GENERIC DISTRACTIONS AND STRATA OF HILBERT SCHEMES DEFINED BY THE CASTELNUOVO-MUMFORD REGULARITY

by

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ABSTRACT

Consider the standard graded polynomial ring in n variables over a field k and fix the Hilbert function of a homogeneous ideal. In the nineties Bigatti, Hulett, and Pardue showed that the Hilbert scheme consisting of all the homogeneous ideals with such a Hilbert function contains an extremal point which simultaneously maximizes all the graded Betti numbers. Such a point is the unique lexsegment ideal associated to the fixed Hilbert function.

For such a scheme, we consider the individual strata defined by all ideals with Castelnuovo-Mumford regularity bounded above by m. In 1997 Mall showed that when k is of characteristic 0 there exists an ideal in each nonempty strata with maximal possible Betti numbers among the ideals of the strata. In chapter 4 of this thesis we provide a new construction of Mall's ideal, extend the result to fields of any characteristic, and show that these ideals have other extremal properties. For example, Mall's ideals satisfy an equation similar to Green's hyperplane section theorem.

The key technical component needed to extend the results of Mall is discussed in Chapter 3. This component is the construction of a new invariant called the distraction-generic initial ideal. Given a homogeneous ideal $I \subset S$ we construct the associated distraction-generic initial ideal, D-gin$_<(I)$, by iteratively computing initial ideals and general distractions. The result is a monomial ideal that is strongly stable in any characteristic and which has many properties analogous to the generic initial ideal of I.

1. INTRODUCTION

Throughout this thesis S will refer to the standard graded polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k. For a fixed Hilbert function $\mathbf{H}$ the Hilbert scheme consisting of all the homogeneous ideals of S with Hilbert function $\mathbf{H}$ will be denoted $\mathrm{Hilb}(\mathbf{H})$. Within the above Hilbert scheme one can consider the individual strata which are defined by the subsets of $\mathrm{Hilb}(\mathbf{H})$ whose elements have Castelnuovo-Mumford regularity bounded above by m. Such strata will be denoted $\mathrm{Hilb}(\mathbf{H}, m)$. One can also consider the Hilbert scheme and associated strata where ever element is saturated. Such schemes and strata will be denoted $\mathrm{Hilb}^{\mathrm{sat}}(\mathbf{H})$ and $\mathrm{Hilb}^{\mathrm{sat}}(\mathbf{H}, m)$, respectively. One of the motivations behind this work was to extend results of Mall proven in the early nineties. Mall showed that over fields of characteristic 0 when $\mathrm{Hilb}(\mathbf{H}, m)$ (respectively $\mathrm{Hilb}^{\mathrm{sat}}(\mathbf{H})$) is nonempty it contain an ideal with maximal possible Betti numbers among the ideals of the strata. In this thesis these results are extended to fields of any characteristic.

The organization of this thesis is as follows. Chapter 2 discusses preliminary information, such as Betti numbers, Hilbert functions and series, generic initial ideals, and distractions of monomial ideals. One of the key results recalled in this section is the following theorem of Bayer and Stillman with regards to the generic initial ideal.

Theorem 1.0.1 (Bayer-Stillman [1])**.** *Let $I \subset S$ be a graded ideal and $<$ the graded reverse lexicographical order. Then I and $gin_<(I)$ have the same depth, regularity, and projective dimension.*

Next, the first part of Chapter 3 discusses the interactions between the distractions of monomial ideals and initial ideals. The following is shown for a fixed monomial ideal I and a fixed upper triangular distraction matrix $\mathscr{T}$

Theorem 1.0.2. *Let $I \subset S$ be a fixed monomial ideal and $\mathscr{T}$ the fixed upper triangular distraction matrix. Then I and $in_{revlex}(D_{\mathscr{T}}(\bar{I}))$ have the same extremal Betti numbers, regularity, and projective dimension.*

The next portion of Chapter 3 uses the results on distractions to build the **distraction-generic initial ideal**. This distraction-generic initial ideal associates to a given homogeneous ideal I a monomial ideal, denoted $\text{D-gin}_<(I)$, that is strongly stable in any characteristic and has many properties similar to that of the generic initial ideal. The following is the main result of this chapter. It is an analogue of the theorem of Bayer-Stillman mentioned early.

Theorem 1.0.3. *Let I be a homogeneous ideal of S. Then I and $\text{D-gin}_<(I)$ have the same Hilbert function, the Betti numbers of $\text{D-gin}_<(I)$ are greater than or equal to the Betti numbers of I, and when using the graded reverse lexicographical order the depth, regularity, projective dimension, and extremal Betti numbers of I and $\text{D-gin}_<(I)$ are equal.*

Note that the contents of Chapter 3 are based on joint work with Giulio Caviglia. Lastly Chapter 4 focuses on proving that the results of Mall hold in any characteristic. The main result of Mall is the following

Theorem 1.0.4 (Mall [2])**.** *Assume k is a field of characteristic 0 and $\boldsymbol{H}$ a fixed Hilbert function. For all m, provided $\mathit{Hilb}(\boldsymbol{H}, m) \neq \emptyset$, there exists an ideal $J \in \mathit{Hilb}(\boldsymbol{H}, m)$ such that $\beta_{\mathrm{ij}}(J) \geq \beta_{\mathrm{ij}}(I)$ for all $\mathrm{i}, \mathrm{j} \in \mathbb{N}$ and all $I \in \mathit{Hilb}(\boldsymbol{H}, m)$.*

Mall also has an analogous result for $\mathrm{Hilb}^{\mathrm{sat}}(\mathbf{H}, m)$.

Theorem 1.0.5 (Mall [2])**.** *Assume k is a field of characteristic 0 and $\boldsymbol{H}$ a fixed Hilbert function. For all m, provided $\mathit{Hilb}^{sat}(\boldsymbol{H}, m) \neq \emptyset$, then there exists an ideal $J \in \mathit{Hilb}^{sat}(\boldsymbol{H}, m)$ such that $\beta_{\mathrm{ij}}(J) \geq \beta_{\mathrm{ij}}(I)$ for all $\mathrm{i}, \mathrm{j} \in \mathbb{N}$ and all $I \in \mathit{Hilb}^{sat}(\boldsymbol{H}, m)$.*

In the first part of Chapter 4 an explicit construction of the Mall ideal is defined starting from a strongly stable ideal I. This construction, $\mathrm{M}(I, m)$, is shown to be a strongly stable ideal with the same Hilbert function and larger Betti numbers than I. The ideal $\mathrm{M}(I, m)$ is then used to prove the main results of Chapter 4, namely the following.

Theorem 1.0.6. *Assume k is a field of any characteristic and $\boldsymbol{H}$ a fixed Hilbert function. For all m, provided $\mathit{Hilb}(\boldsymbol{H}, m) \neq \emptyset$, then there exists an ideal $J \in \mathit{Hilb}(\boldsymbol{H}, m)$ such that $\beta_{\mathrm{ij}}(J) \geq \beta_{\mathrm{ij}}(I)$ for all $\mathrm{i}, \mathrm{j} \in \mathbb{N}$ and all $I \in \mathit{Hilb}(\boldsymbol{H}, m)$ also in this set.*

also

Theorem 1.0.7. *Assume k is a field of any characteristic and $\boldsymbol{H}$ a fixed Hilbert function. For all m, provided $Hilb^{sat}(\boldsymbol{H}, m) \neq \emptyset$, then there exists an ideal $J \in Hilb^{sat}(\boldsymbol{H}, m)$ such that $\beta_{\mathsf{ij}}(J) \geq \beta_{\mathsf{ij}}(I)$ for all $\mathrm{i}, \mathrm{j} \in \mathbb{N}$ and all $I \in Hilb^{sat}(\boldsymbol{H}, m)$ also in this set.*

The proofs of both of the above statements require a reduction to the case of a strongly stable monomial ideal, which is done via the distraction-generic initial ideal.

2. PRELIMINARIES

In this chapter we will cover basic information, facts, and definitions related to graded rings, projective dimension, Betti numbers, local cohomology, Castelnuovo-mumford regularity, Hilbert functions, Gröbner bases, generic initial ideals, and distractions.

2.1 Graded Rings

For a short introduction to the subject of graded rings and algebras see [3].

Definition 2.1.1 ([3])**.** Let R be a commutative ring. R is graded over $\mathbb{Z}$ if there exists a family of subgroups $\{R_{\mathrm{i}}\}_{\mathrm{i}\geq 0}$ such that

1. As a direct sum of abelian groups $R = \oplus_{\mathrm{i}=0}^{\infty} R_{\mathrm{i}}$.

2. $R_{\mathrm{i}}R_{\mathrm{j}} \subseteq R_{\mathrm{i}+\mathrm{j}}$ for all $\mathrm{i}, \mathrm{j} \in \mathbb{Z}$.

Note that each R_{i} is an R-module. One can similarly define graded vector spaces and graded modules. A graded A-algebra refers to a graded ring R that is also an A-algebra. Next we need define what is meant by a homogeneous element and homogeneous submodule.

Definition 2.1.2. Let M be a graded module of R. An element $x \in M$ is homogeneous if $x \in M_{\mathrm{i}}$ for some i. A submodule $N \subseteq M$ is called homogeneous if it is generated by homogeneous elements.

We can also have graded homomorphisms between modules.

Definition 2.1.3. Let R be a graded ring and M, N graded R-modules. Let $f : M \to N$ be an R-module homomorphism. Then f is said to be a **graded homorphism of degree** i if $f(M_d) \subseteq N_{d+\mathrm{i}}$ for all d.

Definition 2.1.4. Let R be a graded ring and M, N graded R-modules. A homogeneous module homomorphism $f : N \to N$ is a graded module homorphism that maps every homogeneous element of degree d in N to a homogeneous element of degree d in M.

Note that homogeneous module homomorphisms are graded homomorphism of degree 0. For Noetherian rings we have the following theorem.

Theorem 2.1.5. *Let R be a commutative graded ring. Then R is a Noetherian ring if and only if R_0 is Noetherian and R is finitely generated as an R_0-algebra.*

Throughout this thesis we will mainly be concerned with the standard graded ring $S = k[x_1, \ldots, x_n]$ where k is a field. Note that S is called standard graded when S is a graded k-algebra with all the variables having degree one. We let $S_{\rm i}$ be the set consisting of all homogeneous polynomials of degree i, in particular $S_0 = k$. Thus a k-basis for $S_{\rm i}$ consists of all the monomials of degree i. Hence $S = \bigoplus_{\rm i} S_{\rm i}$.

2.2 Invariants of Graded Ideals

The purpose of this section is to give a short background on projective dimension, Betti numbers, local cohomology modules, Castelnuovo-Mumford regularity, and Hilbert function of ideals.

2.2.1 Projective Dimension

Recall that for a finitely generated module M of a ring R the projective resolution, $\mathbf{P}$, of M is an exact chain complex

$$\mathbf{P} : \ldots \to P_r \to \cdots \to P_1 \to P_0 \to M \to 0.$$

where every $P_{\rm i}$ is a projective module. When every $P_{\rm i}$ in the resolution are free modules, such a resolution is called a free resolution. A resolution is called graded when all the maps in the resolution are homogeneous module homomorphisms. A finite projective resolution of M is

$$\mathbf{P} : 0 \to P_t \to \cdots \to P_0 \to M \to 0$$

The **projective dimension** of M is defined as $\operatorname{projdim}(M) = t$, where t is the smallest integer such that the resolution is finite. If M does not have a finite projective resolution, then $\operatorname{projdim}(M) = \infty$.

Furthermore, as every free module is a projective module, in the case of polynomial rings over a field Hilbert's Syzygy Theorem gives us the following

Theorem 2.2.1. *Every finitely generated graded module M of the ring S has a graded free resolution of length less than or equal to n. Hence projdim$(M) \leq n$.*

In a sense, the projective dimension of M is a measure of how close M is to being a projective module. It follows from the definition that that $\operatorname{projdim}(M) = 0$ if and only if M is a projective module. As we are working over S the Quillen-Suslin Theorem tells us

Theorem 2.2.2 (Quillen-Suslin)**.** *Let P be a finitely generated projective module over S. Then P is free.*

Hence the projective dimension of M over S also gives a measurement of how close M is to being a free module of S.

2.2.2 Betti Numbers

Recall that for M, where M is any finitely generated graded S-module, you can construct a minimal graded free resolution. In other words there exist a resolution,

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \longrightarrow 0$$

where each F_{i} is a graded free module, the maps are graded of degree zero, and furthermore, these free modules can be chosen so that the resolution has minimal length. This minimal length can be shown to be equivalent to the fact for every i, the basis of $F_{\mathrm{i}+1}$ maps into a minimal system of generators of the kernel of the map f_{i}. Each free module F_{i} can be written as $F_{\mathrm{i}} = \bigoplus_{\mathrm{j}} S(-\mathrm{j})^{\beta_{\mathrm{ij}}}$. These β_{ij} are known as the graded Betti numbers of M. The β_{ij} in the above minimal resolution can be written in terms of the Tor functor which leads to the following definition. Note that $\operatorname{Tor}_{\mathrm{i}}^S(k, M)$ is also a graded S-module.

Definition 2.2.3. The graded Betti numbers of a finitely generated graded module M are defined to be $\beta_{\mathrm{ij}}^S(M) = \dim_k(\operatorname{Tor}_{\mathrm{i}}^S(k, M)_{\mathrm{j}})$.
The ith Betti number of M is defined to be $\beta_{\mathrm{i}}^S(M) = \sum_{\mathrm{j}} \beta_{\mathrm{ij}}$, which is equivalent to $\beta_{\mathrm{i}}^S(M) = \dim_k(\operatorname{Tor}_{\mathrm{i}}^S(k, M))$.

We will use the notation β_{ij} when it is clear from context which ring and module we are referring to. It is well known that the length of any minimal graded free resolution is equal to the projective dimension of M.

Betti numbers are often visually illustrated through the use of a Betti table, a table where the entry in the i-th column and j-th row is $\beta_{\mathrm{i,i+j}}$.

Betti table of a module M

	0	1	$\cdots$	i	$\cdots$	projdim(M)
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\cdots$	$\beta_{\mathrm{i,i}}$	$\cdots$	$\vdots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\cdots$	$\beta_{\mathrm{i,i+1}}$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
j	$\beta_{0,\mathrm{j}}$	$\beta_{1,\mathrm{j+1}}$	$\cdots$	$\beta_{\mathrm{i,i+j}}$	$\cdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$

When considering ideals of polynomial rings, both graded Betti numbers and Betti numbers are invariant under extensions of the base field. In other words, if $k \subset F$ is an extension of k to the field F one can extend the ring S to $\bar{S} = S \otimes_k F$. Every S-module M can be extended to a $\bar{S}$-module $\bar{M}$ in a similar manner. This operation is faithfully flat, hence under this extension the graded Betti numbers of M as an S-module are equal to those of $\bar{M}$ as an $\bar{S}$-module. For instance when k is finite you can extend to an infinite field $k \subset F$ but this extension maintains the equality between the Betti numbers of a module M of S and the Betti numbers of the module $\bar{M}$ of $\bar{S}$.

One important type of Betti numbers are the extremal Betti numbers defined by Bayer, Charalambous, and Popescu [4].

Definition 2.2.4. The graded Betti number $\beta_{\mathrm{i,i+j}} \neq 0$ of a finitely generated S-module M is called **extremal** if $\beta_{k,k+\ell} = 0$ for all pairs of integers $(k, \ell) \neq (\mathrm{i}, \mathrm{j})$ with $k \geq \mathrm{i}$ and $\ell \geq \mathrm{j}$.

In fact, the position of the extremal Betti numbers can be found by considering the outside corners of the nonzero graded Betti numbers when viewed on a Betti table. See [5] for an illustration of this.

2.2.3 Local Cohomology

Local cohomology was first introduced by Alexander Grothendieck. While there are multiple equivalent ways to define the local cohomology modules of a fixed module, we shall only mention one.

The local cohomology of a module M is always calculated with respect to an ideal, say $I \subseteq S$. In this thesis because we are considering the case of graded modules over polynomial rings we will use the homogeneous maximal ideal $I = m = (x_1, \ldots, x_n)$. To define local cohomology we need $\Gamma_I(M) = \{x \in M \mid \exists n \in \mathbb{N} \text{ such that } I^n x = 0\}$.

