'initialized' memory can be written onto by lifting the bottom of the well that needs to be unoccupied above the barrier and then lowering it again, the final state being assured to be the one in which the particle is in the undisturbed well. Since all previously gained entropy is lost, external work input of at least $\Delta W=T \Delta S$, where $T$ is the temperature of
the surroundings, is necessary. Thus, entire process of erasure and writing requires at least $T \Delta S=k T \ln 2$ amount of external work input, which is eventually dissipated as heat.

There have been numerous experimental attempts to test Landauer's principle using the picture just presented. In 2012, Berut et. al. [2] investigated Landauer's principle by looking at colloidal particles pinned down in double-well potential. Their experiment was the first experiment to quantify the small amounts of energy released when a single bit of information is erased. There have been other high precision experiments [3], [4] that demonstrated the validity of Landauer's principle in conventional computation processes.

Landauer's principle is technologically significant for the following reason. Over the last 70 years, computing technology has progressed rapidly, as evidenced by the continuous applicability of Moore's exponential increase law to the growth of concentration of computing power per unit area of integrated circuit chips. This means that computers need to dissipate increasing amounts of energy per unit area and so simultaneously designs have to be made more energy efficient. As seen in Figure 1.1, in near future semiconductor computing efficiency is predicted to increase but is bound above by physical limits that are more stringent than allowed by Landauer's principle [5]. In theory, by inventing other technologies we could still continue to make conventional computing more and more energy efficient, but finally Landauer's principle will make further improvement impossible.

Looking ahead to circumventing the final limit on energy efficiency imposed by Landauer's principle, the paradigm of reversible computing [7] proposes to remove the central condition, irreversibility, driving Landauer's principle. The idea is to design computation around gates which are one-to-one, producing a unique output for every possible input, and reversible, exchanging the input-output data pair when run in reverse. There is no information erasure and no known finite minimum energy bound for computation to occur. In practice, this reversibility of gates has to be paired with lossless, i.e., ballistic propagation of information, leading to an ideal zero dissipation computation paradigm known as Ballistic Reversible Computation.

A further improvement is necessary before Ballistic Reversible Computation can become practical. In the original paradigm, ballistic signals need to arrive together at a reversible gate for computation to occur. This would require complex clocking circuits and other large


Figure 1.1. Energy targets from the 2015 ITRS roadmap: Node energies (red) and gate energies (blue) are shown. The transistor gate energies are predicted to reach the thermal noise limit $\left(100 k_{B} T\right)$ which will limit our scaling capabilities in the future. Beyond this, we will need to either reduce the number of gate operations or reuse the large fractions of the logic signal energy [6].
design overheads to realize. To eliminate this need, the paradigm of Asynchronous Ballistic Reversible Computation (ABRC) has been invented[6], where gates are designed such that input bits can arrive consecutively at arbitrarily delayed intervals for computation to occur. It is this paradigm that we are working to design physical devices for, in collaboration with Drs. Michael Frank and Rupert Lewis at Sandia National Lab.

Specifically, we will consider superconducting networks that implement ABRC. In our design, ballistic bits will be represented by two possible polarizations of topologically stabi-
lized soliton excitations in Josephson Junction (JJ) transmission lines. Our focus will be on designing devices, based on superconducting elements, which can be used to finally construct universal computation gates in the ABRC paradigm. We will discuss the development and characterization of a three terminal rotary device for polarization-dependent chiral routing of fluxons from terminal to terminal.

### 1.2 Topological Quantum Computation and Majorana fermions

While the paradigm of reversible computation tries to improve energy efficiency of computation, yet another paradigm - quantum computation - promises to outperform conventional classical/deterministic bit-based computation when solving some specific yet significant problems.

The difference between deterministic bit-based 'classical' computers and quantum computers fundamentally arise from the objects quantum computers work with - 'qubits' - complex wavefunctions in a two-dimensional Hilbert space, instead of classical bits that classical computers work with, which can only store information in one of two states: 0 or 1 . A qubit can exist in an infinite continuum of complex linear combinations of quantum states $|0\rangle$ and $|1\rangle$ i.e. $\alpha|0\rangle+\beta|1\rangle$, where $\alpha$ and $\beta$ are complex numbers satisfying $|\alpha|^{2}+|\beta|^{2}=1$ and whose overall complex phase is irrelevant. This vastly expanded operation space allows development of quantum computation algorithms, based on unitary transformations from quantum physics, which can outperform classical algorithms for certain tasks. One such task is to factorize a large $m$-digit number [8]. The fastest classical algorithm takes $\sim \exp \left(m^{\frac{1}{3}}\right)$ steps to solve this problem, a daunting task for large $m$. This difficulty underlies the functioning of the dominant public encryption platforms we use today. Meanwhile, Shor [9](1994) showed that quantum computers can perform this factorization in $\sim m^{2} \log m \log \log m$ steps which is considerably faster than classical computation algorithms. This and other results imply that we can solve many complex problems using a quantum computer, which are unsurmountable when using classical computers.

All this is exciting but realizing a quantum computer faces some hurdles. Error correction is an important part of a computer. In the presence of environmental noise, delicate
superpositions of quantum states undergo wavefunction collapse leading to failure of the computation. While environment-induced (usually thermal) errors can affect classical computers, modern designs utilizing appropriate error correction routines can easily assure effectively faultless operation. Modern technologies have yet to produce a quantum computer, even though in theory error correction algorithms have been invented to cure the problem of quantum decoherence [10]-[14]. However, keeping in mind the nascent classical computers and how sensitive they were to the environment, we can be hopeful that in future robust quantum computers will be built that handle errors so well that, like the situation with classical computers now, end users are completely unaware of them.

A central goal for building quantum computers today is thus to prevent environmental decoherence. The challenge is to isolate the computer from the environment well enough such that decoherence is manageable, yet retain enough external control such that users can send inputs and receive outputs. One scheme that promises to make this possible is the idea of topological quantum computation [15], [16]. At the heart of this idea is the existence of novel particles called anyons, more specifically non-abelian anyons, which allow nontrivial unitary transformations to occur amongst a degenerate set of states due to the simple action of moving anyons around each other in complex patterns (braiding). A crucial point is that creating such anyons in conventional matter, composed of 'non'-anyon fermions and bosons, requires the creation of an highly non-local quantum-entangled many-body state. Such high entanglement is expected to be immune to environmental effects, which are local in nature, unless the environmental disturbance has an energy larger than the energy gap separating the anyon states from other excited states.

Where can we find such non-abelian anyons? As it turns out, superconductors can have exotic excitations, called Majorana bound states due to a superficial similarity with Majorana fermions from particle physics [17], which behave like non-abelian anyons [18]. While originally vortices in exotic two dimensional $p+\mathrm{i} p$ superconductors were thought to host these Majorana states [19], it was later argued that such anyons can live at the ends of $p$-wave superconducting wires [20]. This idea has been further distilled to the understanding that if one can create a single fermion mode into which superconductivity is induced by, say, proximity with a conventional superconductor, then the two ends of the one dimensional
channel host Majorana bound states [21]-[23]. This is thought to be an accessible challenge with current experimental technologies and a widespread decade-long effort has been made to study Majorana bound states in superconducting nanowires.

While experiments have reached stunning precision, conclusive establishment of the existence of Majorana bound states with anyonic properties has remained out of reach. It is still the most accessible model of a non-abelian anyon and our group has been developing a Bethe-ansatz solvable model to study the many-body properties of the Majorana bound state. Such a solvable model requires three critical ingredients: (a) it needs to include local attractive interactions; (b) it is a fermionic model with exactly one species of fermion; and (c) it needs to possess hard wall boundary conditions. Solvable models have previously been constructed to study the Majorana bound state, but none have been able to meet all three criteria listed above. We will report some of our progress in the last chapter of this dissertation.

### 1.3 Brief Outline

This dissertation is organized as follows. Chapter 2 and Chapter 3 focus on superconducting electronics, historical attempts at creating superconducting logic, asynchronous ballistic reversible circuits and the formulation of the rotary device. In Chapter 4, we discuss the Kitaev model of the Majorana bound state, the Bethe-ansatz solvable Lieb-Liniger model and its extension to hard walled boundary conditions, an essential step towards realizing Majorana bound states in the corresponding fermion model.

## 2. BALLISTIC FLUXONS IN JOSEPHSON JUNCTION TRANSMISSION LINES

In this chapter, we will discuss how ballistic fluxons propagating in superconducting circuits arise, which can then be used to implement asynchronous ballistic reversible computing.

### 2.1 Superconductivity

Superconductivity was discovered by Onnes over a century ago [24] and we now have a very successful theory of superconductivity in many conventional substances [25]-[27]. Superconductors, of course, famously have zero resistance. However, they are not just 'very good conductors/metals' but entirely new states of matter - they expel magnetic flux, a phenomenon known as the Meissner effect [28], a property that is at odds with the behavior of metals.

We will use two significant properties of superconductors in what follows. First, a superconductor is characterized at all points inside by a complex order parameter field, $\Psi(\vec{r})$ [25]-[27], which can be thought of as a macroscopic condensate wavefunction of -2e-charged BCS Cooper pairs [26] (see below). Second, the complex phase of the order parameter, $\varphi=\arg (\Psi)$, couples to the magnetic vector potential such that in equilibrium deep inside a superconductor, the gradient of the phase is exactly balanced by the vector potential and so:

$$
\begin{equation*}
\varphi\left(\vec{r}_{2}\right)-\varphi\left(\vec{r}_{1}\right)=-\frac{2 \mathrm{e}}{\hbar} \int_{\vec{r}_{1}}^{\vec{r}_{2}} \vec{A}(\vec{r}) \cdot d \vec{r} . \tag{2.1}
\end{equation*}
$$

Since the phase of a complex number is unique up to an integer multiple of $2 \pi$, by considering a closed loop (i.e., $\vec{r}_{1}=\vec{r}_{2}$ above) that is everywhere inside the superconductor and using Stokes' theorem, we have:

$$
\begin{equation*}
2 m \pi=\frac{2 \mathrm{e}}{\hbar} \Phi_{B}, \quad \Rightarrow \quad \Phi_{B}=m \Phi_{0} . \tag{2.2}
\end{equation*}
$$

Herein, $\Phi_{B}$ is the magnetic flux, $\Phi_{0}=h /(2 \mathrm{e})$ is the universal superconducting flux quantum and $m$ is an integer. If there is a superconductor everywhere inside the loop, $\Phi_{B}=0$ due to the Meissner effect and so only $m=0$ is allowed. However, if there is a hole inside the loop, $\Phi_{B}$ can be nonzero but has to be an integer multiple of the universal flux quantum! This is the remarkable property of flux quantization in superconducting rings that has been observed in experiments [29], [30]. As we will see below, ballistic soliton excitations trapping such quantized fluxes, called 'fluxons', can be created in superconducting transmission lines incorporating Josephson junctions [31], laying the foundation for various computation paradigms in superconducting systems [32].

### 2.2 Josephson Junctions

Central to the construction of fluxon-carrying transmission lines is a novel superconducting device - the Josephson junction. Josephson Junctions ([33], [34]) are formed when two superconductors are sandwiched together with a barrier between them. The barrier can be a thin insulator (superconductor-insulator-superconductor junction or S-I-S), a nonsuperconducting metal (S-N-S), or something that diminishes the superconductivity at the barrier (S-c-S).

Notably, Josephson Junctions carry a superconducting current, typically known as the supercurrent, that can exist even in the absence of an external voltage. The physical reason behind this phenomenon is the flow of BCS Cooper pairs across the barrier (DC Josephson effect) [26], [35]. When an applied voltage is present, the supercurrent oscillates with a welldefined period (AC Josephson effect). To explain these effects, we will now use an intuitive derivation given by Feynman [36].

Consider a Josephson junction made up of two superconductors, as shown in figure 2.1. Let $\Psi_{1}$ and $\Psi_{2}$ be the order parameters/condensate wavefunctions for the superconductors in the left and the right superconductor, respectively. Writing down the Schrodinger equations for the two sides, assuming that the condensate has charge -2 e ,


Figure 2.1. Josephson Junction: Two superconductors, 1 and 2 , joined by a thin barrier, form a Josephson junction (JJ). The condensate wavefunctions in the two superconductors are denoted by $\Psi_{1}$ and $\Psi_{2}$, respectively. Supercurrent flows in the absence of voltage if the complex phases of $\Psi_{1}$ and $\Psi_{2}$ are different, a phenomenon known as the DC Josephson effect. Adding a voltage across the JJ results in oscillatory flow of the supercurrent with the frequency of oscillations universally related to the voltage V [37] - this is the AC Josephson effect.

$$
\begin{align*}
& \mathrm{i} \hbar \frac{\partial \Psi_{1}}{\partial t}=\mu_{1} \Psi_{1}+K \Psi_{2}  \tag{2.3}\\
& \mathrm{i} \hbar \frac{\partial \Psi_{2}}{\partial t}=\mu_{2} \Psi_{2}+K \Psi_{1} \tag{2.4}
\end{align*}
$$

where $K$ is some coupling constant that characterizes the junction and $\mu_{1}, \mu_{2}$ are the chemical potentials of the superconductors. Since there is a potential difference between the two terminals, $\mu_{1}-\mu_{2}=-2 \mathrm{e} V$. Defining the zero of energy to be halfway between $\mu_{1}$ and $\mu_{2}$, we get

$$
\begin{align*}
& \mathrm{i} \hbar \frac{\partial \Psi_{1}}{\partial t}=-\mathrm{e} V \Psi_{1}+K \Psi_{2}  \tag{2.5}\\
& \mathrm{i} \hbar \frac{\partial \Psi_{2}}{\partial t}=+\mathrm{e} V \Psi_{2}+K \Psi_{1} \tag{2.6}
\end{align*}
$$

We now make the substitutions

$$
\begin{align*}
& \Psi_{1}=\sqrt{\rho_{1}} \exp ^{\mathrm{i} \varphi_{1}}  \tag{2.7}\\
& \Psi_{2}=\sqrt{\rho_{2}} \exp ^{\mathrm{i} \varphi_{2}} \tag{2.8}
\end{align*}
$$

where $\varphi_{1}, \varphi_{2}$ are the phases and $\rho_{1}, \rho_{2}$ are the electron densities at the different sides of the junction. This is related to the interpretation of the order parameter as the condensate wavefunction. Then, equating the real and imaginary parts in both equations, we obtain the equations of motion for $\rho$ and $\varphi$.

$$
\begin{align*}
\dot{\rho}_{1} & =+\frac{2}{\hbar} K \sqrt{\rho_{1} \rho_{2}} \sin \delta  \tag{2.9}\\
\dot{\rho}_{2} & =-\frac{2}{\hbar} K \sqrt{\rho_{1} \rho_{2}} \sin \delta  \tag{2.10}\\
\dot{\varphi_{1}} & =-\frac{K}{\hbar} K \sqrt{\frac{\rho_{2}}{\rho_{1}}} \cos \delta+\frac{\mathrm{e} V}{\hbar}  \tag{2.11}\\
\dot{\varphi_{2}} & =-\frac{K}{\hbar} K \sqrt{\frac{\rho_{1}}{\rho_{2}}} \cos \delta-\frac{\mathrm{e} V}{\hbar} \tag{2.12}
\end{align*}
$$

where $\delta=\varphi_{2}-\varphi_{1}$.
These equations encode both AC and DC Josephson effects and can be written in a more compact form. From the first two equations, $\dot{\rho}_{1}=-\dot{\rho}_{2}$, which is the statement of charge conservation. Defining $2 K \rho / \hbar=I_{c}$ as the critical current of the junction, the current from 1 to 2 is given by,

$$
\begin{equation*}
J=I_{c} \sin \delta(t) \tag{2.13}
\end{equation*}
$$

This is the time-dependent Josephson relation. Noting that when $V=0$ in Eq. (2.9), both phases $\varphi_{1,2}$ are time-independent, we see that at zero voltage there is a steady (DC)
supercurrent following Eq. (2.13) with a constant value of $\delta$. We have thus derived the DC Josephson effect.

When $V$ in nonzero and the phases evolve in time, we can combine the phase equations,

$$
\begin{equation*}
\dot{\delta}=\dot{\varphi}_{2}-\dot{\varphi}_{1}=-\frac{2 \mathrm{e} V}{\hbar} . \tag{2.14}
\end{equation*}
$$

When $V=V_{0}$ is constant, $\delta(t)=\delta(0)-\left(2 \mathrm{e} V_{0} / \hbar\right) t$ and so using Eq. (2.13), we see that the supercurrent oscillates (AC) with a frequency $\frac{2 e V_{0}}{h}$. This phenomenon is called the AC Josephson effect and the universal ratio between the AC frequency and voltage, $2 \mathrm{e} / \mathrm{h}=483.6$ $\mathrm{GHz} / \mathrm{mV}$, is known as the Josephson constant and used to define a voltage standard in terms of the very accurately measurable frequency.

In passing, we note that the JJ can be characterized as an inductance. To see this, combine Eqs. (2.13) and (2.14) by taking a time derivative of the former:

$$
\begin{equation*}
V=-\frac{\hbar}{2 \mathrm{e}} \dot{\delta}=-\frac{\hbar}{2 \mathrm{e} I_{c} \cos \delta} \frac{d J}{d t} \equiv-L_{S} \frac{d J}{d t} . \tag{2.15}
\end{equation*}
$$

Thus, the JJ can be characterized by an inductance, the Josephson inductance $L_{s}=\hbar / 2 \mathrm{e} I_{c} \cos \delta$, that depends nonlinearly on the Josephson phase, $\delta$.

In addition to the supercurrent characterized by Eqs. (2.13) and (2.14), realistic Josephson junctions show additional channels of current flow. These effects are captured by the RCSJ model below.

### 2.3 The RCSJ Model of a Josephson Junction

The RCSJ model, i.e., the Resistively and Capacitively Shunted Junction model [38], [39] (figure 2.2), captures the behavior of many realistic Josephson junctions. Within this scheme, the real Josephson junction is modeled as an ideal JJ obeying Eqs. (2.13) and (2.14), connected in parallel with a resistance and a capacitor (Figure 2.2, we neglect the noise source term for our purposes). The resistance accounts for additional conventional current carried
by other charge carriers such as thermally excited quasiparticles. The capacitance accounts for charge accumulation at the two superconductor-barrier interfaces.


Figure 2.2. Schematic diagram of the RCSJ model: Real Josephson junctions are modeled by an ideal JJ connected in parallel with a conductor (equivalently, a resistor) and a capacitor. The main driving current source is $I$, while the model is often augmented by a fluctuating current source $I_{F}$. We neglect $I_{F}$ in this dissertation. The Josephson inductance, $L_{S}=\hbar / 2 \mathrm{e} I_{c} \cos \delta$, depends nonlinearly on the JJ phase $\delta$ and characterizes the ideal JJ. The resistance is given by the inverse of the conductance $G$ [40].

Using Kirchoff's law, the current through the entire JJ can be written as follows:

$$
\begin{equation*}
I=I_{S}+I_{N}+I_{D} \tag{2.16}
\end{equation*}
$$

where $I_{S}, I_{N}$ and $I_{D}$ correspond to the currents through the ideal JJ, the resistance and the capacitance respectively.

When written in terms of different voltage and phase parameters, this equation becomes,

$$
\begin{equation*}
I=I_{c} \sin \delta+\frac{V}{R_{N}}+C \frac{d V}{d t} \tag{2.17}
\end{equation*}
$$

Using the voltage-phase relationship, Eq. (2.14), in Eq. (2.17),

$$
\begin{equation*}
I=I_{c} \sin \delta+\frac{1}{R_{N}} \frac{\Phi_{0}}{2 \pi} \frac{d \delta}{d t}+C \frac{\Phi_{0}}{2 \pi} \frac{d^{2} \delta}{d t} \tag{2.18}
\end{equation*}
$$

where $\Phi_{0}$ is the superconducting flux quantum. This is the fundamental expression for JJ current in the RCSJ model, replacing Eq. (2.13).