Definition 2.2.5. The ith local cohomology module of M, with respect to I, is $H^{\mathrm{i}}_I(M)$, where $H^{\mathrm{i}}_I(-)$ is the ith derived functor of Γ_I.

Remember that as Γ_I is a left exact functor there is the equivalence $H^0_I(M) = \Gamma_I(M)$. An important fact of the local cohomology is that the local cohomology can also be computed by using the radical of the ideal I.

Proposition 2.2.6. *Let I be an ideal of S. Then $H^{\mathrm{i}}_I(M) = H^{\mathrm{i}}_{\sqrt{I}}(M)$.*

Notice that the above proposition means that if I and J are two ideals with $\sqrt{I} = \sqrt{J}$ then $H^{\mathrm{i}}_I(M) = H^{\mathrm{i}}_J(M)$. Furthermore, as we are in a Noetherian ring we know for two ideals $I, J \subseteq S$ that $\sqrt{I} = \sqrt{J}$ if and only if $I^n \subseteq J$ for some $n \in \mathbb{N}$ and $J^r \subseteq I$ for some $r \in \mathbb{N}$. For monomial ideals in S this gives us an easy way to tell when the local cohomology modules coincide.

2.2.4 Castelnuovo-Mumford Regularity

The Castelnuovo-Mumford was first used, although not explicitly defined, by Castelnuovo in [6].The first explicit definition came from Mumford (see [7]), which led to this invariant being known as the Castelnuovo-Mumford regularity. We shall first introduce a definition of regularity for modules.

Definition 2.2.7. Let M be a finitely generated graded S-module. The regularity of M is

$$\operatorname{reg}(M) = \max\{\mathrm{j} : H^{\mathrm{i}}_{m}(M)_{\mathrm{j}-\mathrm{i}} \neq 0 \text{ for some index i}\}.$$

Note in the above definition that $H^{\mathrm{i}}_{m}(-)$ is the local cohomology taken with respect to the homogeneous maximal ideal of S. Furthermore, as we are in the case of a graded polynomial ring the local cohomology modules are graded S-modules which means that $H^{\mathrm{i}}_{m}(M)_{\mathrm{j}-\mathrm{i}}$ is well defined. In the case of M being a finitely generated graded S module there is an equivalent definition of the regularity of M where one replaces $H^{I}_{m}(M)_{\mathrm{j}-1}$ with $\mathrm{Tor}_{\mathrm{i}}(k, M)_{\mathrm{i}+\mathrm{j}}$. This in turn gives us the following definition for the regularity of an S-module in terms of its Betti numbers.

Definition 2.2.8 (Castelnuovo-Mumford Regularity)**.** Let M be a finitely generated graded module of S. The regularity of M is $\operatorname{reg}(I) = \max\{\mathrm{j} : \beta_{\mathrm{i},\mathrm{i}+\mathrm{j}}(M) \neq 0 \text{ for some i}\}$.

Notice that in our example of the Betti table from earlier this definition of regularity allows us to say that the last row of the Betti table of a module M will occur at the regularity of M.

2.2.5 Hilbert Functions and Polynomials

Recall that the polynomial ring S is a standard graded ring where the degree of $x_{\mathrm{i}} = 1$ for all $1 \leq \mathrm{i} \leq n$ and S can be written as $S = \bigoplus_{\mathrm{i}\geq 0} S_{\mathrm{i}}$ where each S_{i} is the space of polynomials in degree i. Every finitely generated graded S-module can be decomposed in the same way, i.e. $M = \bigoplus_{\mathrm{i}\geq 0} M_{\mathrm{i}}$ where each graded component of M is a finite-dimensional k-vector space. The Hilbert function is then defined by using the dimension of these vector spaces.

Definition 2.2.9. The **Hilbert function** of the module M, $HF_M(-)$ is a function on $\mathbf{Z}$ where $HF_M(\mathrm{i}) = \dim_k(M_{\mathrm{i}})$ as a k-vector space.

From the Hilbert function comes the Hilbert series.

Definition 2.2.10. The **Hilbert series** is the formal Laurent series

$$HS_M(t) = \sum_{\mathrm{i}\in\mathbb{Z}} HF_M(\mathrm{i})t^{\mathrm{i}}.$$

An important fact is that for large enough degree the Hilbert function behaves as a polynomial. For a standard graded k-algebra the following result is well known.

Theorem 2.2.11. *Let k be a field, R a standard graded k-algebra, and M a nonzero finitely generated graded R-module of dimension d. Then*

- *There exists a Laurent polynomial of $\mathbb{Z}[t, t^{-1}]$, denoted $Q_M(t)$, where $Q_M(1) > 0$ and $HS_M(t) = \frac{Q_M(t)}{(1-t)^d}$.*
- *There exists a polynomial in $\mathbb{Q}[x]$, denoted $P_M(x)$, called the **Hilbert polynomial**, such that $HF_M(\mathrm{i}) = P_M(\mathrm{i})$ for all $\mathrm{i} > \deg(Q_M) - d$.*

In the case of the polynomial ring, the Hilbert function and Hilbert series of the ring can be written easily.

Theorem 2.2.12. *Let $S = k[x_1, \ldots, x_n]$ with the standard grading.*

- *HF_S is a polynomial function of degree $n-1$ where $HF_S(\mathrm{i}) = \binom{n+\mathrm{i}-1}{n-1}$.*
- *$HS_S(t) = \frac{1}{(1-t)^n}$.*

Another important property of Hilbert functions and Hilbert series is that they are additive on short exact sequences.

2.3 Gröbner Bases

To understand computations that involve monomial ideals in polynomial rings one has to first understand monomial orders and Gröbner bases.

Definition 2.3.1 ([5])**.** A monomial order on S is a total order $<$ on the monomials of S such that $1 < u$ for every monomial $u \in S$ not equal to 1, and if $u, v \in S$ are monomials with $u < v$ then $uw < vw$ for every monomial $w \in S$.

While any ordering of the variables in S is allowed, we will follow convention and only consider monomial orders such that $x_1 > x_2 > \cdots > x_n$. Two particular monomial orderings that will be important later are the graded lexicographic order and the graded reverse lexicographic order.

Definition 2.3.2 (Graded lexicographic order (lex))**.** We let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ where $a_{\mathrm{i}}, b_{\mathrm{i}} \in \mathbb{Z}_{\geq 0}$ for every i. Let $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ and $x^{\mathbf{b}} = x_1^{b_1} \cdots x_n^{b_n}$. The **graded lexicographic order** (lex for short) is defined by setting $x^{\mathbf{a}} <_{\mathrm{lex}} x^{\mathbf{b}}$ if either (1) $\sum_{\mathrm{i}=1}^{n} a_{\mathrm{i}} < \sum_{\mathrm{i}=1}^{n} b_{\mathrm{i}}$ or (2) $\sum_{\mathrm{i}=1}^{n} a_{\mathrm{i}} = \sum_{\mathrm{i}=1}^{n} b_{\mathrm{i}}$ and the leftmost nonzero component of the vector $\mathbf{a} - \mathbf{b}$ is negative.

Definition 2.3.3 (Graded reverse lexicographic order (revlex))**.** Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ where $a_{\mathrm{i}}, b_{\mathrm{i}} \in \mathbb{Z}_{\geq 0}$ for every i. Let $x^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ and $x^{\mathbf{b}} = x_1^{b_1} \cdots x_n^{b_n}$. The **Graded reverse lexicographic order** (revlex for short) is defined by setting $x^{\mathbf{a}} <_{\mathrm{revlex}} x^{\mathbf{b}}$ if either (1) $\sum_{\mathrm{i}=1}^{n} a_{\mathrm{i}} < \sum_{\mathrm{i}=1}^{n} b_{\mathrm{i}}$ or (2) $\sum_{\mathrm{i}=1}^{n} a_{\mathrm{i}} = \sum_{\mathrm{i}=1}^{n} b_{\mathrm{i}}$ and the rightmost nonzero component of the vector $\mathbf{a} - \mathbf{b}$ is a positive number.

One of the first steps towards standardizing the concept of monomial orders was done by Macaulay in [8]. By ordering the monomials in a polynomial ring, Macaulay was able to compare graded ideals to monomial ideals and make conclusions about the Hilbert function of the graded ideals. Macaulay's work then influenced Gröbner who used Macaulay's work to explicitly compute bases of certain rings. Gröbner's then student, Buchberger, further expanded the computation of bases using monomial orderings in his thesis (see [9] for an English translation of this thesis). Buchberger came up with an criterion and an algorithm, now known as Buchberger's algorithm, that allowed for the explicit computation of bases of ideals in polynomial rings. These bases are now known as Gröbner bases and are a key tool in the study of ideals of polynomial rings. For more history on the subject see the notes in [5] and [10]. We will now give a short introduction to Gröbner bases using the following definitions from [5].

Definition 2.3.4. Let $f \in S$ be a nonzero polynomial. The **initial monomial** of f with respect to $<$, denoted $\mathrm{in}_<(f)$, is the biggest monomial ordered with respect to $<$ among the monomials belonging to the support of f.

This leads to the following definition of the initial ideal of any given ideal $I \subseteq S$.

Definition 2.3.5. Let $I \subseteq S$ be a nonzero ideal. The **initial ideal** of I with respect to $<$ is defined as

$$\text{in}_<(I) = (\{\text{in}_<(I) : 0 \neq f \in I\}).$$

Gröbner bases can then be defined in terms of the initial ideal.

Definition 2.3.6 ([5])**.** Let I be a nonzero ideal of S. A finite set of nonzero polynomials $\{g_1, \ldots, g_s\}$, $g_i \in I$, is a **Gröbner basis** of I with respect to $<$ if the ideal $\text{in}_<(I)$ is generated by the monomials $\{\text{in}_<(g_1), \ldots, \text{in}_<(g_s)\}$.

A key fact is that for every ideal I and every monomial order $<$, a Gröbner basis with respect to $<$ of I will always exist. This means that problems about ideals in polynomial rings can often be reduced to the case of monomial ideals. Furthermore, you can explicitly find a Gröbner basis of an ideal with respect to a given monomial order through the use of a process known as Buchberger's algorithm. A primary step within Buchberger's algorithm is the computation of polynomials known as S-polynomials.

Definition 2.3.7 ([5])**.** For two nonzero polynomials $f, g \in S$ the **S-polynomial** is the polynomial

$$S(f,g) = \frac{\text{lcm}(\text{in}_<(f), \text{in}_<(g))}{c_f \text{in}_<(f)} f - \frac{\text{lcm}(\text{in}_<(f), \text{in}_<(g))}{c_g \text{in}_<(g)} g$$

where c_f and c_g are the coefficients of $\text{in}_<(f)$ in f and $\text{in}_<(g)$ in g, respectively.

Recall that the division algorithm over S allows you to write any nonzero polynomial $f \in S$ with respect to a set on nonzero polynomials $g_1, \ldots, g_r \in S$ as $f = f_1 g_1 + f_2 g_2 + \cdots + f_r g_r + f'$ where $f' \in S$ and the f_i's satisfy the following conditions:

1. if f' is a nonzero polynomial then none of the monomials in the support of f' belong to the ideal $(\text{in}_<(g_1), \ldots, \text{in}_<(g_r))$ and

2. if f_i is nonzero then there is the inequality $\text{in}_<(f) \geq \text{in}_<(f_i g_i)$.

Such an expression for f is called a **standard expression** of f with respect to the g_i's with f' as the remainder. Alternatively, one can say that f **reduces** to f' with respect to the

g_{i}'s.

Remember that every algorithm needs a stopping criterion, and Buchberger's Algorithm is no different. This leads to Bucherger's Criterion, which outlines when a system of generators of an ideal is a Gröbner basis for that ideal.

Theorem 2.3.8 (Buchberger's Criterion)**.** *Let I be a nonzero ideal of $k[x_1, \ldots, x_n]$ and $G = \{g_1, \ldots, g_s\}$ a system of generators of I. Then G is a Gröbner basis of I if and only if the following condition is satisfied:*

- *For all* $\mathrm{i} \neq \mathrm{j}$*, the S-polynomial* $S(g_{\mathrm{i}}, g_{\mathrm{j}})$ *reduces to 0 with respect to* $g_1, \ldots, g_s$*.*

With this criterion in place we can now discuss Buchberger's algorithm which allows you to start with the generators of an ideal and find a Gröbner basis for that ideal with respect to a given term order. The process is straightforward.

Start with a set $\{g_{\mathrm{i}}\}_{1 \leq \mathrm{i} \leq s}$ of generators for a given ideal $I \subseteq S$.

- Compute the S-polynomials $S(g_{\mathrm{i}}, g_{\mathrm{j}})$ for every index i and j.
- If all these polynomials reduce to 0 with respect to $\{g_{\mathrm{i}}\}_{1 \leq \mathrm{i} \leq s}$, then the Buchberger criterion tells us that our system of generators was a Gröbner basis. If they don't all reduce to 0, then one of the polynomials $S(g_{\mathrm{i}}, g_{\mathrm{j}})$ has a nonzero remainder.
- Label this remainder as g_{s+1}, add it to the list of generators, and repeat the above process until all the S-polynomials reduce to 0 with respect to the current list of polynomials.

As we are working over a Noetherian ring the Buchberger algorithm will always give a Gröbner basis in a finite number of steps. To see this, notice that each step of the algorithm gives the strict inclusion of monomial ideals $(\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s)) \subsetneq (\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_s), \mathrm{in}_<(g_{s+1}))$. If this process did not terminate we would have an infinite ascending chain of ideals, which is not possible in a Noetherian ring.

The fact that the Buchberger algorithm will always return a Gröbner basis in a finite number of steps is what allows one to always find a Gröbner basis with respect to a given monomial order for a set ideal. Furthermore, notice that extending the base field via a flat extension of scalars will not change any of the initial monomials or coefficients that appear in Buchberger's algorithm. Hence if $\bar{k}$ is a field extension of k and $I \subseteq S \subseteq \bar{S}$ then the Gröbner basis of I with respect to a given order will calculated over S will also be a Gröbner basis of the ideal $I\bar{S} \subseteq \bar{S}$.

2.4 Invariants and Initial Ideals

When one wishes to study upper bounds for the homological invariants of ideals $I \subseteq S$ one can often reduce to the case of monomial ideals. Such a reduction allows one to take advantage of the combinatorial properties of monomial ideals. To illustrate how this reduction works, we shall consider initial ideals. The first important property of initial ideals has to do with the Hilbert function.

Proposition 2.4.1 (Macaulay)**.** *Let I be a graded ideal of S and $<$ a monomial order. Then S/I and $S/in_<(I)$ have the same Hilbert function.*

Furthermore, by reducing to the initial ideal we have the following equalities and upper bounds of invariants. The proof the result below is a standard argument that uses the upper semicontinuity of those invariants in flat families.

Proposition 2.4.2. *Let I be a homogeneous ideal of S and $<$ a monomial order. Then*

- $\beta_{\mathrm{ij}}(I) \leq \beta_{\mathrm{ij}}(in_<(I))$ *for all* i *and* j.
- $dim(S/I) = dim(S/in_<(I))$.
- $HF(S/I) = HF(S/in_<(I))$.
- $projdim(S/I) \leq projdim(S/in_<(I))$.
- $reg(S/I) \leq reg(S/in_<(I))$.
- $depth(S/I) \geq depth(S/in_<(I))$.

As a corollary one has:

Corollary 2.4.3. *Let I be a homogeneous ideal of S. If $S/in_<(I)$ is Cohen-Macaulay then S/I is Cohen-Macaulay. If $S/in_<(I)$ is Gorenstein then S/I is Gorenstein.*

2.5 Generic Initial Ideals

When working over polynomial rings it is common to reduce the case of an arbitrary ideal to case of monomial ideals. Such reductions can be done via the initial ideal but sometimes it is possible to reduce even further. In fact when working with monomial ideals over polynomial rings, the best case scenario is a reduction to a strongly stable ideal due to the useful combinatorial properties of such ideals. Over fields of characteristic 0, this is typically done via a reduction to the generic initial ideal.