Rewriting this,

$$
\begin{equation*}
\left(\frac{\hbar}{2 \mathrm{e}}\right) C \frac{d^{2} \delta}{d t}+\left(\frac{\hbar}{2 \mathrm{e}}\right) \frac{1}{R_{N}} \frac{d \delta}{d t}+I_{c}\left(\sin \delta-\frac{I}{I_{c}}\right)=0 \tag{2.19}
\end{equation*}
$$

Multiplying by $\frac{\hbar}{2 \mathrm{e}}$, using the fact that the Josephson coupling energy $E_{J 0}=\frac{\hbar I_{c}}{2 \mathrm{e}}$ and defining normalized current $\mathrm{i}=I / I_{c}$,

$$
\begin{equation*}
\left(\frac{\hbar}{2 \mathrm{e}}\right)^{2} C \frac{d^{2} \delta}{d t}+\left(\frac{\hbar}{2 \mathrm{e}}\right)^{2} \frac{1}{R_{N}} \frac{d \delta}{d t}+E_{J 0} \frac{d}{d \delta}(1-\cos \delta-\mathrm{i} \delta)=0 . \tag{2.20}
\end{equation*}
$$

This equation of motion can be interpreted in terms of a particle of mass $M$ and damping $\eta$ moving in a potential $U$,

$$
\begin{equation*}
M \frac{d^{2} x}{d t^{2}}+\eta \frac{d x}{d t}+\nabla U=0 \tag{2.21}
\end{equation*}
$$

Comparing equations 2.20 and 2.21, we can make the associations

$$
\begin{align*}
M & =\left(\frac{\hbar}{2 \mathrm{e}}\right)^{2} C  \tag{2.22}\\
\eta & =\left(\frac{\hbar}{2 \mathrm{e}}\right)^{2} \frac{1}{R_{N}}  \tag{2.23}\\
U & =E_{J 0}(1-\cos \delta-\mathrm{i} \delta) \tag{2.24}
\end{align*}
$$

The energy potential in the above equation is often interpreted as the energy of the ideal Josephson junction:

$$
\begin{equation*}
E_{\mathrm{pot}}=E_{J 0}\left(1-\cos \delta-\frac{I}{I_{c}} \delta\right) \tag{2.25}
\end{equation*}
$$

and is often referred to as the tilted washboard potential because of its shape, visualized in Figure 2.3.


Figure 2.3. Analogy between gauge-invariant phase difference and damped motion of a particle in a tilted washboard potential: Increasing the current leads to the tilt of the potential. The mass of the particle $M=\left(\frac{\hbar}{2 \mathrm{e}}\right)^{2} C$ and the damping $\eta=\left(\frac{\hbar}{2 \mathrm{e}}\right)^{2} \frac{1}{R_{N}}$ control the motion of the particle [40].

### 2.3.1 Underdamped and Overdamped Josephson Junctions

Equation (2.20) can be made dimensionless in terms of reduced units $\tau \equiv t / \tau_{c}=$ $t /\left(2 \mathrm{e} I_{c} R_{N} / \hbar\right), \mathrm{i}(\tau)=I(t) / I_{c}$ and the Stewart-McCumber parameter $\beta_{C}$,

$$
\begin{equation*}
\beta_{C} \frac{d^{2} \delta}{d \tau^{2}}+\frac{d \delta}{d \tau}+\sin \delta-\mathrm{i}(\tau)=0 \tag{2.26}
\end{equation*}
$$

Herein,

$$
\begin{equation*}
\beta_{C}=\frac{2 \mathrm{e}}{\hbar} I_{c} R_{N}^{2} C \tag{2.27}
\end{equation*}
$$

which is may be compared with square of the quality factor of an equivalent LCR circuit near $\delta=0$. Clearly, $\beta_{C} \ll 1$ implies that the damping term controls the JJ dynamics - this is thus referred to as the 'overdamped' case. The opposite scenario, $\beta_{C} \gg 1$, leads to the inertial term being the deciding factor and the junction is then called 'underdamped'.


Figure 2.4. Overdamped and Underdamped Josephson Junctions:The I-V characteristics of an (a) overdamped and (b) underdamped Josephson Junction. The underdamped case has a hysteretic behavior whereas the overdamped case does not show any hysteresis [40].

Junctions with smaller capacitance or resistance are thus 'overdamped'. In this scenario, using the analogy shown in Figure 2.3, when $I<I_{c}$, the 'particle' quickly comes to rest at an equilibrium point given by the ideal JJ relation. Only a DC supercurrent flows and the voltage drop is zero. When $I>I_{c}$, the tilt is strong enough that the highly damped particle can pass from well to well in a quasi-steady fashion. Thus, there is an AC supercurrent but the DC component is dissipative and passes through the resistor. This gives rise to the current-voltage curve seen in the left panel of Figure 2.4.
'Underdamped' JJs, with $\beta_{C} \gg 1$, show a novel behavior. When $I>I_{c}$, the system behaves similarly to the overdamped case above. When $I<I_{c}$, the behavior depends on initial conditions. If the initial voltage is low, the system sits at an equilibrium point obeying the usual DC Josephson relation, Eq. (2.13). However, if the initial voltage and thus 'velocity' is high enough, the system has enough kinetic energy to tip continuously from well to well, with the kinetic energy overcoming the potential energy barrier after every well so that the particle coninuously moves down the incline. In this scenario, the junction has an AC supercurrent but the DC component passes through the resistor. The underdamped

JJ can thus have two qualitatively different states - 'superconducting' and 'resistive'! This hysteretic behavior leading to two possible states has been utilized as logic states in some older superconducting computing circuits. The behavior is sketched in the right panel of Figure 2.4.

### 2.4 Fluxons in Josephson Junction Transmission Lines

We are now ready to discuss the novel dynamics of JJ transmission lines, example shown in Figure 2.5. Equations governing Josephson phase evolution along the line can be be obtained by using the Kirchoff's current conservation law, the RCSJ current law Eq. (2.18) and the flux-magnetic field relation Eq. (2.1). This procedure [41] leads to what is known as the discrete sine-Gordon equation:


Figure 2.5. JJ transmission lines: Schematic of a JJ transmission line. Fluxons are topologically stable excitations where the JJ phase changes by an integer multiple of $2 \pi$ going from the far left to the far right. (Image credit: Dr. Dewan Woods.)

$$
\begin{equation*}
\ddot{\delta}_{\mathrm{j}}+\Gamma \dot{\delta}_{\mathrm{j}}+\sin \delta_{\mathrm{j}}=\lambda^{2}\left(\delta_{\mathrm{j}+1}-2 \delta_{\mathrm{j}}+\delta_{\mathrm{j}-1}\right)+\mathrm{i}_{\mathrm{j}}^{\mathrm{ext}} \tag{2.28}
\end{equation*}
$$

where $\Gamma=1 / \beta_{c}^{2}$ is the damping parameter, time has been rescaled by the inverse of the 'Josephson plasma frequency' $\tau_{p}=\omega_{p}^{-1}=\sqrt{\Phi_{0} C /\left(2 \pi I_{c}\right)}, \lambda^{2}=\Phi_{0} /\left(2 \pi I_{c} L\right)$ is the discreteness parameter and $\mathrm{i}_{\mathrm{j}}{ }^{\mathrm{ext}}$ is proportional to the external current injected into the $\mathrm{j}^{\text {th }}$ junction. The role of $\lambda$ is that when it is large, the Josephson phase varies slowly in space and so
this discrete-space continuous-time equation can be converted into a more tractable fully continuous equation:

$$
\begin{equation*}
\partial_{t t} \delta(x, t)-\partial_{x x} \delta(x, t)+\sin \delta(x, t)=-\Gamma \partial_{t} \delta(x, t)+\gamma(x, t) \tag{2.29}
\end{equation*}
$$

Herein, we have introduced the quasi-continuous spatial coordinate $x=\mathrm{j} / \lambda$ and defined the driving current field, $\gamma$.

When there is no damping or driving, the phase dynamics at large $\lambda$ is simply given by the exactly solvable sine-Gordon equation:

$$
\begin{equation*}
\partial_{t t} \delta(x, t)-\partial_{x x} \delta(x, t)+\sin \delta(x, t)=0 \tag{2.30}
\end{equation*}
$$

This is a significant simplification because the sine-Gordon equation is known to possess soliton solutions. These are isolated localized ballistic excitations whose form and behaviour is essentially controlled by the interplay between the nonlinear $\sin \delta$ term and the remaining linear terms. This equation can be shown to lead to a conserved energy and momentum. Remarkably, multiple solitons can pass through each other, retaining their energies!

Two families of solitons are particularly interesting to us:

$$
\begin{equation*}
\delta_{ \pm}(x, t)=4 \arctan \left\{\exp \left( \pm \frac{x-x_{0}-v t}{\sqrt{\left.1-v^{2}\right)^{2}}}\right)\right\}, \quad|v|<1 \tag{2.31}
\end{equation*}
$$

These are popularly known as kinks/antikinks respectively, since these solutions require the Josephson phase to change by $\pm 2 \pi$ as one travels from far left to far right. This special topological requirement means that if one initially creates such a traveling soliton in a system obeying the 'pure' sine-Gordon equation, Eq. (2.30), and then slowly 'turns on' the dissipative and driving terms in Eq. (2.29), the localized soliton solution cannot disappear since the Josephson phase still has to change by $\pm 2 \pi$ from far left to far right! Thus, solitons whose phases change by a nonzero multiple of $2 \pi$ across the region of interest continue to exist in the more realistic JJ transmission line, Figure 2.5.

We are mostly interested in the specific solutions, Eq. (2.31), since these are correspond to the elementary topologically stable solitons in the JJ transmission lines. We have seen that the superconducting phase accumulation around a closed loop is equal to the magnetic flux through the loop, equal to one superconducting flux quantum per $2 \pi$ phase change. Drawing a large loop running through the upper and lower superconductor lines in Figure 2.5, and closing them far away, we see that the kink/antikink carries one superconducting flux quantum with it (the two solutions carry magnetic flux in opposite directions, out of or into the paper, respectively). For this reason, these solitons are known as fluxons/antifluxons [31], [32]. They are the ballistic 'bit' carriers in the circuits we numerically study in the next chapter.

## 3. FLUXON-BASED ASYNCHRONOUS BALLISTIC REVERSIBLE COMPUTATION

In this chapter we introduce a fluxon-based superconducting computation paradigm, the RSFQ computation paradigm, which is a prominent forerunner of modern superconducting computation circuits. We then discuss fluxon-based asynchronous reversible ballistic computation and describe the design of a working rotary circuit: the rotary is a critical device for implementing asynchronous ballistic reversible computing in superconducting circuits, that routes fluxons in a circular fashion with direction selected by fluxon polarity.

### 3.1 RSFQ Logic: The Original Fluxon-based Logic

Likharev et. al. [42] developed the first seriously pursued superconducting circuits that implemented fluxon-based computation using a system of logic known as RSFQ (Rapid Single Flux Quantum) logic. They used overdamped junctions, biased by external currents to keep them in a critical state such that highly damped (i.e., not ballistic) fluxons were moved through the circuits with high control. Of course, this was a dissipative paradigm where power was dissipated maintaining driving currents which provided enough energy to make up for the large damping, i.e., the average on the right hand of Eq. (2.29) was zero over the fluxon width.


Figure 3.1. RSFQ transmission line: The RSFQ transmission line made from overdamped Josephson junctions biased with an external current. (a) Equivalent circuit and (b) Progression of the SFQ pulse through different Josephson junctions ( $I_{b}=0.75 I_{c}, L I_{c}=0.5 \Phi_{0}$ ) [42].

An example RSFQ transmission line circuit is shown in Figure 3.1. A SFQ pulse (a fluxon in this driven-dissipative system) arriving from source A travels to the right junction containing JJ J1. J1 is overdamped and the bias current ( $I_{b}<I_{c}$ ) is very close to the critical current. Applying the particle in a washboard potential analogy (Figure 2.3), the bias current holds the 'particle' at the metastable point of tipping over to either of the two wells on either side. The SFQ pulse or $I_{\mathrm{i} n}$ adding to the bias current makes 'particle' tip over to the next well, direction provided by the sign of change of the current, and then come to rest right at the next tipping point (since it is overdamped). In the JJ, this translates to a $2 \pi$ change of the JJ phase, which matches the change required by the passing of an SFQ pulse.

Meanwhile, the superconducting strips have really low inductance ( $L 1, L 2$ and $L 3$ are much less than $L \sim \Phi_{0} / I_{c}$ ) and so cannot hold a static fluxon in a single loop. For example, in Figure 3.1, the loop comprising of $J 1, L 1$ and $J 2$ cannot hold an entire fluxon. The SFQ pulse is forced by this incompatibility to move forward through the successive JJs, leading to fluxon motion in the direction set by initial conditions.

There are significant advantages of using RSFQ logic [43]. The most attractive feature is that this family of circuits can achieve high clock frequencies above 100 GHz ! Also the theoretical power consumption, $P_{\text {diss }} \tau \simeq 10^{-18} \mathrm{~J}$ per bit, is low compared to the semiconductor counterparts. However, keeping power consumption this low requires exquisite control of driving currents, for example setting driving currents to zero in regions where the fluxon is not present. Overall, this and other criteria necessitated significant overhead in circuit design to distribute these driving "power-clock" signals throughout the system. Viable RSFQ circuits have not yet been announced in the public domain.

### 3.2 Asynchronous Ballistic form of Reversible Logic (ABRL)

In contrast to the RSFQ paradigm, we now concentrate on the ballistic approach to computing. As discussed in Chapter 1, in this approach information is conveyed by spatially and temporally localized pulses that are free to move ballistically under their own inertia along the system. The energy can be recycled multiple times and used for carrying out a
large number of operations. One of the first models for ballistic computing was suggested by Fredkin and Toffoli [7], also known as the Billiard Ball Model, where perfect dissipationless elastic billiard balls were used to represent information and gates worked by elastically colliding and routing the balls! This model cannot be implemented literally since it requires extreme precise control of trajectories, including their arrival times at gates, and errors worsened exponentially through the process (due to inherently chaotic dynamics).

The spatial precision issue can be rectified by having the pulses travel in a constrained path, for example, as fluxons in high-quality JJ transmission lines. The circumvention of the temporal precision can be done if the the devices do not require synchronicity (Figure 3.2) by design so that the precise timing of the pulses is not important in determining the action of a gate. This lead to the creation of Asynchronous Ballistic Revesible Logic (ABRL) [6] that incorporates asynchrony, reversibility and ballistic propagation, and is expected to be realizable using superconducting circuits utilizing underdamped JJ transmission line fluxons/antifluxons as ballistic data bits. We will now discuss some of our research into implementing superconducting circuits that can function as ABRL devices.


Figure 3.2. Synchronous vs Asynchronous: Synchronous ballistic circuits need precise alignment of incoming data pulses (left). Asynchronous ballistic circuits can function with any delay between consecutive pulses [6].

### 3.3 ABRL devices

The initial set of devices (up to three terminals) that we are targeting to design are listed in [6]. We will focusing on two nontrivial devices in the list that have been designed: the

Reversible Memory cell designed by our collaborators [44] and the JJ Rotary that we have designed a first working circuit for.

### 3.3.1 Reversible Memory Cell

The Reversible Memory (RM) cell is one of the simplest devices in the ABRL class that conserves the fluxon energy while also reading/writing data using flux polarization. It is a one-port, two-state device which allows the polarity swap of the stored fluxon after a forward moving fluxon reaches the port (figure 3.3). As a memory cell, it functions as a device capable of reading and writing the data simultaneously.


Figure 3.3. Reversible Memory cell: Sketch of the circuit implementing a RM cell. The incoming fluxon should swap its polarity with the stored fluxon after the interaction. As a memory cell, it is useful in reading out the stored bit and writing a new bit simultaneously [44].

Our collaborators at Sandia National Lab have theoretically identified a circuit capable of performing this operation [44]. The transmission line was created using parallel JJ arrays, although without any bias currents (i.e., truly ballistic unlike previous RFSQ designs). The transmission line is terminated with a JJ (critical current $I_{c}$ ) and an inductance (L) in parallel, their values chosen such that they can store exactly one magnetic flux quantum and no more (i.e., $\Phi_{0}<L I_{c}<2 \Phi_{0}$ ). An initial circulating current, $I$, can be used to set up a fluxon/antifluxon in the memory cell (thus $|L I|=\Phi_{0}$ ). The results of a numerical WRSpice
calculation is shown in Figure 3.4. The fluxon swaps its polarity with the stored fluxon if they are of opposite polarity. If the polarities of the fluxons match, the incoming fluxon reflects back with the same polarity without affecting the stored fluxon.


Figure 3.4. WRSpice simulation for RM cell: The top figure shows the RM Cell circuit design, reproduced from the Xic design interface. The memory cell part of the circuit, to the far right, consists of a JJ connected parallel to an inductor with a circulating current in the loop. The value of the inductor and the critical current are chosen such that $\Phi_{0}<L I_{c}<2 \Phi_{0}$ and $|L I|=\Phi_{0}$. Plots (b)-(g) show the dynamics of currents in the corresponding labeled regions of the upper transmission line in the top image. The current is given by the spatial derivative of the JJ phase, which is of opposite signs for a fluxon or an antifluxon passing by the region (see Eq. (2.31)). The fluxon/antifluxon motion direction is deduced from the relative times of arrival at different regions of the transmission line. The left column shows what happens when the stored fluxon has a sign opposite to the incoming fluxon, when the fluxon-antifluxon pair trade places upon collision. The right column shows what happens when the incoming fluxon has the same polarity as the stored fluxon - it simply gets reflected without change in polarity. Thus, this circuit shows the desired behavior of a reversible memory cell. (Image credit: Dr. Dewan Woods.)

### 3.3.2 The JJ Rotary

Three-terminal devices relevant to ABRL computation can be realized by a combination of a stateless device, the Rotary, and a stateful device, the Flipping Rotary [6] (Figure 3.5). The Rotary routes the pulses coming to a terminal to the next in a cyclic fashion, with the direction (e.g., clockwise) being fixed and opposite for fluxons and antifluxons. The clockwise and the anticlockwise Rotaries are just time-reversed variants of each other.

The Flipping Rotary is a 3-terminal device that respects both time-reversal and D3 symmetries (treating the 3 terminals symmetrically). It functions like the Rotary, except that it changes its state every time a pulse passes through the junction, for example, transitions from a clockwise rotary to an anticlockwise rotary as shown in Figure 3.5.


Figure 3.5. Rotary and Flipping Rotary: Diagrams for 3-terminal ABRC devices. (a) The rotary routes the fluxons in a circular fashion, clockwise or anticlockwise according to fluxon polarity. (b) The flipping rotary functions the same as a rotary, however it also switches its state from clockwise to anticlockwise and vice-versa each time a fluxon is routed through the junction [6].

We now present a working circuit realized by our group that implements the action of a Rotary. The full schematic of the circuit, drawn using Xic graphic editor [45], is shown in Figure 3.6. A selection of designs attempted are presented in Appendix-B, while additional custom computer codes that were used to perform parameter sweeps, etc. are reproduced in Appendix-C.

### 3.3.3 Parts of the Rotary Circuit

We will now describe, one by one, the different parts of the Rotary circuit.


Figure 3.6. Rotary circuit: Circuit design in Xic. There are 3 branches, input, bottom and right branch. The input fluxon is generated from the DC to SFQ converter at the beginning of the input branch. The inductors L1, L2, L3, L4, L7, L8 are dummy inductors that are placed between the LJJ20 to measure the current through the transmission line.

PieceWise Linear pulse (PWL)

The circuit is initialized by a current source at the extreme left of Figure 3.7. This input goes to the DCSFQ part of the circuit which generates the SFQ pulse. The setting in the WRSpice [46] format is given by
pwl (0 0 25p $-0.07 \mathrm{~m} 75 \mathrm{p} 0.35 \mathrm{~m} 900 \mathrm{p} 0 \mathrm{r})$,
where the numbers signify the value of the DC pulse at different times. In this case, the pulse start with 0 at 0 seconds, then decreases to -0.07 mV at 25 picoseconds, increasing linearly to reach 0.35 mV at 75 picoseconds and finally coming back to 0 at 900 picoseconds. The (optional) last parameter ' $r$ ' stands for repeat that generates successive pulses every 900 picoseconds.


Figure 3.7. PieceWise Linear Pulse generator: The pulse generator is used to feed a DC current to the DC to SFQ converter. The parameters inside the pwl function describe the piecewise pulse. The odd entries denote the time and the even entries denote the DC amplitude.

## DC to Single Flux Quantum (DCSFQ) converter

The DCSFQ or DC to Single Flux Quantum converter is used to convert the incoming DC pulse generated by the PWL generator to a single flux quantum propagating forward. The DCSFQ circuit used in our calculations is shown in figure 3.8 and taken from the SUNY RSFQ Cell Library [47].

## The JJ Transmission Lines

As discussed earlier, a JJ transmission line consists of a series of JJs connected in parallel by superconducting wires which possess some inductance per segment. The LJJ unit used in our circuit can be seen in Figure 3.9. The JJ in the rotary are of the jjk type, that are also used in these LLJ segments. The $\mathbf{j j k}$ JJ is defined as

$$
. \text { model } j \mathrm{jk} \text { jj (rtype }=0, \operatorname{cct}=1, \operatorname{vg}=2.8 \mathrm{~m}, \quad \text { icrit }=1.5 \mathrm{u}, \quad \text { cap }=60 \mathrm{f}),
$$

where the critical current is $1.5 \mu \mathrm{~A}$, gap voltage is 2.8 mV and capacitance is 60 fF . The rtype decides the shunt resistance which has been set to infinity for our case. The inductance values are chosen in such a way that the product $L I_{c}<\Phi_{0}$ and no flux quantum gets trapped in these units. These LJJ units are repeated 20 times to get the DLJJ20 (Figure 3.10) which is then used in our rotary circuit. We do not have the bias current that was present in


Figure 3.8. DC to SFQ converter: Circuit schematic in Xic. The DCSFQ converter uses the DC pulse from the power source and converts it into a SFQ pulse moving forward. This design has been taken from the SUNY RSFQ Cell Library.
earlier RSFQ circuits and the SFQ pulse can travel ballistically through the transmission line without any overhead or requiring any synchronicity.

## Rotary Junction

Finally, the Rotary junction that is our new contribution, is comprised of three JJs connected to each of the branches as shown in figure 3.11. Note that these JJs are similar to the ones used in our transmission line with infinite shunt resistance. Later, we will also discuss JJs with finite resistances.

### 3.3.4 Fluxon Path in the Working Rotary Circuit

Figure 3.12 shows the working Rotary circuit in action. The successive plots show the currents measured at the dummy inductors L1-6 in Figure 3.6: Plots 1, 4, 6 correspond to the


Figure 3.9. JJ transmission line unit: A unit cell of the JJ transmission line, also called a DLJJ segment. The parameters of the inductances and the JJ are chosen in such a way that the product $L I_{c}<\Phi_{0}$. This ensures that these units do not trap any fluxons.


Figure 3.10. DLJJ 20: The DLJJ segment is repeated 20 times to get DLJJ 20. The transmission lines in our circuits are built by joining successive DLJJ 20 units by superconducting wires.
dummy inductors furthest from the rotary in the left, right and bottom branches respectively; plots $2,3,5$ correspond to the dummy inductors near the rotary in the left, right and bottom branches respectively.

Figure 3.12 can now be used to read out what happens after a fluxon is launched into the left branch (the pulse launcher is set to launch only a single fluxon without repeating). The fluxon reaches the rotary and is routed to the bottom branch. It reflects from the grounded ends of the branch and passes through the rotary junction again. Only this time, it goes to


Figure 3.11. Rotary junction closeup: The Rotary junction consists of 3 Josephson Junctions connected across each of the attached JJ transmission lines. The JJ has the same parameters as the ones used in the transmission line (some variation is allowed).
the right branch. Thus the fluxon follows an anticlockwise trajectory and reaches back to the input branch. We can see that the fluxon takes 1.4 ns to come back to the source.

If we run the simulation for a longer time (Figure 3.13), we find that the fluxon does not move to the bottom branch after moving through the input branch again. Instead, it chooses to go to the right branch in a clockwise fashion. The fluxon then goes back to the input branch without affecting the bottom branch. We can see that the fluxon becomes wider and hence slower after passing through the rotary junction for the first time. We believe that the energy of the fluxon plays a crucial role in deciding its movement in the output branches. Other factors being tuned (not shown here) are by-hand asymmetric delays introduced in the JJ transmission lines.

### 3.4 Time Reversal Property of the Rotary

The Rotary junction routes the incoming fluxon to one of the branches depending on the polarity of the fluxon. This can be seen in figure 3.14, where an incoming antifluxon shows the complete opposite behavior to the fluxon. Instead of moving to the bottom branch, the


Figure 3.12. Fluxon path: The fluxon gets routed to the bottom branch (L7, L8), gets reflected from the grounded ends and moves to the right branch (L3, L4). This anticlockwise circulation as it reflects back from the right branch to the input branch (L1, L2).
chirality of the fluxon routes it to the right branch. The subsequent path is chosen in a clockwise fashion as opposed to anticlockwise in the previous case.

We can test the time reversal symmetry of the rotary junction by flipping the wires close to the rotary junction. Our hypothesis is that the chirality of the incoming fluxon decides whether the fluxon is routed clockwise or anticlockwise. By flipping the superconducting lines, we are converting a fluxon to an antifluxon and vice versa. Figure 3.15 shows that as the (now) antifluxon encounters the rotary junction, it gets routed to the right branch, similar to the antifluxon case and opposite to the fluxon case discussed above.

We can solidify our previous investigations into the time reversal symmetry of the Rotary by combining the previous two cases, i.e. antifluxon with flipped wires. An antifluxon should be routed to the right branch if the wires were not flipped as shown in Figure 3.14. We find


Figure 3.13. Fluxon path for longer times: The fluxon moves a complete circle from input branch, bottom branch, right branch and back to the input branch. As the fluxon encounters the rotary junction for the fourth time, it gets routed to the right branch instead of the bottom branch.


Figure 3.14. Antifluxon input: If the polarity of the input fluxon is reversed, the rotary functions as a clockwise rotary. The antifluxon is now routed to the right branch first and then to the bottom branch.
that due to the flipped wire transformation, the antifluxon is converted to a fluxon at the Rotary and gets routed anticlockwise to the bottom branch, as can be seen in Figure 3.16.


Figure 3.15. Fluxon input with flipped lines near the Rotary junction: If the wires connecting the input branch to the rotary junction are reversed, the fluxon gets converted to an antifluxon before entering the junction. Since the junction behavior only depends on the chirality of the incoming fluxon, it gets routed similar to an antifluxon even though the input fluxon was of a different polarity.


Figure 3.16. Antifluxon input with flipped lines near the Rotary junction: If we combined the previous two transformations: reversed polarity of the incoming fluxon and flipped wires near the rotary junction, we find an anticlockwise rotary.

### 3.5 Rotary Behavior vs. Input Fluxon Energy

We have investigated how the energy of the input fluxon decides whether the fluxon passes through the rotary junction or not. We vary the energy of the fluxon by adding a finite resistance between the power source and the ground at the fluxon source (Figure 3.17). This ensures that the fluxon is slows down before encountering the LJJ20s, the energy lost depending on the value of the resistance. The loss becomes especially significant once the resistance matches the impedance of the JJ transmission line ( $\sim 16 \Omega$, see Appendix-A).


Figure 3.17. Varying Fluxon Energy: Adding a resistance across the LJJ transmission line reduces the energy of the incoming fluxon.

Before we characterize Rotary behavior vs. energy of the input fluxon, we need a scheme to measure its energy after the damping by the resistor in Figure 3.17. We do this by measuring the voltage and loop current across one cross-section of the JJ transmission line, and integrating it over the time of passage of the fluxon across it. Doing this, we have measured the non-linear variation of the fluxon energy with the resistance. The results of our calibration calculations are shown in Figure 3.18.

Our results can be qualitatively summarized as follows, in terms of a higher and lower energy threshold whose values sensitively depend on circuit parameters. Above the higher energy threshold, the fluxon/antifluxon behaves in the the manner summarized in the section


Figure 3.18. Energy as a function of resistance: The energy of the fluxon varies non-linearly with the resistance in Figure 3.17.
above. As the energy dips below this higher threshold but remains above the lower threshold, opposite chiral behavior is observed at the Rotary. Finally, below the lower energy threshold, the fluxon/antifluxon is simply 'eaten' by the Rotary.

## 4. TOWARDS EXACT MODELS OF TOPOLOGICAL QUANTUM COMPUTATION

In this chapter, we will consider developing an exactly solvable model to help uncover a manybody description of the braiding properties of Majorana bound states in superconducting wires. Specifically, we first introduce the Kitaev superconducting wire model underlying most current experimental projects that are being carried out to realize non-abelian anyonic Majorana bound states in superconductors. We will then introduce the Lieb-Liniger model for the one-dimensional Bose gas and discuss how it can be adapted for interacting fermions in one dimension. Finally, we will show how hard wall boundary conditions can be introduced and discuss next steps towards realizing a solvable model for studying Majorana bound state.

### 4.1 Majorana Fermions in Particle Physics and Condensed Matter

In 1928, Paul Dirac discovered the relativistic field theoretical equations for electrons[48]. A counter-intuitive consequence of this description was that the energy spectrum was not bounded from below. Dirac proposed to rectify this problem by positing that negative energy states are all occupied in 'empty' vacuum. The notion that vacuum consisted of an infinite number of filled states was quite disturbing and much effort was expended to resolve this (then) apparently unphysical result.

As part of this effort, Ettore Majorana proposed a new quantum field theory, which he named "symmetric theory of electrons and positrons" [17] where, loosely speaking, particles were required to be their own antiparticles:

$$
\begin{equation*}
\gamma_{\mathrm{i}}^{\dagger}=\gamma_{\mathrm{i}} \tag{4.1}
\end{equation*}
$$

His theory did not have Dirac's filled Fermi sea as the particle and antiparticles in Dirac's theory combined in a specific manner to give rise to the Majorana particles.

While Majorana fermions have never been seen in nature as fundamental particles in free space, condensed matter systems do allow the Majorana condition, Eq. (4.1), to be realized. Superconductors have Bogoliubov excitations that are complex superpositions of holes and
particles - they are a prime target for trying to realize Majorana physics. Indeed, isolated quasiparticles in exotic $p+\mathrm{i} p$ superconductors can satisfy the Majorana condition - these are known as Majorana bound states and usually arise in regions where the superconducting order parameter is made to vanish, such as in vortices [19] or, as discussed below, at the ends of superconducting wires [20]-[23]. Moreover, they are predicted to satisfy two remarkable properties: they occur at zero energy, i.e., their presence signifies the presence of a macroscopically degenerate ground state; braiding two such bound states by, say, moving vortices around each other, creates a unitary transformation superposing these two ground states [18]. These are hallmarks of a system that can be utilized for topological quantum computation [15], [16] and thus a topic of considerable focus of experimental research.

Despite the intense decades-long attention focused on the realization of Majorana bound states in superconducting one dimensional electrons, Majorana bound states with nonabelian braiding properties are yet to be demonstrated. In the foregoing, we will outline Kitaev's original (mean field) model and initial steps towards realizing a many-body Bethe ansatz solvable model that can mimic the basic requirements for realizing Majorana bound states.

### 4.2 Kitaev Quantum Wire

In 2001, Kitaev came up with a theoretical model of a superconducting wire [20] where a single flavor/spin of fermions interact with a p-wave superconducting interaction. The Hamiltonian is:

$$
\begin{equation*}
H_{1}=\sum_{\mathrm{j}}\left[-w\left(a_{\mathrm{j}}^{\dagger} a_{\mathrm{j}+1}+a_{\mathrm{j}+1}^{\dagger} a_{\mathrm{j}}\right)-\mu\left(a_{\mathrm{j}}^{\dagger} a_{\mathrm{j}}-\frac{1}{2}\right)+\Delta a_{\mathrm{j}} a_{\mathrm{j}+1}+\Delta^{*} a_{\mathrm{j}+1}^{\dagger} a_{\mathrm{j}}^{\dagger}\right], \tag{4.2}
\end{equation*}
$$

wherein $w$ is the tunneling amplitude, $\mu$ a chemical potential, $\Delta=|\Delta| \mathrm{e}^{\mathrm{i} \theta}$ is the superconducting pairing amplitude, and $a_{\mathrm{j}}^{\dagger}$ creates a spinless fermion at site j in the one-dimensional chain.

Kitaev solved this mean-field quadratic model, in the presence of an boundary, by decomposing each site's fermion operator into two Majorana operators, represented in a visually
(a)

(b)


Figure 4.1. Majorana bound states in Kitaev wires: Schematic illustration of the ground state of the Kitaev wire Hamiltonian for (a) $\mu \neq 0,|\Delta|=$ $w=0$, (b) $|\Delta|=w>0, \mu=0[49]$. In the first case, the pairing of Majoranas happens at the same site that leads to a unique ground state, separated from the excited state by a gap; In the latter case, the Majoranas pair with the other site Majoranas and hence there are two of them left unpaired. This gives rise to 2 Majorana zero modes and a doubly degenerate ground state separated from Bogoliubov excitations by a gap.
appealing fashion in Figure 4.1. Depending upon parameters, the inner Majorana operators in Figure 4.1 are bound together, leaving unpaired and localized Majorana bound states at the ends. This occurs when $|\Delta|=w>0, \mu=0$. Subsequent research activity has led to many realistic proposals and experiments that realize the Kitaev chain ([21], [22]).

The Kitaev model and subsequent realizations employ a mean field Bogoliubov de-Gennes approach. As we will discuss below, there is evidence that such an approach yields qualitatively incorrect results for one dimensional Bosonic systems. Thus, there is reason to believe that calculations that go beyond mean field theory may reveal qualitatively new details about Majorana bound states in supercondutors. Our long term goal is to approach this problem via an exact solution and calculate experimentally measurable signatures like the tunneling spectrum or the consequences of braiding operations.

### 4.3 An Invitation: Mean Field vs Exact Solution for a Bose Gas

Lieb and Liniger developed an exactly solvable model of the Bose gas [50] in one dimension, discussed below. Using this, they showed that while the mean-field calculation is qualitatively accurate up to a certain strength of interactions, for stronger interactions it
fails to reproduce an entire new branch of excitations in the interacting Bose gas as revealed by the exactly solved model (Figure 4.2).


Figure 4.2. Excitations in an Interacting One-dimensional Bose gas: The Figure shows the excitation spectrum of a short-range interacting Bose gas with interaction strength $\gamma=c / \rho=0.787$ (for definition of $c$ see Eq. (4.3)), with the solid curves derived from the exact Lieb-Liniger solution [51] and the dashed line showing the corresponding mean field prediction of Bogoliubov's theory. The mean field prescription does yield the first type of excitation with reasonable accuracy but completely fails to capture the emergence of second type of excitation [51].

From Figure 4.2 we see that Bogoliubov's mean-field prescription fails to calculate a novel branch of excitations present in the system. Is there such a remarkable departure from meanfield results in the one dimension superconducting wire which are supposed to host Majorana bound states at its ends? We discuss the Lieb-Liniger model (interacting bosons with periodic boundaries) and Gaudin's work (interacting bosons with hard walled boundaries) below and discuss how extend these boson-based models to the domain of spinless fermions with attractive interactions, which are expected to shown superconducting correlations and realize Majorana bound states at the ends of the wire.

### 4.4 The Lieb-Liniger model

The Lieb-Liniger model [50] consists of Bosons interacting with each other through a repulsive Dirac delta potential, satisfying periodic boundary conditions. The corresponding many-body stationary Schrödinger equation is given by:

$$
\begin{equation*}
\left[-\sum_{\mathrm{i}=1}^{N} \frac{\partial^{2}}{\partial x_{\mathrm{i}}^{2}}+2 c \sum_{<\mathrm{i}, \mathrm{j}\rangle} \delta\left(x_{\mathrm{i}}-x_{\mathrm{j}}\right)\right] \psi=E \psi, \tag{4.3}
\end{equation*}
$$

with repulsion strength $c>0$ and $<\mathrm{i}, \mathrm{j}>$ denoting nearest neighbors. This problem can be thought as a Kronig-Penny model where interactions come into effect only when two particles touch each other.

The intrinsic boundary conditions are exactly same as in the case of a single particle moving in the presence of Dirac delta potential. The singular nature of the potential leads to a discontinuity in the spatial derivative of the many-body wavefunction whenever two coordinates have the same value. These 'boundary' conditions are derived in AppendixD.1):

1. $\psi$ is continuous i.e. $\left.\psi\right|_{x_{\mathrm{j}}=x_{k}^{+}}=\left.\psi\right|_{x_{\mathrm{j}}=x_{k}^{-}}$.
2. $\left[\left.\left(\frac{\partial}{\partial x_{\mathrm{j}}}-\frac{\partial}{\partial x_{k}}\right)\right|_{x_{\mathrm{j}}=x_{k}^{+}}-\left.\left(\frac{\partial}{\partial x_{\mathrm{j}}}-\frac{\partial}{\partial x_{k}}\right)\right|_{x_{\mathrm{j}}=x_{k}^{-}}\right] \psi=\left.2 c \psi\right|_{x_{\mathrm{j}}=x_{k}}$.

Since the bosonic wavefunction is symmetric, we can rewrite the second boundary condition as

$$
\begin{equation*}
\left[\left.\left(\frac{\partial}{\partial x_{\mathrm{j}}}-\frac{\partial}{\partial x_{k}}\right)\right|_{x_{\mathrm{j}}=x_{k}^{+}}\right] \psi=\left.c \psi\right|_{x_{\mathrm{j}}=x_{k}} . \tag{4.4}
\end{equation*}
$$

Due to the this symmetry, we can partition the full Hilbert space $\mathbb{H}$ into subspaces. We can then solve the problem in one of the subspaces, say

$$
\mathbb{H}_{1}: 0 \leq x_{1} \leq x_{2} \leq x_{3} \leq \ldots \ldots \leq x_{N-1} \leq x_{N} \leq L
$$

and the result will be applicable to the full Hilbert space. Thus, we can define

$$
\begin{aligned}
\chi\left(x_{1}, x_{2}, x_{3}, ., x_{\mathrm{i}} \ldots, x_{N-1}, x_{N}\right)= & \sum_{Q} \Theta\left(x_{Q(2)}-x_{Q(1)}\right) \Theta\left(x_{Q(3)}-x_{Q(2)}\right) \ldots \Theta\left(x_{Q(N)}-x_{Q(N-1)}\right) \\
& \psi\left(x_{Q(1)}, x_{Q(2)}, \ldots, x_{Q(N-1)}, x_{Q(N)}\right)
\end{aligned}
$$

where $\chi\left(y_{1}, y_{2}, \ldots\right)=0$ unless $0 \leq y_{1} \leq y_{2} \ldots \leq y_{N} \leq L$.
The periodic boundary condition which is given by either of the two equations,

$$
\begin{array}{r}
\chi\left(0, x_{2}, x_{3}, \ldots, x_{N-1}, x_{N}\right)=\chi\left(L, x_{2}, x_{3}, \ldots, x_{N-1}, x_{N}\right), \\
\chi\left(x_{1}, x_{2}, x_{3}, ., x_{\mathrm{i}} \ldots, x_{N-1}, x_{N}\right)=\chi\left(x_{1}, x_{2}, x_{3}, ., x_{\mathrm{i}}+L \ldots, x_{N-1}, x_{N}\right),
\end{array}
$$

is a major point in defining the whole problem here. Note that this does not belong to the space $\mathbb{H}_{1}$. Thus, we need to tweak it a bit. After some manipulation, we get

$$
\psi\left(0, x_{2}, x_{3}, \ldots, x_{N-1}, x_{N}\right)=\psi\left(x_{2}, x_{3}, \ldots ., x_{N-1}, x_{N}, L\right)
$$

This is same as our earlier condition because of periodicity and this belongs to the subspace $\mathrm{H}_{1}$ which is our area of interest.