Theorem 2.5.1 ([5]). *Assume $|k| = \infty$. Let $I \subseteq S$ be a homogeneous ideal and $<$ a monomial order on S. Then there exists a nonempty Zariski open subset $U \subseteq GL_n(k)$ such that $in_<(\alpha I) = in_<(\alpha' I)$ for all $\alpha, \alpha' \in U$.*
*The ideal $in_<(\alpha I)$ is called the **generic initial ideal** of I with respect to $<$ and will be denoted $gin_<(I)$.*

For a definition of generic initial ideals when $|k| < \infty$ see the remark after Definition 2.1 in [11] or the discussion in chapter 3 of this thesis after Theorem 3.1.9. Generic initial ideals are always Borel fixed (see [12] and [1]), i.e. fixed under the action of the Borel subgroup of $\mathrm{GL}_n(k)$. When the characteristic of k is 0, a monomial ideal being Borel fixed is equivalent to that monomial ideal being strongly stable.

Definition 2.5.2. A monomial ideal $I \subset S$ is strongly stable if $x_{\mathrm{i}}(u/x_{\mathrm{j}}) \in I$ for all monomials $u \in I$ and all $\mathrm{i} < \mathrm{j}$ such that x_{j} divides u.

To illustrate this property, consider the following example.

Example 2.5.3. Consider the polynomial ring $k[x_1, x_2, x_3]$, k any characteristic. The ideal $(x_1x_3) \subset k[x_1, x_2, x_3]$. Then $x_2(x_1x_3/x_3) = x_1x_2 \notin (x_1x_3)$ which means that this ideal is not strongly stable.
The ideal $(x_1^2, x_1x_2, x_1x_3) \subset k[x_1, x_2, x_3]$ is strongly stable, however.

Note that when checking for strong stability, it is enough to check for the property on the generators of an ideal.

Strong stability and being Borel fixed are not equivalent over fields of characteristic p. While a strongly stable ideal will always be Borel fixed, when the characteristic of k is not 0 there will be Borel fixed ideals that are not strongly stable (see [10]).

Example 2.5.4. Consider the polynomial ring $k[x_1, x_2]$ where the characteristic of k is 2. Then the ideal (x_1^2, x_2^2) is Borel fixed but is not strongly stable.

Thanks to Bayer and Stillman the following is known about the homological invariants of generic initial ideals with respect to the revlex order.

Theorem 2.5.5 (Bayer-Stillman [1]). *Let $I \subseteq S$ be a graded ideal and "$<$" the revlex order. Then*

1. *$projdim(I) = projdim(gin_<(I))$.*

2. *$depth(S/I) = depth(S/gin_<(I))$.*

3. *S/I is Cohen-Macaulay if and only if $S/gin_<(I)$ is Cohen-Macaulay.*

4. *$reg(I) = reg(gin_<(I))$.*

2.6 Distractions and Polarizations

We first recall the definition of a distraction matrix and distraction of an ideal from Definition 2.1 and Corollary 2.10 of [13].

Definition 2.6.1 (Distraction Matrix [13]). We let $\mathscr{L} = (\ell_{\mathrm{ij}} \mid \mathrm{i} = 1, \ldots, n, \mathrm{j} \in \mathbb{Z}_{>0})$ be an infinite matrix with entries $\ell_{\mathrm{ij}} \in S_1$ with the following two properties:

1. The equality $\langle \ell_{1\mathrm{j}_1}, \ldots, \ell_{n\mathrm{j}_n} \rangle = S_1$ holds for every $\mathrm{j}_1, \ldots, \mathrm{j}_n \in \mathbb{Z}_{>0}$.

2. There exists an integer $N \in \mathbb{Z}_{>0}$ such that $\ell_{\mathrm{ij}} = \ell_{\mathrm{i}N}$ for every $\mathrm{j} > N$.

Then $\mathscr{L}$ is a **distraction matrix**.

In the next chapter of this thesis we will need a very particular type of distraction matrix which corresponds to upper triangular changes of coordinates. Therefore we define a triangular distraction matrix as follows.

Definition 2.6.2 (Triangular Distraction)**.** A distraction matrix $\mathscr{L}$ is **triangular** if for every i, j, the support of ℓ_{ij} is contained in $\{x_1, \ldots, x_{\mathrm{i}}\}$.

Remark 2.6.3. It follows from the definition that a triangular distraction matrix will also have the property that $x_{\mathrm{i}} \in \operatorname{supp}(\ell_{\mathrm{ij}})$ for all i and all j.

For any distraction matrix it is important to understand how the matrix acts on the individual monomials of an ideal $I \subseteq S$. Definition 2.2 of [13] states the following.

Definition 2.6.4 ([13])**.** Let $\mathscr{L}$ be a distraction matrix and $t = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Then $D_{\mathscr{L}}(t) = \prod_{\mathrm{i}=1}^{n} \left(\prod_{\mathrm{j}=1}^{a_{\mathrm{i}}} \ell_{\mathrm{ij}} \right)$ is called the **$\mathscr{L}$-distraction** of t.

Note that the above definition of a distraction of a monomial allows you to define the distraction of a vector subspace of S, which can then be used to justify the definition of a distraction of a monomial ideal as follows

Definition 2.6.5 ([13])**.** For a graded monomial ideal $I \subseteq S$ the distraction of I with respect to a distraction matrix $\mathscr{L}$ is

$$D_{\mathscr{L}}(I) = \oplus_d D_{\mathscr{L}}(I_d).$$

Several key properties of distractions are stated below. These statements can be found in Corollary 2.10 and Corollary 2.20 of [13].

Theorem 2.6.6 ([13])**.** *Let $\mathscr{L}$ be a distraction matrix and I a monomial ideal in S. Then*

1. *$D_{\mathscr{L}}(I)$ is a homogeneous ideal of S.*

2. *If $I = (t_1, \ldots, t_r)$ then $D_{\mathscr{L}}(I) = (D_{\mathscr{L}}(t_1), \ldots, D_{\mathscr{L}}(t_r))$.*

3. *$HF(I) = HF(D_{\mathscr{L}}(I))$.*

4. *The graded Betti numbers of I and $D_{\mathscr{L}}(I)$ are the same.*

Since the previous theorem gives equality between the Hilbert function and Betti numbers of an ideal and the distraction of the ideal, it gives information about the invariants that can be computed by using the Hilbert function and Betti numbers. In particular, extremal Betti numbers, projective dimension, and Castelnuovo-Mumford regularity will be the same for both an ideal and its distraction. It is worth noting that the distraction of a monomial ideal is related to the concept of the polarizing a monomial ideal and then specializing it. This allows the previous theorem to be justified using polarizations.

Definition 2.6.7 (Polarization [5]). Let $I \subseteq S$ be a monomial ideal where $I = (u_1, \ldots, u_r)$ is a minimal generating set of monomials. Write $u_{\mathrm{i}} = \prod_{\mathrm{j}=1}^{n} x_{\mathrm{j}}^{a_{\mathrm{ij}}}$ for $1 \leq \mathrm{i} \leq r$. For each $1 \leq \mathrm{j} \leq n$ choose α_{j} as $\alpha_{\mathrm{j}} = \max\{a_{\mathrm{ij}} : 1 \leq \mathrm{i} \leq r\}$. Define S^p to be the polynomial ring

$$S^p = S[x_{11}, x_{12}, \ldots, x_{1\alpha_1}, x_{21}, \ldots, x_{2\alpha_2}, \ldots, x_{n1}, \ldots, x_{n\alpha_n}].$$

Let $P(I) \subseteq S^p$ be the squarefree monomial ideal $P(I) = (v_1, \ldots, v_r)$ where $v_{\mathrm{i}} = \prod_{\mathrm{j}=1}^{n} \prod_{t=1}^{a_{\mathrm{ij}}} x_{\mathrm{j}t}$ for $1 \leq \mathrm{i} \leq r$. Then v_{i} is the polarization of u_{i} and $P(I)$ is the polarization of I.

Remark 2.6.8. As before we let $\mathrm{HF}(-)$ denote the Hilbert series. It is well known that $HS\,(S^p/P(I)) = HS\,(S/I)\,\frac{1}{(1-z)^{\sum \alpha_{\mathrm{i}}}}$. It then follows from part two of Theorem 2.6.6 that $x_{\mathrm{ij}} - \ell_{\mathrm{ij}}$ is a regular sequence for $S^p/P(I)$. Furthermore we see that $D_{\mathscr{L}}(I)$ is just the image of $P(I)$ after modding out such a regular sequence.

3. DISTRACTION-GENERIC INITIAL IDEALS

3.1 General and Generic Distractions

In this section we study generic distractions and the interaction of initial ideals with such operations. In doing so we will construct ideals associated to a given ideal with properties similar to those of generic initial ideals. Note that some of the results of this chapter are taken from [14].

In order to focus the discussion and simplify notation for this section we will fix a monomial ideal $I \subseteq S$, where I is generated in degree d or less, and set $M \gg d$. While in this section it will be sufficient to have $M \geq d$, in later sections M will be defined in terms of the regularity of $\operatorname{lex}(I)$. One reason for fixing M as such is given in the next remark.

Remark 3.1.1. If a monomial ideal J is generated in degree less than or equal to d and $\mathscr{L}$, $\mathscr{L}'$ are two distraction matrices agreeing on the first M columns, where $M \geq d$, then clearly $D_{\mathscr{L}}(J) = D_{\mathscr{L}'}(J)$.

The distraction matrices defined further on in this section depend on M. This choice of M allows us to treat the distraction matrices considered here as finite matrices.

Notation 3.1.2. We extend the field k by setting

$$k^* = k\left(\alpha_{\mathrm{i},\mathrm{j},h} \mid 1 \leq \mathrm{i} \leq n, 1 \leq \mathrm{j} \leq M, 1 \leq h \leq n\right)$$

and then extend the scalars of S by defining the new ring

$$S^* = S \otimes_k k^*.$$

Our ideal I can then be extended to S^* by setting

$$I^* = I \otimes_k k^* = IS^* \subseteq S^*.$$

We define the **generic distraction matrix** $\mathscr{G}$ and the **generic triangular distraction matrix** $\mathscr{T}$ over k^* as follows:

$$\mathscr{G} = \Big(G|X\Big) \quad \text{and} \quad \mathscr{T} = \Big(T|X\Big),$$

where

$$G = [g_{\mathrm{ij}}]_{\mathrm{i}=1,\mathrm{j}=1}^{n,M}, \quad g_{\mathrm{ij}} = \sum_{h=1}^{n} \alpha_{\mathrm{i,j},h} x_h,$$

$$T = [t_{\mathrm{ij}}]_{\mathrm{i}=1,\mathrm{j}=1}^{n,M}, \quad t_{\mathrm{ij}} = \sum_{h=1}^{\mathrm{i}} \alpha_{\mathrm{i,j},h} x_h,$$

$$\text{and } X = [\chi_{\mathrm{ij}}]_{\mathrm{i}=1,\mathrm{j}=1}^{n,\infty} \text{ where } \chi_{\mathrm{ij}} = x_{\mathrm{i}}.$$

Note that the behavior of the above distraction matrices will depend strictly on the first M columns of the matrix. As such, when referencing the action of just the first M columns of $D_{\mathscr{G}}$ and $D_{\mathscr{T}}$ on an ideal we will use D_G and D_T, respectively.

Recall that $<$ indicates a monomial order such that $x_1 > x_2 > \cdots > x_n$. If a particular monomial order is needed, it will be specified in the theorem statement. It is important to note that the generic triangular matrix $\mathscr{T}$ is needed in the next section in place of the generic distraction matrix $\mathscr{G}$. This is because when ever $J \subseteq S^*$ is a strongly stable monomial ideal and $<$ the lexicographical monomial order, then $\mathrm{in}_<(D_{\mathscr{G}}(J))$ will always be different from J unless $J = \mathrm{lex}(J)$. In particular, for $r \gg 0$ it happens that $(\mathrm{in}_< \circ D_{\mathscr{G}})^r(J)$ will always be $\mathrm{lex}(J)$. This follows from Pardue's proof of the Bigatti-Hulett-Pardue inequality on Betti numbers in [15].

Given the generic triangular distraction matrix $\mathscr{T}$ a question then becomes how does the distraction of an ideal interact with the initial ideal. As mentioned in the previous chapter, one highly studied class of ideals is that of Borel fixed ideals. Recall that over fields of characteristic 0, Borel fixed is equivalent to being strongly stable. As this equivalence does not hold over fields of characteristic p, over such fields there are Borel fixed ideals that are not strongly stable. No matter the characteristic of the base field, however, the initial ideal of a Borel fixed ideal will always be a Borel fixed ideal. And further, the generic initial ideal is always Borel fixed. However, the distraction of a Borel fixed ideal is not generally Borel fixed even after taking an initial ideal.

Example 3.1.3. Let $I = (x_1^3, x_2^3) \subset k[x_1, x_2]$ where k is of characteristic 3 and $<$ is the graded reverse lexographic order. Note that I is a Borel fixed ideal. Let

$$\mathscr{D} = \begin{pmatrix} x_1 & x_1 & x_1 & x_1 & \cdots \\ x_2 & x_2 & x_1 + x_2 & x_2 & \cdots \end{pmatrix}.$$

Then $D_{\mathscr{D}}(I) = (x_1^3, x_2^2(x_1 + x_2))$ and $\mathrm{in}_<(D_{\mathscr{D}}(I)) = (x_1^3, x_1 x_2^2, x_2^5)$ which is not Borel fixed in characteristic 3.

Notice in the above distraction that although $\mathscr{D}$ is a triangular distraction, it is not a generic triangular distraction. The question then becomes under what conditions is the distraction of Borel fixed ideal Borel fixed and under what conditions the initial ideal of a distraction of Borel fixed ideal is also Borel fixed.
For what follows we need to define the concept of a Zariski open set of distraction matrices. Define $\mathbb{D}_{nM}(k)$ as the set all of all distraction matrices on $k[x_1, \ldots, x_n]$ where $\mathscr{L} \in \mathbb{D}_{nM}(k)$ has the property that $\ell_{\mathrm{ij}} = \ell_{\mathrm{i}M}$ for every j $> M$ and for all i. Let $\mathbb{T}_{nM} \subset \mathbb{D}_{nM}$ be the subset of all triangular distraction matrices in $\mathbb{D}_{nM}$.

Definition 3.1.4. Assume that k is an infinite field. Then $\mathbb{D}_{nM}(k)$ can be viewed as the affine space $\mathbb{A}^{n \cdot n \cdot M}(k)$ and $\mathbb{T}_{nM}$ as the affine space $\mathbb{A}^{\binom{n+1}{2} \cdot M}(k)$. We say that a property (p) **holds for a general distraction** if there exists a nonempty Zariski open set $\mathcal{U}$ of $\mathbb{D}_{nM}(k)$ (respectively, $\mathbb{T}_{nM}(k)$) such that the property (p) is constant on $\mathcal{U}$.