Figure 4.3. Partitioning the Hilbert Space: The left Figure shows a partition of the Hilbert space when particle coordinates are arranged in the specific order shown. Periodic boundary conditions transform the line into a ring, this section of Hilbert space is denoted in the main text by $\mathbb{H}_{1}$. Using Boson symmetry, the wavefunction for all other permutations of coordinates has the same value as the wavefunction for the arrangement shown.

### 4.5 Bethe Ansatz Solution to the Lieb-Liniger Model

Hans Bethe came up with a many body wavefunction ansatz for the scattering problem in one dimension. He noticed that the only nontrivial thing that can happen in 1D point scattering is the exchange of momenta between the particles as shown in Figure 4.4. His ansatz wavefunction form can be written as a superposition of plane waves summed over all the permutations of the wavevectors of the individual particles:

$$
\begin{equation*}
\psi\left(x_{1}, x_{2}, x_{3}, \ldots, x_{N-1}, x_{N}\right)=\sum_{P} A_{P} \exp \left(\mathrm{i} \sum_{\mathrm{j}=1}^{N} k_{P(\mathrm{j})} x_{\mathrm{j}}\right) \tag{4.5}
\end{equation*}
$$

in the subset $\mathbb{H}_{1} \subseteq \mathbb{H}$. Here, $A_{P}$ is the scattering amplitude and $k_{P(\mathrm{j})}$ are the wavevectors for different particles denoted by $P(\mathrm{j})$.


Figure 4.4. Scattering in one dimension: In one dimension, pairwise scattering can only interchange the momenta or leave them unchanged. This follows from imposing both momentum and energy conservation. This underlies the use of permutations of $k$ values in the Bethe ansatz wavefunction.

To calculate the coefficients $A_{P}$, we consider two different permutations,

$$
\begin{aligned}
\mathbb{P} & :(P(1), P(2), \ldots, P(\mathrm{j}-1), P(\mathrm{j}), P(\mathrm{j}+1), \ldots ., P(N)), \\
\mathbb{P}^{\prime} & :(P(1), P(2), \ldots, P(\mathrm{j}-1), P(\mathrm{j}+1), P(\mathrm{j}), \ldots \ldots, P(N)) .
\end{aligned}
$$

We can apply (4.4) on these two permutations(Appendix-D.2) to find the relation between $A_{\mathrm{P}}$ and $A_{\mathbb{P}^{\prime}}$,

$$
\begin{align*}
\frac{A_{\mathrm{P}}}{A_{\mathrm{P}^{\prime}}} & =\frac{k_{P(\mathrm{j})}-k_{P(\mathrm{j}+1)}-\mathrm{i} c}{k_{P(\mathrm{j})}-k_{P(\mathrm{j}+1)}+\mathrm{i} c}  \tag{4.6}\\
& =\exp \left[-\mathrm{i} \theta\left(k_{P(\mathrm{j})}-k_{P(\mathrm{j}+1)}\right)\right]
\end{align*}
$$

where

$$
\theta(k)=2 \arctan \frac{c}{k} .
$$

We find that as the particle scatters with the nearest neighbor, it acquires a phase change $\theta$ which is a function of the difference between the momenta. We can also show a fascinating property, distinct from our simple picture of the non-interacting Bose gas, that all the wavevectors have to be different for $c \neq 0$, otherwise the wavefunction goes to zero. This can be shown by analyzing the different types of permutation that can exist. If two $k^{\prime} s$ are same, swapping them returns a negative coefficient of the same magnitude which cancels the positive one. Other coefficients can also be broken down into the interchange of these equal $k^{\prime} s$ and hence cancel out.

### 4.6 Bethe Equations

We have not yet exhausted all boundary conditions in the problem. The periodicity of the system yields:

$$
\psi\left(x_{1}, x_{2}, x_{3}, ., x_{\mathrm{i}} \ldots, x_{N-1}, x_{N}\right)=\psi\left(x_{1}, x_{2}, x_{3}, ., x_{\mathrm{i}}+L \ldots, x_{N-1}, x_{N}\right)
$$

The following calculations have been done a bit differently than the original paper. More details can be found in [52]. As we discussed, the periodicity condition takes the wavefunction out of the subspace $\mathbb{H}_{1}$. We now consider a subspace

$$
\mathbb{H}_{2}: x_{2} \leq x_{3} \leq \ldots \ldots . \leq x_{N-1} \leq x_{N} \leq x_{1}
$$

Now, $x_{1}+L$ lies in this subspace. Here, a particle when going from $x_{\mathrm{i}}$ to $x_{\mathrm{i}}+L$, scatters with each particle acquiring phases from each of them corresponding to their momentum difference. This can be seen as follows.


Figure 4.5. Scattering in a ring: A particle on a ring scatters with every other particle gathering phase shifts to come back at the same place. The periodic boundary condition suggests that the sum of phase shifts should sum up to unity.

We write

$$
\begin{equation*}
\psi\left(x_{1}, x_{2}, x_{3}, \ldots, x_{N-1}, x_{N}\right)=\sum_{P} A_{P} \exp \left(\mathrm{i} \sum_{\mathrm{j}=1}^{N} k_{Q(\mathrm{j})} x_{\mathrm{j}}\right) \tag{4.7}
\end{equation*}
$$

where $Q(2)=P(1), Q(3)=P(2), .$. and so on. Now, equating $\psi\left(x_{1}, x_{2}, x_{3}, \ldots, x_{N-1}, x_{N}\right)$ in $\mathbb{H}_{\mathbb{1}}$ with $\psi\left(x_{1}, x_{2}, x_{3}, \ldots ., x_{N-1}, x_{N}\right)$ in $\mathbb{H}_{2}$,

$$
\sum_{P} A_{Q} \exp \left(\mathrm{i} \sum_{\mathrm{j}=1}^{N} k_{Q(\mathrm{j})} x_{\mathrm{j}}\right)=\sum_{P} A_{P} \exp \left(\mathrm{i} \sum_{\mathrm{j}=1}^{N} k_{Q(\mathrm{j})} x_{\mathrm{j}}\right) .
$$

Comparing term by term, we get

$$
\begin{equation*}
A_{P} \exp \left(\mathrm{i} k_{Q_{(1)}} L\right)=A_{Q} \tag{4.8}
\end{equation*}
$$

Also, from (4.6), we get

$$
\begin{equation*}
\frac{A_{P}}{A_{Q}}=\prod_{n=2}^{N} \exp \left[-\mathrm{i} \theta\left(k_{Q(1)}-k_{Q(n)}\right)\right] . \tag{4.9}
\end{equation*}
$$

Combining (4.8), (4.9) and generalizing,

$$
\begin{equation*}
1=\exp ^{\mathrm{i} k_{\mathrm{j}} L} \prod_{\substack{n=1 \\ n \neq \mathrm{j}}}^{N} \exp ^{-\mathrm{i} \theta\left(k_{\mathrm{j}}-k_{n}\right)} \tag{4.10}
\end{equation*}
$$

Taking the log of eq. 4.10, we find

$$
\begin{equation*}
k_{\mathrm{j}} L=2 \pi I_{\mathrm{j}}+\sum_{\substack{n=1 \\ n \neq \mathrm{j}}}^{N} \theta\left(k_{\mathrm{j}}-k_{n}\right), \tag{4.11}
\end{equation*}
$$

where $I_{\mathrm{j}}$ are integers and come out as a result of the periodicity of the exponential function. Equation (4.11) is a collection of N equations that are known as Bethe equations and $k_{\mathrm{j}}^{\prime} s$ are called Bethe roots. Care should be taken in interpreting these Bethe roots as the true momenta of the particles. Only the total momentum, the sum of these asymptotic momenta, is conserved as the total physical momentum of the system. For the case of free Boson gas $(c=0)$, we get back the quantization condition i.e. $k_{\mathrm{j}}=\frac{2 \pi I_{\mathrm{j}}}{L}$. The solution of these Bethe equations gives us the eigenstates and the corresponding eigenenergies. The quantum numbers $I_{\mathrm{j}}$ play an important role in determining the energies, as we will see later.

### 4.7 Ground State of the Lieb-Liniger Model

A trivial limit that we can apply to our problem is the infinite interaction limit or the impenetrable limit $(c \rightarrow \infty)$. The bosons now transform into free fermions. As expected, $\theta=\pi$ for this case, to give the required antisymmetry (Figure 4.7).

## NOTE :

$$
\begin{aligned}
\frac{A_{\mathbb{P}}}{A_{\mathbb{P}^{\prime}}} & =\frac{k_{P(\mathrm{j})}-k_{P(\mathrm{j}+1)}-\mathrm{i} c}{k_{P(\mathrm{j})}-k_{P(\mathrm{j}+1)}+\mathrm{i} c} \\
& =\exp \left[-\mathrm{i} \theta\left(k_{P(\mathrm{j})}-k_{P(\mathrm{j}+1)}\right)\right]
\end{aligned}
$$

For $c \rightarrow 0$,

$$
\frac{A_{\mathbb{P}}}{A_{\mathbb{P}^{\prime}}}=1 \Longrightarrow \exp ^{-\mathrm{i} \theta_{k}}=1
$$

(Free particle quantization condition)

For $c \rightarrow \infty$,

$$
\frac{A_{\mathbb{P}}}{A_{\mathbb{P}^{\prime}}}=-1 \Longrightarrow \theta=\pi
$$

(Free fermion condition)

We have to be a bit attentive when considering the cases of odd and even number of particles in the impenetrable limit.

## 1. N is odd

The sum on the right hand side in Eq. (4.11) gives an even multiple of $\pi$ and we get back the free particle quantization condition. The lowest energy state corresponds to filling the $I_{\mathrm{j}}^{\prime} s$ equidistantly around 0 .

$$
\begin{equation*}
I_{\mathrm{j}}=\frac{-(N-1)}{2}, \frac{-(N-3)}{2}, \ldots \ldots,-1,0,1, \ldots \ldots, \frac{(N-3)}{2}, \frac{(N-1)}{2} \tag{4.12}
\end{equation*}
$$

Comparing it with Lieb's original paper, we see that indeed this choice makes $n_{\mathrm{j}}=1$ (Appendix-E).

## 2. $\mathbf{N}$ is even

The sum on the right hand side in (4.11) gives $(N-1) \pi$ and the quantization condition changes to

$$
\begin{equation*}
k_{\mathrm{j}} L=2 \pi\left(I_{\mathrm{j}}+\frac{1}{2}\right) . \tag{4.13}
\end{equation*}
$$

For this case, we can choose $I_{\mathrm{j}}^{\prime} s$ to be half-odd integers to make it $k_{\mathrm{j}}^{\prime} s$ even multiples of $2 \pi$ (Free fermions in periodic boundary condition). The only price we have to pay is that we have to shift our $\theta_{k}$ by $\pi$.

$$
\begin{equation*}
I_{\mathrm{j}}=\frac{-N}{2}, \frac{-(N-2)}{2}, \ldots \ldots,-\frac{1}{2}, \frac{1}{2}, \ldots \ldots, \frac{(N-2)}{2}, \frac{N}{2} \tag{4.14}
\end{equation*}
$$

If we think about the fermion problem, this quantization condition (4.13) is strange as it corresponds to anti-periodic boundary conditions. The bosons do not care as their wavefunction is the modulus of fermionic wavefunction. The ground state of an even number of fermions with periodic boundary conditions is actually "two fold degenerate", with an extra fermion at one of the Fermi points. For bosons, we just add a $\pi$ to the Bethe equations. Even if reduce $c$ from $\infty$, as long as $k_{\mathrm{j}}^{\prime} s$ evolve smoothly, we can use this assignment of $I_{\mathrm{j}}$.

In the thermodynamic limit, the discrete summation can be approximated with

$$
\sum_{k}(\ldots \ldots) \rightarrow L \int_{-K}^{K} f(k) d k
$$

where $L f(k) d k=$ number of $k_{\mathrm{j}}$ in the interval $d k$ and $K=k_{N}=-k_{1}$. This transforms the Bethe equations Eq. (4.11) in the continuum limit,

$$
\begin{equation*}
k L=2 \pi I+L \int_{-K}^{K} \theta\left(k-k^{\prime}\right) f\left(k^{\prime}\right) d k^{\prime} . \tag{4.15}
\end{equation*}
$$

Differentiating with respect to $k$ and using the fact that $\frac{d I}{d k}=L f(k)$ (Density of states),

$$
\begin{equation*}
1=2 \pi f(k)+\int_{-K}^{K} \theta^{\prime}\left(k-k^{\prime}\right) f\left(k^{\prime}\right) d k^{\prime} \tag{4.16}
\end{equation*}
$$

where $\theta^{\prime}(k)=-\frac{2 c}{k^{2}+c^{2}}$. This is fixed by the density

$$
\begin{equation*}
\rho=\frac{N}{L}=\int_{-K}^{K} f(k) d k . \tag{4.17}
\end{equation*}
$$

Now, we have two equations and two unknowns $(f(k)$ and $K)$. Eqn. (4.16) is an inhomogeneous Friedholm equation of second type and can be solved numerically. The results below have been calculated on Mathematica.

Solving for $f(k)$, we find that $f(k)$ is symmetric around 0 . We can calculate the total momentum and total energy of the system,

$$
\begin{gather*}
P_{t o t}=L \int_{-K}^{K} k f(k) d k  \tag{4.18}\\
E_{t o t}=L \int_{-K}^{K} k^{2} f(k) d k \tag{4.19}
\end{gather*}
$$

Obviously, the total momentum of the ground state is 0 since $f(k)$ is even. If we fix the density $\rho$, we can define everything in terms of a dimensionless parameter $\gamma=\frac{c}{\rho}$,

$$
\begin{aligned}
E_{t o t} / L & \propto \frac{N}{L^{3}} \\
& =N \rho^{3} \mathrm{e}(\gamma)
\end{aligned}
$$



Figure 4.6. Ground state energy: Comparison between the exact ground state energy and the Bogoliubov approximated energy. The Bogoliubov mean field result agrees with the exact result for lower values of the interaction strength and breaks down as the interaction increases. This behaviour is expected since the fluctuations increase with the increase in the interaction strength leading to the departure from the mean field result.

Bogoliubov's mean field result comes out to be :

$$
\mathrm{e}(\gamma)=\gamma\left[1-\frac{4}{3 \pi} \sqrt{\gamma}\right]
$$

A check would be to calculate the energy in the $\gamma \rightarrow \infty$ limit,

$$
\begin{aligned}
f(k) & =\frac{1}{2 \pi}, K=k_{F} . \\
\Longrightarrow \rho & =\frac{k_{F}}{\pi} .
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{e}(\gamma \rightarrow \infty) & =\frac{1}{\rho^{3}} \int_{-K}^{K} k^{2} f(k) d k \\
& =\frac{1}{\rho^{3}} \int_{-k_{F}}^{k_{F}} \frac{1}{2 \pi} k^{2} d k \\
& =\frac{\pi^{2}}{3}
\end{aligned}
$$

This is the energy of the gas in the so-called Tonks-Girardeau limit[53].

### 4.8 The Excitation Spectrum of the Lieb-Liniger Model

For bosons, we consider two type of excitations from the ground state seeking inspiration from the impenetrable limit. In this limit, there is a Fermi surface and we can consider excitations around it but we should keep in mind that we are still dealing with bosons.

1. Type $\mathbb{I}$ (particle): A particle is excited from the edge of the Fermi surface.
2. Type III(hole): A particle is excited from within the Fermi surface to the edge of the Fermi surface.

### 4.8.1 Type I excitation

We go to the impenetrable limit, and imagine making a hole in the distribution of roots. We then move one particle above the 'Fermi surface' (largest of the roots). This will cause all the other roots to change slightly from their ground state values as they are inter-connected by Bethe's equations. Once we remove the largest root $k_{N}$ to a higher value than $k_{F}$, it should stand apart from other roots.

For example: the case of $N=7$ is shown in the Figure 4.8.
We can think of this as the ground state of an $N-1$ particle problem, with $N-1$ roots being affected by the presence of another root at $k_{I}$. We have to make adjustments such that when $k_{I}$ joins others, we get back the original $N$ particle ground state. This is because of the difference of $\pi$ when considering odd and even number of particles.


Figure 4.7. Excitations in a Bose gas: Two types of excitations in the Bose gas: (a) Type $I$ - This is akin to the conventional particle excitation in a Fermi gas, where particles in the highest occupied states are promoted to the lowest unoccupied states. Since we are dealing with bosons, the largest of the Bethe roots are excited to the next larger value. (b) Type II - There can be a hole created in the bulk of the Bethe roots and the Bethe roots arrange themselves due to the interconnectedness by Bethe equations. This also gives an excitation energy which is directly dependent on which Bethe root we try to remove.


Figure 4.8. Making a hole in the distribution of the roots: The Bethe roots arrange themselves as we increase the last Bethe root above $k_{F}$. This is a typical particle excitation.

So, for $N=7$, we shuffle the $N=6$ ground state. As seen in the Figure 4.9, we see that since $N=7$ is odd, the particles are shifted by $\pi$. This now can be generalized to any value of $c$.

Suppose the shifts are $\delta k_{\mathrm{i}}=\frac{w_{\mathrm{i}}}{L}$ (following the notation used in Lieb's original paper[51]),

$$
\begin{align*}
\delta k_{\mathrm{j}} L+\pi & =\sum_{s=1}^{N-1} \theta^{\prime}\left(k_{\mathrm{j}}-k_{s}\right)\left(\delta k_{\mathrm{j}}-\delta k_{s}\right)+\theta\left(k_{\mathrm{j}}-k_{I}\right), \\
w_{\mathrm{j}}+\pi & =\frac{2 c}{L} \sum_{s=1}^{N-1} \frac{\left(w_{s}-w_{\mathrm{j}}\right)}{c^{2}+\left(k_{s}-k_{\mathrm{j}}\right)^{2}}-\theta\left(q-k_{\mathrm{j}}\right), \tag{4.20}
\end{align*}
$$



Figure 4.9. Ground state re-assignment for $N=6$ : We emulate the excitation by reassigning the ground state of $N=6$ particles under the effect of anothe root $k_{I}$. Since we want the roots to be compatible with the Bethe equations, we shift the roots by an additional $\pi$ to the left for the case when $k_{I}$ joins the rest of the roots and makes the number of particles odd. In other language, we are changing $I_{N}$ in the Bethe equation to $I_{N}+Z$, where $Z$ is any positive non-zero integer.
where $q=k_{I}$. The last term arises because we add the excited particle to the $\mathrm{N}-1$ particle ground state. Promoting summation to an integral in the thermodynamic limit,

$$
\begin{equation*}
w(k)+\pi=2 c \int_{-K}^{K} \frac{(w(r)-w(k))}{c^{2}+(r-k)^{2}} f(r) d r-\theta(q-k), \tag{4.21}
\end{equation*}
$$

where $f(k)$ is the distribution function of $k^{\prime} s$ in the ground state.
Defining $w(k) f(k) \equiv J(k), 4.21$ becomes

$$
\begin{equation*}
w(k)+\pi=2 c \int_{-K}^{K} \frac{J(r)}{c^{2}+(r-k)^{2}} d r-2 c w(k) \int_{-K}^{K} \frac{f(r) d r}{c^{2}+(r-k)^{2}}-\theta(q-k) . \tag{4.22}
\end{equation*}
$$

But we know from 4.16,

$$
\begin{gathered}
2 c \int_{-K}^{K} \frac{f(r) d r}{c^{2}+(r-k)^{2}}=[2 \pi f(k)-1] \\
\Longrightarrow w(k)+\pi=2 c \int_{-K}^{K} \frac{J(r)}{c^{2}+(r-k)^{2}} d r-w(k)[2 \pi f(k)-1]-\theta(q-k) .
\end{gathered}
$$

Finally,

$$
\begin{equation*}
2 \pi J(k)=2 c \int_{-K}^{K} \frac{J(r)}{c^{2}+(r-k)^{2}} d r-\pi-\theta(q-k) \tag{4.23}
\end{equation*}
$$

Now, for the momentum of the excited state,

$$
\begin{array}{rlr}
p & =\left(\sum_{\mathrm{j}=1}^{N-1} k_{\mathrm{j}}^{\prime}+q\right)-\sum_{\mathrm{j}=1}^{N} k_{\mathrm{j}}, & \\
& =\sum_{\mathrm{j}=1}^{N-1} k_{\mathrm{j}}+\frac{1}{L} \sum_{\mathrm{j}=1}^{N-1} w_{\mathrm{j}}+q, & \\
& =\frac{1}{L} \sum_{\mathrm{j}=1}^{N-1} w_{\mathrm{j}}+q . & \text { [ground state momentum is } 0 \text { ] } \\
p & =\int_{-K}^{K} J(r) d r+q . & \tag{4.24}
\end{array}
$$

The excitation energy can be calculated in the same way as the momentum,

$$
\begin{align*}
\epsilon_{1} & =\sum_{\mathrm{j}=1}^{N-1}\left(k_{\mathrm{j}}^{\prime}\right)^{2}-E_{0}(N)+q^{2} \\
& =\sum_{\mathrm{j}=1}^{N-1}\left(k_{\mathrm{j}}+\frac{1}{L} w_{\mathrm{j}}\right)^{2}-E_{0}(N)+q^{2} \\
& =\sum_{\mathrm{j}=1}^{N-1}\left(k_{\mathrm{j}}^{2}+2 \frac{w_{\mathrm{j}} k_{\mathrm{j}}}{L}+O\left(\frac{1}{L^{2}}\right)\right)-E_{0}(N)+q^{2} \\
& =\sum_{\mathrm{j}=1}^{N-1}\left(k_{\mathrm{j}}^{2}-E_{0}(N)\right)+q^{2}+\sum_{\mathrm{j}=1}^{N-1} 2 \frac{w_{\mathrm{j}} k_{\mathrm{j}}}{L} . \\
\epsilon_{1} & =-\mu+q^{2}+2 \int_{-K}^{K} r J(r) d r \tag{4.25}
\end{align*}
$$

where $\mu$ is the chemical potential. To find $\epsilon_{1}(p)$, we need to eliminate the parameter $q$ from 4.24 and 4.25. The terms with $J(r)$ are a result of other $k^{\prime} s$ being pushed around. In the limit $\gamma \rightarrow \infty, J(r) \rightarrow 0$.