Theorem 3.1.5. *Under the assumptions of Definition 2.3, there exists a nonempty Zariski open set $\mathcal{U}$ of $\mathbb{T}_{nM}(k)$ (respectively $\mathbb{D}_{nM}(k)$) such that $in_<(D_{\mathscr{U}}(I))$ is constant as a function of $\mathscr{U}$ on $\mathcal{U}$ and that $in_<(D_{\mathscr{U}}(I))$ is equal to $in_<(D_{\mathscr{T}}(I^*)) \cap S$ (respectively $in_<(D_{\mathscr{G}}(I^{)}) \cap S$).*

Proof. We shall prove the above statement for $\mathbb{T}_{nM}(k)$ as the proof for $\mathbb{D}_{nM}(k)$ is similar. Compute a Gröbner basis for $\mathrm{in}_<(D_{\mathscr{T}}(I^*))$ using the Buchberger algorithm. Then collect all elements in k^* which appear as nonzero coefficients of every possible term in every possible polynomial used and needed in the Buchberger algorithm. Notice that there are only a finite number of such coefficients and as these coefficients live in k^* they are rational polynomial functions in the elements $\alpha_{\mathrm{ij}h}$. Denote the numerators and denominators of these coefficients as $p_1, \dots, p_q \in k[\alpha_{\mathrm{ij}h}]_{1\le \mathrm{i}\le n, 1\le \mathrm{j}\le M, 1\le h\le n}$. Now define $\mathcal{U} = \{\mathscr{U} \in \mathbb{T}_{nM}(k) \mid p_1(\underline{\mu}) \neq 0, \dots, p_q(\underline{\mu}) \neq 0\}$ where the ij entry of $\mathscr{U}$ is $\sum_{h=1}^{\mathrm{i}} \mu_{\mathrm{ij}h} x_h$ and $p(\underline{\mu})$ is the evaluation of $p \in k[\alpha_{\mathrm{ij}h}]_{1\le \mathrm{i}\le n, 1\le \mathrm{j}\le M, 1\le h\le n}$ at the $\mu_{\mathrm{ij}h}$'s. Now $\mathcal{U}$ is a nonempty Zariski open set since it is the nonzero locus of a finite set of nonzero polynomials. For every $\mathscr{U} \in \mathcal{U}$, consider $\mathrm{in}_<(D_{\mathscr{U}}(I^*))$ and its Gröbner basis. The Buchberger algorithm will produce the same initial ideal for any choice of $\mathscr{U} \in \mathcal{U}$ since the initial monomials of such a Gröbner basis will be the same as the initial monomials of the Gröbner basis of $D_{\mathscr{T}}(I^*)$. $\square$

Remark 3.1.6. Notice that the computation of $\mathrm{in}_<(D_{\mathscr{G}}(I^*))$ and $\mathrm{in}_<(D_{\mathscr{T}}(I^*))$ depend only on the characteristic of the base field k. This follows from the fact that the scalars used in such calculations are inside $\mathbb{Q}(\alpha_{\mathrm{ij}h})$ when $\mathrm{char}(k) = 0$ and inside $\mathbb{Z}_p(\alpha_{\mathrm{ij}h})$ when $\mathrm{char}(k) = p$.

We now need the following lemma.

Lemma 3.1.7. *Let $J \subseteq k[x_1, \dots, x_n]$ be a monomial ideal and $F \subseteq k$ a field, not necessarily infinite. Let $\mathcal{B}(k) \subset GL_n(k)$ be the Borel subgroup of $GL_n(k)$ and $\mathcal{B}(F) \subset GL_n(F)$ be the Borel subgroup of $GL_n(F)$. Then J is fixed under the action of $\mathcal{B}(F)$ if and only if J is fixed under the action of $\mathcal{B}(k)$.*

Proof. First, note that since $F \subseteq k$ we have that $\mathcal{B}(F) \subseteq \mathcal{B}(k)$. Thus one of the above implications is clearly satisfied. When $\mathrm{char}(k) = 0$ we have that $\mathrm{char}(F) = 0$. In characteristic 0 Borel fixed is equivalent to being strongly stable. Hence if J is fixed under $\mathcal{B}(F)$ then J is a strongly stable monomial ideal and is fixed under $\mathcal{B}(k)$.
When $\mathrm{char}(k) = p$ we have that $\mathrm{char}(F) = p$. We observe that the proof of Theorem 15.23 from [10] also works when the field considered is finite. It follows that as J is fixed under $\mathcal{B}(F)$ it must be p-Borel. As J is p-Borel, we have that J is fixed under $\mathcal{B}(k)$. $\square$

Before we prove the next statement, however, notation for the rank of a vector space is needed. Let $d \in \mathbb{N}$ be a fixed degree. Write a monomial basis for the vector space of homogeneous polynomials in degree d as $m_1 = x_1^d$, $m_2 = x_1^{d-1}x_2$, $\cdots$, $m_s = x_n^d$, ordered with respect to $<$. For a given vector space $\mathcal{W}$ generated by homogenous polynomials of degree d write $\mathcal{W} = \langle w_1, \ldots, w_p\rangle$ where $w_{\mathrm{i}} = \sum a_{\mathrm{ij}}m_{\mathrm{j}}$ for all i. Denote by A the matrix $(a_{\mathrm{ij}})_{1\le \mathrm{i}\le p, 1\le \mathrm{j}\le s}$ of size p by s.

Definition 3.1.8. For a vector space $\mathcal{W}$ as given above, we define the **rank vector** of $\mathcal{W}$, $r(\mathcal{W})$, as the vector whose ith entry $r_{\mathrm{i}}(\mathcal{W})$, is the rank of the submatrix of A consisting of the first i columns of A.

One sees by Gaussian elimination that $m_1 \in \mathrm{in}(\mathcal{W})$ if and only if $r_1 \neq 0$ and in general that $m_{\mathrm{i}+1} \in \mathrm{in}_<(\mathcal{W})$ if and only if $r_{\mathrm{i}+1}(\mathcal{W}) > r_{\mathrm{i}}(\mathcal{W})$. It follows that $r_{\mathrm{i}}(\mathcal{W}) = r_{\mathrm{i}}(\mathrm{in}_<(W))$ and that the rank vector is independent of the choice of generators of the vector space.

It is also important to understand how the rank vector of $\mathrm{in}_<(\mathcal{W})$ and the rank vector of $\mathcal{W}$ change under the action of Borel matrices. Recall that for any monomial order $<$ there exists a weight $\underline{w} = (w_1, \ldots, w_n) \in \mathbb{N}_+^n$ such that the ordering of monomials in S_d under $<$ is the same ordering of monomials in S_d under $\underline{w}$. You can then extend S to $\tilde{S} = S[t]$, the homogenization of S where t is given weight 0. Then $\mathcal{W} \subseteq S_d$ can be extended to the vector space $\tilde{\mathcal{W}} = \langle \tilde{v} \mid v \in \mathcal{W}\rangle \subseteq \tilde{S}_d$ where $\tilde{v} = t^a \cdot v(t^{-w_1}x_1, \ldots, t^{-w_n}x_n)$ and a is the largest weight of a monomial in the support of v. Specializing back to S with $t = 1$ gives back $\mathcal{W}$ while specializing back with $t = 0$ gives $\mathrm{in}_<(\mathcal{W})$. In general, for all i where the ith rank is not 0 this gives the inequality $r_{\mathrm{i}}(\tilde{\mathcal{W}}) \ge r_{\mathrm{i}}(\tilde{\mathcal{W}}_{t=0}) = r_{\mathrm{i}}(\mathrm{in}_<(\mathcal{W}))$.

Under the action of a matrix g we then have that $r_{\mathrm{i}}(g(\tilde{\mathcal{W}}_{t=0})) = r_{\mathrm{i}}((g\tilde{\mathcal{W}})_{t=0}) \le r_{\mathrm{i}}(g\tilde{\mathcal{W}})$. Furthermore, by collecting all the nonzero minors of the matrix $\tilde{A}$ associated with $\tilde{\mathcal{W}}$ we can define a nonempty Zariski open set $\mathcal{U}$ of $k = \mathbb{A}^1$ where those minors do not vanish. This then gives that for all $\alpha \in \mathcal{U}$ we have $r_{\mathrm{i}}(\tilde{\mathcal{W}}) = r_{\mathrm{i}}(\tilde{\mathcal{W}}_{t=\alpha})$ where we can write $\tilde{\mathcal{W}}_{t=\alpha} = \lambda\mathcal{W}$, where λ is the diagonal matrix whose diagonal entries are precisely $\alpha^{-w_1}, \ldots, \alpha^{-w_n}$. Hence for a matrix g, $r_{\mathrm{i}}(g\lambda\mathcal{W}) \ge r_{\mathrm{i}}(g(\mathrm{in}_<(\mathcal{W})))$.

With this information we can now prove the following theorem.

Theorem 3.1.9. *Let I be the monomial ideal fixed at the start of this section. Then the ideal $\mathrm{in}_<(D_{\mathscr{F}}(I^*))$ (respectively $\mathrm{in}_<(D_{\mathscr{G}}(I^*))$) is Borel fixed.*

Proof. Since $\mathrm{in}_<(D_{\mathscr{T}}(I^*))$ is a monomial ideal, by the previous lemma it is sufficient to show that is fixed under the action of $\mathcal{B}(k)$. Further, by Remark 3.1.6 and Lemma 3.1.7 we can extend our field k and thereby assume that k is infinite as the monomial generators of $\mathrm{in}_<(D_{\mathscr{T}}(I^*))$ and the Borel fixed condition do not change under field extensions.

To show that $\mathrm{in}_<(D_{\mathscr{T}}(I^*))$ is Borel fixed, it is enough to show that for every degree d the vector space $(\mathrm{in}_<(D_{\mathscr{T}}(I^*)))_d$ is Borel fixed. By the previous discussion of rank of a vector space one has for all $\mathscr{L} \in \mathbb{T}_{nM}(k)$ and for all $1 \leq \mathrm{i} \leq s$ that $r_{\mathrm{i}}\left((D_{\mathscr{L}}(I))_d\right) \leq r_{\mathrm{i}}\left((D_{\mathscr{T}}(I^*))_d\right)$. This follows because every non zero minors used to compute $r_{\mathrm{i}}\left((D_{\mathscr{L}}(I))_d\right)$ correspond to a non zero minor of $(D_{\mathscr{T}}(I^*))_d$.

Now for the sake of contradiction assume that $\mathrm{in}_<(D_{\mathscr{T}}(I^*))$ is not Borel fixed. Then by Lemma 3.1.7 there exists a $b \in \mathcal{B}_n(k)$ such that $b(\mathrm{in}_<(D_{\mathscr{T}}(I^*))) \neq \mathrm{in}_<(D_{\mathscr{T}}(I^*))$. As k is an infinite field there exists $\mathscr{L} \in \mathbb{T}_{nM}(k)$ with $\mathrm{in}_<(D_{\mathscr{L}}(I)) = \mathrm{in}_<(D_{\mathscr{T}}(I^*)) \cap S$. But by our choice of b we also have that $b(\mathrm{in}_<((D_{\mathscr{L}}(I))_d)) \neq \mathrm{in}_<((D_{\mathscr{L}}(I))_d)$. As $b \in \mathcal{B}_n(k)$, b is an upper triangular matrix. So for every monomial vector space $\mathcal{W} \subseteq S_d$ we have that $r_{\mathrm{i}}(b\mathcal{W}) \geq r_{\mathrm{i}}(\mathcal{W})$ for $1 \leq \mathrm{i} \leq s$. This fact combined with $b(\mathrm{in}_<((D_{\mathscr{L}}(I))_d)) \neq \mathrm{in}_<((D_{\mathscr{L}}(I))_d)$ means that there exists an index i where $r_{\mathrm{i}}(b(\mathrm{in}_<((D_{\mathscr{L}}(I))_d))) > r_{\mathrm{i}}(\mathrm{in}_<((D_{\mathscr{L}}(I))_d)) = r_{\mathrm{i}}(D_{\mathscr{T}}(I^*)_d)$. We then also have that $r_{\mathrm{i}}(b(\lambda D_{\mathscr{L}}(I))_d) \geq r_{\mathrm{i}}(b(\mathrm{in}_<((D_{\mathscr{L}}(I))_d)))$ where λ is the diagonal matrix found by homogenizing and specializing. This is a contradiction because the rank with respect to $\mathscr{T}$ are larger than or equal to the rank using a distraction with entries in k.

□

Example 3.1.10. Let $I \subseteq S$ be a principle monomial ideal, $I = (u)$. Then the only Borel fixed ideal with the same Hilbert function as I is the ideal $(x_1^{\deg(u)})$. Hence $\mathrm{in}_<(D_{\mathscr{G}}(I^*)) = \mathrm{in}_<(D_{\mathscr{T}}(I^*)) = (x_1^{\deg(u)})$.

Recall that over an infinite field k, the generic initial ideal of an ideal $J \subset S$ is defined to be $\mathrm{gin}_<(J) = \mathrm{in}_<(gJ)$ where g is an element of a nonempty Zariski open setset $\mathcal{U} \subset \mathrm{GL}_n(k)$. The subset $\mathcal{U}$ is a special subset such that $\mathrm{in}_<(gJ) = \mathrm{in}_<(g'J)$ for all $g, g' \in \mathcal{U}$.

It is however, not necessary to require that the base field k is infinite in order to define the generic initial ideal. In general, one can extend the field k to an infinite field $\tilde{k} = k(\mathbf{y})$ and S to $\tilde{S} = \tilde{k}[x_1, \ldots, x_n]$ in a similar way to the extension laid out in the notation at the start of the section, where $\mathbf{y}$ is a finite number of algebraically independent indeterminates. Setting $\tilde{J} = J \otimes_k \tilde{k} \subseteq \tilde{S}$ one can then compute the generic initial ideal of J over $\tilde{S}$ and contract back to S. In other words, $\text{gin}_<(J) = \text{in}_<(g\tilde{I}) \cap S$. Thus one gets the usual generic initial ideal without requiring the base field to be infinite.

Recall that a monomial ideal I being of **Borel type** means that all the associated primes of I are of the form $(x_1, \ldots, x_{\mathrm{i}})$ for some i. This is equivalent to $x_n, x_{n-1}, \ldots, x_1$ being a filter regular sequence. Bayer and Stillman proved that Borel fixed ideals are of Borel type in [1]. We now recall the following theorem about ideals of Borel type. Parts 1 and 2 of the following statement are due to Bayer and Stillman in [1]. Part 3 is due to Bayer, Charalambous, and Popescu in [4].

Theorem 3.1.11. *Let I be a graded ideal such that $in_{revlex}(I)$ is of Borel type. Then*

1. *$reg(I) = reg(in_{revlex}(I))$.*

2. *$projdim(I) = projdim(in_{revlex}(I))$.*

3. *I and $in_{revlex}(I)$ have the same extremal Betti numbers.*

Recall that if a monomial ideal is Borel fixed then it is of Borel type (see [5]). Hence we get the following statement.

Theorem 3.1.12. *Let $I \subset k[x_1, \ldots, x_n]$ be a fixed monomial ideal and $\mathscr{T}$ (respectively $\mathscr{G}$) be the distration matrix defined at the start of the section. Then I and $in_{revlex}(D_{\mathscr{T}}(I^*))$ (respectively $in_{revlex}(D_{\mathscr{G}}(I^*))$) have the same extremal Betti numbers, regularity, and projective dimension.*

Proof. As extending by scalars is faithfully flat, we have that I and I^* have the same standard graded Betti numbers. By properties of distraction matrices, I^* and $D_{\mathscr{T}}(I^*)$ have same extremal betti numbers, regularity, and projective dimension. By Theorem 3.1.9 we know that $\text{in}_{\text{revlex}}(D_{\mathscr{T}}(I^*))$ is Borel fixed. So then applying Theorem 3.1.11 to $D_{\mathscr{T}}(I^*)$ gives the theorem. ∎

3.2 The Distraction-Generic Initial Ideal

In this section we will fix a Hilbert function $\mathbf{H}$. Consider a homogeneous (but not necessarily monomial) ideal $I \in \text{Hilb}(\mathbf{H})$. The lexsegment ideal whose Hilbert function is $\mathbf{H}$ will have the largest regularity out of all ideals in $\text{Hilb}(\mathbf{H})$. For more information about the lexsegment ideal, see the beginning of Chapter 4. As we will be iteratively be computing distractions and initial ideals, we set $M = \text{reg}(\text{lex}(I))$ and then follow the notation laid out in Notation 3.1.2.

Remember that we will use $<$ to indicate a monomial order such that $x_1 > x_2 > \cdots > x_n$. If a particular monomial order is needed, it will be specified in the theorem statement.

The next portion of this section will be spent proving the following statement.

Proposition 3.2.1. *Using the assumptions and notations at the start of this section and I a homogeneous ideal in S, $(in_< \circ D_{\mathscr{T}})^r\,(gin_<(I^*))$ is eventually constant as a function of r.*

The above proposition is important because as a consequence of this proposition, we can define the following ideal which has similar properties to that of the generic initial ideal, but, as will be shown later in this section, is always strongly stable in characteristic p.

Definition 3.2.2 (D-gin)**.** Let I be a homogeneous ideal in S,

$$\text{D-gin}_<(I) = ((\text{in}_< \circ D_{\mathscr{T}})^r\,(\text{gin}_<(I^*))) \cap S$$

for r sufficiently large.

We first need to set up the following ordering on monomial ideals.

Notation 3.2.3. Let U be the set of all monomial ideals of S^* (equivalently of S). Define a total order, $\succ$, on the elements of U from $>$ as follows.

- Set $\underline{x}^{\underline{a}} \prec_m \underline{x}^{\underline{b}}$ when either (1) the total degree $|\underline{a}| > |\underline{b}|$ or (2) the total degree $|\underline{a}| = |\underline{b}|$ and $\underline{x}^{\underline{a}} < \underline{x}^{\underline{b}}$ in the monomial order.

- For $J, J' \in U$ fix a minimal set of monomial generators $J = (m_1, \ldots, m_s)$ where $m_1 \ {}_m{\succ} m_2 \ {}_m{\succ} \cdots {}_m{\succ} m_s$ and fix a minimal set of monomial generators $J' = (m'_{1'}, m'_{2'}, \ldots, m'_{s'})$ where $m'_{1'} \ {}_m{\succ} m'_{2'} \ {}_m{\succ} \cdots {}_m{\succ} m'_{s'}$.

  Then $J \succ J'$ if $J \subsetneq J'$ or there exists an index i such that $m_{\mathrm{j}} = m'_{\mathrm{j}}$ when $\mathrm{j} < \mathrm{i}$ and $m_{\mathrm{i}} \ {}_m{\succ} m'_{\mathrm{i}}$.

- Write $J \succeq J'$ whenever $J = J'$ or $J \succ J'$.

Proposition 3.2.4. *If J is a strongly stable monomial ideal of S^* then $D_{\mathscr{T}}(J) = J$.*

Proof. Without loss of generality we can assume that J is generated in a single degree d. We induct on the number, r, of minimal monomial generators of J. If $r = 1$ then J is generated by a power of x_1 and clearly $D_{\mathscr{T}}(J) = J$.

Without loss of generality write $J = (v_1, v_2 \ldots, v_r)$ where the v_{i}'s are ordered decreasing with respect to revlex. Under this ordering $(v_1, \ldots, v_{r-1})$ is strongly stable and therefore $(v_1, \ldots, v_{r-1})$ fixed under $D_{\mathscr{T}}$ by the inductive hypothesis. Hence $D_{\mathscr{T}}(J) = (v_1, \ldots, v_{r-1}) + D_{\mathscr{T}}(v_r)$.

Now $v_r = x_1^{a_1} \cdots x_n^{a_n}$ and since $\mathscr{T}$ is triangular, all the monomials in the support $D_{\mathscr{T}}(v_r)$ are either a scalar multiple of v_r or obtained by a repeated use of a strongly stable exchange of variables.

Notice that $\{v_r\} \subseteq \operatorname{supp}(D_{\mathscr{T}}(v_r)) \subseteq \{v_r\} \cup (v_1, \ldots, v_{r-1})$ where the last inclusion follows from the fact that $\mathscr{T}$ is triangular and J is strongly stable.

Thus $D_{\mathscr{T}}(J) \subseteq J$ and the equality follows as $HF_J = HF_{D_{\mathscr{T}}(J)}$. □

Proposition 3.2.5. *If J is a nonzero monomial ideal of S^* generated in degree less than or equal to M and J is not strongly stable then in$_<(D_{\mathscr{T}}(J)) \succ J$*

Proof. Write $J = (v_1, \ldots, v_r)$ where the v_{i}'s are ordered $v_1 \ {}_m{\succ} v_2 \ {}_m{\succ} \cdots {}_m{\succ} v_r$. Notice that this ordering gives us that $\deg(v_1) \le \deg(v_2) \le \cdots \le \deg(v_r)$. Let i be the smallest index such that $(v_1, \ldots, v_{\mathrm{i}})$ is not strongly stable. If $\mathrm{i} = 1$ then $v_1 \neq x_1^{\deg(v_1)}$ but $\mathrm{in}_<(D_{\mathscr{T}}(v_1)) = (x_1^{\deg(v_1)})$. Therefore $x_1^{\deg(v_1)}$ is the largest monomial generator of $\mathrm{in}_<(D_{\mathscr{T}}(J))$ with respect to $\succ$. Hence $\mathrm{in}_<(D_{\mathscr{T}}(J)) \succ J$.

Now assume $\mathrm{i} > 1$. Then $(v_1, \ldots, v_{\mathrm{i}-1})$ is a strongly stable ideal and by the previous theorem

$D_{\mathscr{T}}((v_1, \ldots, v_{i-1})) = (v_1, \ldots, v_{i-1})$. By assumption $(v_1, \ldots, v_i)$ is not strongly stable. Hence there exists indices $1 \leq t < s \leq n$ such that x_s divides v_i and $\frac{v_i}{x_s}x_t \notin (v_1, \ldots, v_{i-1})$. Write $v_i = x_1^{a_1} \cdots x_n^{a_n}$ and $D_{\mathscr{T}}(v_i) = \prod_{i=1}^{n} \left(\prod_{j=1}^{a_i} g_{ij} \right)$. By the definition of $\mathscr{T}$ the support of $D_{\mathscr{T}}(v_i)$ consists of all the monomials that can be obtained by repeated applications of strongly stable exchanges on v_i.
Thus $\frac{v_i}{x_s}x_t \in \operatorname{Supp}(D_{\mathscr{T}}(v_i))$. Let w_i be the largest monomial contained in the set $\{\operatorname{Supp}(D_{\mathscr{T}}(v_i)) \setminus (v_1, \ldots, v_{i-1})\}$ with respect to ${}_m\succ$. Then $\deg(w_i) = \deg(v_i)$ and we have $w_i \, {}_m\succeq \frac{v_i}{x_s}x_t \, {}_m\succ v_i$. Now $\{v_1, \cdots, v_{i-1}, w_i\} \subseteq \operatorname{in}_<(D_{\mathscr{T}}(J))$. Furthermore $v_1, \ldots, v_{i-1}, w_i$ are among the minimal generators of $D_{\mathscr{T}}(J)$ because $D_{\mathscr{T}}(v_{i+1}), \ldots, D_{\mathscr{T}}(v_r)$ have degrees equal to or larger than $\deg(v_i)$. Hence $\operatorname{in}_<(D_{\mathscr{T}}(J)) \succ J$. $\square$

The proof of proof of Proposition 3.2.1 then follows from the previous propositions.

Proof of Proposition 3.2.1. For given Hilbert function there exist only finitely many monomial ideals in $k[x_1, \ldots, x_n]$ with that Hilbert function. Since I is fixed every monomial ideal with the same Hilbert function as I will be generated in degree less than or equal to M. There are only finitely many such ideals. Therefore r in Proposition 3.2.1 must be finite as otherwise the previous propositions would give an infinitely ascending chain of monomial ideals with respect to $\succ$ all having Hilbert function $\mathbf{H}$. $\square$

As an immediate corollary we have

Corollary 3.2.6. *D-gin$_<(I)$ is a strongly stable monomial ideal.*

Proof. By Proposition 3.2.1 we know that D-gin$_<(I)$ is well defined and constant for all $r' > r \in \mathbb{N}$ for some fixed r. So by the previous propositions it must be strongly stable. $\square$

3.2.1 Homological Invariants and the Distraction-Generic Initial Ideal

Our construction D-gin$_<(I)$ has many properties which are similar to that of the generic initial ideal. Of prime importance are the following.

Theorem 3.2.7. *Let I be a homogeneous ideal of S.*

1. *D-gin$_<(I) = I$ if and only if I is strongly stable.*

2. *When char(k) = 0 we have D-gin$_<(I)$ = gin$_<(I)$.*

3. *I and D-gin$_<(I)$ have the same Hilbert function.*

4. $\beta_{\mathrm{ij}}(I) \leq \beta_{\mathrm{ij}}(D\text{-}gin_<(I))$.

5. *When $<$ is the degree revlex order, then reg(S/I) = reg(D-gin$_<(S/I)$), projdim(S/I) = projdim(S/D-gin$_<(I)$), depth(S/I) = depth(S/D-gin$_<(I)$), and moreover the extremal Betti numbers of I and D-gin$_<(I)$ are the same.*

For part of the proof of statement (5) in the above theorem and the proof of statement (4) in the next corollary we will need the Auslander-Buchsbaum formula. The general form of the Auslander-Buchsbaum formula is as follows.

Theorem 3.2.8 (Auslander-Buchsbaum)**.** *Let M be a finitely generated graded module of a commutative local Noetherian ring R such that projdim$(M) < \infty$. Then*

$$depth(M) + projdim(M) = dim(R).$$

There is also a specialized version of the Auslander-Buchsbaum formula for polynomial rings.

Theorem 3.2.9 (Auslander-Buchsbaum for Polynomial Rings)**.** *Let M be a finitely generated graded S-module, where S is the polynomial ring $S = k[x_1, \ldots, x_n]$. Then*

$$projdim(M) + depth(M) = n.$$

Now for the proof of above theorem.

Proof. 1) Since I is strongly stable, $\mathrm{gin}_<(I) = I$ and $\mathrm{in}_<(I) = I$. Hence by the previous proposition, $\text{D-gin}_<(I) = I$.
2) As $\mathrm{gin}_<(I)$ is a strongly stable monomial ideal in characteristic 0 it immediately follows that $\text{D-gin}_<(I) = \mathrm{gin}_<(I)$.
3) For any ideal I, $\mathrm{in}_<(I)$, $\mathrm{gin}_<(I)$, and $D_{\mathscr{G}}(I)$ all have the same Hilbert function. Hence I

and D-gin$_<(I)$ have the same Hilbert function.
4) Distractions, initial ideals, and generic initial ideals all satisfy upper semi-continuity for Betti numbers. Hence $\beta_{\mathrm{ij}}(I) \leq \beta_{\mathrm{ij}}(\text{D-gin}_<(I))$.
5) The equality of the regularity, projective dimension, and extremal Betti numbers follows from Theorem 3.1.12. The equality of the depth follows from the equality of the projective dimension and the Auslander-Buchsbaum formula. □

Remark 3.2.10. Notice that although the above inequality $\beta_{\mathrm{ij}}(I) \leq \beta_{\mathrm{ij}}(\text{D-gin}_<(I))$ is phrased for the graded Betti numbers since the j Betti number is a sum of graded Betti numbers we also have the ineqality $\beta_{\mathrm{j}}(I) \leq \beta_{\mathrm{j}}(\text{D-gin}_<(I))$ for all j.

Parts 3, 4, and 5 of the above theorem will be crucial for the use of the distraction-generic initial ideal in the proof of the main result in the next chapter. The inequality of the Betti numbers is especially of note since it is unknown if other constructions of strongly stable generic ideals in characteristic p, such as the zero-generic initial ideal defined by Caviglia and Sbarra in [11], preserve this pointwise inequality.

As a corollary of Theorem 3.2.7 we have the following.

Corollary 3.2.11. *Let I be a homogeneous ideal $I \subseteq S$.*

1. *dim$(S/I) =$ dim$(S/D\text{-}gin_<(I))$.*
2. *projdim$(S/I) \leq$ projdim$(S/D\text{-}gin_<(I))$.*
3. *reg$(S/I) \leq$ reg$(S/D\text{-}gin_<(I))$.*
4. *depth$(S/I) \geq$ depth$(S/D\text{-}gin_<(I))$.*

Proof. For statement (1) the equality $\dim(S/I) = \dim(S/\text{D-gin}_<(I))$ is a direct consequence of the fact that I and D-gin$_<(I)$ have the same Hilbert function. As we have previously proved that $\beta_{\mathrm{ij}}(I) \leq \beta_{\mathrm{ij}}(\text{D-gin}_<(I))$, this gives us the two inequalities (2) $\text{projdim}(S/I) \leq \text{projdim}(S/\text{D-gin}_<(I))$ and (3) $\text{reg}(S/I) \leq \text{reg}(S/\text{D-gin}_<(I))$.
Lastly the inequality (4) $\text{depth}(I) \geq \text{depth}(S/\text{D-gin}_<(I))$ follows from the Auslander-Buchsbaum formula for polynomial rings in conjunction with statement (2).

□

We also have the following relation between S/I and $S/\text{D-gin}_<(I)$ in regards to being Cohen-Macaulay. Recall that an S-module is Cohen-Macaulay if and only if the dimension of the modules is equal to the depth of the module.

Theorem 3.2.12. *Let I be a homogeneous ideal of S.*

1. *If $S/D\text{-}gin_<(I)$ is Cohen-Macaulay, then S/I is Cohen-Macaulay.*
2. *If $S/D\text{-}gin_<(I)$ is Gorenstein then S/I is Gorenstein.*
3. *When $<$ is the revlex order we have that $S/D\text{-}gin_<(I)$ is Cohen-Macaulay if and only if S/I is Cohen-Macaulay.*

Proof. For the first statement $S/\text{D-gin}_<(I)$ is Cohen-Macaulay if and only if $\dim(S/\text{D-gin}_<(I)) = \text{depth}(S/\text{D-gin}_<(I))$. Statements (1) and (2) from 3.2.11 then give the chain of inequalities

$$\dim(S/I) = \dim(S/\text{D-gin}_<(I)) = \text{depth}(S/\text{D-gin}_<(I)) \leq \text{depth}(I) \leq \dim(S/I)$$

giving that $\dim(S/I) = \text{depth}(S/I)$. Hence S/I is Cohen-Macaulay.
For statement (2) recall that $S/\text{D-gin}_<(I)$ being Gorenstein is equivalent to $S/\text{D-gin}_<(I)$ being Cohen-Macaulay and the last non-zero Betti number of $S/\text{D-gin}_<(I)$ being equal to 1. Hence if $S/\text{D-gin}_<(I)$ is Gorenstein then S/I is Cohen-Macaulay. As S is a polynomial ring the last nonzero Betti number of S/I occurs at the same spot of the free resolution of S/I as the last nonzero Betti number of $S/\text{D-gin}_<(I)$ occurs in its free resolution. By the inequality $\beta_{\text{j}}(I) \leq \beta_{\text{j}}(\text{D-gin}_<(I))$ the last nonzero Betti number of S/I must be 1 giving us that S/I is Gorenstein as well.
Lastly to get statement (3) notice that under the revlex order we have $\text{projdim}(I) = \text{progdim}(\text{D-gin}_<(I))$. Using the Auslander-Buchsbaum formula we have that the depth of S/I and $S/\text{D-gin}_<(I)$ are the same. Hence S/I being Cohen-Maulay implies that $S/\text{D-gin}_<(I)$ is also Cohen-Macaulay. □

Some other minor things to note are as follows.

Remark 3.2.13. Consider the case $S = k[x_1, x_2]$, $I \subseteq S$ a homogeneous ideal. Since D-gin$_<(I)$ is strongly stable it must be a lexsegment ideal as every strongly stable ideal in a polynomial ring in two variables is a lexsegment ideal.

3.2.2 Saturation and the Distraction-Generic Initial Ideal

The purpose of this section is to consider how the saturation of an ideal interacts with the distraction-generic initial ideal. It is motivated by the following combination of results found in [16].

Proposition 3.2.14. *Let $I \subseteq S = k[x_1, \dots, x_n]$ be a homogenous ideal, k a field of characteristic 0, I^{sat} its saturation with respect to the homogeneous maximal ideal, and "$<$" the revlex order. Then $gin_<(I^{sat}) = (gin_<(I))^{sat}$.*

Although Green in [16] only considers fields of characteristic 0, following a proof similar to Theorem 3.2.22 below, one can see that the above proposition is also true without the characteristic 0 assumption on k.

Recall some basic facts about saturations of ideals.

Definition 3.2.15. Let $I \subset S$ be a homogeneous ideal. Then the **saturation** of I is defined as $I^{\text{sat}} = (I :_S \mathbf{m}^\infty) = \cup_{\text{i} \geq 0}(I :_S \mathbf{m}^k)$, where $\mathbf{m}$ denote the homogeneous maximal ideal, i.e. $\mathbf{m} = (x_1, \dots, x_n)$.
I is called **saturated** if $(I :_S \mathbf{m}) = I$.

Note that the image of the ideal $(I :_S \mathbf{m})$ in S/I is also known as the **socle** of S/I. When it is clear which ring the saturation is being taken with respect to, the underscore on the colon may be dropped.
When I is a strongly stable monomial ideal we have the following (see [10]).