The chemical potential $\mu$ is defined as the energy difference between $N$ and $N-1$ particles.

$$
\begin{aligned}
\mu & =\frac{\partial E_{0}}{\partial N}=\frac{\partial}{\partial N}\left(N \rho^{2} \mathrm{e}(\gamma)\right) \\
& =3 \rho^{2} \mathrm{e}(\gamma)+\rho^{3} \frac{\partial \rho}{\partial \gamma} \frac{\partial \gamma}{\partial N} \\
& =3 \rho^{2} \mathrm{e}(\gamma)-\rho^{2} \gamma \dot{\rho}
\end{aligned}
$$

### 4.8.2 Type III excitation

Similar to the previous case in the impenetrable limit, we think of $N+1$ particle ground state to be shuffled by the presence of a hole. For example: $N=7$, we go to the $N=8$ ground state and reassign the roots.


Figure 4.10. Ground state re-assignment for $N=8$ : The hole excitation is carried out by reassigning the roots of $N=8$ particles and removing one of them. The roots are now shifted to the right by an additional $\pi$ as we are going from $N=8$ to $N=7$ by removing a particle. In other language, we are changing $I_{\mathrm{j}}$ in the Bethe equation to $I_{\mathrm{j}+1}$, where j is the position of the hole.

The signs of $\pi$ and $\theta\left(k-k_{I I}\right)$ change from 4.23 in this case to accommodate an extra particle instead of one less particle. The shifts are now,

$$
\begin{aligned}
k_{\mathrm{j}}^{\prime} & =k_{\mathrm{j}}+\frac{1}{L} w_{\mathrm{j}}, & \mathrm{j} \leq \mathrm{i} ; \\
k_{\mathrm{j}}^{\prime} & =k_{\mathrm{j}+1}+\frac{1}{L} w_{\mathrm{j}}, & \mathrm{j}>\mathrm{i} ;
\end{aligned}
$$

where $k_{\mathrm{i}}=q$ is the root that is shifted. Going the same way as done for Type $\mathbb{I}$ excitations, we get the following equations,

$$
\begin{align*}
2 \pi J(k) & =2 c \int_{-K}^{K} \frac{J(r)}{c^{2}+(r-k)^{2}} d r+\pi+\theta(q-k),  \tag{4.26}\\
p & =\int_{-K}^{K} J(r) d r-q,  \tag{4.27}\\
\epsilon_{2} & =\mu-q^{2}+2 \int_{-K}^{K} r J(r) d r . \tag{4.28}
\end{align*}
$$

Bogoliubov's excited state sprectrum is given by:

$$
\frac{\epsilon(p)}{\rho^{2}}=\left|\frac{p}{\rho}\right|\left[\left(\frac{p}{\rho}\right)^{2}+4 \gamma\right] .
$$

If you notice in Figure 4.11, the Bogoliubov result fits the first type of excitations perfectly for a specified value of $\gamma$. But this second type of excitation is completely absent from Bogoliubov's prescription.

### 4.9 Experimental Validation of the Lieb-Liniger Solution

The first experimental challenge was to build a one dimension(1D) gas of bosons. Thankfully, this can be done using real, 3D particles. It can be proved, from the Schrodinger equation for 3D particles in a long cylindrical container, that the one-dimensional Lieb-Liniger model accurately describes the low energy states. The mathematical description has been formulated for the ground states[54] as well as the excited states[55]. The recent advancements in ultracold atom experiments where the fermions or bosons are trapped in 1D have


Figure 4.11. Excited state spectrum: Comparison between the exact excited state energies and the Bogoliubov excitations. The Bogoliubov spectrum agrees with the Type I excitation energy for specific value of $\gamma=c / \rho=0.786$. The Type II excitation is absent from the mean field theory for any value of $\gamma$. This was an important revelation that came out from the exact solution.
led to an enhanced knowledge of strong correlation effects and quantum statistics[56]. The particles in the waveguides are limited to move in a single direction due to the tight confinement in the transverse directions. These particles can then be characterized as quasi one-dimensional systems. The effective interaction in these systems can be controlled by varying the confinement parameters[57]-[59].

Recent experiments have tested a lot of theoretical predictions such as momentum distribution profiles[60], [61], the ground state of the Tonks-Girardeau gas[62],quantum correla-tions[63]-[67], elementary excitations and dark solitons[68], [69], thermalization and quantum dynamics[70]-[72] and many more.

### 4.10 Hard Wall Boundary Condition: Bethe Ansatz

Since we are ultimately interested in adapting the fermionic version of an attractive LiebLiniger model to investigate Majorana bound states at the ends of the wire, we discuss below
the procedure for replacing periodic boundary conditions in the Lieb-Liniger model to fixed or hard-walled boundary condition.

This is the archetypal particle in a box problem generalized to many particles. The boundary condition now becomes[Figure 4.12],

$$
\begin{equation*}
\psi\left(0, x_{2}, \ldots, x_{N-1}, x_{N}\right)=0=\psi\left(L, x_{2}, \ldots, x_{N-1}, x_{N}\right) \tag{4.29}
\end{equation*}
$$



Figure 4.12. Hard wall: Hard wall boundaries signify that the wavefunction goes to zero there. Such boundaries are essential for realizing Majorana bound states in exactly solvable models of the Kitaev superconducting wire.

The wavefunction now goes to zero at he boundaries. We still encounter the fact that $\psi\left(L, x_{2}, \ldots, x_{N-1}, x_{N}\right)$ does not belong to the subspace $R_{1}$. This problem was first solved by Gaudin[73] in 1971.

For the hard boundary case, a big difference from the original periodic model is that particles at the edges can be reflected back by the hard walls, flipping their momenta. This occurs in tandem with the exchange of momenta between particles as shown in Figure 4.13.


Figure 4.13. Scattering with hard walls: The scattering in the bulk remains the same with hardwalled boundaries. The scattering with the hardwall at the ends reverses the sign of the momentum. Permutations on the k values now mixes the negative values of k acquired at the boundaries so that the Bethe ansatz now becomes a permutation over $\epsilon_{\mathrm{i}} k_{\mathrm{i}}$, where $\epsilon_{\mathrm{i}}$ can be $+/-$.

The ansatz now becomes[74],

$$
\begin{equation*}
\psi_{\left\{\epsilon_{i}, k_{i}\right\}}\left(x_{1}, x_{2}, \ldots, x_{N}\right)=\sum_{\epsilon_{1}, \ldots, \epsilon_{N}} \sum_{P} \epsilon_{1} \ldots \ldots \epsilon_{N} A\left(\epsilon_{1} k_{P_{1}} \ldots \ldots . \epsilon_{N} k_{P_{N}}\right) \exp ^{\mathrm{i}\left(\epsilon_{1} k_{P_{1}} x_{1}+\ldots .+\epsilon_{N} k_{P_{N}} x_{N}\right)}, \tag{4.30}
\end{equation*}
$$

where the sum extends over all the permutations $P$ and all signs $\epsilon_{\mathrm{i}}= \pm$. The $\pm$ signs refer to the particles moving right or left. The wavefunction still belongs to the domain $R_{1}$.

### 4.11 Hard Wall Boundary Condition: Bethe Equations

The Bethe equations can be easily generalized by first solving the toy problem with 2 particle(Appendix-E.2). We find that for $N$ particles,

$$
\begin{equation*}
\exp \left(\mathrm{i} 2 k_{\mathrm{j}} L\right)=\prod_{\substack{l=1 \\ l \neq \mathrm{j}}}^{N} \frac{\left(k_{\mathrm{j}}-k_{l}+\mathrm{i} c\right)\left(k_{\mathrm{j}}+k_{l}+\mathrm{i} c\right)}{\left(k_{\mathrm{j}}-k_{l}-\mathrm{i} c\right)\left(k_{\mathrm{j}}+k_{l}-\mathrm{i} c\right)} \tag{4.31}
\end{equation*}
$$

Taking the logarithm[75] of eqn. 4.31,

$$
\begin{equation*}
k_{\mathrm{j}} L=\pi I_{\mathrm{j}}-\sum_{\substack{l=1 \\ l \neq \mathrm{j}}}^{N}\left(\arctan \left(\frac{k_{\mathrm{j}}+k_{l}}{c}\right)+\arctan \left(\frac{k_{\mathrm{j}}-k_{l}}{c}\right)\right) . \tag{4.32}
\end{equation*}
$$

Defining $\Theta(k) \equiv-\arctan \left(\frac{k}{c}\right)$,

$$
\begin{equation*}
k_{\mathrm{j}} L=\pi I_{\mathrm{j}}+\sum_{\substack{l=1 \\ l \neq \mathrm{j}}}^{N}\left(\Theta\left(k_{\mathrm{j}}+k_{l}\right)+\Theta\left(k_{\mathrm{j}}-k_{l}\right)\right) . \tag{4.33}
\end{equation*}
$$

These are the Bethe equations for the case of hard walled boundary conditions.
The quantum numbers $I_{\mathrm{j}}$ have to be calculated by going to the $c \rightarrow \infty$ limit. We notice that although the definition of the phases are a bit different than the periodic case, we do not face the problem with odd or even number of particles. The equation has $I_{\mathrm{j}}$ in multiples of $\pi$, so we can just take the $I_{\mathrm{j}}$ to be same for both odd and even $N$.

Also, since we are considering hard walled boundary conditions, we get the product of sines in the impenetrable limit. Hence, we need to consider only positive values of the wavevectors. For the ground state in the impenetrable limit,

$$
I_{\mathrm{j}}=1,2, \ldots, N
$$

We will use these values to calculate the ground state energy for any non-zero value of $c$.

### 4.12 Hard Wall Boundary Condition: Ground State

The ground state energy can be calculated once we have found out the Bethe roots.
Again, we take the thermodynamic limit and convert the sum into an integral,

$$
\begin{equation*}
k L=\pi I+L \int_{0}^{K}\left(\Theta\left(k+k^{\prime}\right)+\Theta\left(k-k^{\prime}\right)\right) f\left(k^{\prime}\right) d k^{\prime} \tag{4.34}
\end{equation*}
$$

where $K$ is the largest of the roots.


Figure 4.14. Ground state energy in the presence of hard walls: Comparison between the exact ground state energies for the periodic and hard wall boundary conditions in the thermodynamic limit, and the Boguliubov result for the ground state energy. The ground state energy for the hardwall case matches with that of the periodic case. The boundary conditions converge to the same result when we work in the thermodynamic limit.

Taking the derivative with respective to $k$ and using the same arguments as in the periodic case,

$$
\begin{equation*}
1=\pi f(k)+c \int_{0}^{K} f(r) d r\left[\frac{1}{c^{2}+(k-r)^{2}}+\frac{1}{c^{2}+(k+r)^{2}}\right] \tag{4.35}
\end{equation*}
$$

This is fixed by the density

$$
\begin{equation*}
\rho=\frac{N}{L}=\int_{0}^{K} f(k) d k, \tag{4.36}
\end{equation*}
$$

and the energy is calculated as

$$
\begin{equation*}
E_{t o t} / L=\int_{0}^{K} k^{2} f(k) d k=N \rho^{2} \mathrm{e}(\gamma) . \tag{4.37}
\end{equation*}
$$

We see that in the thermodynamic limit, the ground state energy of the periodic case matches with that of the hard wall case as shown in Figure 4.14. Our aim is to look at the 1D superconductor which is the 1D Fermi gas with interactions. In this section, we focus on the spinless or spin-polarized fermions.

### 4.13 Fermions With P-wave Interactions

Since the spins are taken out of the account, the spatial part of the fermion wavefunction has to be antisymmetric. This poses a problem as we cannot use our beloved delta function potential with this system. The solution although can be derived from the scattering physics. The next scattering potential after the s-wave is the p-wave potential. We are interested in what happens to the fermions in an attractive potential (the other case is not as interesting as the attractive case). The potential for this has to be antisymmetric so as to balance the antisymmetricity of the wavefunction. We will use the pseudopotential associated with p-wave scattering [76]-[78],

$$
\begin{equation*}
V\left(x_{\mathrm{i}}, x_{\mathrm{j}}\right)=-2 c\left(\frac{\partial}{\partial x_{\mathrm{i}}}-\frac{\partial}{\partial x_{\mathrm{j}}}\right) \delta\left(x_{\mathrm{i}}-x_{\mathrm{j}}\right)\left(\frac{\partial}{\partial x_{\mathrm{i}}}-\frac{\partial}{\partial x_{\mathrm{j}}}\right) . \tag{4.38}
\end{equation*}
$$

The boundary conditions turn out to be quite different from the $\delta$ potential that we are used to. Also, we have the antisymmetry condition for fermions,

$$
\psi(x)=-\psi(-x) .
$$

The boundary condition comes out to be [78],

$$
\begin{equation*}
\psi\left(x_{\mathrm{i}}-x_{\mathrm{j}}=0^{+}\right)=-c \psi^{\prime}\left(x_{\mathrm{i}}=x_{\mathrm{j}}\right) . \tag{4.39}
\end{equation*}
$$

Notice that this is exactly the same boundary condition on the wavefunction derivative which we encountered in the case of hard-wall bosons but with the parameter $-\frac{1}{c}$. The Bethe equations can be found out exactly by substituting this in our earlier case.

### 4.14 Bound states in Lieb-Liniger model[79]

Until now, we have only considered the possibility of a repulsive interaction between the bosons (the original Lieb-Liniger model), but there is a possibilty of getting bound states when $(c<0)$. This is a direct consequence of bound state formation for an attractive delta potential well. Let us look at this for two-particle case. We can always write the Bethe ansatz by separating into center of mass coordinates and relative coordinates.

$$
\begin{align*}
\psi\left(x_{1}, x_{2}\right) & =\Theta\left(x_{1}-x_{2}\right)\left[A_{12} \mathrm{e}^{\mathrm{i}\left(k_{1} x_{1}+k_{2} x_{2}\right)}+A_{21} \mathrm{e}^{\mathrm{i}\left(k_{2} x_{1}+k_{1} x_{2}\right)}\right] \\
& +\Theta\left(x_{2}-x_{1}\right)\left[A_{12} \mathrm{e}^{\mathrm{i}\left(k_{2} x_{1}+k_{1} x_{2}\right)}+A_{21} \mathrm{e}^{\mathrm{i}\left(k_{1} x_{1}+k_{2} x_{2}\right)}\right]  \tag{4.40}\\
& =\mathrm{e}^{\mathrm{i}\left(k_{1} x_{1}+k_{2} x_{2}\right)}\left[A_{12} \Theta\left(x_{1}-x_{2}\right)+A_{21} \Theta\left(x_{2}-x_{1}\right)\right] \\
& +\mathrm{e}^{\mathrm{i}\left(k_{2} x_{1}+k_{1} x_{2}\right)}\left[A_{21} \Theta\left(x_{1}-x_{2}\right)+A_{12} \Theta\left(x_{2}-x_{1}\right)\right] \tag{4.41}
\end{align*}
$$

Introducing the center of mass and relative coordinates,

$$
\begin{array}{rlrl}
X & \equiv \frac{x_{1}+x_{2}}{2}, & x \equiv \frac{x_{1}-x_{2}}{2} \\
K \equiv k_{1}+k_{2}, & k \equiv k_{1}-k_{2} . \tag{4.43}
\end{array}
$$

We can write the wavefunction as

$$
\psi(X, x)=\mathrm{e}^{\mathrm{i} K X} \begin{cases}A_{12} \mathrm{e}^{\mathrm{i} k x}+A_{21} \mathrm{e}^{-\mathrm{i} k x}, & x>0  \tag{4.44}\\ A_{21} \mathrm{e}^{\mathrm{i} k x}+A_{12} \mathrm{e}^{-\mathrm{i} k x}, & x<0\end{cases}
$$

Using the standard condition on derivatives generated by the delta potential (integrating the Schrodinger equation around the discontinuity), we get

$$
\begin{align*}
\mathrm{i} k\left(A_{12}-A_{21}\right)-\mathrm{i} k\left(A_{21}-A_{12}\right) & =2 c\left(A_{12}+A_{21}\right) \\
\left(A_{12}-A_{21}\right) k+\mathrm{i} c\left(A_{12}+A_{21}\right) & =0 \tag{4.45}
\end{align*}
$$

Now, coming back to the bound state solutions which should have complex momenta. Since we want a solution that does not diverge at $\infty, \operatorname{Im}(K)=0$ and hence K has to be purely real. If we choose $\operatorname{Im}(k)>0$, then $A_{21}=0$ to preserve the finiteness of the wavefunction. This means that the $\operatorname{Re}(k)=0$ and $k$ has to be purely imaginary. Hence, for $c<0$,

$$
\begin{align*}
& A_{21}=0 \Longrightarrow k=-\mathrm{i} c  \tag{4.46}\\
& A_{12}=0 \Longrightarrow k=\mathrm{i} c \tag{4.47}
\end{align*}
$$

The scattering phase diverges when these above conditions happen. This divergence is to counter the exponentially growing part of the wavefunction,

$$
\begin{array}{r}
\frac{A_{12}}{A_{21}}=\frac{\mathrm{i} k+c}{\mathrm{i} k-c}, \\
\Longrightarrow A_{12}=A_{21} \frac{\mathrm{i} k+c}{\mathrm{i} k-c}, \tag{4.49}
\end{array}
$$

where the right hand side goes as $0 \times \infty$. This represents a bound state of two particles with momenta

$$
\begin{equation*}
k_{1,2}=\frac{K \pm \mathrm{i} c}{2}, \quad \operatorname{Im}(K)=0 \tag{4.50}
\end{equation*}
$$

with energy $E=\frac{K^{2}}{2}-\frac{c^{2}}{2}$ and total momentum $K$. The bound state wavefunction looks like,

$$
\begin{equation*}
\psi_{\text {bound }}\left(x_{1}, x_{2}\right)=\mathrm{e}^{\mathrm{i} K\left(x_{1}+x_{2}\right) / 2} \mathrm{e}^{\mathrm{c}\left|x_{1}-x_{2}\right| / 2} \tag{4.51}
\end{equation*}
$$

For the case of three particles and $c<0$, we have two complex momentum solutions:

$$
\begin{array}{ll}
k_{1}=\alpha-\mathrm{i} c, & k_{2}=\alpha+\mathrm{i} c, \\
k_{3}=\beta  \tag{4.53}\\
k_{1}=k_{3}-\mathrm{i} c, & k_{2}=k_{3}+\mathrm{i} c, \\
\operatorname{Im}\left(k_{3}\right)=0
\end{array}
$$

These have been found using the same considerations that we took for the two particle case i.e. finiteness of the wavefunction and complex momenta. The first one is a two-particle bound state scattering with a third independent particle. The second one is a three-particle bound state and is called a "string" of 3 particles. A string of n-particles has the same real part of the momentum for each particle and the momentum is equispaced symmetrically with respect to the real axis.