Proposition 3.2.16. *Let $I \subseteq S$ be a strongly stable monomial ideal. Then for all $t \geq 0$ and all $1 \leq \text{i} \leq n$*

$$I :_S x_{\text{i}}^t = I :_S (x_1, \dots, x_{\text{i}})^t$$

and further

$$I :_S x_{\text{i}}^\infty = I :_S (x_1, \dots x_{\text{i}})^\infty.$$

Note that for strongly stable ideals the above proposition in particular gives the equalities

$$I :_S \mathbf{m}^\infty = I :_S x_n^\infty$$

and

$$I :_S \mathbf{m} = I :_S x_n.$$

Hence for two strongly stable saturated ideals we have the following way to check ideal equality.

Lemma 3.2.17. *Let $I, J \subseteq S$ be two strongly stable saturated ideals. Assume that $I_d = J_d$ for all $d >> 0$. Then $I = J$.*

Proof. We do a decreasing induction on a degree $\mathrm{i} \leq d$. Take a monomial $u \in J_{\mathrm{i}-1}$. Then $x_n u \in J_{\mathrm{i}} = I_{\mathrm{i}}$. As I is a strongly stable saturated ideal $(I : x_n) = I$. Hence $u \in I$ and therefore $u \in I_{\mathrm{i}-1}$. Similarly every $v \in I_{\mathrm{i}-1}$ is also in $J_{\mathrm{i}-1}$. Hence $I = J$. □

Now recall the definition of an associated prime.

Definition 3.2.18. Let R be a Noetherian ring and $I \subset R$ an ideal. The **associated primes** of I, denoted by $\mathrm{Ass}(I)$, are the prime ideals $P \subset R$ such that there exists a nonzero element $x \in R/I$ with $P = \{a \in R | ax = 0\}$.

The following proposition is well known.

Proposition 3.2.19. *Let $I \subseteq S$ be homogeneous monomial ideal. Then I is saturated if and only if $\mathbf{m}$ is not an associated prime of I.*

In the case of R being a Noetherian local ring, we recall the following (see [10]).

Proposition 3.2.20. *Let R be a Noetherian local ring with maximal ideal $\mathbf{m}$. Then depth$(R) = 0$ if and only if $\mathbf{m}$ is an associated prime of R.*

Putting this all together gives for a compact proof for the following statement.

Proposition 3.2.21. *Assume $I \subseteq S$ is a homogeneous ideal and $<$ is the revlex order. Then I is saturated if and only if D-gin$_<(I)$ is saturated.*

Proof. By Theorem 3.2.7 since the monomial order is the revlex order we have that $\text{projdim}(S/I) = \text{projdim}(S/\text{D-gin}_<(I))$. By the Auslander-Buchsbaum formula it follows that $\text{depth}(S/I) = \text{depth}(S/\text{D-gin}_<(I))$. If I is saturated then $\text{depth}(S/I) > 0$ which implies that $\text{depth}(S/\text{D-gin}_<(I)) > 0$. So by the previous theorem $\text{D-gin}_<(I)$ is a saturated ideal. Alternatively, if $\text{D-gin}_<(I)$ is saturated then $\text{depth}(S/\text{D-gin}_<(I)) > 0$ implying that $\text{depth}(S/I) > 0$ and hence I is saturated. □

Furthermore we have an result for the distraction-generic ideal analogous to the theorem stated at the beginning of the section.

Theorem 3.2.22. *Let $I \subseteq S$ be a homogeneous ideal, I^{sat} its saturation with respect to the homogeneous maximal ideal, and $<$ the revlex order. Then $\text{D-gin}_<(I^{sat}) = \text{D-gin}_<(I)^{sat}$.*

Proof. As noted earlier in this section as both $\text{D-gin}_<(I^{\text{sat}})$ and $\text{D-gin}_<(I)^{\text{sat}}$ are strongly stable saturated ideals it suffices to show that $\text{D-gin}_<(I^{\text{sat}})_r = (\text{D-gin}_<(I)^{\text{sat}})_r$ for $r >> 0$. However for some sufficiently large degree r we will have that $I_r = (I^{\text{sat}})_r$ and that $(\text{D-gin}_<(I)^{\text{sat}})_r = \text{D-gin}_<(I)_r$. Hence in some large degree there is the equality

$$\text{D-gin}_<(I^{\text{sat}})_r = \text{D-gin}_< I_r = \text{D-gin}_<(I)_r = (\text{D-gin}_<(I))^{\text{sat}}_r$$

Where the equality $\text{D-gin}_< I_r = \text{D-gin}_<(I)_r$ follows from the definition of the Distraction-generic initial ideal. □

Further note that it should be possible to expand the definition of $\text{D-gin}_<$ so that the results presented in this chapter hold for homogeneous finitely generated modules M of S and not just ideals of S.

4. THE MALL IDEAL

A natural question that arises when studying polynomial rings is whether or not there are any ideals that exhibit extremal properties. To this end one can fix the Hilbert function $\mathbf{H}$ of a homogeneous ideal and consider the Hilbert Scheme, $\text{Hilb}(\mathbf{H})$, associated to $\mathbf{H}$ consisting of all the graded ideals in S with Hilbert function $\mathbf{H}$. As an example of an ideal with extremal properties, for any given ideal $I \subset S$, there exists a unique ideal $\text{lex}(I)$, known as the lexsegment ideal, such that I and $\text{lex}(I)$ have the same Hilbert function ([8]). In order to understand the lexsegment ideal it is important to first understand what a lexsegment is.

Definition 4.0.1. A vector space $\mathcal{L} \subseteq S_d$ is called a lexsegment if it is generated by monomials and whenever $u \in \mathcal{L}$ and $v \in S_d$ with $v \geq u$ in the lex ordering, we have that $v \in \mathcal{L}$.

A monomial ideal $I \subset S$ is called a **lexsegment** ideal if for every d the monomials in I_d form a lexsegment. We use $\text{lex}(I)$ or $\text{lex}(\mathbf{H})$ to indicate the unique lexsegment ideal with the same Hilbert function as I (by abuse of notation we with use lex of a vector space to refer to the lexsegment with the same dimension as a given vector space). Note also that lexsegment ideals are strongly stable ideals. Furthermore, the following was shown by Bigatti [17], Hulett [18], and Pardue [15],

Theorem 4.0.2 (Bigatti-Hulett-Pardue)**.** *Fix a Hilbert function $\boldsymbol{H}$ and let lex($\boldsymbol{H}$) denote the unique lexsegment ideal corresponding to $\boldsymbol{H}$. Then the graded Betti numbers of lex($\boldsymbol{H}$) are pointwise the largest among all ideals $I \in$ Hilb($\boldsymbol{H}$).*

However, for a given ideal I the regularity of I and $\text{lex}(I)$ may be different. This then leads to the definition of a strata of a Hilbert scheme. For a fixed regularity m, the Hilbert strata defined by m of $\text{Hilb}(\mathbf{H})$ is the set of all ideals in $\text{Hilb}(\mathbf{H})$ with regularity less than or equal to m. Such a strata will be denoted by $\text{Hilb}(\mathbf{H}, m)$.

In his paper "Betti Numbers, Castelnuovo Mumford Regularity, and Generalizations of Macaulay's Theorem" [2] Daniel Mall shows that when the characteristic of k is zero and $\text{Hilb}(\mathbf{H}, m) \neq \emptyset$ there exists an ideal $J \in \text{Hilb}(\mathbf{H}, m)$ with the property that $\beta_{qd}(J) \geq \beta_{qd}(I)$ for all $q, d \in \mathbb{N}$ and all $I \in \text{Hilb}(\mathbf{H}, m)$. In other words, over a field of characteristic 0 for a fixed regularity and Hilbert function you can find an ideal that is extremal with respect to graded Betti numbers. Note that in the case of m being equal to or larger than the regularity of the lexsegment ideal this ideal will be the lexsegment ideal. Mall's proof, however, relies on the reduction to the case of a strongly stable monomial ideal using the generic initial ideal, which is why his proof only holds when the characteristic of the base field is 0. Mall also has an analogous result over fields of characteristic 0 for the saturated Hilbert strata, $\mathrm{Hilb}^{\mathrm{sat}}(\mathbf{H}, m)$, of the saturated Hilbert scheme, $\mathrm{Hilb}^{\mathrm{sat}}(\mathbf{H})$, where $\mathrm{Hilb}^{\mathrm{sat}}(\mathbf{H})$ consists of the set whose elements are all saturated homogeneous ideals with a fixed Hilbert function $\mathbf{H}$. By using the distraction-generic initial ideal from the previous section, Mall's results can be expanded to fields of any characteristic.

4.1 Hilbert Strata and the Mall Ideal

This section will focus on explicitly constructing Mall's ideal and showing that it is a strongly stable ideal contained in $\mathrm{Hilb}(\mathbf{H}, m)$. Both this construction and the expansion of Mall's result will use the following extension of Green's Hyperplane Theorem ([19]). It was proven by Herzog and Popescu ([20]) in characteristic 0 and by Gasharov ([21]) in any characteristic.

Theorem 4.1.1 ([19] [20] [21]). *Assume $|k| = \infty$ and let $I \subseteq S$ be a homogeneous ideal with Hilbert function $\boldsymbol{H}$ and ℓ a general linear form. Then for all d*

$$HF(S/(I + (\ell^d))) \leq HF(S/(lex(I) + (x_n^d))).$$

Furthermore when I is strongly stable, ℓ can be chosen to be x_n.

4.1.1 Decomposing Strongly Stable Ideals

As the distraction-generic initial ideal from the previous chapter allows us to reduce to the case of a strongly stable monomial ideal no matter what characteristic we are in, we will build Mall's ideal by starting with the reduction to the strongly stable monomial ideal case. Our construction, however, will use the following decomposition of monomial ideals and vector spaces.

Remark 4.1.2. Let I be a monomial ideal, $I \subseteq S = k[x_1, \ldots, x_n]$.
I can be written as

$$I = I_{[0]} \oplus I_{[1]} \cdot x_n \oplus \cdots \oplus I_{[\mathrm{j}]} \cdot x_n^{\mathrm{j}} \oplus \cdots$$

where each $I_{[\mathrm{j}]}$ is an ideal of $\bar{S} = k[x_1, \ldots, x_{n-1}]$.
For $I_d \subset I$ we will write

$$I_d = I_{d,[0]} \oplus I_{d,[1]} \cdot x_n \oplus \cdots \oplus I_{d,[\mathrm{j}]} \cdot x_n^{\mathrm{j}} \oplus \cdots$$

In the preceding remark notice that $I_{[0]}$ will be a monomial ideal of $\bar{S}$ generated by all the monomials of I that do not contain x_n. Similarly $I_{[1]}$ will be a monomial ideal of $\bar{S}$ that is generated by the set of monomials $w \in \bar{S}$, where $wx_n = u$, $u \in I$. In general, $I_{[\mathrm{j}]}$ will be a monomial ideal that is generated by the set of monomials $w \in \bar{S}$, where $wx_n^{\mathrm{j}} = u$, $u \in I$. Furthermore, for any monomial $u \in I_{[\mathrm{j}]}$ notice that $ux_n^{\mathrm{j}} \in I$ which means that $ux_n^{\mathrm{j}+1} \in I$. Hence $I_{[\mathrm{j}]} \subseteq I_{[\mathrm{j}+1]}$.

This decomposition leads to the following property for strongly stable monomial ideals.

Proposition 4.1.3. *Let $I \subseteq S$ be a monomial ideal. I being a strongly stable ideal is equivalent to each $I_{[\mathrm{j}]}$ being a strongly stable ideal of $\bar{S}$ and for each index $\mathrm{j} > 0$, $I_{[\mathrm{j}]}\bar{m} \subseteq I_{[\mathrm{j}-1]}$, where $\bar{m} = (x_1, \ldots, x_{n-1}) \subset \bar{S}$.*

Proof. Assume I is a strongly stable ideal of S and that $I_{[\mathrm{j}]}$ is not strongly stable. Then there would exist a monomial $u \in I_{[\mathrm{j}]}$ and indices s, t such that $s < t < n$ and x_t divides u but $x_s(u/x_t) \notin I_{[\mathrm{j}]}$. But this would imply that $x_s(u/x_t)x_n^{\mathrm{j}} \notin I$, which contradicts I being strongly stable. To see that $I_{[\mathrm{j}]}\bar{m} \subseteq I_{[\mathrm{j}-1]}$, notice that for any $w = ux_n^{\mathrm{j}} \in I$, u not divisible by x_n, we must have that $x_{\mathrm{i}}ux_n^{\mathrm{j}-1} \in I$ for any index $\mathrm{j} < n$. As $u \in I_{[\mathrm{j}]}$ this gives $I_{[\mathrm{j}]}\bar{m} \subseteq I_{[\mathrm{j}-1]}$. For the other direction, let $u \in I$ be a monomial. Then $u = wx_n^{\mathrm{j}}$ for some index j. As $I_{[\mathrm{j}]}$ is strongly stable, for any indices s, t with $s < t < n$ and x_t dividing w we have that $x_s(w/x_t) \in I_{[\mathrm{j}]}$ which means that $x_s(u/x_t) \in I$. Furthermore, as $I_{[\mathrm{j}]}\bar{m} \subseteq I_{[\mathrm{j}-1]}$ we have that $x_{\mathrm{i}}w \in I_{[\mathrm{j}-1]}$ for every index $\mathrm{i} < n$ and hence $x_{\mathrm{i}}(u/x_n) = x_{\mathrm{i}}wx_n^{\mathrm{j}-1} \in I$. So I is a strongly stable monomial ideal.

□

We can also discuss the growth of strongly stable ideals from one degree to the next.

Remark 4.1.4. Let I be a strongly stable ideal generated in degree d. If we consider I in degree $d+a$ for $a \in \mathbb{N}$ then we can write I_{d+a} as

$$I_{d+a} = I_{d,[0]} \cdot \bar{m}^a \oplus I_{d,[0]}\bar{m}^{a-1} \cdot x_n \oplus \cdots \oplus I_{d,[0]} \cdot x_n^a \oplus I_{d,[1]} \cdot x_n^{a+1} \oplus \cdots \oplus I_{d,[\mathrm{j}]} \cdot x_n^{a+\mathrm{j}} \oplus \cdots$$

where $\bar{m} = (x_1, \ldots, x_{n-1})\bar{S}$

We can decompose vector spaces in the same manner.

Remark 4.1.5. For V a monomial vector space inside S_d, for any degree d,

1) V can be written as

$$V = V_{[0]} \oplus V_{[1]} \cdot x_n \oplus \cdots \oplus V_{[\mathrm{j}]} \cdot x_n^{\mathrm{j}} \oplus \cdots$$

where each $V_{[\mathrm{j}]}$ is an vector space of $\bar{S}_d \subset S_d$.

2) V being a strongly stable vector space is equivalent to each $V_{[\mathrm{j}]}$ being a strongly stable vector space of $\bar{S}$ and for each index j > 0, $V_{[\mathrm{j}]} \cdot \bar{S}_1 \subseteq V_{[\mathrm{j}-1]}$, where $\bar{S}_1$ is the degree one component of $\bar{S}$.

The proofs of the above statements are similar to the proofs of the ideal case.

Remark 4.1.6. For I a strongly stable ideal use the above notation to write $I_{\mathrm{j}} = I_{\mathrm{j},[0]} \oplus I_{\mathrm{j},[1]}x_n \oplus \cdots$. As I is a standard graded ideal, and hence a graded module, $I_{\mathrm{j}-1}S_1 \subseteq I_{\mathrm{j}}$. In particular, as I is strongly stable we can note that $I_{\mathrm{j}-1}x_n \subseteq I_{\mathrm{j},[1]}x_n \oplus \cdots$.

4.1.2 Building the Mall Ideal

For this section we will fix a regularity m. Given a strongly stable $I \subseteq S$ generated in degree less than or equal to m (equivalently $\mathrm{reg}(I) \leq m$) we recursively define $\mathrm{M}(I, m)$ as follows

Definition 4.1.7 (Mall Ideal of I)**.** Let n be the number of variables in S.

1) $\mathrm{M}(I, m) = I$ when $n = 1$.

2) For $n > 1$ Let I_m be the m-graded component of I, which is a strongly stable vector

space. Then $I_m = I_{m,[0]} \bigoplus I_{m,[1]}x_n \bigoplus \cdots \bigoplus I_{m,[m]}x_n^m$ (where some of the summands may be zero) and $\left(I_{m,[1]} \bigoplus I_{m,[2]}x_n \bigoplus \cdots \bigoplus I_{m,[m]}x_n^{m-1}\right) = (I : x_n)_{m-1}$. Further note that $(I : x_n^{\infty})_{[0]} = \overline{(I : x_n^{\infty})} \subset \bar{S}$. We can then define $\mathrm{M}(I, m)$ as follows.

$$\mathrm{M}(I, m) := \bigoplus_{d<m} (\mathrm{lex}(I))_d \bigoplus \left(\bar{M} \bigoplus L\right).$$

Where $(\bar{M} \oplus L)$ is the ideal generated by the direct sum of the vector spaces $\bar{M}$ and L. $\bar{M} = \mathrm{M}((I : x_n^{\infty})_{[0]}, m)_m$, $L = \mathrm{lex}\left((I : x_n)_{m-1}\right) \cdot x_n$, and $\mathrm{M}((I : x_n^{\infty})_{[0]}, m)_m$ is calculated in the ring $k[x_1, \ldots, x_{n-1}]$ and extended to S.

Proposition 4.1.8. *With the above setup, $M(I, m)$ is a strongly stable ideal that has the same Hilbert function as I and reg$(M(I, m)) \leq m$, $\beta_{\mathrm{ij}}(I) \leq \beta_{\mathrm{ij}}(M(I, m))$ for all* $\mathrm{i}, \mathrm{j} \in \mathbb{N}$.

For ease of reading we will break the proof of Proposition 4.1.8 into multiple pieces. First we shall consider the statement that $\mathrm{M}(I, m)$ is a strongly stable ideal.

Stability of Mall Ideal. We will proceed by induction on the number of variables of S. As $\mathrm{M}(I, m) = I$ for $n = 1$ and in one variable each monomial ideal is lex, this statement is true for $n = 1$.

Now assume our statement holds for rings of $n - 1$ variables and consider S, a polynomial ring over n variables. As $\mathrm{M}(I, m)$ is generated by a direct sum of monomial vector spaces, it will be a monomial vector space. First, we show that $\mathrm{M}(I, m)$ is an ideal, or equivalently that the product of the vector space S_1 and the graded component of degree d is contained into the graded component of degree $d + 1$ for all d.

By definition of the Mall ideal $\mathrm{M}(I, m)_d = \mathrm{lex}(I)_d$ for $d \leq m - 1$. So the above is true for $d < m - 1$.

Now to see that $\mathrm{M}(I, m)_{m-1} \cdot S_1 \subseteq \mathrm{M}(I, m)_m$. By definition $\mathrm{M}(I, m)_{m-1} = \mathrm{lex}(I)_{m-1}$ and the degree m component is $\mathrm{M}(I, m)_m = \mathrm{M}((I : x_n^{\infty})_{[0]}, m)_m \bigoplus \mathrm{lex}\left((I : x_n)_{m-1}\right) \cdot x_n$. As vector spaces we have $\mathrm{lex}(I)_{m-1} \cdot S_1 = \left(Z_{[0]} \oplus Z_{[1]}x_n \oplus \cdots \oplus Z_{[m-1]}x_n^{m-1}\right) \cdot S_1$. As $\mathrm{lex}(I)_{m-1}$ is the

$m-1$ component of a lex ideal, each $Z_{[\mathrm{i}]}$ is itself a lexsegment, and hence this is a strongly stable vector space. So

$$\mathrm{lex}(I)_{m-1} \cdot S_1 = \left(Z_{[0]} \oplus Z_{[1]}x_n \oplus \cdots \oplus Z_{[m-1]}x_n^{m-1}\right) \cdot S_1$$

$$= Z_{[0]}\bar{S}_1 \oplus Z_{[0]}x_n \oplus Z_{[1]}x_n^2 \oplus \cdots \oplus Z_{[m-1]}x_n^m$$

To see this directly without using Remark 4.1.4, first notice that $Z_{[0]} \cdot S_1 \subseteq Z_{[0]}\bar{S}_1 \oplus Z_{[0]}x_n$. As each $Z_{[\mathrm{j}]}$ is strongly stable, we also have $Z_{[\mathrm{j}]}x_n^{\mathrm{j}} \cdot S_1 \subseteq Z_{[\mathrm{j}-1]}x_n^{\mathrm{j}} \oplus Z_{[\mathrm{j}]}x_n^{\mathrm{j}+1}$. This gives us the inclusion

$$\left(Z_{[0]} \oplus Z_{[1]}x_n \oplus \cdots \oplus Z_{[m-1]}x_n^{m-1}\right) \cdot S_1$$

$$\subseteq Z_{[0]}\bar{S}_1 \oplus Z_{[0]}x_n \oplus Z_{[1]}x_n^2 \oplus \cdots \oplus Z_{[m-1]}x_n^m$$

For the other inclusion, we have $Z_{[\mathrm{j}]}x_n^{\mathrm{j}+1} \subseteq Z_{[\mathrm{j}]}x_n^{\mathrm{j}} \cdot S_1 \oplus Z_{[\mathrm{j}+1]}x_n^{\mathrm{j}+1} \cdot S_1$ giving

$$\left(Z_{[0]} \oplus Z_{[1]}x_n \oplus \cdots \oplus Z_{[m-1]}x_n^{m-1}\right) \cdot S_1$$

$$\supseteq Z_{[0]}\bar{S}_1 \oplus Z_{[0]}x_n \oplus Z_{[1]}x_n^2 \oplus \cdots \oplus Z_{[m-1]}x_n^m$$

Note that since $Z_{[0]}$ is lex inside $\bar{S}$, it is also lex in $\bar{S}_{m-1}$. Next, write $\mathrm{M}(I,m)_m = \mathrm{M}(I,m)_{m,[0]} \oplus \mathrm{M}(I,m)_{m,[1]}x_n \oplus \cdots \mathrm{M}(I,m)_{m,[m]}x_n^m$, where possibly some of these summands are 0. Then by construction of $\mathrm{M}(I,m)_m$ we see that $\mathrm{M}(I,m)_{m,[0]} = \mathrm{M}((I:x_n^\infty)_{[0]}, m)_m$ and the tail of $\mathrm{M}(I,m)_m$, $\mathrm{M}(I,m)_{m,[1]}x_n \oplus \cdots \mathrm{M}(I,m)_{m,[m]}x_n^m$, corresponds to $\mathrm{lex}\left((I:x_n)_{m-1}\right) \cdot x_n$. As $Z_{[0]}$ are those elements of $\mathrm{lex}(I)_{m-1}$ of degree $m-1$ in I, $\mathrm{M}(I,m)_{m,[1]}$ are those elements of degree $m-1$ in $\mathrm{lex}\left((I:x_n)_{m-1}\right)$, and both $Z_{[0]}$ and $\mathrm{M}(I,m)_{m,[1]}$ are lex in $\bar{S}$ we have $Z_{[0]} \subseteq \mathrm{M}(I,m)_{m,[1]}$.

So now we show $Z_{[0]}x_n \oplus Z_{[1]}x_n^2 \oplus \cdots \oplus Z_{[m-1]}x_n^m \subseteq \mathrm{lex}\left((I:x_n)_{m-1}\right) \cdot x_n$ and $Z_{[0]}\bar{S}_1 \subseteq \mathrm{M}((I:x_n^\infty)_{[0]}, m)_m$. Note that $Z_{[0]}x_n \oplus Z_{[1]}x_n^2 \oplus \cdots \oplus Z_{[m-1]}x_n^m \subseteq \mathrm{lex}\left((I:x_n)_{m-1}\right) \cdot x_n$ is equivalent to $Z_{[0]} \oplus Z_{[1]}x_n \oplus \cdots \oplus Z_{[m-1]}x_n^{m-1} = \mathrm{lex}(I)_{m-1} \subseteq \mathrm{lex}\left((I:x_n)_{m-1}\right)$.

But this is true since $\mathrm{lex}(I)_{m-1} \subseteq \mathrm{lex}(I_{m-1}) \subseteq \mathrm{lex}((I:x_n)_{m-1})$.

Now to see that $Z_{[0]}\bar{S}_1 \subseteq \mathrm{M}((I:x_n^\infty)_{[0]}, m)_m$. We will do this by showing that $Z_{[0]} \subseteq$

$\mathrm{M}(I,m)_{m,[1]} \subseteq \mathrm{lex}(I_{m,[1]})$. By induction, $\mathrm{M}((I : x_n^{\infty})_{[0]}, m) \subseteq \overline{S}$ is a strongly stable ideal. Notice that since $(I : x_n)_{m-1,[0]} = I_{m,[1]} \subseteq (I : x_n^{\infty})_{[0]}$ by construction we have $\mathrm{lex}(I_{m,[1]}) \cdot \bar{S}_1 \subseteq \mathrm{M}((I : x_n^{\infty})_{[0]}, m)_m$. By Green's Hyperplane Theorem we can state that as vector spaces $\dim\left((S/I + \ell)_d\right) \leq \dim\left((S/\mathrm{lex}(I) + x_n)_d\right)$. Equivalently, as vector spaces we have $\dim((I + \ell)_d) \geq \dim((\mathrm{lex}(I) + (x_n))_d)$ which means $\dim(I_{d,[0]}) \geq \dim(\mathrm{lex}(I)_{d,[0]})$. When I is strongly stable we can take $x_n = \ell$ which means $\mathrm{lex}_{\overline{S}}(I_{d,[0]}) \supseteq (\mathrm{lex}_{\overline{S}}(I))_{d,[0]}$. But as $\mathrm{M}((I : x_n^{\infty})_{[0]}, m)_{m-1} = \mathrm{lex}_{\overline{S}}(I_{m,[1]})$ there is the containment $\mathrm{M}(I,m)_{m,[1]} = \mathrm{lex}((I : x_n)_{m-1})_{[0]} \subseteq \mathrm{lex}(I_{m,[1]})$. So $Z_{[0]}\overline{S}_1 \subseteq \mathrm{lex}(I_{m,[1]}) \cdot \bar{S}_1 \subseteq \mathrm{M}((I : x_n^{\infty})_{[0]}, m)_m$.
So $\mathrm{M}(I,m)$ is an ideal.
To see that it is a strongly stable ideal, it is enough to check that it is a strongly stable vector space. As a vector space is strongly stable if and only if its summands are strongly stable vector spaces and closed under multiplication by $\bar{S}_1$, it is enough to check that $\mathrm{M}(I,m)_{m,[1]} \cdot \bar{S}_1 \subseteq \mathrm{M}(I,m)_{m,[0]}$. But this again follows from Green's Hyperplane Theorem. □

The next piece we shall consider is the Hilbert function.

Hilbert Function of the Mall Ideal. For the Hilbert Function, notice that the statement holds for $n = 1$ variables and assume by induction it holds for $n-1$ variables. In n variables, induct also on the degree d of the Hilbert function. Note $\mathrm{M}(I,m)_{d+1} = \mathrm{M}(I,m)_d \cdot S_1 = \mathrm{M}(I,m)_{d,[0]} \cdot \bar{S}_1 \oplus \mathrm{M}(I,m)_d \cdot x_n$. and that for $d < m$, $M(I,m)_d = \mathrm{lex}(I)_d$, which makes the base case of induction on d true. So we can assume that for dimension less than or equal to d, the Hilbert functions are equal. Then by induction on the degree, $I_d \cdot x_n$ and $\mathrm{M}(I,m)_d \cdot x_n$ have the same dimension. By induction on the number of variables, $\mathrm{M}((I : x_n^{\infty})_{[0]}, m)$ has the same Hilbert function as $(I : x_n^{\infty})_{[0]}$. But as the regularity of $I \leq m$, in degree greater than or equal to m the Hilbert function of $(I : x_n^{\infty})_{[0]}$ is the same as the Hilbert function of $I_{[0]}$. So the dimension of $\mathrm{M}(I,m)_{d,[0]} \cdot \overline{S}_1$ is the same as $I_{[0]} \cdot \overline{S}_1$, giving us equal Hilbert functions. □

For the regularity, recall that for a strongly stable monomial ideal $I \subset S$ the regularity of I is less than or equal to the degree of the largest generator of I.

Regularity of the Mall Ideal. Notice that $\mathrm{M}(I, m)$ is a strongly stable ideal generated in degree less than or equal to m. Hence $\operatorname{reg}(\mathrm{M}(I, m)) \leq m$. $\square$

For the inequality of the graded Betti numbers we need the following extension of Green's Hyperplane Theorem.

Theorem 4.1.9. *Let B_1 and B_2 be two strongly stable vector spaces generated in degree d and let Q_1 and Q_2 be the ideals generated in S by B_1 and B_2, respectively. Assume further that the Hilbert functions are equal, i.e. $HF(Q_1) = HF(Q_2)$. Then $HF(Q_1+(x_n)) = HF(Q_2+(x_n))$.*

Proof. We prove this by induction on d. For $d = 0$ and $d = 1$ the statement is obvious. So assume that $d > 1$ First, note that Q_1 and Q_2 are both strongly stable monomial ideals as they were generated from strongly stable vector spaces. Hence the minimal graded free resolution of both Q_1 and Q_2 are linear resolutions.

$$0 \to F_{r_1} \to F_{r_1-1} \to \cdots \to F_1 \to F_0 \to Q_1$$

$$0 \to F'_{r_2} \to F'_{r_2-1} \to \cdots \to F'_1 \to F'_0 \to Q_2$$

Each free module is of the form $F_{\mathrm{i}} = \oplus_{\mathrm{j}} S(-\mathrm{j})^{\beta_{\mathrm{ij}}}$ and $F'_{\mathrm{i}} = \oplus_{\mathrm{j}} S(-\mathrm{j})^{\beta'_{\mathrm{ij}}}$ where each of these sums is a finite direct sum. Furthermore, as both Q_1 and Q_2 are generated in degree d and have a linear resolution we know that $\beta_{\mathrm{i,i+j}} = 0$ for all $\mathrm{j} \neq d$. Hence $F_{\mathrm{i}} = \oplus_{\mathrm{j}} S(-\mathrm{j})^{\beta_{\mathrm{ij}}} = S(-(\mathrm{i}+d))^{\beta_{\mathrm{i,i}+d}}$ and $F'_{\mathrm{i}} = \oplus_{\mathrm{j}} S(-\mathrm{j})^{\beta'_{\mathrm{ij}}} = S(-(\mathrm{i}+d))^{\beta'_{\mathrm{i,i}+d}}$. Using the additivity of the Hilbert series and the fact that S is a polynomial ring over a field, one gets that

$$HS_{Q_1}(t) = HS_S(t) \sum_{\mathrm{ij}} (-1)^{\mathrm{i}} \beta_{\mathrm{ij}} t^{\mathrm{j}} = \frac{1}{(1-t)^n} \sum_{\mathrm{ij}} (-1)^{\mathrm{i}} \beta_{\mathrm{ij}} t^{\mathrm{j}} = \frac{1}{(1-t)^n} \sum_{\mathrm{i}} (-1)^{\mathrm{i}} \beta_{\mathrm{i,i}+d} t^{\mathrm{i}+d}$$

and

$$HS_{Q_2}(t) = HS_S(t) \sum_{\mathrm{ij}} (-1)^{\mathrm{i}} \beta'_{\mathrm{ij}} t^{\mathrm{j}} = \frac{1}{(1-t)^n} \sum_{\mathrm{ij}} (-1)^{\mathrm{i}} \beta'_{\mathrm{ij}} t^{\mathrm{j}} = \frac{1}{(1-t)^n} \sum_{\mathrm{i}} (-1)^{\mathrm{i}} \beta'_{\mathrm{i,i}+d} t^{\mathrm{i}+d}$$

As both the projective dimension and regularity of Q_1 and Q_2 are finite, both of the above sums are finite. By the equality of the Hilbert functions it follows that $\beta_{\mathrm{i,i}+d} = \beta'_{\mathrm{i,i}+d}$ and

$r_1 = r_2$.
It follows that the socles of the algebras S/Q_1 and S/Q_2 have the same dimension. But as Q_1 and Q_2 are strongly stable ideals generated in degree d the socles of those algebras have dimension equal to the monomials of Q_1 (respectively Q_2) that are divisible by x_n. This forces equality between the Hilbert function of $Q_1 + (x_n)$ and $Q_2 + (x_n)$ in degree d. We can now consider the ideals generated by $d+1$ components of Q_1 and Q_2, respectively. As these are both ideals generated in a single degree by a strongly stable vector space the proof is done by induction.