The repulsive Lieb-Liniger model does not lead to any bound state. Although the attractive Lieb-Liniger model has string solutions of length $n$ characterized by their quasi-momenta

$$
\begin{equation*}
k_{\mathrm{j}}=\frac{K}{n}-\mathrm{i} \frac{n+1-2 \mathrm{j}}{2} c, \quad \mathrm{j}=1 \ldots n, \tag{4.54}
\end{equation*}
$$

and total momentum $K$ and energy (calculated by squaring and summing the individual momenta)

$$
\begin{equation*}
E=\frac{K^{2}}{n}-\frac{n\left(n^{2}-1\right)}{12} c^{2} . \tag{4.55}
\end{equation*}
$$

As seen in the case of bound states, the string solutions have a lower energy compared to the non-string like solutions in the repulsive case. The ground state for N particles has a zero momentum string that has all the particles and its energy diverging $O\left(N^{3}\right)$. The ground state energy needs to scale linearly with the system size for the thermodynamical limit to be valid. This means that our attractive Lieb-Liniger model is unstable in the thermodynamic limit. The first excitations of the attractive Lieb-Liniger are one string state of N particles with a finite total momentum. For a 2 string solution, we can have a $M$ particle string and a N-M particle string.

### 4.14.1 Two-string Bound Solutions

NOTE: The bound state momenta and the energies given below are only when $L \rightarrow \infty$. For details, see the appendix section G.

Let us look at the attractive case of bosons. Let us assume that the ground state consists of only two particle strings. Using 4.54, we see that the imaginary parts of the momenta of the 2 particle strings would be $\pm \mathrm{i} c / 2$. The real parts (called as "string centers") although have to be determined by the Bethe equations. These calculations have already been done in the literature[80]. Since we are dealing with strings of only two particles, the expressions become really simple.

The string centers satisfy the reduced Bethe equations[80],

$$
\begin{equation*}
\mathrm{j} \lambda_{\alpha}^{\mathrm{j}} L-\sum_{k, \beta} \Phi_{\mathrm{j} k}\left(\lambda_{\alpha}^{\mathrm{j}}-\lambda_{\beta}^{k}\right)=2 \pi I_{\alpha}^{\mathrm{j}} \tag{4.56}
\end{equation*}
$$

where $\mathrm{j}($ or k$)=$ number of particles in a string, $\alpha($ or $\beta)=1,2, \ldots N_{\mathrm{j}}$ labels the strings of given length, $\Phi_{\mathrm{j} k}\left(\lambda_{\alpha}^{\mathrm{j}}-\lambda_{\beta}^{k}\right)=2 \arctan \left(2 \frac{\lambda_{\alpha}^{\mathrm{j}}-\lambda_{\beta}^{k}}{|c| \mathrm{j}}\right)$.

For our case, $\mathrm{j}=2$ for all the strings and there are $\mathrm{N} / 2$ strings in total. This implies that the string center satisfy the normal Bethe equations with the phase shift given by $2 \arctan \left(\frac{\lambda_{\alpha}^{j}-\lambda_{\beta}^{k}}{|c|}\right)$.

The total momentum of the ground state is then

$$
\begin{equation*}
P=\sum_{\alpha} 2 \lambda_{\alpha}^{(2)} \tag{4.57}
\end{equation*}
$$

and the total energy is,

$$
\begin{equation*}
E_{0}=\frac{P^{2}}{2}-\frac{N}{2} \frac{c^{2}}{2} \tag{4.58}
\end{equation*}
$$

For getting the string centers, we have to solve the eq.4.14.1 for $\mathrm{j}=2, \alpha=1,2, \ldots, N / 2$,

$$
\begin{equation*}
2 \lambda_{\alpha}^{(2)} L-\sum_{\beta \neq \alpha} \Phi\left(\lambda_{\alpha}^{(2)}-\lambda_{\beta}^{(2)}\right)=2 \pi I_{\alpha}^{(2)}, \tag{4.59}
\end{equation*}
$$

where the $I_{\alpha}^{(2)}$ are even or odd half integers depending on whether $N$ is even or odd. As we can see from the Bethe equations, the ground state would have the string centers to be equally distributed around 0 . Hence, the total momentum $P$ would has to be 0 making the energy, $E_{0}=-\frac{N}{2} \frac{c^{2}}{2}$. Taking only the 2 particle strings as we have done, removes the non-extensivity of the ground state as now $E / N=$ finite.

### 4.14.2 Future Outlook

It turns out that delta function potentials cannot capture the formation of antisymmetric bound states as required for realizing the Lieb-Liniger model version of the Kitaev wire. We are actively pursuing remedies that address this issue. With this delicate issue addressed, it is a matter of numerical prowess to discover what happens when hard wall boundary conditions are introduced and conditions are favourable for constructing Majorana bound states.

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## A. IMPEDANCE CALCULATION FOR THE LJJ

The impedance of the Josephson Junction (JJ) is generally calculated as follows [81]:

$$
\begin{equation*}
Z_{J J}=\left[\mathrm{i}\left(\omega C_{J}-1 / \omega L_{J}(0)\right)\right]^{-1}, \tag{A.1}
\end{equation*}
$$

where $C_{J}$ and $L_{J}$ are the JJ capacitance and the intrinsic inductance in the small signal limit when the phase difference along the junction approaches 0 .

The value of the intrinsic inductance is given by the expression [82]:

$$
\begin{equation*}
L_{J}(I)=\frac{\Phi_{0}}{2 \pi \sqrt{I_{C}^{2}-I^{2}}} \tag{A.2}
\end{equation*}
$$

where $\Phi_{0}=2.07 \times 10^{-15} \mathrm{~Wb}$ is the flux quantum, $I_{C}$ is the critical current of the JJ and $I$ is the superconducting current. The JJs in the transmission line $(\mathbf{j j k})$ have $I_{C}=1.5 \mu \mathrm{~A}$ and $I=0$, which gives $L_{J} \sim 220 \mathrm{pH}$.

The next bit is calculating the impedance itself. For jjk, the parameters are: $C_{J}=60$ $\mathrm{fF}, L_{J}=220 \mathrm{pH}, \omega=\pi / \tau, \tau=20 \mathrm{ps}$, we find that

$$
\begin{equation*}
\left|Z_{\mathrm{jj} k}\right| \text { (transmission line) } \sim 51 \Omega \tag{A.3}
\end{equation*}
$$

The inductors in the transmission line have an impedance

$$
\begin{equation*}
Z_{L}=\mathrm{i} \omega L=\mathrm{i} 1.23 \Omega, \tag{A.4}
\end{equation*}
$$

for the 7.845 pH inductors.
The effective impedance of the LJJ can be calculated by solving a recursion relation in terms of the $Z_{J J}$ and $Z_{L}$, as shown in the figure A.1.

The recursive relation comes out to be

$$
\begin{equation*}
\left(\left[Z_{L J J}+2 Z_{L}\right]^{-1}+\left[Z_{J J}\right]^{-1}\right)^{-1}+2 Z_{L}=Z_{L J J} \tag{A.5}
\end{equation*}
$$



Figure A.1. Recursive nature: Diagram showing the recursive relation in LJJ. The effective impedance of the LJJ can be calculated by equating the impedance of the left and right diagrams [81].
which when simplified gives us

$$
\begin{equation*}
\left|Z_{L J J}\right|=2 \sqrt{Z_{L}\left(Z_{L}+Z_{J J}\right)} \sim 16 \Omega . \tag{A.6}
\end{equation*}
$$

This is precisely the reason that when we add a resistance at the end of one of the branches that matches this impedance, the fluxon dies down. This can be seen in figure A.2, where the fluxon does not reflect back if we have a resistance of $16 \Omega$ at the end of the transmission line.


Figure A.2. Impedance matching: The current traces showing the fluxon paths as it encounters the branch end. These transmission lines have a resistance at the end with values (Left) $1 \mu \Omega$, (Middle) $16 \Omega$ and (Right) $1 G \Omega$. When the resistance is very small, the fluxon is reflected back with the same polarity (closed circuit). When the resistance is extremely large, it resembles an open circuit termination and the fluxon is reflected with a reversed polarity. If the resistance value matches the impedance of the transmission line, the resistance gets absorbed.

## B. PROMISING CIRCUITS

Along with the working circuit discussed in the section 3.3.2, there were a few circuits that showed promise. These circuits could be explored further and compared with our present working circuit in terms of efficiencies and designing capability.

## B. 1 Rotary with current sources

Most of the older SFQ circuits have bias currents that allow the fluxon to travel forward. Hence, we tried working with a rotary circuit (Figure B.1) that is symmetric with respect to the two branches and has a DC current source attached to each of them. The resistors were added later to block the reflections from the ends as the fluxon was bouncing back and forth.


Figure B.1. Rotary circuit with current sources: The branches 'bottom' (L16, L17) and 'right' (L6, L7) have been supplied with additional DC currents $(4 \mu A)$. The ends have a resistor which has a value matching the impedance of the LJJ's to block the fluxon from reflecting back.

As we vary the values of the bias current, we find the critical current value that allows the fluxon to travel to one of the branches (Figure B.2). We also noticed that if the bias current is between $1.5 \mu \mathrm{~A}$ to $4 \mu \mathrm{~A}$, the fluxon only travels to the bottom branch. A sweep through the bias current gives us the critical value required for the fluxon to travel to the other branches. The fluxon does not enter the branches for a bias current $\leq 1.22 \mu \mathrm{~A}$.


Figure B.2. Rotary in action: (a) Positive Bias $(4 \mu A$ ), (b) Negative Bias $(-4 \mu A)$. When we change the sign of the biasing source, the direction of the rotary is flipped. For negative bias, the fluxon enters the right branch instead of the bottom branch.

## B.1.1 Phases of JJ in the Rotary junction

The Josephson junctions corresponding to different branches and their phases are shown in figure B.3. For the bias current $(4 \mu A)$, as the fluxon travels bottom branch, we can see that the JJs of the input branch and bottom branch see a phase change of $2 \pi$. The right branch JJ does not show any change which can be seen in figure B.3a.

Near the critical value of the bias current, we find that the JJ of the input branch shows a peak but drops down immediately. For Bias $=1.23 \mu A$, the JJ of the input and second branch go up to $2 \pi$, whereas for Bias $=1.22 \mu A$, the JJs do not switch. This can be seen in figures B.3b, B.3c.


Figure B.3. Critical value of bias: Difference in the phases of JJs as a function of the bias current. We can see that above a critical bias current, the fluxon chooses a branch. Below the critical value, the fluxon does not enter any branch. The current traces are for the input branch, branch 2 and branch 3 respectively. The JJ phase plots are also in the same order.

## B.1.2 Effect of terminating resistance

To see the influence of the resistor on the fluxon travel trajectory, we remove the bias sources from the ends of both branches.

Looking at the figure B.4, we notice that the reflection of the fluxon can be suppressed by adding a finite value of the resistor at the ends of output branches. Changing the value of the resistor does not affect the result as the peaks traveling in the output branch keeps getting


Figure B.4. Terminating with resistors: Including finite resistors at the ends of branches suppresses the reflection of the fluxon. The value of the zero resistance is chosen to be a very small value because WRSpice does not allow absolute zero values for initializing these devices.
wider and wider (figure B.4a). This is not a soliton behavior and this is the precise reason why any value of resistance is able to suppress it by damping the current in the circuit.

## B.1.3 Addition of a circulating current

The reason why the bottom branch gets chosen when the fluxon enters from the input branch is because of the polarity of the fluxon field, as was seen in the working circuit discussed in section 3.3.2. Due to this chirality, the JJ attached to the bottom branch gets activated first and the fluxon travels along the LJJs in this branch. If there is a a circulating current in the rotary loop (Figure B.5) that reverses this direction, this can lead to the fluxon entering the other branch (right branch).

Note that without the circulating current, the circuit is still that of the working rotary with the current sources fixed at $4 \mu A$ and the resistances set to $15 \Omega$. As we increase the circulating current value, the fluxon in the bottom branch keeps getting slower. After a


Figure B.5. Circulating current in the loop: The inductors are given a very small finite value $(0.001 \mathrm{pH})$ and a current can be set using the ic parameter of WRSpice. Positive (negative) value leads to clockwise (counterclockwise) current direction.
critical value of the circulating current, the fluxon enters the right branch instead of the bottom branch (Figure B.6d).

## B.1.4 Replacing the source with circulating current

The circulating current suggests that removing the current source should not affect the rotary, although the current values will be different because of this change. Let us try that. For no circulating current, the fluxon does not enter any of the branches.

## Clockwise

As seen in the case where we only had the current sources, increasing the circulating current above a critical threshold leads to fluxon entering the right branch (Figure B.7b).


Figure B.6. Loop current: Current traces showing the direction of fluxon as a function of the circulating current amplitude. We can see that as we increase the clockwise circulating current, the fluxon slows down and then after a critical value, moves to Branch 3.

## Counter-Clockwise

Intuitively, we expect that reversing the direction of the circulating current should force the fluxon to travel in the other branch (bottom branch). This is what we observe in figure


Figure B.7. Clockwise current:Current traces depicting the role of the circulating current in forcing the fluxon to travel in the other branches. A positive value of the circulating current above a certain threshold leads to the fluxon moving to the right branch.
B.8a. Notice that the current values are different from the ones found when we had a clockwise current. This is because, in one case (counter-clockwise), the current adds up to the existing fluxon current whereas, in the other case, it opposes the current (clockwise). Increasing the current destroys the fluxon in the bottom branch as seen in figure B.8b.


Figure B.8. Anticlockwise current:Current traces depicting the role of the circulating current in forcing the fluxon to travel in the other branches. A negative value of the circulating current leads to the fluxon moving to the bottom branch and a further increase forcing the fluxon to reflect in the input branch.

## B. 2 Symmetric rotary

The previous circuits have a built-in chirality that makes it route the pulses in a circular fashion. We looked at a symmetric version of the circuit that does not have a chirality. The full circuit can be seen in figure B. 10 and the zoomed version of the rotary junction in figure B.11.

The rotary junction consists of 3 Josephson Junctions (JJs) with configuration different from the JJs used in the transmission line. The transmission line uses JJ of the type,

```
.model jjk jj(rtype=0, cct=1, vg=2.8m, icrit=1.5u, cap=60f),
```

whereas the rotary junction contains JJ of the type,

```
.model jjk2 jj(rtype=0, cct=1, vg=2.8m, icrit=15u, cap=60f),
```

that has a critical current $I_{c}^{\mathrm{jjk} 2}=10 I_{c}^{\mathrm{jjk}}$. There are 4 inductors in rotary junction: $L 1, L 2$, $L 6$ and $L 7$ in the circuit. These inductors have a 300 pH inductance with $L 1=0 \mathrm{pH}$. The output branches end with a $15 \Omega$ resistance to suppress the reflection of the fluxon.


Figure B.9. Built-in chirality: The working rotary circuit has a chirality with respect to the connection of the branches to the rotary junction.

## B.2.1 Motivation for junctions

Wustmann et. al. ([83]) use a JJ in the interface that has critical currents and capacitance different from the ones used in the transmission line JJ values. The interface has 3 JJs that have variable parameters used to control the flow as shown in figure B.12. This can be used to make a rotary junction that has JJs with different values of critical currents and capacitances.

In the following, we will enumerate the steps taken to achieve a working rotary circuit.

1. Step 1: Changing the JJs in the rotary junction

As seen in section B.2.1, changing the interface JJs can affect the fluxon progression. If we only change B1 to a different JJ (.model jjk 2 jj (rtype $=0$, cct $=1$, $\mathrm{vg}=2.8 \mathrm{~m}$, icrit $=15 \mathrm{u}$, cap $=600 \mathrm{f})$ ) where $I_{c}^{\text {new }} / I_{c}=C^{n e w} / C=10$, we find that it allows fluxon to move in one of the branches as seen in figure B.13. Only the first six plots in the figure show the direction of fluxon in the branches.


Figure B.10. Symmetric circuit: A symmetric version of the rotary circuit. One of the legs of the branches is grounded and the other leg is connected to the rotary junction. The ends of the branches are connected with a resistor that matches the impedance of the transmission line impedance.

If we change all of the JJs to jjk 2 , it produces a symmetric output as shown in figure B.14.
2. Step 2: Introducing a current in the rotary loop

We noticed that the current in the rotary starts oscillating due to finite value of inductor and the capacitance attached to the JJ (finite value of LC). To rectify this so that we have a smooth trace, we need to extend the time period of the oscillations. We can do this by increasing the value of the inductor or capacitance, but we will choose inductors in this case.

We find that we can get the fluxon to travel in different branches as shown in figure B.15a. But there is still a problem! The fluxon is still getting reflected back to the input branch and we have to eliminate this reflection.
3. Step 3: Changing the capacitance of the rotary JJs


Figure B.11. Symmetric junction closeup: Zoomed in version of figure B.10. There are 4 inductors in the junction: L1, L2, L6 and L7 with L1 being set to 0 . There are 3 JJs : B0, B1 and B2.


Figure B.12. Wustmann junction: An interface between two LJJs which has variable parameters. Depending on the value of the critical currents and capacitances of the interface JJ, Wustmann et. al. observe different behaviors for the fluxon propagation [83].


Figure B.13. jjk2 rotary: Current traces showing fluxon propagation in different branches when B2 is changed from jjk to jjk 2 . The fluxon prefers to move towards the changed JJ branch.


Figure B.14. All jjk2 junction: Current traces in the dummy inductors when all the rotary JJs are changed to jjk2. The fluxon does not move into any of the branches and keeps shuttling back and forth in the input branch.


Figure B.15. Addition of loop current: Current traces with $L 2=L 6=$ $L 7=300 \mathrm{pH}$ and loop current $\pm 10 \mu A$. The JJs are jjk2. The current polarity decides the path of the fluxon as it passes through the rotary junction.

In the reference [83], they use a lower value of capacitance in the interface junction to allow the fluxon to move forward. Hence, we will reduce the capacitance back to the capacitance of the JJs used in the transmission line i.e. $60 f F$.

As expected, this removes the reflection. The plots can be seen in figure B.17. The negative current plot has a fluxon and not an anti-fluxon. The reversed behavior is due to the positioning of the dummy inductor that detects the current sign depending on the orientation.

## B.2.2 Final working circuit

The final circuit shows symmetric behavior when there is no current in the loop (figure B.17a). The fluxon does not enter any of the branches and gets reflected back to the input branch. When the loop current is $10 \mu A$, fluxon enters the branch with inductors $L 14$ and $L 15$ (figure B.17b). As the loop current is reversed, fluxon enters the other branch with inductors $L 10$ and $L 11$ (figure B.17c).