□

Furthermore we have the following.

Theorem 4.1.10. *For any $I_1, I_2 \in Hilb(\boldsymbol{H}, m)$ where I_1 and I_2 are strongly stable we have $M(I_1, m) = M(I_2, m)$.*

Proof. First, notice that when $n = 1$ we have $S = k[x_1]$ and $\mathrm{M}(I, m) = I$. Since I is strongly stable monomial ideal with a fixed Hilbert function and regularity, I is unique with $I = (x_1^r)$ for some $r \in \mathbb{N}$.
When $n = 2$ we are in the case of $S = k[x_1, x_2]$. In this polynomial ring every strongly stable ideal is a lexsegment ideal. As the lexsegment ideal is unique for a fixed Hilbert function, we have that $\mathrm{M}(I, m) = I$ and is unique.
For $n \geq 3$ we have $S = k[x_1, \ldots, x_n]$. If $\mathrm{Hilb}(\mathbf{H}, m)$ contains only one strongly stable ideal I then $\mathrm{M}(I, m)$ is trivially unique. Otherwise, assume that there is more than one strongly stable ideal, say $I_1, I_2 \in \mathrm{Hilb}(\mathbf{H}, m)$ and that the statement holds for the Mall ideal in $n-1$ variables. Set $Q = \mathrm{M}(I_1, m)$ and $P = \mathrm{M}(I_2, m)$. For $d < m$ we have that $Q_d = P_d$ as $HF(Q) = HF(P)$ and both P and Q are lexsegments in degree d by the definition of the Mall ideal.
For $d = m$, We have

$$Q_m = \mathrm{M}((I_1 : x_n^{\infty})_{[0]}, m)_m \bigoplus \mathrm{lex}\left((I_1 : x_n)_{m-1}\right) \cdot x_n$$

and

$$P_m = \mathrm{M}((I_2 : x_n^{\infty})_{[0]}, m)_m \bigoplus \operatorname{lex}((I_2 : x_n)_{m-1}) \cdot x_n$$

Note that while $(I_1 : x_n^{\infty})_{[0]}$ and $(I_2 : x_n^{\infty})_{[0]}$ may have different Hilbert functions, these Hilbert functions will agree in degree m with the Hilbert functions of $(I_1)_{[0]}$ and $(I_2)_{[0]}$ respectively. After going module x_n, by Theorem 4.1.9 the Hilbert functions of $(I_1)_{[0]}$ and $(I_2)_{[0]}$ will also agree in degree m and above. Hence by induction we have that $\mathrm{M}((I_1 : x_n^{\infty})_{[0]}, m)_m = \mathrm{M}((I_2 : x_n^{\infty})_{[0]}, m)_m$ and $\operatorname{lex}((I_1 : x_n)_{m-1}) \cdot x_n = \operatorname{lex}((I_2 : x_n)_{m-1}) \cdot x_n$. □

In other words, for a fixed Hilbert function $\mathbf{H}$ and regularity m, the Mall ideal is unique.

Remark 4.1.11. Notice that if I is a strongly stable ideal with Hilbert function $\mathbf{H}$ and regularity m with corresponding Mall ideal $\mathrm{M}(I, m)$. Then $\mathrm{M}(I, m)/(x_n) \subset k[x_1, \ldots, x_{n-1}]$ is also a Mall ideal. This follows from the definition of the Mall ideal as in degree $d < m$ the Mall ideal is a lexsegment and the property holds for lexsegment ideals and in degree m the Mall ideal depends only on the generators of degree m.

Now to consider the inequality of the Betti numbers.

Graded Betti Numbers of the Mall Ideal. Remember that I is a strongly stable ideal. Recall that I and $J = \mathrm{M}(I, m)$ have the same Hilbert function. It is easy to see that

$$\mathrm{HF}((I + (x_n))/(x_n)) \geq \mathrm{HF}((J + (x_n))/(x_n))$$

with equalities, thanks to Theorem 4.1.9 for all degrees $d \geq m$. Furthermore, for $n = 1$ and $n = 2$ the inequality holds trivially as the Mall ideal will be the lexsegment ideal. So now consider $n \geq 3$. Since x_n is a non-zero divisor for I and J we have the equality

$$\beta_{\mathrm{ij}}^{S}(I) = \beta_{\mathrm{ij}}^{S}(I/x_n I) = \beta_{\mathrm{ij}}^{\bar{S}}\left(I_{[0]} \oplus \frac{I_{[1]}}{I_{[0]}}(-1) \oplus \frac{I_{[2]}}{I_{[1]}}(-2) \oplus \cdots\right)$$

and analogous equalities for J. Now since J is equal to $\operatorname{lex}(I)$ in all degrees $d < m$, the inequalities on the Betti numbers β_{ij} are clear for all $\mathrm{j} < \mathrm{i} + m$ thanks to the Bigatti-Hulett Pardue theorem. When $\mathrm{j} \geq \mathrm{i} + m$ the desired inequalities follow by observing first that

$$\beta_{\mathrm{ij}}^{\bar{S}}\left(I_{[0]}\right) \leq \beta_{\mathrm{ij}}^{\bar{S}}\left(\mathrm{M}(I_{[0]}, m)\right) = \beta_{\mathrm{ij}}^{\bar{S}}\left(J_{[0]}\right)$$

where the first inequality is a consequence of the induction on n. And also by observing that in degrees $d \geq m$ the $\bar{S}$ modules

$$\frac{I_{[1]}}{I_{[0]}}(-1) \oplus \frac{I_{[2]}}{I_{[1]}}(-2) \oplus \cdots$$

and

$$\frac{J_{[1]}}{J_{[0]}}(-1) \oplus \frac{J_{[2]}}{J_{[1]}}(-2) \oplus \cdots$$

have the same Hilbert functions and are both annihilated by $\bar{m}$, thus contributing in an identical manner to $\beta_{\mathrm{ij}}^{S}(I)$ and $\beta_{\mathrm{ij}}^{S}(J)$ respectively. □

4.2 The Results of Mall Over Fields of Any Characteristic

In his paper [2] Mall has two main theorems with regards to homogeneous ideals. The first is as follows.

Theorem 4.2.1 (Mall [2]). *Assume k is a field of characteristic 0 and fix a Hilbert function $\boldsymbol{H}$. For all m, provided Hilb$(\boldsymbol{H}, m) \neq \emptyset$, then there exists an ideal $J \in$ Hilb$(\boldsymbol{H}, m)$ such that $\beta_{\mathrm{ij}}(J) \geq \beta_{\mathrm{ij}}(I)$ for all $\mathrm{i}, \mathrm{j} \in \mathbb{N}$ and all $I \in$ Hilb$(\boldsymbol{H}, m)$.*

His second theorem is similar, but for saturated homogeneous ideals.

Theorem 4.2.2 (Mall [2]). *Assume k is a field of characteristic 0 and fix a Hilbert function $\boldsymbol{H}$. For all m, provided Hilb$^{sat}(\boldsymbol{H}, m) \neq \emptyset$, then there exists an ideal $J \in$ Hilb$^{sat}(\boldsymbol{H}, m)$ such that $\beta_{\mathrm{ij}}(J) \geq \beta_{\mathrm{ij}}(I)$ for all $\mathrm{i}, \mathrm{j} \in \mathbb{N}$ and all $I \in$ Hilb$^{sat}(\boldsymbol{H}, m)$.*

By using the distraction-generic initial ideal we are able to extend these results to fields of characteristic p.

Definition 4.2.3. Let $I \subset S$ be a homogeneous ideal of S with $I \in \text{Hilb}(\mathbf{H}, m)$ for a fixed Hilbert function $\mathbf{H}$. Let $<$ be the revlex order. Then $\text{M}(I, m)$ is defined as $\text{M}(I, m) = \text{M}(\text{D-gin}_<(I), m) \in \text{Hilb}(\mathbf{H}, m)$.

Note that under the revlex order, the distraction-generic ideal of an ideal I has the same regularity as I. We now have the following result.

Theorem 4.2.4. *Assume k is a field of any characteristic and fix a Hilbert function $\boldsymbol{H}$. For all m, provided $Hilb(\boldsymbol{H}, m) \neq \emptyset$, then there exists an ideal $J \in Hilb(\boldsymbol{H}, m)$ such that $\beta_{\text{ij}}(J) \geq \beta_{\text{ij}}(I)$ for all $\text{i}, \text{j} \in \mathbb{N}$ and all $I \in Hilb(\boldsymbol{H}, m)$.*

Proof. For any arbitrary ideal $I \in \text{Hilb}(\mathbf{H}, m)$ the ideal $\text{D-gin}_{revlex}(I)$ is a strongly stable monomial ideal with $HF(I) = HF(\text{D-gin}_{revlex}(I))$ and $\text{reg}(I) = \text{reg}(\text{D-gin}_{revlex}(I))$. Furthermore, as we proved in the previous chapter $\beta_{\text{ij}}(I) \leq \beta_{\text{ij}}(\text{D-gin}_{revlex}(I))$. Hence we only need to show that there exists a monomial ideal whose graded Betti numbers are larger than those of all the strongly stable ideals in $\text{Hilb}(\mathbf{H}, m)$. However, by Theorem 4.1.10 we know that for any strongly stable $I_1, I_2 \in \text{Hilb}(\mathbf{H}, m)$ that $\text{M}(I_1, m) = \text{M}(I_2, m)$. Furthermore, as we have shown in the previous section for any strongly stable ideal I we have $\text{HF}(I) = \text{HF}(\text{M}(I, m))$, $\text{reg}(\text{M}(I, m)) \leq m$, and $\beta_{\text{ij}}(\text{M}(I, m)) \geq \beta_{\text{ij}}(I)$. Hence we are done. $\square$

Recall that in general the lexsegment ideal corresponding to a given saturated ideal is not necessarily saturated. Therefore we must also define the saturated Mall ideal.

Definition 4.2.5. Let I be a saturated homogeneous ideal of S. Let $\tilde{I} = \text{D-gin}_{revlex}(I) \cap \bar{S}$. Then we define $\text{M}_{\text{sat}}(I, m) = \text{M}(\tilde{I}, m) \cdot S$, where $\text{M}(\tilde{I}, m)$ is computed inside $k[x_1, \ldots, x_{n-1}] = \bar{S}$.

Note that in the above definition one can assume without loss of generality that the base field k is infinite. Hence such a general linear form ℓ will always exist. It then follows from the above definition of $\text{M}_{\text{sat}}(I, m)$ that x_n is a nonzerodivisor of $S/\text{M}_{\text{sat}}(I, m)$.

Remark 4.2.6. Note that for a strongly stable saturated ideal I we have that $\text{HF}(\tilde{I}) = \text{HF}(\text{M}(\tilde{I}, m))$ which means that $\text{HF}(I) = \text{HF}(\text{M}_{\text{sat}}(I, m))$. Further $\text{reg}(\tilde{I}) \leq m$ gives that $\text{reg}(\text{M}(\tilde{I}, m)) \leq m$. Hence $\text{reg}(\text{M}_{\text{sat}}(I, m)) \leq m$ and $\text{M}_{\text{sat}}(I, m)$ is a strongly stable saturated ideal.

We now have the following.

Theorem 4.2.7. *Assume k is a field of any characteristic and fix a Hilbert function $\boldsymbol{H}$. For all m, provided $Hilb^{sat}(\boldsymbol{H}, m) \neq \emptyset$, then there exists an ideal $J \in Hilb^{sat}(\boldsymbol{H}, m)$ such that $\beta_{\mathrm{ij}}(J) \geq \beta_{\mathrm{ij}}(I)$ for all $\mathrm{i}, \mathrm{j} \in \mathbb{N}$ and all $I \in Hilb^{sat}(\boldsymbol{H}, m)$.*

Proof. For any arbitrary ideal $I \in \mathrm{Hilb}^{\mathrm{sat}}(\mathbf{H}, m)$ the ideal $\text{D-gin}_{revlex}(I)$ is a saturated strongly stable monomial ideal with the equalities $HF(I) = HF(\text{D-gin}_{revlex}(I))$ and $\mathrm{reg}(I) = \mathrm{reg}(\text{D-gin}_{revlex}(I))$, Furthermore, as we proved in the previous chapter there is the pointwise inequality between Betti numbers $\beta_{\mathrm{ij}}(I) \leq \beta_{\mathrm{ij}}(\text{D-gin}_{revlex}(I))$. Hence we only need to show that there exists a saturated monomial ideal whose graded Betti numbers are larger than those of all the strongly stable ideals in $\mathrm{Hilb}^{\mathrm{sat}}(\mathbf{H}, m)$. Let $I \in \mathrm{Hilb}^{\mathrm{sat}}(\mathbf{H}, m)$ be an arbitrary strongly stable ideal. As previously stated by the definition of $\mathrm{M}_{\mathrm{sat}}(I, m)$ we have that x_n is a nonzerodivisor of $S/\mathrm{M}_{\mathrm{sat}}(I, m)$, $\mathrm{HF}(I) = \mathrm{HF}(\mathrm{M}_{\mathrm{sat}}(I, m))$, $\mathrm{reg}(\mathrm{M}_{\mathrm{sat}}(I, m)) \leq m$, and $\mathrm{M}_{\mathrm{sat}}(I, m)$ is a strongly stable saturated ideal. As going modulo x_n does not change the Betti numbers of a strongly stable saturated ideal $\beta_{\mathrm{ij}}(\tilde{I}) \leq \beta_{\mathrm{ij}}(\mathrm{M}(\tilde{I}, m))$. It follows that $\beta_{\mathrm{ij}}(I) \leq \beta_{\mathrm{ij}}(\mathrm{M}_{\mathrm{sat}}(I, m))$ and we are done.

□

4.3 Green's Hyperplane Section Theorem

Recall the statement of Green's Hyperplane Section theorem, in its strongest form, which was introduced at the beginning of this chapter.

Theorem 4.3.1 ([19] [20] [21]). *Assume $|k| = \infty$ and let $I \subseteq S$ be a homogeneous ideal with Hilbert function $\boldsymbol{H}$ and ℓ a general linear form. Then for all $d \geq 1$*

$$HF(S/(I + (\ell^d))) \leq HF(S/(lex(I) + (x_n^d))).$$

Furthermore when I is strongly stable, ℓ can be chosen to be x_n.

The extremal properties of the Mall ideal give us the following result, which is similar to the strongest form of the hyperplane theorem.

Theorem 4.3.2. *Let $I \subseteq S$ be a strongly stable ideal. Then*

$$HF(S/(I + (x_n^d))) \leq HF(S/(M(I, m) + (x_n^d))).$$

Proof. For degree t, $t < m$ this follows immediately from Green's Hyperplane section theorem as $\mathrm{M}(I, m)_t = \mathrm{lex}(I)_t$ for $t < m$. For degree $t = m$ when $d = 1$ we have equality between $\mathrm{HF}(S/(I + (\ell)))$ and $\mathrm{HF}(S/(\mathrm{lex}(I) + (x_n)))$ in degree m due to Theorem 4.1.9. When $d > 1$ this theorem follows again from Green's Hyperplane Section theorem due to the construction of the Mall ideal, because in particular we have

$$\mathrm{HF}(S/((I : x_{n-1})_{m-1} + (x_n^d))) \leq \mathrm{HF}(S/(\mathrm{lex}((I : x_{n-1})_{m-1}) + (x_n^d))).$$

But as noted in the definition of the Mall ideal, $I_m = I_{m,[0]} \oplus (I : x_n)_{m-1} \cdot x_n$. As the Hilbert functions of $I_{m,[0]}$ and $\bar{M}$ are identical the inequality follows.

For $d > m$ this statement follows from the above reasoning in conjunction with Remark 4.1.4. $\square$

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