(a) Loop Current $=10 \mu A$

(b) Loop Current $=-10 \mu A$

WSPRCE

(c) Loop Current $=0 \mu \mathrm{~A}$

Figure B.16. Capacitance correction: Current traces with $L 2=L 6=$ $L 7=300 \mathrm{pH}$ and loop current $\pm 10 \mu A$ and $0 \mu A$. The JJs are jjk2 with $C=$ $60 f F$. The changed capacitance eliminates the reflection to the input branch.


Figure B.17. Working symmetric circuit: Current traces with $L 2=L 6=$ $L 7=300 \mathrm{pH}$ and loop current $0 \mu A$ and $\pm 10 \mu A$. The fluxon does not enter any of the output branches when there is no current in the rotary loop and gets reflected back. A positive value of loop current allows the fluxon to travel in one of the branches whereas a negative value directs it to the other branch.

## B.2.3 Working range of impedance

Let us calculate the impedance of the circuit now that we have a different junction circuit. We will calculate the intrinsic inductance of the JJ ( $\mathbf{j} \mathbf{j k 2}$ ) first.Using equation A. 2 and the fact that we have a supercurrent of $10 \mu A$ flowing through the JJ, we have

$$
\begin{equation*}
L_{J}=\frac{\Phi_{0}}{2 \pi \sqrt{I_{C}^{2}-10^{-5}}} \sim 29.47 \mathrm{pH} \tag{B.1}
\end{equation*}
$$

for a critical current $I_{C}=15 \mu \mathrm{~A}$. Choosing the value of $C_{J}=300 \mathrm{fF}$ (perfect rotary) and plugging the value of $L_{J}$ in equation A.1,

$$
\begin{equation*}
\left|Z_{J J}\right|=5.92 \Omega \tag{B.2}
\end{equation*}
$$

Since we have two 110 pH inductors in series as well ( 220 pH combined), we have

$$
\begin{equation*}
\left|Z_{L}\right|=34.56 \Omega \tag{B.3}
\end{equation*}
$$

The effective impedance of a single leg of rotary is

$$
\begin{equation*}
\left|Z_{r o t}\right|=\left|Z_{J J}\right|+\left|Z_{L}\right|=40.48 \Omega \tag{B.4}
\end{equation*}
$$

The impedance of the JJ jjk 2 is affected by three parameters: $I, I_{C}$ and $C_{J}$. To calculate the working range, we fix two of the paramters $I=10 \mu A$ and $C_{J}=300 \mathrm{fF}$. After sweeping the different values, the range comes out to be

$$
\begin{equation*}
13 \mu A \leq I_{C} \leq 17 \mu A \tag{B.5}
\end{equation*}
$$

with an uncertainty of $0.1 \mu \mathrm{~A}$. This translates to an impedance range of

$$
\begin{equation*}
39.1 \Omega \leq\left|Z_{r o t}\right| \leq 43.3 \Omega \tag{B.6}
\end{equation*}
$$

## B.2.4 Failure of loop current reset

A reversible rotary should work in such a way that it can route any number of pulses. The current circuit is incapable of doing that as the loop current value does not revert to its original value after interacting with the incoming flux. This means that sending additional pulses would not be directed towards any particular branch. This can be seen in figure B.18. The rotary junction had a similar behavior in the previous circuit as well.


Figure B.18. Fluxon path for long times: Fluxon path and rotary junction phases for multiple input pulses. We find that the rotary junction does not reset simply and the loop current relaxes to zero once the first pulse passes through. The next pulses do not get directed to any of the output branches.

Moreover, the inductor $L 0$ which is a dummy inductor connected to the rotary end from the left side, starts conducting a non-zero current after the pulse.

This is due to the fact that we have 3 grounded terminals near the rotary junction. The grounded terminals along with one of the rotary legs form a loop that has $L I_{c}>\Phi_{0}$. We


Figure B.19. Trapped fluxon: Inductor $L 0$ conducting a non-zero current even when the pulse has left the rotary. Due to the high value of inductance in the rotary junction, the product $L I_{c}>\Phi_{0}$. This leads to a trapped fluxon in the loop formed by the grounded terminals and the rotary junction legs.


Figure B.20. Junction node diagram: Diagrammatic view of Node 1 before and after the pulse impact. The currents in the rotary legs are different because the dummy inductor $L 0$ has a non-zero current flowing through.
have a fluxon trapped in this loop after the pulse travels to the output branch that affects the future routing of pulses.

## C. CODE FOR AUTOMATING THE CIRCUIT SIMULATION

Our circuits were designed using a software Xic (cite Xic website). The simulations were carried out using WRSpice which is a computational library for solving differential equations of electronic circuits. Although good for individual calculations, WRSpice is not a great choice for doing parameter sweeps along with a restrictive plotting software. We used a Python wrapper (pyWRSpice) (cite pyWRSpice github) to carry out the simulations and used Matplotlib for plotting.

## C. 1 Using pyWRSpice for automation

The circuit file can be generated as a text file from Xic using the dump button. We use the circuit details in defining our circuit script that will be used in our simulation. The code given below is for our working circuit.

```
# Circuit script for generating the Xic circuit
script_wrspice = """
* Generated by Xic from cell DJ-energy
4.tran {dt}ps {time}ns uic
.options maxdata=4134303
.model jjMIT jj(rtype=1,cct=1,icon=10m, vg=2.8m, icrit=1m, cap=700f)
BO 1 4 29 jjk
B1 5 1 30 jjk
B2 5 4 31 jjk
IO 0 11 pwl({tzero} {zero} {tfirst}p {first}m {tsecond}p {second}m {tthird
    }p {third}m)
LO 10 13 100 pH
L1 12 15 0pH
L2 14 14 16 0pH
L3 17 19 0pH
L4 18 22 0pH
L5 21 0 0pH
L6 20 0 0pH
L7 23 24 0pH
L8 25 26 0pH
```

```
20 L9 28 0 0pH
21 L10 27 0 0pH
22 X0 11 10 dcsfq.xic
23 X1 13 0 12 2 dljj20.xic
24 X2 15 2 14 3 dljj20.xic
25 X3 16 3 1 4 dljj20.xic
26 X4 1 5 17 6 dljj20.xic
27 X5 19 6 18 7 dljj20.xic
28 X6 22 7 21 20 dljj20.xic
29 X7 4 5 8 23 dljj20.xic
30 X8 8 24 9 25 dljj20.xic
31 X9 9 26 28 27 dljj20.xic
32 .subckt dljj-seg.xic LT LB RT RB
33 BO 5 6 7 jjk ics=1.5uA
34 LO LT 5 7.845pH
35 L1 5 RT 7.845pH
36 L2 LB 6 7.845pH
37 L3 6 RB 7.845 pH
38 .ends dljj-seg.xic
39 . subckt dljj20.xic LT LB RT RB
40 XO LT LB 5 25 dljj-seg.xic
41 X1 5 25 6 26 dljj-seg.xic
42 X2 6 26 7 27 dljj-seg.xic
43 X3 7 27 8 28 dljj-seg.xic
44 X4 8 28 9 29 dljj-seg.xic
45 X5 9 29 10 30 dljj-seg.xic
46 X6 10 30 11 31 dljj-seg.xic
47 X7 11 31 12 32 dljj-seg.xic
48 X8 12 32 13 33 dljj-seg.xic
49 X9 13 33 14 34 dljj-seg.xic
50 X10 14 34 15 35 dljj-seg.xic
51 X11 15 35 16 36 dljj-seg.xic
52 X12 16 36 17 37 dljj-seg.xic
53 X13 17 37 18 38 dljj-seg.xic
54 X14 18 38 19 39 dljj-seg.xic
55 X15 19 39 20 40 dljj-seg.xic
```

```
X16 20 40 21 41 dljj-seg.xic
X17 21 41 22 42 dljj-seg.xic
X18 22 42 23 43 dljj-seg.xic
X19 23 43 24 44 dljj-seg.xic
X20 24 44 RT RB dljj-seg.xic
.ends dljj20.xic
.subckt dcsfq.xic IN OUT
B1 6 7 14 jjr ics=0.17125mA
B2 10 13 17 jjr ics=0. 17125mA
B3 11 12 16 jjr ics=0. 1475mA
B4 8 9 15 jjr ics=0.245mA
I1 0 3 0.373mA
L1 IN 5 3.3528pH
L2 5 6 1.2936 pH
L3 5 11 1. 2672 pH
L4 7 4 1. 1352 pH
L5 11 0 3.5904pH
L6 7 10 0.2112 pH
L7 4 8 1.7424pH
L8 8 OUT 2.112pH
LPI1 3 4 0.0792 pH
LPJ2 13 0 0.1848pH
LPJ3 12 10 0.6864 pH
LPJ4 9 0 0.132 pH
.ends dcsfq.xic
1.model jjr jj(rtype=1, cct=1, icon=10m, vg=2.8m, delv=0.08m,
82 + icrit=1m, vshunt=0.5mV cap=1.31p)
.model jjk jj(rtype=0, cct=1, vg=2.8m, icrit=1.5u, cap=60f)
89 .save @I1.X0[c]
```

83
84
85
87
88
90
91

```
* Control section
.control
run
set filetype=binary
write {output_file} L1#branch L2#branch L3#branch L4#branch L7#branch L8#
        branch
    . endc
" " "
```

We can define a function getData that takes the circuit script and use pyWRSpice to get the output data.

```
def getData(dt = 0.2, time = 1, power_amp_list=[0, -0.07, 0.35, 0],
    power_time_list = [0, 25, 50, 900], res = 0, circuit = script_wrspice):
    engine = simulation.WRWrapper(script_wrspice,
    command = "/usr/local/xictools/bin/wrspice") # Typical for Unix
        dat1 = engine.run(circuit, dt=dt, time=time, zero = power_amp_list[0],
        first= power_amp_list[1], second = power_amp_list[2], third =
        power_amp_list[3], tzero = power_time_list[0], tfirst= power_time_list
        [1], tsecond = power_time_list[2], tthird = power_time_list [3])
        df = dat1.to_array()
        return df
```

We can also define a function drawPlots to draw the plots from the output data.

```
def drawPlots(df, num_of_curr, num_of_vol, title, path_name, fsize = (25,
    16), ylabel_loc=0.07, yrange= = - 2, 15], num_of_xticks = 10):
        ,',Draws the fluxon plots for different values of the loop currents,',
```

```
plt.rcParams['figure.dpi'] = 300
```

plt.rcParams['figure.dpi'] = 300
plt.rcParams['savefig.dpi'] = 300
plt.rcParams['savefig.dpi'] = 300
SMALL_SIZE = 8
SMALL_SIZE = 8
MEDIUM_SIZE = 10
MEDIUM_SIZE = 10
BIGGER_SIZE = 12
BIGGER_SIZE = 12
plt.rc('font', size=fsize[0] + 1) \# controls default text
plt.rc('font', size=fsize[0] + 1) \# controls default text
sizes
sizes
plt.rc('axes', titlesize=SMALL_SIZE) \# fontsize of the axes title
plt.rc('axes', titlesize=SMALL_SIZE) \# fontsize of the axes title
plt.rc('axes', labelsize=fsize[0] + 1) \# fontsize of the x and y
plt.rc('axes', labelsize=fsize[0] + 1) \# fontsize of the x and y
labels
labels
plt.rc('xtick', labelsize=fsize[1]*20/16) \# fontsize of the tick
plt.rc('xtick', labelsize=fsize[1]*20/16) \# fontsize of the tick
labels
labels
plt.rc('ytick', labelsize=fsize[1]*20/16) \# fontsize of the tick
plt.rc('ytick', labelsize=fsize[1]*20/16) \# fontsize of the tick
labels
labels
plt.rc('legend', fontsize=SMALL_SIZE) \# legend fontsize
plt.rc('legend', fontsize=SMALL_SIZE) \# legend fontsize
plt.rc('figure', titlesize=BIGGER_SIZE) \# fontsize of the figure
plt.rc('figure', titlesize=BIGGER_SIZE) \# fontsize of the figure
title
title
label_list = ['L1','L2', 'L3', 'L4', 'L7', 'L8']
label_list = ['L1','L2', 'L3', 'L4', 'L7', 'L8']
colors = plt.rcParams["axes.prop_cycle"]()
colors = plt.rcParams["axes.prop_cycle"]()
ts = df[0]
ts = df[0]
fig, axs = plt.subplots(num_of_curr + num_of_vol, figsize=fsize,
fig, axs = plt.subplots(num_of_curr + num_of_vol, figsize=fsize,
sharex=True, sharey=False)
sharex=True, sharey=False)
fig.suptitle(title, fontsize='x-large')
fig.suptitle(title, fontsize='x-large')
for i, data in enumerate(df):
for i, data in enumerate(df):
if i>0:
if i>0:
\# Get the next color from the cycler
\# Get the next color from the cycler
c = next(colors)["color"]
c = next(colors)["color"]
axs[i-1].set_xticks(np.linspace(0.0, time, int(time*
axs[i-1].set_xticks(np.linspace(0.0, time, int(time*
num_of_xticks + 1)), rotation='vertical')

```
num_of_xticks + 1)), rotation='vertical')
```

        axs[i-1].set_xlim(left=0)
        if i<=num_of_curr:
        axs [i-1].plot(ts*1e9, data*1e6, color=c)
        axs [i-1]. axhline (0, ls='--', color='b')
        axs [i-1].set_ylim([yrange[0], yrange[1]])
        axs [i-1].set_ylabel(f'\{label_list[i-1]\}')
        else:
        axs [i-1].plot(ts*1e9, data*1e3, color=c)
        axs[i-1].axhline(0, ls='--', color='b')
        axs[i-1].set_ylabel('Voltage [mV]')
    ```
fig.text(ylabel_loc,0.5, "Current [uA]", ha="center", va="center",
rotation=90)
plt.xlabel("Time [ns]", labelpad=50)
plt.savefig(path_name)
plt.show()
```

Once we are done setting up all the functions, we can do a sweep for our parameter values. For example, we can do a sweep over all the different values of time resolution to see whether the simulation is missing something for larger resolutions.

```
time = 5
for delta_t in numpy.linspace(0.1, 0.5, 5):
    df = getData(dt=delta_t, time=time)
```


## D. SCATTERING CALCULATIONS

## D. 1 Separation of variables

Let us look at the Schrodinger equation for two particles.

$$
\begin{equation*}
\left[-\left(\frac{\partial}{\partial x_{1}^{2}}+\frac{\partial}{\partial x_{2}^{2}}\right)+2 c \delta\left(x_{1}-x_{2}\right)\right] \psi\left(x_{1}, x_{2}\right)=E \psi\left(x_{1}, x_{2}\right) \tag{D.1}
\end{equation*}
$$

Defining new variables with respect to the center of mass and relative coordinates,

$$
\begin{array}{r}
K=k_{1}+k_{2}, k=k_{1}-k_{2}, \\
X=\left(x_{1}+x_{2}\right) / 2, x=x_{1}-x_{2} .
\end{array}
$$

Changing the derivatives accordingly,

$$
\begin{aligned}
\frac{\partial}{\partial x_{1}} & =\frac{\partial}{\partial X} \cdot \frac{1}{2}+\frac{\partial}{\partial x}, \\
\frac{\partial}{\partial x_{2}} & =\frac{\partial}{\partial X} \cdot \frac{1}{2}-\frac{\partial}{\partial x} .
\end{aligned}
$$

Rewriting the Schrodinger equation in terms of new variables,

$$
\begin{equation*}
\left[-\left(\frac{1}{2} \frac{\partial}{\partial X^{2}}+2 \frac{\partial}{\partial x^{2}}\right)+2 c \delta(x)\right] \psi\left(x_{1}, x_{2}\right)=E \psi\left(x_{1}, x_{2}\right) . \tag{D.2}
\end{equation*}
$$

Now we can use the separation of variables,

$$
\psi\left(x_{1}, x_{2}\right)=A(X) B(x)
$$

The equation D. 2 can then be written as

$$
\begin{equation*}
-\frac{1}{A} \frac{\partial A}{\partial X^{2}}-4 \frac{1}{B} \frac{\partial B}{\partial x^{2}}+4 c \delta(x)=2 E \tag{D.3}
\end{equation*}
$$

Thus,

$$
\begin{aligned}
-\frac{1}{A} \frac{\partial A}{\partial X^{2}} & =K^{2} \\
-4 \frac{1}{B} \frac{\partial B}{\partial x^{2}}+4 c \delta(x) & =2 E-K^{2}
\end{aligned}
$$

The first equation tells us that the center of mass stays free even in the presence of this potential. The second equation gives us the Schrodinger equation in terms of the relative coordinates and the relative potential.

$$
\begin{equation*}
-\frac{\partial B}{\partial x^{2}}+c \delta(x) B=\left(\frac{k}{2}\right)^{2} B \tag{D.4}
\end{equation*}
$$

The condition on derivatives due to this potential can be found out by integrating the above equation around the region of singularity,

$$
\begin{equation*}
\left.\psi^{\prime}\right|_{x \rightarrow 0^{+}}-\left.\psi^{\prime}\right|_{x \rightarrow 0^{-}}=\int_{-\epsilon}^{\epsilon} c \delta(x) \psi(x) d x=c \psi(0) . \tag{D.5}
\end{equation*}
$$

Reverting back to the original coordinate gives us the general condition,

$$
\begin{equation*}
\left[\left.\left(\frac{\partial}{\partial x_{\mathrm{j}}}-\frac{\partial}{\partial x_{k}}\right)\right|_{x_{\mathrm{j}}=x_{k}^{+}}-\left.\left(\frac{\partial}{\partial x_{\mathrm{j}}}-\frac{\partial}{\partial x_{k}}\right)\right|_{x_{\mathrm{j}}=x_{k}^{-}}\right] \psi=\left.2 c \psi\right|_{x_{\mathrm{j}}=x_{k}} . \tag{D.6}
\end{equation*}
$$

## D. 2 Scattering coefficients

Applying the boundary condition 4.4 on permutations $\mathbb{P}$ and $\mathbb{P}^{\prime}$ i.e. on $x_{\mathrm{j}}$ and $x_{\mathrm{j}+1}$,

$$
\begin{equation*}
A_{\mathbb{P}}\left[\mathrm{i}\left(k_{P(\mathrm{j}+1)}-k_{P(\mathrm{j})}\right)\right]+A_{\mathbb{P}^{\prime}}\left[\mathrm{i}\left(k_{P(\mathrm{j})}-k_{P(\mathrm{j}+1)}\right)\right]=c\left(A_{\mathbb{P}}+A_{\mathbb{P}^{\prime}}\right) \tag{D.7}
\end{equation*}
$$

which gives us the required condition 4.6,

$$
\frac{A_{\mathrm{P}}}{A_{\mathbb{P}^{\prime}}}=\frac{k_{P(\mathrm{j})}-k_{P(\mathrm{j}+1)}-\mathrm{i} c}{k_{P(\mathrm{j})}-k_{P(\mathrm{j}+1)}+\mathrm{i} c} .
$$

## E. LIEB'S CALCULATIONS

We outline the original calculations done by Lieb in his paper [50] in this section.

## E. 1 Bethe ansatz in the thermodynamic limit

The Bethe equations can also be written as

$$
\begin{equation*}
(-)^{(N-1)} \mathrm{e}^{-\mathrm{i} k_{\mathrm{j}} L}=\exp \left[\mathrm{i} \sum_{s=1}^{N} \theta_{s \mathrm{j}}\right] \tag{E.1}
\end{equation*}
$$

where $\theta_{s \mathrm{j}}=-2 \arctan \left(\frac{k_{s}-k_{\mathrm{j}}}{c}\right)$.
Define,

$$
\begin{equation*}
\delta_{\mathrm{j}} \equiv\left(k_{\mathrm{j}+1}-k_{\mathrm{j}}\right) L=\sum_{s=1}^{N}\left(\theta_{s, \mathrm{j}}-\theta_{s, \mathrm{j}+1}\right)+2 \pi n_{\mathrm{j}} . \tag{E.2}
\end{equation*}
$$

For the ground state, $n_{\mathrm{j}}=1$ which minimizes the $\delta^{\prime} s$ and allows $k$ 's to be as compactly distributed as possible. Eq. (4.16) is then derived by looking at the difference $\theta\left(k_{s}-k_{\mathrm{j}}\right)-\theta\left(k_{s}-k_{\mathrm{j}+1}\right)$ and then putting the thermodynamic limit.

After calculating the equations, we do a change of variables,

$$
k \equiv K x ; \quad c \equiv K \lambda ; \quad f(K x) \equiv g(x) .
$$

Eqns. (4.16), (4.17) and (4.18) then become

$$
\begin{gather*}
1+2 \lambda \int_{-1}^{1} \frac{g(x) d x}{\lambda^{2}+(x-y)^{2}}=2 \pi g(y)  \tag{E.3}\\
\gamma \int_{-1}^{1} g(x) d x=\lambda  \tag{E.4}\\
\mathrm{e}(\gamma)=\frac{\gamma^{3}}{\lambda^{3}} \int_{-1}^{1} g(x) x^{2} d x \tag{E.5}
\end{gather*}
$$

respectively, where $\gamma=\frac{c}{\rho}$ is a dimensionless quantity. (E.3) is an inhomogeneous Freidholm equation of the second kind.

To solve these equations and get the ground state spectrum, we need to

1. Solve (E.3) for a fixed value of $\lambda$.
2. Find $\lambda$ as a function of $\gamma$ using (E.4).
3. Use (E.5) gives the energy spectrum.

Now we move on to the excited state spectrum. The first excited state had the following equations to solve:

$$
\begin{aligned}
2 \pi J(k) & =2 c \int_{-K}^{K} \frac{J(r)}{c^{2}+(r-k)^{2}} d r-\pi-\theta(q-k), \\
p & =\int_{-K}^{K} J(r) d r+q \\
\epsilon_{1} & =-\mu+q^{2}+2 \int_{-K}^{K} r J(r) d r .
\end{aligned}
$$

Defining new variables,

$$
k=K x ; \quad J(K x)=\mathrm{j}(x) ; \quad q=K s ; \quad c=K \lambda .
$$

The equations now become,

$$
\begin{align*}
2 \pi \mathrm{j}(x) & =2 \lambda \int_{-1}^{1} \frac{\mathrm{j}(y)}{\lambda^{2}+(x-y)^{2}} d y-\pi+2 \arctan \left(\frac{s-x}{\lambda}\right),  \tag{E.6}\\
\frac{p}{\rho} & =\frac{\gamma}{\lambda} \int_{-1}^{1} \mathrm{j}(y) d y+s  \tag{E.7}\\
\frac{\epsilon_{1}}{\rho^{2}} & =\gamma \frac{\partial \epsilon_{0}(\gamma) / \partial \lambda}{\partial \gamma / \partial \lambda}-3 \epsilon_{0}(\gamma)+\frac{\gamma^{2}}{\lambda^{2}}\left[s^{2}+2 \int_{-1}^{1} y \mathrm{j}(y) d y\right] . \tag{E.8}
\end{align*}
$$

The value of $\gamma$ as a function of $\lambda$ was already calculated with regards to the ground state spectrum. The parameter " $s$ " can be eliminated from E. 8 by using E. 7 and we get $\epsilon_{1}(p)$.

Similarly, the other excitation comes out to be

$$
\begin{align*}
2 \pi \mathrm{j}(x) & =2 \lambda \int_{-1}^{1} \frac{\mathrm{j}(y)}{\lambda^{2}+(x-y)^{2}} d y+\pi-2 \arctan \left(\frac{s-x}{\lambda}\right)  \tag{E.9}\\
\frac{p}{\rho} & =\frac{\gamma}{\lambda} \int_{-1}^{1} \mathrm{j}(y) d y-s  \tag{E.10}\\
\frac{\epsilon_{2}}{\rho^{2}} & =3 \epsilon_{0}(\gamma)-\gamma \frac{\partial \epsilon_{0}(\gamma) / \partial \lambda}{\partial \gamma / \partial \lambda}+\frac{\gamma^{2}}{\lambda^{2}}\left[2 \int_{-1}^{1} y \mathrm{j}(y) d y-s^{2}\right] \tag{E.11}
\end{align*}
$$

## E. 2 Friedholm solver

The Friedholm equations of the second kind were solved on Mathematica XI. The code for the solver was taken from a StackExchange thread. This solver can solve the equations of the form:

$$
\begin{equation*}
f(x)-\lambda \int_{a}^{b} K(x, y) f(y) d y=g(x) \tag{E.12}
\end{equation*}
$$

## F. BETHE'S HARDWALL EQUATIONS FOR 2 PARTICLES

Using the Bethe ansatz for the hardwall case 4.30, the wavefunction at its full form

$$
\begin{align*}
\psi\left(x_{1}, x_{2}\right)= & A\left(k_{1}, k_{2}\right) \exp \left(\mathrm{i}\left[k_{1} x_{1}+k_{2} x_{2}\right]\right)+A\left(k_{2}, k_{1}\right) \exp \left(\mathrm{i}\left[k_{2} x_{1}+k_{1} x_{2}\right]\right) \\
& -A\left(-k_{1}, k_{2}\right) \exp \left(\mathrm{i}\left[-k_{1} x_{1}+k_{2} x_{2}\right]\right)-A\left(k_{1},-k_{2}\right) \exp \left(\mathrm{i}\left[k_{1} x_{1}-k_{2} x_{2}\right]\right) \\
& -A\left(k_{2},-k_{1}\right) \exp \left(\mathrm{i}\left[k_{2} x_{1}-k_{1} x_{2}\right]\right)-A\left(-k_{2}, k_{1}\right) \exp \left(\mathrm{i}\left[-k_{2} x_{1}+k_{1} x_{2}\right]\right) \\
& +A\left(-k_{1},-k_{2}\right) \exp \left(\mathrm{i}\left[-k_{1} x_{1}-k_{2} x_{2}\right]\right)+A\left(-k_{2},-k_{1}\right) \exp \left(\mathrm{i}\left[-k_{2} x_{1}-k_{1} x_{2}\right]\right) . \tag{F.1}
\end{align*}
$$

The boundary condition on the derivative results in a relation between permutations,

$$
\begin{equation*}
\frac{A\left(\ldots, \epsilon_{\mathrm{j}} k_{\mathrm{j}}, \ldots, \epsilon_{l} k_{l}, \ldots\right)}{A\left(\ldots, \epsilon_{l} k_{l}, \ldots, \epsilon_{\mathrm{j}} k_{\mathrm{j}}, \ldots\right)}=\frac{\epsilon_{\mathrm{j}} k_{\mathrm{j}}-\epsilon_{l} k_{l}+\mathrm{i} c}{\epsilon_{\mathrm{j}} k_{\mathrm{j}}-\epsilon_{l} k_{l}-\mathrm{i} c} . \tag{F.2}
\end{equation*}
$$

So, for our two particle case,

$$
\begin{aligned}
A\left(k_{1}, k_{2}\right) & =\left(\frac{k_{1}-k_{2}+\mathrm{i} c}{k_{1}-k_{2}-\mathrm{i} c}\right) A\left(k_{2}, k_{1}\right), \\
A\left(-k_{1}, k_{2}\right) & =\left(\frac{k_{1}+k_{2}-\mathrm{i} c}{k_{1}+k_{2}+\mathrm{i} c}\right) A\left(k_{2},-k_{1}\right), \\
A\left(k_{1},-k_{2}\right) & =\left(\frac{k_{1}+k_{2}+\mathrm{i} c}{k_{1}+k_{2}-\mathrm{i} c}\right) A\left(-k_{2}, k_{1}\right), \\
A\left(-k_{1},-k_{2}\right) & =\left(\frac{k_{1}-k_{2}-\mathrm{i} c}{k_{1}-k_{2}+\mathrm{i} c}\right) A\left(-k_{2},-k_{1}\right) .
\end{aligned}
$$

This leaves us with 4 unknown coefficients. Writing the wavefunction again,

$$
\begin{align*}
\psi\left(x_{1}, x_{2}\right)= & A\left(k_{1}, k_{2}\right) \exp \left(\mathrm{i}\left[k_{1} x_{1}+k_{2} x_{2}\right]\right)+\left(\frac{k_{1}-k_{2}-\mathrm{i} c}{k_{1}-k_{2}+\mathrm{i} c}\right) A\left(k_{1}, k_{2}\right) \exp \left(\mathrm{i}\left[k_{2} x_{1}+k_{1} x_{2}\right]\right) \\
& -A\left(-k_{1}, k_{2}\right) \exp \left(\mathrm{i}\left[-k_{1} x_{1}+k_{2} x_{2}\right]\right)-\left(\frac{k_{1}+k_{2}+\mathrm{i} c}{k_{1}+k_{2}-\mathrm{i} c}\right) A\left(-k_{1}, k_{2}\right) \exp \left(\mathrm{i}\left[k_{2} x_{1}-k_{1} x_{2}\right]\right) \\
& -A\left(k_{1},-k_{2}\right) \exp \left(\mathrm{i}\left[k_{1} x_{1}-k_{2} x_{2}\right]\right)-\left(\frac{k_{1}+k_{2}-\mathrm{i} c}{k_{1}+k_{2}+\mathrm{i} c}\right) A\left(k_{1},-k_{2}\right) \exp \left(\mathrm{i}\left[-k_{2} x_{1}+k_{1} x_{2}\right]\right) \\
& +A\left(-k_{1},-k_{2}\right) \exp \left(\mathrm{i}\left[-k_{1} x_{1}-k_{2} x_{2}\right]\right)+\left(\frac{k_{1}-k_{2}+\mathrm{i} c}{k_{1}-k_{2}-\mathrm{i} c}\right) A\left(-k_{1},-k_{2}\right) \exp \left(\mathrm{i}\left[-k_{2} x_{1}-k_{1} x_{2}\right]\right) \tag{F.3}
\end{align*}
$$

We can use the hardwall boundary condition now,

$$
\psi\left(0, x_{2}\right)=0=\psi\left(x_{1}, L\right)
$$

Using the first part and equating the coefficients of the exponents, we get

$$
\begin{aligned}
A\left(k_{1}, k_{2}\right) & =A\left(-k_{1}, k_{2}\right), \\
A\left(k_{1}, k_{2}\right)\left(\frac{k_{1}-k_{2}-\mathrm{i} c}{k_{1}-k_{2}+\mathrm{i} c}\right) & =A\left(-k_{1}, k_{2}\right)\left(\frac{k_{1}+k_{2}-\mathrm{i} c}{k_{1}+k_{2}+\mathrm{i} c}\right), \\
A\left(-k_{1}, k_{2}\right)\left(\frac{k_{1}+k_{2}+\mathrm{i} c}{k_{1}+k_{2}-\mathrm{i} c}\right) & =A\left(-k_{1},-k_{2}\right)\left(\frac{k_{1}-k_{2}+\mathrm{i} c}{k_{1}-k_{2}-\mathrm{i} c}\right), \\
A\left(k_{1},-k_{2}\right) & =A\left(-k_{1},-k_{2}\right) .
\end{aligned}
$$

Writing everything in terms of $A\left(k_{1}, k_{2}\right)$,

$$
\begin{align*}
\psi\left(x_{1}, x_{2}\right)=A\left(k_{1}, k_{2}\right)[ & \left(k_{1}-k_{2}+\mathrm{i} c\right)\left(k_{1}+k_{2}-\mathrm{i} c\right) \exp \left(\mathrm{i}\left[k_{1} x_{1}+k_{2} x_{2}\right]\right) \\
& +\left(k_{1}-k_{2}-\mathrm{i} c\right)\left(k_{1}+k_{2}-\mathrm{i} c\right) \exp \left(\mathrm{i}\left[k_{2} x_{1}+k_{1} x_{2}\right]\right) \\
& -\left(k_{1}-k_{2}+\mathrm{i} c\right)\left(k_{1}+k_{2}-\mathrm{i} c\right) \exp \left(\mathrm{i}\left[-k_{1} x_{1}+k_{2} x_{2}\right]\right) \\
& -\left(k_{1}+k_{2}-\mathrm{i} c\right)\left(k_{1}-k_{2}+\mathrm{i} c\right) \exp \left(\mathrm{i}\left[k_{2} x_{1}-k_{1} x_{2}\right]\right) \\
& -\left(k_{1}+k_{2}+\mathrm{i} c\right)\left(k_{1}-k_{2}-\mathrm{i} c\right) \exp \left(\mathrm{i}\left[k_{1} x_{1}-k_{2} x_{2}\right]\right) \\
& -\left(k_{1}-k_{2}-\mathrm{i} c\right)\left(k_{1}+k_{2}-\mathrm{i} c\right) \exp \left(\mathrm{i}\left[-k_{2} x_{1}+k_{1} x_{2}\right]\right) \\
& +\left(k_{1}+k_{2}+\mathrm{i} c\right)\left(k_{1}-k_{2}-\mathrm{i} c\right) \exp \left(-\mathrm{i}\left[k_{1} x_{1}+k_{2} x_{2}\right]\right) \\
& \left.+\left(k_{1}-k_{2}-\mathrm{i} c\right)\left(k_{1}+k_{2}-\mathrm{i} c\right) \exp \left(-\mathrm{i}\left[k_{2} x_{1}+k_{1} x_{2}\right]\right)\right] \tag{F.4}
\end{align*}
$$

Now, moving on to the second part of the hardwall boundary condition i.e. $\psi\left(x_{1}, L\right)=0$,

$$
\begin{aligned}
& \exp \left(\mathrm{i} k_{1} L\right)\left(k_{1}-k_{2}-\mathrm{i} c\right)\left(k_{1}+k_{2}-\mathrm{i} c\right)-\exp \left(-\mathrm{i} k_{1} L\right)\left(k_{1}+k_{2}+\mathrm{i} c\right)\left(k_{1}-k_{2}+\mathrm{i} c\right)=0 \\
& \exp \left(\mathrm{i} k_{2} L\right)\left(k_{1}-k_{2}+\mathrm{i} c\right)\left(k_{1}+k_{2}-\mathrm{i} c\right)-\exp \left(-\mathrm{i} k_{2} L\right)\left(k_{1}+k_{2}+\mathrm{i} c\right)\left(k_{1}-k_{2}-\mathrm{i} c\right)=0 .
\end{aligned}
$$

The Bethe equation come out as

$$
\begin{align*}
& \exp \left(\mathrm{i} 2 k_{1} L\right)=\frac{\left(k_{1}-k_{2}+\mathrm{i} c\right)\left(k_{1}+k_{2}+\mathrm{i} c\right)}{\left(k_{1}-k_{2}-\mathrm{i} c\right)\left(k_{1}+k_{2}-\mathrm{i} c\right)}  \tag{F.5}\\
& \exp \left(\mathrm{i} 2 k_{2} L\right)=\frac{\left(k_{2}-k_{1}+\mathrm{i} c\right)\left(k_{2}+k_{1}+\mathrm{i} c\right)}{\left(k_{2}-k_{1}-\mathrm{i} c\right)\left(k_{2}+k_{1}-\mathrm{i} c\right)} \tag{F.6}
\end{align*}
$$

This can be genralized for an N -particle system by carefully observing the relations between the coefficients.

## G. STRING SOLUTION OF TWO PARTICLES

Let us check whether the Bethe equations are satisfied by the string solutions (4.50) of the two particles.

$$
\begin{align*}
\mathrm{e}^{\mathrm{i} k_{1} L} & =\left(\frac{k_{1}-k_{2}+\mathrm{i} c}{k_{1}-k_{2}-\mathrm{i} c}\right)  \tag{G.1}\\
\mathrm{e}^{\mathrm{i} k_{2} L} & =\left(\frac{k_{2}-k_{1}+\mathrm{i} c}{k_{2}-k_{1}-\mathrm{i} c}\right) \tag{G.2}
\end{align*}
$$

The solution as in 4.50 is given by

$$
\begin{equation*}
k_{1}=K / 2+\mathrm{i} c / 2 \quad k_{2}=K / 2-\mathrm{i} c / 2 \tag{G.3}
\end{equation*}
$$

Putting these in G. 1 and G.2, we get

$$
\begin{align*}
\mathrm{e}^{c L / 2} & =0  \tag{G.4}\\
\mathrm{e}^{-c L / 2} & =\infty \tag{G.5}
\end{align*}
$$

which are only satisfied when $L \rightarrow \infty$ or $c \rightarrow-\infty$.

## G. 1 Single string solution for 3 particles in a hardwall

A 3 particle solution can be taken as:

$$
\begin{equation*}
k_{1}=\alpha-\mathrm{i} \Lambda, \quad k_{2}=\alpha+\mathrm{i} \Lambda, \quad k_{3}=\alpha \tag{G.6}
\end{equation*}
$$

The Bethe equations with these momenta are:

$$
\begin{align*}
\mathrm{e}^{\mathrm{i} 2(\alpha-\mathrm{i} \Lambda) L} & =\left(\frac{2 \Lambda-c}{2 \Lambda+c}\right)\left(\frac{\Lambda-c}{\Lambda+c}\right)\left(\frac{2 \alpha+\mathrm{i} c}{2 \alpha-\mathrm{i} c}\right)\left(\frac{2 \alpha-\mathrm{i} \Lambda+\mathrm{i} c}{2 \alpha-\mathrm{i} \Lambda-\mathrm{i} c}\right)  \tag{G.7}\\
\mathrm{e}^{\mathrm{i} 2(\alpha+\mathrm{i} \Lambda) L} & =\left(\frac{2 \Lambda+c}{2 \Lambda-c}\right)\left(\frac{\Lambda+c}{\Lambda-c}\right)\left(\frac{2 \alpha+\mathrm{i} c}{2 \alpha-\mathrm{i} c}\right)\left(\frac{2 \alpha+\mathrm{i} \Lambda+\mathrm{i} c}{2 \alpha+\mathrm{i} \Lambda-\mathrm{i} c}\right)  \tag{G.8}\\
\mathrm{e}^{\mathrm{i} 2 \alpha L} & =\left(\frac{2 \alpha-\mathrm{i} \Lambda+\mathrm{i} c}{2 \alpha-\mathrm{i} \Lambda-\mathrm{i} c}\right)\left(\frac{2 \alpha+\mathrm{i} \Lambda+\mathrm{i} c}{2 \alpha+\mathrm{i} \Lambda-\mathrm{i} c}\right) \tag{G.9}
\end{align*}
$$

Putting the value from G. 9 in G. 7 and G.8,

$$
\begin{align*}
\mathrm{e}^{2 \Lambda L} & =\left(\frac{2 \Lambda-c}{2 \Lambda+c}\right)\left(\frac{\Lambda-c}{\Lambda+c}\right)\left(\frac{2 \alpha+\mathrm{i} c}{2 \alpha-\mathrm{i} c}\right)\left(\frac{2 \alpha+\mathrm{i} \Lambda-\mathrm{i} c}{2 \alpha+\mathrm{i} \Lambda+\mathrm{i} c}\right)  \tag{G.10}\\
\mathrm{e}^{-2 \Lambda L} & =\left(\frac{2 \Lambda-c}{2 \Lambda+c}\right)\left(\frac{\Lambda-c}{\Lambda+c}\right)\left(\frac{2 \alpha+\mathrm{i} c}{2 \alpha-\mathrm{i} c}\right)\left(\frac{2 \alpha-\mathrm{i} \Lambda-\mathrm{i} c}{2 \alpha-\mathrm{i} \Lambda+\mathrm{i} c}\right) \tag{G.11}
\end{align*}
$$

The only way G. 10 and G. 11 are consistent with each other is if,

$$
\begin{equation*}
\frac{(2 \alpha+\mathrm{i} c)^{2}}{(2 \alpha-\mathrm{i} c)^{2}}=\frac{(2 \alpha+\mathrm{i} c)^{2}+\Lambda^{2}}{(2 \alpha-\mathrm{i} c)^{2}+\Lambda^{2}} \tag{G.12}
\end{equation*}
$$

The only solution to the above equation is $\Lambda=0$ in which case there is no 3 string bound state solution.

